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JUMP THEOREMS FOR THE BOCHNER-MARTINELLI INTEGRAL IN DOMAINS WITH A PIECEWISE SMOOTH BOUNDARY

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Abstract

Jump theorems for the Bochner-Martinelli integral in domains with a piecewise smooth boundary are obtained. Moreover, theorem for the Bochner-Martinelli integral in domains with a piecewise smooth boundary is proved for continuous functions and also for functions from the class $\mathcal{L}^p$.

Keywords: Bochner-Martinelli integral, Cauchy integral, domain with a piecewise smooth boundary.

Mathematics Subject Classification (2010): 17A32, 17A70, 17B30.

Introduction

The Bochner-Martinelli integral representation for holomorphic functions of many complex variables appeared in the works by Martinelli and Bochner ([21, 22, 18]) in the early 40's of the last century. It was the first essentially multidimensional representation in which integration took place over the whole boundary of the domain. This integral representation has a universal kernel not depending on the type of the domain, just like the Cauchy kernel in $\mathbb{C}^1$. But in the space $\mathbb{C}^n$ at $n > 1$, the Bochner-Martinelli kernel was a harmonic, not holomorphic, function. For a long time, this circumstance prevented the widespread use of the Bochner-Martinelli integral in multidimensional complex analysis. Martinelli and Bochner used their presentation to prove the Hartogs (Osgood-Brown) theorem on the erasure of compact singularities of holomorphic functions in $\mathbb{C}^n$ at $n > 1$.

The jump problem was one of the first problems considered for the Bochner-Martinelli integral (see [20, 6, 4, 19]). It turned out that the smoothness properties of the function and the boundaries of the domain for it are the same as for the Cauchy integral. As a rule, it is simpler to prove the jump theorem than the Sokhotsky-Plemelj formula and, in addition, the difference $F^+ - F^-$ can have a limit on the boundary ∂D, but the functions F^+ and F^- do not have it. Therefore, the jump theorem holds for wider classes of functions than the Sokhotsky-Plemelj formulas.

The boundary behavior of the Bochner-Martinelli integral has been studied by many authors by analogy with the Cauchy integral for domains with the smooth boundary under various additional assumptions [20, 6, 4, 19, 7, 12, 16, 17]. These studies have been based on the fact that the Bochner-Martinelli integral is the sum of the double layer potential and the tangent derivative of the simple layer potential. Therefore, the jump in the Bochner-Martinelli integral coincides with the integrand, but behaves when approaching the boundary of the domain as the Cauchy integral (an integral of Cauchy type), i.e. slightly worse than the potential of the double layer. So the Bochner-Martinelli integral combines the properties of the Cauchy integral and the double layer potential.

In the present paper, we study the boundary behavior of the Bochner-Martinelli integral in domains with a piecewise smooth boundary. For boundaries with conical points, its behavior was considered in [11], and for domains whose boundary contains conical edges it was considered in [5].

1 The jump theorem for the Bochner-Martinelli integral of continuous functions

In this section, we study the boundary behavior of the Bochner-Martinelli integral of continuous functions in domains with a piecewise smooth boundary. Here it is proved that the Bochner-Martinelli type integral of continuous functions behaves at the boundary points where smoothness is violated, just as in the smooth case. The proved jump theorem shows that, the difference $F^+ - F^-$ can have a limit on ∂D, while the functions F^+ and F^- can have no limit. Therefore, this jump theorem is valid for wider classes of functions than the Sokhotsky-Plemelj formulas.

Let D be a bounded domain in $\mathbb{C}^n$ with a piecewise smooth boundary. The function F is defined by the following formula

$$F(z) = \int\limits_{\partial D} f(\zeta) U(\zeta, z), \ z \notin \partial D, \tag{1}$$

i.e. F is a Bochner-Martinelli type integral, $f \in \mathcal{C}(\partial D)$. Recall that $U(\zeta, z)$ is a closed differential form (of the type $(n, n-1)$) having the form

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} \, d\bar{\zeta}[k] \wedge d\zeta,$$

where $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$, and $d\bar{\zeta}[k] = d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{k-1} \wedge d\bar{\zeta}_{k+1} \wedge \cdots \wedge d\bar{\zeta}_n$.

We denote by F^+ the integral (1) for points $z \in D$, and by F^- – the integral (1) for points $z \notin \overline{D}$.

Consider a straight circular two-sheeted cone $V_{z^0} = V^+_{z^0} \cup V^-_{z^0}$ with a vertex at the point $z^0 \in \partial D$ with a sufficiently small angle β between the axis and the generatrix such that there exists a neighborhood Ω_{z^0}, for which $V^+_{z^0} \cap \Omega_{z^0} \subset \overline{D}$, $\ V^-_{z^0} \cap \Omega_{z^0} \subset (\mathbb{C}^n \backslash D)$. Existence of such a cone follows from piecewise smoothness of ∂D. This cone consists of lines that are not tangent to each smooth component of ∂D at the point z^0. Take two points $z^+ \in V_{z^0} \cap D$ and $z^- \in V_{z^0} \cap (\mathbb{C}^n \setminus \overline{D})$ such that $a|z^+ - z^0| \leqslant |z^- - z^0| \leqslant b|z^+ - z^0|$, where a And b are some constants independent of $z^\pm$, $0 < a \leqslant b < \infty$.

Theorem 1. *If $f \in \mathcal{C}(\partial D)$, then the limit*

$$\lim_{z^\pm \to z^0} (F^+(z^+) - F^-(z^-)) = f(z^0) \tag{2}$$

exists for any point $z^0 \in \partial D$ and is reached uniformly (the angle β, and constants a and b are fixed).

Theorem 1 for domains with a smooth boundary is proved in [4] (see also [7, 19]). On the plane $\mathbb{C}$, it turns into a jump theorem for an integral of Cauchy type.

Proof. If smoothness is not violated at a point $z^0 \in \partial D$, then the statement of the theorem follows from the result of [4].

Consider the points from ∂D where smoothness is violated. Let the boundary D be not smooth at the point $z^0 \in \partial D$. We denote by $B(z^0, \varepsilon)$ a ball with the radius $\varepsilon > 0$ centered at the point $z^0 \in \partial D$. Then

$$D \cap B(z^0, \varepsilon_0) = \{z \in B(z^0, \varepsilon_0) : \rho_1(z) < 0, ..., \rho_l(z) < 0\},$$

$l \leqslant n$, where $\rho_1(z^0) = ... = \rho_l(z^0) = 0$, $\rho_j \in \mathcal{C}^1(B(z^0, \varepsilon_0))$, ρ_j is real-valued functions. We have $\partial D \cap B(z^0, \varepsilon_0) = \Gamma$, $\Gamma = \Gamma_1 \cup ... \cup \Gamma_l$ where

$$\Gamma_j = \{z \in B(z^0, \varepsilon_0) : \rho_j(z) = 0,\ \rho_k(z) \leqslant 0,\ k \neq j\},$$

Γ_j are smooth surfaces with a piecewise boundary.

