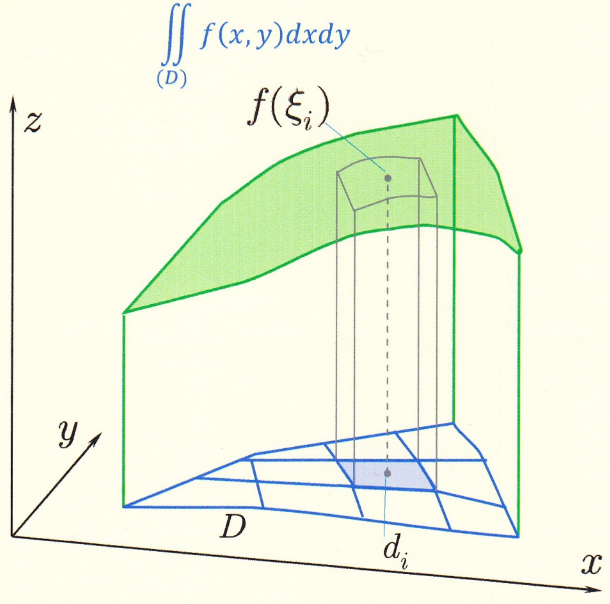
S.A.IMOMKULOV, Z.SH.IBRAGIMOV, R.A.SHARIPOV

INTEGRAL



O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS
TA'LIM VAZIRLIGI

URGANCH DAVLAT UNIVERSITETI

S.A.Imomkulov, Z.Sh.Ibragimov R.A.Sharipov

INTEGRAL

(uslubiy qo'llanma)

Ushbu uslubiy qo'llanma Urganch davlat universiteti o'quv-uslubiy kengashining 2018-yil 25-iyundagi 8-sonli yig'ilishi qaroriga asosan nashr qilishga tavsiya etilgan.

Urganch – 2018

C 18
I 48
S.A.Imomkulov, Z.Sh.Ibragimov R.A.Sharipov. Integral. Uslubiy qo'llanma. Mas'ul muharrir **M.Vaisova**. O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi Urganch davlat universiteti. Urganch, Urganch davlat universiteti noshirlik bo'limi, 2018-yil, 132 bet.

КБК 72.3
UO'K 517.373(075.8)

Uslubiy qo'llanma oliy ta'lim muassasalari talabalari uchun mo'ljallangan bo'lib, matematika fani bo'yicha aniqmas integral, aniq integral, karrali integrallar, egri chiziqli integrallar va Grin formulasi va uning tatbiqlariga oid masalalarni yechishga mo'ljallangan masalalardan tashkil topgan. Har bir mavzuga doir masalalardan namunalar izoh berilib, yechimlari yoritilgan.

Mas'ul muharrir **M.Vaisova**, f.-m.f.n., dotsent.

Taqrizchilar: **A.A.Atamuratov**, f.-m.f.n., dotsent,
Y.R.Saidov, f.-m.f.n., dotsent.

ISBN: 978-9943-5180-2-5

SO'ZBOSHI

Hozirgi kunda Respublikamizda ta'lim sohasida olib borilayotgan islohotlar talabalar uchun zamon talabiga javob beradigan darslik, o'quv va uslubiy qo'llanmalar yaratishni taqozo qiladi. Ushbu uslubiy qo'llanma integrallarni hisoblash va tatbiqlariga bag'ishlangan bo'lib, talabalarning mustaqil mushohada qilish qobiliyatini shakllantirishga, ijodiy fikrlash qobiliyatini oshirishga yordam beradi.

Qadim zamonlardan beri odamlar dehqonchilik qilinadigan ekin maydonlariningyuzalarini o'lchashuchun ularniyuza birligi sifatida olingan kvadratlarga bo'lib tashlaydilar va bu bo'laklarni qo'shib, ekin maydoni yuza miqdorini keltirib chiqaradilar. Arximed bu usuldan geometrikfiguralarning yuzalari va hajmlarini hisoblashda foydalangan. Masalan, doira yuzasini hisoblashda muntazam ko'pyoqlardan foydalanadi. Arximedning matematika fani rivojiga qo'shgan eng katta hissasi "Doirani o'lchash haqida" nomli asaridagi aylana uzunligining diametriga nisbatini hisoblagani bo'lgan. Bu ishida u π sonini taqribiyaniqlagan. Buning uchun aylanaga tashqi va ichki muntazam 96 tomonliburchaklar chizib, ularning tomonlari uzunligini hisoblab, π sonini $3\frac{10}{71} < \pi < 3\frac{1}{7}$ aniqlikda baholagan.

Greklar Arximedgacha aylana va ko'pburchak yuzalarini, prizma va silindr, piramida va konus hajmlarinihisoblashni bilishgan. Ammo Arximed yuza va hajmlarni hisoblashning umumiy metodini yaratdi. U figuralarni elementar figuralar yordamida "qamrab olish" usulini qo'lladi. Bu usul bilan Arximed ungacha hech kim hisoblay olmagan sfera sirti va shar hajmini hisoblagan. Arximedning bu g'oyalari hozirgi zamon "Differensial va integral hisob" fanining asosida yotadi.

I-BOB

ANIQMAS INTEGRAL

Matematik analiz kursidan ma'lumki (a, b) intervalda uzluksiz bo'lgan har qanday funksiya boshlang'ich funksiyaga ega, yani aniqmas integrali mavjud. Demak aniqmas integralni mavjudlik masalasi nazariy jihatdan hal qilingan. Amaliy jahatdan esa, berilgan funksiyaning aniqmas integralini aniq bir funksiyalar oilasi bo'lishligini ko'rsatish masalasi, qoyilishiga ko'ra bu doimo ochiq masaladir. Biz ko'proq amaliy ahamiyatga ega bo'lgan elementar funksiyalarni aniqmas integrallarini hisoblash usullarini o'rganamiz.

Asosiy elementar funksiyalar: o'zgarmas, darajali, ko'rsatkichli, logarfimik, trigonometrik va teskari trigonometrik funksiyalar ustida chekli sondagi arifmetik amallarni bajarishdan va ularning kompozitsiyalaridan hosil qilingan funksiyaga elementar fuksiya deyiladi.

Bizga ma'lumki har qanday elementar funksiyaning hosilasi yana elementar fuksiya bo'ladi, lekin har qanday elementar funksiyaning boshlang'ich funksiyasi elementar funksiya bo'lavermaydi. Masalan $y = \exp(-x^2)$ funksiyaning boshlang'ich funksiyasi elementar funksiyalar sinfiga tegishli emas. Agar bunday funksiyalarning aniqmas integrallari muhim tadbiqlarga ega bo'lgan hollarda ularning zarur xossalari maxsus usullar yordamida o'rganiladi (masalan, [8] ga qarang).

Demak shunday ekan, bizni asosiy fazifamiz elementar funksiyalar sinfidan boshlang'ich funksiyasi elementar funksiya boladigan funksiyalar sinflarini ajratishdan iboratdir.

Shu maqsadda, uslubiy qo'llanmani boshlang'ich funksiyasi bevosita hisoblanadigan sodda elementar funksiyalarni aniqmas integrallari jadvalini keltirish bilan boshladik. Undan so'ng integrallahsning asosiy usuli bo'lgan, o'zgaruvchini almashtirish yordamida berilgan elementar funksiyani yangi o'zgaruvchining sodda elementar funksiyasiga keltirib integrallash usulini va bo'laklab integrallash usullini bayon qildik.

Boshlang'ich funksiyasi yana elementar funksiya bo'ladigangan eng sodda va qulay fuksiyalar oilasi bu Ratsional funksiyalar oilasidir. Har qanday ratsional funksiyani qo‘yidagicha $\frac{A}{(x-a)^k}$ va $\frac{Ax+B}{(x^2+px+q)^k}$ oddiy kasrlarni chekli yig‘indisi ko‘rinishida tasvirlash munkin. Demak, ularni integralalsh shu oddiy kasrlarni integralashdan iboratdir. Keyingi 1.4-1.9 -paragraflarda ratsional fuksiyalar va almashtirish yordamida ratsional funksiyaga keladigan funksilarning aniqmas integrallarini topish usullari talqin qilingan. 1.10-paragrafda ba’zi bir elementar bo‘lmagan funksiyalarni aniqmas integrallari hisoblab ko‘rsatilgan.

1.1-§. Sodda funksiyalarning aniqmas integrallari jadvali

1. Aniqmas integralning ta'rifi. $f(x)$ funksiya (a,b) intervalda aniqlangan bo'lib, $F'(x) = f(x)$, $x \in (a,b)$, bo'lsa, u holda $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi deyiladi.

Agar $F_1(x)$ va $F_2(x)$ funksiyalar $f(x)$ funksiyaning boshlang'ich funksiyalari bo'lsalar, u holda ular bir biridan o'zgarmasga farq qiladilar. Ya'ni $F_2(x) = F_1(x) + C$ bo'ladi.

Ta'rif. Agar $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, u holda $\{F(x) + C\}$ funksiyalar oilasiga $f(x)$ funksiyaning aniqmas integrali deyiladi va quyidagicha belgilanadi.

$$\int f(x)dx = F(x) + C.$$

2. Aniqmas integralning asosiy xossalari.

a) $d[\int f(x)dx] = f(x)dx$;

b) $\int d\Phi(x) = \Phi(x) + C$;

v) $\int Af(x)dx = A\int f(x)dx (A = \text{const}; A \neq 0)$;

g) $\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$

3. Sodda funksiyalarning aniqmas integrallari jadvali.

I. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C (\mathrm{n} \neq -1)$.

II. $\int \frac{dx}{x} = \ln|x| + C (x \neq 0)$.

III. $\int \frac{dx}{1+x^2} = \begin{cases} arctgx + C \\ -arcctgx + C \end{cases}$.

IV. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left\{\frac{1+x}{1-x}\right\} + C$.

V. $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C \\ -\arccos x + C \end{cases}$.

VI. $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\left|x + \sqrt{x^2 \pm 1}\right| + C$.

VII. $\int a^x dx = \frac{a^x}{\ln a} + C (a > 0, a \neq 1)$; $\int e^x\, dx = e^x + C$.

VIII. $\int \sin x dx = -\cos x + C$

XII. $\int shxdx = chx + C$

IX. $\int \cos x dx = \sin x + C$

XIII. $\int chxdx = shx + C$

X. $\int \frac{dx}{\sin^2 x} = -ctgx + C$

XIV. $\int \frac{dx}{sh^2 x} = -cth + C$

XI. $\int \frac{dx}{\cos^2 x} = tgx + C$

XV. $\int \frac{dx}{ch^2 x} = thx + C$

Jadval yordamida hisoblashga misollar.

1-misol. $\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}} dx$ – integralni hisoblang.

Yechish.

$$\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}} dx = \int \frac{\sqrt{x}}{\sqrt[4]{x}} dx - 2\int \frac{\sqrt[3]{x^2}dx}{\sqrt[4]{x}} + \int \frac{dx}{\sqrt[4]{x}} =$$

$$= \int x^{\frac{1}{4}} dx - 2\int x^{\frac{5}{12}} dx + \int x^{-\frac{1}{4}} dx = -\frac{4}{3\sqrt[4]{x^3}} + \frac{12\cdot 2}{7\sqrt[12]{x^7}} - \frac{4}{5\sqrt[4]{x^5}} + C$$

2-misol. $\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}} dx$ – integralni hisoblang.

Yechish.

$$\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}} dx = \int \frac{\sqrt{1+x^2}}{\sqrt{(1+x^2)(1-x^2)}} dx + + \int \frac{\sqrt{1-x^2}}{\sqrt{(1-x^2)(1+x}}$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1+x^2}} dx = \arcsin x + \ln\left|x+\sqrt{1+x^2}\right| + C$$

3-misol. $\int \frac{dx}{(x-a)^n}$ – integralni hisoblang.

Yechish.

$$\int \frac{dx}{(x-a)^n} = \frac{1}{1-n}\frac{1}{(x-a)^{n-1}} + C.$$

4-misol. $\int \frac{dx}{\sqrt{x^2\pm a^2}}$ – integralni hisoblang.

Yechish.

$$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \int \frac{d(x/a)}{\sqrt{(x/a)^2\pm 1}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C.$$

5-misol. $\int \frac{dx}{a^2+x^2}$ – integralni hisoblang.

Yechish.

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{d(x/a)}{1+(x/a)^2} = \begin{cases} \frac{1}{a} arctg(x/a) + C \\ -\frac{1}{a} arcctg(x/a) + C \end{cases}$$

Mustaqil ishlash uchun misollar

6-misol. $\int \frac{1+\cos^2 x}{1+\cos 2x} dx$

7-misol. $\int \frac{dx}{\sqrt{3+3x^2}}$

8-misol. $\int \frac{(1-x)^2}{x\sqrt{x}} dx$

9-misol. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$

10-misol. $\int 2\sin\frac{x}{2} dx$

11-misol. $\int \frac{(1+x)^2}{x(1+x^2)} dx$

12-misol. $\int tg^2 x dx$

13-misol. $\int \frac{\sqrt{x}-x^3e^x+x^2}{x^3} dx$

14-misol. $\int \frac{dx}{\cos 2x+\sin^2 x}$

15-misol. $\int (\arcsin x + \arccos x)\, dx$

1.2-§. O‘zgaruvchini almashtirish usuli

Bizdan $f(x)$ – elementar funksiyaning

$$\int f(x)dx$$

–aniqmas integralini topish talab qilinsin. Agar $f(x)$ funksiyaning boshlang‘ich funksiyasi elementar funksiya ekanligi bizga ma’lum bo‘lsa ham, uni bevosita topolmasak, u holda biz o‘zgaruvchini aniq bir almashtirish orqali funksiyani boshlang‘ich funksiyasi bevosita topiladigan sodda funksiyaga almashtirib olishga harakat qilamiz. Bu usul hamma vaqt ham ijobiy natija beravermasada, boshlang‘ich funksiyasi elementar funksiya bolishi ma’lum bo‘lgan funksiyalarni integrallashda birdan - bir eng samarali usuldir.

$$x = \varphi(t) \tag{1.1}$$

deb olib, integral ostidagi ifodada o‘zgaruvchini almashtiramiz. Bu yerda $\varphi(t)$ uzluksiz funksiya bo‘lib, uzluksiz hosilaga va teskari funksiyaga ega. U holda

$$f(x)dx = f(\varphi(t))\varphi'(t)dt$$

va

$$\int f(x)dx = \int f[\varphi(t)]\,\varphi'(t)dt \qquad (1.2)$$

bo‘ladi. Agar $g(t) = f(\varphi(t))\varphi'(t)$ funksiya boshlang‘ich funksiyasi bevosita topiladigan sodda elementar funksiya bo‘lsa, uning boshlangich funksiyasi argumenti t o‘rniga uning (1.1) tenglikka asosan x orqali ifodasi $t = \varphi^{-1}(x)$ qo‘yib, $f(x)$ funksiyani boshlang‘ich funksiyasini hosil qilamiz.

Integrallashda o‘zgaruvchini almashtirishni ba’zan $x = \varphi(t)$ ko‘rinishda emas, balki $t = f(x)$ ko‘rinishda tanlash ma’qulroq bo‘ladi. Buni misolda ko‘rsatamiz. Ushbu

$$\int \frac{f'(x)dx}{f(x)}$$

ko‘rinishdagi integralni hisoblash kerak bo‘lsin. Bu yerda

$$f(x) = t$$

deb qabul qilish qulay, u vaqtda

$$f'(x)dx = dt,$$

$$\int \frac{f'(x)dx}{f(x)} = \int \frac{dt}{t} + C = \ln|t| + C = \ln|f(x)| + C.$$

Endi misollar ishlab ko‘rsatamiiz.

1-misol. $\int \frac{\cos x}{\sqrt[3]{\sin x}}dx$– integralni hisoblang. Bu yerda $t = sinx$ almashtirishni bajaramiz va $dt = \cos x dx$, demak,

$$\int \frac{\cos x}{\sqrt[3]{\sin x}}dx = \int \frac{dt}{\sqrt[4]{t}} = \int t^{-\frac{1}{4}}dt = \frac{4t^{3/4}}{3} + C = \frac{4}{3}\sin^{3/4}x + C.$$

2-misol. $\int \frac{xdx}{(1+x^2)^{\frac{1}{2}}}$ – integralni hisoblang. Bu yerda $t = 1 + x^2$ almashtirishni bajaramiz va

$$\int \frac{xdx}{(1+x^2)^{\frac{1}{2}}} = \frac{1}{2}\int \frac{d(1+x^2)}{(1+x^2)^{\frac{1}{2}}} = \frac{1}{2}\int \frac{dt}{t^{\frac{1}{2}}} = \sqrt{t} + C = \sqrt{1+x^2} + C$$

natijaga ega bo‘lamiz.

3-misol. $\int x^2\sqrt[3]{1+x^3}dx$ –integralni hisoblang. Bu yerda $1 + x^3 = t^3$ almashtirishni bajaramiz va $dx = \frac{t^2}{(t^3-1)^{2/3}}dt$, demak

$$\int x^2\sqrt[3]{1+x^3}dx = \int (t^3-1)^{2/3}t\frac{t^2}{(t^3-1)^{2/3}}dt = \int t^3dt = \frac{1}{4}t^4 + C =$$
$$= \frac{1}{4}(1+x^3)^{4/3} + C.$$

4-misol. $\int \frac{dx}{x\ln x\ln(\ln x)}$ –integralni hisoblang. Bu yerda $t = \ln(\ln x)$ almashtirishni bajaramiz va

$$\int \frac{dx}{x\ln x\ln(\ln x)} = \int \frac{d\ln x}{\ln x\ln(\ln x)} = \int \frac{de^t}{te^t} = \int \frac{dt}{t} =$$
$$= \ln|t| + C = \ln|\ln(\ln x)| + C.$$

natijaga ega bo‘lamiz.

5-misol. $\int \frac{x^4dx}{1+x^{10}}$ –integralni hisoblang. Bu yerda $t = x^5$ almashtirishni bajaramiz va

$$\int \frac{x^4dx}{1+x^{10}} = \frac{1}{5}\int \frac{dt}{1+t^2} = \frac{1}{5}arctgt + C = \frac{1}{5}arctgx^5 + C.$$

Mustaqil ishlash uchun misollar

6-misol. $\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx.$

7-misol. $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}dx$

8-misol. $\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}dx$

9-misol. $\int \frac{\ln tgx}{\sin x\cos x}dx$

10-misol. $\int \frac{e^{2x}}{\sqrt[4]{e^x+1}}dx$

11-misol. $\int \frac{\sqrt{1+\ln x}}{x\ln x}dx$

12-misol. $\int \frac{dx}{\sqrt{1+e^x}}$

13-misol. $\int \frac{dx}{1+\sqrt[3]{x+1}}$

14-misol. $\int \sqrt{1+\cos^2 x}\cdot \sin 2x\, cos2xdx$

15-misol. $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}dx$

1.3-§. Bo‘laklab integrallash usuli

$u(x)$ va $\vartheta(x)$ funksiyalar uzluksiz differensiallanuvchi funksiyalar bo‘lsin. U holda

$$d(u\cdot\vartheta) = ud\vartheta + \vartheta du.$$

Bundan, tenglikni har ikkala tomonini integrallab

$$u\vartheta=\int ud\vartheta+\int\vartheta du$$

yoki

$$\int ud\vartheta=u\vartheta-\int\vartheta du \qquad (1.3)$$

formulani hosil qilamiz. Bu formula ***bolaklab integrallash formulasi*** deyiladi. Bu formula ko'pincha integral ostidagi ifoda $ud\vartheta$ ko'rinishida tasvirlaganda $d\vartheta$ differensialdan ϑ funksiyani toppish bilan

$$\int vdu$$

integralni hisoblash

$$\int udv$$

integralni bevosita hisoblashdan osonroq bo'lgan hollarda qo'llaniladi. Integral belgisi ostidagi ifodani u va dv ko'paytuvchilarga qulay tarzda ajratishni bilish malakasi masalalar yechish jarayonida hosil bo'ladi. u va dv ko'paytuvchilarni qanday tanlashni bir necha misolda ko'rsatamiz.

1-misol. $\int x^a \ln x dx$ – integralni hisoblang (bunda $a\neq -1$).

Yechish.

$$\int x^a\ln x dx=\left|\begin{array}{l} u=\ln x\\ d\vartheta=xdx,\\ \vartheta=\frac{1}{a+1}x^{a+1}\end{array}\right|=\frac{1}{a+1}x^{a+1}\ln x-\frac{1}{a+1}\int x^{a+1}d\ln x=$$

$$=\frac{1}{a+1}x^{a+1}\ln x-\frac{1}{a+1}\int x^{a+1}\frac{1}{x}dx=\frac{1}{a+1}x^{a+1}\ln x-\frac{1}{a+1}\int x^a\,dx=$$

$$=\frac{1}{a+1}x^{a+1}\ln x-\frac{1}{(a+1)^2}x^{a+1}+C.$$

2-misol. $\int x^n\, e^x dx$ integralni hisoblang ($n\geq 1$).

Yechish.

$$\int x^n e^x dx = \begin{vmatrix} u = x^n, \\ d\vartheta = e^x dx, \\ \vartheta = e^x \end{vmatrix} = x^n e^x - n\int x^{n-1} e^x dx = x^n e^x - nx^{n-1} +$$

$$+n(n-1)\int x^{n-2} e^x dx = \ldots = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} x^{n-k} e^x + C.$$

3-misol. $\int e^{ax}\cos bx dx$ – integralni hisoblang.

Yechish.

$$F(x) = \int e^{ax}\cos bx dx = \begin{vmatrix} u = e^{ax}, & du = ae^{ax}dx, \\ d\vartheta = \cos bx dx, & \frac{1}{b} d(\sin bx) \end{vmatrix} =$$

$$= \frac{1}{b} e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx dx == \begin{vmatrix} u = e^{ax}, & du = ae^{ax}dx, \\ d\vartheta = \sin bx dx, & -\frac{1}{b} d(\cos bx) \end{vmatrix} =$$

$$= \frac{1}{b} e^{ax}\sin bx + \frac{a}{b^2} e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx dx.$$

Shunday qilib qo‘yidagi tenglikka ega bo‘ldik:

$$F(x) = \frac{1}{b} e^{ax}\sin bx + \frac{a}{b^2} e^{ax}\cos bx - \frac{a^2}{b^2} F(x)$$

Bu tenglikdan $F(x)$ funksiyani topib,

$$\int e^{ax}\cos bx dx = \frac{a\cos bx + b\sin bx}{a^2 + b^2} e^{ax} + C$$

natijaga ega bo‘lamiz.

4-misol. $\int \arcsin x dx$ – integralni hisoblang.

Yechish.

$$\int \arcsin x dx = \begin{vmatrix} u = \arcsin x, \\ d\vartheta = dx, \\ \vartheta = x \end{vmatrix} = x\arcsin x - \int x d(\arcsin x) =$$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x\arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} d(1-x^2) =$$

$$= x\arcsin x + \sqrt{1-x^2} + C.$$

5-misol. $\int \sqrt{1-x^2}\, dx$– integralni hisoblang.

$$F(x)=\int\sqrt{1-x^2}\,dx=\int\frac{1-x^2}{\sqrt{1-x^2}}dx=\int\frac{dx}{\sqrt{1-x^2}}-\int\frac{x^2dx}{\sqrt{1-x^2}}=$$

$$=\arcsin x-\int\frac{x^2dx}{\sqrt{1-x^2}}=\arcsin x+\frac{1}{2}\int\frac{xd(1-x^2)}{\sqrt{1-x^2}}=\left|\begin{array}{c}u=x,\\ d\vartheta=\frac{1}{2}\frac{d(1-x^2)}{\sqrt{1-x^2}},\\ \vartheta=\sqrt{1-x^2}\end{array}\right|$$

$$\arcsin x+x\sqrt{1-x^2}-\int\sqrt{1-x^2}dx=\arcsin x+x\sqrt{1-x^2}-F(x).$$

Bu tenglikdan $F(x)$ funksiyani topib,

$$\int\sqrt{1-x^2}\,dx=\frac{1}{2}(\arcsin x+x\sqrt{1-x^2})+C$$

natijaga ega bo‘lamiz.

Mustaqil ishlash uchun misollar

Quyidagi integrallarni bo‘laklab integrallash formulasini qo‘llab hisoblang.

6-misol. $\int x^2e^x\sin x dx.$

7-misol. $\int\sqrt{a^2-x^2}\,dx.$

8-misol. $\int e^{ax}\sin bxdx.$

9-misol. $\int\frac{xarctgx}{\sqrt{1+x^2}}dx.$

10-misol. $\int x^5e^{x^2}dx$

11-misol. $\int(arctgx)^2dx$

12-misol. $\int x^2sh2xdx$

13-misol. $\int\frac{x^2}{(1+x^2)^2}dx$

14-misol. $\int xtg^2xdx$

15-misol. $\int\ln^2xdx$

1.4-§. Ratsional funksiyalar. Eng sodda ratsional funksiyalar va ularni integrallash

Ushbu

$$P_n(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n,\quad a_0\neq 0,$$

formula bilan berilgan funksiyaga $n-$darajali ko‘phad deyiladi. Ikkita ko‘phadning nisbatiga esa ratsional funksiya deyiladi:

$$\frac{P_n(x)}{Q_m(x)}=\frac{a_0x^n+a_1x^{n-1}+\ldots+a_n}{b_0x^m+b_1x^{m-1}+\ldots+b_m}$$

Bu yerda $P_n(x)$ va $Q_m(x)$ ko‘phadlar o‘zaro qisqarmaydigan, yani umumiy ildizga ega bo‘lmagan ko‘phaddirlar.

Agar $n \geq m$ bo‘lsa, suratni maxrajga bo‘lib, berilgan ratsional funksiyani biror ko‘phad bilan boshqa bir ratsional funksiya yig‘indisi ko‘rinishida tasvirlash mumkin:

$$\frac{P_n(x)}{Q_m(x)} = P_{n-m}(x) + \frac{P_k(x)}{Q_m(x)},$$

bu yerda $k < m$.

1-misol. $R(x) = \frac{x^4-3}{x^2+2x+1}$ ratsional funksiya berilgan bo‘lsin.

Suratni maxrajiga bo‘lamiz va qo‘yidagi

$$\frac{x^4 - 3}{x^2 + 2x + 1} = x^2 - 2x + 3 - \frac{4x - 6}{x^2 + 2x + 1}$$

natijaga ega bo‘lamiz.

Darajali funksiyaning boshlang‘ich funksiyasi bizga ma’lum bo‘lganligi sababli, ko‘phadlarni integrallash bizga qiyin emas. Demak, ratsional funksiyalarni aniqmas integrallarini hisoblashni o‘rganar ekanmiz, biz faqat suratdagi ko‘phadning darajasi maxrajdagi ko‘phadning darajasidan kichik bo‘lgan holni o‘rganishimiz yetarli.

