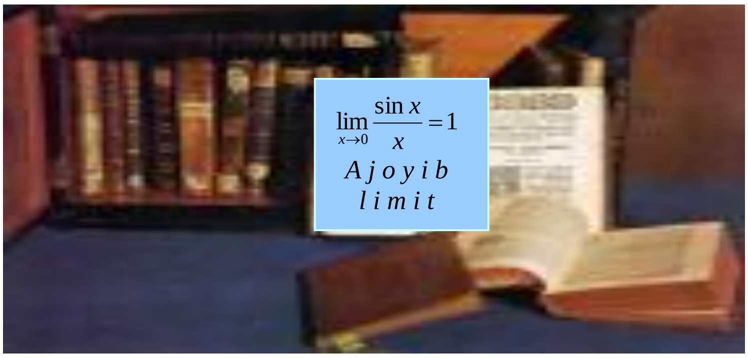
O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

Jizzax Politexnika Instituti

«Oliy matematika» kafedrasi

Limitlar

(O'quv qo'llanma)



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Mazkur o’quv qo’llanma “Oliy matematika” fani o’quv dasturi asosida tayyorlandi va “Oliy matematika” kafedrasining (21.12.2005 yil, №4), “Aftomexanika” fakulteti (18.01.2006 yil, №6) ilmiy – uslubiy kengashlari tomonidan eshitilib institut uslubiy kengashiga tasdiqlash uchun tavsiya etildi.

Jizzax politexnika institut (27.01.2006 yil, №5) ilmiy – uslubiy kengashi tomonidan nashrga tavsiya etilgan.

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Egamqulov Sh va Berdiyorov A “”Limitlar” (o’quv qo’llanma) - Jizzax-2006, 62 bet

A N N O T A T S I Y A

Mazkur o’quv qo’llanma texnika oliy o’quv yurtlarining «Oliy matematika» fani o’quv dasturi asosida yozilgan bo’lib, texnika oliy o’quv yurtlari fakultetlari talabalari uchun mo’ljallangan. Undan **«Limitlar»** bo’limini o’rganishda boshqa o’quv yurti talabalari qo’shimcha adabiyot sifatida foydalanishlari mumkin.

Har bir mavzuga doir misollarning yechimi va mustaqil bajarish uchun topshiriqlar keltirilgan.

M u h a r r i r: **t.f.n.dots. T.Abduazizov**

K I R I S H.

Ushbu o’quv qo’llanma **“Matematik tahlil”** ning asosiy tushunchalari orasida markaziy o’rinni ishg’ol etadigan, shu bilan birga, turli ixtisosliklarning barcha mutaxasislik fanlarida keng qo’llashga to’g’ri keladigan **“Limitlar”** nazariyasiga bag’ishlangan.

Limit tushunchasining ta’rifi birinchi marta Djon Vallis (1616-1705 ingliz matematigi) “Cheksiz kichik miqdorlarning arifmetikasi” da (1665 yil) yozilgan. I.Nyuton (1642-1727 – ingliz fizigi va matematigi) o’zining mashhur **“Natural filosofiyaning matematik boshlang’ichlari”** da (1686-1687) birinchi va oxirgi nisbatlar (yoki yig’indi) usulini e’lon qildi; bunda limitlar nazariyasining boshlang’ich elementlarini ko’rish mumkin. Lekin yangi hisobni limit tushunchasidan foydalanib asoslash mumkinligi XVII asr mashhur matematiklaridan birortasining ham hayoliga kelgan emas; shunday qilinganda yangi hisoblashga qarshi chiqishlarga javob berilgan bo’lar edi. Bu ma’noda Eyler **«Differensial hisob»** (1755) so’z boshida limit haqida ochiq gapiradi, lekin kitobning biror yerida ham bu tushunchadan foydalanilmaydi !

O.L. Koshi (1789-1857 fransuz analisti) ning **«Algebraik analizi»** (1821) va uning navbatdagi nashrlari bu ko’rsatilgan masalada burilish hosil qildi; matematik tahlilning hammasini qat’iy tuzib chiqish uchun Koshi qo’lida asosiy kuch sifatida xizmat qilgan limitlar nazariyasi birinchi marta bu asarlarda taraqqiy ettirildi. Koshi bajargan yo’l tahlil boshlangandagi barcha noaniqliklarni (tumanni) tarqatdi va umum tomonidan tan olindi.

Shu bilan birga Koshi qilgan ishlarni boshqa olimlar ham bajargan, masalan, ular ichida B.Bolsano (1781-1848 chex faylasufi va matematigi) maxsus o’rin egallaydi.

Limit tushunchasining turli-tuman shakllarda uchrab turishi, bu shakllarning hammasini bitta ko'rinishga keltirish masalasini qo'yadi. Bu maqsadga erishishning ikki yo'li bor: yo (masalan Shatunovskiy va Mur-Smit izidan borib) **«Tartiblangan o'zgaruvchi»** limitining eng umumiy ta'rifini berish yoki G.M.Fixtengolts usuli – har qanday limitni eng sodda holda – nomerlangan qiymatlar ketma-ketligini qabul qiladigan o'zgaruvchining limitiga keltirish kerak. Birinchi yo'l yangi o'rganuvchilarga qiyin, shu sababli biz ikkinchi yo'lni tanladik. Limitning har bir yangi shakliga ta'rifni avvalo ketma-ketliklar limiti yordamida berib, nihoyat uning ketidan **«ε - δ tilida»** ifodaladik.

«lim» belgi qisqartirilgan lotincha **«Limes»** so'zining birinchi uchta harifidir, u o'zbek tilida marra, chegara (limit) ma'nosini anglatadi.

I – BOB. SONLAR KETMA – KETLIGINING LIMITI.

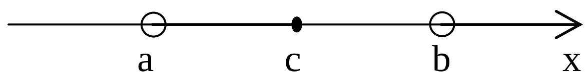
Funksiya limiti matematik tahlilning muhim tushunchalaridan biri. Dastlab, nuqtaning atrofi, sonli to'plamning limit nuqtasi tushunchalarini, so'ngra natural argumentli funksiyalar (sonlar ketma-ketligi) ning limitini qaraymiz.

1-§. Nuqtaning atrofi.

Terminologiyaga doir quyidagi izohni beramiz. Haqiqiy sonlarni sonlar o'qidagi nuqtalar bilan tasvirlaganimiz sababli, nuqta deganda sonni tushunamiz va aksincha. Bunda geometrik termin arifmetik ma'noga ega; nuqta so'zi bunda son so'zi o'rnida ishlatiladi.

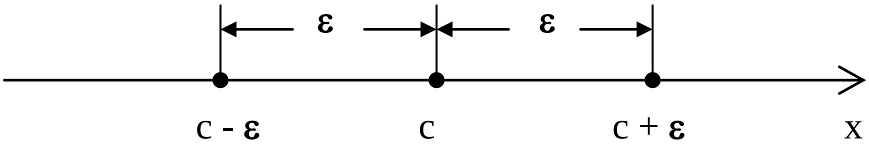
Yuqorida aytib o'tilganlarni e'tiborga olib, quyidagi ta'rifni beramiz.

Ta'rif. *C nuqtaning atrofi deb, o'rtasi C nuqta bo'lgan istalgan (a, b) intervalga aytiladi. (1-shakl)*



1 – s h a k l.

Har bir C nuqtaning cheksiz ko'p atrofi bo'ladi, chunki u cheksiz ko'p intervalning o'rtasidir. Arifmetika nuqtai nazaridan qaraganda C nuqtaning atrofini $(c-\varepsilon,\ c+\varepsilon)$ interval shaklida tasvirlash mumkin, bunda ε ixtiyoriy musbat son yoki ikkinchi xil aytganda $c-\varepsilon < x < c+\varepsilon$ $(\varepsilon > 0)$ shartni qanoatlantiruvchi x sonlarning to'plamidir (2-shakl).



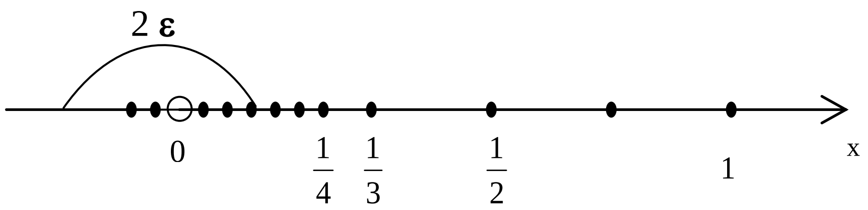
2 – s h a k l.

Masalan, koordinatasi 4 ga teng nuqtaning atroflari (4-**ε**, 4+**ε**) intervaldan iborat; **ε** ga istalgan musbat qiymatlarni berib, koordinatasi 4 ga teng nuqtaning hamma atroflarini hosil qilamiz. Tanlangan **ε** uchun 4-**ε** < x < 4 + **ε**, yoki | x–4 | < **ε** bajarilsa, x nuqta ko'rsatilgan atrofga tegishli bo'ladi. **ε** ni atrofning radiusi deb, atrofning o'zi esa **ε** atrof deb ataladi.

2 – §. Sonli to'plamning limit nuqtasi.

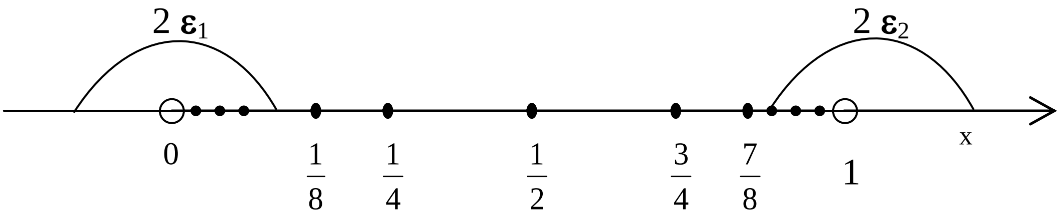
Sonlarni O x sonlar o'qida nuqtalar bilan tasvirlaymiz.

1°. M_1 sonlar to'plami (3-shakl): $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$



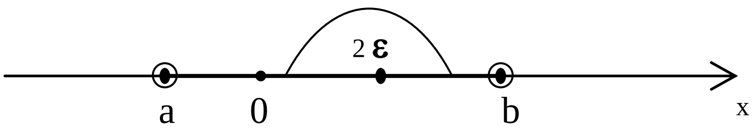
3 – s h a k l.

2°. M_2 sonlar to'plami (4-shakl): $1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \ldots, \frac{1}{2^k}, 1-\frac{1}{2^k}, \ldots$



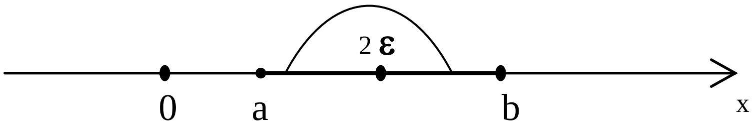
4 – s h a k l.

3°. M_3 : (a , b) interval (5-shakl)



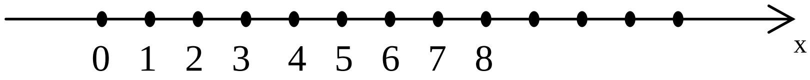
5 – s h a k l.

4^o. $M_4 : [a \; ; \; b]$ segment (6-shakl)



6 – s h a k l.

5^o. M_5 : 1,2,3, …, n, … (7-shakl)



7 – s h a k l.

Ta'rif. *Agar **a** nuqtaning istalgan atrofida **M** sonlar to'plamiga tegishli cheksiz ko'p nuqtalar yotsa **a** nuqta **M** sonlar to'plamining limit nuqtasi deb ataladi.*

Chunonchi, O nuqta $\mathbf{M_1}$ to'plamning limit nuqtasidir, chunki **O** nuqtaning istagan atrofida $\mathbf{M_1}$ to'plamga qarashli nuqtalar cheksiz ko'pdir (3-shakl). **O** nuqtaning o'zi $\mathbf{M_1}$ to'plamga kirmaydi.

$\mathbf{M_2}$ to'plam **O** nuqta va **1** nuqtadan iborat ikkita limit nuqtaga ega (4-shakl). **O** limit nuqta to'plamga kirmaydi; **1** limit nuqta $\mathbf{M_2}$ to'plamga kiradi. Shuning uchun **1** nuqta $\mathbf{M_2}$ to'plamning nuqtasidir.

(a ; b) intervalning barcha nuqtalari $\mathbf{M_3}$ to'plamning limit nuqtalaridir.

Intervalning **a** va **b** uchlari ham uning limit nuqtalaridir (5- chizma), ammo **a** va **b** nuqtalar intervalga kirmaydi.

[a , b] segmentning hamma nuqtalari bu segmentning limit nuqtalaridir. **[a , b]** segmentning o'ziga kirmagan limit nuqtalari bo'lmaydi.

$\mathbf{M_5}$ to'plamning limit nuqtalari yo'q.

Shuningdek, sonlarning istagan chekli to'plami ham limit nuqtalarga ega emas.

3 – §. Sonlar ketma-ketligi tushunchasi.

Aytaylik , N={1,2,3, …} to'plamda biror $f(n)$ funksiya berilgan bo'lsin. Bu funksiya qiymatlarini $\mathbf{x_n}$ bilan belgilaymiz.

$f(n) = x_n$ (1). ($f(1) = x_1$, $f(2) = x_2$, … , $f(n) = x_n$, …).

Qaralayotgan funksiya qiymatlaridan tashkil topgan ushbu x_1 , x_2 , … , x_n, , … to'plam *sonlar ketma – ketligi* deyiladi.