Consider the difference

$$F^+(z^+) - F^-(z^-) = \int\limits_{\partial D} (f(\zeta) - f(z^0)) U(\zeta, z^+) -$$

$$- \int\limits_{\partial D} (f(\zeta) - f(z^0)) U(\zeta, z^-) + f(z^0) \int\limits_{\partial D} (U(\zeta, z^+) - U(\zeta, z^-)).$$

Since

$$\int\limits_{\partial D} (U(\zeta, z^+) - U(\zeta, z^-)) = 1,$$

it is enough for us to prove that

$$\lim_{z^\pm \to z^0} \int\limits_{\partial D} (f(\zeta) - f(z^0))(U(\zeta, z^+) - U(\zeta, z^-)) = 0.$$

One can pass to the limit under the integral sign in

$$\int\limits_{\partial D \setminus \Gamma} (f(\zeta) - f(z^0))(U(\zeta, z^+) - U(\zeta, z^-)).$$

Since $z^0 \notin \partial D \setminus \Gamma$,

$$\lim_{z^{\pm}\to z^0} \int\limits_{\partial D\setminus\Gamma} (f(\zeta)-f(z^0))(U(\zeta,z^+)-U(\zeta,z^-))=0.$$

It remains to consider this integral over the set Γ. Since $\Gamma=\Gamma_1\cup...\cup\Gamma_l$, it is enough to show that for any Γ_j, $j=1,...,l$,

$$\lim_{z^{\pm}\to z^0} \int\limits_{\Gamma_j} (f(\zeta)-f(z^0))(U(\zeta,z^+)-U(\zeta,z^-))=0. \tag{3}$$

For any Γ_j, make a unitary transformation into $\mathbb{C}^n$ and a shift such that the point z^0 passes to zero, and the tangent plane to Γ_j at z^0 passes to the plane $T=\{w\in\mathbb{C}^n : \operatorname{Im} w_n=0\}$. Moreover, the surface Γ_j in a neighborhood of zero will be given by the system of equations

$$\begin{cases} \zeta_1 &= & w_1, \\ \vdots & & \\ \zeta_{n-1} &= & w_{n-1}, \\ \zeta_n &= & u_n+i\varphi(w), \end{cases}$$

where $w=(w_1,\cdots,w_{n-1},u_n)\in T$, a function $\varphi\in\mathcal{C}(W)$, W is a neighborhood of zero of the plane T, and $\varphi(w)=o(|w|)$ at $w\to 0$. Denote by $\tilde{z}^{\pm}$ the projections of points $z^{\pm}$ on the axis $\operatorname{Im} w_n$. Then

$$|z^{\pm}-\tilde{z}^{\pm}|\leqslant|\tilde{z}^{\pm}|\operatorname{tg}\beta,$$

$$|z^{\pm}|\leqslant\frac{\tilde{z}^{\pm}}{\cos\beta},\quad a|\tilde{z}^{\pm}|\cos\beta\leqslant|\tilde{z}^{-}|\leqslant\frac{b|\tilde{z}^{+}|}{\cos\beta}. \tag{4}$$

Fix $\varepsilon>0$ and take a $(2n-1)$-dimensional ball B' in the plane T centered at 0 with the radius ε such that:

1) $B'\subset W$,

2) $|w-\tilde{z}^{\pm}|\leqslant C|\zeta(w)-z^{\pm}|$ for $w\in B'$ where C is a constant independent of the point $z^0=0$.

Here $B'=B(z^0,\varepsilon)\cap T$, and Γ'_j is the projection of Γ_j onto T. By virtue of the condition 2) imposed when choosing the ball B' and the inequality $|\zeta(w)|\leqslant C|w|\leqslant C|w-\tilde{z}^{\pm}|$, we obtain

$$\left|\frac{\bar{\zeta}_k}{|\zeta-z^+|^{2n}}-\frac{\bar{\zeta}_k}{|\zeta-z^-|^{2n}}\right|=$$

$$=\left|\frac{1}{|\zeta-z^+|}-\frac{1}{|\zeta-z^-|}\right|\sum_{i=0}^{2n-1}\frac{|\zeta_k|}{|\zeta-z^+|^i|\zeta-z^-|^{2n-i-1}}= \tag{5}$$

$$=||\zeta-z^+|-|\zeta-z^-|||\zeta_k|\sum_{i=0}^{2n-1}\frac{1}{|\zeta-z^+|^{i+1}|\zeta-z^-|^{2n-i}}\leqslant$$

$$\leqslant C_1 C^{2n} \sum_{i=0}^{2n-1} \frac{|z^+| + |z^-|}{|w - \tilde{z}^+|^i |w - \tilde{z}^-|^{2n-i}}.$$

We can assume that $a_1 = a\cos\beta < 1$. Then taking into account inequalities (4), we get

$$|w - \tilde{z}^{\pm}| \geqslant |w - a_1 \tilde{z}^{\pm}|.$$

Therefore, we have from (5)

$$\left| \frac{\bar{\zeta}_k}{|\zeta - z^+|^{2n}} - \frac{\bar{\zeta}_k}{|\zeta - z^-|^{2n}} \right| \leqslant \frac{d|\tilde{z}^{\pm}|}{|w - a_1 \tilde{z}^{\pm}|^{2n}},$$

where d depends only on a, b, C,n C_1, β. Similarly,

$$\left| \frac{\bar{z}_k}{|\zeta - z^+|^{2n}} - \frac{\bar{z}_k}{|\zeta - z^-|^{2n}} \right| \leqslant$$

$$\leqslant \frac{|\bar{z}_k|}{|\zeta - z^+|^{2n}} + \frac{|\bar{z}_k|}{|\zeta - z^-|^{2n}} \leqslant \frac{d_1 |\tilde{z}^+|}{|w - a_1 \tilde{z}^+|^{2n}}.$$

And, finally $d\,\sigma_j \leqslant d_2\, dS$ where dS is an element of the surface T, and $d\sigma_j$ is an element of Γ_j, also d_2 does not depend on the point z^0. One can pass to the limit under the integral sign in the integral (2.3) over $\Gamma_j \setminus (\Gamma_j \cap B(z^0, \varepsilon))$:

$$\lim_{z^{\pm} \to z^0} \int\limits_{\Gamma_j \setminus (\Gamma_j \cap B(z^0,\varepsilon))} (f(\zeta) - f(0))(U(\zeta, z^+) - U(\zeta, z^-)) = 0.$$

Therefore it remains to consider the integral

$$\left| \int\limits_{\Gamma_j \cap B(z^0,\epsilon)} (f(\zeta) - f(0))(U(\zeta, z^+) - U(\zeta, z^-)) \right| \leqslant$$

$$\leqslant d_3 \int\limits_{\Gamma'_j \cap B'} \frac{|(f(\zeta(w)) - f(0))||\tilde{z}^+|}{(|w|^2 + a_1^2 |\tilde{z}^+|^2)^n}\, d\,S. \qquad (6)$$

