Applied Mathematics and Computation 317 (2018) 150–159

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Optimal quadrature formulas with derivatives for Cauchy type singular integrals

D.M. Akhmedov, A.R. Hayotov*, Kh.M. Shadimetov

Institute of Mathematics, Uzbek Academy of Sciences, 29, Do'rmon yo'li Street, Tashkent 100125, Uzbekistan

ARTICLE INFO

MSC:
MSC65D32

Keywords:
Optimal quadrature formulas
The extremal function
Sobolev space
Optimal coefficients
Cauchy type singular integral

ABSTRACT

In the present paper in $L_2^{(m)}(0,1)$ space the optimal quadrature formulas with derivatives are constructed for approximate calculation of the Cauchy type singular integral. Explicit formulas for the optimal coefficients are obtained. Some numerical results are presented.

1. Introduction. Statement of the problem

Many problems of science and engineering are naturally reduced to singular integral equations. Moreover plane problems are reduced to one dimensional singular integral equations (see [10]). The theory of one dimensional singular integral equations is given, for example, in [7,11]. It is known that the solutions of such integral equations are expressed by singular integrals. Therefore approximate calculation of singular integrals with high exactness is actual problem of numerical analysis.

We consider the following quadrature formula

$$\int_0^1 \frac{\varphi(x)}{x-t}\,dx \cong \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N} C_\alpha[\beta]\varphi^{(\alpha)}(x_\beta), \tag{1.1}$$

with the error functional

$$\ell(x) = \frac{\varepsilon_{[0,1]}(x)}{x-t} - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\delta^{(\alpha)}(x-x_\beta) \tag{1.2}$$

where $0<t<1$, $C_\alpha[\beta]$ are the coefficients, x_β ($\in[0,\ 1]$) are the nodes, N is a natural number, $n=\overline{0,m-1}$, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval [0, 1], δ is the Dirac delta function, φ is a function of the space $L_2^{(m)}(0,1)$. Here $L_2^{(m)}(0,1)$ is the Sobolev space of functions with a square integrable mth generalized derivative and equipped with the norm

$$\|\varphi|L_2^{(m)}(0,1)\| = \left\{\int_0^1 (\varphi^{(m)}(x))^2\,dx\right\}^{1/2}$$

* Corresponding author.
E-mail addresses: hayotov@mail.ru, abdullo_hayotov@mail.ru (Kh.M. Shadimetov).

http://dx.doi.org/10.1016/j.amc.2017.09.009

and $\{\int_0^1 (\varphi^{(m)}(x))^2\,dx\}^{1/2} < \infty$.

Since the functional ℓ of the form (1.2) is defined on the space $L_2^{(m)}(0,1)$ it is necessary to impose the following conditions (see [20])

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, 2, \ldots, m-1. \tag{1.3}$$

Hence it is clear that for existence of the quadrature formulas of the form (1.1) the condition $N \geq m-1$ has to be met.

The difference

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)\,dx = \int_0^1 \frac{\varphi(x)}{x-t}\,dx - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N} C_\alpha[\beta]\varphi^{(\alpha)}(x_\beta) \tag{1.4}$$

is called *the error* of the formula (1.1).

By the Cauchy–Schwarz inequality

$$|(\ell, \varphi)| \leq \left\|\varphi|L_2^{(m)}\right\| \cdot \left\|\ell|L_2^{(m)*}\right\|$$

the error (1.4) of the formula (1.1) on functions of the space $L_2^{(m)}(0,1)$ is reduced to finding the norm of the error functional ℓ in the conjugate space $L_2^{(m)*}(0,1)$.

Obviously the norm of the error functional ℓ depends on the coefficients and the nodes of the quadrature formula (1.1). The problem of finding the minimum of the norm of the error functional ℓ by coefficients and nodes is called *the S.M. Nikol'skii problem*, and the obtained formula is called *the optimal quadrature formula in the sense of Nikol'skii*. This problem was first considered by S.M. Nikol'skii [12], and continued by many authors, see e.g. [13] and references therein. Minimization of the norm of the error functional ℓ by coefficients when the nodes are fixed is called *Sard's problem* and the obtained formula is called *the optimal quadrature formula in the sense of Sard*. First this problem was investigated by A. Sard [14].

There are several methods of construction of optimal quadrature formulas in the sense of Sard such as the spline method, φ-function method (see e.g. [3,9]) and Sobolev's method which is based on construction of discrete analogs of a linear differential operator (see e.g. [19,20]).

The main aim of the present paper is to construct optimal quadrature formulas in the sense of Sard of the form (1.1) in the space $L_2^{(m)}(0,1)$ by the Sobolev method for approximate integration of the Cauchy type singular integral. This means to find the coefficients $C_\alpha[\beta]$ which satisfy the following equality

$$\|\mathring{\ell}|L_2^{(m)*}\| = \inf_{C_\alpha[\beta]} \|\ell|L_2^{(m)*}\|. \tag{1.5}$$

Thus, in order to construct optimal quadrature formulas in the form (1.1) in the sense of Sard we have to consequently solve the following problems.