Avvalo biz qo‘yidagi eng sodda ratsional funksiyalarni aniqmas integrallarini bizga ma’lum bo‘lgan sodda funksiyalarning integrallash jadvali o‘garuvchini almashtirish usuli va bo‘laklab integralash formulasi yordamida hisoblab topamiz.

I. $\frac{A}{x-a}$,

II. $\frac{A}{(x-a)^k}$ ($k \geq 2$ – butun musbat son)

III. $\frac{Ax+B}{x^2+px+q}$ (maxrajning ildizlari kompleks sonlar, ya’ni $\frac{p^2}{4} - q < 0$).

IV. $\frac{Ax+B}{(x^2+px+q)^k}$ ($k \geq 2$ – butun musbat son, maxrajning ildizlari kompleks sonlar).

I, II va III tipdagi ratsional funksiyalarni 1.1-§ keltirilgan integrallash jadvali yordamida integrallaymiz:

I. $\int \frac{A}{x-a} dx = A\ln|x - a| + C$.

II. $\int \frac{A}{(x-a)^k} dx = A \int (x - a)^{-k} dx = A \frac{(x-a)^{-k+1}}{-k+1} + C = \frac{A}{(1-k)(x-a)^{k-1}} + C$.

III. $\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{x^2+px+q}dx = \frac{A}{2}\ln|x^2+px+q| +$
$\left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p}{4}\right)^2} =$
$\frac{A}{2}\ln|x^2+px+q| + \frac{2B-Ap}{\sqrt{4q-p^2}} arctg \frac{2x+p}{\sqrt{4q-p^2}} + C$

IV. tipdagi ratsional funksiyalarni integrallash murakkabroq hisoblashni talab qiladi. Agar shu tipdagi integral berilgan boʻlsa oʻzgaruvchini almashtirish usuli va boʻlaklab integrallash formulasini qoʻllaymiz (qoʻshimcha maʼlumotlar uchun [3]va [5] ga qarang) :

$\int \frac{Ax+B}{(x^2+px+q)^k}dx$ – integralni qoʻyidagicha qoʻshiluvchilarga ajratamiz:

$$\int \frac{Ax+B}{(x^2+px+q)^k}dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{(x^2+px+q)^k}dx$$
$$= \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^k}$$

Birinchi integralni $x^2+px+q=t,\ (2x+p)dx=dt$ almashtirish yordamida hisoblaymiz:

$$\int \frac{2x+p}{(x^2+px+q)^k}dx = \int \frac{dt}{t^k} = \int t^{-k}dt = \frac{t^{-k+1}}{1-k} + C = \frac{1}{(1-k)(x^2+px+q)^{k-1}} + C.$$

Ikkinchi integralni qoʻyidagicha koʻrinishda yozib olamiz:

$$I_k = \int \frac{dx}{(x^2+px+q)^k} = \int \frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^k} = \int \frac{dt}{(t^2+m^2)^k}$$

bu yerda

$$x+\frac{p}{2}=t,\ dx=dt,\quad q-\frac{p^2}{4}=m^2$$

Endi oxirgi integralni qoʻyidagicha qoshiluvchilarga ajratamiz:

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{(t^2+m^2)-t^2}{(t^2+m^2)^k}dt = \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}} - \frac{1}{m^2}\int \frac{t^2}{(t^2+m^2)^k}dt.$$

Bu yerda oxirgi integralni boʻlaklab,

$$\int \frac{t^2dt}{(t^2+m^2)^k} = -\frac{1}{2(k-1)}\left[t\frac{1}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right]$$

natijaga ega bo‘lamiz. Bu ifodani yuqoridagi tenglikka qo‘ysak, quyidagi rekurent formulani hosil bo‘ladi:

$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}} + \frac{1}{m^2}\frac{1}{2(k-1)}\left[\frac{t}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right] =$
$= \frac{t}{2m^2(k-1)(t^2+m^2)^{k-1}} + \frac{2k-3}{2m^2(k-1)} I_{k-1}.$

Shu jarayonni davom ettirib, chekli qadamdan so‘ng ma’lum integral

$$I_1 = \int \frac{dt}{t^2+m^2} = \frac{1}{m} arctg\frac{t}{m} + C$$

ga yetib boramiz. So‘ngra t va m o‘rniga ularning qiymatlarini qo‘yib, IV integrallarning x va berilgan A, B, p, q sonlar orqali ifodasiga ega bo‘lamiz.

2-misol.

$\int \frac{x-1}{(x^2+2x+3)^2}dx = \int \frac{\frac{1}{2}(2x+2)+(-1-1)}{(x^2+2x+3)^2}dx = \frac{1}{2}\int \frac{dt}{t^2+2} - \frac{1}{2}\int \frac{t^2}{(t^2+2)^2}dt =$
$= \frac{1}{2}\frac{1}{\sqrt{2}} arctg\frac{t}{\sqrt{2}} - \frac{1}{2}\int \frac{t^2dt}{(t^2+2)^2}$

Oxirgi integralga $x + 1 = t$ almashtirishni qo‘llanamiz:

$\int \frac{t^2dt}{(x^2+2x+3)^2} = \int \frac{dx}{[(x+1)^2+2]^2}\int \frac{dt}{(t^2+2)^2} = \frac{1}{2}\int \frac{(t^2+2)-t^2}{(t^2+2)^2}dt = \frac{1}{2}\int \frac{dt}{t^2+2} - \frac{1}{2}\int \frac{t^2}{(t^2+2)}dt =$
$= \frac{1}{2}\frac{1}{\sqrt{2}} arctg\frac{t}{\sqrt{2}} - \frac{1}{2}\int \frac{t^2dt}{(t^2+2)^2}$

Oxirgi integralni qaraymiz:

$$\int \frac{t^2dt}{(t^2+2)^2} = \frac{1}{2}\int \frac{td(t^2+2)}{(t^2+2)^2} = -\frac{1}{2}\int td\left(\frac{1}{t^2+2}\right) =$$
$$= -\frac{1}{2}\frac{t}{t^2+2} + \frac{1}{2}\int \frac{dt}{t^2+2} = -\frac{t}{2(t^2+2)} + \frac{1}{\sqrt{2}} arctg\frac{t}{\sqrt{2}}$$

(ixtiyoriy o‘zgarmasni hozircha yozmaymiz; uni faqat oxirgi natijada hisobga olamiz).

Demak,

$$\int \frac{dx}{(x^2+2x+3)^2} = \frac{1}{2\sqrt{2}} arctg\frac{x+1}{\sqrt{2}}$$
$$-\frac{1}{2}\left[-\frac{x+1}{2(x^2+2x+3)} + \frac{1}{2\sqrt{2}} arctg\frac{x+1}{\sqrt{2}}\right]$$

Oxirgi natija bunday bo‘ladi:

$$\int \frac{x-1}{(x^2+2x+3)^2}dx = -\frac{x+2}{2(x^2+2x+3)^2} + \frac{\sqrt{2}}{4} arctg\frac{x+1}{\sqrt{2}} + C.$$

1.5 - §. Ratsional funksiyalarni eng sodda ratsional funksiyalarga ajratish

Endi har qanday ratsional funksiyani I, II, III va IV ko'rinishdagi sodda funksiyalarning chekli yig'indisi ko'rinishida yozish mumkinligini ko'rsatamiz.

Ushbu ratsional funksiya

$$\frac{P_n(x)}{Q_m(x)} = \frac{a_0x^n + a_1x^{n-1}+\ldots+a_n}{b_0x^m + b_1x^{m-1}+\ldots+b_m}, \qquad (n < m),$$

berilgan bo'lsin.

1-teorema. *Koeffisiyentlari haqiqiy sonlardan iborat* $Q_m(x)$ *ko'phad karraliklari bilan qo'shib hisoblaganda* k *ta haqiqiy va* $2s$ *ta kompleks ildizlarga ega bo'lsa, u holda bu ko'phad uchun qo'yidagicha yoyilma o'rinli:*

$$Q_m(x) = a_0(x-c_1)^{k_1}(x-c_2)^{k_2}\ldots(x-c_\alpha)^{k_\alpha} \times$$
$$\times (x^2+p_1x+q_1)^{s_1}(x^2+p_2x+q_2)^{s_2}\ldots(x^2+p_\beta x+q_\beta)^{s_\beta}, \tag{1.4}$$

bu yerda $k_1+k_2+\ldots+k_\alpha = k, s_1+s_2+\ldots+s_\beta = s$, $c_1, c_2, \ldots, c_\alpha$ *–ko'phadning haqiqiy ildizlari va* $x^2+p_lx+q_l = (x-d_l)(x-\bar{d}_l)$ *bolib,* d_l *va uning qo'shmasi* $\bar{d}_l$*ko'phadning kompleks ildizlaridir,* $l = \overline{1,\beta}$.

Bu teoremadan foydalanib quyidagi teorema aniqmas koeffisientlarni topish metodi yordamida isbotlanadi.

2-teorema. Agar

$$Q_m(x) = a_0(x-c_1)^{k_1}(x-c_2)^{k_2}\ldots(x-c_\alpha)^{k_\alpha} \times$$
$$\times (x^2+p_1x+q_1)^{s_1}(x^2+p_2x+q_2)^{s_2}\ldots(x^2+p_\beta x+q_\beta)^{s_\beta}$$

bo'lsa, u holda $R(x) = \frac{P_n(x)}{Q_m(x)}$ *– ratsional funksiya quyidagicha sodda ratsional funksiyalarga yoyiladi*

$$R(x) = \sum_{j=1}^{\alpha}\sum_{r=1}^{k_j}\frac{A_j^r}{(x-c_j)^r} + \sum_{l=1}^{\beta}\sum_{r=1}^{s_l}\frac{B_l^r x + C_l^r}{(x^2+p_lx+q_l)^r}. \tag{1.5}$$

$R(x)$ aniq bir ratsional funksiya bo'lgan holda (1.5) formuladagi A_j^r,B_l^r ,C_l^r koeffitsientlarni qo'yidagicha usul bilan topamiz: (1.5)

formulaning o‘ng tomonidagi kasrlarni umumiy maxrajga keltirgandan so‘ng tenglikning o‘ng va chap qismlarining suratlarida hosil bo‘lgan ko‘phadlar aynan teng bo‘lishi kerak. Bu tengliklar A_j^r, B_l^r, C_l^r nomalumlarning chiziqli tenglamalar sistemasidan iborat bo‘ladi. Shu sistemani yechib ularni topamiz.
Bu teoremalarning isbotlarini [2], [3], [5] adabiyotladan o‘rganish mumkin.

1-Misol. $R(x)=\frac{x^2+2}{(x+1)^3(x-2)}$ ratsional funksiyani sodda ratsional funksiyalarga yoying.

Yechish. ([5] ga qarang) 2- teoremaga ko‘ra quyidagicha formal yoyilma o‘rinli:

$$\frac{x^2+2}{(x+1)^3(x-2)}=\frac{A}{(x+1)^3}+\frac{A_1}{(x+1)^2}+\frac{A_2}{x+1}+\frac{B}{x-2}$$

Bu tenglikning o‘ng tomonini umumiy maxrajga keltirib va suratlarni tenglashtirib, ushbuni hosil qilamiz:

$$\begin{aligned}&x^2+2\\&=A(x-2)+A_1(x+1)(x-2)+A_2(x+1)^2(x-2)\\&+B(x+1)^3\end{aligned}$$

yoki

$$\begin{aligned}x^2+2=(A_2+B)+x^3+(A_1+3B)x^2+(A-A_1-3A_2+3B)x\\+(-2A-2A_1-2A_2+B)\end{aligned}$$

x^3, x^2, x^1, x^0 (ozod had) yonidagi koeffitsientlarni tenglashtirib, noma’lum koeffitsientlarni aniqlash uchun ushbu chiziqli tenglamalar sistemasini hosil qilamiz:

$$\begin{aligned}0&=A_2+B,\\1&=A_1+3B,\\0&=A-A_1-3A_2+3B,\\2&=-2A-2A_1-2A_2+B,\end{aligned}$$

Bu sistemani yechib, quyidagilarni topamiz:

$$A=-1,\quad A_1=\frac{1}{3},\quad A_3=-\frac{2}{9},\ B=\frac{2}{9}$$

Natijada quyidagi tenglikka ega bo‘lamiz:

$$\frac{x^2+2}{(x+1)^3(x-2)}=-\frac{1}{(x+1)^3}+\frac{1}{3(x+1)^2}+\frac{2}{9(x+1)}+\frac{2}{(x-2)}.$$

2-Misol. $R(x) = \frac{x}{1+x^4}$- ratsional funksiyani sodda ratsional funksiyalarga yoying.

Yechish.

Bu yerda maxrajdagi koʻphadning ildizlari -1 ning toʻrtinchi darajali ildizlaridir, ya’ni kompleks sonlar. U quyigagicha koʻpaytuvchilarga ajraladi:

$$1 + x^4 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)).$$

Demak,

$$\frac{x}{1+x^4} = \frac{1}{2\sqrt{2}}\left(\frac{1}{x^2 - \sqrt{2}x + 1} - \frac{1}{x^2 + \sqrt{2}x + 1}\right).$$

1.6-§. Ratsional funksiyalarni integrallash

Bizdan

$$\int \frac{P_n(x)}{Q_m(x)} dx$$

integralni hisoblash talab etilsin (qoʻshimcha ma’lumot uchun [3] va [5]ga qarang).

1- hol. Maxrajning ildizlari haqiqiy va har xil, ya’ni

$$Q_m(x) = (x - c_1)(x - c_2)\ldots(x - c_m)$$

boʻlsin. Bu holda $\frac{P_n(x)}{Q_m(x)}$ ratsional funksiya I tipdagi eng sodda kasrlarga ajraladi:

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{x - c_1} + \frac{A_2}{x - c_2} + \ldots + \frac{A_m}{x - c_m}$$

va bu tenglikning har ikkala tomonini integrallasak:

$$\int \frac{P_n(x)}{Q_m(x)} dx = \int \frac{A_1}{x - c_1} dx + \int \frac{A_2}{x - c_2} dx + \ldots + \int \frac{A_m}{x - c_m} dx$$
$$= A_1 \ln|x - c_1| + A_2 \ln|x - c_2| + \ldots + A_m \ln|x - c_m| + C$$

2-hol. Maxrajning ildizlari haqiqiy va ulardan ba’zilari karrali:

$$Q(x) = (x - c_1)^{k_1} - (x - c_2)^{k_2} \ldots (x - c_m)^{k_m}$$

Bu holda $\frac{P_n(x)}{Q_m(x)}$ ratsional funksiya I va II tipdagi sodda ratsional funksiyalarga ajraladi.

1-misol. $\int \frac{x^2+2}{(x+1)^3+(x-2)} dx$ integralni hisoblang.

Yechish.

$$\int \frac{x^2+2}{(x+1)^3+(x-2)}dx = -\int \frac{dx}{(x+1)^3} + \frac{1}{3}\int \frac{dx}{(x+1)^2} - \frac{2}{9}\int \frac{dx}{x+1} + \frac{2}{9}\int \frac{dx}{x-2} =$$

$$= \frac{1}{2}\frac{1}{(x+1)^2} - \frac{1}{3(x+1)} - \frac{2}{9}\ln|x+1| + \frac{2}{9}\ln|x-2| + C =$$

$$= -\frac{2x-1}{6(x+1)^2} + \frac{2}{9}\ln\left|\frac{x-2}{x+1}\right| + C$$

3-hol. Maxrajning ildizlari faqat kompleks sonlardan ibora . Bu holda $\frac{P_n(x)}{Q_m(x)}$ ratsional funksiya III va IV tipdagi ratsional funksiyalarga yoyiladi va integrali shu tipdagi ratsional funksiyalar, logarfimik fuksiyalar hamda “arctg”lar yigindilaridan iborat bo‘ladi.

2-misol. $\int \frac{x^2}{1+x^4}dx$ **–** integralni hisoblang.

Yechish. Bu misolda maxrajdagi ko‘phad ildizlari -1 sonining to‘rtinchi darajali ildizlari, yani kompleks sonlar.

$$\int \frac{x^2}{1+x^4}dx = \frac{1}{2\sqrt{2}}\left(\int \frac{xdx}{x^2-\sqrt{2}x+1} - \int \frac{xdx}{x^2+\sqrt{2}x+1}\right) =$$

$$= \frac{1}{4\sqrt{2}}\left(\int \frac{d((x-\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2)}{(x-\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2} - \int \frac{d((x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2)}{(x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}\right) +$$

$$+\frac{1}{4}\left(\int \frac{d(x-\frac{\sqrt{2}}{2})}{(x-\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2} + \int \frac{d(x+\frac{\sqrt{2}}{2})}{(x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}\right) =$$

$$= \frac{1}{4\sqrt{2}}\ln\frac{(x-\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}{(x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}$$

$$+\frac{1}{2\sqrt{2}}\left(arctg\left(\frac{x-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) + arctg\left(\frac{x+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)\right) + C.$$

4-hol. Maxrajning ildizlari orasida kompleks ildizlar bor . Bu holda $\frac{P_n(x)}{Q_m(x)}$ ratsional funksiya I, II , III va IV tipdagi sodda ratsional funksiyalarga yoyiladi va integrali shu tipdagi ratsional funksiyalar, logarfimik funksiyalar hamda “arctg” lar yigindilaridan iborat bo‘ladi.

Mustaqil ishlash uchun misollar

3-misol. $\int \frac{2x+3}{(x-2)(x+5)}dx$

4-misol. $\int \frac{x}{(x+1)(x+2)(x+3)}dx$

5-misol. $\int \frac{x^3}{x^3+5x^2+6x}dx$

6-misol. $\int \frac{x^4}{x^4+5x^3+4}dx$

7-misol. $\int \frac{x}{x^2-3x+2}dx$

8-misol. $\int \frac{x^2+5x+4}{x^4+5x^2+4}dx$

9-misol. $\int \frac{dx}{(x+1)(x^2+1)}$

10-misol. $\int \frac{x}{(x-1)^2(x^2+2x+2)}dx$

11-misol. $\int \frac{dx}{1+x^4}$

12-misol. $\int \frac{dx}{1-x^6}$

1.7-§. Trigonometrik funksiyalarning ba’zi sinflarini integrallash

Biz yuqorida faqat algebraik, yani ratsional va irratsional elementar funk siyalarning integrallarini hisoblash usullarini o‘rgandik. Endi bu paragrafda ba’zi bir trigonometrik funksiyalar integrallarini hisoblash usullarini o‘rganamiz ([5] ga qarang).

Ushbu

$$\int R\,(sinx, cosx)dx \tag{1.6}$$

integral berilgan bo‘lsin, bu yerda

$$R(\sin x, \cos x) = \frac{\sum_{k+s=0}^{n} a_{ks}\sin^k x\cos^s x}{\sum_{p+q=0}^{m} b_{pq}\sin^p x\cos^q x}.$$

Bu integral

$$tg\frac{x}{2} = t \tag{1.7}$$

almashtirish yordami bilan ratsional funksiyaning integraliga keltirilishi mumkinligini ko‘rsatamiz. sin x va cos x funksiyalarning $tg\frac{x}{2}$ bilan, ya’ni t bilan ifodalaymiz:

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2},$$

$$\cos x=\frac{\cos^2\frac{x}{2}\sin^2\frac{x}{2}}{1}=\frac{\cos^2\frac{x}{2}\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}.$$

hamda

$$x=2arctgt,\quad dx=\frac{2dt}{1+t^2}$$

bo‘ladi. Shunday qilib, sin$x$, cos$x$ va dx lar t bilan ratsional ifodalanadi. Ratsional funksiyalarning kompozitsiyasi yana ratsional funksiya bo‘lgani uchun hosil qilingan ifodalarni (1.6) integralga qo‘yib, ratsional funksiyaning integralini hosil qilamiz:

$$\int R\,(\sin x,\cos x)dx=\int R\left(\frac{t}{1+t^2},\frac{1-t}{1+t^2},\right)\frac{tdt}{1+t^2},$$

1-misol. Ushbu

$$\int\frac{dx}{\sin x}$$

integralni qaraymiz. Yuqorida yozilgan formulalarga asosan:

$$\int\frac{dx}{\sin x}=\int\frac{\frac{2dt}{1+t^2}}{\frac{2t^2}{1+t^2}}=\int\frac{dt}{t}\ln|t|+C=\ln\left|tg\frac{x}{2}\right|+C.$$

Ko‘rilgan almashtirish $R\,(cos\,x, sin\,x)$ ko‘rinishdagi har qanday funksiyani integrallash uchun imkon beradi. Shuning uchun uni ba’zan universal trigonometrik almashtirish deb ataydilar. Lekin praktikada bu almashtirish ko‘pincha ancha murakkab ratsional funksiyaga olib keladi. Shuning uchun univer sal almashtirish bilan bir qatorda ba’zi hollar uchun maqsadga tez olib keladigan boshqa almashtirishlarni ham bilish foydalidir.

1) Agar integral $\int R(\sin x)\cos xdx$ ko‘rinishda bo‘lsa, u holda $\sin x=t$,

$\cos xdx=dt$ almashtirish bu integralni $\int R(t)dt$ ko‘rinishga olib keladi.

2) Agar integral $\int R(\cos x)\sin xdx$ ko‘rinishda bo‘lsa, u holda $\cos x=t$,

$\sin x dx = -dt$ almashtirish yordamida bu in tegral $-\int R(t)dt$ ko‘rinishga keladi.

3) Agar integral ostidagi funksiya faqat tgx ga bog‘liq bo‘lsa, u holda $tgx = t$, $x = arctgt$, $dx = \frac{dt}{1+t^2}$ almashtirish yordamida bu integral ratsional funksiyaning integraliga keltiriladi:

$$\int R\,(tgx)dx = \int R\,(t)\frac{dt}{1+t^2}$$

4) Agar integral ostidagi funksiya $R(\sin x, \cos x)$ ko‘rinishida bo‘lsa, ammo $\sin x$ va $\cos x$ faqat juft darajalarga kirsa, u holda $tgx = t$ almashtirish tadbiq etiladi, chunki $\sin^2 x$ va $\cos^2 x$ lar tg x bilan ratsional ifodalanadi:

$$\cos^2 x = \frac{1}{1+tg^2 x} = \frac{1}{1+t^2},$$
$$\sin^2 x = \frac{tg^2 x}{1+tg^2 x} = \frac{t^2}{1+t^2},$$
$$dx = \frac{dt}{1+t^2}.$$

Buday almashtirishdan keyin biz ratsional funksiyaning integralini hosil qilamiz.

2-misol. Ushbu

$$\int \frac{\sin^3 x}{2+\cos x}dx$$

integral hisoblansin. Bu integral $\int R\,(\cos x)\sin x dx$ ko‘rinishga keltirish oson. Haqiqatan,

$$\int \frac{\sin^3 x}{2+\cos x}dx = \int \frac{\sin^2 x \sin x dx}{2+\cos x} = \int \frac{1-\cos^2 x}{2+\cos x}\sin x dx$$

Bu holda $\cos x = z, sinxdx = -dz$ almashtirishni bajaramiz:

$$\int \frac{\sin^3 x}{2+\cos x}dx = \int \frac{1-z^2}{2+z}(-dz) = \int \frac{z^2-1}{z+2}dz = \int \left(z-2+\frac{3}{z+2}\right)dz =$$
$$= \frac{z^2}{2} - 2z + 3\ln(z+2) + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln(\cos x + 2) + C$$

3-misol. $\int \frac{dx}{2-\sin^2 x}$ integral hisoblang.

Yechish. Bu yerda tg $x = t$ almashtirishni bajaramiz.

$$\int \frac{dx}{2-\sin^2 x} = \int \frac{dt}{\left(2-\frac{t^2}{1+t^2}\right)(1+t^2)} = \int \frac{dt}{2+t^2}$$

$$= \frac{1}{\sqrt{2}} arctg \frac{t}{\sqrt{2}} + C == \frac{1}{\sqrt{2}} arctg \left(\frac{tgx}{\sqrt{2}}\right) + C$$

Mustaqil ishlash uchun misollar

5-misol. $\int \frac{dx}{\sin^2 x \sqrt[4]{ctgx}}$ **10-misol.** $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}$

6-misol. $\int \frac{dx}{\cos x}$ **11-misol.** $\int \frac{dx}{2\sin x - \cos x + 5}$

7-misol. $\int \frac{dx}{(\arcsin x)^2 \sqrt{1-x^2}}$ **12-misol.** $\int \frac{\cos^2 x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2}$

8-misol. $\int \sin^2 x \cos^4 x dx$ **13-misol.** $\int \sin x \sin \frac{x}{2} \sin \frac{x}{3} dx$

9-misol. $\int \frac{\sin^2 x}{1+\sin^2 x} dx$ **14-misol.** $\int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx$

1.8-§. Ba'zi bir irratsional funksiyalarni integrallash. Eyler almashtirishlari

Har qanday irratsional funksiyadan olingan integral ele mentar funksiyalar orqali ifodalanavermaydi. Bu va bundan keyingi paragraflarda integrallari oʻzgaruvchilarni almashtirish yordami bilan ratsional funksiyalar integrallariga keltiriladigan irratsional funksiyalarni qaraymiz ([5] ga qarang) .

I. $\int R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right) dx$ integralni qaraymiz. Bunda R – oʻz argumentlariga nisbatan ratsional funksiya.

$\frac{m}{n}, \ldots, \frac{r}{s}$ kasrlarning umumiy maxraji k boʻlsin. Quyidagicha

$$x = t^k, \qquad dx = kt^{k-1} dt$$

almashtirishni bajaramiz.

Bu holda x argumentning barcha kasrli darajasi $t$ning butun darajasi bilan ifoda etiladi. Demak, integral ostidagi funksiya t argumentning ratsional fuksiyasiga aylanadi.

1-misol. Ushbu integral hisoblansin:

$$\int \frac{x^{1/2}dx}{x^{3/4}+1}$$
.

Yechish. $\frac{1}{2}, \frac{3}{4}$ kasrlarning umumiy maxraji 4 bo'ladi; shuning uchun $x = t^4, dx = 4t^3dt$ almashtirishni bajaramiz; u holda;

$$\int \frac{x^{\frac{1}{2}}dx}{x^{\frac{3}{4}}+1} = 4\int \frac{t^2}{t^3+1}dt = 4\int \frac{t^5}{t^3+1}dt = 4\int \left(t^2 - \frac{t^2}{t^3+1}\right)dt =$$

$$= 4\int t^2\,dt - 4\frac{t^2}{t^3+1}dt = 4\frac{t^3}{3} - \frac{4}{3}ln|t^3+1| + C = \frac{4}{3}\left[x^{3/4} - ln\left|x^{3/4}+1\right|\right.$$

II. $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right]dx$ ko'rinishdagi integralni qaraymiz. Bu integral

$$\frac{ax+b}{cx+d} = t^k$$

almashtirish yordami bilan ratsional funksiyaning integraliga keltiriladi, bunda k bilan $\frac{m}{n}, \ldots \frac{r}{s}$ kasrlaning umumiy maxraji belgilangan.