The expression $\dfrac{|\tilde{z}^+|}{(|w|^2 + a_1^2|\tilde{z}^+|^2)^n}$ is (up to a constant) the Poisson kernel for a half-space. Since f is continuous on ∂D, choose for any $\delta > 0$ a ball B' of the radius ε such that $|(f(\zeta(w)) - f(0))| < \delta$ for $w \in B'$ (moreover, ε can be chosen not depending on the point $z^0 = 0$). Therefore we obtain from (6) that

$$\left| \int\limits_{\Gamma_j \cap B(z^0,\epsilon)} (f(\zeta) - f(0))(U(\zeta, z^+) - U(\zeta, z^-)) \right| \leqslant$$

$$\leqslant d_4\delta \int\limits_{\Gamma_j'\cap B'} \frac{a_1|\tilde{z}^+|}{(|w|^2 + a_1^2|\tilde{z}^+|^2)^n}\, dS \leqslant$$

$$\leqslant d_4\delta \int\limits_{T} \frac{a_1|\tilde{z}^+|}{(|w|^2 + a_1^2|\tilde{z}^+|^2)^n}\, dS.$$

The last integral is equal to a constant independent of $\tilde{z}^+$ (see, for example, [14]).

To complete the proof it remains to observe that

$$\lim_{z^\pm\to z^0} \int\limits_{\Gamma_j\cap B(z^0,\varepsilon)} (f(\zeta) - f(0))(U(\zeta, z^+) - U(\zeta, z^-)) = 0,$$

moreover, this limit is reached uniformly, since the radius of the ball $B(z^0, \varepsilon)$ does not depend on z^0.

The proof of Theorem 1 shows that it can be formulated fro integrable functions.

Theorem 2. *If z^0 is the Lebesgue point of a function $f \in \mathcal{L}^1(\partial D)$, then*

$$\lim_{z^\pm\to z^0}(F^+(z^+) - F^-(z^-)) = f(z^0).$$

Theorem 2 is proved similarly to Theorem 1.

Corollary 1. *If, under the conditions of Theorem 1, a function F^+ is continuously extended onto $\overline{D}$, then F^- is also continuously extended onto $\mathbb{C}^n \setminus D$, and vice versa.*

Proof. Now choose the cones V_z so that the axes of them form a continuous family, i.e. the axes of the cones depend continuously on z. Let a function F^- be continuously extended on $\mathbb{C}^n \setminus D$. We need to show that F^+ is continuously extended onto $\overline{D}$.

Let $z^0 \in \partial D$ and $z \to z^0$, $z \in D$. We need to show that

$$\lim_{z\to z^0} F^+(z) = f(z^0) - F^-(z^0).$$

Consider sufficiently small $\varepsilon > 0$ and the ball B_{z^0} centered at the point z^0 of the radius ε. Through the point $z \in D \cap B_{z^0}$, draw a straight line l parallel to the axis of the cone V_{z^0} until it intersects with the boundary. We denote by z' the intersection point of l with ∂D, and by $z'' \in (\mathbb{C}^n \setminus \overline{D})$ – a point lying on this line and symmetric to z with respect to z'. For sufficiently small ε, the points z and z'' lie in the cone $V_{z'}$. Then

$$|z - z'| \leqslant |z - z^0|,\ |z' - z^0| \leqslant |z - z^0|,$$

$$|F^+(z) - (f(z^0) - F^-(z^0))| \leqslant$$

$$\leqslant |F^+(z) - (f(z^0) - F^-(z^0)) + f(z') - f(z') + F^-(z'') - F^-(z'')| \leqslant$$

$$\leqslant |F^+(z) - F^-(z'') - f(z')| + |F^-(z'') + F^-(z^0)| +$$

$$+ |f(z') - f(z^0)| < \varepsilon$$

at sufficiently small $|z - z^0|$ since due to Theorem 1, $|F^+(z) - F^-(z'') - f(z')|$ is estimated by a value tending to zero at $\varepsilon \to 0$.

2 Jump theorem of the Bochner-Martinelli integral for functions of the class $\mathcal{L}^p$

In this section, we prove the jump theorem of the Bochner-Martinelli integral for functions of the class $\mathcal{L}^p$ in domains with a piecewise smooth boundary.

Let, as before, D be a bounded domain in $\mathbb{C}^n$ with the piecewise smooth boundary ∂D, a function $f \in \mathcal{L}^p(\partial D), 1 \leqslant p < \infty$. As the defining functions of D, we take $\rho_j \in \mathcal{C}^1(\mathbb{C}^n)$, $j = 1, ..., s_0$. Then

$$D = \{z \in \mathbb{C}^n : \rho_1(z) < 0, ..., \rho_{s_0}(z) < 0\},$$

$$\tilde{\Gamma}_j = \{z \in \partial D : \rho_j(z) = 0\},\ \partial D = \bigcup_{j=1}^{s_0} \tilde{\Gamma}_j.$$

Here $\tilde{\Gamma}_j$ are smooth surfaces with a piecewise smooth boundary. Denote by $\nu_j(\zeta)$ the unit vector of the external normal to the surface $\tilde{\Gamma}_j$ at the point ζ.

Theorem 3. *If $F(z)$ is the integral of the form (1), then*

$$\lim_{\varepsilon \to +0} \sum_{j=1}^{s_0} \int\limits_{\tilde{\Gamma}_j} |F(z - \varepsilon\nu_j(z)) - F(z + \varepsilon\nu_j(z)) - f(z)|^p d\sigma = 0,$$

in addition

$$\sum_{j=1}^{s_0} \int\limits_{\tilde{\Gamma}_j} |F(z - \varepsilon\nu_j(z)) - F(z + \varepsilon\nu_j(z))|^p d\sigma \leqslant C \int\limits_{\partial D} |f|^p d\sigma \tag{7}$$

where the constant C does not depend on f and ε (for sufficiently small ε, the point $z - \varepsilon\nu_j(z) \in D$, and the point $z + \varepsilon\nu_j(z) \in (\mathbb{C}^n \setminus \overline{D})$).

Proof. Since

$$\lim_{\varepsilon \to +0} \sum_{j=1}^{s_0} \int\limits_{\tilde{\Gamma}_j} |F(z - \varepsilon\nu_j(z)) - F(z + \varepsilon\nu_j(z)) - f(z)|^p d\sigma =$$

$$= \lim_{\varepsilon \to +0} \int\limits_{\tilde{\Gamma}_1} |F(z - \varepsilon\nu_1(z)) - F(z + \varepsilon\nu_1(z)) - f(z)|^p d\sigma + \ ... +$$

$$+ \lim_{\varepsilon \to +0} \int\limits_{\tilde{\Gamma}_{s_0}} |F(z - \varepsilon\nu_{s_0}(z)) - F(z + \varepsilon\nu_{s_0}(z)) - f(z)|^p d\sigma,$$

it is enough to prove

$$\lim_{\varepsilon \to +0} \int\limits_{\tilde{\Gamma}_1} |F(z - \varepsilon\nu_1(z)) - F(z + \varepsilon\nu_1(z)) - f(z)|^p d\sigma = 0$$

for one $\tilde{\Gamma}_j$, for example, for $\tilde{\Gamma}_1$. The remaining terms are estimated similarly.