Problem 1. Find the norm of the error functional (1.2) of the quadrature formula (1.1) in the space $L_2^{(m)*}(0,1)$.

Problem 2. Find the coefficients $C_\alpha[\beta]$ which satisfy the equality (1.5).

Many works are devoted to the problem of approximate integration of Cauchy type singular integrals (see, for instance, [2,4,6,8,10,15,17,18] and references therein).

The rest of the paper is organized as follows. In Section 2 using a concept of extremal function we find the norm of the error functional (1.2). Section 3 is devoted to successive minimization of $\|\ell\|^2$ with respect to the coefficients $C_\alpha[\beta]$. In Section 4 we give some definitions and known results which we use in the proof of the main results. In Section 5 we give the algorithm for construction of optimal quadrature formulas of the form (1.1). Explicit formulas for coefficients of the optimal quadrature formulas of the form (1.1) are found in Section 5 for the cases $m = 1, 2, 3$. Finally, in Section 6 some numerical results which confirm our theoretical results are presented.

2. The extremal function and the expression for the error functional norm

To solve Problem 1, i.e., for finding the norm of the error functional (1.2) in the space $L_2^{(m)}(0,1)$ a concept of the extremal function is used [19,20]. The function ψ_ℓ is said to be *the extremal function* of the error functional (1.2) if the following equality holds

$$(\ell, \psi_\ell) = \|\ell|L_2^{(m)*}\|\ \|\psi_\ell|L_2^{(m)*}\|. \tag{2.1}$$

In the space $L_2^{(m)}$ the extremal function ψ_ℓ of a functional ℓ is found by S.L. Sobolev [19,20]. This extremal function has the form

$$\psi_\ell(x) = (-1)^m \ell(x) * G_m(x) + P_{m-1}(x), \tag{2.2}$$

where

$$G_m(x) = \frac{|x|^{2m-1}}{2\cdot(2m-1)!} \tag{2.3}$$

is a solution of the equation

$$\frac{d^{2m}}{dx^{2m}} G_m(x) = \delta(x), \tag{2.4}$$

$P_{m-1}(x)$ is a polynomial of degree $m-1$, and $*$ is the operation of convolution, i.e.

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy.$$

It is well known [19] that for any functional ℓ in $L_2^{(m)*}$ the equality

$$\begin{aligned}\|\ell|L_2^{(m)*}\|^2 &= (\ell, \psi_\ell) = (\ell(x), (-1)^m \ell(x) * G_m(x)) \\ &= \int_{-\infty}^{\infty} \ell(x)\left((-1)^m \int_{-\infty}^{\infty} \ell(y) G_m(x-y)\,dy\right) dx\end{aligned}$$

holds [19].

Applying this equality to the error functional (1.2) and taking into account (2.2) we obtain the following

$$\begin{aligned}\|\ell\|^2 = (-1)^m \Bigg[&\sum_{k=0}^{n}\sum_{\alpha=0}^{n}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N} (-1)^k C_k[\gamma] C_\alpha[\beta] \frac{(h\beta - h\gamma)^{2m-1-\alpha-k}\operatorname{sgn}(h\beta - h\gamma)}{2(2m-1-\alpha-k)!} \\ &- 2\sum_{\alpha=0}^{n}\sum_{\beta=0}^{N} (-1)^\alpha C_\alpha[\beta] \int_0^1 \frac{(x-h\beta)^{2m-1-\alpha}\operatorname{sgn}(x-h\beta)}{2(2m-1-\alpha)!\,(x-t)}\,dx \\ &+ \int_0^1\int_0^1 \frac{(x-y)^{2m-1}\operatorname{sgn}(x-y)}{2(2m-1)!(x-t)(y-t)}\,dx\,dy \Bigg],\end{aligned} \tag{2.5}$$

where $\operatorname{sgn} x$ is the signum function.

Thus Problem 1 is solved for quadrature formulas of the form (1.1) in the space $L_2^{(m)}(0,1)$.

Further we consider Problem 2.

3. Minimization of the norm of the error functional $\ell(x)$

Now we consider the minimization problem of the norm (2.5) of the error functional ℓ under conditions (1.3).

It should be noted that minimization of $\|\ell\|^2$ by $C_\alpha[\beta]$, $\alpha = \overline{0,n}, \beta = \overline{0,N}$ is very hard. Here we suggest successive minimization of $\|\ell\|^2$ by $C_\alpha[\beta]$, i.e. first we consider the case $m=1$ and the expression (2.5) of $\|\ell\|^2$ we minimize by $C_0[\beta]$. Further we consider the case $m=2$, and using the obtained values for $C_0[\beta]$, the expression (2.5) of $\|\ell\|^2$ we minimize by $C_1[\beta]$. After that in the case $m=3$, using the obtained values of $C_0[\beta]$ and $C_1[\beta]$, the expression (2.5) for $\|\ell\|^2$ we minimize by $C_2[\beta]$ and so on.