2-misol. Ushbu integral hisoblansin:

$$\int \frac{\sqrt{x+4}}{x}dx.$$

Yechish. $\mathrm{x} + 4 = t^2 + 4, x = t^2 - 4, \quad dx = 2tdt$ almashtirishni bajaramiz:

Bu holda

$$\int \frac{\sqrt{x+4}}{x}dx = 2\int \frac{t^2}{t^2-4}dt = 2\int \left(1 + \frac{4}{t^2-4}\right)dt - 2\int dt - 2\int dt + 8\int$$

$$= 2t + 2\ln\left|\frac{t-2}{t+2}\right| + C = 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.$$

III. Eyler almashtirishlari ([5] ga qarang). Ushbu

$$\int \boldsymbol{R}\,(\boldsymbol{x}, \sqrt{\boldsymbol{a}\boldsymbol{x}^2 + \boldsymbol{b}\boldsymbol{x} + \boldsymbol{c}})\boldsymbol{d}\boldsymbol{x} \tag{1.8}$$

ko'rinishdagi integralni qaraymiz. Bunda $a \neq 0$.

Bunday integral Eylerning quyidagi almashtirishlari yordami bilan yangi o‘zgaruvchi ratsional funksiyasining integraliga keltiriladi.

1. Eylerning birinchi almashtirishi. Agar $a > 0$ bo‘lsa, u holda

$$\sqrt{ax^2 + bx + c} = \pm\sqrt{a}x + t$$

deb olamiz (Aniqlik uchun hamma vaqt $\sqrt{a}$ oldida $+$ ishorani olish mumkin). Bu holda

$$ax^2 + bx + c = ax^2 + 2\sqrt{a}x + t^2,$$

bundan $x$ni $t$ning ratsional funksiyasi kabi aniqlaymiz:

$$x = \frac{t^2 - c}{b - 2\sqrt{a}t}$$

(demak, dx ham t bilan ratsional ifodalanadi), demak,

$$\sqrt{ax^2 + bx + c} = \sqrt{a}x + t = \sqrt{a}\,\frac{t^2 - c}{b - 2t\sqrt{a}} + t$$

ya’ni $\sqrt{ax^2 + bx + c}$, ifoda t ning ratsional funksiyasi bo‘ladi. $\sqrt{ax^2 + bx + c}$ va dx lar t bilan ratsional ifoda etilgani sababli berilgan (1.8) integral t ning ratsional funksiyasidan olingan integralga aylanadi.

3-misol. $\int \frac{dx}{\sqrt{4x^2+x+1}}$ **- integralni hisoblang.**

Yechish.

$$\int \frac{dx}{\sqrt{4x^2 + x + 1}} = \left| \begin{array}{c} \sqrt{4x^2 + x + 1} = t + 2x \\ x = \dfrac{t^2 - 1}{1 - 4t} \\ dx = \dfrac{-4t^2 + 2t - 4}{(1 - 4t)^2} dt \end{array} \right| =$$

$$= \int \frac{(-4t^2 + 2t - 4)dt}{(1 - 4t)^2\left(t + 2\left(\frac{t^2 - 1}{1 - 4t}\right)\right)} =$$

$$= 2\int \frac{(2t^2 - t + 2)dt}{(1 - 4t)(2t^2 - t + 2)} = 2\int \frac{dt}{1 - 4t} = -\frac{1}{2}\int \frac{d(1 - 4t)}{1 - 4t} =$$

$$= -\frac{1}{2}\ln|1 - 4t| + C = -\frac{1}{2}\ln\left|1 - 4\sqrt{4x^2 + x + 1} - 8x\right| + C.$$

4-misol. $\int \frac{dx}{\sqrt{x^2+c}}$ – integral hisoblang.

Yechish. Bu yerda $a = 1 > 0$ bo'lgani uchun $\sqrt{x^2 + c} = -x + t$ deb olamiz, bu holda

$$\sqrt{x^2 + c} = -x^2 + 2xt + t^2$$

bundan

$$x = \frac{t^2 - c}{2t}.$$

Demak,

$$dx = \frac{t^2 + c}{2t^2} dt, \sqrt{x^2 + c} = -x + t = -\frac{t^2 - c}{2t} + t = \frac{t^2 + c}{2t}.$$

Dastlabki integralga qaytamiz:

$$\int \frac{dx}{\sqrt{x^2 + c}} = \int \frac{\frac{t^2 + c}{2t^2}}{\frac{t^2 + c}{2t}} = \int \frac{dt}{t} = \ln|t| + C_1 = \ln\left|x + \sqrt{x^2 + c}\right| + C_1.$$

2. Eylerning ikkinchi almashtirishi. Agar $c > 0$ bo'lsa, bu holda

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}.$$

deb olamiz; u holda (aniqlik uchun hamma vaqt $\sqrt{c}$ oldida + ishorani olish mumkin)

$$ax^2 + bx + c = x^2 t^2 + 2t\sqrt{c} + c.$$

Bundan x o'zgaruvchi t ning ratsional funksiyasi kabi aniqlanadi:

$$x = \frac{2\sqrt{c}t - b}{a - t^2}.$$

dx va $\sqrt{ax^2 + bx + c}$ ham t orqali ratsional ifoda etilgani sababli x, $\sqrt{ax^2 + bx + c}$ va dx ning qiymatlarini $\int R\,(x, \sqrt{ax^2 + bx + c})dx$ integralga qo'ysak, bu integral ham t ning ratsional funksiyasidan olingan integralga keladi.

5-misol. $\int \frac{dx}{\sqrt{4 + 2x - x^2}}$ - integralni hisoblang.

Yechish.

$$\int \frac{dx}{\sqrt{4+2x-x^2}} = \left| \begin{array}{c} \sqrt{4+2x-x^2} = tx+2 \\ x = \dfrac{2-4t}{t^2+1} \\ dx = \dfrac{4t^2-4t-4}{t^2+1}dt \end{array} \right|$$

$$\int \frac{\frac{4(t^2-t-1)}{(t^2+1)^2}}{\frac{t(2-4t)}{t^2+1}+2}dt = -2\int \frac{dt}{t^2+1} = -2arctgt + C =$$

$$= -2arctg\frac{\sqrt{4+2x-x^2}-2}{x} + C.$$

6-misol. $\int \frac{\left(1-\sqrt{1+x+x^2}\right)^2}{x^2(1+x+x^2)}dx$- integralni hisoblang

Yechish. $\sqrt{1+x+x^2} = xt+1$ deb olamiz. Bu holda

$1+x+x^2 = x^2t^2+2xtx+1;$ $\quad x = \frac{2t-1}{1-t^2},$ $\quad dx = \frac{2t^2-2t+2}{(1-t^2)^2}dt;$

$\sqrt{1+x+x^2} = xt+1 = \frac{t^2-t+1}{1-t^2};$ $\quad 1-\sqrt{1+x+x^2} = \frac{-2t^2+t}{1-t^2}.$

bo‘ladi. Bularni berilgan integralga qo‘yamiz va

$$\int \frac{\sqrt{1+x-x^2}^2}{x^2\sqrt{1+x+x^2}}dx = \int \frac{(-2t^2+t)^2(1-t^2)^2(1-t^2)(2t-2t+2)}{(1-t^2)^2(-2t-1)^2(t^2-t+1)(1-t^2)^2}dt =$$

$$= 2\int \frac{t^2}{1-t^2}dt = -2t + \ln\left|\frac{1+t}{1-t}\right| + C = -\frac{2\left(\sqrt{1+x+x^2}-1\right)}{x} +$$

$$+\ln\left|\frac{x+\sqrt{1+x+x^2}-1}{x-\sqrt{1+x+x^2}+1}\right| + C = -\frac{2\left(\sqrt{1+x+x^2}-1\right)}{x} +$$

$$+\ln\left|2x+2\sqrt{1+x+x^2}+1\right| + C$$

natijaga ega bo‘lamiz.

3. Eylerning uchinchi almashtirishi. ax^2+bx+c uchhadning haqiqiy ildizlari α va β bo‘lsin, bu holda $ax^2+bx+c = (x-\alpha)t$ deb olamiz;

$$ax^2 + bx + c = a(x-\alpha)(x-\beta)$$

bo'lgani uchun

$$\sqrt{a(x-\alpha)(x-\beta)} = (x-\alpha)t,$$
$$a(x-\alpha)(x-\beta) = (x-\alpha)^2t^2,$$
$$a(x-\beta) = (x-\alpha)t^2,$$

Bundan x ni t ning funksiyasi kabi aniqlaymiz;

$$x = \frac{a\beta - at^2}{a-t^2}.$$

dx va $\sqrt{ax^2+bx+c}$ ham t bilan ratsional bog'liq bo'lgani sababli berilgan integral t ning ratsional funksiyasidan oli nadigan integralga almashadi.

7- m i s o l. $\int \frac{dx}{\sqrt{x^2+3x-4}}$ – integralni hisoblang..

Yechish. $x^2 + 3x - 4 = (x+4)(x-1)$ bo'lgani sababli

$$\sqrt{(x+4)(x-1)} = (x+4)t$$

deb olamiz. U holda

$$(x+4)(x-1) = (x+4)^2t^2, \quad x-1 = (x+4)t^2,$$
$$x = \frac{1+4t^2}{1-t^2}, dx = \frac{10t}{(1-t^2)^2}dt,$$
$$\sqrt{(x+4)(x-1)} = \left[\frac{1+4t^2}{1-t^2} + 4\right]t = \frac{5t}{1-t^2}$$

bo'ladi. Bularni berilgan integralga qo'yamiz va $\int \frac{dx}{x^2+3x-4} =$
$\int \frac{10t(1-t^2)}{(1-t^2)^2 5t}dt = \int \frac{2}{1-t^2}dt = \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{1+\sqrt{\frac{x-1}{x+4}}}{1-\sqrt{\frac{x-1}{x+4}}}\right| + C =$
$\ln\left|\frac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right| + C$ natijaga ega bo'lamiz.

Mustaqil ishlash uchun misollar

8-misol. $\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx$

9-misol. $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$

10-misol. $\int \frac{xdx}{(1-x^2)\sqrt{1-x^2}}$

11-misol. $\int \frac{dx}{x+\sqrt{x^4+2x^2-1}}$

12-misol. 4. $\int \frac{(x^2-1)dx}{(x^2+1)\sqrt{x^2+1}}$

13-misol. $\int \frac{dx}{x\sqrt[6]{1+x^6}}$

14-misol. $\int \frac{dx}{x^3 \sqrt[5]{1+\frac{1}{x}}}$ **15-misol.** $\int \frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$

1.9- §. Binomial formula bilan berilgan funksiyani integrallash

$x^m(a+bx^n)^p$, $(\ m,n,p \in Q)$, – binomial formula bilan berilgan funksiyani integrallash masalsi Chebishev tomonidan o‘rganilgan bo‘lib, faqat quyidagi uchta holdagina uning aniqmas integrali elementar funksiyalar oilasiga tegishli bo‘lishligi isbotlangan([1], [4] va [5] ga qarang):

1) $p \in Z$ bo‘lsa, $p > 0$ bo‘lgan holda $(a+bx^n)^p$ ifodani Nyuton binomi formulasi bo‘yicha yoyib, darajali funksiyalarni integrallash masalasiga keltiramiz , $\boldsymbol{p < 0}$ bo‘lgan holda esa $x = t^N$ almashtirishni qo‘llasak binomial ifoda bilan berilgan funksiya t o‘garuvchining ratsional funksiyasi bo‘ladi (bunda N berilgan $\boldsymbol{m}$ va n kasrlarning umumiy maxraji).

2) $p = \frac{k}{s}$- kasr son, lekin $\frac{m+1}{n}$ – butun son bo‘lsa, u holda $a + bx^n = t^N$ almashtirish qo‘llanilgandan so‘ng binomial ifoda bilan berilgan funksiya t o‘zgaruvchining ratsional funksiyasi bo‘ladi. (bunda N berilgan $\boldsymbol{p}$ kasrning maxraji).

3) $p = \frac{k}{s}$, $\frac{m+1}{n}$ – kasr sonlar bo‘lib, $\frac{m+1}{n} + p$–son bo‘lsa, u holda $ax^{-n} + b = t^N$ almashtirish qo‘llanilgandan so‘ng binomial ifoda bilan berilgan funksiya t o‘zgaruvchining ratsional funksiyasi bo‘ladi (bunda N berilgan $\boldsymbol{p}$ kasrning maxraji).

Demak, faqat shu hollardagina uning aniqmas integrali elementar funksiyalar oilasiga tegishli bo‘ladi va uni aniq hisoblab topish mumkin.

1-misol. $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2} dx$ integralni hisoblang.

Yechish. $\boldsymbol{m = \frac{1}{2}, n = \frac{1}{3}}$, $\boldsymbol{p = -2.}$ Bu holda $x = t^6$almashtirishni qo‘llaymiz.

$$\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2} dx = \int x^{\frac{1}{2}}(1+x^{\frac{1}{3}})^{-2} dx = \left| \begin{matrix} x = t^6, \\ dx = 6t^5 dt \end{matrix} \right| =$$

$$= \int (t^6)^{\frac{1}{2}}(1+(t^6)^{\frac{1}{3}})^{-2}\, 6t^5 dt = 6\int \frac{t^8 dt}{(1+t^2)^2} =$$

$$= 6\int(t^4 - 2t^2 + 3)dt - 6\int\frac{4t^2+3}{(1+t^2)^2}dt =$$

$$= 6\int(t^4 - 2t^2 + 3)dt - 6\int\frac{4(1+t^2)}{(1+t^2)^2}dt + 6\int\frac{dt}{(1+t^2)^2} =$$

$$= \frac{6}{5}t^5 - 4t^3 + 18t - 21arctgt + \frac{3t}{(1+t^2)} = \frac{6}{5}x^{\frac{5}{6}} - 4x^{\frac{1}{2}} + 18x^{\frac{1}{6}} -$$

$$-21arctgx^{\frac{1}{6}} + \frac{3x^{\frac{1}{6}}}{1+x^{\frac{1}{3}}} + C.$$

2-misol. $\int x^3(1-x^2)^{-\frac{3}{2}}dx$ integralni hisoblang.

Yechish. $\boldsymbol{m = 3, n = 2, p = -\frac{3}{2}; \ \frac{m+1}{n} = 2}$. Bu holda $1 - x^2 = t^2$almashtirishni qo'llaymiz.

$$\int x^3(1-x^2)^{-\frac{3}{2}}dx = \begin{vmatrix} 1 - x^2 = t^2 \\ x^2 = 1 - t^2 \\ xdx = -tdt \end{vmatrix} =$$

$$= \int x^2\frac{xdx}{(\sqrt{1-x^2})^3} = \int(1-t^2)\frac{-tdt}{\left(\sqrt{t^2}\right)^3} = -\int\frac{(1-t^2)dt}{t^2} =$$

$$= -\int t^{-2}dt + \int tdt = \frac{1}{t} + \frac{t^2}{2} = \frac{1}{\sqrt{1-x^2}} + \frac{1-\mathrm{x}^2}{2} + C.$$

Mustaqil ishlash uchun misollar

1-misol. $\int \boldsymbol{x\sqrt{1+x^4}dx}$. **2-misol.** $\int \boldsymbol{x^5(1+x^2)^{\frac{2}{3}}dx}$.

3-misol. $\int\boldsymbol{\frac{\sqrt{4+\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx}$. **4-misol.** $\int\boldsymbol{\frac{dx}{x^2\sqrt[3]{(1+x^3)^2}}}$.

5-misol. $\int\boldsymbol{\frac{\sqrt{x}}{1+\sqrt[3]{x^2}}dx}$.

1.10-§. Har xil funksiyalarni integrallashga misollar

1. $\int\frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx$ (bunda $P_n(x)$ n –darajali ko'phad) ko'rishishdagi integral ushbu formula bilan topiladi:

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = (A_1x^{n-1} + A_2x^{n-2} + \ldots + A_n)\sqrt{ax^2+bx+c} +$$
$$+B\int \frac{dx}{\sqrt{ax^2+bx+c}},$$
bu yerda $A_1, A_2, \ldots, A_n, B$lar o‘zgarmaslar bo‘lib, ular yuqoridagi tenglikni differentsiallab $\sqrt{ax^2+bx+c}$ ga ko‘paytirish va x ning bir xil darajalarining koeffitsentlarini solishtirish natijasida topiladi.

$\int \mathrm{P}_n(x)\sqrt{ax^2+bx+c} \cdot dx$ ko‘rinishdagi integral ham xuddi yuqoridagiga o‘xshash yo‘l bilan topiladi:

$$\int P_n(x)\sqrt{ax^2+bx+c} \cdot dx$$
$$= \int \frac{(ax^2+bx+c)P_n(x)}{\sqrt{ax^2+bx+c}} = (A_1x^{n+1} + A_2x^{n} + \ldots + A_{n+2})\sqrt{ax^2+bx+c} + B\int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

2. $\int \frac{Ax+B}{(x-a)\sqrt{ax^2+bx+c}}dx$ integralni $x-a=\frac{1}{t}$ almashtirish bajarib topish mumkin.

1-misol. $\int \frac{2x+5}{\sqrt{9x^2+6x+2}}dx$– integralni hisoblang.

Yechish.

$$\int \frac{2x+5}{\sqrt{9x^2+6x+2}}dx = (Ax+B)\sqrt{9x^2+6x+2} + D\int \frac{dx}{\sqrt{9x^2+6x+2}} =$$

$$= \left| \begin{array}{l} \frac{2x+5}{\sqrt{9x^2+6x+2}} = A(\sqrt{9x^2+6x+2}) + \\ +(Ax+B)\frac{18x+6}{2\sqrt{9x^2+6x+2}} + \frac{D}{\sqrt{9x^2+6x+2}}; \\ 2x+5 = A(9x^2+6x+2) + (Ax+B)(9x+3) + D \\ 2x+5 = 9Ax^2 + 6Ax + 2A + 9Ax^2 + 3B + 3Ax + 9Bx + D \\ 2x+5 = 18Ax^2 + x(9A+9B) + (2A+3B+D) \\ 18A = 0,\ \ 9(A+B) = 2,\ \ 2A+3B+D = 5. \\ A = 0,\ \ B = \frac{2}{9},\ \ D = \frac{13}{3}. \end{array} \right|$$
$$=$$

$$= \int \frac{2x+5}{\sqrt{9x^2+6x+2}}dx = \frac{2}{9}\sqrt{9x^2+6x+2} + \frac{13}{3}\int \frac{dx}{\sqrt{9x^2+6x+2}} =$$

$$= \frac{2}{9}\sqrt{9x^2+6x+2} + \frac{13}{3}\int \frac{dx}{\sqrt{(3x+1)^2+1}}$$

$$= \frac{2}{9}\sqrt{9x^2+6x+2} + \frac{13}{3}\ln\left|(3x+1) + \sqrt{(3x+1)^2+1}\right| + C.$$

2-misol. $\int \frac{x^3}{\sqrt{1+2x-x^2}}dx$ – integralni hisoblang.

Yechish.

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}dx$$

$$= (Ax^2+Bx+C)\sqrt{1+2x-x^2} + D\int \frac{dx}{\sqrt{2-(x-1)^2}} =$$

$$= \left| \begin{array}{l} \frac{x^3}{\sqrt{1+2x-x^2}} = (Ax^2+Bx+C)\frac{2-2x}{2\sqrt{1+2x-x^2}} + \\ +(\sqrt{1+2x-x^2})(2Ax+B) + \frac{D}{\sqrt{2-(x-1)^2}}; \\ x^3 = (Ax^2+Bx+C)(1-x) + (2Ax+B)(1+2x-x^2) + D; \\ x^3 = -3Ax^3 + x^2(5A-2B) + x(3B-C+2A) + (C+B+D); \\ -3A = 1,\ \ 5A-2B = 0,\ \ 3B-C+2A = 0. \\ A = -\frac{1}{3},\ \ B = -\frac{5}{6},\ \ C = -\frac{19}{3},\ \ D = 4. \end{array} \right| =$$

$$= \int \frac{x^3}{\sqrt{1+2x-x^2}}dx = (-\frac{1}{3}x^2 - \frac{5}{6}x - \frac{19}{3})\sqrt{1+2x-x^2} + 4\int \frac{d}{\sqrt{2-(}}$$

$$= -\frac{1}{6}(2x^2+5x+38)(\sqrt{1+2x-x^2}) + 4\arcsin\frac{x-1}{\sqrt{2}} + C.$$

3-misol. $\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$ – integralni hisoblang.

Yechish.

$$\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}} = \qquad \left| \begin{matrix} x+1=\frac{1}{t} \\ dx=-\frac{1}{t^2}dt \end{matrix} \right| =$$

$$= \int \frac{dx}{(x+1)^5\sqrt{(x+1)^2-1}} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t^5}\sqrt{\frac{1}{t^2}-1}} = -\int \frac{t^4dt}{\sqrt{1-t^2}} =$$

$$= (At^3+Bt^2+Ct+D)\sqrt{1-t^2} + E\int \frac{dt}{\sqrt{1-t^2}} =$$

$$= \left| \begin{matrix} (3At^2+2Bt+C)\sqrt{1-t^2} + (At^3+Bt^2+Ct+D)\frac{-2t}{2\sqrt{1-t^2}} + \frac{E}{\sqrt{1-t^2}}. \\ -t^4 = (3At^2+2Bt+C)(1-t^2) - (At^3+Bt^2+Ct+D)t + E = \\ = -4At^4 - 3Bt^3 + t^2(3A-2C) + t(2B-D) + C + E. \\ -4A=-1, \ -3B=0, \ 3A-2C=0, \ 2B-D=0, \ C+E=0. \\ A=\frac{1}{4}, \ B=0, \ C=\frac{3}{8}, \ D=0, \ E=-\frac{3}{8}. \end{matrix} \right|$$

$$=$$

$$= \int \frac{dx}{(x+1)^5\sqrt{x^2+2x}} = (\frac{1}{4}t^3 + \frac{3}{8}t)(\sqrt{1-t^2} - \frac{3}{8})\int \frac{dt}{\sqrt{1-t^2}} =$$

$$= \left(\frac{1}{4(x+1)^3} + \frac{3}{8(x+1)}\right)\sqrt{1-\frac{1}{(x+1)^2}} - \frac{3}{8}\arcsin\frac{1}{x+1} + C.$$

3. Ba’zi elementar bo‘lmagan oddiy funksiyalarni integrallash:

4-misol. $\int |x|dx$– integralni hisoblang.

Yechish.

$$\int |x|dx = \frac{1}{2}x|x| + C$$

5-misol.$\int x|x|dx$ – integralni hisoblang.

Yechish. Bo‘laklab integrallash formulasini qo‘llaymiz va qo‘yidagi

$$\int x|x|dx \ = \left| \begin{matrix} d\vartheta = \frac{1}{2}d(x|x|) \\ u = x \end{matrix} \right| = \frac{1}{2}x^2|x| - \frac{1}{2}\int x|x|\,dx$$

tenglikka ega bo‘lamiz. Bu tenglikdan berilgan integralni topib,

$$\frac{3}{2}\int x|x|\,dx = \frac{1}{2}x^2|x| + C; \quad \int x|x|\,dx = \frac{1}{3}x^2|x| + C$$

natijaga ega bo‘lamiz.

6-misol. $\int (x+|x|)^2 dx$– integralni hisoblang.

Yechish.

$$\int (x+|x|)^2 dx = \int (x^2 + 2x|x| + |x|^2) dx =$$

$$= \frac{2}{3}(x^3 + x^2|x|) + C.$$

7-misol. $\int (|1+x| + |1-x|) dx$– integralni hisoblang.

Yechish.

$$\int (|1+x| + |1-x|) dx =$$

$$= \frac{1}{2}((1+x)|1+x| + (1-x)|1-x|) + C.$$

8-misol. $\int \mathrm{sgn} x dx$– integralni hisoblang.

Yechish.

$$\int \mathrm{sgn} x dx = |x| + C$$

8-misol. $\int e^{-|x|} dx$– integralni hisoblang.

Yechish.

$$\int e^{-|x|} dx = (\mathrm{sgn} x) e^{-|x|} + C$$

9-misol. $\int \max(1, x^2) dx$– integralni hisoblang.

Yechish.

$$\int \max(1, x^2) dx = \begin{cases} \frac{1}{3}x^3 + \frac{4}{3} + C, & x \le -1 \\ x + C, & -1 < x < 1 \\ \frac{1}{3}x^3 + \frac{2}{3} + C, & x > 1 \end{cases}$$

10-misol. $\int [x] dx$– integralni hisoblang.

Yechish.

$$\int [x] dx = x[x] - \frac{[x]([x]+1)}{2} + C$$

.

Mustaqil ishlash uchun misollar

11-misol. $\int \frac{x^2+4x}{\sqrt{x^2+2x+2}}dx$. 12-misol. $\int \frac{x^2dx}{\sqrt{1-2x-x^2}}$.

13-misol. $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$. 14-misol. $\int \frac{x^2-x+1}{\sqrt{x^2+2x+2}}dx$.

15-misol. $\int \{x\}dx$. 16-misol. $\int [x]|\sin\pi x|dx$.

17-misol. $\int f(x)dx$,, qaysikim, $f(x) = \begin{cases} 1 - x^2, & agar \ |x| \le 1, \\ 1 - |x|, & agar \ |x| > 1. \end{cases}$

8-misol. $\int \mathrm{sgn}(\sin x)dx$.

II-BOB.

ANIQ INTEGRAL

2.1-§ Aniq integralning ta'rifi

$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsin.$[a;b]$ kesmani ixtiyoriy ravishda

$$a = x_0, x_1, x_2, \cdots, x_{i-1}, x_i, \cdots, x_{n-1}, x_n = b$$

nuqtalar yordamida n ta bo'lakka bo'lamiz. Bu bo'laklarning har birini, shuningdek, uning uzunligini $\Delta x_i (i = 1,2, \cdots, n)$ orqali belgilaylik, yani

$$\Delta x_1 = x_1 - x_0,\ \Delta x_2 = x_2 - x_1, \Delta x_3 = x_3 - x_2, ...,$$
$$\Delta x_i = x_i - x_{i-1}, ..., \Delta x_n = x_n - x_{n-1}.$$

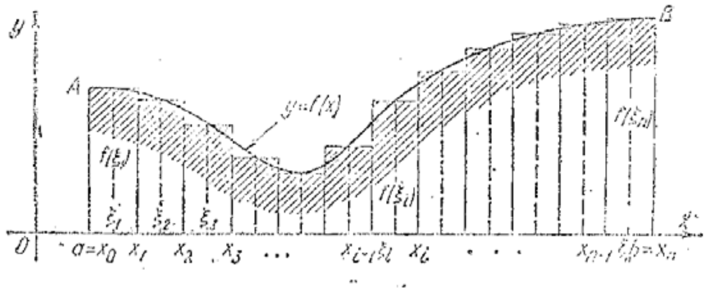
Bu bo'laklarning har biridan ixtiyoriy $\xi_1,\ \xi_2,\ \xi_3,\ \dots\ ,\ \xi_i,\ \dots\ ,$ $\xi_n$nuqtalarni olamiz va

$$S_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \tag{2.1}$$

yig'indini tuzamiz.