Denote $z^+ = z - \varepsilon\nu_1(z)$, and $z^- = z + \varepsilon\nu_1(z)$. For each point $\zeta \in \Gamma_1$, choose a ball $B(\zeta, r)$ with the radius r independent of ζ such that for $z \in \Gamma_1 \cap B(\zeta, r)$, the inequality

$$|\zeta - z^{\pm}|^2 \geqslant k(|w-\zeta|^2 + \varepsilon^2)$$

holds (k does not depend on ζ and ε) at $\varepsilon < \frac{r}{2}$, where w is the projection of z onto the tangent plane T_ζ to the surface Γ_1 at the point ζ. This can always be done due to the fact that

$$||\zeta - w| - |\zeta - z|| \leqslant |w - z| = o(|\zeta - w|)$$

at $w \to \zeta$ (see the proof of Theorem 1). We have

$$\int\limits_{\tilde{\Gamma}_1} |F(z^+) - F(z^-) - f(z)|^p d\sigma =$$

$$= \int\limits_{\tilde{\Gamma}_1} d\sigma(z) \left| \int\limits_{\tilde{\Gamma}_1} (f(\zeta) - f(z))(U(\zeta, z^+) - U(\zeta, z^-)) \right|^p \leqslant$$

$$\leqslant \int\limits_{\tilde{\Gamma}_1} d\sigma(z) \left(\int\limits_{\tilde{\Gamma}_1} |U(\zeta, z^+) - U(\zeta, z^-)| \right)^{p-1} \int\limits_{\tilde{\Gamma}_1} |f(\zeta) - f(z)|^p \times$$

$$\times |U(\zeta, z^+) - U(\zeta, z^-)|$$

by virtue of the Jensen inequality (see, for example, [15]) applied to the integral

$$\left(\int\limits_{\tilde{\Gamma}_1} |f(\zeta) - f(z)||U(\zeta, z^+) - U(\zeta, z^-)| d\sigma \right)^p .$$

The integral

$$\int\limits_{\tilde{\Gamma}_1} |U(\zeta, z^+) - U(\zeta, z^-)|$$

was estimated in Theorem 1, where it was shown that it is bounded by the constant independent of ε, and the integral

$$\int\limits_{\tilde{\Gamma}_1} d\sigma(z) \int\limits_{\tilde{\Gamma}_1} |f(\zeta) - f(z)|^p |U(\zeta, z^+) - U(\zeta, z^-)| \leqslant$$

$$\leqslant C_1 \sum_{m=1}^{n} \int\limits_{\tilde{\Gamma}_1} d\sigma(\zeta) \int\limits_{\tilde{\Gamma}_1} |f(\zeta) - f(z)|^p \left| \frac{\bar{\zeta}_m - \bar{z}_m^+}{|\zeta - z^+|^{2n}} - \frac{\bar{\zeta}_m - \bar{z}_m^-}{|\zeta - z^-|^{2n}} \right| d\sigma(z).$$

If $z \in B(\zeta, r) \cap \Gamma_1$, then

$$\left|\frac{\bar{\zeta}_m - \bar{z}_m}{|\zeta - z^+|^{2n}} - \frac{\bar{\zeta}_m - \bar{z}_m}{|\zeta - z^-|^{2n}}\right| =$$

$$= |\bar{\zeta}_m - \bar{z}_m)|||\zeta - z^+| - |\zeta - z^-|| \sum_{j=0}^{2n-1} \frac{1}{|\zeta - z^+|^{j+1}|\zeta - z^-|^{2n-j}} \leqslant$$

$$\leqslant \frac{6\varepsilon n}{k^n(|w - \varepsilon|^2 + \varepsilon^2)^n},$$

and

$$\left|\frac{\varepsilon \nu_{1_m}}{|\zeta - z^+|^{2n}} + \frac{\varepsilon \nu_{1_m}}{|\zeta - z^-|^{2n}}\right| \leqslant \frac{2\varepsilon}{k^n(|w - \zeta|^2 + \varepsilon^2)^n}.$$

Then

$$\int\limits_{\tilde{\Gamma}_1 \cap B(\zeta, r)} |f(\zeta) - f(z)|^p \left|\frac{\bar{\zeta}_m - \bar{z}_m^+}{|\zeta - z^+|^{2n}} - \frac{\bar{\zeta}_m - \bar{z}_m^-}{|\zeta - z^-|^{2n}}\right| d\sigma(z) \leqslant$$

$$\leqslant d \int\limits_{T_\zeta \cap B(\zeta, r)} |f(\zeta) - f(z(w))|^p \frac{\varepsilon}{(|w - \zeta|^2 + \varepsilon^2)^n} dS(w) = d \cdot I_1.$$

Introducing the variable $t = \frac{\varepsilon}{w - \zeta}$ $(t \in \mathbb{R}^{2n-1})$, we obtain

$$I_1 = \int\limits_{\{\varepsilon|t| < r\}} \frac{|f(\zeta) - f(z(\zeta + \varepsilon t))|^p}{(|t|^2 + 1)^n} dS(t),$$

and the integral

$$I_\varepsilon(t) = \int\limits_{\tilde{\Gamma}_1} |f(\zeta) - f(z(\zeta + \varepsilon t))|^p d\sigma(\zeta)$$

tends to zero at $\varepsilon \to +0$ for the fixed t. Moreover,

$$I_\varepsilon(t) \leqslant A\|f\|^p_{\mathcal{L}^p}.$$

Therefore

$$\int\limits_{\tilde{\Gamma}_1} d\sigma(\zeta) \int\limits_{\{\varepsilon|t| < r\}} \frac{|f(\zeta) - f(z(\zeta + \varepsilon t))|^p}{(|t|^2 + 1)^n} dS(t) =$$

$$= \int\limits_{\{\varepsilon|t| < r\}} \frac{I_\varepsilon(t)}{(|t|^2 + 1)^n} dS(t) \leqslant \int\limits_{\mathbb{R}^{2n-1}} \frac{I^*_\varepsilon(t)}{(|t|^2 + 1)^n} dS(t)$$

where $I^*_\varepsilon(t) = I_\varepsilon(t)$ in the ball $\{t : \ \varepsilon|t| < r\}$ and $I^*_\varepsilon(t) = 0$ outside this ball. In the last integral, we can pass to the limit under the integral sign for $\varepsilon \to +0$ by virtue of the Lebesgue theorem on bounded convergence.