Next we realize this successive minimization. Here we use the Lagrang method. We consider the function

$$\Phi(\mathbf{C}, \lambda) = \|\ell\|^2 - 2(-1)^m \sum_{p=0}^{m-1} \lambda_p(\ell, x^p),$$

where $\|\ell\|^2$ is defined by (2.5) and

$$\mathbf{C} = (C_0[0], C_0[1], \ldots, C_0[N], C_1[0], C_1[1], \ldots, C_1[N], \ldots, C_{m-1}[0], C_{m-1}[1], \ldots, C_{m-1}[N]),$$

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}).$$

We consider *the case* $m=1$ then the quadrature formula (1.1) has the form

$$\int_0^1 \frac{\varphi(x)}{x-t}\,dx \cong \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) \tag{3.1}$$

and $\|\ell\|^2$ depends only on $C_0[\beta]$ $(\beta = \overline{0,N})$.

Equating to zero partial derivatives of $\Phi(\mathbf{C}, \lambda)$ by $C_0[\beta]$ and λ_0 we get the following system of linear equations

$$\sum_{\gamma=0}^{N} C_0[\gamma] \frac{(h\beta - h\gamma)\ \operatorname{sgn}(h\beta - h\gamma)}{2} + \lambda_0 = F_0(h\beta), \tag{3.2}$$

$$\beta = 0, 1, \ldots, N,$$

$$\sum_{\gamma=0}^{N} C_0[\gamma] = g_0, \tag{3.3}$$

where

$$F_0(h\beta) = \int_0^1 \frac{(x-h\beta)\ \mathrm{sgn}(x-h\beta)}{2(x-t)}\,dx = \frac{1}{2}\Big[1-2(h\beta)+(t-h\beta)\big(\ln(t-t^2)-2\ln|h\beta-t|\big)\Big],$$

$$g_0 = \int_0^1 \frac{1}{x-t}\,dx = \ln\frac{1-t}{t}.$$

Further we consider *the case* $m=2$. In this case the quadrature formula (1.1) takes the form

$$\int_0^1 \frac{\varphi(x)}{x-t}\,dx \cong \sum_{\beta=0}^{N}\Big(C_0[\beta]\varphi(h\beta)+C_1[\beta]\varphi'(h\beta)\Big) \tag{3.4}$$

and expression (2.5) of $\|\ell\|^2$ depends on $C_0[\beta]$ and $C_1[\beta]$. Then using the solution $C_0[\beta]$ and λ_0 of system (3.2)–(3.3), equating to zero partial derivatives of the function $\Phi(\mathbf{C}, \lambda)$ by $C_1[\beta]$ and λ_1 we get

$$\sum_{\gamma=0}^{N} C_1[\gamma]\frac{(h\beta-h\gamma)\ \mathrm{sgn}(h\beta-h\gamma)}{2} - \lambda_1 = F_1(h\beta), \tag{3.5}$$

$$\beta = 0,1,\ldots,N,$$

$$\sum_{\gamma=0}^{N}\Big(C_0[\gamma](h\gamma)+C_1[\gamma]\Big) = g_1, \tag{3.6}$$

where

$$F_1(h\beta) = -f_1(h\beta) + \sum_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)^2\ \mathrm{sgn}(h\beta-h\gamma)}{4}, \tag{3.7}$$

$$f_1(h\beta) = -\int_0^1 \frac{(x-h\beta)^2\ \mathrm{sgn}(x-h\beta)}{4(x-t)}\,dx - \frac{1}{4}\Big[3(h\beta)^2-2(h\beta)(1+t)+\frac{1}{2}+t(t-h\beta)^2\Big(\ln(t-t^2)-2\ln|h\beta-t|\Big)\Big],$$

$$g_1 = \int_0^1 \frac{x}{x-t}\,dx = 1+t\ln\frac{1-t}{t}. \tag{3.8}$$

In *the case* $m=3$ the quadrature formula (1.1) has the form

$$\int_0^1 \frac{\varphi(x)}{x-t}\,dx \cong \sum_{\beta=0}^{N}\Big(C_0[\beta]\varphi(h\beta)+C_1[\beta]\varphi'(h\beta)+C_2[\beta]\varphi''(h\beta)\Big) \tag{3.9}$$

and $\|\ell\|^2$, defined by equality (2.5), depends on $C_0[\beta]$, $C_1[\beta]$ and $C_2[\beta]$. Then using solutions $C_0[\beta]$ and λ_0 of system (3.2)–(3.3) and $C_1[\beta]$, λ_1 of system (3.5)–(3.6), equating to zero partial derivatives of $\Phi(\mathbf{C}, \lambda)$ by $C_2[\beta]$ and λ_2 we have the following system of linear equations

$$\sum_{\gamma=0}^{N} C_2[\gamma]\frac{(h\beta-h\gamma)\ \mathrm{sgn}\,(h\beta-h\gamma)}{2} + \lambda_2 = F_2(h\beta), \tag{3.10}$$

$$\beta = 0,1,\ldots,N,$$

$$\sum_{\gamma=0}^{N}\Big(C_0[\gamma](h\gamma)^2+2C_1[\gamma](h\gamma)+2C_2[\gamma]\Big) = g_2, \tag{3.11}$$