S_n yig'indi $f(x)$funksiyaning $[a;b]$ kesmadagi ***integral yig'indisi*** deyaladi. Geometrik nuqtai nazardan (2.1) integral yig'indi 1-rasmda shtrixlab ko'rsatilgan barcha to'g'ri to'rtburchaklar yuzlarining yig'indisini beradi (1-rasmda).

Ta'rif. $S_n$integral yig'indisining Δx_i kesmalarning eng kattasi nolga intilgandagi (bunda bo'laklarga bo'lish soni n cheksizlikka intiladi) limiti $f(x)$ funksiyadan $[a, b]$ kesmadan olingan ***aniq integral*** deyiladi. Ya'ni,



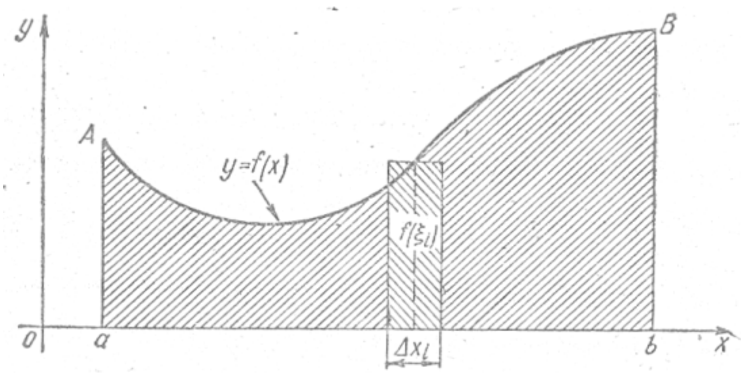
1-rasm.

$$\int_a^b f(x)dx = \lim_{max\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i \qquad (2.2).$$

Agar $f(x)$ funksiya $[a;b]$ da uzluksiz bo‘lsa, u holda (2.2) limit mavjud bo‘lib, u $[a;b]$ kesmani qism kesmalarga bo‘lish usuliga va ξ_i nuqtalarning tanlanishga bo‘g‘liq bo‘lmaydi. Agar $[a;b]$ da $f(x) \geq 0$ bo‘lsa, u holda (2.2) aniq integral **aABb** egri chiziqli trapetsiyaning yuzidan, ya’ni

$$x = a,\ x = b,\ y = 0\ va\ y = f(x)$$

chiziqlar bilan chegaralangan feguraning yuzidan iborat bo‘ladi (2-rasm).



2-rasm.

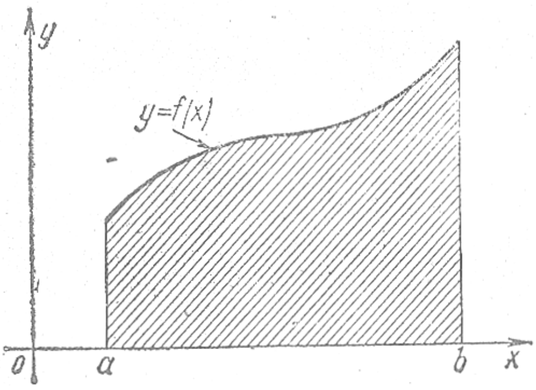
(2.2) da $\int -$ integral belgisi, $f(x)-$ integral ostidagi funksiya, $x-$ integrallash o‘zgaruvchisi interval *integrallash intervali,* $\ll a \gg$ *va* $\ll b \gg$sonlari *integrallashning quyi* va *yuqori chegaralari* deyiladi.

1- eslatma. Integrallash o‘zgaruvchisi x faqat integrallanayotgan funksiya turini ko‘rsatish uchun xizmat qiladi,shuning uchun uni istalgan boshqa harf bilan almashtirishimiz mumkin, ya’ni aniq integral integrallash o‘zgaruvchisini belgilashga bog‘liq emas (3-,4-rasmlar).

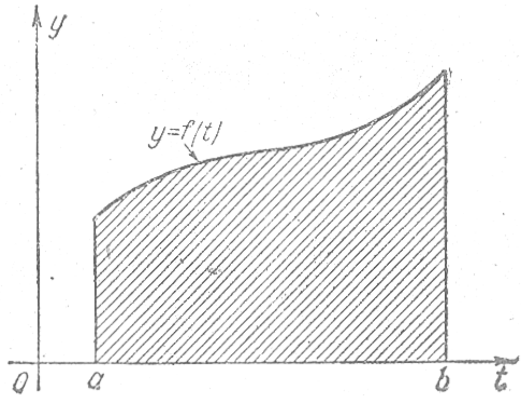
2- eslatma. Integralga ta’rif berishda$f(x)$ funksiya $[\mathrm{a;b}]$ oraliqda uzluksiz bo‘lishini talab qilgan edik. Shuni qayt qilib o‘tish kerakki, uzlukli funksiyalar orasida integrallanuvchi bo‘lganlari ham, integrallanuvchi bo‘lmaydiganlari ham bor, masalan, agar$y = f(x)$ funksiya $[a;b]$ da bo‘lakli-uzluksiz[1] va chegaralangan

[1]$[a;b]$ intervalda funksiya 1-turdagi chekli sondagi uzulish nuqtalariga ega bo‘lsa, bu funksiya $[a;b]$ intervalda ***bo‘lakli-uzluksiz*** deyiladi.

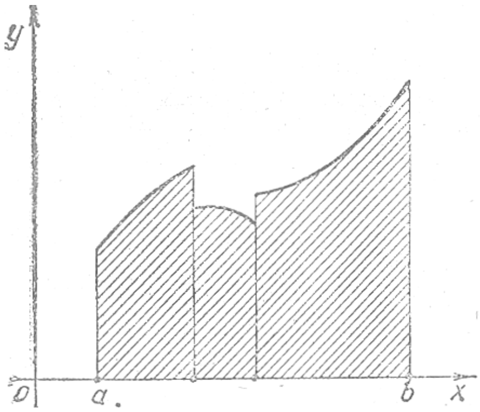
bo‘lsa, u holda funksiya berilgan integralga integrallanuvchi bo‘ladi (5-rasm).



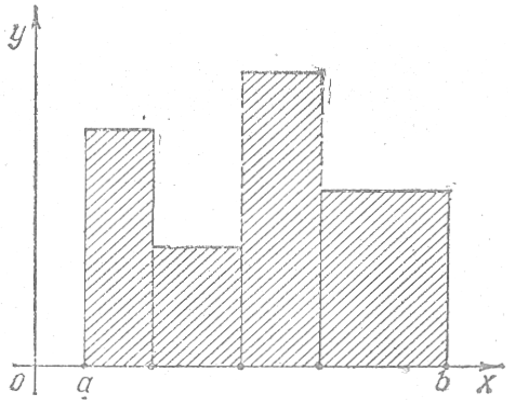
3-rasm.



4-rasm.



5-rasm.



6-rasm.

6- rasmda pog‘onali funksiyaning $[a;b]$ da integrali qanday aniqlanish ko‘rsatilgan. Paragraf oxirida quyidagini aytib o‘taylik. Agar $y = f(x)$ funksiyaning aniqmas integrali ya’ni funksiya (boshlang‘ich funksiyasi:

$$F(x) + C$$

bo‘lsa, u holda bu funksiyaning $[a;b]$ intervaldagi aniq integrali son bo‘ladi. Yuqorida ko‘rdikki, bu son (agar $[a;b]$ da $f \geq 0$ bo‘lsa) yuzini ifodalaydi, kelgusida esa u jism hajmini, massasini, ishini, bosimni, kuch momentini ifodalashi mumkinligini ko‘ramiz.

2.2- § Aniq integralning asosiy xossalari.

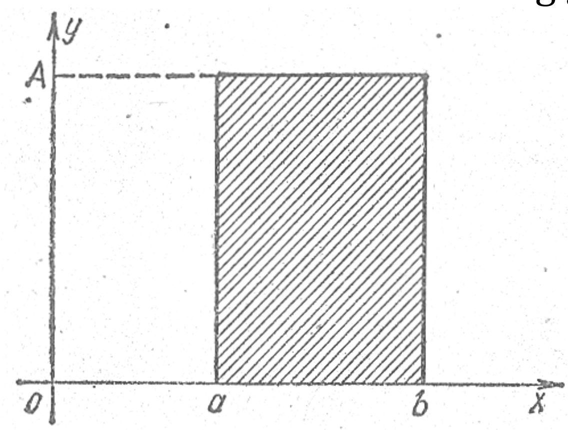
1-xossa. O‘zgarmas ko‘paytuvchini aniq integral belgisi tashqarisiga chiqarish mumkin, ya’ni

$$\int_a^b Af(x)dx = A\int_a^b f(x)dx$$

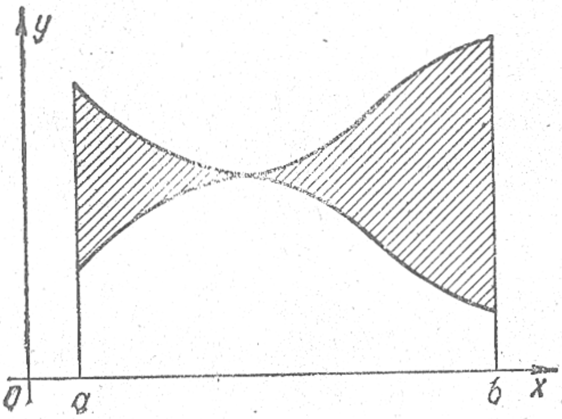
bu yerda $A = const$.
Xususan, agar $[a; b]$da$f(x) = 1$, bo‘lsa

$$\int_a^b A dx = A(a - b).$$

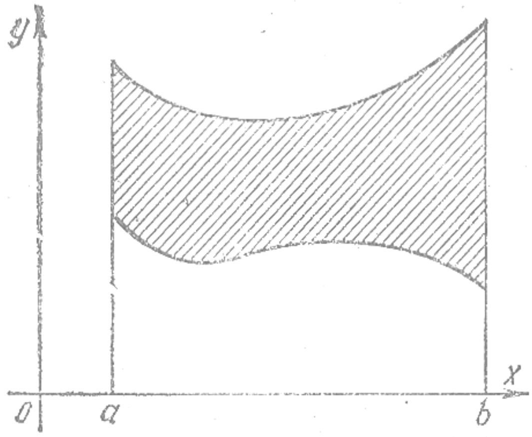
7-rasmda bu formulaning geometrik tasviri keltirilgan.



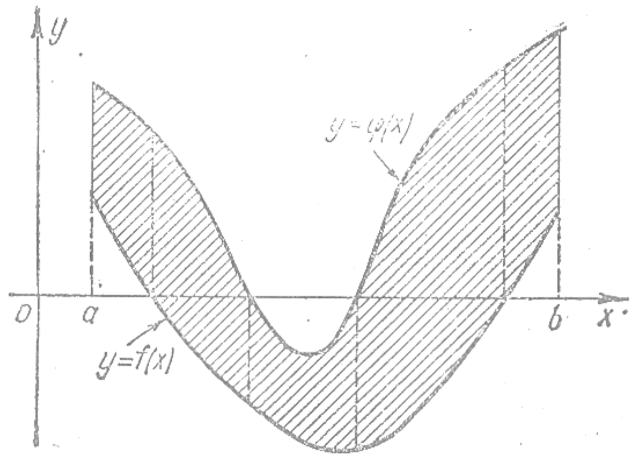
7-rasm.



8-rasm.



9-rasm.



10-rasm.

2-xossa. Chekli sondagi funksiyalar algebrik yig‘indisining aniq integrali qo‘shiluvchilar aniq integrallarining algebrik yig‘ndisiga teng, ya’ni

$$\int_a^b [f(x) + \varphi(x) - g(x)]\, dx = \int_a^b f(x)dx + \int_a^b \varphi(x)dx - \int_a^b g(x)dx$$

3-xossa. Agar $[a; b]$ (bu yerda $(a < b)$ da $f(x) \leq \varphi(x)$ bo‘lsa, u holda

$$\int_a^b f(x)dx \leq \int_a^b \varphi(x)dx$$

bo‘ladi, ya’ni katta funksiya katta integralga ega bo‘ladi (bitta intervalning o‘zida, chapdan o‘ngga tomon hisoblaganda) (8,9 va 10-rasm).

4-xossa. (Aniq integralni baholash haqida). Agar $f(x)$ funksiyaning $[a;b](a \leq b)$ dagi eng katta va eng kichik qiymatlari m va M bo‘lsa, u holda

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

11-rasmda bu xossaning geometrik tasviri keltirilgan. Agar $f(x) \geq 0$ bo‘lsa,

$$S_{aA_1B_1b} \leq \int_a^b f(x)dx \leq S_{aA_2B_2b}.$$

Ushbu

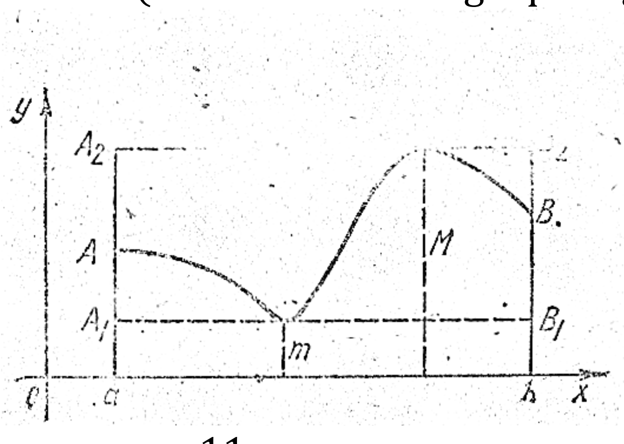
$$\mu = \frac{1}{b-a}\int_a^b f(x)dx$$

kattalik $f(x)$ funksiyaning $[a;b]$ dagi ***o‘rtacha qiymati*** deyiladi.

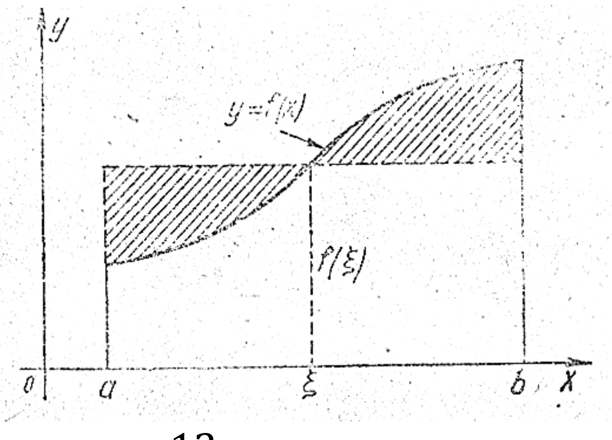
5-xossa. (O‘rtacha qiymat haqida.) Agar $y = f(x)$ funksiya $[a;b]$ da uzluksiz bo‘lsa, u holda bu kesmada hech bo‘lmaganda shunday bitta ξ nuqta topiladiki, unda

$$\int_a^b f(x)dx = f(\xi)\,(b-a)$$

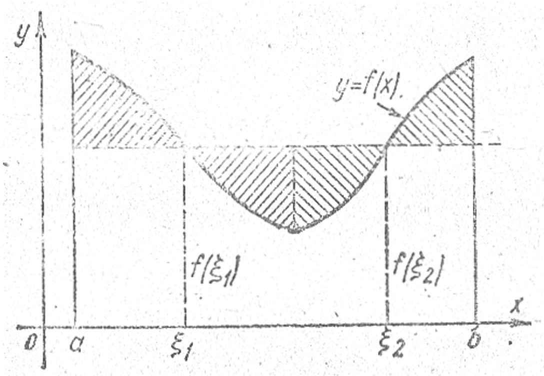
bo‘ladi (12 va 13-rasmlarga qarang).



11-rasm.



12-rasm.

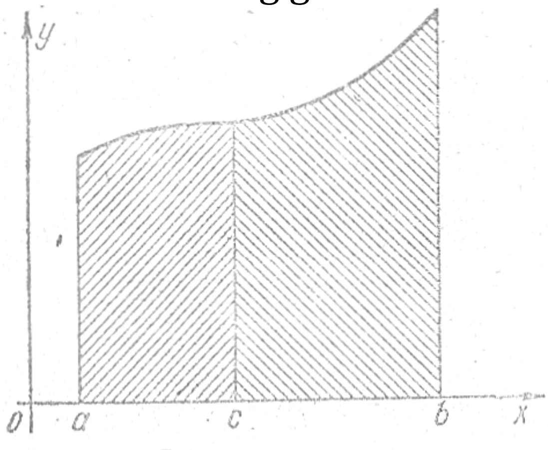


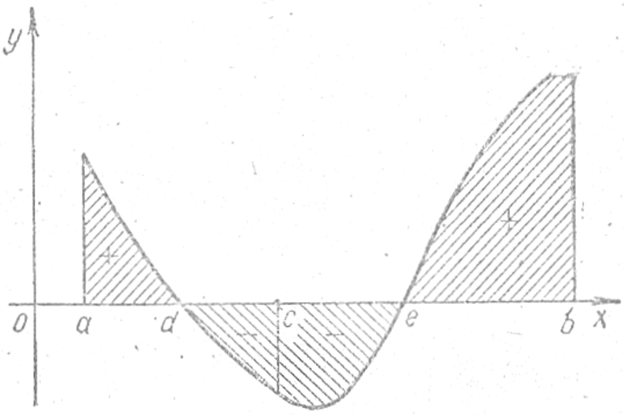
13-rasm.

6-xossa. Istalgan uchta $a, b, c,$ son uchun quyidagi tenglik o‘rinli:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

bu yerda uchala integral mavjud deb faraz qilinadi. 14 va 15-rasmlar bu formulaning geometrik tasviri bo‘ladi.





14-rasm. 15-rasm.

1-izoh.$\int_a^b f(x)dx$ integral tushunchasini kiritishda $a>b$ bo‘lganda ta’rifga ko‘ra

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

deb faraz qilgan edik.

2-izoh. Agar $a = b$ bo‘lsa, ta’rifga ko‘ra $x = a$ nuqtada chegaralangan istalgan funksiya uchun

$$\int_a^a f(x)dx = 0$$

tenglik o‘rinli bo‘ladi.

3-izoh .Agar $[a; b]$ da $f(x) < 0$ va $a < b$ bo‘lsa, u holda

$$\int_a^b f(x)dx < 0$$

Masalan, $[d; e]$ intervalda 15-rasm

$$\int_d^e f(x)dx < 0$$

bo‘ladi.

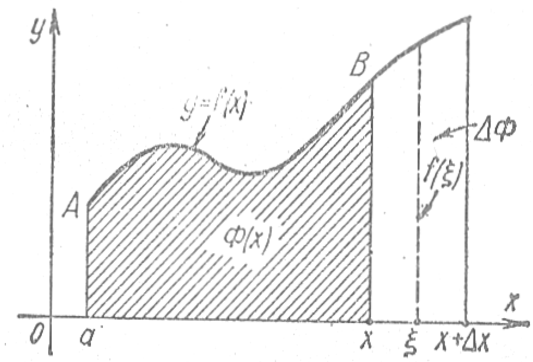
2.3-§ Aniq integralni hisoblash.

1. **Yuqori chegarasi o‘zgaruvchi bo‘lgan integral**. Hozir aniq integralni hisoblashning ajoyib formulasi- Nyuton- Leybnits formulasini keltirib chiqarishga imkon beradigan teoremani bayon qilamiz.

Agar $f(x)$ uzluksiz funksiya va

$$F(x) = \int_a^x f(t)dt\,, \quad F' = f(x)$$

bo‘lsa, u holda, yani aniq integraldan yuqori chegarasi bo‘yicha olinganhosila integral ostidagi funksiyaga teng, bu yerda integrallash o‘zgaruvchisi o‘rniga yuqori chegaraning qiymati qo‘yiladi. 16-rasmda bu formulani geometrik tasviri keltirilgan.



16-rasm.

Agar funksiyanining yuqori chegarasi x ning funksiyasi bo‘lsa, ya’ni

$$F(x) \int_a^{\varphi(x)} f(t)dt \qquad (2.3)$$

bo‘lsa, u holda(3.1)ekanligini isbot qilish mumkin.

2. Nyuton- Leybnits formulasi. Agar $F(x)$ funksiya $[a; b]$ da uzluksiz f(x) funksiyaning boshlang‘ichi bo‘lsa, u holda qo‘ydagi formula o‘rinli bo‘ladi:

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a) \qquad (2.4).$$

Nyuton- Leybnits formulasi deb ataluvchi (2.4) formula aniqmas integral bilan aniq integral o‘rtasidagi bog‘lanishni ifodalaydi.

Agar ilgari integral yig‘ndining (1.1-§ dagi (1.2) formulaga qarang) ba’zan uzundan-uzoq hisoblashlar bilan bog‘liq limitni hisoblashimiz zarur bo‘lsa, endi Nyuton- Leybnitsformulasiga ko‘ra $f(x)$ funksiyaning birorta $F(x)$ boshlang‘ichfunksiyani topishimiz kifoya. Ana shu $F(b) - F(a)$ ayirma bizga aniq integralning aniq qiymatini beradi.

3. Aniq integralda o‘zgaruvchisini almashtirish. Agar aniq integralni hisoblashda birdaniga $F(x)$ boshlang‘ich funksiyani topishga qiynalsak, $x = \varphi(t)$ almashtirishyordamida yangi *t*o‘zgaruvchiga o‘tamiz. Bu holda shu formula o‘rinli bo‘ladi:

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t) \cdot \varphi'(t)dt] (3.3),$$

bu yerda $\varphi(t)$va $\varphi'(t)[\alpha;\beta]$ kesmadagi uzluksiz funksiyalar, $a = \varphi(t), b = \varphi'(t)$.

4. Bo‘laklab integrallash. Agar $u(x)\ va\ v(x)$ lar $[a;b]$ kesmada differensialanuvchi funksiyalar bo‘lsa, u holda (3.3) formula o‘rinlidir.

3.1-misol:

$$\int_{-2}^{1} x^2 dx$$

integralni integral yig‘indilarning limiti sifatida hisoblang.

Yechish. Bu yerda$f(x) = x^2,\ a = -2,\ b = 1, [-2;1]$ oraliqni n ta teng qisimga bo‘lamiz: $\Delta x_1 = \Delta x_2 = \Delta x_3 = \cdots = \Delta x_i = \ldots = \Delta x_n$, u holda: $\Delta x_i = \frac{b-a}{n} = \frac{1-(-2)}{n} = \frac{3}{n}$. $\xi_1, \xi_2, \ldots, \xi_i, \ldots \xi_n$ nuqtalar uchun $i = 1,2,3, \ldots, n$ bo‘lganda intervallarning oxirlari Δx_i larni olamiz, ya’ni

$$\xi_1 = -2 + \frac{3}{n}, \xi_2 = -2 + \frac{3}{n}, \xi_3 = -2 + 3 \cdot \frac{3}{n}, \ldots, \xi_n = -2 + n \cdot \frac{3}{n} \quad .$$

$f(\xi)(i = 1,2,3, \ldots, n)$ larni hisoblaymiz:

$$\sum_{i=1}^{n}(-2+i\cdot\frac{3}{n})^2\cdot\frac{3}{n}=\frac{3}{n}\sum_{i=1}^{n}(-2+i\cdot\frac{3}{n})^2=$$

$$=\frac{3}{n}\left[4n-4\cdot\frac{3}{n}(1+2+3+\cdots+n)+\frac{9}{n^2}(1^2+2^2+3^2+\cdots+n^2)\right]$$

$$=$$

$$=\frac{3}{n}\left(4n-\frac{12}{n}\sum_{k=1}^{n}k+\frac{9}{n^2}\sum_{k=1}^{n}k^2\right)$$

n ta daslabki natural sonning yig‘ndisi $1+n$ ga teng (arifmetik progerssiya) n ta natural sonning kvadratlari yig‘indisi, ma’lumki

$$\frac{n(n+1)(2n+1)}{6}$$

ga teng. Bu qiymatlarni integral yig‘indiga qo‘yib, topamiz:

$$\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\frac{3}{n}\left[4n-4\cdot\frac{3}{n}\frac{1+n}{2}\cdot n+\frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$=12-18\cdot\frac{1+n}{n}+\frac{9}{2}\cdot\frac{(n+1)(2n+1)}{n^2}.$$

U holda

$$\int_{-2}^{1}x^2dx=\lim_{n\to\infty}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$$

$$=\lim_{n\to\infty}\left[12-18\left(1+\frac{1}{n}\right)+\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$=12-18+\frac{9}{2}\cdot 2=3,$$

ya’ni

$$\int_{-2}^{1}x^2dx=3.$$

3.2-misol. $\int_1^5 x^3dx$ integralni integral yig‘indilarning limiti sifatida hisoblang.