(1) ketma-ketlikni tashkil etgan x_n (n = 1,2,3, …) sonlar uning *hadlari* deyiladi: x_1– ketma – ketlikning birinchi hadi, x_2 – ketma-ketlikning ikkinchi hadi va hokazo, x_n– ketma – ketlikning n – hadi (yoki umumiy hadi). (1) ketma-ketlik qisqacha x_n yoki $\{x_n\}$ kabi belgilanadi.

Ko'p holda ketma-ketliklarning umumiy hadi formula bilan ifodalanadi. Uning barcha hadlari shu formula orqali topiladi.

M a s a l a l a r.

1. $x_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$
2. $x_n = n$: $1, 2, 3, \dots, n, \dots$
3. $x_n = 1$: $1, 1, 1, \dots, 1, \dots$
4. $x_n = (-1)^{n+1}$: $1, -1, 1, \dots, (-1)^{n+1}, \dots$

Biror $\{x_n\}$: x_1, x_2, … , x_n, … ketma- ketlik berilgan bo'lsin.

1-ta'rif. *Agar shunday o'zgarmas M son mavjud bo'lsaki, $\{x_n\}$ ketma – ketlikning har bir hadi shu sondan katta bo'lmasa, ya'ni $\forall n \in N$ uchun $x_n \le M$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ yuqoridan chegaralangan ketma – ketlik deyiladi.*

2-ta'rif. *Agar shunday o'zgarmas m son mavjud bo'lsaki, $\{x_n\}$ ketma – ketlikning har bir hadi shu sondan kichik bo'lmasa, ya'ni $\forall n \in N$ uchun $x_n \geq m$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ quyidan chegaralangan ketma – ketlik deyiladi.*

3-ta'rif. *Agar ketma – ketlik ham quyidan, ham yuqoridan chegaralangan bo'lsa, ya'ni shunday o'zgarmas m va M sonlar topilsaki, $\forall n \in N$ uchun $m \leq x_n \leq M$ tengsizliklar o'rinli bo'lsa, $\{x_n\}$ chegaralanagan ketma – ketlik deyiladi.*

Misollar.

1. $x_n = \frac{n+1}{n}$ ketma – ketlik quyidan va yuqoridan chegaralangan, chunki $x_n = \frac{n+1}{n} > 1$, ya'ni ketma-ketlik quyidan chegaralangan. Ikkinichi tomondan, $\frac{n+1}{n} = 1 + \frac{1}{n}$ ga egamiz, bu yerda $\frac{1}{n}$ to'g'ri kasr, demak, $1 + \frac{1}{n} < 2$, ya'ni ketma – ketlik yuqoridan chegaralangan. (m=1, M=2).

2. $x_n = \frac{1}{2^{n-1}}$ ketma – ketlik quyidan chegaralangan, chunki ketma – ketlikning har bir hadi O dan kichik emas (m=0).

3. 0, -1, -2, … ,-n, … ketma – ketlik yuqoridan chegaralangan, chunki ketma – ketlikning har bir hadi O dan katta emas. (M=0).

4- ta'rif. *Agar $\{x_n\}$ ketma – ketlikning hadlari quyidagi $x_1 \le x_2 \le \ldots \le x_n \le \ldots (x_1 < x_2 < \ldots < x_n < \ldots)$ tengsizliklarni qanoatlantirsa, ya'ni $\forall n \in N$ uchun $x_n \le x_{n+1} (x_n < x_{n+1})$ bo'lsa, $\{x_n\}$ o'suvchi (qat'iy o'suvchi) ketma – ketlik deyiladi.*

5-ta'rif. *Agar $\{x_n\}$ ketma – ketlikning hadlari quyidagi $x_1 \ge x_2 \ge \ldots \ge x_n \ge \ldots$ $(x_1 > x_2 > \ldots > x_n > \ldots)$ tengsizliklarni qanoatlantirsa, ya'ni $\forall n \in N$ uchun $x_n \ge x_{n+1}$ $(x_n > x_{n+1})$ bo'lsa, $\{x_n\}$ kamayuvchi (qat'iy kamayuvchi) ketma-ketlik deyiladi.*

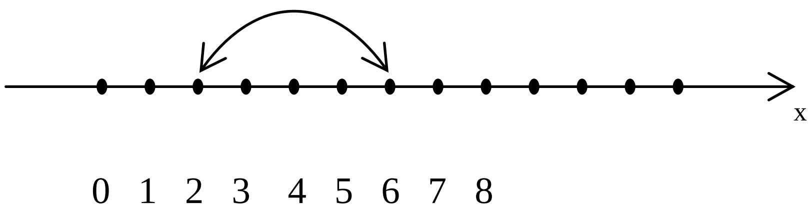
O'suvchi (qat'iy o'suvchi), kamayuvchi (qat'iy kamayuvchi) ketma – ketliklar *monoton ketma – ketliklar* deyiladi.

M i s o l. $x_n = \frac{1}{n^2 - 1}$ ketma – ketlik monoton kamayuvchi ketma – ketlikdir, chunki kamayuvchi ketma – ketlik uchun $x_{n+1} < x_n \Rightarrow \frac{x_{n+1}}{x_n} < 1$, tengsizlik bajariladi. Ketma – ketlikning (n+1) hadini yozamiz: $x_{n+1} = \frac{1}{(n+1)^2 - 1} = \frac{1}{n^2 + 2n}$. U holda $\frac{x_{n+1}}{x_n} = \frac{n^2 - 1}{n^2 + 2n} < 1$, chunki $\forall n \in N$ uchun $n^2 - 1 < n^2 + 2n$ bo'ladi. Berilgan ketma – ketlik kamayuvchidir.

4 – §. Sonlar ketma – ketligining limiti

Biror $\{x_n\}$: $x_1, x_2, \ldots, x_n, \ldots$ ketma – ketlik hamda biror a nuqta (son) berilgan bo'lsin. Bu ketma – ketlikning hadlari a nuqtaning biror atrofiga tegishli bo'ladimi, tegishli bo'lsa, nechta hadi tegishli bo'ladi – shularni aniqlash ketma – ketlikning limiti tushunchasini kiritishda muhim ahamiyatga ega.

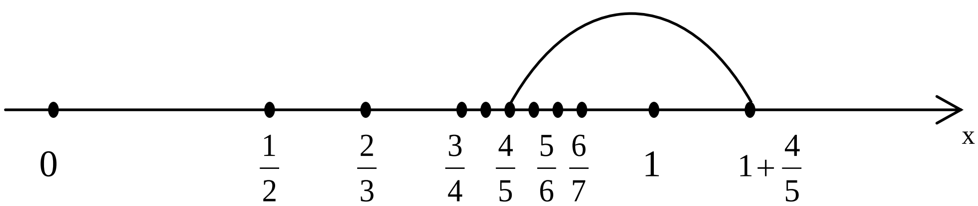
M i s o l l a r. 1. Ushbu $\{x_n\}=\{n\}$: 1,2,3, … , n, … ketma – ketlikni hamda a=4 nuqtaning (4-2; 4+2)=(2 ; 6) atrofi ($\boldsymbol{\varepsilon}$ =2) olinsa, unda ketma – ketlikning 3 ta hadi (3;4;5-hadlari) shu atrofga tegishli bo'ladi. (8-shakl).



8 – s h a k l.

Agar a=2 nuqta olinsa va uning $\left(-\frac{1}{4};\frac{1}{4}\right)$ atrofi qaralsa, unda berilgan x_n=n ketma – ketlikning bitta ham hadi shu atrofga tegishli bo'lmasligini ko'ramiz.

2. Ushbu $\{x_n\}=\left\{\frac{n}{n+1}\right\}$: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ ketma – ketlikda a=1 nuqtaning $\left(1-\frac{4}{5};1+\frac{4}{5}\right)=\left(\frac{1}{5};\frac{9}{5}\right)\left(\varepsilon=\frac{4}{5}\right)$ atrofi olinsa, unda berilgan ketma – ketlikning 4-hadidan boshlab keyingi barcha hadlari shu atrofga tegishli bo'ladi. (9-shakl).



9 – s h a k l.

Yuqorida keltirilgan misollardan ko'rinadiki, biror nuqta atrofga ketma – ketlikning chekli sondagi hadlari tegishli bo'lishi, biror hadidan boshlab keyingi barcha hadlari, jumladan ketma – ketlikning barcha hadlari (cheksiz sondagi hadlari) tegishli bo'lishi, bitta ham hadi tegishli bo'lmasligi mumkin ekan.

T a' r i f. Agar a nuqtaning ixtiyoriy (a-**ε**, a+**ε**) atrofi (∀**ε**>0) olinganda ham $\{x_n\}$ ketma – ketlikning biror hadidan boshlab, keyingi barcha hadlari shu atrofga tegishli bo'lsa, a son $\{x_n\}$ ketma – ketlikning limiti deyiladi va $\lim_{n\to\infty} x_n = a$ $\left(yoki \quad \lim x_n = a \quad yoki \quad x_n \to a\right)$ kabi belgilanadi.

Bu holda $\{x_n\}$ ketma – ketlik a ga intiladi deb ham yuritiladi. $\{x_n\}$ ketma – ketlikning biror hadidan boshab keyingi barcha hadlari a nuqtaning ixtiyoriy (a-**ε**, a+**ε**) atrofiga tegishliligi, ∀**ε** >0 son olinganda ham shunday natural n_0 son topilib, barcha $n > n_0$ uchun $a-\varepsilon<x_n< a+\varepsilon$ tengsizliklarning o'rinli bo'lishidan iboratdir.

Ravshanki, $a-\varepsilon < x_n< a + \varepsilon \Leftrightarrow -\varepsilon < x_n - a < \varepsilon \Leftrightarrow |x_n - a| < \varepsilon$.

Bu hol ketma – ketlik limitini quyidagicha ta'riflash imkonini beradi.

T a' r i f. *Agar* ∀**ε** *>0 son olinganda ham shunday natural n_0 son ($n_0 \in N$) topilsaki, barcha $n>n_0$ uchun $|x_n - a|<$* **ε** *tengsizlik bajarilsa, a son $\{x_n\}$ ketma – ketlikning limiti deyiladi va yuqoridagidek* $\lim_{n\to\infty} x_n = a$ *kabi belgilanadi.*

1–i z o h. O’zgarmas miqdor c ko’pincha hamma qiymatlari bir xil: x=c bo’lgan o’zgaruvchi miqdor deb qaraladi.

O’zgarmas miqdorning limiti shu o’zgarmas miqdorning o’ziga teng, chunki $\forall \varepsilon > 0$ da $|x - c| = |c - c| = 0 < \varepsilon$ tengsizlik doimo bajariladi.

2–i z o h. Limitning ta’rifidan o’zgaruvchi miqdor ikkita limitga ega bo’la olmasligi kelib chiqadi. (Isboti [2], 29-bet)

3–i z o h. Har bir o’zgaruvchi miqdor limitga ega deb o’ylash yaramaydi.

5 – §. Cheksiz kichik hamda cheksiz katta miqdorlar.

Agar o’zgaruvchi miqdor a limitga intilsa, u holda a o’zgarmas son ekani limitning ta’rifidan ko’rinadi. Ammo, **«intiladi»** tushunchasi o’zgaruvchi miqdorning boshqacha o’zgarish usulini tavsiflash uchun ham ishlatiladi.

Agar oldindan berilgan har bir musbat M son uchun shunday $n_0 \in N$ son topilsaki, barcha $n > n_0$ uchun $|x_n| > M$ tengsizlik o’rinli bo’lsa, $\{x_n\}$ ketma – ketlikning limiti (∞) deb qaraladi va $\lim\limits_{n\to\infty} x_n = \infty$ yoki $x_n \to \infty$ kabi belgilanadi.

Agar har qanday M > 0 son berilganda ham shunday $n_0 \in N$ son topilsaki, barcha $n > n_0$ uchun $x_n > M$ ($x_n < -M$) tengsizlik o’rinli bo’lsa, $\{x_n\}$ ketma – ketlikning limiti $+\infty$ ($-\infty$) deb qaraladi.

T a' r i f. *Agar* $\{x_n\}$ *ketma – ketlikning limiti cheksiz* $\lim_{n\to\infty} x_n = \infty$, *bo'lsa, u holda* $\{x_n\}$ *cheksiz katta miqdor deyiladi.*

Masalan, $x_n=2n$ *ketma – ketlik cheksiz katta miqdor bo'ladi, chunki* $\lim_{n\to\infty} 2n = \infty$

T a' r i f. *Agar* $\{x_n\}$ *ketma – ketlikning limiti 0 ga teng bo'lsa,* $\lim_{n\to\infty} x_n = 0$, *u holda* $\{x_n\}$ *cheksiz kichik miqdor deyiladi.*

Masalan, $x_n = \dfrac{1}{n+1}$ ketma – ketlik cheksiz kichik miqdor bo'ladi, chunki

$$\lim_{n\to\infty} \frac{1}{n+1} = 0.$$

T a' r i f. *Agar* $\{x_n\}$ *ketma – ketlikning limiti chekli son bo'lsa, uni yaqinlashuvchi ketma – ketlik, agar ketma – ketlikning limiti cheksiz yoki ketma – ketlik limitga ega bo'lmasa, uni* *uzoqlashuvchi ketma – ketlik* *deyiladi.*

T e o r e m a. $\{x_n\}$ *ketma – ketlikning a limitga ega bo'lishi uchun* $\alpha_n = x_n - a$ *cheksiz kichik miqdor bo'lishi zarur va etarli. Demak,* $\{x_n\}$ *ketma – ketlikning limiti a bo'lsa, uning umumiy hadi* x_n *ni* $x_n = a + \alpha_n$ *ko'rinishda yozish mumkin, bunda* α_n *cheksiz kichik miqdor va aksincha.*

M i s o l. $\{x_n\} = \left\{\dfrac{n}{n+1}\right\}$ ketma – ketlikda: $x_n = \dfrac{n}{n+1} = 1 - \dfrac{1}{n+1}$.