where

$$F_2(h\beta) = f_2(h\beta) - \sum_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)^3\ \mathrm{sgn}(h\beta-h\gamma)}{12} + \sum_{\gamma=0}^{N} C_1[\gamma]\frac{(h\beta-h\gamma)^2\ \mathrm{sgn}(h\beta-h\gamma)}{4},$$

$$
\begin{aligned}
f_2(h\beta) &= \int_0^1 \frac{(x-h\beta)^3 \ \mathrm{sgn}(x-h\beta)}{12(x-t)} \,\mathrm{d}x \\
&= \frac{1}{12}\Big[-\frac{11}{3}(h\beta)^3 + (5t+3)(h\beta)^2 - \Big(2t^2+3t+\frac{3}{2}\Big)(h\beta) + t^2 + \frac{t}{2} \\
&\quad + \frac{1}{3} + (t-h\beta)^3\Big(\ln(t-t^2) - 2\ln|h\beta - t|\Big)\Big], \\
g_2 &= \int_0^1 \frac{x^2}{x-t}\,\mathrm{d}x = \frac{1}{2} + t + t^2 \ln\frac{1-t}{t}.
\end{aligned}
$$

Suppose, continuing by this way, for the cases $m = 1, 2, ..., k-1$ we found $C_0[\beta], C_1[\beta], ..., C_{k-2}[\beta]$ and $\lambda_0, \lambda_1, ..., \lambda_{k-2}$. We consider *the case* $m = k$. Then square of the norm (2.5) of the error functional ℓ of quadrature formulas (1.1) depends on $C_0[\beta], C_1[\beta], ..., C_{k-2}[\beta]$ and $C_{k-1}[\beta]$. Further using the obtained solutions $C_0[\beta], C_1[\beta], ..., C_{k-2}[\beta]$ and $\lambda_0, \lambda_1, ..., \lambda_{k-2}$ of corresponding systems, equating to zero partial derivatives of the function $\Phi(\mathbf{C}, \lambda)$ by $C_{k-1}[\beta]$ and λ_{k-1} we arrive to the system of linear equations

$$
\begin{aligned}
&\sum_{\gamma=0}^{N}(-1)^{k-1}C_{k-1}[\gamma]\frac{(h\beta-h\gamma)\ \mathrm{sgn}\,(h\beta-h\gamma)}{2} + (k-1)!\lambda_{k-1} = F_{k-1}(h\beta), \\
&\beta = 0, 1, ..., N, \\
&\sum_{\gamma=0}^{N}C_0[\gamma](h\gamma)^{k-1} + \sum_{i=1}^{k-1}\sum_{\gamma=0}^{N}C_i[\gamma](k-1)(k-2)...(k-i)(h\gamma)^{k-1-i} = g_{k-1},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
F_{k-1}(h\beta) &= f_{k-1}(h\beta) - \sum_{l=0}^{k-2}\sum_{\gamma=0}^{N}(-1)^{l}C_l[\gamma]\frac{(h\beta-h\gamma)^{k-l}\ \mathrm{sgn}(h\beta-h\gamma)}{2(k-l)!}, \\
f_{k-1}(h\beta) &= \int_0^1 \frac{(-1)^{k-1}(x-h\beta)^k \ \mathrm{sgn}(x-h\beta)}{2k!(x-t)}\,\mathrm{d}x, \\
g_{k-1} &= \int_0^1 \frac{x^{m-1}}{x-t}\,\mathrm{d}x.
\end{aligned}
$$

Further we solve systems (3.2)–(3.3), (3.5)–(3.6) and (3.10)–(3.11), i.e. we find optimal coefficients of quadrature formulas of the form (3.1), (3.4) and (3.9).

4. Preliminaries

In this section we give some definitions and formulas that we need to prove the main results. Here we use the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [19,20]. For completeness we give some definitions.

Assume that the nodes x_β are equally spaced, i.e., $x_\beta = h\beta,\ h = \frac{1}{N},\ N = 1, 2, ...,$ functions $\varphi(x)$ and $\psi(x)$ are real-valued and defined on the real line $\mathbb{R}$.

Definition 1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β.

Definition 2. *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$
[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta)\cdot\psi(h\beta),
$$

if the series on the right hand side of the last equality converges absolutely.

Definition 3. *The convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product*

$$
\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta-h\gamma)] = \sum_{\gamma=-\infty}^{\infty}\varphi(h\gamma)\cdot\psi(h\beta-h\gamma).
$$

The Euler–Frobenius polynomials $E_k(x)$, $k = 1, 2, \dots$ are defined by the following formula [5]

$$
E_k(x) = \frac{(1-x)^{k+2}}{x}\left(x\frac{d}{dx}\right)^k \frac{x}{(1-x)^2},
$$

$E_0(x) = 1.$

The discrete analog $D_m(h\beta)$ of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ satisfies the following equality

$$hD_m(h\beta) * G_m(h\beta) = \delta(h\beta), \tag{4.1}$$

where $G_m(h\beta) = \frac{|h\beta|^{2m-1}}{2(2m-1)!}$, $\delta(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1.