It remains to consider the integral

$$\int\limits_{\tilde{\Gamma}_1} d\sigma(\zeta) \int\limits_{\tilde{\Gamma}_1 \backslash B(\zeta,r)} |f(\zeta) - f(z)|^p \left| \frac{\bar{\zeta}_m - \bar{z}_m^+}{|\zeta - z^+|^{2n}} - \frac{\bar{\zeta}_m - \bar{z}_m^-}{|\zeta - z^-|^{2n}} \right| d\sigma(z).$$

Since $|\zeta - z| \geqslant r$, we get $|\zeta - z^{\pm}| \geqslant ||\zeta - z| - |z - z^{\pm}|| \geqslant r - \varepsilon > \frac{r}{2}$. Then

$$\left| \frac{\bar{\zeta}_m - \bar{z}_m}{|\zeta - z^+|^{2n}} - \frac{\bar{\zeta}_m - \bar{z}_m}{|\zeta - z^-|^{2n}} \right| \leqslant$$

$$\leqslant |\bar{\zeta}_m - \bar{z}_m||z^+ - z^-| \sum_{i=0}^{2n-1} \frac{1}{|\zeta - z^+|^{j+1}|\zeta - z^-|^{2n-j}} \leqslant d_1\varepsilon,$$

and

$$\left| \frac{\varepsilon \nu_{1_m}}{|\zeta - z^+|^{2n}} + \frac{\varepsilon \nu_{1_m}}{|\zeta - z^-|^{2n}} \right| \leqslant d_2\varepsilon,$$

i.e.

$$\int\limits_{\tilde{\Gamma}_1} d\sigma(\zeta) \int\limits_{\tilde{\Gamma}_1 \backslash B(\zeta,r)} |f(\zeta) - f(z)|^p \left| \frac{\bar{\zeta}_m - \bar{z}_m^+}{|\zeta - z^+|^{2n}} - \frac{\bar{\zeta}_m - \bar{z}_m^-}{|\zeta - z^-|^{2n}} \right| d\sigma(z) \leqslant d_3\varepsilon \left(\int\limits_{\tilde{\Gamma}_1} |f|^p d\sigma \right)^2 .$$

The inequality (7) is proved similarly. Theorem 3 for domains with smooth boundaries is cited at functions $f \in \mathcal{L}^1(\partial D)$ in [8], at $f \in \mathcal{L}^p(\partial D)$ – in [9], and it belongs to Kytmanov A.M.

3 Jump theorem for the complex normal derivative of the Bochner-Martinelli integral

In this section, we prove the jump theorem for a $\bar{\partial}$-normal derivative of the Bochner-Martinelli integral in domains with a piecewise twice smooth boundary.

Let D be a bounded domain in $\mathbb{C}^n$ with a piecewise smooth boundary ∂D from the class $\mathcal{C}^2$. Denote by $\Omega(\overline{D})$ a neighborhood of the domain D, by Ω_{z^0} – a neighborhood of the point $z^0 \in \partial D$. As the defining functions of D, we take $\rho_j \in \mathcal{C}^2(\mathbb{C}^n)$, $j = 1, ..., s_0$. Then

$$D = \{z \in \Omega(\bar{D}) : \rho_1(z) < 0, ..., \rho_{s_0}(z) < 0\},$$
$$\tilde{\Gamma}_j = \{z \in \Omega(\overline{D}) : \rho_j(z) = 0,\ j = 1, ..., s_0\},$$

$\partial D \cap \Omega_{z^0} = \Gamma,\ \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_l$, where

$$\Gamma_j = \{z \in \Omega_{z^0} : \rho_j(z) = 0, \rho_k(z) \leq 0, k \neq j, k = 1, \ldots, l\},$$

Γ_j are smooth surfaces with a piecewise smooth boundary, $\rho_j(z)$ is the distance from the point z to Γ_j, i.e. $\rho_j(z) = -\mathrm{dist}(z, \Gamma_j)$ if $z \in D$. Since $\partial D \in \mathcal{C}^2$, we can state the following (see [1], and also [3]):

a) there exists a neighborhood Ω of the surface $\tilde{\Gamma}_j$ such that $\rho_j \in \mathcal{C}^2(\Omega)$;

b) $|\mathrm{grad}\rho_j| = \sqrt{\sum\limits_{k=1}^{n} \left|\frac{\partial \rho_j}{\partial z_k}(z)\right|^2} \equiv \dfrac{1}{2}$ in Ω;

c) $\dfrac{\partial \rho_j}{\partial z_k}(z^{\pm}) = \dfrac{\partial \rho_j}{\partial z_k}(z^0)$, $\dfrac{\partial \rho_j}{\partial \bar{z}_k}(z^{\pm}) = \dfrac{\partial \rho_j}{\partial \bar{z}_k}(z^0)$, $k = 1, \ldots, n$, $z^{\pm} \in \Omega$, points z^+ and z^- are on the normal to $\tilde{\Gamma}_j$ at z^0, and $|z^+ - z^0| = |z^- - z^0|$.

In this case $(\rho_j)_k = 2\dfrac{\partial \rho_j}{\partial z_k}$, and $(\rho_j)_{\bar{k}} = 2\dfrac{\partial \rho_j}{\partial \bar{z}_k}$ where $(\rho_j)_k = \dfrac{\partial \rho_j}{\partial z_k}/|\,\mathrm{grad}\,\rho_j|$.

In what follows, we assume that the function ρ defining the domain is chosen so that it is a function from the class $\mathcal{C}^2$ in the neighborhood of the boundary of D, and the level surfaces $\{z : \rho(z) = \pm\varepsilon\}$ of this function for $\varepsilon > 0$ are smooth.

Consider an open straight circular two-sheeted cone $V_{z^0} = V^+_{z^0} \cup V^-_{z^0}$ which is not tangent to all $\Gamma_1, ..., \Gamma_l$ having the vertex at the point $z^0 \in \partial D$ with sufficiently small angle β between the axis and the generatrix such that there exists a neighborhood Ω_{z^0}, for which $V^+_{z^0} \cap \Omega_{z^0} \subset \bar{D}$, $V^-_{z^0} \cap \Omega_{z^0} \subset (\mathbb{C}^n \backslash D)$. Take two points $z^+ \in V_{z^0} \cap D$ and $z^- \in V_{z^0} \cap (\mathbb{C}^n \backslash \overline{D})$ such that they are on the one straight line passing through z^0 and satisfy the condition $|z^+ - z^0| = |z^- - z^0|$.

Transform the $\bar{\partial}$-normal derivative $\bar{\partial}_n F$. This derivative is a restriction of the form (see [7])

$$*\bar{\partial} F = 2^{1-n} i^n \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial F}{\partial \bar{z}_k} dz[k] \wedge d\bar{z}$$

on the domain boundary, i.e.

$$\bar{\partial}_n F = \sum_{k=1}^{n} \frac{\partial F}{\partial \bar{z}_k} \rho_k = 2 \sum_{k=1}^{n} \frac{\partial F}{\partial \bar{z}_k} \frac{\partial \rho}{\partial z_k}$$

where F is the Bochner-Martinelli integral.