Ravshanki, $\alpha_n = \dfrac{1}{n+1}$ cheksiz kichik miqdor. Demak, berilgan ketma – ketlikning limiti 1 ga teng: $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$.

1–Lemma. Ikki cheksiz kichik miqdor yig'indisi yana cheksiz kichik miqdor bo'ladi.

2–Lemma. Chegaralangan ketma – ketlik bilan cheksiz kichik miqdor ko'paytmasi cheksiz kichik miqdor bo'ladji. (Lemmalar isboti [4] 86-betda berilgan.)

Endi cheksiz kichik hamda cheksiz katta miqdorlar orasidagi bog'lanishni keltiramiz:

1^0. Agar $\{x_n\}$ $(x_n \neq 0, \quad n = 1,2,...)$ cheksiz kichik miqdor bo'lsa, $\left\{\dfrac{1}{x_n}\right\}$ cheksiz katta miqdor bo'ladi.

2^0. Agar $\{x_n\}$ cheksiz katta miqdor bo'lsa, $\left\{\dfrac{1}{x_n}\right\}$ cheksiz kichik miqdor bo'ladi.

6 – §. Yaqinlashuvchi ketma – ketliklar va ularning xossalari.

Ketma – ketlik limitining mavjudligi (ya'ni yaqinlashuvchi) haqidagi masala muhim masalalardan biridir. Bu masalani hal qilib beruvchi teoremalar mavjud. Biroq ular matematik tahlilning nozik faktlariga asoslanib isbotlanadi. (Isboti [3], 194-bet).

1–teorema. Agar $\{x_n\}$ ketma – ketlik o'suvchi bo'lib, yuqoridan chegaralangan bo'lsa, ketma – ketlik chekli limitga ega (ya'ni yaqinlashuvchi) bo'ladi.

2–teorema. Agar $\{x_n\}$ ketma – ketlik kamayuvchi bo'lib, quyidan chegaralangan bo'lsa, ketma – ketlik chekli limitga ega (ya'ni yaqinlashuvchi) bo'ladi.

T a' r i f. Agar $\forall \varepsilon > 0$ son olganda ham shunday $n_0 \in N$ topilsaki, barcha $n > n_0$, barcha $m > n_0$ uchun $|x_n - x_m| < \varepsilon$ tengsizlik bajarilsa, $\{x_n\}$ *fundamental ketma-ketlik* deyiladi.

Har qanday yaqinlashuvchi ketma – ketlik fundamental ketma – ketlik bo'ladi.

3–teorema. (Koshi teoremasi). Agar $\{x_n\}$ ketma – ketlik fundamental ketma – ketlik bo'lsa, u yaqinlashuvchi bo'ladi.

Yaqinlashuvchi ketma – ketliklar qator xossalarga ega. Bu xossalarni isbotsiz keltiramiz.

1⁰. Agar $\{x_n\}$ ketma – ketlik yaqinlashuvchi bo'lsa, uning limiti yagona bo'ladi.

2⁰. Agar $\{x_n\}$ ketma – ketlik yaqinlashuvchi bo'lsa, chegaralangan bo'ladi.

3⁰. Agar $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar yaqinlashuvchi bo'lsa, u holda $\{x_n \pm y_n\}$ ketma – ketlik ham yaqinlashuvchi va $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n$ bo'ladi.

4⁰. Agar $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar yaqinlashuvchi bo'lsa, u holda $\{x_n \cdot y_n\}$ ketma – ketlik ham yaqinlashuvchi va $\lim\limits_{n\to\infty}(x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$ bo'ladi.

N a t i j a. Agar $\{x_n\}$ ketma – ketlik yaqinlashuvchi bo'lsa, $\{c \cdot x_n\}$ ketma – ketlik ham yaqinlashuvchi va $\lim\limits_{n\to\infty} c \cdot x_n = c \cdot \lim\limits_{n\to\infty} x_n$ bo'ladi, bu yerda c – o'zgarmas son.

5⁰. Agar $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar yaqinlashuvchi bo'lib, $y_n \neq 0 \quad (n = 1,2,3,...)$ va $\lim\limits_{n\to\infty} y_n \neq 0$ bo'lsa, u holda $\left\{\frac{x_n}{y_n}\right\}$ ketma – ketlik ham yaqinlashuvchi va $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$ bo'ladi.

6⁰. Agar $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar yaqinlashuvchi bo'lib, ∀n∈N da $x_n \leq y_n \quad (x_n \geq y_n)$ bo'lsa, u holda $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n \left(\lim\limits_{n\to\infty} x_n \geq \lim\limits_{n\to\infty} y_n\right)$ bo'ladi.

7⁰. Agar $\{x_n\}$ va $\{z_n\}$ ketma – ketliklar yaqinlashuvchi va $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = a$ bo'lib $\forall n \in N$ da $x_n \le y_n \le z_n$ bo'lsa, u holda $\{y_n\}$ ketma – ketlik ham yaqinlashuvchi va $\lim\limits_{n\to\infty} y_n = a$ bo'ladi.

8⁰. Agar $\{x_n\}$ ketma – ketlik yaqinlashuvchi bo'lib, $\lim\limits_{n\to\infty} x_n = a$ bo'lsa, u holda $x_n = a + \alpha_n$ bo'ladi va aksincha, bunda α_n cheksiz kichik miqdor.

7 – §. Sonlar ketma-ketliklari limitini hisoblash.

Sonlar ketma – ketligi mavzusining asosiy masalalaridan biri uning limitini topishdan iborat. Ketma – ketliklarning limitlarini topishda ta'rifdan va ketma – ketlik limitining xossalaridan foydalaniladi.

1 – m i s o l. Ketma – ketlik limitining ta'rifidan foydalanib isbotlang.

a) $\lim x_n = 1$, agar $x_n = \dfrac{2n-1}{2n+1}$; b) $\lim x_n = \dfrac{3}{5}$, agar $x_n = \dfrac{3n^2+1}{5n^2-1}$ bo'lsa va qaysi n nomerdan boshlab $\left| x_n - \dfrac{3}{5} \right| < 0{,}01$ tengsizlik bajarilishi ko'rsatilsin.

Y e c h i s h. a) Har qanday $\varepsilon > 0$ uchun shunday $N(\varepsilon)$ son topiladiki, $n > N(\varepsilon)$ da $|x_n - 1| < \varepsilon$ tengsizlik bajariladi. Buning uchun absalyut qiymat ayirmasini topamiz: $\left| \dfrac{2n-1}{2n+1} - 1 \right| = \left| \dfrac{-2}{2n+1} \right| = \dfrac{2}{2n+1}$

Demak, $|x_n - 1| < \varepsilon$ tengsizlik bajariladi, agar $\frac{2}{2n+1} < \varepsilon$, bundan $n > \frac{1}{\varepsilon} - \frac{1}{2}$. Shuning uchun $N(\varepsilon)$ sifatida $\frac{1}{\varepsilon} - \frac{1}{2}$ sonning butun qismi, ya'ni $N = \mathrm{E}\left(\frac{1}{\varepsilon} - \frac{1}{2}\right)$ ni olamiz.

Shunday qilib, har qanday $\varepsilon > 0$ uchun N son topiladiki, n>N bo'lganda $|x_n - 1| < \varepsilon$ bajariladi, bu esa $\lim\limits_{n\to\infty} \frac{2n-1}{2n+1} = 1$.

b) Endi $\left|x_n - \frac{3}{5}\right|$ absalyut miqdor farqini topamiz: $\left|\frac{3n^2+1}{5n^2-1} - \frac{3}{5}\right| = \frac{8}{5(5n^2-1)}$.

$\varepsilon > 0$ berilgan **n** ni shunday tanlaymizki $\frac{8}{5(5n^2-1)} < \varepsilon$ tengsizlik bajarilsin.

Bu tengsizlikni yechib: $n^2 > \frac{8}{25\varepsilon} + \frac{1}{5} \Rightarrow n > \frac{1}{5}\sqrt{\frac{8+5\varepsilon}{\varepsilon}}$.

$N = \mathrm{E}\left(\frac{1}{5}\sqrt{\frac{8+5\varepsilon}{\varepsilon}}\right)$ deb, $n > N$ da $|x_n - 3{,}5| < \varepsilon$.

Agar $\varepsilon = 0{,}01$ bo'lsa, $N = \mathrm{E}\left(\frac{1}{5}\sqrt{\frac{8+5\varepsilon}{\varepsilon}}\right) = \mathrm{E}\left(\frac{1}{5}\sqrt{805}\right) = 5$, bundan ketma – ketlikning 6-hadidan boshlab barcha hadlari $\left(\frac{3}{5} - 0{,}01; \quad \frac{3}{5} + 0{,}01\right)$ intervalda yotadi.

2 – m i s o l. Ushbu 0,1; 0,11; 0,111; … ketma – ketlikning limiti $\frac{1}{9}$ ga teng.

I s b o t. Buning uchun $\frac{1}{9} - 0{,}1 = \frac{1}{9\cdot 10} < \frac{1}{10}$; $\frac{1}{9} - 0{,}11 = \frac{1}{9\cdot 100} < \frac{1}{100}$ … , $\frac{1}{9} - \underbrace{0{,}11\ldots1}_{\text{n ta raqam}} = \frac{1}{9\cdot 10^n} < \frac{1}{10^n}$ ekanligini e'tiborga olib, $\frac{1}{9} - a_n$ ayirma 0,11… davriy kasrda o'nli kasr xonalarini yetarli darajada ko'proq olish bilan ixtiyoriy kichik $\varepsilon > 0$ sondan ham kichik bo'lishi mumkin ekanligini ko'rish oson. Demak, berilgan ketma – ketlik limiti $\frac{1}{9}$ ga teng.

3 – m i s o l.

a) $f(x) = x^3$ funksiya $x \to 0$ da (ya'ni nol atrofida);

b) $f(x) = (x+3)^2$ funksiya $x = -3$ nuqta atrofida;

c) $f(x) = \sin(x-a)$ funksiya $x = 2$ nuqta atrofida va shuningdek 2-kΠ nuqtalardan istalgan birining atrofida cheksiz kichik funksiyalardir.

4 – m i s ol. $\lim_{n\to\infty}\left(\frac{2n^3}{2n^2+3} + \frac{1-5n^2}{5n+1}\right)$ ni toping.

Y e c h i s h. Agar yig'indining limiti xossasidan foydalansak, har bir qo'shiluvchi cheksiz katta miqdor bo'lib, chegaralanmagan. Shuning uchun ular limitga ega emas; kasrlarni qo'shsak:

$$x_n = \frac{2n^3 - 13n^2 + 3}{10n^3 + 2n^2 + 15n + 3}.\ \text{Bundan}\ \lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{2 - \frac{13}{n} + \frac{3}{n^3}}{10 + \frac{2}{n} + \frac{15}{n^2} + \frac{3}{n^3}} = \frac{1}{5}$$

I z o x. Limitni hisoblashda kasrning surat va maxrajini **n** ning eng yuqori darajasi (n=3) ga bo'ldik.

5 – m i s o l. $\lim_{n\to\infty}\frac{5n^2-3}{2n^2+n+2} = \lim_{n\to\infty}\frac{5 - \frac{3}{n^2}}{2 + \frac{1}{n} + \frac{2}{n^2}} = \frac{5}{2} = 2{,}5$.

8 – §. O'z bilimini sinash uchun savollar va topshiriqlar.

1. **«Lim»** belgisi nimani anglatadi ?

2. Nuqtaning atrofi deganda nimani tushunasiz ?

3. Sonli to'plamning limit nuqtasi.

4. Sonlar ketma – ketligi deb nimaga aytiladi ?

5. Chegaralangan ketma – ketlik.

6. O'suvchi va kamayuvchi ketma – ketliklar.

7. Sonlar ketma – ketligining limiti.

8. Cheksiz kichik va cheksiz katta miqdorlar.

9. Cheksiz kichik miqdor limiti.

10. Cheksiz katta miqdor limiti.

11. Yaqinlashuvchi ketma – ketlik xossalari.

12. Sonlar ketma-ketliklari limitini hisoblash usullari.

13. Berilgan ketma – ketliklarning qaysilari o'suvchi ($\uparrow$), kamayuvchi ($\downarrow$) ekanligini aniqlang:

1) $\{x_n\}=\left\{\dfrac{n}{n+1}\right\}$ 2) $\{x_n\}=\left\{\dfrac{n^2}{n^2+2}\right\}$ 3) $\{x_n\}=\left\{\dfrac{2n}{n^2+1}\right\}$ 4) $\{x_n\}=\left\{\dfrac{3n+5}{2n+1}\right\}$

14. Berilgan ketma – ketliklarning qaysilari chegaralangan ?

1) $\{x_n\}=\{3n-1\}$; 2) $\{x_n\}=\left\{\dfrac{1}{n^2}\right\}$ 3) $\{x_n\}=\left\{\dfrac{n(n+1)}{3}\right\}$; 4) $\{x_n\}=\left\{\dfrac{1}{n^3-1}\right\}$;

5) $\{x_n\}=\left\{\dfrac{2}{n(n+1)}\right\}$; 6) $\{x_n\}=\left\{\dfrac{n}{n+1}\right\}$; 7) $\{x_n\}=\left\{\left(-\dfrac{1}{2}\right)^n\right\}$; 8) $\{x_n\}=\left\{\dfrac{2^n}{2^n-1}\right\}$.