It should be noted that the operator $D_m(h\beta)$ was first introduced and investigated by S.L. Sobolev [19]. In the work [16] the discrete analog $D_m(h\beta)$ of the differential operator $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$, which satisfies Eq. (4.1), was constructed and the following result was proved.

Lemma 4.1. *The discrete analog of the differential operator* $\mathrm{d}^{2m}/\mathrm{d}x^{2m}$ *has the form*

$$D_m(h\beta) = p\begin{cases} \sum_{k=1}^{m-1} A_k q_k^{|\beta|-1} & \text{for} \quad |\beta| \geq 2, \\ 1 + \sum_{k=1}^{m-1} A_k & \text{for} \quad |\beta| = 1, \\ C + \sum_{k=1}^{m-1} \frac{A_k}{q_k} & \text{for} \quad \beta = 0, \end{cases}$$

where

$$p = \frac{(2m-1)!}{h^{2m}}, \quad A_k = \frac{(1-q_k)^{2m+1}}{E_{2m-1}(q_k)}, \quad C = -2^{2m-1},$$

$E_{2m-1}(q)$ is the Euler–Frobenius polynomial of degree $2m-1$, q_k are the roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$, $|q_k| < 1$, h is a small positive parameter.

Furthermore several properties of the discrete argument function $D_m(h\beta)$ were proved in [16,19,20]. Here we give the following properties of $D_m(h\beta)$.

Lemma 4.2. *The discrete argument function $D_m(h\beta)$ and the monomials $(h\beta)^k$ are related to each other as follows*

$$\sum_{\beta=-\infty}^{\infty} D_m(h\beta)(h\beta)^k = \begin{cases} 0 & \textit{when} \quad 0 \leq k \leq 2m-1, \\ (2m)! & \textit{when} \quad k = 2m. \end{cases}$$

In particular, when $m = 1$ from Lemma 4.1 we get the discrete analog $D_1(h\beta)$ of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ which has the form

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geq 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0. \end{cases} \tag{4.2}$$

Now, based on Lemma 4.2, we get the following properties of the operator $D_1(h\beta)$

$$D_1(h\beta) * 1 = 0, \qquad D_1(h\beta) * (h\beta) = 0, \tag{4.3}$$

$$hD_1(h\beta) * \frac{|h\beta|}{2} = \delta(h\beta). \tag{4.4}$$

where $\delta(h\beta)$ is the discrete delta function.

5. The main results

In this section we solve systems (3.2)–(3.3), (3.5)–(3.6) and (3.10)–(3.11).

It should be noted that in the process of solution of these systems only the discrete analog $D_1(h\beta)$ of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ is used and each of these systems is reduced to the system of two linear equations with two unknowns as shown below in the proof of Theorem 5.2.

We note that in the work [15] was only considered the case $m = 1$, i.e. there were constructed optimal quadrature formulas for singular integrals of Cauchy type over the interval $[-1, 1]$ with weights 1 and $1/\sqrt{1-x^2}$ in the space $L_2^{(1)}(-1, 1)$ and the results given as Theorem 1 and Theorem 2, respectively. From Theorem 1 of [15] by linear transformation which transforms the interval $[-1, 1]$ to the interval $[0, 1]$ the following is obtained.

Theorem 5.1. *In the space $L_2^{(1)}(0, 1)$ when $t \neq h\beta$, $\beta = 0, 1, ..., N$, the coefficients of optimal quadrature formulas of the form (3.1) are defined as follows*

$$C_0[0] = \frac{1}{h}\left[(h-t)\ln\frac{|h-t|}{t} - h\right],$$

$$C_0[\beta] = \frac{1}{h}\left[(h(\beta+1)-t)\ln\frac{|h(\beta+1)-t|}{|h\beta-t|} + (h(\beta-1)-t)\ln\frac{|h(\beta-1)-t|}{|h\beta-t|}\right],$$
$$\beta = 1, 2, ..., N-1,$$

$$C_0[N] = \frac{1}{h}\left[(t-h(N-1))\ln\frac{1-t}{|h(N-1)-t|}+h\right].$$

Now, using Theorem 5.1, we solve system (3.5)–(3.6).

The following holds

Theorem 5.2. *In the space $L_2^{(2)}(0,1)$ when $t\neq h\beta$, $\beta=0,1,\dots,N$, the coefficients of optimal quadrature formulas of the form (3.4) have the form*

$$C_1[0] = \frac{1}{2h}\left[t(h-t)\ln\frac{|h-t|}{t}+\frac{h}{2}(h-2t)\right],$$

$$\begin{aligned} C_1[\beta] = \frac{1}{2h}\Big[& h(h(\beta+1)-t)\ln\frac{|h(\beta+1)-t|}{|h\beta-t|} - h(h(\beta-1)-t) \\ & \times \ln\frac{|h(\beta-1)-t|}{|h\beta-t|} - (h(\beta+1)-t)^2\ln\frac{|h(\beta+1)-t|}{|h\beta-t|} \\ & - (h(\beta-1)-t)^2\ln\frac{|h(\beta-1)-t|}{|h\beta-t|}+h^2\Big], \end{aligned}$$

$$\beta = 1,2,\dots,N-1,$$

$$C_1[N] = \frac{1}{2h}\left[(t-1)(t-1+h)\ln\frac{1-t}{|1-h-t|}+\frac{1}{2}(h^2-2h(1-t))\right].$$