Theorem 4. *If $f \in \mathcal{C}(\partial D)$, then the relation*

$$\lim_{z^{\pm} \to z^0} (\bar{\partial}_n F^+(z^+) - \bar{\partial}_n F^-(z^-)) = 0$$

holds for the integral F of the form (1). This limit is reached uniformly with respect to the point $z^0 \in \partial D$ if the angle β is fixed. If $\bar{\partial}_n F^+(z^+)$ is continuously extended to $\overline{D}$, then $\bar{\partial}_n F^-(z^-)$ is also continuously extended on $\mathbb{C}^n \setminus D$, and vice versa.

For domains with a smooth boundary from the class $\mathcal{C}^2$, Theorem 4 is proved in [10] (see also [7, Chapter 2]).

Proof. Consider the points from ∂D where smoothness is violated. Let the boundary D be not smooth at the point $z^0 \in \partial D$.

Consider the difference

$$\bar{\partial}_n F^+(z^+) - \bar{\partial}_n F^-(z^-) = \bar{\partial}_n \int_{\partial D} f(\zeta) U(\zeta, z^+) -$$

$$-\bar{\partial}_n \int\limits_{\partial D} f(\zeta)U(\zeta, z^-) = \bar{\partial}_n \int\limits_{\partial D\setminus \Omega_{z^0}} f(\zeta)(U(\zeta, z^+)-$$

$$-U(\zeta, z^-)) + \bar{\partial}_n \int\limits_{\partial D\cap \Omega_{z^0}} f(\zeta)(U(\zeta, z^+) - U(\zeta, z^-)).$$

In the integral

$$\bar{\partial}_n \int\limits_{\partial D\setminus \Omega_{z^0}} f(\zeta)(U(\zeta, z^+) - U(\zeta, z^-)),$$

one can pass to the limit under the integral sign since $z^0 \notin \partial D \setminus \Omega_{z^0}$.

$$\lim_{z^\pm \to z^0} \bar{\partial}_n \int\limits_{\partial D\setminus \Omega_{z^0}} f(\zeta)(U(\zeta, z^+) - U(\zeta, z^-)) = 0.$$

It remains to consider this integral over the set $\partial D \cap \Omega_{z^0} = \Gamma$. Since $\Gamma = \Gamma_1 \cup ... \cup \Gamma_l$, it is sufficient to show that

$$\lim_{z^\pm \to z^0} (\bar{\partial}_n F_j^+(z^+) - \bar{\partial}_n F_j^-(z^-)) = 0,$$

where

$$F_j(z) = \int\limits_{\Gamma_j} f(\zeta)U(\zeta, z), \quad z \notin \Gamma_j,$$

for all $j = 1, ..., l$.

If $f \equiv \text{const}$, then $\bar{\partial}_n F \equiv 0$. Thus, we can assume that $f(z^0) = 0$ at the point $z^0 \in \partial D$. The restriction of the kernel $U(\zeta, z)$ on ∂D has the form (see [7, §4])

$$\frac{(n-1)!}{\pi^n} \sum_{k=1}^{n} \frac{\partial \rho}{\partial \bar{\zeta}_k} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} d\sigma,$$

where $d\sigma$ is an element of the surface ∂D.

Hence,

$$\bar{\partial}_n F_j^+(z^+) - \bar{\partial}_n F_j^-(z^-) =$$

$$= -\frac{(n-1)!}{\pi^n} \int\limits_{\Gamma_j} f(\zeta) \sum_{k=1}^{n} \frac{\partial \rho(z)}{\partial z_k} \frac{\partial \rho}{\partial \bar{\zeta}_k} \left(\frac{1}{|\zeta - z^+|^{2n}} - \frac{1}{|\zeta - z^-|^{2n}} \right) d\sigma_j +$$

$$+\frac{n!}{\pi^n} \int\limits_{\Gamma_j} f(\zeta) \left[\frac{\sum\limits_{k=1}^{n} \frac{\partial \rho}{\partial z_k}(\zeta_k - z_k^+) \sum\limits_{m=1}^{n} \frac{\partial \rho}{\partial \zeta_m}(\bar{\zeta}_m - \bar{z}_m^+)}{|\zeta - z^+|^{2n+2}} - \right.$$

$$\left. - \frac{\sum\limits_{k=1}^{n} \frac{\partial \rho}{\partial z_k}(\zeta_k - z_k^-) \sum\limits_{m=1}^{n} \frac{\partial \rho}{\partial \zeta_m}(\bar{\zeta}_m - \bar{z}_m^-)}{|\zeta - z^-|^{2n+2}} \right] d\sigma_j,$$

here $d\sigma_j$ is an element of the surface Γ_j.

Denote the first integral by I_1, and the second one by I_2.

For a given Γ_j, make a unitary transformation into $\mathbb{C}^n$ and a shift such that the point z^0 is turned to 0, the plane tangent to Γ_j at z^0 – to the plane $T = \{w \in \mathbb{C}^n : \operatorname{Im} w_n = 0\}$. In this case, the surface Γ_j in a neighborhood of zero will be given by a system of equations

$$\begin{cases} \zeta_1 & = & w_1, \\ \vdots & & \\ \zeta_{n-1} & = & w_{n-1}, \\ \zeta_n & = & u_n + i\varphi(w), \end{cases}$$

where $w = (w_1, \cdots, w_{n-1}, u_n) \in T$, a function $\varphi(w)$ is from the class $\mathcal{C}^2$ in a neighborhood W of zero. Denote by $\tilde{z}^{\pm}$ the projections of the points $z^{\pm}$ onto the axis $\operatorname{Im} w_n$, and $\tilde{z}^{\pm} = (0, ..., 0, \pm i y_n)$. The surface Γ_j is the Lyapunov surface with the Hölder index that is 1, therefore the following estimates are valid (see [2, §27], [13, §7]):

$$|\varphi(w)| \leqslant C|w|^2, \ w \in W, \tag{8}$$

$$\left|\frac{\partial\varphi}{\partial u_\rho}\right| \leqslant C_1|w|, \ p = 1, ..., n, \ \left|\frac{\partial\varphi}{\partial v_\rho}\right| < C_1|w|, \ p = 1, ..., n-1,$$

where $u_\rho = \operatorname{Re} w_p, v_\rho = \operatorname{Im} w_p$. This implies that

$$\left|\frac{\partial\rho}{\partial\zeta_k}(\zeta(w))\right| \leqslant C_2|w|, \quad \left|\frac{\partial\rho}{\partial\bar{\zeta}_k}(\zeta(w))\right| \leqslant C_2|w|, \tag{9}$$

$w \in W, \ k = 1, ..., n-1$ since

$$\frac{\partial\varphi}{\partial w_p} = \frac{\frac{\partial\rho}{\partial w_p}}{\frac{\partial\rho}{\partial y_n}}, \quad \text{a} \quad 0 < \tilde{C}_3 \leqslant \left|\frac{\partial\rho}{\partial y_n}\right| \leqslant C_3 \quad \text{at} \quad w \in W.$$

It should be noted that constants do not depend on the considered point z^0. And finally

$$|\zeta(w)| \leqslant C_4|w|. \tag{10}$$

Fix $\varepsilon > 0$ and take in the plane T a $(2n-1)$-dimensional ball B' centered at zero with the radius ε such that:

1) $B' \subset W$;
2) $|f(\zeta(w))| < \varepsilon$ at $w \in B'$;
3) a set $\{z \in \mathbb{C}^n : (z_1, ... z_{n-1}, \operatorname{Re} z_n) \in B', |\operatorname{Im} z_n| < a\} \subset W$;
4) $C(2|y_n| + C|w|^2) \leqslant d < 1$ for $|y_n| < a$, and $w \in B'$ (the constant C is defined in (8)).
5) $|\zeta - z^{\pm}| \geqslant c|\zeta - \tilde{z}^{\pm}|$ for $\zeta \in B'$ where c is a constant not depending on ζ and $z^{\pm}$.