15. Ushbu 1) $\{\alpha_n\}=\left\{\dfrac{(-1)^n+2}{n}\right\}$; 2) $\{\alpha_n\}=\left\{\dfrac{1}{2^n+1}\right\}$; 3) $\{\alpha_n\}=\left\{\dfrac{n^2}{n^4+4}\right\}$;

4) $\{\alpha_n\}=\left\{\dfrac{1}{n^2}\right\}$; 5) $\{\alpha_n\}=\left\{\dfrac{1}{n^2+1}\right\}$; 6) $\{\alpha_n\}=\left\{\dfrac{5}{n^2+4}\right\}$ ketma – ketliklar cheksiz ketma – ketliklar ekanini isbot qiling.

16. Quyidagi ketma – ketliklarning limitlarini hisoblang:

1) $\lim\limits_{n\to\infty}\dfrac{3n+1}{7n-1}$; 2) $\lim\limits_{n\to\infty}5\cdot\dfrac{n+1}{n}$; 3) $\lim\limits_{n\to\infty}\dfrac{2n^2+3n-4}{3n^2-4n+1}$; 4) $\lim\limits_{n\to\infty}\dfrac{1-n}{n^2+n+1}$;

5) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+3n+1}+n}{2n+3}$; 6) $\lim\limits_{n\to\infty}\dfrac{5+n-3n^2}{4-n+2n^2}$; 7) $\lim\limits_{n\to\infty}\dfrac{5n+3}{3n^2-2n+1}$;

8) $\lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)}{n^2+1}$; 9) $\lim\limits_{n\to\infty}\dfrac{(n+2)(n+3)(n+4)}{n^3+n+1}$; 10) $\lim\limits_{n\to\infty}\dfrac{(2n-1)(n-2)(n-3)}{3n^3+2n^2+n}$;

17. Hisoblang $\lim\limits_{n\to\infty}x_n$, agar: 1) $x_n=\left(\dfrac{3n^2+n-2}{4n^2+2n+7}\right)^2$; 2) $x_n=\dfrac{1^2+2^2+...+n^2}{5n^3+n+1}$;

3) $x_n=\dfrac{1+2+...+n}{n^2}$; 4) $x_n=\left(\dfrac{2n^3+2n^2+1}{4n^3+7n^2+3n+4}\right)^4$; 5) $x_n=\sqrt[n]{5n}$;

6) $x_n=\sqrt[n]{n^8}$; 7) $x_n=\sqrt[n]{n^5}$; 8) $x_n=\sqrt[n]{6n+3}$;

18. Quyidagilarni isbotlang: 1) $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$; 2) $\lim\limits_{n\to\infty}\dfrac{a^n}{n!}=0$; (a>0).

3) $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$, (a>0). 4) $x_n=3^{\sqrt[3]{n}}$ ketma–ketlikning umumiy hadi $n\to\infty$ da cheksiz katta ekanligini; 5) $x_n=\dfrac{1}{n^k}$, (k>0) ketma – ketlikning umumiy hadi cheksiz kichik ketma – ketlik ekanligini. 6) $\dfrac{2}{3}-0{,}6$; $\dfrac{2}{3}-0{,}66$; … ; $\dfrac{2}{3}-0{,}\underbrace{66...6}_{n\ \text{raqam}}$; ayirmalarni tuzib, $\lim\limits_{n\to\infty}0{,}\underbrace{666...6}_{n\ \text{raqam}}=\dfrac{2}{3}$ ekanini.

II – B O B

F U N K S I Y A   L I M I T I.

Biz I bobda sonlar ketma – ketligi va uning limitini o’rgandik. Endi haqiqiy argumentli funksiya limiti va ularning xossalari bilan tanishamiz.

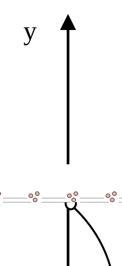
1 – §. Funksiya limiti ta’riflari.

$f(x)$ funksiya X to’plamda berilgan bo’lib, a nuqta X to’plamning limit nuqtasi bo’lsin (umuman aytganda a nuqta X to’plamga tegishli bo’lishi shart emas).

<u>***1 – t a’ r i f.***</u> *Agar* ***b*** *nuqtaning har qanday* **ε** *atrofida doimo* ***a*** *nuqtaning shunday* ***δ*** *atrofi topilsaki, unda* ***x*** *argumentning ana shu atrofga tegishli istagan qiymati uchun* $f(x)$ *funksiyaning qiymati* ***b*** *nuqtaning* **ε** *atrofiga tegishli bo’lsa,* ***x*** *o’zgaruvchi* ***a*** *ga intilganda* ***b*** *son* $f(x)$ <u>*funksiyaning limiti deyiladi*</u> *va* $\lim\limits_{x\to a} f(x) = b$ *kabi belgilanadi.*

Avvalo bu ta’rifning geometrik ma’nosini tekshirib ko’ramiz.

x argumentning barcha qiymatlari $f(x)$ funksiyaning tegishli qiymatlariga akslantiriladi. **Oy** o’qda $f(x)$ funksiya qiymatlarini ko’rsatuvchi nuqtalar to’plamini hosil qilamiz. **Oy** o’qda ordinatasi $b = \lim f(x)$ ga teng nuqtani olamiz. Argumentning biror qiymatiga to’g’ri kelgan **b** nuqta funksiyaning qiymati bo’lishi mumkin, lekin bu nuqta funksiya qiymatlarining to’plamiga tegishli bo’lmasligi ham mumkin. (10-shakl).



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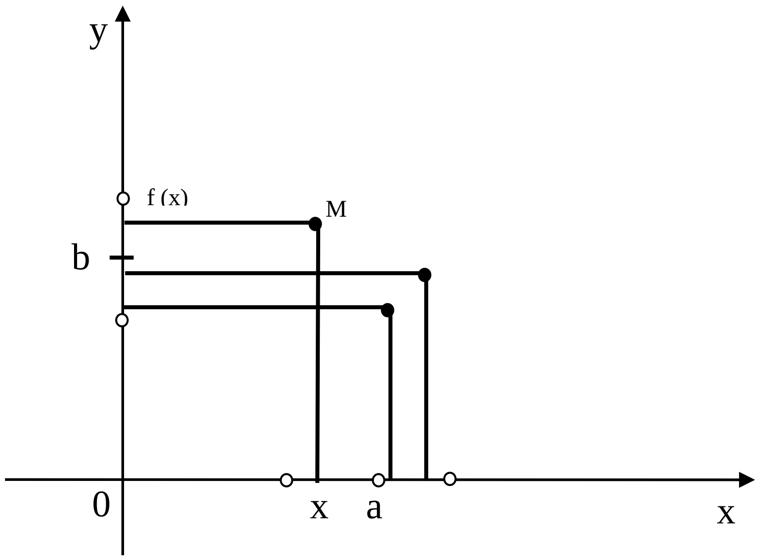
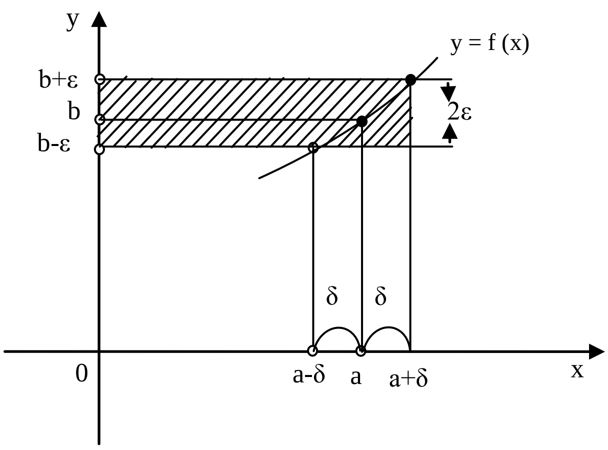
10 – s h a k l.



11 – s h a k l.



12 – s h a k l.

x argument **a** ga intilganda $f(x)$ funksiya uchun **b** nuqta (son) limit bo'lishi yoki bo'lmasligini aniqlash uchun:

1) **b** nuqtaning **ε** radiusli atrofini ixtiyoriy ravishda tanlab olish kerak, bunda **ε** istagan (shu jumladan, istagancha kichik) musbat son (10-shakl) va

2) **a** nuqta atrofining shunday **δ** radiusini izlab topish kerakki, argumentning **δ** atrofiga tushgan hamma qiymatlari funksiyaning **ε** atrofiga albatta tushadigan qiymatlarini (balki $f(a)$ qiymatdan boshqa qiymatlarini) aniqlaydigan bo'lsin. (11-shakl).

Qisqacha aytganda, **a** nuqtaning **δ** atrofidagi argumentning $x \neq a$ hamma qiymatlari **b** nuqtaning **ε** atrofiga qarashli nuqtalarga aks etiladi (11-shaklda funksiyaning grafigi ko'rsatilmagan).

Shuni ham esda tutish zarurki, argument qiymatlaridan iborat to'plamning limit nuqtasi **a** dir, demak, u shu to'plamga tegishli yoki tegishli bo'lmasligi mumkin; birinchi holda $f(a)$ mavjud, ya'ni ma'noga ega, ikkinchi holda esa $f(a)$ ma'noga ega emasdir.

2 – t a' r i f.	*Agar x to'plamning nuqtalaridan tuzilgan **a** ga yaqinlashuvchi har qanday $\{x_n\}$ ketma – ketlik olganda ham, funksiya qiymatlaridan iborat $\{f(x_n)\}$ ketma – ketlik yagona (chekli yoki cheksiz) **b** limitga intilsa, shu **b** ga $f(x)$ funksiyaning **a** nuqtadagi (**x** ning **a** ga intilgandagi) limiti deyiladi va $\lim\limits_{x \to a} f(x) = b$ kabi belgilanadi.*

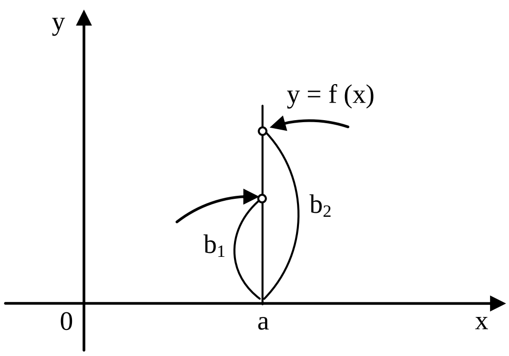
Funksiya limitiga berilgan bu ta'rif ***Geyne ta'rifi*** deyiladi.

Eslatma. Agar **a** ga intiluvchi ikkita $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar olinganda mos $\{f(x_n)\}$ va $\{f(y_n)\}$ ketma – ketliklarning limiti turlicha bo'lsa, u holda $f(x)$ funksiya $x \to a$ da limitga ega bo'lmaydi.

<u>3 – t a’ r i f.</u> *Agar* $\forall \varepsilon > 0$ *son uchun shunday* ***δ(ε)>0*** *son topilsaki, argument* ***x*** *ning* $0 < |x - a| < \delta$ *tengsizlikni qanoatlantiruvchi barcha qiymatlarida* $|f(x) - b| < \varepsilon$ *tengsizlik bajarilsa,* ***b*** *son* $f(x)$ *funksiyaning* ***a*** *nuqtada* $(x \to a)$ *limiti deyiladi va* $\lim_{x \to a} f(x) = b$ *kabi belgilanadi. Funksiya limitiga berilgan bu ta’rif* ***Koshi*** *yoki* ***“ε– δ ” ta’rifi*** *deyiladi.*

Agar $x \to a$ da $f(x) \to b$ bo’lsa, u holda $y = f(x)$ funksiyaning grafigida bu quyidagicha tasvirlanadi (12-shakl) $|x - a| < \delta$ tengsizlikdan $|f(x) - b| < \varepsilon$ tengsizlik chiqar ekan, u holda bu, **a** nuqtadan **δ** dan yiroq bo’lmagan masofada turuvchi barcha **x** nuqtalar uchun $y = f(x)$ funksiya grafigining **M** nuqtalari $y = b - \varepsilon$ va $y = b + \varepsilon$ to’g’ri chiziqlar bilan chegaralangan, yoki **2ε** bo’lgan yo’l ichida yotadi.

<u>4 – t a’ r i f.</u> *Agar* ***x*** *biror* ***a*** *sondan kichik qiymatlarnigina qabul qilib, shu* ***a*** *songa intilanda* $f(x)$ *funksiya* b_1 *limitga intilsa, u holda* $\lim_{x \to a-0} f(x) = b_1$ *yoziladi va* b_1 *ga* $f(x)$ *funksiyaning* ***a*** *nuqtadagi <u>chap limiti</u> deyiladi. Agar* ***x*** *faqat* ***a*** *dan katta qiymatlarnigina qabul qilsa, u holda* $\lim_{x \to a+0} f(x) = b_2$ *yoziladi va* b_2 *ga funksiyasining* ***a*** *nuqtadagi <u>o’ng limiti</u> deyiladi. (13-shakl)*



13 – s h a k l.

Agar o'ng limit va chap limit mavjud va teng, ya'ni $\mathbf{b_1=b_2=b}$ bo'lsa, u holda **b**, limitning yuqorida berilgan ma'nosida **a** nuqtadagi limitning o'zi bo'lishini isbotlash mumkin.

Funksiyaning o'ng va chap limitlariga uning <u>bir tomonli limitlari</u> deyiladi.