Proof. Suppose $C_1[\beta]=0$ when $\beta<0$ and $\beta>N$. Then, using Definition 3, we can rewrite the system (3.5)–(3.6) in the following convolution form

$$C_1[\beta]*\frac{|h\beta|}{2}-\lambda_1 = F_1(h\beta), \quad \beta=0,1,\dots,N, \tag{5.1}$$

$$\sum_{\gamma=0}^{N}\Big(C_0[\gamma](h\gamma)+C_1[\gamma]\Big) = g_1, \tag{5.2}$$

Denoting by

$$u_1(h\beta) = C_1[\beta]*\frac{|h\beta|}{2}-\lambda_1, \tag{5.3}$$

using (4.2), (4.3) and (4.4), we obtain

$$C_1[\beta] = hD_1(h\beta)*u_1(h\beta). \tag{5.4}$$

In order to calculate the convolution (5.4) we need to determine the function $u_1(h\beta)$ for all integer values of β. From equality (5.1) we get that $u_1(h\beta)=F_1(h\beta)$ when $\beta=0,1,2,\dots,N$. Now we need to find representation of the function $u_1(h\beta)$ when $\beta<0$ and $\beta>N$.

Taking into account that $C_1[\beta]=0$ when $\beta<0$ and $\beta>N$ for $u_1(h\beta)$ we get the following

$$u_1(h\beta) = \begin{cases} -\frac{h\beta}{2}(g_1-\nu_1)+a_1^-, & \beta\le 0, \\ F_1(h\beta), & 0\le\beta\le N, \\ \frac{h\beta}{2}(g_1-\nu_1)-a_1^+, & \beta\ge N, \end{cases} \tag{5.5}$$

where $\nu_1=\sum_{\gamma=0}^{N}C_0[\gamma](h\gamma)$ is known (since the coefficients $C_0[\beta]$ are the coefficients given in Theorem 5.1) and

$$a_1^- = \mu_1-\lambda_1, \quad a_1^+ = \mu_1+\lambda_1,$$

here $\mu_1=\frac{1}{2}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)$ and λ_1 are unknowns. If we find unknowns a_1^- and a_1^+ then from the last system of equations we have

$$\mu_1 = \frac{1}{2}(a_1^++a_1^-), \quad \lambda_1 = \frac{1}{2}(a_1^+-a_1^-). \tag{5.6}$$

Now from (5.5) for a_1^+ and a_1^- when $\beta=0$ and $\beta=N$ we get the following

$$a_1^+ = \frac{1}{2}\Big(g_1-\nu_1\Big)-F_1(1), \quad a_1^- = F_1(0),$$

where $F_1(0)$ and $F_1(1)$ are obtained from (3.7) putting $\beta=0$ and $\beta=N$, respectively, and g_1 is defined by (3.8). This means that we obtained the explicit form of the function $u_1(h\beta)$.

Further, using (4.2) and (5.5) from (5.4) calculating the convolution $hD_1(h\beta)*u_1(h\beta)$ for $\beta=\overline{0,N}$ we get

$$C_1[\beta]=hD_1(h\beta)*u_1(h\beta)=h\sum_{\gamma=-\infty}^{\infty}D_1(h\beta-h\gamma)u_1(h\gamma)$$

$$=h\Bigg[\sum_{\gamma=0}^{N}D_1(h\beta-h\gamma)F_1(h\gamma)+\sum_{\gamma=1}^{\infty}D_1(h\beta+h\gamma)\left(\frac{h\gamma}{2}\Big(g_1-\nu_1\Big)+a_1^-\right)$$
$$+\sum_{\gamma=1}^{\infty}D_1(h(N+\gamma)-h\beta)\left(\frac{1+h\gamma}{2}\Big(g_1-\nu_1\Big)-a_1^+\right)\Bigg]. \tag{5.7}$$

From (5.7) for $\beta=0$ we get

$$C_1[0]=\frac{1}{h}\Big[F_1(1)-F_1(0)+\frac{h}{2}(g_1-\nu_1)\Big], \tag{5.8}$$

for $\beta=1,\dots,N-1$ we have

$$C_1[\beta]=\frac{1}{h}\Big[F_1(h(\beta-1))-2F_1(h\beta)+F_1(h(\beta+1))\Big], \tag{5.9}$$

and for $\beta=N$ we obtain

$$C_1[N]=\frac{1}{h}\Big[F_1(1-h)-F_1(1)+\frac{h}{2}(g_1-\nu_1)\Big]. \tag{5.10}$$

Using equalities (3.7) and (3.8) from (5.8), (5.9) and (5.10), after some simplifications, we get explicit formulas for coefficients $C_1[\beta]$, $\beta=0,1,\dots,N$, which are given in the statement of Theorem 5.2. Theorem 5.2 is proved.