Yechish. Bu yerda $f(x)=x^3$, $a=1$, $b=5$; $[1;5]$ kesmani shunday qismlarga bo‘lamizki, bo‘linish nuqtalari absissalari quyidagicha:

$$x_0 = 1,\ x_1 = \sqrt[n]{5}, x_1 = \sqrt[n]{5^2}, \dots, x_{i-1} = \sqrt[n]{5^{i-1}},\ \ x_i = \sqrt[n]{5^i}, \dots, x_n = \sqrt[n]{5^n} = 5.$$

$\xi_1, \xi_2, \dots, \xi_i, \dots, \xi_n$ geometrik prograssiyani tashkil etsin. N ta hadlar uchun $[x_{i-1}; x_i], (i = 1,2,3, \dots, n)$ intervallarning chap oxirlarini olamiz, ya'ni

$$\xi_1 = 1, \xi_2 = \sqrt[n]{5}, \xi_3 = \sqrt[n]{5^2}, \dots, \xi_n = \sqrt[n]{5^{n-1}}.$$

U holda

$$f(\xi)_1 = 1,\ f(\xi)_2 = \sqrt[n]{5^3},\qquad f(\xi)_3 = \sqrt[n]{5^6},\qquad \dots\ ,\ f(\xi)_n = \sqrt[n]{5^{3(n-1)}};$$

$$\Delta x_1 = \sqrt[n]{5} - 1, \Delta x_2 = \sqrt[n]{5^2} - \sqrt[n]{5}, \Delta x_3 = \sqrt[n]{5^3} - \sqrt[n]{5^2}; \dots, \Delta x_n = \sqrt[n]{5^n} - \sqrt[n]{5^{n-1}}\ .$$

Demak, interval yig'indisi quydagicha bo'ladi:

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = 1\left(\sqrt[n]{5} - 1\right) + \sqrt[n]{5^3}\left(\sqrt[n]{5^2} - \sqrt[n]{5}\right) + \dots + \sqrt[n]{5^{3(i-1)}}\left(\sqrt[n]{5^i} - \sqrt[n]{5^{i-1}}\right) + \dots + \sqrt[n]{5^{3(n-1)}}\left(\sqrt[n]{5^n} - \sqrt[n]{5^{n-1}}\right)$$

$$= \left(\sqrt[n]{5} + \sqrt[n]{5^5} + \sqrt[n]{5^9} + \dots + \sqrt[n]{5^{4n-3}}\right) - \left(1 + \sqrt[n]{5^4} + \sqrt[n]{5^8} + \dots + \sqrt[n]{5^{4(n-1)}}\right)$$

$$= \frac{\sqrt[n]{5}\left(\sqrt[n]{5^{4n}} - 1\right)}{\sqrt[n]{5^4} - 1} - \frac{1\left(\sqrt[n]{5^{4n}} - 1\right)}{\sqrt[n]{5^4} - 1} = \frac{(5^4 - 1)(\sqrt[n]{5} - 1}{\sqrt[n]{5^4} - 1}$$

$$= \frac{624\left(\sqrt[n]{5} - 1\right)}{\left(\sqrt[n]{5} - 1\right)\left(\sqrt[n]{5} + 1\right)\left(\sqrt[n]{5^2} + 1\right)} = \frac{624}{\left(\sqrt[n]{5} + 1\right)\left(\sqrt[n]{5^2} - 1\right)},$$

ya'ni

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \frac{624}{\left(5^{\frac{1}{n}} + 1\right)\left(5^{\frac{2}{n}} + 1\right)}.$$

n→ ∞ da limitga o'tamiz:

$$\int_1^5 x^2\, dx = \lim_{n\to\infty} f(\xi_i)\Delta x_i = \lim_{n\to\infty} \frac{624}{(5^{\frac{1}{n}} + 1)(5^{\frac{2}{n}} + 1)} = \frac{624}{2 \cdot 2} = 156.$$

3.3-misol. Qo'ydagi integrallarni Nyuton- Leybnits formulasiga ko'ra hisoblaymiz:

$$a)\int_{-2}^{3} x^4\,dx; \qquad b)\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{sin^2 x}.$$

Yechish. $a)$ Bu yerda $f(x) = x^4$ funksiyaning boshlanig‘ich funksiyasi $F(x) = \frac{1}{5}x^5$ ga teng. 3-§ dagi (3.2) formulaga ko‘ra topamiz:

$$\int_{-2}^{3} x^4\,dx = \frac{1}{5}x^4\Big|_{-2}^{3} = \frac{1}{5}(3^5 - (-2^5)) = \frac{1}{5}(243 + 32) = 55\,.$$

$$b)\; f(x) = \frac{1}{sin^2 x}\;.F(x) = -ctgx.$$

U holda

$$b) = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{sin^2 x} = -\,ctgx\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = -\left(ctg\left(\frac{\pi}{4}\right) - ctg\left(\frac{\pi}{6}\right)\right) = \sqrt{3} - 1.$$

Taqqoslash uchun 2-misolni Nyuton- Leybnits formulasidan foydalanib yechaylik:

$$\int_{1}^{5} x^3 dx = \frac{x^4}{4}\Big|_{1}^{5} = \frac{1}{4}(5^4 - 1^4) = \frac{1}{4}(625 - 1) = 156.$$

3.4- misol. $\int_0^1 \frac{sin\,x}{x}dx$ integralni hisoblang.

Yechish. $f(x) = sinx$ funksiyani Teylor qatoriga yoyamiz

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots .$$

Tenglikning ikkala tomonini x ga bo‘lib yuboramiz va quyidagi natijaga ega bo‘lamiz.

$$\frac{sin\,x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-1)!} + \cdots$$

shundan so‘ng, tenglikning ikkala tomoni integrallab olamiz:

$$\int_0^1 \frac{\sin x}{x} dx = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots + (-1)^{n-1} \frac{x^{2n-2}}{(2n-1)!} + \dots \right) dx$$
$$= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)(2n-1)!}$$
$$+ \dots \Big|_0^1 = 1 - \frac{1}{18} + \frac{1}{120} = 1 - 0{,}055555 + 0{,}00833$$
$$= 0{,}95278.$$

Demak,

$$\int_0^1 \frac{\sin x}{x} dx \approx 0{,}95278$$

ga teng ekan.

3.5- misol. $\int_0^1 e^{-x^2} dx$ integralni $\varepsilon = 0{,}001$ aniqlikda hisoblash talab qilinsin.

$F(x) = e^x$ funksiyaning darajali qatorga yoyilmasi

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

dan foydalanamiz

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

bo‘lishini topamiz.

Bizga ma’lumki,

$$e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!}$$

taqribiy formuladan foydalanib, tenglikning ikkala tomoni integrallab olamiz:

$$\int_0^1 e^{-x^2} dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!}\right) dx\,.$$

Bu taqribiy tenglikning o‘ng tomonidagi integralni hisoblaymiz :

$$\int_0^1 (1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}-\frac{x^{10}}{5!}+\frac{x^{12}}{6!})dx$$
$$= x-\frac{x^3}{3}+\frac{x^5}{2!\cdot 5}-\frac{x^7}{3!\cdot 7}+\frac{x^9}{4!\cdot 9}-\frac{x^{11}}{5!\cdot 11}+\frac{x^{13}}{6!\cdot 13}\Big|_0^1$$
$$= 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}-\frac{1}{1320}+\frac{1}{9360}\approx 0{,}7469$$

Shunday qilib ushbu $\int_0^1 e^{-x^2}dx \approx 0{,}7469$ ga teng ekan.

Endi esa $\int_0^1 e^{-x^2}dx$ ushbu integralni parabola formulasidan foydalanib taqribiy qiymatini hisoblaymiz.[0,1] segmentni

$$x_0 = 0\,, \quad x_1 = \frac{1}{10}\,, \quad x_2 = \frac{2}{10}, \quad x_3 = \frac{3}{10}\,, \quad x_5 = \frac{5}{10}\,,$$
$$x_6 = \frac{6}{10}\,, \quad x_7 = \frac{7}{10}\,, \quad x_8 = \frac{8}{10}, \quad x_9 = \frac{9}{10}\,, \quad x_{10} = 1$$

nuqtalar yordamida 10 ta teng bo'lakka bo'lamiz. Bunda har bir bo'lakning uzunligi $\frac{1}{10}$ga teng bo'ladi. Integral ostidagi $f(x)=e^{-x^2}$ funksiyaning x_i, $(i = 0,1,2 \dots\dots..,10)$nuqtalardagi qiymatlari quyidagicha bo'ladi.Simpson formulasiga asosan ya'ni,

$$\int_a^b f(x)dx \approx \frac{b-a}{6n}\Big[\big(f(x_0)+f(x_{2n})\big)+4(f(x_1)+f(x_2)+\cdots$$
$$+f(x_{2n-1})+2\big(f(x_2)+f(x_4)+\cdots+f(x_{2n-2})\big)\Big]$$

$f(x_0)=f(0)=1{,}0000; f(x_1)=f\left(\frac{1}{10}\right)=0{,}99005;$

$f(x_2)=f\left(\frac{2}{10}\right)=0{,}96079; f(x_3)=f\left(\frac{3}{10}\right)=0{,}91393;$

$f(x_4)=f\left(\frac{4}{10}\right)=0{,}85214\,; f(x_5)=f\left(\frac{5}{10}\right)=0{,}77680;$

$f(x_6)=f\left(\frac{6}{10}\right)=0{,}69768; f(x_7)=f\left(\frac{7}{10}\right)=0{,}61263;$

$f(x_8)=f\left(\frac{8}{10}\right)=0{,}52729; f(x_9)=f\left(\frac{9}{10}\right)=0{,}44486;$

$f(x_{10})=f(1)=0{,}36788.$

Shunday qilib,

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{30}[(1{,}00000 + 0{,}36788)$$
$$+ 4(0{,}99005 + 0{,}91393 + 0{,}77680 + 0{,}61263$$
$$+ 0{,}44486)$$
$$+ 2(0{,}96079 + 0{,}85214 + 0{,}69768 + 0{,}52769)]$$
$$= \frac{1}{30}(1{,}36788 + 6{,}07580 + 14{,}96108) \approx 0{,}74682$$

Demak, $\int_0^1 e^{-x^2}dx \approx 0{,}74682$.

III- bob.

KARRALI INTEGRALLAR

3.1-§. Ikki karrali integral

Oxy tekislikda L chegaralangan D yopiq sohani qaraymiz. D sohada uzluksiz funksiya $z = f(x, y)$ berilgan bo‘lsin. D sohani ixtiyoriy chiziqlar bilan n ta bo‘lakka bo‘lamiz:

$$\Delta s_1, \Delta s_{2,}\ \Delta s_{3,\ldots,}\ \Delta s_n$$

Ularni yuzachalar deb ataymiz. Yangi simvollar kiritmaslik maqsadida
$\Delta s_1, \Delta s_{2,}\ \Delta s_{3,\ldots,}\ \Delta s_n$ orqali bularning nomlarinigina emas, yuzlarini ham belgilaymiz. $\Delta s_{ис}$ yuzalarning har birida P_i nuqta olamiz, u holda n ta nuqta hosil bo‘ladi:

$$P_1, P_2, \ldots, P_n$$

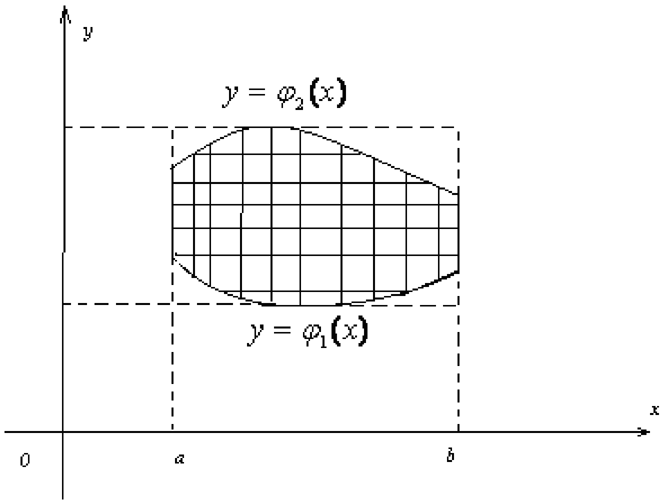
Funksiyaning tanlangan nuqtalardagi qiymatlarini $f(P_1), f(P_2), \ldots, f(P_n)$ bilan belgilaymiz va $f(P_i)\Delta s_i$ ko‘rinishidagi ko‘paytmalarning yig‘indisini tuzamiz:

$$V_n = f(P_1)\Delta s_1 + f(P_2)\Delta s_2 + f(P_n)\Delta s_n = \sum_{i=1}^{n} f(P_i)\Delta s_i$$

Bu yig‘indi D sohada *f(x, y)* funksiya uchun *integral yig‘indi* deb ataladi va bu yig‘indiga mos quyidagi limitga *f(x, y)* funksiyaning D soha bo‘yicha ikkin karrali integrali deyiladi:

$$\lim_{\operatorname{diam}\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i = \iint_D f(x, y)\,dxdy$$

D soha quyidagicha bo‘lsin:



17-rasm

D soha quyidan $y = \varphi_1(x)$ yuqoridan $y = \varphi_2(x)$ egri chiziqlar bilan, yon tomonlardan esa $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan. Bu yerda shuni esda tutish kerakki, $y = \varphi_1(x)$ va $y = \varphi_2(x)$ funksiyalar $[a, b]$ oraliqda aniqlangan bo'lishi talab etiladi, bu oraliqning tashqarisida esa, ya'ni $x < a$ va $x > b$ qiymatlarda $\varphi_1(x)$ va $\varphi_2(x)$ funksiyalarning aniqlangan bo'lishi yoki bo'lmasligi bizni qiziqtirmaydi.

Agar $x$ning $[a\,, b]$ oraliqdagi har bir o'zgarmas qiymati uchun

$$J(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \tag{3.1}$$

integral mavjud bo'lsa, u holda takroriy integral

$$\int_a^b J(x)dx = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \tag{3.2}$$

ham mavjud bo'ladi va quyidagi tenglik o'rinli bo'ladi:

$$\iint_D f(x, y)dD = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \tag{3.3}$$

Endi ikki karrali integralning geometrik ma'nosiga kelsak, (3.3) integral ostki asosi XOY tekisligida yotuvchi D soha bo'lgan, ustki asosi esa $z = f(x, y)$ sirt bilan chegaralangan va yasovchilari $O\,Z$ o'qiga parallel bo'lgan silindrsimon jismning (g'o'laning) hajmini aniqlaydi. Bu yerda shuni yodda tutingki, agar $f(x, y) = 1$ bo'lsa, u holda

$$\iint_D f(x, y)dD = \iint_D dD = (D) \quad (D) - D$$

sohaning yuzi

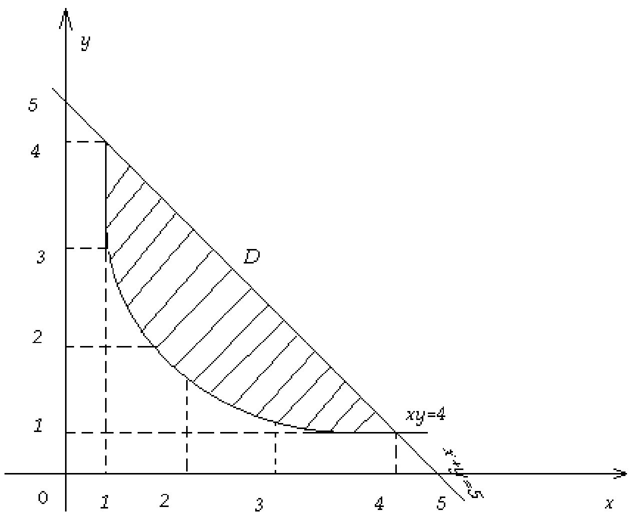
Agar $\varphi_1(x) = c(const)$, $\varphi_2(x) = d(const)$ bo'lsa, u holda D soha $\{[a\,,b] \times [c\,,d]\}$ to'g'ri to'rtburchakdan iborat bo'lib, (3.3) integral ustki asosi $z = f(x,y)$ sirt bilan chegaralangan, qolgan yoqlari $x = a$,$x = b$, $y = c, y = d$ va xOy tekisliklarda yotuvchi parallelopipedning hajmini aniqlaydi.

Endi yuqorida aytilgan mulohazalarni tipik misollar yechishga qo'llaymiz.

Tekis figuraning yuzini hisoblash

1-misol. $xy = 4 \;\; va \;\; x + y = 5$ chiziqlar bilan chegaralangan sohaning yuzini toping.

Yechish. Avvalo berilgan sohani koordinatalar tekisligida tasvirlaymiz.



18-rasm

D sohada x ning eng katta va eng kichik qiymatlarini topish sizga 1-kurs matematik analiz kursidan ma'lum. Buning uchun har ikkala tenglamani birgalikda yechamiz:

$$\begin{cases} xy = 4 \\ x + y = 5 \end{cases} \Rightarrow \begin{cases} \mathrm{x}(5 - x) = 4 \\ y = 5 \; - x \end{cases} \Rightarrow \begin{cases} x^2 - 5x + 4 = 0 \\ y = 5 - x \end{cases} \Rightarrow$$

$$\begin{cases} x_1 = 4;\; x_2 = 1 \\ y = 5 - x \end{cases}$$

Demak, x o'zgaruvchi $[1, 4]$ oraliqdagi qiymatlarni qabul qilar ekan. D sohaning chegaralari $y = \frac{4}{x}$ va $y = 5 - x$ chiziqlaridir. $a = 1;\;\; b = 4$ bo'ladi. (3) fomulada $f(x,y) = 1$, $\varphi_1 = \frac{4}{x}$ va

$\varphi_2(x) = 5 - x$ larni o‘rniga qo‘ysak va D sohaning yuzini S bilan belgilasak, quyidagini hosil qilamiz:

$$S = \iint\limits_D dD$$

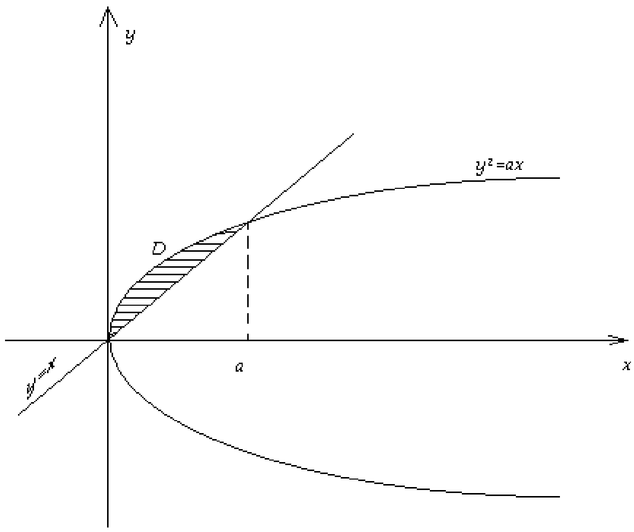
$$= \int_1^4 dx \int_{\frac{4}{x}}^{5-x} dy = \int_1^4 (5 - x - \frac{4}{x})dx = (5x - \frac{x^2}{2} - 4\ln x)\Big|_1^4 =$$

$$= 20 - 8 - 4\ln 4 - 5 + \frac{1}{2} = 7{,}5 - 4\ln 4 \quad (kv \ \ birlik).$$

2-misol*.* $y^2 = ax$ parabola va $y = a$ to‘g‘ri chiziq bilan chegaralangan sohaning yuzini toping*,* $(a > 0)$.

Yechish. Berilgan chiziqlarning grafigini chizamiz.a ning son qiymati aniq berilmagani uchun parabolaning formal (yuzaki) grafigini chizamiz. To‘g‘ri chiziq va parabolaning kesishish nuqtalarini topamiz:

$$\begin{cases} y^2 = ax \\ y = x \end{cases} \Rightarrow \begin{cases} x^2 = ax \\ y = x \end{cases} \Rightarrow \begin{cases} x(x - a) = 0 \\ y = x \end{cases} \Rightarrow \begin{cases} x_1 = 0; \quad x_2 = a \\ y = x \end{cases}$$



19-rasm

Demak, x ning chegaralari 0 va a sonlari ekan. Parabolaning yuqori bo‘lagida uning tenglamasi $y = \sqrt{ax}$ ko‘rinishda bo‘ladi. Yuqoridagi shakldan ko‘rinadiki, $\varphi_1(x) = x$, $\varphi_2(x) = \sqrt{ax}$. Bularni

(3.3) formulaga qo‘ysak va $f(x,y)=1$ ekanini hisobga olsak, quyidagini topamiz:

$$S=\int\int_{a} dD=\int_0^a dx\int_x^{\sqrt{ax}} dy=\int_0^a(\sqrt{ax}-x)dx=\sqrt{a}\cdot\int_0^a\sqrt{x}dx-$$

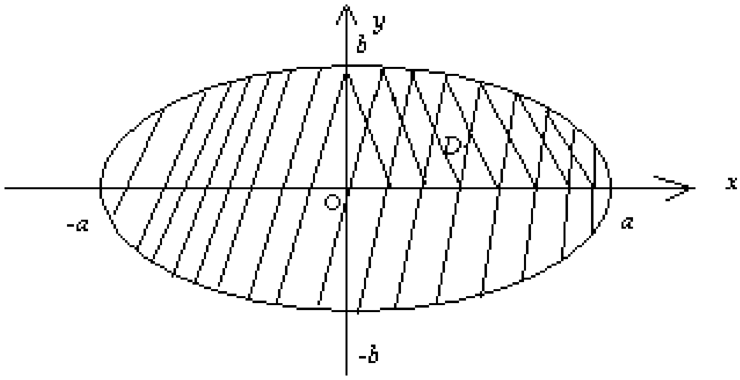
$$-\int_0^a xdx=\sqrt{a}\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^a-\frac{x^2}{2}\Bigg|_0^a=\frac{2}{3}\sqrt{a}\ \cdot a\sqrt{a}-\frac{1}{2}a^2=(\frac{2}{3}-\frac{1}{2})a^2$$

$$=\frac{1}{6}a^2$$

(kv birlik).

3-misol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips bilan chegaralangan sohaning yuzini toping.

Yechish. Ellipsning grafigini chizamiz.



20-rasm

Berilgan D sohaning yuzini hisoblash uchun uning koordinatalar tekisligi I choragida joylashgan qismi yuzini hisoblab, natijani 4 ga ko‘paytirish kifoya. Demak, $(D)=4\cdot(D_1)$

Ellips tenglamasini y ga nisbatan yechamiz

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}\Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2)\Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

Shaklidan ko‘rinadiki, D sohaning yuzi uchun $\varphi_1(x)=0$ va

$$\varphi_2(x)=\frac{b}{a}\sqrt{a^2-x^2}\ ,\ x\in[0,a].$$

Bularni (3) formulada o‘rniga qo‘ysak va $f(x,y)=1$ ekanini hisobga olsak, quyidagini topamiz:

$$S = \iint\limits_{D} dD = 4\iint\limits_{D_1} dD_1 = 4\int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy =$$

$$= 4\int_0^a y\Big|_0^{\frac{b}{a}\sqrt{a^2-x^2}} \cdot dx = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}dx$$

,

bu yerda$x = asint$ belgilash kiritamiz:

$$dx = a\cos tdt,\ x = 0 \ \Rightarrow \ t = 0,\ \ x = a \ \Rightarrow \ t = \frac{\pi}{2}.$$

Bularni o‘rniga qo‘ysak,

$$S = \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t} \cdot a\cos tdt$$

$$= 4b\int_0^{\frac{\pi}{2}} a\cos^2 tdt = 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} =$$

$$= 2ab\int_0^{\frac{\pi}{2}} (1+\cos 2t)dt = 2ab(t + \frac{1}{2}\sin 2t)\Big|_0^{\frac{\pi}{2}} = 2ab\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right)$$

$$= 2ab \cdot \frac{\pi}{2} = \pi ab$$

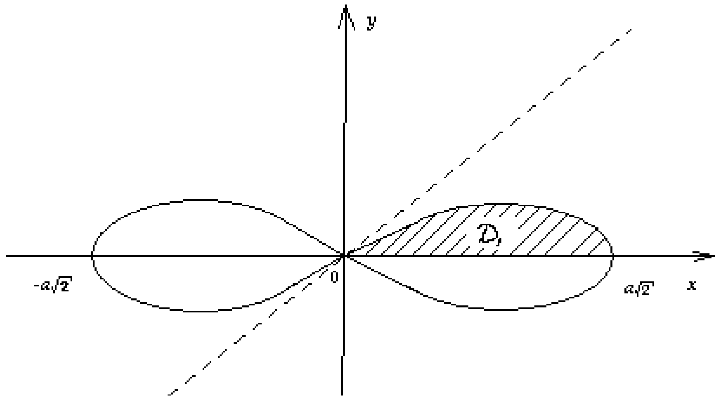
(kv birlik).

4-misol. $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ lemniskata bilan chegaralangan sohaning yuzini toping.

Bu misolni yechishga kirishishdan oldin ma’ruzadan ikki karrali integrallarni hisoblashda qutb koordinatalarini qo‘llanilishi mavzusini takrorlang. Sizga ma’lumki, agar D soha konturining qutb koordinatalaridagi tenglamasi $r = r(\varphi)$ bo‘lsa, (r −qutb o‘qi, φ −qutb burchagi), bu yerda φ o‘zgaruvchi $[\alpha, \beta]$ oraliqda n qiymatlar qabul qilsa, u holda D sohaning yuzi quydagicha aniqlanadi:

$$S=\iint_D dD=\int_\alpha^\beta d\varphi\int_0^{r(\varphi)} rdr$$

Endi 4-misolni yechishga o‘tamiz. Oldin lemniskata grafigini chizamiz:



21-rasm

Lemniskataning qutb koordinatalardagi tenglamasiga o‘tish uchun uning tenglamasidagi *x,y* lar o‘rniga $x=r\cos\varphi\,, y=r\sin\varphi$ larni qo‘yamiz.

$$[r^2(\cos^2\varphi+\sin^2\varphi)]^2=2a^2r^2(\cos^2\varphi-\sin^2\varphi)\Rightarrow$$
$$\Rightarrow r^4=2a^2r^2\cos2\varphi\Rightarrow r^2=2a^2\cos2\varphi\Rightarrow$$
$$\Rightarrow r=a\sqrt{2\cos2\varphi}\qquad(\varphi\in[-\frac{\pi}{4},\frac{\pi}{4}]\cup[\frac{3\pi}{4},\frac{5\pi}{4}])$$

Shaklda ko‘rsatilgan D_1 soha uchun $0\le\varphi\le\frac{\pi}{4}$ oraliqda qiymatlar qabul qiladi.

$S=4\cdot(D_1)$ ekanini hisobga olib, quyidagini topamiz:

$$S=4\iint_{D_1} dD_1=4\int_0^{\frac{\pi}{4}} d\varphi\int_0^{a\sqrt{2\cos2\varphi}} rdr=4\int_0^{\frac{\pi}{4}}\frac{r^2}{2}\bigg|_0^{a\sqrt{2\cos2\varphi}}d\varphi=$$
$$=2\int_0^{\frac{\pi}{4}}2a^2\cos2\varphi d\varphi=2a^2\sin2\varphi\big|_0^{\pi/4}=2a^2(\sin\frac{\pi}{2}-0)=2a^2$$

(kv birlik).