<u>5 – t a' r i f.</u> ***(Geyne ta'rifi).*** *Agar **X** to'plamning nuqtalaridan tuzilgan, har bir hadi **a** dan katta (kichik) bo'lib, **a** ga intiluvchi har qanday $\{x_n\}$ ketma – ketlik olinganda ham mos $\{f(x_n)\}$ ketma – ketlik yagona **b** soniga intilsa, shu **b** son $f(x)$ funksiyaning **a** nuqtadagi <u>o'ng (chap) limiti</u> deyiladi va quyidagicha belgilanadi:* $\lim\limits_{x\to a+0} f(x)=b$ *yoki* $f(a+0)=b$ $\left(\lim\limits_{x\to a-0} f(x)=b\right.$ *yoki* $\left.f(a-0)=b\right)$.

<u>6 – t a' r i f.</u> ***(Koshi ta'rifi).*** *Agar $\forall\varepsilon>0$ son uchun shunday **δ>0** son topilsaki, argument **x** ning tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, **b** son $f(x)$ funksiyaning **a** nuqtadagi o'ng (chap) limiti deyiladi va quyidagicha belgilanadi:* $\lim\limits_{x\to a+0} f(x)=b$ *yoki* $f(a+0)=b$ $\left(\lim\limits_{x\to a-0} f(x)=b\right.$ *yoki* $\left.f(a-0)=b\right)$.

<u>Teorema</u>. *Funksiya limiti uchun berilgan **Geyne** va **Koshi** (2- va 3-ta'riflar) ta'riflari o'zaro ekvivalentdir.(Isboti [3], 204-bet).*

Biz yuqorida $f(x)$ funksiya $x\to a$ dagi chekli **b** limitga ega bo'lishining **Koshi** ta'rifini (3-ta'rif) keltirdik. $b=\infty(b=+\infty,\quad b=-\infty)$ bo'lgan holda funksiya limitining **Koshi ta'rifi** quyidagicha ifodalanadi.

7 – t a' r i f. Agar $\forall\varepsilon > 0$ son uchun shunday **δ > 0** son topilsaki, **x** argumentning $0 < |x - a| < \delta$ tengsizliklarni qanoatlantiruvchi barcha qiymatlarida $|f(x)| > \mathrm{E}$ $(f(x) > \mathrm{E}; \quad -f(x) > \mathrm{E})$ tengsizlik bajarilsa, $f(x)$ funksiyaning **a** nuqtadagi limiti $\infty(+\infty, -\infty)$ deyiladi va $\lim\limits_{x\to a} f(x)\left(\lim\limits_{x\to a} f(x) = +\infty; \quad \lim\limits_{x\to a} f(x) = -\infty\right)$ kabi belgilanadi.

Endi $x \to \infty(x \to +\infty; \quad x \to -\infty)$ da funksiya limiti tushunchasini keltiramiz.

8 – t a' r i f. (Geyne ta'rifi). Agar **x** to'plamning nuqtalaridan tuzilgan har qanday cheksiz katta (musbat cheksiz katta; manfiy cheksiz katta) $\{x_n\}$ ketma – ketlik olganda ham mos $\{f(x_n)\}$ ketma – ketlik yagona **b** ga intilsa, **b** son $f(x)$ funksiyaning $x \to \infty$ dagi $(x \to +\infty; \quad x \to -\infty)$ limiti deyiladi va $\lim\limits_{x\to a} f(x) = b\left(\lim\limits_{x\to +\infty} f(x) = b; \quad \lim\limits_{x\to -\infty} f(x) = b\right)$ kabi belgilanadi.

9 – t a' r i f. (Koshi ta'rifi). Agar $\forall\varepsilon > 0$ son uchun shunday $M(\varepsilon) > 0$ dan topilsaki, **x** argumentning $|x| > M(\varepsilon)(x > M; \quad -x > M)$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, **b** son $f(x)$ funksiyaning $x \to \infty(x \to +\infty; \quad x \to -\infty)$ dagi limiti deyiladi va $\lim\limits_{x\to \infty} f(x) = b\left(\lim\limits_{x\to +\infty} f(x) = b; \quad \lim\limits_{x\to -\infty} f(x) = b\right)$ kabi belgilanadi.

Bu ta'rif ba'zan *"**ε – δ**" ta'rifi deb* ham nomlanadi.

M i s o l l a r.

1. Limitning **Geyne** ta’rifidan foydalanib $\lim\limits_{x\to 2}\frac{3x+1}{5x+4}=\frac{1}{2}$ tenglik to’g’riligini ko’rsating.

Y e c h i l i s h i: $\{x_n\}$ ketma – ketlik:

1) $f(x)=\frac{(3x+1)}{(5x+4)}$ $\left(yani \quad x_n\neq-\frac{4}{5}\right)$ funksiyaning mavjudlik sohasiga tegishli bo’lishi; 2) 2 soniga ketma – ketlik yaqinlashuvchi ya’ni $\lim\limits_{n\to\infty}x_n=2$ shartlarini qanoatlantiruvchi **x** ning qiymatlaridan iborat.

U holda funksiya qiymatlaridan iborat ketma – ketlik $f(x_n)=\frac{3x_n+1}{5x_n+4}$ bo’ladi. Chekli limitga ega bo’lgan ketma – ketliklar haqidagi teoremalarga asosan $\lim\limits_{n\to\infty}f(x_n)=\lim\limits_{n\to\infty}\frac{3x_n+1}{5x_n+4}=\frac{\lim(3x_n+1)}{\lim(5x_n+4)}=\frac{6+1}{10+2}=\frac{1}{2}$.

Funksiya limitining ta’rifiga asosan: $\lim\limits_{x\to 2}\frac{3x+1}{5x+4}=\frac{1}{2}$

2. Funksiya limitining **Koshi** ta’rifi (ya’ni “ε – δ” ta’rifidan foydalanib) asosida $\lim\limits_{x\to 1}(3x-8)=-5$ ekanligini ko’rsating.

Y e c h i s h. “ε – δ” ta’rifiga asosan $\forall\varepsilon>0$ uchun shunday δ>0 topiladiki $|x-1|<\delta$ tengsizlikdan $|f(x)-(-5)|=|f(x)+5|<\varepsilon$ tengsizlikka kelamiz. Boshqacha aytganda $|3x-8+5|=3|x-1|<\varepsilon$ tengsizlikni yechish kerak. Bu esa $|x-1|<\frac{\varepsilon}{3}=\delta$ bo’lganda $|f(x)+5|<\varepsilon$ bajariladi. Bundan $\lim\limits_{x\to 1}(3x-8)=-5$ tenglik bajariladi.

3. $\lim\limits_{x\to 1}\sin\dfrac{1}{x-1}$ ning limiti mavjud emasligini ko'rsating.

<u>Y e c h i s h.</u> Ikki ketma – ketlik $x_n = 1+\dfrac{1}{\pi n}$ va $y_n = 1+\dfrac{2}{(4n+1)\pi}$ $(n=1,2,...)$ lar

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 1$.

Funksiya qiymatlaridan iborat ketma – ketliklar quyidagicha:

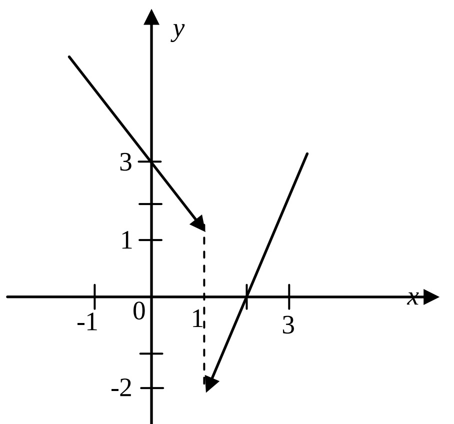
$$f(x_n)=\sin\frac{1}{1+\frac{1}{(n\pi)}-1}=\sin n\pi=0 \text{ va } f(y_n)=\sin\frac{1}{1+\frac{2}{[(4n+1)\pi]}-1}=\sin\frac{4\pi+1}{2}\pi=$$

$=\sin\left(2\pi n+\dfrac{\pi}{2}\right)=1$ bo'lib, $\lim\limits_{x_n\to 1} f(x_n)=0$ va $\lim\limits_{y_n\to 1} f(y_n)=1$, ya'ni $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar har xil limitlarga ega bo'lgani uchun $x\to 1$ da $\sin\dfrac{1}{x-1}$ funksiyaning limiti mavjud emasligini ko'rsatadi.

4. $f(x)=\begin{cases}-2x+3, & agar \quad x\le 1\\ 3x-5, & agar \quad x>1\end{cases}$ funksiyaning $x\to 1$ dagi bir tomonlama limitlarini toping.

<u>Y e c h i s h.</u> $x\le 1$ bo'lsa, $f(x)-2x+3$ bo'lib, $f(1-0)=\lim\limits_{x\to 1-0} f(x)=1$ – chap limit.

Agar x >1 bo'lsa, $f(x)=3x-5$ bo'lib, $f(1+0)=\lim\limits_{x\to 1+0} f(x)=-2$ – o'ng limit (14-shakl)



14 – s h a k l.

2 – §. Chekli limitga ega bo'lgan funksiyalarning xossalari.

Chekli limitga ega bo'lgan funksiyalar qator xossalarga ega bo'lib, bu xossalarni o'rganishda asosan funksiya limiti ta'riflaridan foydalaniladi.

$f(x)$ funksiya **X** to'plamda berilgan, **a** esa **X** ning limit nuqtasi bo'lsin.

1^0. Agar $f(x)$ funksiyaning **a** nuqtada limiti mavjud bo'lsa, bu limit yagonadir.

2^0. Agar $\lim\limits_{x\to a} f(x)=b$ bo'lib, b>p (b<q) bo'lsa, u holda **a** ning yetarli kichik atrofidan olingan $x(x\neq a)$ ning qiymatlarida $f(x)>p(f(x)<q)$ bo'ladi.

3^0. Agar $\lim\limits_{x\to a} f(x)=b\neq\infty$ bo'lsa, u holda **a** ning yetarlicha kichik atrofidan olingan $x(x\neq a)$ ning qiymatlarida $f(x)$ funksiya chegaralangan bo'ladi.

4^0. Agar $\lim\limits_{x\to a} f_1(x)=b_1$, $\lim\limits_{x\to a} f_2(x)=b_2$ bo'lib, x argumentning $0<|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $f_1(x)\leq f_2(x)$ tengsizlik o'rinli bo'lsa, u holda $b_1\leq b_2$ tengsizlik o'rinli bo'ladi.

5^0. Agar **x** argumentning $0<|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $f_1(x)\leq f(x)\leq f_2(x)$ tengsizlik o'rinli bo'lib, $\lim\limits_{x\to a} f_1(x)=\lim\limits_{x\to a} f_2(x)=b$ bo'lsa, u holda $\lim\limits_{x\to a} f(x)$ – mavjud va u ham **b** ga teng.

6^0. Agar $\lim\limits_{x\to a} f_1(x)=b_1$, $\lim\limits_{x\to a} f_2(x)=b_2$ bo'lsa, u holda $f_1(x)\pm f_2(x)$, $f_1(x)\cdot f_2(x)$, $\dfrac{f_1(x)}{f_2(x)}(f_2(x)\neq 0)$ funksiyalar ham limitga ega va

$\lim\limits_{x\to a}(f_1(x)\pm f_2(x))=b_1\pm b_2$, $\lim\limits_{x\to a}(f_1(x)\cdot f_2(x))=b_1\cdot b_2$, $\lim\limits_{x\to a}\dfrac{f_1(x)}{f_2(x)}=\dfrac{b_1}{b_2}$ $(b_2\neq 0)$

munosabatlar o'rinli.

7^0. Agar $\lim\limits_{x\to a} f(x)$ mavjud bo'lsa, u holda $\lim\limits_{x\to a}(k\cdot f(x))$ ham majud va u $k\cdot\lim\limits_{x\to a} f(x)$ ga teng *(k-const)*, ya'ni $\lim\limits_{x\to a}(k\cdot f(x))=k\lim\limits_{x\to a} f(x)$.

8^0. Agar $\lim\limits_{x\to a} f(x)$ mavjud va chekli bo'lsa, u holda $\lim\limits_{x\to a}[f(x)]^m$ ham mavjud *(m∈N)* va $\lim\limits_{x\to a}[f(x)]^m=\left[\lim\limits_{x\to a} f(x)\right]^m$ munosabat o'rinli bo'ladi.

Faraz qilaylik $\{x_n\}$ to'plamda $t=\varphi(x)$ funksiya aniqlangan va bu funksiya qiymatlaridan iborat $\{t\}$ to'plamda $y=f(t)$ funksiya aniqlangan bo'lib, ular yordamida murakkab $y=f(\varphi(x))$ funksiya hosil qilingan bo'lsin.

9^0. Agar 1) $\lim\limits_{x\to a}\varphi(x)=c$ bo'lib, **a** nuqtaning shunday *(a – ε, a + ε)* atrofi mavjud bo'lsaki, bu atrofdan olingan barcha **x** lar uchun $\varphi(x)\neq c$ bo'lsa, 2) **c** nuqta **T** to'plamning limit nuqtasi bo'lib, $\lim\limits_{t\to c} f(t)=b$ bo'lsa, u holda $x\to a$ da murakkab funksiya $f(\varphi(x))$ limitga ega va $\lim\limits_{x\to a} f(\varphi(x))=b$ bo'ladi.

M i s o l l a r.