Condition 5) is provided by the relations

$$|\zeta - \tilde{z}^{\pm}| \leqslant |\zeta - z^{\pm}| + |z^{\pm} - \tilde{z}^{\pm}| \leqslant |\zeta - z^{\pm}| + |\tilde{z}^{\pm}| \operatorname{tg}\beta \leqslant$$

$$\leqslant |\zeta - \tilde{z}^{\pm}| + \operatorname{tg}\beta(|w - \zeta| + |\zeta - \tilde{z}^{\pm}| \leqslant$$

$$\leqslant (1 + \operatorname{tg}\beta)|\zeta - z^{\pm}| + \operatorname{tg}\beta|\varphi(w)| \leqslant C'|\zeta - z^{\pm}|, \ \ C' = \frac{1}{c},$$

$$|\varphi(w)| = o(|w|) = o(|\zeta - z^{\pm}|), \ \ w \to 0.$$

Since $\tilde{z}^{\pm} = (0, ...0, \pm i y_n)$, the identity

$$|\zeta(w) - \tilde{z}^{\pm}|^2 = |w|^2 + (\pm y_n - \varphi(w))^2$$

is valid, what implies

$$|\zeta - \tilde{z}^{\pm}| = \frac{1}{|w - \tilde{z}^{\pm}|^2} \frac{1}{\left(1 - \frac{\pm 2\varphi y_n - \varphi^2}{|w - \tilde{z}^{\pm}|^2}\right)}, \text{ but}$$

$$\frac{| \pm 2\varphi y_n - \varphi^2|}{|w - \tilde{z}^{\pm}|^2} \leqslant \frac{C|w|^2(2|y_n| - C|w|^2)}{|w|^2 + y_n^2} \leqslant$$

$$\leqslant C(2|y_n| + C|w|^2) \leqslant d < 1$$

at $|y_n| \leqslant a$, $w \in B'$.

Hence,

$$\frac{1}{1 - \frac{\pm 2\varphi y_n - \varphi^2}{|w - \tilde{z}^{\pm}|^2}} = \sum_{k=0}^{\infty} \frac{(\pm 2\varphi y_n - \varphi^2)^k}{|w - \tilde{z}^{\pm}|^{2k}} = 1 + \frac{\pm 2\varphi y_n - \varphi^2}{|w - \tilde{z}^{\pm}|^2} h(w, z),$$

and the function $h(w, z)$ is uniformly bounded at $w \in B'$, $|y_n| \leqslant a$. That's why

$$\frac{1}{|\zeta - \tilde{z}^{\pm}|^{2n}} = \frac{1}{|w - \tilde{z}^{\pm}|^{2n}} \left(1 + \frac{(\pm 2\varphi y_n - \varphi^2) h_1(z, w)}{|w - \tilde{z}^{\pm}|^2}\right), \tag{11}$$

$$\frac{1}{|\zeta - \tilde{z}^{\pm}|^{2n+2}} = \frac{1}{|w - \tilde{z}^{\pm}|^{2n+2}} \left(1 + \frac{(\pm 2\varphi y_n - \varphi^2) h_2(z, w)}{|w - \tilde{z}^{\pm}|^2}\right), \tag{12}$$

and the functions h_1, h_2 are uniformly bounded at $w \in B'$, $|y_n| \leqslant a$.

Estimate the integral I_1 over the surface Γ_j.

$$\left|\frac{1}{|\zeta - z^+|^{2n}} - \frac{1}{|\zeta - z^-|^{2n}}\right| \leqslant \left|\frac{1}{|\zeta - z^+|^{2n}} - \frac{1}{|\zeta - \tilde{z}^+|^{2n}}\right| +$$

$$+ \left|\frac{1}{|\zeta - \tilde{z}^+|^{2n}} - \frac{1}{|\zeta - \tilde{z}^-|^{2n}}\right| + \left|\frac{1}{|\zeta - z^-|^{2n}} - \frac{1}{|\zeta - \tilde{z}^-|^{2n}}\right|$$

Using (11) and (12), we obtain

$$\left|\frac{1}{|\zeta - \tilde{z}^+|^{2n}} - \frac{1}{|\zeta - \tilde{z}^-|^{2n}}\right| \leqslant \frac{2(|2\varphi y_n| + \varphi^2)}{|w - \tilde{z}^+|^{2n+2}} |h_1| \leqslant$$

$$\leqslant C_5 \frac{(2|y_n| + C|w|^2)}{(|w|^2 + y_n^2)^n},$$

and the estimate for

$$\left|\frac{1}{|\zeta - z^+|^{2n}} - \frac{1}{|\zeta - \tilde{z}^+|^{2n}}\right| + \left|\frac{1}{|\zeta - \tilde{z}^+|^{2n}} - \frac{1}{|\zeta - \tilde{z}^-|^{2n}}\right|$$

is performed in the same way as in Section 1.

Further, taking into account that $d\sigma_j \leqslant C_6 dS$, where dS is an element of the plane T, and $d\sigma_j$ is an element of the surface Γ_j, we get

$$|I_1| = \frac{(n-1)!}{\pi^n}\left|\int\limits_{\Gamma_j} f(\zeta)\sum_{k=1}^{n}\frac{\partial\rho}{\partial z_k}\frac{\partial\rho}{\partial\bar{\zeta}_k}\left(\frac{1}{|\zeta - z^+|^{2n}} - \frac{1}{|\zeta - z^-|^{2n}}\right)d\sigma_j\right| \leqslant$$

$$\leqslant \varepsilon C_7 \int\limits_{B'} \frac{(2|y_n| + C|w|^2)}{(|w|^2 + y_n^2)^n} dS.$$

The integral

$$\int\limits_{B'} \frac{|y_n|}{(|w|^2 + y_n^2)^n} dS \leqslant \int\limits_{T} \frac{|y_n|}{(|w|^2 + y_n^2)^n} dS = \text{const}, \quad \text{and}$$

$$\int\limits_{B'} \frac{|w|^2}{(|w|^2 + y_n^2)^n} dS \leqslant \int\limits_{B'} \frac{1}{(|w|^2 + y_n^2)^{n-1}} dS.$$

Introducing the polar coordinate system in the ball B', we obtain $dS = |w|^{2n-2} d|w| \wedge d\omega$ where $d\omega$ is an element of the unit sphere in $\mathbb{R}^{2n-1}$. Then

$$\int\limits_{B'} \frac{1}{(|w|^2 + y_n^2)^{n-1}} dS = \sigma_{2n-1} \int\limits_{0}^{R} \frac{|w|^{2n-2}}{(|w|^2 + y_n^2)^{n-1}} d|w| \leqslant R\sigma_{2n-1}.$$

Here R is the radius of the ball B', and σ_{2n-1} is the area of the unit sphere in $\mathbb{R}^{2n-1}$. Therefore $|I_1| \leqslant C_8\varepsilon$ where the constant C_8 does not depend on z^0 and y_n.