Now using, Theorems 5.1 and 5.2, we get the following result for the coefficients of the quadrature formula (3.9), i.e. we get the solution of the system (3.10)–(3.11).

Theorem 5.3. *In the space $L_2^{(3)}(0,1)$ when $t\neq h\beta$, ($\beta=0,1,2,\dots,N$), the coefficients of the optimal quadrature formulas of the form (3.9) have the form*

$$C_2[0]=\frac{1}{12h}\Big[-\frac{h^3}{6}+2ht(h-t)-t(h-t)(2t-h)\ln\frac{|h-t|}{t}\Big],$$

$$C_2[\beta]=\frac{1}{6h}\Big[2h^2(t-h\beta)+\Big(2(t-h\beta)^3+h^2(t-h\beta)\Big)\ln|t-h\beta|$$
$$-\left((t-h(\beta+1))^3+\frac{h^2}{2}(t-h(\beta+1))+\frac{3h}{2}(t-h(\beta+1))^2\right)\ln|h(\beta+1)-t|$$
$$-\left((t-h(\beta-1))^3+\frac{h^2}{2}(t-h(\beta-1))-\frac{3h}{2}(t-h(\beta-1))^2\right)\ln|h(\beta-1)-t|\Big],$$

$$\beta=1,2,\dots,N-1,$$

$$C_2[N]=\frac{1}{12h}\Big[\frac{h^3}{6}-2h^2+2h+2h^2t+2ht^2-4ht+(t-1)(t-1+h)(2(t-1)+h)\ln\frac{1-t}{|1-h-t|}\Big].$$

Theorem 5.3 is proved similarly as Theorem 5.2.

Remark 1. It should be noted that when singular point t coincides with nodes $h\beta$ (i.e. when $t=h\beta$ for $\beta=1,2,\dots,N-1$) of the quadrature formulas (3.1), (3.4) and (3.9) then in Theorems 5.1, 5.2 and 5.3 the corresponding coefficients $C_\alpha[\beta]$, $\alpha=0,1,2$, equals zero.

6. Numerical results

In this section we give some numerical results in order to show numerically convergence of our optimal quadrature formulas (3.1), (3.4) and (3.9), with coefficients given correspondingly in Theorems 5.1, 5.2 and 5.3, in dependence on the values of N and m. Furthermore, here we compare numerical results of our quadrature formulas with numerical results of the optimal quadrature formula constructed in [1] in the space $L_2^{(2)}(0,1)$.

Now as function φ in the formula (1.1) we take the monomial x^6, i.e. $\varphi=x^6$. Then for the function $\varphi=x^6$ the absolute value of the error (1.4) is:

$$\left|\left(\ell,x^6\right)\right|=\left|\int_0^1\frac{x^6}{x-t}\,dx-\sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}C_\alpha[\beta]((h\beta)^6)^{(\alpha)}\right|. \tag{6.1}$$

Table 1
The numerical results for $|(\ell, x^6)|$ when $m = 1, 2, 3,\ t = 0.0015,\ 0.15, 0.45, 0.65, 0.75, 0.95$ and $N = 10$.

t	$m = 1$	$m = 2$	$m = 3$
0.0015	0.00626707	0.00003402	0.00002509
0.15	0.00788350	0.00003792	0.00003380
0.45	0.01453324	0.00018749	0.00004759
0.65	0.01951954	0.00072082	0.00004164
0.75	0.01899824	0.00117041	0.00002843
0.95	0.01752485	0.00278585	0.00006152

Table 2
The numerical results for $|(\ell, x^6)|$ when $m = 1, 2, 3,\ t = 0.0015,\ 0.151, 0.451, 0.651, 0.751, 0.951$ and $N = 100$.

t	$m = 1$	$m = 2$	$m = 3$
0.0015	0.00006262	3.35×10^{-9}	2.51×10^{-9}
0.151	0.00007926	1.38×10^{-8}	3.67×10^{-9}
0.451	0.00016918	2.36×10^{-7}	6.91×10^{-9}
0.651	0.00029129	6.93×10^{-7}	8.51×10^{-9}
0.751	0.00035608	1.05×10^{-6}	8.56×10^{-9}
0.951	0.00022846	2.09×10^{-6}	2.53×10^{-9}

Table 3
The numerical results for the error of the formula (6.2) of [1] when $N = 10$ and 100.

t	$N = 10$	t	$N = 100$
0.0015	0.00070762	0.0015	7.21×10^{-7}
0.15	0.00083110	0.151	8.48×10^{-7}
0.45	0.00083110	0.451	1.30×10^{-6}
0.65	0.00214868	0.651	2.07×10^{-6}
0.75	0.00295468	0.751	2.92×10^{-6}
0.95	0.01296825	0.951	1.66×10^{-5}

Using formulas for the optimal coefficients $C_\alpha[\beta]$, $\alpha = 0, 1, 2$ given in Theorems 5.1, 5.2 and 5.3 from (6.1) when $m = 1, 2, 3$ and for some values of t we get numerical results which are presented in Tables 1 and 2 correspondingly for $N = 10$ and $N = 100$.