1. $\lim_{x\to 1}(3x^3+5x^2-x+2)$ limitni ko'phadning limitini topish qoidasiga ko'ra hisoblanadi.

$$\lim_{x\to 1}(3x^3+5x^2-x+2)=3\cdot 1^3+5\cdot 1^2-1+2=9$$

2. $\lim_{x\to 2}\frac{x^2+x-2}{x^2-2}$ limitning maxraji x = 2 da noldan farqli bo'lgani uchun kasr– rasional funksiyaning limitini hisoblash qoidasiga ko'ra topamiz:

$$\lim_{x\to 2}\frac{x^2+x-2}{x^2-2}=\frac{2^2+2-2}{2^2-2}=\frac{4}{2}=2$$

3. $\lim_{x\to 2}\frac{5}{4x-8}$ limitda bo'luvchining limiti nolga teng:

$\lim_{x\to 2}(4x-8)=0$. Demak, bo'linmaning limiti haqidagi xossani qo'llab bo'lmaydi, chunki 4x – 8 ifoda $x\to 2$ da cheksiz kichik miqdordir, unga teskari miqdor $\frac{1}{4x-8}$ esa cheksiz katta miqdordir. Shuning uchun $x\to 2$ da $\frac{1}{4x-8}\cdot 5$ ko'paytma cheksiz katta miqdor, ya'ni $\lim_{x\to 2}\frac{5}{4x-8}=\infty$.

4. $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{6x}{x^3-8}\right)$ ifoda $x\to 2$ da ikkita cheksiz katta miqdorning ayirmasidan iboratdir. Kasrlarni ayirib, surat va maxraji $x\to 2$ da nolga intiladigan kasrni hosil qilamiz. Kasrni $x-2$ ga qisqartirib, quyidagiga ega bo'lamiz:

$$\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{6x}{x^3-8}\right)=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x^2+2x+4)}=$$

$$=\lim_{x\to 2}\frac{x-2}{x^2+2x+4}=\frac{2-2}{4+4+4}=\frac{0}{12}=0$$

5. $\lim\limits_{x\to\infty}\dfrac{x^4-2x^2+3}{2x^4+1}$ limitni hisoblash uchun kasrning surat va maxrajini argumentning eng yuqori darajasiga, ya'ni x^4 bo'lamiz:

$$\lim_{x\to\infty}\frac{x^4-2x^2+3}{2x^4+1}=\lim_{x\to\infty}\frac{1-\dfrac{2}{x^2}+\dfrac{3}{x^4}}{2+\dfrac{1}{x^4}}=\frac{1-\dfrac{2}{\infty}+\dfrac{3}{\infty}}{2+\dfrac{1}{\infty}}=\frac{1}{2}.$$

3 – §. Ajoyib limitlar.

Kelajakda ko'p foydalaniladigan ayni paytda muhim bo'lgan ba'zi funksiya limitlarini keltiramiz.

1. Agar **x** radian o'lchovi bilan berilgan bo'lsa, $\lim_{x\to 0}\frac{\sin x}{x}=1$ (1) munosabat o'rinli, ya'ni $\frac{\sin x}{x}$ funksiyaning $x\to 0$ dagi limiti **x** ning **0** ga intilish qonuniga bog'liq emas. Shuning uchun (1) ga – *birinchi ajoyib limit* deyiladi.

Ravshanki, $0<x<\frac{\pi}{2}\left(-\frac{\pi}{2}<x<0\right)$ oraliqda olingan ixtiyoriy **x** larda $0<\sin x<x<tgx$ tengsizliklar o'rinli.

Endi $\sin x<x<tgx$ tengsizliklarni $\sin x$ ga bo'lib, $1<\frac{x}{\sin x}<\frac{1}{\cos x}$ va undan $\cos x<\frac{\sin x}{x}<1 \Rightarrow 0<1-\frac{\sin x}{x}<1-\cos x$.

$1-\cos x=2\sin^2\frac{x}{2}$ va $0<x<\frac{\pi}{2}$ da $\sin^2\frac{x}{2}<\sin\frac{x}{2}$ larni e'tiborga olsak, $1-\cos x<2\sin\frac{x}{2}<2\cdot\frac{x}{2}=x$ munosabat o'rinli bo'ladi.

Demak, ixtiyoriy $0<x<\frac{\pi}{2}$ da $0<1-\frac{\sin x}{x}<x$. Bundan $\left|\frac{\sin x}{x}-1\right|<|x|$ tengsizlik o'rinli bo'lishi kelib chiqadi.

$\forall\varepsilon>0$ sonni olib, unga ko'ra $\delta>0$ sonni (uni olingan **ε** va $\frac{\pi}{2}$ sonlardan kichik qilib) olinsa, u holda $|x-0|=|x|<\delta$ bo'lganda $\left|\frac{\sin x}{x}-1\right|<|x|<\varepsilon$ bo'ladi. Bu esa $\lim\limits_{x\to0}\frac{\sin x}{x}=1$ bo'lishini bildiradi.

(1) dan quyidagi tengliklarning to'g'riligini isbotlash qiyin emas:

$$\lim_{x\to0}\frac{x}{\sin x}=1 \quad (2) \qquad \lim_{x\to0}\frac{\sin kx}{x}=k \quad (3) \qquad \lim_{x\to0}\left(\frac{\sin x}{x}\right)^k=1 \quad (4)$$

$$\lim_{x\to0}\frac{tgx}{x}=1 \quad (5) \qquad \lim_{x\to0}\frac{\arcsin x}{x}=1 \quad (6) \qquad \lim_{x\to0}\frac{arctgx}{x}=1 \quad (7)$$

2. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$ (8) tenglik o'rinli ekanligini ko'rsatamiz.

Faraz qilaylik, **x > 1** bo'lsin. **x** ning butun qismini **n** orqali belgilasak, u holda $n\le x<n+1$ bo'lib, bundan esa $\frac{1}{n+1}<x\le\frac{1}{n}$ tengsizliklarga ega bo'lamiz. Bu tengsizliklardan $\left(1+\frac{1}{n+1}\right)^n<\left(1+\frac{1}{x}\right)^x<\left(1+\frac{1}{n}\right)^{n+1}$ (9) tengsizliklar kelib chiqadi.

$\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n=e$, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}=e$ hamda (9) tengsizliklardan foydalanib chekli limitga ega bo'lgan funksiya xossalariga ko'ra $x\to\infty$ da $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$ tenglikka ega bo'lamiz.

Endi $x<-1$ bo'lsin. $x=-y$ belgilash kiritsak, u holda:

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y}=\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y}=$$

$$=\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)=e\cdot1=e$$ boladi.

Demak, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

N a t i j a. $\lim_{x\to0}(1+x)^{\frac{1}{x}}=e$ (10) tenglik o'rinlidir.

Haqiqatdan ham $\frac{1}{x}=y$ belgilash natijasida $\lim_{x\to0}(1+x)^{\frac{1}{x}}=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}$ bo'lib, $\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}=e$ munosabatdan $\lim_{x\to0}(1+x)^{\frac{1}{x}}=e$ kelib chiqadi.

(8) va (10) formulalarga ikkinchi ajoyib limit deyiladi. **e=2,7182818 …** irasional son bo'lib, asosi **e** ga teng bo'lgan logarifmga natural logarifm deyiladi, ya'ni $\log_a x=\log_e x=\ln x$; $\lg x=M\ln x, M=0{,}43429...$

Masalalarni yechishda ***«ajoyib limitlar»*** deb ataluvchi ushbu limitlardan foydalanishga to'g'ri keladi:

$$\lim_{x\to0}\frac{\log_a(1+x)}{x}=\log_a e \quad (11) \qquad \lim_{x\to0}\frac{\ln(1+x)}{x}=1 \quad (12)$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\ln_a \quad (13) \qquad \lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha \quad (14)$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\ln e=1 \quad (15) \qquad \lim_{x\to 0}\frac{e^x-1}{x}=\ln e=1 \quad (16)$$

Bularda $a\neq 1,\quad a>0$.

Logarifmik funksiyaning aniqlanish sohasida uzluksizligidan $\lim_{x\to x_0} f(x)=f\left(\lim_{x\to x_0} x\right)$ (17) tenglik o'rinli bo'lishligini e'tiborga olib topamiz:

$$\lim_{x\to x_0}\frac{\log_a(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\log_a(1+x)=\lim_{x\to 0}\left[\log_a(1+x)^{\frac{1}{x}}\right]=$$

$$=\log_a\left[\lim_{x\to a}(1+x)^{\frac{1}{x}}\right]=\log_a e.$$

Demak, $\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e$ (11) tenglik bajariladi.

Xususan, $\lim_{x\to 0}\frac{\ln(1+x)}{x}=\ln e=1$ (15)

(13) limitni hioblash uchun $a^x-1=t$ deb almashtirish olamiz. Ravshanki, $x\to 0$ da $t\to 0$ va

$$a^x-1=t \Rightarrow a^x=1+t \Rightarrow \log_a a^x=\log_a(1+t) \Rightarrow x\log_a a=$$

$= \log_a(1+t) => x = \log_a(1+t)$. Natijada ushbu tenglikka ega bo'lamiz: $\lim\limits_{x\to a} \frac{a^x - 1}{x} = \lim\limits_{t\to 0} \frac{t}{\log_a(1+t)} = \lim\limits_{t\to 0} \frac{1}{\frac{\log_a(1+t)}{t}}$.

Agar $\lim\limits_{t\to 0} \frac{1}{\frac{\log_a(1+t)}{t}} = \frac{1}{\lim\limits_{t\to 0} \frac{\log_a(1+t)}{t}} = \frac{1}{\log_a e} = \ln a$ bo'lishini hisobga olsak, u holda $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \ln a (a \neq 1, \quad a > 0)$ (13) ekanligini topamiz.

Yuqoridagiga o'xshash mulohaza bilan (14) tenglikning to'g'riligini ko'rsatish mumkin.

Misollar.

1) $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{bx}}{\sin cx-\sin dx}=\lim\limits_{x\to 0}\dfrac{a\cdot\dfrac{e^{ax}-1}{ax}-b\cdot\dfrac{e^{bx}-1}{bx}}{c\cdot\dfrac{\sin cx}{cx}-d\cdot\dfrac{\sin dx}{dx}}=\dfrac{a-b}{c-d}$

$$\lim_{\alpha\to 0}\frac{e^{\alpha}-1}{\alpha}=1;\qquad \lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1$$

2) $\lim\limits_{x\to a}\dfrac{\ln x-\ln a}{x-a}=\lim\limits_{t\to 0}\dfrac{\ln(t+a)-\ln a}{t}=\lim\limits_{t\to 0}\dfrac{\ln\left(1+\dfrac{t}{a}\right)}{t}=\dfrac{1}{a}$

$x-a=t$ bo'lsin u holda $x=t+a$ va $x\to a$ da $t\to 0$

3) $\lim\limits_{x\to 2}\dfrac{\sin(x-2)}{x^2-4}=\lim\limits_{x\to 2}\dfrac{1}{x+2}\cdot\dfrac{\sin(x-2)}{x-2}=\lim\limits_{x\to 2}\dfrac{1}{x+2}\cdot\lim\limits_{x\to 2}\dfrac{\sin(x-2)}{x-2}=\dfrac{1}{4}$

4) $\lim\limits_{x\to 1}x^{\frac{1}{x-1}}=\lim\limits_{x\to 1}\left[1+(x-1)\right]^{\frac{1}{x-1}}=e$

4 – §. Ekvivalent cheksiz kichik funksiyalar.

1 – t a' r i f. Agar $\lim\limits_{x\to a}\alpha(x)=0$ bo'lsa, u holda $\alpha(x)$ $x\to a$ da cheksiz kamayuvchi funksiya deyiladi. Ba'zan *«cheksiz kichik»* ham deyiladi.

2 – t a' r i f. Agar $\alpha(x)$ va $\beta(x)$ lar cheksiz kichik bo'lib, $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}=0$ bo'lsa, u holda $\alpha(x)$ ni $\beta(x)$ ga nisbatan yuqori tartibli cheksiz kichik deyiladi va $\alpha(x)=0(\beta(x))$ ko'rinishda yoziladi.

3 – t a' r i f. Agar $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}=c$ $(c\neq 0, \quad c\neq\infty)$ bo'lsa, u holda $\alpha(x)$ va $\beta(x)$ lar bir xil tartibli cheksiz kichiklar deyiladi.

Agar c=1 bo'lsa, u holda $\alpha(x)$ va $\beta(x)$ lar ekvivalent cheksiz kichiklar deyiladi va $\alpha(x)$ ~ $\beta(x)$ lar ko'rinishda yoziladi.

T e o r e m a. Agar x=a nuqta atrofida $\alpha(x)$, $\beta(x)$, $\alpha_1(x)$, $\beta_1(x)$ lar cheksiz kichik va $\alpha(x)$ ~ $\alpha_1(x)$, $\beta(x)$ ~ $\beta_1(x)$ bo'lsa, u holda $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}=\lim\limits_{x\to a}\frac{\alpha_1(x)}{\beta_1(x)}$ bo'ladi.

Q o i d a. Agar $x\to a$ da $\alpha(x)$ cheksiz kamayuvchi funksiyaning $\beta(x)$ cheksiz kamayuvchi funksiyaga nisbatining limiti 1 soniga teng bo'lsa, u holda $\alpha(x)$ va $\beta(x)$ cheksiz kamayuvchi funksiyalar ekvivalent bo'ladi, aks holda ekvivalent bo'lmaydi.

Misollar.

$x \to 0$ da quyidagi funksiyalarning qaysilari ekvivalent cheksiz kichikdir:

1) $\alpha(x) = \sqrt[n]{1+x} - 1$ va $\beta(x) = \frac{1}{n}x$ 2) $\alpha(x) = \frac{\sqrt{1+x+x^2} - 1}{\sin 2x}$ va $\beta(x) = \sin x$?

Yechilishi.