Show that the form of the integral I_2 does not change under a unitary transformation. Indeed, the distance does not change, therefore the functions ρ, $d\sigma$, and $|\zeta - z^0|$ do not change. Consider the expression:

$$\sum_{k=1} \frac{\partial\rho}{\partial z_k}(\zeta_k - z_k).$$

Let a unitary transformation be given with the help of the matrix $A = \|a_{j,k}\|_{j,k=1}^{n}$, i.e.

$$z'_k = \sum_{j=1}^{n} a_{jk} z_j, \ \ k = 1, ..., n,$$

and the inverse transformation – with the help of the matrix $B = \|b_{j,k}\|_{j,k=1}^n$. Then

$$\sum_{k=1}^{n} a_{kj} b_{sk} = \delta_{js}$$

where δ_{js} is the Kronecker symbol. Therefore

$$\sum_{k=1}^{n} \frac{\partial \rho}{\partial z_k}(\zeta_k - z_k) = \sum_{k,j,s=1}^{n} \frac{\partial \rho}{\partial z'_j} a_{kj} b_{sk} (\zeta'_k - z'_k) =$$

$$= \sum_{j,s=1}^{n} \frac{\partial \rho}{\partial z'_j} \delta_{js} (\zeta'_s - z'_s) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z'_j} (\zeta'_j - z'_j).$$

By the same way, one can show that the form of the sum

$$\sum_{k=1}^{n} \frac{\partial \rho}{\partial \bar{\zeta}_k} (\bar{\zeta}_k - \bar{z}_k)$$

does not change. Thus, the form of the integral I_2 is invariant at unitary transformations. Then

$$\sum_{k=1}^{n} \frac{\partial \rho}{\partial z_k}(0)(\zeta_k - z_k^{\pm}) \sum_{m=1}^{n} \frac{\partial \rho}{\partial \bar{\zeta}_m} (\bar{\zeta}_m - \bar{z}_m^{\pm}) =$$

$$= -\frac{i}{2}(\zeta_n - z_n^{\pm}) \sum_{m=1}^{n} \frac{\partial \rho}{\partial \bar{\zeta}_m} (\bar{\zeta}_m - \bar{z}_m^{\pm}) =$$

$$= -\frac{i}{2} \sum_{m=1}^{n-1} \frac{\partial \rho}{\partial \bar{\zeta}_m} \bar{\zeta}_m (\zeta_n - z_n^{\pm}) - \frac{i}{2} \frac{\partial \rho}{\partial \bar{\zeta}_n} (u_n^2 + \varphi^2 + y_n^2 \mp 2\varphi y_n).$$

The integral I_2 over the surface Γ_j is divided into three integrals:

$$I'_2 = \frac{in!}{\pi^n} \int\limits_{B'} f(\zeta(w)) \left(\frac{2\varphi y_n \frac{\partial \rho}{\partial \bar{\zeta}_n} + \sum\limits_{m=1}^{n-1} \frac{\partial \rho}{\partial \bar{\zeta}_m} \bar{\zeta}_m y_n}{|w - \tilde{z}^{+}|^{2n+2}} \right) d\sigma'_j,$$

$$I_2^{\pm} = \pm \frac{in!}{2\pi^n} \int\limits_{B'} f(\zeta(w)) \left[\frac{\partial \rho}{\partial \bar{\zeta}_n} (u_n^2 + \varphi^2 + y_n^2 \pm 2\varphi y_n) + \right.$$

$$\left. + \sum_{m=1}^{n-1} \frac{\partial \rho}{\partial \bar{\zeta}_m} \bar{\zeta}_m (\zeta_n - \tilde{z}_n^{\pm}) \right] \frac{(\pm 2\varphi y_n - \varphi^2)\eta_2}{|w - \tilde{z}^{+}|^{2n+4}} d\sigma'_j,$$

where $d\sigma'_j$ is the image of $d\sigma_j$ under the mapping $w \to \zeta(w)$, and η_2 is defined in (12). Using (11) – (12), we obtain

$$|I'_2| \leqslant M_1 \varepsilon \int\limits_{B'} \frac{M_2 |w|^2 |y_n| + M_3 |w|^2 |y_n|}{|w - \tilde{z}^{+}|^{2n+2}} dS \leqslant$$

$$\leqslant M_4\varepsilon \int\limits_T \frac{|y_n|}{(|w|^2+y_n^2)^n} dS = M_5\varepsilon.$$

Integrals

$$|I_2^{\pm}| \leqslant M_6\varepsilon \int\limits_{B'} \frac{|w|^2(2|y_n|+C|w|^2)(M_7|w|^2+M_8|w|^2|y_n|+M_9y_n^2)}{(|w|^2+y_n^2)^{n+2}} dS \leqslant$$

$$\leqslant M_{10}\varepsilon \int\limits_{B'} \frac{|y_n|}{(|w|^2+y_n^2)^n} dS + M_{11}\varepsilon \int\limits_{B'} \frac{dS}{(|w|^2+y_n^2)^{n-1}} \leqslant M_{12}\varepsilon.$$

Choose now cones V_z such that the cone axis form a continuous family. Let the function $\overline{\partial}_n F^-$ be continuously extended on $\mathbb{C}^n \backslash D$. We need to show that the function $\overline{\partial}_n F^+$ is continuously extended on $\overline{D}$.

Let $z^0 \in \partial D$ and $z \to z^0$, $z \in D$. Consider sufficiently small $\varepsilon > 0$ and the ball B_{z^0} centered at the point z^0 with the radius ε. Draw through $z \in D \cap B_{z^0}$ a straight line parallel to the axis of the cone V_{z^0} until it intersects with the boundary. Denote the intersection point by z', and denote by $z'' \in (\mathbb{C}^n \setminus \overline{D})$ the point lying on this straight line and symmetric to z. For sufficiently small ε, points z and z'' lie in the cone $V_{z'}$. Then

$$|\overline{\partial}_n F^+(z) - \overline{\partial}_n F^-(z^0)| \leqslant |\overline{\partial}_n F^+(z) - \overline{\partial}_n F^-(z'')| +$$

$$+ |\overline{\partial}_n F^-(z'') - \overline{\partial}_n F^-(z')| + |\overline{\partial}_n F^-(z') - \overline{\partial}_n F^-(z^0)|.$$

Each of the terms on the right-hand side of the inequality is estimated by a quantity tending to zero as $\varepsilon \to 0$.

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