5-misol. Egri chiziqli koordinatalar kiritish yordamida quyidagi chiziqlar bilan chegaralangan sohaning yuzini hisoblang*:*

$$y^2=2px\,,\ \ y^2=2qx\,,\ \ x^2=2ry\ \ ,\ x^2=2sy.$$

Bu yerda $0<p<q\ \ ,\quad 0<r<s\ .$

Yechish. Bu misolni yechishga kirishishdan oldin ma'ruzadan "ikki karrali integrallarda o'zgaruvchilarni almashtirish" mavzusini takrorlang. Bu yerda yangi ξ , η egri chiziqli koordinatalarni quyidagicha kiritamiz:

$$y^2 = 2\xi x \quad (p \le \xi \le q) \qquad x^2 = 2\eta y \quad (r \le \eta \le s)$$

Bu tenglamalardagi x,y larni ξ , η lar orqali ifodalaymiz:

$$\left.\begin{matrix} x^2 = 2\eta y \\ x = \frac{y^2}{2\xi} \end{matrix}\right\} \Rightarrow \begin{cases} \frac{y^4}{4\xi^2} = 2\eta y \\ x = \frac{y^2}{2\xi} \end{cases} \Rightarrow \begin{cases} y^4 = 8\xi^2\eta y \\ x = \frac{y^2}{2\xi} \end{cases} \Rightarrow \begin{cases} y = 2\sqrt[3]{\xi^2\eta} \\ x = \frac{4\sqrt[3]{\xi^4\eta^2}}{2\xi} \end{cases} \Rightarrow x = 2 \cdot \sqrt[3]{\xi\eta^2}; \quad y = 2\sqrt[3]{\xi^2\eta}.$$

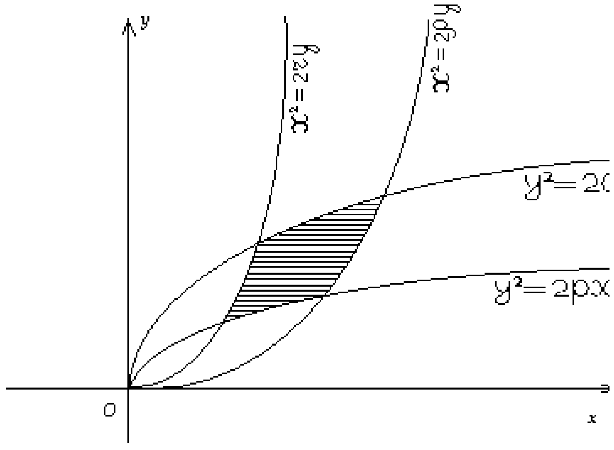
Endi x va y larning ξ , η lar bo'yicha xususiy hosilalarini hisoblaymiz:

$$\frac{\partial x}{\partial \xi} = \frac{2}{3}(\xi\eta^2)^{-\frac{2}{3}} \cdot \eta^2 \;; \qquad \frac{\partial x}{\partial \eta} = \frac{2}{3}(\xi\eta^2)^{-\frac{2}{3}} \cdot 2\xi\eta \;;$$

$$\frac{\partial y}{\partial \xi} = \frac{2}{3}(\xi^2\eta)^{-\frac{2}{3}} \cdot 2\xi\eta \;; \qquad \frac{\partial y}{\partial \eta} = \frac{2}{3}(\xi^2\eta)^{-\frac{2}{3}} \cdot \xi^2 \;;$$

Bularni $I(\xi,\eta) = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$ yakobianda o'rniga qo'yamiz va determinantning xossasiga ko'ra uning yo'llaridagi umumiy ko'paytuvchilarni determinant ishorasidan tashqariga chiqaramiz:

$$I(\xi,\eta) = \frac{4}{9}\xi\eta(\xi\eta^2)^{-\frac{2}{3}} \cdot (\xi^2\eta)^{-\frac{2}{3}} \cdot \begin{vmatrix} \eta & 2\xi \\ 2\eta & \xi \end{vmatrix} = \frac{4}{9}\xi\eta(\xi\eta)^{-2} \cdot (-3\xi\eta) = -\frac{4}{3}$$



22-rasm

Bu yerda shuni esda tutish kerakki, yangi ξ, η oʻzgaruvchilar boʻyicha integrallash sohasi $\Delta = \{[p, q] \times [r, s]\}$ toʻgʻri toʻrtburchagi boʻladi. Demak,

$$S = \iint_D dD = \iint_\Delta |I(\xi, \eta)|\, d\xi d\eta = \frac{4}{3} \int_p^q d\xi \int_r^s d\eta = \frac{4}{3}(q - p)(s - r)$$

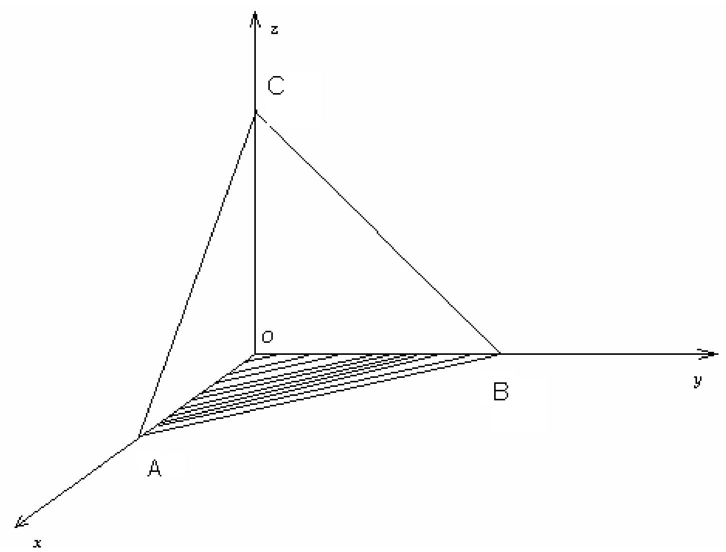
Yuqorida yechilgan misollarda shunday xulosaga kelamiz: D sohaning konturi (chegarasi) silliq yopiq egri chiziq boʻlgan hollarda qutb koordinatalari kiritish, egri chiziqli toʻrtburchak boʻlgan hollarda esa egri chiziqli koordinatalar yordamida oʻzgaruvchilarni almashtirish maqsadga muvofiqdir.

Jismning hajmini hisoblash

6-misol. Koordinatalar tekisliklari va $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ tekisligi bilan chegaralangan jismning hajmini toping.

Bu misolni yechishga kirishishdan oldin “ikki karrali integrallar yordamida jismning hajmini hisoblash” mavzusini ma’ruzadan takroran oʻqing.

Yechish.



23-rasm

Qaralayotgan jism uchburchakli piramida boʻlib, uning uchta yoqlari koordinata tekisliklarida yotadi.

D sohaning chegaralari $x = 0\,, y = 0$ va $\frac{x}{a} + \frac{y}{b} = 1$ to‘g‘ri chiziqlardir.
Demak,

$$D = \{x, y\colon 0 \le x \le a\,, \qquad 0 \le y \le b(1 - \frac{x}{a})\,\}$$

$$z = f(x, y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right).$$

Bularni (3) formulaga qo‘ysak va qaralayotgan jism hajmini V bilan belgilasak, quyidagini hosil qilamiz:

$$V = \iint\limits_D f(x, y) dD = \iint\limits_D c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dxdy = \int\limits_0^a dx \int_0^a dx \int\limits_0^a c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy =$$

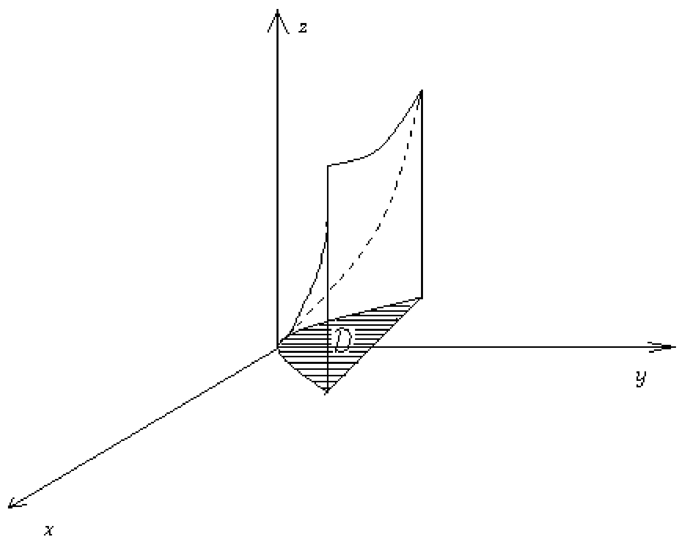
$$= c\int\limits_0^a \left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)\Bigg|_0^{b - \frac{bx}{a}} \cdot dx = \int\limits_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{bx}{a}\left(1 - \frac{x}{a}\right) - \left(\frac{b^2\left(1 - \frac{x}{a}\right)^2}{2b}\right)\right] dx$$

$$= bc\int\limits_0^a \frac{(1 - \frac{x}{a})^2}{2} dx = -abc \cdot \frac{\left(1 - \frac{x}{a}\right)^3}{6}\Bigg|_0^a = \frac{abc}{6}$$

(kub birlik).

7-misol. $z = x^2 + y^2$ aylanma paraboloid, $y = x^2$ silindr va $y = 1$ tekisliklar bilan chegaralangan jismning hajmini toping.

Yechim.



24-rasm

D soha quyidagicha bo‘ladi:

$$D = \{(x,y): -1 \le x \le 1\ ;\ x^2 \le y \le 1\},$$
$$f(x,y) = x^2 + y^2.$$
Bularni (3) tenglikda oʻrniga qoʻyamiz:
$$V \iint\limits_D f(x,y)dD = \iint\limits_D (x^2+y^2)dxdy$$
$$= \int_{-1}^{1} dx \int_{x^2}^{1} (x^2+y^2)dy = \int_{-1}^{1} \left(x^2y + \frac{y^3}{3}\right)\Bigg|_{x^2}^{1} dx =$$
$$= \int_{-1}^{1} (x^2 + \frac{1}{3} - x^4 - \frac{x^6}{3})dx = \left(\frac{x^3}{3} + \frac{x}{3} - \frac{x^5}{5} - \frac{x^7}{21}\right)\Bigg|_{-1}^{1} = 2\left(\frac{1}{3} + \frac{1}{3} - \frac{1}{5} - \frac{1}{21}\right)$$
$$= 2 \cdot \frac{70-21-5}{105} = \frac{88}{105}\ ;\ \ (kub\ birlik).$$
8-misol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidning hajmini toping.

Yechish. Berilgan tenglamadan z ni aniqlaymiz:
$$z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$
Bu yerda D sohaning konturi $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellips boʻladi.

Oʻzgaruvchilarni $x = ar\cos\varphi$, $y = br\sin\varphi$ formulalar yordamida almashtiramiz.

Natijada $D' = \{(\varphi, r) :\ 0 \le \varphi \le 2\pi; 0 \le r \le 1\}$
$$\frac{\partial x}{\partial\varphi} = -ar\sin\varphi\ ;\quad \frac{\partial x}{dr} = a\cos\varphi$$
$$\frac{\partial y}{\partial\varphi} = br\cos\varphi\ \ ;\ \ \frac{\partial y}{\partial r} = b\sin\varphi$$
Yakobian $I(\varphi, r) = abr\begin{vmatrix} -\sin\varphi & \cos\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} = ab\mathrm{r}(-\sin^2\varphi - \cos^2\varphi) =$
$-abr$; $|I(\varphi, r)| = abr$.

Ellipsoidning XOY tekisligidan yuqorida joylashgan boʻlagining hajmini hisoblab, natijani 2 ga koʻpaytirsak, uning hajmi kelib chiqadi.

$$\frac{1}{2}V = \iint\limits_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}dxdy$$

$$= c\iint\limits_{D'} \sqrt{1 - r^2\cos^2\varphi - r^2\sin^2\varphi} \cdot |I(\varphi, r)| \cdot d\varphi dr =$$

$$= c\int_0^1 dr\int_0^{2\pi} \sqrt{1 - r^2} \cdot abrd\varphi = abc \cdot 2\pi\int_0^1 \sqrt{1 - r^2}r \cdot dr =$$

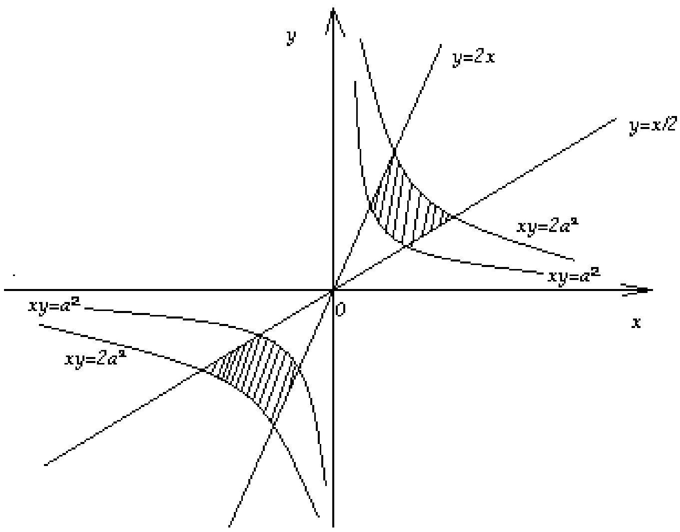
$$= -\pi abc \cdot \left.\frac{(1 - r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 = \frac{2}{3}\pi abc$$

(kub birlik).

Bundan $V = \frac{4}{3}\pi abc$ kelib chiqadi.

9-misol. $xy = a^2$, $xy = 2a^2$ silindrlar, $y = \frac{x}{2}$, $z = 0$, $y = 2x$ tekisliklar hamda $z = \frac{x^2}{p} + \frac{y^2}{q}$ paraboloid bilan chegaralangan jismning hajmini toping.

Yechish. Integrallash sohasini XOY koordinatalar tekisligida tasvirlaymiz:



25-rasm

Shaklda ko‘rsatilgan sohalar simmetrik bo‘lgani uchun va paraboloid tenglamasida

$z=\frac{(-x)^2}{p}+\frac{(-y)^2}{q}=\frac{x^2}{p}+\frac{y^2}{q}$ tenglik o‘rinli bo‘lgani uchun $x>0,\ y>0$ bo‘lgan sohada jism hajmini hisoblab natijani 2 ga ko‘paytirsak, izlanayotgan jismning hajmi kelib chiqadi.

Egri chiziqli koordinatalarni

$$xy=ua^2, 1\le u\le 2;$$
$$y=vx\,,\ \frac{1}{2}\le v\le 2$$

tengliklar yordamida kiritamiz. Bu tengliklardan $x,\ y$ o‘zgaruvchilarnilarni aniqlaymiz: $x=a\sqrt{\frac{u}{v}}\,, y=a\sqrt{uv}$. Bundan

$$\frac{\partial x}{\partial u}=\frac{a}{2\sqrt{uv}},\quad \frac{\partial x}{\partial v}=-\frac{au}{2v\sqrt{uv}}$$
$$\frac{\partial y}{\partial u}=\frac{av}{2\sqrt{uv}},\quad \frac{\partial y}{\partial v}=\frac{au}{2\sqrt{uv}}$$

xususiy hosilalarni aniqlaymiz. Bularni $I(u,v)$ yakobianda o‘rniga qo‘ysak va umumiy ko‘paytuvchilarni determinant ishorasidan tashqariga chiqarsak, quyidagi kelib chiqadi:

$$I(u,v)=\frac{a^2}{4uv}\begin{vmatrix}1 & -\frac{u}{v}\\ v & u\end{vmatrix}=\frac{a^2}{4uv}\cdot 2u=\frac{a^2}{2v}.$$

Natijada

$$\frac{1}{2}V=\iint\limits_{D} f(x,y)dD=\iint\limits_{D\prime} f(x(u,v),y(u,v))\,|I(u,v)|dudv=$$
$$=\int\limits_{\frac{1}{2}}^{2}dv\int\limits_{1}^{2}\left(\frac{a^2u}{pv}+\frac{a^2uv}{q}\right)\frac{a^2}{2v}du=\frac{a^4}{2}\int\limits_{1}^{2}udu\int\limits_{\frac{1}{2}}^{2}\left(\frac{1}{pv}+\frac{v}{q}\right)\frac{1}{v}dv=$$
$$=\frac{a^2}{4}u^2\Big|_1^2\cdot\int\limits_{\frac{1}{2}}^{2}\left(\frac{1}{pv^2}+\frac{1}{q}\right)dv=\frac{3a^2}{4}\left(-\frac{1}{pv}+\frac{v}{q}\right)\Big|_{\frac{1}{2}}^{2}=$$
$$=\frac{3a^4}{4}\left(-\frac{1}{2p}+\frac{2}{q}+\frac{2}{p}-\frac{1}{2q}\right)=\frac{3a}{4}\left(\frac{3}{2p}+\frac{3}{2q}\right)=\frac{9a^4}{8}\left(\frac{1}{p}+\frac{1}{q}\right)$$
$$=\frac{9a^4(p+q)}{8pq}$$

(kub birlik).

Bundan

$$V = \frac{9a^4(p+q)}{4pq}$$

kelib chiqadi.

Mustaqil ishlash uchun misollar

Quydagi chiziqlar bilan chegaralangan sohalarning yuzlarini toping.

1. $y = x^2$, $\quad y = 8 - x^2$;
2. $y^2 = 2x$, $\quad x - y - 1 = 0$;
3. $(x^2 + y^2)^2 = 2ax^4$, $(a > 0)$ ning yuzini toping,

$x = r\cos\varphi, y = r\sin\varphi$ qutb qoordinatalariga yoki $x = ar\cos^\alpha\varphi$, $y = br\sin^\alpha\varphi$ umumlashgan qutb koordinatalariga o'tib quyidagi chiziqlar bilan chegaralangan sohalar yuzalarini hisoblang:

4. $(x^2 + y^2)^2 = 2ax^3$
5. $(x^2 + y^2)^3 = a^2(x^4 + y^4)$astroidaning yuzini toping. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Ko'rsatma: $x = r\cos^3\varphi$, $\quad y = r\sin^3\varphi$ almashtirish bajaring;

6. $(x^2 + y^2)^3 = 4a^2x^2y^2$;
7. $x^2 + y^2 = ax$, $x^2 + y^2 = by$, $M\left(\frac{b}{2}, \frac{a}{2}\right) \in S$;
8. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{xy}{c^2}$;
9. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = x^2 + y^2$;
10. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{c^2}$;
11. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^3 = \frac{x^4}{h^4} + \frac{y^4}{k^4}$;
12. $\frac{x^4}{a^4} + \frac{y^4}{b^4} = \frac{x^2y}{c^3}$;
13. $\frac{x^4}{a^4} + \frac{y^4}{b^4} = \frac{x^2}{h^2} + \frac{y^2}{k^2}$;
14. $\frac{x^6}{a^6} + \frac{y^6}{b^6} = \frac{x^4}{h^4} + \frac{y^4}{k^4}$;
15. $\left(\frac{x}{a} + \frac{y}{a}\right)^2 = \frac{x}{a} - \frac{y}{b}$, $y = 0$.

Mos o'zgaruvchilarni almashtirib quyidagi chiziqlar bilan chegaralangan yuzani toping.

16. $xy = p$, $xy = q$, $y^2 = ax$, $y^2 = bx$, $0 < p < q$, $0 < a < b$.
17. $xy = p$, $xy = q$, $y = \alpha x$, $y = \beta x$, $0 < p < q$, $0 < \alpha < \beta$.
18. $y^2 = py$, $x^2 = qy$, $y = \alpha x$, $y = \beta x$, $0 < p < q$, $0 < \alpha < \beta$.
19. $y = ax^3$, $y = bx^3$, $y^2 = px$, $y^2 = qx$, $0 < a < b$, $0 < p < q$.

20. $y=\frac{x^5}{a^4},\ y=\frac{x^5}{b^4},\ x=\frac{y^5}{c^4},\ x=\frac{y^5}{d^4},\ x>0,\ y>0,\ 0<a<b,\ 0<c<d.$

Quyidagi sirtlar bilan chegaralangan jismlarning hajmlarini toping.

21. $z=1+x+y;\ z=0;\quad x+y=1,\ x=0,\ y=0.$

22. $y=\sqrt{x}\ ;\quad y=2\sqrt{x}$ silindrlar $z=0$ va $x+z=6$ tekisliklar.

23. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1\ ;\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

24. $z=xy\ ,\ x^2=y\ ,\ x^2=2y\ ,\ y^2=x\ ,\ z=0$

25. $0\le z\le x^2,\ x+y\le 5,\ x-2y\ge 2,\ y\ge 0.$

26. $x+y+z\le a,\ 3x+y\ge a,\ 3x+2y\le 2a,\ y\ge 0,\ z\ge 0.$

27. $x+y\le 1,\ z\le x^2+y^2,\ x\ge 0,\ y\ge 0,\ z\ge 0.$

28. $x+y\le a,\ 0\le 2bz\le y^2,\ x\ge 0,\ y\ge 0.$

29. $z^2\le 2px,\ y\le x\le a,\ y\ge 0.$

30. $z^2\ge 2px,\ z^2\ge 2qy,\ 0\le z\le a,\ x\ge 0,\ y\ge 0.$

31. $y^2\le 2q(a-x),\ z^2\le 2px.$

32. $0\le z\le xy,\ y\le x\le 1.$

33. $x^2+y^2\le a^2,\ z\ge 0,\ x+y+z-4a\ge 0.$

34. $x^2+y^2\le a^2,\ x+y+z\le a,\ z\ge 0.$

35. $x^2+y^2\le a^2, 0\le az\le a^2-2y^2.$

36. $(x^2+y^2)^2\le a^2(x^2-y^2),\ 0\le bz\le x^2+y^2.$

37. $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ \ \frac{x^2}{a^2}+\frac{z^2}{b^2}\le 1.$

38. $4x\ge y^2,\ 4y\ge x^2,\ 0\le z\le y.$

39. $z\ge 0,\ x+z\le 1,\ x\ge y^2.$

40. $ax\le x^2+y^2\le 2ax,\ 0\le bz\le x^2y^2.$

Javoblar:

Ikki karrali integrallar yordamida tekis figuraning yuzini va jismning hajmini hisoblash.

3. $\frac{5}{8}\pi a^2$ 4. $\frac{5\pi a^2}{8}$ 5. $\frac{3}{8}\pi a^2$ 6. $\frac{1}{2}\pi a^2$ 7. $\frac{a^2\pi}{8}+\frac{b^2-a^2}{4}arctg\frac{a}{b}-\frac{ab}{4}$ 8. $\frac{a^2b^2}{2c^2}$ 9. $\frac{\pi}{2}ab(a^2+b^2)$ 10. $\frac{\pi}{2}\frac{a^3b}{c^2}$ 11. $\frac{3}{8}\pi ab\left(\frac{a^4}{h^4}+\frac{b^4}{k^4}\right)$ 12. $\frac{\pi\sqrt{2}}{16}\frac{a^5b^3}{c^6}$ 13. $\frac{\pi\sqrt{2}}{2}\left(\frac{a^2}{h^2}+\frac{b^2}{k^2}\right)$ 14. $\frac{2}{3}\pi ab\left(\frac{a^4}{h^4}+\frac{b^4}{k^2}\right)$ 15. $\frac{1}{12}ab$ 16. $\frac{1}{3}(q-p)\ln\frac{b}{a}$ 17. $\frac{1}{2}(q-p)\ln\frac{b}{a}$ 18. $\frac{1}{6}(q^2-p^2)(b^3-a^3)$ 19. $\frac{5}{12}\left(q^{\frac{4}{5}}-p^{\frac{4}{5}}\right)\left(a^{-\frac{3}{5}}-b^{-\frac{3}{5}}\right)$ 20. $\frac{2}{3}(a-b)(c-d)$ 21. $\frac{5}{6}$ 22. $\frac{48\sqrt{6}}{5}$ 23. $\frac{4}{3}\pi abc(2\sqrt{2}-1)$

24. $\frac{3}{4}$ 25. 32 26. $\frac{a^3}{18}$ 27. $\frac{1}{6}$ 28. $\frac{a^4}{24b}$ 29. $\frac{2}{5}a^2\sqrt{2ap}$ 30. $\frac{1}{20}\frac{a^5}{pq}$ 31. $\pi a^2\sqrt{pq}$ 32. $\frac{1}{8}$ 33. 34. $\frac{a^3}{12}(9\pi+10)$ 35. $a^3\left(\frac{\pi}{4}+1\right)$ 36. $\frac{\pi a^4}{4b}$ 37. $\frac{16ab^3}{3}$ 38. $\frac{88}{5}$ 39. $\frac{8}{15}$ 40. $\frac{45\pi a^4}{32}$

3.2-§. Uch karrali integral

Fazoda S yopiq sitr bilan chegaralangan biror yopiq V soha berilgan bo‘lsin. V sohada biror $f(x,y,z)$ uzluksiz funksiya aniqlangan bo‘lsin, bunda x,y,z soha nuqtasining to‘g‘ri burchakli koordinatalari. Aniqlik uchun $f(x,y,z)\geq 0$ bo‘lgan holda biz bu funksiyani qandaydir bir moddaning V sohaga taqsimlanish zichligi deb hisoblashimiz mumkin. Δv_1 simvol bilan sohaning o‘zinigina emas, balki uning hajmini ham belgilab, V sohani ixtiyoriy ravishda Δv_1 sohada ixtiyoriy P_i nuqtani tanlab olamiz va f funksiyaning bu nuqtadagi qiymatini $f(P_i)$ bilan belgilaymiz. Integral yig‘indini

$$\sum f(P_i)\Delta v_i$$

ko‘rinishida tuzamiz va bunda Δv_1 ning eng katta diametrini nolga intiladigan qilib, Δv_1 sohalarning sonini cheksiz orttirib boramiz. Agar $f(x,y,z)$ funksiya uzluksiz bo‘lsa, yuqoridagi integral yig‘indining limiti mavjud bo‘ladi, bunda integral yig‘indining limiti ikki karrali integralni ta’riflagandagi ma’noda tushuniladi. V sohani bo‘lish usuliga ham, P_i nuqtani tanlash usuligayam bog‘liq bo‘lmagan bu limit $\iiint_V f(P)dv$ simvol bilan belgilaniladi va uch *karrali intregral* deb ataladi. Shunday qilib, ta’rifga ko‘ra:

$$\lim_{\text{diam}\Delta v_{\text{紹}}\to 0}\sum f(P_i)\Delta v_i=\iiint\limits_V f(P)dv$$

yoki

$$\iiint\limits_V f(P)dv=\iiint\limits_V f(x,y,z)dx\,dy\,dz$$

V uch o‘lchovli chegaralangan yopiq soha (jism) bo‘lib, u silliq yoki bo‘lakli-silliq sirt bilan chegaralangan. Agar V soha quyidan $z=f_1(x,y)$ yuqoridan $z=f_2(x,y)$ sirtlar bilan chegaralangan bo‘lsa, bundan tashqari V sohada x, y lar orasida $\varphi_1(x) \leq y \leq \varphi_2(x)$ tengsizlik o‘rinli bo‘lib, x esa shu sohada $[a, b]$ oraliqda qiymatlar qabul qilsa, u holda agar $f(x, y, z)$ funksiyadan V soha bo‘yicha olingan uch karrali integral mavjud bo‘lsa, quyidagi tenglik o‘rinli bo‘ladi:

$$\iiint\limits_V f(x,y,z)dv = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)dz \qquad (3.4)$$

agar $f(x, y, z) = 1$ bo‘lsa (4) tenglik Vsohaning hajmini aniqlaydi:

$$\iiint\limits_V dv = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{f_1(x,y)}^{f_2(x,y)} dz \qquad (3.5)$$

(5) tenglik o‘ng tomonini z bo‘yicha integrallasak,

$$\iiint\limits_V dv = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} (f_2(x,y) - f_1(x,y))dy$$

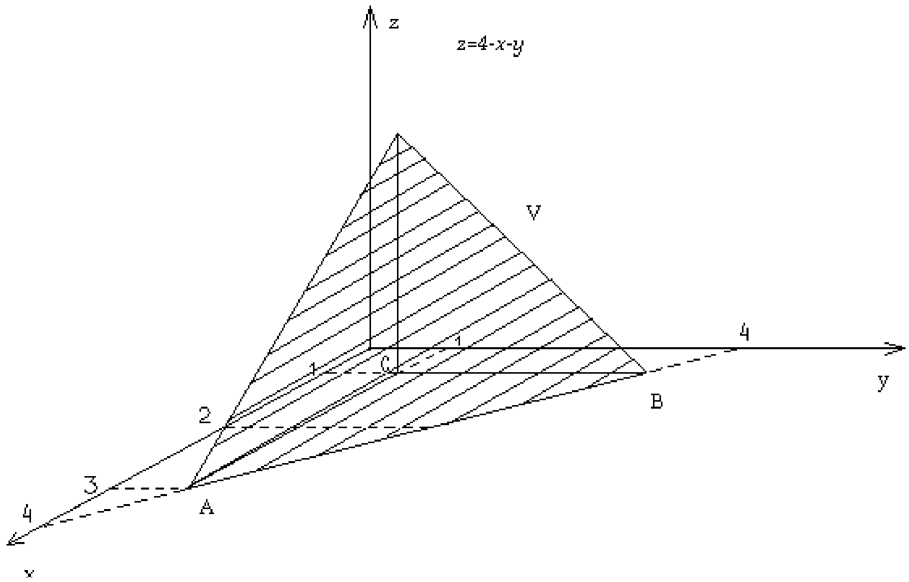
kelib chiqadi. Bu tenglikning o‘ng tomonini bizga tanish bo‘lgan ikki karrali integralning takroriy integrallar orqali ifodalanishidir. Uch karrali integrallarning geometrik tadbiqlariga doir misollar yechishda juda zarur bo‘lgan quyidagi vavzularni ma’ruzadan qaytarib chiqing:

1. Uch karrali integrallarni hisoblashda sferik va silindrik koordinatalardan foydalanish.
2. Uch karrali integrallarda o‘zgaruvchilarni almashtirish.