1) $$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = \lim_{x \to a} \frac{\sqrt[n]{1+x} - 1}{\frac{1}{n} \cdot x} = \left(\frac{0}{0} \right) = \left\| \begin{matrix} \sqrt[n]{1+x} = t, & 1+x = t^n, & x = t^n - 1 \\ x \to 0\, da & t \to 1 & \end{matrix} \right\| =$$

$$= \lim_{t \to 1} \frac{t-1}{\frac{1}{n}(t^n - 1)} = \left(\frac{0}{0} \right) = \lim_{t \to 0} \frac{(t-1) \cdot n}{(t-1)(t^{n-1} + t^{n-2} + \ldots + t + 1)} = \frac{n}{n} = 1$$, demak, $x \to 0$

da $\sqrt[n]{1+x} - 1 \sim \frac{1}{n} \cdot x$

2) $$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = \lim_{x \to 0} \frac{\frac{\sqrt{1+x+x^2} - 1}{\sin 2x}}{\sin x} = \lim_{x \to 0} = \frac{\sqrt{1+x+x^2} - 1}{\sin x \cdot \sin 2x} = \left(\frac{0}{0} \right) =$$

$$= \lim_{x \to 0} \frac{x(1+x)}{\sin x \cdot \sin 2x \sqrt{1+x+x^2} + 1} = \left(\frac{0}{0} \right) = \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{1}{\sqrt{1+x+x^2} + 1} \cdot \lim_{x \to 0} \frac{1+x}{\sin 2x} =$$

$= 1 \cdot \frac{1}{2} \cdot \infty = \infty$, demak, o'zaro ekvivalent emas.

5 – §. Ekvivalent cheksiz kichik funksiyalar jadvali.

Agar $\lim\limits_{x\to a} f(x) = k$, $0 < |k| < \infty$ bo'lsa, u holda $f(x)\alpha(x) \sim k\alpha(x)$.

Agar $\alpha(x) \sim \psi(x)$, $\beta(x) \sim \psi(x)$ bo'lsa, u holda $\alpha(x) \sim \beta(x)$.

$x \to a$ da $\alpha(x)$ *«cheksiz kichik»* uchun ekvivalent cheksiz kichik funksiyalar jadvali:

1. $\sin\alpha(x) \sim \alpha(x)$;
2. $tg\alpha(x) \sim \alpha(x)$;
3. $1 - \cos\alpha(x) \sim \dfrac{[\alpha(x)]^2}{2}$;
4. $\arcsin\alpha(x) \sim \alpha(x)$;
5. $arctg\alpha(x) \sim \alpha(x)$;
6. $\ln[1 + \alpha(x)] \sim \alpha(x)$;
7. $a^{\alpha(x)} - 1 \sim \alpha(x)\ln a (a > 0)$;
8. $e^{\alpha(x)} - 1 \sim \alpha(x)$;
9. $[1 + \alpha(x)]^p - 1 \sim p\alpha(x)$;
10. $\sqrt[n]{1 + \alpha(x)} - 1 \sim \dfrac{\alpha(x)}{n}$;

M i s o l l a r.

1. $x \to 0$ da $\sin\sqrt{x\sqrt{x}} \sim \sqrt{x^2 + \sqrt{x^3}}$ bo'lishligini ko'rsating.

Y e c h i s h: Jadvaldagi 1 – formulaga asosan:

$\sin\sqrt{x\sqrt{x}} \sim \sqrt{x\sqrt{x}} = x^{\frac{3}{4}}$; $\sqrt{x^2+\sqrt{x^3}} = x^{\frac{3}{4}}\sqrt{1+x^{\frac{1}{2}}} \sim x^{\frac{3}{4}}$. Bundan $\sin\sqrt{x\sqrt{x}} \sim \sqrt{x^2+\sqrt{x^3}}$ to'g'riligi kelib chiqadi.

2. Quyida berilgan *«cheksiz kichik»* larni ekvivalentlariga almashtiring:

a) $a\sin\alpha - b\alpha^3$; b) $(1-\cos\alpha)^2 + 12\alpha^3 + 2\alpha^4 + 5\alpha^5$;

Y e c h i s h:

a) $\alpha(x)$ va $\beta(x)$ *«cheksiz kichik»* lar har xil tartibli, ya'ni $\alpha(x) = a\sin\alpha$ miqdor 1–tartibli, $\beta(x) = (-b\alpha^3)$ – 3–tartibli bo'lgani uchun:

$a\sin\alpha + (-b\alpha^3) \sim 3\sin\alpha \sim 3\alpha$.

b) $(1-\cos\alpha)^2 + 12\alpha^3 + 2\alpha^4 + 5\alpha^5 = 4\sin^4\frac{\alpha}{2} + 12\alpha^3 + 2\alpha^4 + 5\alpha^5$ qo'shiluvchilarning eng kichik tartibli $12\alpha^3$ bo'lgani uchun: $(1-\cos\alpha)^2 + 12\alpha^3 + 2\alpha^4 + 5\alpha^5 \sim 12\alpha^3$.

3. Quyidagi limitlarni ekvivalent cheksiz kichiklardan foydalanib hisoblang:

a) $$\lim_{x\to 0}\frac{\sin kx}{\ln(1+kx)} = \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{kx}{kx} = 1$$

(Jadvalga asosan: $\sin kx \sim kx$ va $\ln(1+kx) \sim kx$)

b) $$\lim_{x\to 0}\frac{\sqrt[4]{1+x^2}-1}{\ln\cos x} = \lim_{x\to 0}\frac{\frac{x^2}{4}}{\ln\left[1+(\cos x-1)\right]} = \frac{1}{4}\lim_{x\to 0}\frac{x^2}{\cos x - 1} = -\frac{1}{4}\lim_{x\to 0}\frac{x^2}{\frac{x^2}{2}} = -\frac{1}{2}$$

6 – §. Limitlarni hisoblash yo'llari.

Funksiyaning limiti uning argumentining intilgan sonida aniqlangan bo'lishiga bog'liq emas. Amalda esa funksiya limitini topishda bu munosabat katta ahamiyatga ega.

I. Agar berilgan $f(x)$ funksiya elementar bo'lib, **x** intilgan son uning aniqlanish sohasiga tegishli bo'lsa, u holda funksiyaning limiti $f(x)$ ning **x** intilgan son qiymatidagi xususiy qiymatiga teng bo'ladi, ya'ni $\lim\limits_{x\to a} f(x) = f(a)$.

1 – m i s o l .

$\lim\limits_{x\to 2}\dfrac{x^2-3x+4}{x-1} = \dfrac{4-6+4}{1} = 2$, chunki elementar funksiya bo'lib, argument intilgan son uning aniqlanish sohasiga kirganligi uchun uning limiti funksiyaning argumenti intilgan son qiymatidagi xususiy qiymatiga teng.

Agar funksiyada argument ∞ ga yoki uning aniqlanish sohasiga tegishli bo'lmagan songa intilsa, bu holda funksiya limitini topishda alohida tekshirish olib borish kerak bo'ladi.

Yuqorida bayon qilingan limitlar xossalariga suyanib, quyidagi ko'p uchraydigan limitlar topilgan:

1. $\lim\limits_{x\to\infty} ax = \infty$

3. $\lim\limits_{x\to\infty}\dfrac{a}{x} = 0$

2. $\lim\limits_{x\to -0}\dfrac{a}{x} = -\infty$

4. $\lim\limits_{x\to+\infty} a^x = \begin{cases} 0, & agar \quad a<1 \\ +\infty, & agar \quad a>1 \\ \infty, & agar \quad a<-1 \end{cases}$

5. $\lim\limits_{x\to\infty} \frac{x}{a} = \infty$

6. $\lim\limits_{x\to+0} \frac{a}{x} = +\infty$

7. $\lim\limits_{x\to 0} \frac{a}{x} = \infty$

8. $\lim\limits_{x\to-\infty} a^x = \begin{cases} 0, & agar \quad |a|<1 \\ +\infty, & agar \quad 0<a<1 \\ \infty, & agar \quad -1<a<0 \end{cases}$

9. $\lim\limits_{x\to\infty} \log_a x = \begin{cases} +\infty, & agar \quad a>1 \\ -\infty, & agar \quad 0<a<1 \end{cases}$

10. $\lim\limits_{x\to+0} \log_a x = \begin{cases} -\infty, & agar \quad a>1 \\ +\infty, & agar \quad 0<a<1 \end{cases}$

Bu oddiy limitlardan formula tariqasida foydalanish mumkin, ularda qatnashgan $a > 0$ o'zgarmas sondir.

Izox. *$a < 0$ bo'lganda **x** faqat butun son qiymatlarini qabul qilishi mumkin, **x** ning hamma qiymatlari uchun $a < 0$ bo'lganda a^x aniqlanmagan.*

Funksiya limitini topishda $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty\cdot\infty$, 1^∞, 0^∞ kabi aniqmasliklarni ***«ochib»*** limitlarni hisoblash limitlar nazariyasining asosiy vazifasidir.

Bunda misollarga qarab, ma'lum algebraik va trigonometrik almashtirishlar bajarib, so'ngra limitlarni hisoblaymiz.

II. $x \to a$ yoki $x \to \infty$ da $f(x)$ funksiya ikki cheksiz kichik miqdorning nisbatidan $\left(\frac{0}{0}\right)$ iborat bo'lgan hol.

2 – m i s o l. $\lim\limits_{x\to 3}\frac{x^2-5x+6}{3x+9}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 3}\frac{(x-2)(x-3)}{3(x-3)}=\lim\limits_{x\to 3}\frac{x-2}{3}=\frac{1}{3}$

3 – m i s o l. $\lim\limits_{x\to 0}\frac{x}{\sqrt{2-x}-\sqrt{2+x}}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}{2-x-2-x}=$

$=-\lim\limits_{x\to 0}\frac{\sqrt{2-x}+\sqrt{2+x}}{2}=-\frac{\sqrt{2}+2}{2}$; Bundagi 2–misolda $ax^2+bx+c=a(x-x_1)(x-x_2)$ dan foydalanildi, x_1, x_2 lar $ax^2+bx+c=0$ kvadrat tenglamaning ildizlaridir;

3–misolda esa kasrning surat va maxrajini $\left(\sqrt{2-x}+\sqrt{2+x}\right)$ ga ko'paytirib, maxrajdagi irrasionallikni yo'qotamiz, so'ngra **x** ga qisqartirdik.

4 – m i s o l. $\lim\limits_{x\to 0}\frac{\sin\frac{x}{5}}{x}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{\sin\frac{x}{5}}{5\cdot\frac{x}{5}}=\frac{1}{5}$;

III. $x\to a$ yoki $x\to\infty$ da $f(x)$ funksiya ikki cheksiz katta miqdorning nisbatidan $\left(\frac{\infty}{\infty}\right)$ iborat bo'lgan hol.

5 – m i s o l. $\lim\limits_{x\to\infty}\frac{2x^4-5x^3+x^2}{x^4+1}=\left(\frac{\infty}{\infty}\right)=\lim\limits_{x\to\infty}\frac{2-\frac{5}{x}+\frac{1}{x^2}}{1+\frac{1}{x^4}}=\frac{2-\frac{5}{\infty}+\frac{1}{\infty}}{1+\frac{1}{\infty}}=2$;

Bu misolda $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmaslikni ochish uchun surat va maxrajini noma'lumning eng katta darajasiga bo'ldik.

IV. $x \to a$ yoki $x \to \infty$ da $f(x)$ funksiya cheksiz kichik va cheksiz katta miqdorlar ko'paytmasi $(0 \cdot \infty)$ dan iborat bo'lgan hol.

Bu hol ma'lum almashtirishlar yordamida I**I** yoki **III** holga keladi.

6 – m i s o l. $\lim\limits_{x \to \infty} x \cdot arcctgx = (\infty \cdot 0) = \lim\limits_{\alpha \to 0} \alpha \cdot ctg\alpha = \lim\limits_{\alpha \to 0} \frac{\alpha \cos\alpha}{\sin\alpha} =$

$arcctgx = \alpha$ ***deb belgilasak,*** $x = ctg\alpha$ ***ga ega bo'lamiz, bundan*** $x \to \infty$ ***da*** $\alpha \to 0$

$$= \lim_{\alpha \to 0} \cos\alpha \cdot \lim_{\alpha \to 0} \frac{\alpha}{\sin\alpha} = 1 \cdot 1 = 1$$

V. $x \to a$ yoki $x \to \infty$ da $f(x)$ funksiya ikki cheksiz katta miqdorlar ayirmasi $(\infty - \infty)$ dan iborat bo'lgan hol.

Bu holda funksiyani kasr bilan almashtirilsa, II yoki III hollardan biriga keladi.

7 – m i s o l. $\lim\limits_{x \to \infty} \left(\frac{x^3}{3x^2 - 4} - \frac{x^2}{3x + 2} \right) = (\infty - \infty) = \lim\limits_{x \to \infty} \frac{2x^3 + 4x^2}{9x^3 + 6x^2 - 12x - 8} =$

$$= \lim_{x \to \infty} \frac{2 + \frac{4}{x}}{9 + \frac{6}{x} - \frac{12}{x^2} - \frac{8}{x^3}} = \frac{2}{9}$$

VI. $x \to a$ yoki $x \to \infty$ da $f(x)$ funksiya asosi 1 ga, ko'rsatkichi ∞ ga intiladigan daraja (1^{∞}) bo'lgan hol.

Bunday funksiyalarning limitini topishda 2 – ajoyib limitdan foydalaniladi.