The numerical results of Tables 1 and 2 show that the error of the optimal quadrature formulas decreases as m and N increase.

In the work [1], in the space $L_2^{(2)}(0, 1)$, for the Cauchy type singular integral the optimal quadrature formulas of the form

$$\int_0^1 \frac{\varphi(x)}{x-t} dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(h\beta) \tag{6.2}$$

was constructed. From Theorem 1 of [1] when $N = 10$ and $N = 100$ for corresponding values of t from Tables 1 and 2 and for the function $\varphi = x^6$ we get numerical results for the error of the optimal quadrature formula (6.2) given in Table 3.

Comparison of the third column of Table 1 with the second column of Table 3 and the third column of Table 2 with forth column of Table 3, correspondingly, shows that the optimal quadrature formula (1.1) with derivatives of the present paper more exactly calculate the Cauchy type singular integral than the optimal quadrature formula (6.2) of [1].

7. Conclusion

Thus, in the present paper in the Sobolev space $L_2^{(m)}(0, 1)$, the algorithm of construction of optimal quadrature formulas with derivatives for approximate calculation of the Cauchy type singular integral is given using discrete analog of the operator $\frac{d^2}{dx^2}$. Here this algorithm is realized for the cases $m = 1,\ 2$ and 3.

Acknowledgment

We are very grateful to the reviewer for remarks and suggestions, which have improved the quality of the paper.

References

[1] D.M. Akhmedov, Optimal quadrature formulas for the Cauchy type singular integral in the Sobolev space $L_2^{(2)}(0, 1)$, Uzbek Math. J. 2 (2011) 48–56. 1707.00242v1 [math.NA].
[2] S.M. Belotserkovskii, I.K. Lifanov, Numerical Methods in Singular Integral Equations, Nauka, Moscow, 1985. in Russian.
[3] P. Blaga, G. Coman, Some problems on optimal quadrature, Stud. Univ. Babe-Bolyai Math. 52 (4) (2007) 21–44.

[4] Z.K. Eshkuvatov, N.M.A. Long, M. Abdulkawi, Numerical evaluation for Caushy type singular integrals on the interval, J. Comput. Appl. Math. 233 (8) (2010) 1995–2001.
[5] F.G. Frobenius, On Bernoulli numbers and Euler polynomials in German, Berl. Ber. 1910 (1910) 809–847.
[6] B.G. Gabdulkhaev, Cubature formulas for many dimensional singular integrals I, Izvestiya Vuzov. Math. 4 (1975) 3–13.
[7] F.D. Gakhov, Boundary Problems, in Russian, Moscow, Nauka, 1977, p. 640.
[8] M.I. Israilov, K.M. Shadimetov, Weight optimal quadrature formulas for singular integrals of the Cauchy type, Dokl. AN RUz 8 (1991) 10–11.
[9] F. Lanzara, On optimal quadrature formulae, J. Inequal. Appl. 5 (2000) 201–225.
[10] I.K. Lifanov, The Method of Singular Equations and Numerical Experiments, TOO "Yanus", Moscow, 1995, p. 520. in Russian.
[11] N.I. Muskhelishvili, Singular Integral Equations, Nauka, Moscow, 1968, p. 512. in Russian.
[12] S.M. Nikol'skii, Concerning estimation for approximate quadrature formulas, Uspekhi Mat.Nauk 5 2 (36) (1950) 165–177. In Russian.
[13] S.M. Nikol'skii, Quadrature Formulas, Nauka, Moscow, 1988. in Russian.
[14] A. Sard, Best approximate integration formulas; best approximation formulas, Am. J. Math 71 (1949) 80–91.
[15] K.M. Shadimetov, Optimal quadrature formulas for singular integrals of the Cauchy type, Dokl. AN RUz 6 (1987) 9–11. In Russian.
[16] K.M. Shadimetov, The Discrete Analogue of the Operator and Its Properties. Voprosy vichislitelnoy i prikladnoy matematiki. Tashkent, 1985, pp. 22–35. arXiv: 1001.0556[math.NA] 4 2010.
[17] K.M. Shadimetov, A.R. Hayotov, D.M. Akhmedov, Optimal quadrature formulas for the Cauchy type singular integral in the Sobolev space $L_2^{(2)}(-1,1)$, Am. J. Numer. Anal. 1 (1) (2013) 22–31, doi:10.12691/ajna-1-1-4.
[18] K.M. Shadimetov, A.R. Hayotov, D.M. Akhmedov, Optimal quadrature formulas for Cauchy type singular integrals in Sobolev space, Appl. Math. Comput. 263 (2015) 302–314.
[19] S.L. Sobolev, Introduction to the Theory of Cubature Formulas, Nauka, Moscow, 1974. in Russian.
[20] S.L. Sobolev, V.L. Vaskevich, The Theory of Cubature Formulas, Kluwer Academic Publishers Group, Dordrecht, 1997.