Endi tipik misollarni yechishga o‘tamiz.

1-misol. $x - 1 = 0$, $y - 1 = 0$, $z = 0$ *va* $x + y + z = 4$ tekisliklar bilan chegaralangan jismning (piramidaning) hajmini toping.

Yechish.



26-rasm

$x + y + z = 4$ tenglamada $z = 0$ qo‘ysak, $x + y = 4$ bo‘ladi. Bu XOY tekislikda (AB) to‘g‘ri chiziq tenglamasidir. $x + y = 4$ tenglamada $y = 1$ qo‘ysak, A nuqtaning absissasi kelib chiqadi $x + 1 = 4 \Rightarrow x = 3$. Demak, V sohada: $1 \le x \le 3$; $1 \le y \le 4 - x$; $0 \le z \le 4 - x - y$. Bularni (3.5) formulaga qo‘ysak,

$$\iiint\limits_{V} dv = \int_1^3 dx \int_1^{4-x} dy \int_0^{4-x-y} dz \int_1^3 dx \int_1^{4-x} dy \int_1^{4-x-y} dz$$

$$= \int_1^3 dx \int_1^{4-\mathrm{x}} (4 - \mathrm{x} - \mathrm{y})\mathrm{dy} =$$

$$= \int_1^3 \left(4y - xy - \frac{y^2}{2}\right)\Bigg|_1^{4-x} dx$$

$$= \int_1^3 \left[4(4 - x) - x(4 - x) - \frac{(4 - x)^2}{2} - 4 + x + \frac{1}{2}\right] dx =$$

$$= \int_1^3 (16 - 4x - 4x + x^2 - 8 + 4x - \frac{1}{2}x^2 + x - 3{,}5)dx$$

$$= \int_1^3 \left(\frac{1}{2}x^2 - 3x + 4{,}5\right) dx =$$

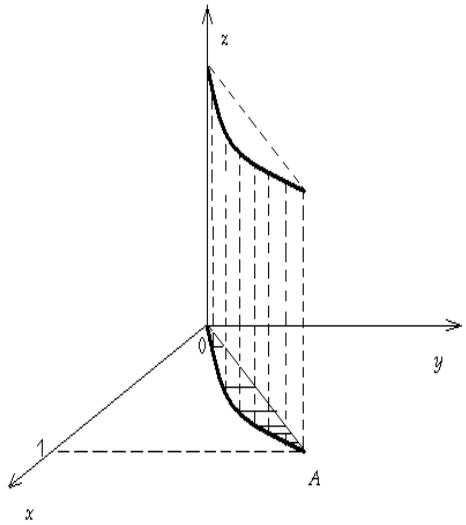
$$= \left(\frac{x^3}{6} - \frac{3x^2}{2} + 4{,}5x\right)\Bigg|_1^3 = \frac{27}{6} - \frac{27}{2} + 13{,}5 - \frac{1}{6} + \frac{3}{2} - 4{,}5$$

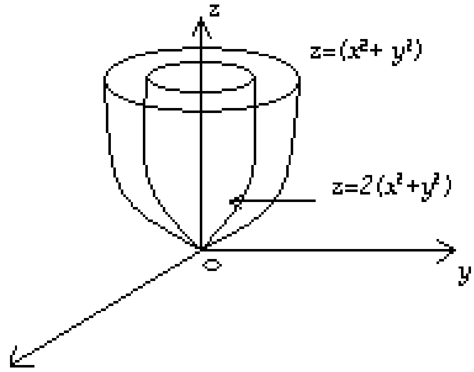
$$= 13{,}5 - 12 - \frac{1}{6} = 1{,}5 - \frac{1}{6} = \frac{4}{3}$$

kub birlik bo‘ladi.

2-misol. $z = x^2 + y^2$, $z = 2(x^2 + y^2)$, $y = x$, $y = x^2$ sirtlar bilan chegaralangan jismning hajmini toping.

Yechish. Berilgan sirtlarning dastlabki ikkitasi aylanma paraboloid bo‘lib, uchinchisi Oz o‘qidan o‘tuvchi tekislikdir, $y = x^2$ esa yasovchilari Oz o‘qiga parallel bo‘lgan parabolik silindr ekani bizga ma’lum.





27-rasm

Demak, tasvirlangan jismning $z = x^2 + y^2$ va $z = 2(x^2 + y^2)$ aylanma paraboloidlar orasiga olingan bo‘lagining hajmini hisoblash talab etiladi. $\begin{cases} y = x \\ y = x^2 \end{cases}$ sistemadan x ni topamiz:

$$\begin{cases} y = x \\ y = x^2 \end{cases} \Rightarrow \begin{cases} x^2 = x \\ y = x^2 \end{cases} \Rightarrow \begin{cases} x^2 - x = 0 \\ y = x^2 \end{cases} \Rightarrow \begin{cases} x_1 = 0, x_2 = 1 \\ y = x^2 \end{cases}$$

Bundan $V = \{x, y, z \,:\, 0 \le x \le 1\,;\; x^2 \le y \le x\,;\; x^2 + y^2 \le z \le 2(x^2 + y^2)\}$. Bularni (3.5) formulada o‘rniga qo‘ysak, quydagi kelib chiqadi:

$$V=\iiint_V dv=\int_0^1 dx\int_{x^2}^{x} dy\int_{x^2+y^2}^{2(x^2+y^2)} dz=\int_0^1 dx\int_{x^2}^{x}(x^2+y^2)dy=$$

$$=\int_0^1\left(x^2y+\frac{y^3}{3}\right)\Bigg|_{x^2}^{x} dx=\int_0^1\left(x^3+\frac{x^3}{3}-x^4-\frac{x^6}{3}\right)dx$$

$$=\int_0^1\left(\frac{4}{3}x^3-x^4-\frac{x^6}{3}\right)dx=\left(\frac{x^4}{3}-\frac{x^5}{5}-\frac{x^7}{21}\right)\Bigg|_0^1=$$

$$=\frac{1}{3}-\frac{1}{5}-\frac{1}{21}=\frac{35-21-5}{105}=\frac{9}{105}=\frac{3}{55}\text{ (kub birlik).}$$

3-misol. $(x^2+y^2+z^2)^3=a^3xyz$ sirt bilan chegaralangan jismning hajmini toping.

Yechish. Tenglikning chap tomoni kvadratlar yig‘indisi bo‘lgani uchun manfiy bo‘la olmaydi. Demak, tenglik o‘rinli bo‘lishi uchun xyz ko‘paytma manfiy bo‘lmasligi kerak. $x\ y\ z\Rightarrow$

$\Rightarrow$ 1) $x\geq 0\,, y\geq 0\,, z\geq 0\,;$ 2) $x\geq 0\,, y\leq 0\,, z\leq 0$

3) $x\leq 0\,, y\geq 0\,, z\leq 0;$ 4) $x\leq 0\,, y\leq 0\,, z\geq 0$

Berilgan jism kordinatalar sistemasining 4ta oktantida yotar ekan. Agar $M(x,y,z)$ qaralayotgan jismning ixtiyoriy nuqtasi bo‘lsa, unga OX o‘qiga nisbatan simmetrik $M_1(x,-y,-z)$ nuqta, OY o‘qiga nisbatan simmetrik $M_2(-x,y,-z)$ nuqta va OZ o‘qiga nisbatan simmetrik $M_3(-x,-y,z)$ nuqtalar mavjuddir. Bundan berilgan jismning yuqorida ko‘rsatilgan 4 ta oktantdagi bo‘laklarining kattaliklari bir hil ekani kelib chiqadi. Shuning uchun jisimning 1 oktantdagi bo‘lagining hajmini hisoblab, natijani 4 ga ko‘paytirsak, berilgan jismning hajmi kelib chiqadi. Sferik koordinatalar kiritamiz:

$$x=r\sin\psi\cos\varphi\,;\qquad y=r\sin\psi\sin\varphi\,;\qquad z=r\cos\psi$$

1- oktantda $0\leq\psi\leq\frac{\pi}{2}$ va $0\leq\varphi\leq\frac{\pi}{2}$ bo‘ladi. Yakobian $|I(r,\varphi,\psi)|=r^2\sin\psi$ sferik koordinatalarni berilgan tenglamada o‘rniga qo‘ysak, quydagiga ega bo‘lamiz:

$$[r^2\sin^2\psi(\cos^2\varphi+\sin^2\varphi)+r^2\cos^2\psi]^3=a^3r^3\sin^2\psi\cos\varphi\cos\psi\sin\psi\Rightarrow$$

$$\Rightarrow\quad r^3=a^3\sin^2\psi\cos\varphi\cos\psi\sin\varphi\Rightarrow\qquad r=a\sqrt[3]{\sin^2\varphi\cos\varphi\cos\psi\sin\psi}$$

Demak

$$V=\iiint\limits_V dV=4\int\limits_0^{\frac{\pi}{2}}d\psi\int\limits_0^{\frac{\pi}{2}}d\varphi\int\limits_0^{a\sqrt[3]{sin^2\psi cos\varphi cos\psi sin\varphi}}r^2 sin\psi dr=$$

$$=\frac{4}{3}\int\limits_0^{\frac{\pi}{2}}d\psi\int\limits_0^{\frac{\pi}{2}}a^3 sin^3\psi cos\varphi\ cos\psi\ sin\varphi\ d\varphi=\frac{4}{3}a^3\int\limits_0^{\frac{\pi}{2}}cos\varphi sin\varphi\ d\varphi\cdot$$

$$\cdot\int\limits_0^{\frac{\pi}{2}}sin^3\psi cos\psi d\psi=\frac{4}{3}a^3\left[\frac{sin^2\varphi}{2}\bigg|_0^{\frac{\pi}{2}}\cdot\frac{sin^4\varphi}{4}\bigg|_0^{\frac{\pi}{2}}\right]=\frac{4}{3}a^3\cdot\frac{1}{2}\cdot\frac{1}{4}=\frac{a^3}{6}$$

(kub birlik).

4-misol $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ sirt bilan chegaralangan jismning hajmini toping.

Yechish. Tenglikning har ikkala tomoni kvadratlar yig‘indisi bo‘lgani uchun berilgan jisimning hamma oktantlardagi bo‘laklarining kattaliklari bir xil bo‘ladi.

Shuning uchun jisimning 1 oktantdagi bo‘lagining hajmini hisoblab , natijani 8 ga ko‘paytirsak , berilgan jisimning hajmi kelib chiqadi. Umumlashgan sferik koordinatalardan foydalanamiz,

$x=arsin\psi cos\varphi$; $y=brsin\psi sin\varphi$;

$$z=crcos\varphi\quad\left(0\le\varphi\le\frac{\pi}{2}\ ;\ 0\le\psi\le\frac{\pi}{2}\right)$$

Yakobian $|I(r,\varphi,\psi)|=abcr^2 sin\varphi$

$x\ ,y\ ,z$ lar uchun sferik koordinatalarni berilgan tenglamaga qo‘yib quyidagini topamiz:

$$[r^2sin^2\varphi(cos^2\psi+sin^2\psi)+r^2cos^2\varphi]^2=r^2sin^2\varphi(cos^2\psi+sin^2\psi)\Rightarrow$$

$$\Rightarrow\quad r^4=r^2sin^2\varphi\ \Rightarrow\ r^2sin^2\varphi\ \Rightarrow r=sin\varphi$$

Bundan

$$V=\iiint\limits_V dV=8\int\limits_0^{\frac{\pi}{2}}d\psi\int\limits_0^{\frac{\pi}{2}}d\varphi\int\limits_0^{sin\varphi}abcr^2 sin\varphi dr=8abc$$

(kub birlik).

5-misol. $(x+2y+z)^2+(y+2z)^2+(2x+z)^2=R^2$ ellipsoidning hajmini toping .

Yechish. O‘zgaruvchilarni $\begin{cases} x+2y+z=u \\ y+2z=\vartheta \\ 2x+z=\omega \end{cases}$ formulalar yordamida almashtiramiz. Natijada ellipsoidning tenglamasi $u^2+v^2+\omega^2=R^2$ ko‘rinishni oladi, bu esa $u\,\vartheta\,\omega$ koordinatalar sistemasida sfera tenglamasidir. Yuqoridagi almashtirishlar orqali V soha (ellipsoid) V' sohaga (sharga) o‘tadi

$$V'=\Big\{(u,\vartheta,\omega)\in R^3 : -R\le u\le R\,;\ -\sqrt{R^2-u^2}\le\vartheta$$
$$\le\sqrt{R^2-u^2}\,;\quad |\omega|\le\sqrt{R^2-u^2-\vartheta^2}\Big\}$$

Endi tenglamalar sistemasini yechib x, y, z larni aniqlaymiz:

$$\Delta=\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}=1+8-2=7;$$

$$\Delta_x=\begin{vmatrix} u & 2 & 1 \\ \vartheta & 1 & 2 \\ \omega & 0 & 1 \end{vmatrix}=u-2\vartheta+3\omega;$$

$$\Delta_y=\begin{vmatrix} 1 & u & 1 \\ 0 & \vartheta & 2 \\ 2 & \omega & 1 \end{vmatrix}=\vartheta+4u-2\vartheta-2\omega=4u-\vartheta-2\omega;$$

$$\Delta_z=\begin{vmatrix} 1 & 2 & u \\ 0 & 1 & \vartheta \\ 2 & 0 & \omega \end{vmatrix}=\omega+4\vartheta-2u.$$

Bundan

$$x=\frac{\Delta x}{\Delta}=\frac{1}{7}(u-2\vartheta+3\omega),$$
$$y=\frac{\Delta y}{\Delta}=\frac{1}{7}(4u-\vartheta-2\omega),$$
$$z=\frac{\Delta z}{\Delta}=\frac{1}{7}(\omega+4\vartheta-2u),$$

kelib chiqadi.

$$\frac{\partial x}{\partial u}=\frac{1}{7}\ ;\qquad \frac{\partial x}{\partial\vartheta}=-\frac{2}{7}\ ;\qquad \frac{\partial x}{\partial\omega}=\frac{3}{7}\ ;$$
$$\frac{\partial y}{\partial u}=\frac{4}{7}\ ;\qquad \frac{\partial y}{\partial\vartheta}=-\frac{1}{7}\ ;\qquad \frac{\partial y}{\partial\omega}=-\frac{2}{7};$$

$$\frac{\partial z}{\partial u} = -\frac{2}{7}; \qquad \frac{\partial z}{\partial \vartheta} = \frac{4}{7}; \qquad \frac{\partial z}{\partial \omega} = \frac{1}{7}.$$

Yakobian

$$I(u,\vartheta,\omega) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial \vartheta} & \frac{\partial z}{\partial \omega} \end{vmatrix} = \begin{vmatrix} \frac{1}{7} & -\frac{2}{7} & \frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{vmatrix} = \frac{1}{7^3}\begin{vmatrix} 1 & -2 & 3 \\ 4 & -1 & -2 \\ -2 & 4 & 1 \end{vmatrix} = V$$

$$\frac{1}{7^3}(-1-8+48-6+8+8) = \frac{1}{7^3}\cdot 49 = \frac{1}{7}$$

$$= \iiint_V dV$$

$$= \iiint_V |I(u,\vartheta,\omega)|\, du\, d\vartheta\, d\omega$$

$$= \frac{8}{7}\int_0^R du \int_0^{\sqrt{R^2-u^2}} dv \int_0^{\sqrt{R^2-u^2-v^2}} d\omega = \frac{8}{7}\int_0^R du \int_0^{\sqrt{R^2-u^2}} \sqrt{R^2-u^2-\vartheta^2}\, d\vartheta;$$

Lekin $\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi a^2}{4}$ ekani bizga ma'lum. Bu yerda a ni $\sqrt{a^2-u^2}$ ga almashtirsak,

$\int_0^{\sqrt{R^2-u^2}} \sqrt{R^2-u^2-\vartheta^2}\,d\vartheta = \frac{\pi}{4}(R^2-u^2)$ kelib chaqdi .

Natijada $V = \frac{8}{7}\int_0^R \frac{\pi}{4}(R^2-u^2)\,du = \frac{2\pi}{7}\left(R^2u - \frac{u^3}{3}\right)\Big|_0^R = \frac{2\pi}{7}\left(R^3 - \frac{R^3}{3}\right) = \frac{2\pi}{7}\cdot\frac{2R^3}{3} = \frac{4\pi R^3}{21}$ (kub birlik).

Bu yerda ellipsoidning hajmini hisoblashda uning birinchi oktantdagi bo'lagining hajmini hisoblab, natijani 8 ga ko'paytirdik.

Topshiriqlar

Quyidagi jismlar bilan chegaralangan figura hajmini toping.

1.$x = 0$; $y = 0$, $z = 0$, $x + y + z = 2$ tekisliklar.

2.$z = x^2 + y^2$, $z = x^2 + 2y^2$ paraboloidlar, $y = x$, $y = 2x$ va $x = 1$

tekisliklar.
3. $V=\{(x,y,z): x^2+2y^2-z^2 \le a^2,\ x^2+2y^2 \ge 4z^2\}$.
4. $V=\{(x,y,z): 2(x^2+2y^2) \le 2az \le 3a\sqrt{a^2-x^2-2y^2}\}$.
5. $V=\{(x,y,z): x^2+4y^2 \le z^2,\ a^2 \le x^2+4y^2+z^2 \le 9a^2\}$.
6. $V=\{(x,y,z): x^2+y^2+z^2 \le 2Rz,\ z^2tg^2\alpha \le x^2+y^2 \le z^2tg^2\beta,$
$0<\alpha<\beta<\pi/2\}$.
7. $V=\{(x,y,z): x^2+y^2+z^2 \le R^2,\ x^2+y^2+z^2 \le 2Rz\}$.
Quyidagi sirtlar bilan chegaralangan jismlaning hajmlarini hisoblang.
8. $(x^2+y^2+z^2)^2=a^3z$.
9. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{x^2y}{h^3}$.
10. $(x^2+y^2+z^2)^3=a^2z^4$.
11. $(x^2+y^2+z^2)^3=az(x^2+y^2)^2$.
12. $(x^2+y^2+z^2)^3=a^2y^2z^2$.
13. $(x^2+y^2+z^2)^3=a^3(x^3+y^3)$.
14. $(x^2+y^2+z^2)^3=a^3z(x^2-y^2)$.
15. $(x^2+y^2+z^2)^4=a^3z(x^4+y^4)$.
16. $(x^2+y^2+z^2)^2=a^3ze^{-\frac{x^2+y^2}{x^2+y^2+z^2}}$.
17. $(x^2+y^2+z^2)^2=a^6\sin^2\left[\frac{\pi z}{\sqrt{x^2+y^2+z^2}}\right]$.
18. $(x^2+y^2)^2+z^4=za^3$.
19. $(x^2+y^2)^2+z^4=a^3(y-x)$.
20. $(x^2+y^2)^3+z^6=3a^3z^3$.

Javoblar:

Uch karrali integrallar yordamida jismning hajmini hisoblash

2. $\frac{7}{12}$ 3. $\frac{29\pi\sqrt{2}a^3}{192}$ 4. $\frac{\pi\sqrt{2}a^3}{12}(3+2\sqrt{5})$ 5. $\frac{9\pi a^3}{3}(2-\sqrt{2})$ 6. $\frac{4}{3}\pi R^3(\cos^4\alpha-\cos^4\beta)$ 7. $\frac{5\pi R^3}{12}$ 8. $\frac{1}{3}\pi a^3$ 9. $\frac{\pi a^7b^4c}{192h^9}$ 10. $\frac{4\pi a^3}{21}$ 11. $\frac{\pi a^3}{168}$ 12. $\frac{32}{315}a^3$ 13. $\frac{5\sqrt{2}}{24}\pi a^3$ 14. $\frac{1}{3}a^3$ 15. $\frac{\pi a^3}{12}$ 16. $\frac{\pi}{3}(1-e^{-1})a^3$ 17. $\frac{8}{3}a^3$ 18. $\frac{\pi^2a^3}{6}$ 19. $\frac{2}{3}\pi a^3$ 20. $\frac{2\pi^2a^3}{3\sqrt{3}}$

3.3-§. Ikki karrali va uch karrali integrallarning mexanikaga tadbiqlari

Agar (P) biror tekis figura (masalan, plastinka) bo'lib, uning $M(x,y)$ nuqtalaridagi zichligi $\rho(M) = \rho(x,y)$ bo'lsa, u holda (p) figuraning (plastinkaning) massasi

$$m = \iint_P \rho(x,y)dp \qquad (3.6)$$

formula bilan aniqlanadi . Agar (P) figura bir jinsli bo'lsa, ya'ni hamma nuqtalarida zichligi bir xil bo'lsa , u holda $\rho(x,y) = a\ (const)$ bo'ladi, (3.6) formula esa,

$$m = a\iint_P dp \qquad (3.7)$$

ko'rinishni oladi .

OX va OY koordinata o'qlariga nisbatan statik momentlar quyidagi formulalar bilan aniqlanadi.

$$M_x = \iint_P y\rho(x,y)dp \qquad M_y = \iint_P x^2\rho(x,y)dp \qquad (3.8)$$

Inersiya momentlari esa,

$$I_x = \iint_P y^2\rho(x,y)dp, \qquad I_y = \iint_P x^2\rho(x,y)dp, \qquad (3.9)$$

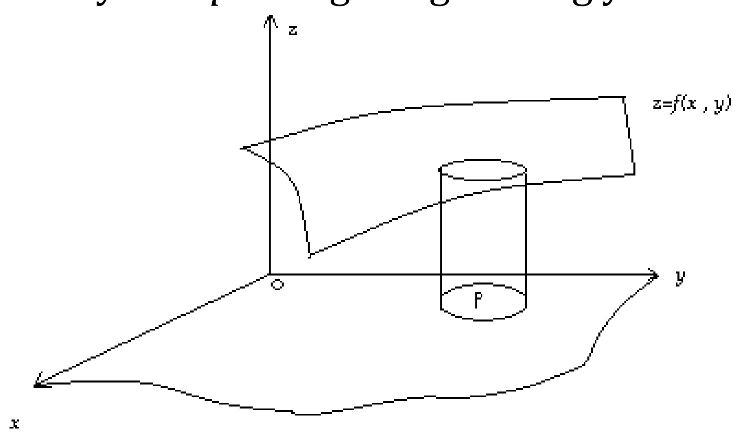
formulalar yordamida topiladi. Qaralayotgan jismning og'irlik markazining koordinatalari quyidagicha topiladi:

$$x_0 = \frac{\iint_P x\rho(x,y)dp}{m}; \qquad y_0 = \frac{\iint_P y\rho(x,y)dp}{m} \qquad (3.10)$$

Agar jism bir jinsli bo'lsa, ya'ni $\rho(x,y) = const$ bo'lsa, (3.10) formulalar soddalashadi va

$$x_0 = \frac{\iint_P xdp}{p}; \qquad y_0 = \frac{\iint_P ydp}{p} \qquad (3.11)$$

ko'rinishini oladi. Bu yerda p berilgan figuraning yuzi.



28-rasm

Yasovchilari Oz o'qiga parallel, ostki asosii xOy tekisligida va ustki asosi esa $z = f(x,y)$ sirt bilan chegaralangan silindrik g'o'lani qaraymiz.