8 – m i s o l. $$\lim_{x\to 0}(1+2x)^{\frac{4}{x}} = (1^{\infty}) = \lim_{x\to 0}(1+2x)^{\frac{4}{2x}\cdot 2} = e^{8}$$

VII. Ekvivalent cheksiz kichiklardan foydalanib yechiladigan hol.

9 – m i s o l. $$\lim_{x\to 0}\frac{\sin 2x + \arcsin^{2} x - arctg^{2}x}{3x} = \lim_{x\to 0}\frac{2x}{3x} = \frac{2}{3}$$

Ekvivalent cheksiz kichik funksiyalar jadvaliga asosan:

$$\sin 2x + \arcsin^{2} x - arctg^{2}x \sim \sin 2x \sim 2x$$

7 – §. O'Z BILIMINI SINASH UCHUN SAVOLLAR VA TOPSHIRIQLAR.

1. Funksiya limiti ta'riflari.

2. Funksiya limitining **Geyne** va **Koshi** ta'riflari va ularning farqi.

3. O'ng va chap limitlar deganda nimani tushunasiz ?

4. Chekli limitga ega bo'lgan funksiyalarning xossalari.

5. Birinchi ajoyib limit.

6. Ikkinchi ajoyib limit.

7. Ekvivalent cheksiz kichik funksiyalar va uning jadvali.

8. Limitlarni hisoblash yo'llarini sanab va tushuntirib bering.

9. Quyidagi limitlarni hisoblang:

a) $\lim\limits_{x\to 2}(2x^3+4x^2-x-4)$; b) $\lim\limits_{x\to 3}\dfrac{x^2-x+6}{x^2-3}$; c) $\lim\limits_{x\to 3}\dfrac{x^4-2x^2}{x}$.

10. limitlarni hisoblang:

a) $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{3x+9}$; b) $\lim\limits_{x\to 2}\dfrac{x^3-7x+6}{x^3-5x^2+2x+8}$; c) $\lim\limits_{x\to\infty}\dfrac{2x^3-2x^2+1}{x^3+4x^2+2x}$;

d) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)$; e) $\lim\limits_{x\to 0}\left(10\sin^2 x+\cos^2 x+\dfrac{x-1}{3x+2}\right)$;

f) $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$; m) $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$, n∈N.

11. Ajoyib limitlardan foydalanib hisoblang:

a) $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{x^2}$; b) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+4}{x^2-3x+7}\right)^x$;

12. Ekvivalent cheksiz kichiklardan foydalanib, quyidagi limitlarni toping:

a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sin 3x}$; b) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x+x^2}$; c) $\lim\limits_{x\to 0}\dfrac{tg^2 2x}{\sin^2\dfrac{x}{3}}$;

III – B O B.

Talabalarning yakka tartibda bajaradigan mustaqil ishlari.

I. $\{x_n\}$ ketma – ketlikning umumiy hadi berilgan, uning dastlabki beshta hadini yozing.

1) $\{x_n\}=\dfrac{\sin\left(\dfrac{n\pi}{2}\right)}{n}$; 2) $\{x_n\}=\sin\left(\dfrac{\pi n}{3}\right)$; 3) $\{x_n\}=2^{-n}\cos n\pi$;

4) $\{x_n\}=\left(1+\dfrac{1}{n}\right)^n$; 5) $\{x_n\}=\left\{\dfrac{1}{3n}\sin\left(\dfrac{n\pi}{2}\right)\right\}$; 6) $\{x_n\}=\cos(2n+1)\pi$;

7) $\{x_n\}=\dfrac{tg\,\dfrac{n\pi}{2}}{n}$; 8) $\{x_n\}=(-1)^n\sin\dfrac{\pi n}{2}$; 9) $\{x_n\}=(-1)^n\cos\dfrac{\pi n}{2}$;

10) $\{x_n\}=(-1)^{n+1}\cos\dfrac{\pi n}{3}$;

II. $\{x_n\}$ ketma – ketlikning limitlari hisoblansin.

1) $\{x_n\}=\left\{\dfrac{4n}{1+2n}\right\}$; 2) $\{x_n\}=\left\{\dfrac{2n-1}{2n+1}\right\}$; 3) $\{x_n\}=\left\{\dfrac{n^3}{1+n^4}\right\}$;

4) $\{x_n\}=\left(\dfrac{1+n^2}{1-n^2}\right)$; 5) $\{x_n\}=\left\{\dfrac{n}{1+\sqrt{n}}\right\}$; 6) $\{x_n\}=\left\{\dfrac{2n+2}{2n-1}\right\}$;

7) $\{x_n\}=\left\{\dfrac{n^2+1}{3n^2+2}\right\}$; 8) $\{x_n\}=\left\{\dfrac{n^3-1}{2n^3+3n^2+1}\right\}$;

9) $\{x_n\}=\left\{\dfrac{n^2+n+1}{(n+1)^2}\right\}$; 10) $\{x_n\}=\left\{\dfrac{1}{2n}+\dfrac{2n}{3n+1}\right\}$;

III. $\lim\limits_{n\to\infty} x_n$ ni toping, agar:

1) $x_n=\dfrac{3n^2+5n+4}{2+n^2}$; 2) $x_n=\dfrac{5n^3+2n^2-3n+7}{4n^3-2n+11}$; 3) $x_n=\dfrac{4n^2-4n+3}{2n^3+3n+4}$;

4) $x_n=\dfrac{(n+1)^2}{2n^2}$; 5) $x_n=\dfrac{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^n}}$; 6) $x_n=\left(1+\dfrac{1}{n}\right)^{n+1}$;

7) $x_n=\dfrac{(n+2)!+(n+1)!}{(n+3)!}$; 8) $x_n=\dfrac{(n+1)^3-(n-1)^3}{(n+1)^2+(n-1)^2}$;

9) $x_n=\dfrac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$; 10) $x_n=\dfrac{n^3-100n^2+1}{100n^2+15n}$;

IV. Berilgan limitlarni hisoblang.

_1 – v a r i a n t._

1) $\lim\limits_{x\to 2}\frac{x^3-8}{x^2+x-6}$ 2) $\lim\limits_{n\to\infty}\frac{\sqrt[3]{n^3+2n-1}}{n+2}$ 3) $\lim\limits_{x\to 2}\frac{x^3-7x+6}{x^3-5x^2+2x+8}$

4) $\lim\limits_{x\to -2}\frac{5x^2+13x+6}{3x^2+2x-8}$ 5) $\lim\limits_{x\to\infty}\left(\sqrt{2x^2+8x+3}-\sqrt{2x^2+4x+3}\right)$

6) $\lim\limits_{x\to 0}\frac{1-\cos 2x}{x\sin x}$ 7) $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^x$

_2 – v a r i a n t._

1) $\lim\limits_{x\to -1}\frac{x^2-x-2}{x^3+1}$; 2) $\lim\limits_{n\to\infty}\frac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt[3]{n^3+1}}$; 3) $\lim\limits_{x\to 1}\frac{x^2+2x-3}{x^2-1}$;

4) $\lim\limits_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$; 5) $\lim\limits_{x\to\infty}\frac{5x^3-7x}{1-2x^3}$; 6) $\lim\limits_{x\to 0}\frac{\sin^2\frac{x}{2}}{x^2}$; 7) $\lim\limits_{x\to\infty}(1+2x)^{\frac{5}{x}}$;

3 – variant.

1) $\lim\limits_{x\to 4}\dfrac{x^2-16}{x^2+x-20}$; 2) $\lim\limits_{n\to\infty}\dfrac{1000n^3+3n^2}{0{,}001n^3-100n^2+1}$; 3) $\lim\limits_{x\to 0}\dfrac{2x^2}{\sqrt{5-x}-\sqrt{5+x}}$;

4) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-2x-3}$; 5) $\lim\limits_{x\to\infty}\dfrac{2x^2+7x-2}{3x^2-x-4}$; 6) $\lim\limits_{x\to 0}\dfrac{\sin\frac{x}{3}}{x}$; 7) $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$;

4 – variant.

1) $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{3x^2-9x}$; 2) $\lim\limits_{n\to\infty}\dfrac{n!}{(n+1)!-n!}$; 3) $\lim\limits_{x\to -1}\dfrac{x+1}{\sqrt{6x^2+3}+3x}$;

4) $\lim\limits_{x\to 2}\dfrac{x^3-8}{2x^2-9x+10}$; 5) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+3x}-1}$; 6) $\lim\limits_{x\to 0}\dfrac{1-\cos 8x}{x^2}$; 7) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{3x-7}$;

5 – variant.

1) $\lim\limits_{x\to 2}\dfrac{x^3-4x^2+7x-6}{x^3-5x^2+2x+8}$; 2) $\lim\limits_{x\to 0}\dfrac{2x^2}{\sqrt{5-x}-\sqrt{5+x}}$; 3) $\lim\limits_{x\to\infty}\left(x^3-6x^2+5x-1\right)$;

4) $\lim\limits_{x\to 3}\dfrac{x^2-6x+1}{3x-9}$; 5) $\lim\limits_{x\to 0}\dfrac{4x^3-2x^2+5x}{3x^2+7x}$; 6) $\lim\limits_{x\to 0}\dfrac{\sin\frac{x}{2}}{x}$; 7) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{3x-7}$;

6 – v a r i a n t.

1) $\lim\limits_{x\to 0}\dfrac{x^2+2x}{x^3+3x}$; 2) $\lim\limits_{x\to -2}\left(\dfrac{1}{x+2}-\dfrac{12}{x^3+8}\right)$; 3) $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2-4x}\right)$;

4) $\lim\limits_{x\to 2}\dfrac{2x^2-7x+6}{x^2-5x+6}$; 5) $\lim\limits_{x\to 1}\dfrac{\sqrt{3+2x}-\sqrt{x+4}}{3x^2-4x+1}$; 6) $\lim\limits_{x\to 0}\dfrac{\sin^2 2x}{x^2}$; 7) $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x+4}\right)^{3x+2}$;

7 – v a r i a n t.

1) $\lim\limits_{x\to 4}\dfrac{3x^2-10x-8}{4x^2+6x-64}$; 2) $\lim\limits_{x\to 4}\dfrac{\sqrt{21+x}-5}{x^3-64}$; 3) $\lim\limits_{x\to\infty}\dfrac{x^2+3x-5}{3x^2+1}$;

4) $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{\sqrt{5-x}-\sqrt{x+1}}$; 5) $\lim\limits_{x\to\infty}\dfrac{3x^4-2x+1}{3x^2+2x-5}$; 6) $\lim\limits_{x\to 0}\dfrac{\sin 7x-\sin 3x}{x\sin x}$; 7) $\lim\limits_{x\to\infty}\left(\dfrac{2x}{2x-3}\right)^{2-5x}$;

8 – v a r i a n t.

1) $\lim\limits_{x\to 3}\dfrac{3x^2-7x-6}{2x^2-7x+3}$; 2) $\lim\limits_{x\to 5}\dfrac{\sqrt{x+4}-3}{\sqrt{x-1}-2}$; 3) $\lim\limits_{x\to\infty}\dfrac{4x^2+8x+5}{5x^3+4}$;

4) $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$; 5) $\lim\limits_{x\to 3}\dfrac{\sqrt{4x-3}-3}{x^2-9}$; 6) $\lim\limits_{x\to 0}\dfrac{tg3x-\sin 3x}{2x^2}$; 7) $\lim\limits_{x\to -\infty}\left(\dfrac{4x+3}{2x-5}\right)^{1+7x}$;

9 – v a r i a n t.

1) $\lim\limits_{x\to 1}\dfrac{4x^4-5x^2+1}{x^2-1}$; 2) $\lim\limits_{x\to\infty}\dfrac{8x^2+4x-5}{4x^2-3x+2}$; 3) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x}-x\right)$;

4) $\lim\limits_{x\to 1}(1-x)\cdot\dfrac{3}{1-x^3}$; 5) $\lim\limits_{x\to 0}\dfrac{3x^2+x}{4x^2-5x}$; 6) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{1-\sin 2x}{\pi-4x}$; 7) $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{2x-1}\right)^{x+2}$;

10 – v a r i a n t.

1) $\lim\limits_{x\to 5}\dfrac{3x^2-6x-45}{2x^2-3x-35}$; 2) $\lim\limits_{x\to -2}\dfrac{x^2+2x}{3x^2+x-10}$; 3) $\lim\limits_{x\to\infty}\dfrac{x-2x^2+5x^4}{x^4+3x^2+2}$;

4) $\lim\limits_{x\to 3}\dfrac{x^3-27}{\sqrt{3x}-x}$; 5) $\lim\limits_{x\to\infty}\dfrac{4x^2-10x+7}{2x^3-3x}$; 6) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x\sin x}$; 7) $\lim\limits_{x\to\infty}\left(\dfrac{x-2}{x+1}\right)^{2x-3}$;

F O Y D A L A N I L G A N   A D A B I Y O T L A R.

1. G.M.Fixtengols. "Matematik analiz asoslari" 1 – jild, "O'qituvchi", Toshkent 1970 yil.

2. N.S.Piskunov. Differensial va integral hisob. 1 – jild, "O'qituvchi", Toshkent 1972 yil.

3. T.Jo'raev va boshqalar. Oliy matematika asoslari. 1 – qism, "O'zbekiston", Toshkent 1995 yil.

4. B.A. Abdalimov. Oliy matematika, "O'qituvchi" Toshkent 1994 yil.

5. Sh.I.Tojiev. Oliy matematikadan masalalar yechish. "O'zbekiston", Toshkent 2002 yil.

6. I.A.Maron.Differensialnoe i integralnoe ischislenie v primerax i zadachax "Nauka", M.1973.

7. T.Sharifova, E.Yo'ldoshev. Matematik analizdan misol va masalalar yechish "O'qituvchi", Toshkent 1996 yil.

M U N D A R I J A.

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