G‘o‘la bir jinsli bo‘lsin, ya’ni $\rho(x,y)=const$. Soddalik uchun $\rho(x,y)=1$ deb olamiz. Shu g‘o‘lani *xy, zx* va *yz* koordinata tekisliklariga nisbatan statik momentlari deb quyidagi formulalar bilan aniqlanuvchi kattaliklarga aytiladi:

$$M_{xy}=\frac{1}{2}\iint_{P} z^2 dp; \qquad M_{zx}=\iint_{P} yzdp; \qquad M_{yz}=\iint_{P} xzdp \tag{3.12}$$

Bular yordamida g‘o‘laning og‘irlik markazining koordinatalarini osongina aniqlay olamiz.

$$x_0=\frac{M_{yz}}{V}=\frac{\iint_{P} xzdp}{V}; \qquad y_0=\frac{M_{xz}}{V}=\frac{\iint_{P} yzdp}{V}; \qquad z_0=\frac{M_{xy}}{V}=\frac{\frac{1}{2}\iint_{P} z^2dp}{V}; \tag{3.13}$$

Shunga o‘xshash silindrik g‘o‘laning koordinata tekisliklariga nisbatan inersiya momentlari quyidagicha aniqlanadi:

$$I_{zx}=\iint_{P} y^2zdp, \qquad I_{yz}=\iint_{P} x^2zdp \tag{3.14}$$

Oz o‘qiga nisbatan inersiya momenti esa

$$I_z=I_{zx}+I_{yz}=\iint_{P}(x^2+y^2)zdp \tag{3.15}$$

formula bilan hisoblanadi. Agar g‘o‘laning yasovchilari *Ox* yoki *Oy* o‘qiga parallel bo‘lsa, u holda (3.12), (3.13), (3.14) va (3.15) formulalarda z o‘zgaruvchi x yoki y bilan almashtiriladi. Agar qaralayotgan jismning zichligi *xOy* tekisligiga parallel qismda yotuvchi nuqtalari uchun $\rho(x,y)$ ga ($\rho(x,y)\neq 1$) teng bo‘lib, *Oz* o‘qiga parallel ustunchalarida o‘zgarmas bo‘lsa, u holda (3.12), (3.13), (3.14) va (3.15) formulalarda integral ostidagi ifoda $\rho(x,y)$ ga ko‘paytiriladi. Bordi-yu jismning hamma nuqtalarida uning zichligi o‘zgaruvchi bo‘lsa, ya’ni zichligi z ga ham bog‘liq bo‘lsa, u holda koordinata tekisliklariga va o‘qlariga nisbatan statik va inersiya momentlarini hisoblash uchun, shuningdek jismning og‘irlik markazining koordinatalarini topish uchun ikki karrali integralning kuchi yetmaydi. Bu holda uch karrali integralga murojaat qilishga to‘g‘ri keladi. Jismning hajmi V bo‘lib, ixtiyoriy $M(x,y,z)$ nuqtaning zichligi $\rho(M)=\rho(x,y,z)$ bo‘lsa, uning massasi

$$m=\iiint\limits_{V}\rho(x,y,z)dV \tag{3.16}$$

formula bilan aniqlanadi.

Koordinata tekisliklariga nisbatan statik momentlari

$$M_{xy} = \iiint\limits_V z\rho(x,y,z)\,dV; \quad M_{zx} = \iiint\limits_V y\rho(x,y,z)\,dV; \quad M_{yz} = \iiint\limits_V x\rho(x,y,z)\,d \qquad (3.17)$$

Og‘irlik markazining koordinatalari

$$x_0 = \frac{\iiint_V x\rho(x,y,z)\,dV}{m};\ y_0 = \frac{\iiint_V y\rho(x,y,z)\,dV}{m};\ z_0 = \frac{\iiint_V z\rho(x,y,z)\,dV}{m} \qquad (3.18)$$

Koordinata o‘qlariga nisbatan inersiya momentlari

$$I_x = \iiint\limits_V (y^2+z^2)\rho(x,y,z)dV\,; \qquad I_y = \iiint\limits_V (z^2+x^2)\rho(x,y,z)dV\,;$$

$$I_z = \iiint\limits_V (x^2+y^2)\rho(x,y,z)dV\,; \qquad (3.19)$$

formula bilan aniqlanadi.

Koordinata tekisliklariga nisbatan inersiya momentlari esa quyidagicha topiladi:

$$I_{xy} = \iiint\limits_V z^2\rho(x,y,z)dV; \qquad I_{xz} = \iiint\limits_V y^2\rho(x,y,z)dV; \qquad I_{zy} = \iiint\limits_V x^2\rho(x,y,z)dV;$$

Qaralayotgan jism massasi $A(\xi,\eta,\zeta)$ nuqtani Nyuton qonuni bo‘yicha tortish kuchini $\bar{F}$ bilan belgilasak, $\bar{F}$ ning o‘qlardagi proyeksiyalari quyidagi formula bilan aniqlanadi: (A nuqta massasi m ga teng bo‘lsa)

$$F_x = k\iiint_V m\frac{x-\xi}{r^3}\rho(x,y,z)dV; \qquad F_y = k\iiint_V m\frac{y-\eta}{r^3}\rho(x,y,z)dV;$$

$$F_z = k\iiint_V m\frac{z-\zeta}{r^3}\rho(x,y,z)dV; \qquad (3.20)$$

Bu yerda

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}\,; \qquad \bar{F} = \{F_x, F_y, F_z\}$$

k- Nyuton koeffitsenti (gravitatsion doimiylik).

Sunga o‘xshash berilgan jismning A nuqtaga potensialini aniqlaymiz:

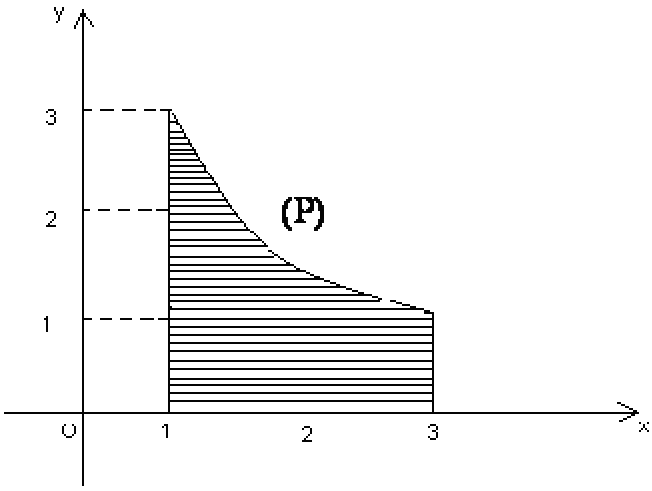
$$W = \iiint_V \frac{\rho(x,y,z)dV}{r} \qquad (3.21)$$

Bu yerda ikkita holga e’tibor berish kerak:

1) Agar $A(\xi,\eta,\zeta)$ nuqta berilgan jismdan tashqarida yotsa, u holda (3.20), (3.21) integrallar xosmas integrallar hisoblanadi, lekin bu holda ham (3.20), (3.21) integrallar mavjuddir. Endi tipik misollar yechishga o‘tamiz.

1-misol. (P) tekis figura egri chiziqli trapetsiya ko‘rinishida bo‘lib, $x = 1,\ x = 3\ va\ y = 0$ to‘g‘ri chiziqlar, $y = \frac{3}{x}$ giperbola bilan chegaralangan bo‘lsin. Uning hamma nuqtalarida zichligi $\rho(x,y) = 1$ bo‘lsa shu tekis figuraning massasini va koordinata o‘qlariga nisbatan statik momentlarni toping.

Yechish.



29-rasm

$P = \{(x,y) \in R^2 : 1 \le x \le 3; \quad 0 \le y \le \frac{3}{x}\}$, (3.6) formulada $\rho(x,y) = 1$ qo‘ysak, figuraning massasi kelib chiqadi.

$$m = \iint\limits_P dp = \int_1^3 dx \int_0^{\frac{3}{x}} dy = \int_1^3 \frac{3}{x} dx = 3\ln x|_1^3 = 3\ln 3$$

Shunga o‘xshash (3) formulalarda ham $\rho(x,y) = 1$ qo‘ysak

$$M_x = \iint\limits_P ydp = \int_1^3 dx \int_0^{\frac{3}{x}} ydy$$

$$= \int_1^3 \frac{y^2}{2}\Big|_0^{\frac{3}{x}} dx = \int_1^3 \frac{9}{2x^2} dx = -\frac{9}{2x}\Big|_1^3 = -\frac{9}{6} + \frac{9}{2} = \frac{9}{2} + \frac{3}{2} = 3$$

$$M_y = \iint\limits_P xdp = \int_1^3 dx \int_0^{\frac{3}{x}} xdy = \int_1^3 x \cdot \frac{3}{x} dx = 3 \cdot 2 = 6$$

Ox va Oy o'qlariga nisbatan statik momentlar kelib chiqadi.

2-misol. $z = 0, z = ky \quad (k > 0)$ tekisliklar $x^2 + y^2 = a^2$ slindr bilan chegaralangan bir jinsli silindrik g‘o‘la kesimi og‘irlik markazining koordinatalarini toping $(y \ge 0)$.

Yechish. Qaralayotgan jism bir jinsli bo‘lgani uchun zichligi $\rho(x,y,z) = 1$ deb olamiz. $y \ge 0$ bo‘lgani uchun $z = ky \ge 0$ bo‘ladi.

Demak, $p = \{ (x,y) \in R^2 : -a \le x \le a; \quad 0 \le y \le \sqrt{a^2 - x^2} \}$ - bu slindrik g‘o‘laning xOy tekisligida yotuvchi asosi. (3.12) formulalarga ko‘ra quyidagilarni topamiz:

$$M_{xy} = \frac{1}{2}\iint\limits_{P} z^2 dp$$

$$= \frac{1}{2}\int\limits_{-a}^{a} dx \int\limits_{0}^{\sqrt{a^2-x^2}} k^2 y^2 dy$$

$$= \frac{k^2}{2}\int\limits_{-a}^{a} \frac{(a^2 - x^2)^{\frac{3}{2}}}{3} dx = \frac{\pi}{16} k^2 a^2$$

$$M_{zx} = \iint\limits_{P} yzdp = k \int\limits_{-a}^{a} dx \int\limits_{0}^{\sqrt{a^2-x^2}} y^2 dy = \frac{k}{3}\int\limits_{-a}^{a} (a^2 - x^2)^{\frac{3}{2}} dx = \frac{\pi}{8} k a^4$$

$$M_{yz} = \iint\limits_{P} xzdp = k \int\limits_{-a}^{a} dx \int\limits_{0}^{\sqrt{a^2-x^2}} xydy =$$

$$= k \int\limits_{-a}^{a} \frac{xy^2}{2}\Bigg|_{0}^{\sqrt{a^2-x^2}} dx = \frac{k}{2}\int\limits_{-a}^{a} (a^2 x - x^3) dx = 0$$

Berilgan jisimning hajmi V bo‘lsa, u holda

$V = \iint_{P} zdp = \int_{-a}^{a} dx \int_{0}^{\sqrt{a^2-x^2}} kydy = \frac{k}{2}\int_{-a}^{a} (a^2 - x^2) dx =$

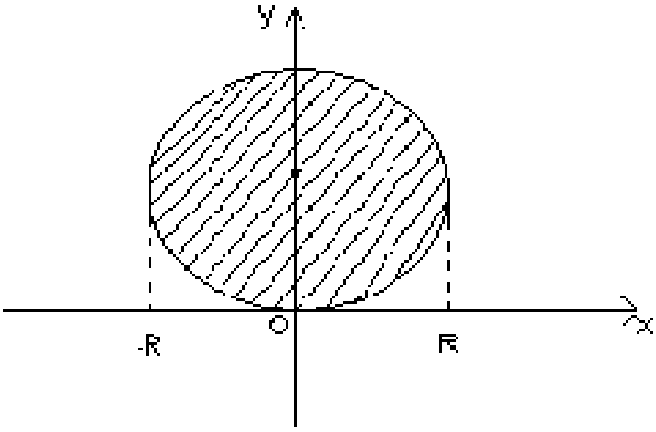
$\frac{k}{2}(a^2 x - \frac{x^3}{3})\Big|_{-a}^{a} = k\,(a^3 - \frac{a^3}{3}) = \frac{2}{3} k a^3$

Bularni (3.13) formulalarga qo‘ysak, og‘rlik markazining koordinatalari kelib chiqadi:

$x_0 = \frac{M_{yz}}{V} = 0; \quad y_0 = \frac{M_{xz}}{V} = \frac{\frac{\pi}{3} k a^4}{\frac{2}{3} k a^3} = \frac{3\pi}{16} a; \quad z_0 = \frac{M_{xy}}{V} = \frac{\frac{\pi}{16} k^2 a^4}{\frac{2}{3} k a^3} = \frac{3\pi}{32} ka.$

3-misol. Tekis figura R radusli doira shaklda bo‘lsa, uning urunmasiga nisbatan inersiya momentini toping.

Yechish. Bu yerda shu narsaga e’tibor qilish kerakki, doira aylanasiga urinmalar soni cheksiz ko‘p, lekin shu urinmalarga nisbatan inersiya momentlari hamma urinmalar uchun bir xil bo‘ladi. Shuning uchun Ox o‘qiga koordinatalar boshida urinuvchi doiraning shu Ox o'qiga nisbatan inersiya momentini aniqlasak, masala yechilgan bo'ladi.



30-rasm

Aylana tenglamasi $x^2 + (y - R)^2 = R^2$, bundan $R - \sqrt{R^2 - x^2} \leq y \leq R + \sqrt{R^2 - x^2}$

$$P = \{ (x, y) \in R^2 : -R \leq x \leq R ; \quad R - \sqrt{R^2 - x^2} \leq y \leq R + \sqrt{R^2 - x^2}\}$$

(3.9) formulaga ko'ra quydagini topamiz $(\rho(x, y) = 1)$:

$$I_x = \iint_P y^2 dP = \int_{-R}^{R} dx \int_{R-\sqrt{R^2-x^2}}^{R+\sqrt{R^2-x^2}} y^2 dy =$$

$$\frac{1}{3}\int_{-R}^{R} \left[\left(R + \sqrt{R^2 - x^2}\right)^3 - \left(R - \sqrt{R^2 - x^2}\right)^3\right] dx =$$

$$-3R(R^2 - x^2) + (R^2 - x^2)^{\frac{3}{2}} dx = \frac{2}{3}\int_0^R 6R^2\sqrt{R^2 - x^2} + 2(R^2 - x^2)^{\frac{3}{2}}\Big]$$

$x = R\sin t$ belgilash kiritamiz.

$$dx = R\cos t dt \;; \quad x = 0 \Rightarrow t = 0 \,; \quad x = R \Rightarrow t = \frac{\pi}{2}$$

Bularni o'rniga qo'ysak,

$$I_x = \frac{4}{3}\int_0^{\frac{\pi}{2}} (3R^2 \cdot R\cos t + R^3\cos^3 t)R\cos t dt$$

$$= \frac{4}{3}R^4\int_0^{\frac{\pi}{2}} (3\cos^2 t + \cos^4 t)dt =$$

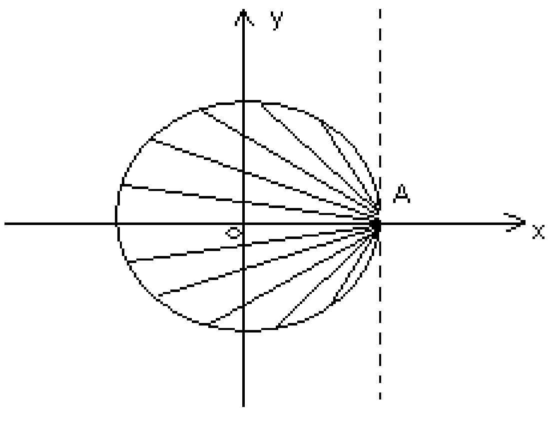
$$\frac{4}{3}R^2\int_0^{\frac{\pi}{2}} \left[3 \cdot \frac{1+\cos 2t}{2} + \left(\frac{1+\cos 2t}{2}\right)^2\right]dt$$

$$= \pi R^4 + \frac{1}{3}R^4\int_0^{\frac{\pi}{2}} \left(1 + \frac{1+\cos 4t}{2}\right)dt =$$

$$= \pi R^4 + \frac{1}{3}R^4 \cdot \frac{3}{2} \cdot \frac{\pi}{2} = \pi R^4 + \frac{\pi R^4}{4} = \frac{5}{4}\pi R^4.$$

4-misol. $\boldsymbol{x^2 + y^2 \leq a^2}$ doiraviy plastinkaning har bir $M(x, y)$ nuqtasidagi zichligi shu nuqtadan $A(a, 0)$ nuqtagacha bo‘lgan masofaga proporsional bo‘lib, plastinkaning markazida $a$ga teng bo‘lsa, uning og‘irlik markazi koordinatlarini toping.

Yechish.



31-rasm

$P = \{\, (x, y) \in R^2: -a \leq x \leq a\,; \quad -\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}\}$.

Shartga ko‘ra $M(x, y)$nuqtadagi zichligi $\rho(M) = |MA| \cdot k$

demak,$\rho(x, y) = k\sqrt{(x-a)^2 + y^2}$, $\rho(0,0) = a$ bo‘lgani uchun

$$k \cdot \sqrt{(0-a)^2 + 0^2} = a \Rightarrow ka = a \Rightarrow k = 1$$

Bundan $\rho(x, y) = k\sqrt{(x-a)^2 + y^2}$ kelib chiqadi.

Plastinkaning massasini topamiz.

$$m=\iint\limits_{P}\rho(x,y)dp=\int\limits_{-a}^{a}dx\int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\sqrt{(x-a)^2+y^2}dy$$

$$\left.\begin{aligned}x-a&=r\cos\varphi\\ y&=r\sin\varphi\end{aligned}\right\}$$

Bu yerda x, y larni qiymatlarini platinka aylanasining tenglamasiga qo‘ysak, r ning φ orqali ifodasi kelib chiqadi.

$$(a+r\cos\varphi)^2+r^2\sin^2\varphi=a^2\Rightarrow a^2+2ar\cos\varphi+r^2(\cos^2\varphi+\sin^2\varphi)$$
$$=a^2\Rightarrow r=-2a\cos\varphi,\quad\left(\frac{\pi}{2}\leq\varphi\leq\frac{3\pi}{2}\right).$$

Qutb boshi *A(a,0)* nuqtada yotadi (shakilga qarang).

Natijada

$$m=\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}}d\varphi\int\limits_{0}^{-2a\cos\varphi}r^2dr$$
$$=\frac{1}{3}\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(-8a^3)\cos^3\varphi d\varphi=-\frac{8a^3}{3}\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(1-\sin^2\varphi)d(\sin\varphi)=$$
$$=-\frac{8a}{3}\left(\sin\varphi-\frac{\sin^3\varphi}{3}\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}=-\frac{8a^2}{3}\left(-1-1+\frac{1}{3}+\frac{1}{3}\right)=-\frac{8a^2}{3}\left(-\frac{1}{3}\right)$$
$$=\frac{32a^2}{9}$$

Endi o‘qlarga (Ox, Oy) nisbatan statik momentlarni aniqlaymiz.

$$M_y=\iint\limits_{P}y\rho(x,y)dp$$
$$=\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}}d\varphi\int\limits_{0}^{-2a\cos\varphi}r^3\sin\varphi dr=\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{16a^4\cos^4\varphi}{4}\sin\varphi d\varphi=$$

$$= -4a^4 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^4\varphi d(\cos\varphi) = -4a^4 \frac{\cos^5\varphi}{5}\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0$$

$$M_y = \iint\limits_{P} y\rho(x,y)dp = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\varphi \int_{0}^{-2a\cos\varphi} (r\cos\varphi + a)r^2 dr =$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{r^4}{4}\cos\varphi + a \cdot \frac{r^3}{3}\right)\Bigg|_0^{-2a\cos\varphi} d\varphi =$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(4a^4\cos^5\varphi - \frac{a}{3}8a^3\cos^3\varphi\right) d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[(4a^4(1-\sin^2\varphi)^2 d\varphi - \frac{8}{3}a^4(1-\sin^2\varphi)\right] d(\sin\varphi)$$

$$= \left[4a^4\left(\sin\varphi - 2\frac{\sin^3\varphi}{3} + \frac{\sin^5\varphi}{5}\right) - \frac{8}{3}a^4\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\right]\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= 4a^4\left(-2 + \frac{4}{3} - \frac{2}{5}\right) -$$

$$-\frac{8}{3}a^4\left(-2 + \frac{2}{3}\right) = 4a^4\frac{-30+20-6}{15} - \frac{8}{3}a^4\left(-\frac{4}{3}\right) =$$

$$= -\frac{64}{15}a^4 + \frac{32}{9}a^4 = \frac{-192+160}{45}a^4 = -\frac{32}{15}a^4$$

Bularni (3.10) formulalarga qo‘yib, og‘irlk markazining koordinatalarini topamiz:

$$x_0 = \frac{M_y}{m} = -\frac{32}{45}a^4 \ : \ \frac{32}{9}a^3 = -\frac{9}{5},$$

$$y_0 = \frac{M_x}{m} = \frac{0}{m} = 0, \qquad \left(-\frac{a}{3};\ 0\right).$$

5-misol. Birlik kub $(0 \le x \le 1;\quad 0 \le y \le 1;\quad 0 \le z \le 1)$ har bir $M(x, y, z)$ nuqtasidagi zichligi $\rho(x, y, z) = x + y + z$ formula bilan berilgan.

Yechish. (II) formulaga koʻra

$$m = \iiint\limits_V \rho(x, y, z)dV$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 (x + y + z)dz$$

$$= \int_0^1 dx \int_0^1 \left(xz + yz + \frac{z^2}{2}\right)\Bigg|_0^1 dy =$$

$$= \int_0^1 dx \int_0^1 \left(x + y + \frac{1}{2}\right) dy = \int_0^1 \left(xy + \frac{y^2}{2} + \frac{1}{2}y\right)\Bigg|_0^1 dx = \int_0^1 (x + 1)dx$$

$$= \left(\frac{x^2}{2} + x\right)\Bigg|_0^1 = \frac{1}{2} + 1 = 1{,}5$$

6-misol. $\boldsymbol{z = x^2 + y^2}$ paraboloid $\boldsymbol{x + y = a}$, $\boldsymbol{x = 0}$; $\boldsymbol{y = 0}$; $\boldsymbol{z = 0}$;
tekisliklar bilan chegaralangan jismning ogʻirlik markazining koordinatalarini toping.

Yechish. Berilgan jism bir jinsli boʻlgani uchun $\rho = (x, y, z) = 1$ deb olamiz. Bu holda jism massasi son jihatdan uning hajmiga teng, ya'ni $m = V$ boʻladi V soha quyidagicha boʻladi:

$$V = \{(x, y, z) \in R^3 :\ 0 \le x \le a\,;\ \ 0 \le y \le a - x, 0 \le z \le x^2 + y^2\}$$

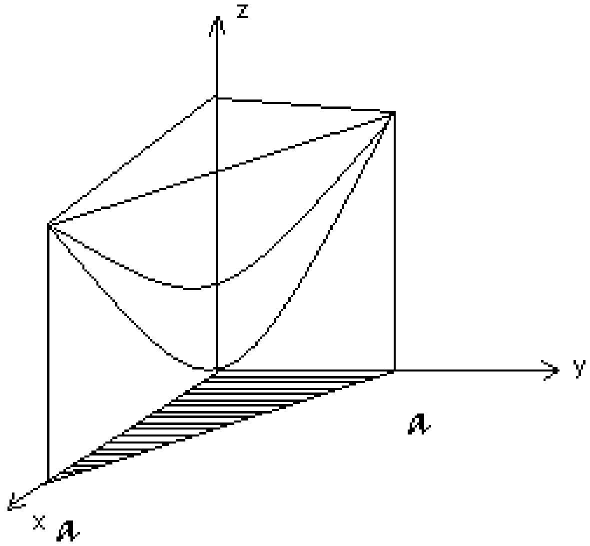
jism massasi

$$m = \iiint\limits_V dV = \int_0^a dx \int_0^{a-x} dy \int_0^{x^2+y^2} dz$$
$$=$$

$$= \int_0^a dx \int_0^{a-x} (x^2+y^2)dy = \int_0^a dx \int_0^{a-x} (x^2+y^2)dy =$$

$$\int_0^a \left(x^2y + \frac{y^3}{3}\right)\Bigg|_0^{a-x} dx = \int_0^a \left(ax^2 - x^3 + \frac{(a-x)^3}{3}\right)dx =$$

$$= \left(a\cdot\frac{x^3}{3} - \frac{x^4}{4} - \frac{(a-x)^4}{12}\right)\Bigg|_0^a = \frac{a^4}{3} - \frac{a^4}{4} + \frac{a^4}{12} = \frac{a^4}{6}$$



32-rasm

Endi (12) formulalar yordamida koordinata tekisliklariga nisbatan statik momentlarini topamiz:

$$M_{xy} = \iiint\limits_V zdV = \int_0^a dx \int_0^{a-\mathrm{x}} dy \int_0^{x^2+y^2} zdz = \int_0^a dx \int_0^{a-x} \frac{(x^2+y^2)^2}{2} dy =$$

$$= \int_0^a dx \int_0^{a-x} \frac{1}{2}(x^4 + 2x^2y^2 + y^4)dy$$

$$= \frac{1}{2}\int_0^a \left(x^4y + 2x^2\cdot\frac{y^3}{3} + \frac{y^5}{5}\right)\Bigg|_0^{a-x} dx =$$

$$=\frac{1}{2}\int_{0}^{a}\left[ax^4-x^5+\frac{2}{3}x^2(a^3-3a^2x+3ax^2-x^3)+\frac{1}{5}(a-x)^5\right]dx$$

$$=\frac{1}{2}\int_{0}^{a}\left[ax^4-x^5+\frac{2}{3}a^3x^2--2a^2x^3+2ax^4-\frac{2}{3}x^5+\frac{1}{5}(a-x)^5\right]dx$$

$$=\frac{1}{2}\left[a\cdot\frac{x^5}{5}-\frac{x^6}{6}+\frac{2}{3}a^3-\frac{x^3}{3}-2a^2\cdot\frac{x^4}{4}+2a\cdot\frac{x^5}{5}-\frac{2}{3}\cdot\frac{x^6}{6}-\right.$$

$$\left.-\frac{1}{5}\cdot\frac{(a-x)^6}{6}\right]\Bigg|_0^a=\frac{1}{2}a^6\left(\frac{1}{5}-\frac{1}{6}+\frac{2}{9}-\frac{1}{2}+\frac{2}{5}-\frac{1}{9}+\frac{1}{30}\right)=\frac{1}{2}a^3\left(\frac{1}{15}+\frac{1}{9}-\frac{1}{10}\right)=\frac{7}{180}a^6.$$

$$M_{zx}=\iiint\limits_V ydV=\int_0^a dx\int_0^{a-x}dy\int_0^{x^2+y^2}ydz=$$

$$=\int_0^a dx\int_0^{a-x}(x^2y+y^3)dy=\int_0^a\left(\frac{x^2y^2}{2}+\frac{y^4}{4}\right)\Bigg|_0^{a-x}dx=$$

$$=\int_0^a\left[\frac{1}{2}a^2x^2-ax^3+\frac{1}{2}x^4(a-x)^4\right]dx=\left(\frac{a^2}{2}\cdot\frac{x^3}{3}-a\cdot\frac{x^4}{4}+\frac{1}{2}\cdot\frac{x^5}{5}-\frac{1}{4}\cdot\frac{(a-x)^5}{5}\right)\Bigg|_0^a==a^5\left(\frac{1}{6}-\frac{1}{4}+\frac{1}{10}+\frac{1}{20}\right)=a^5\left(\frac{3}{20}-\frac{1}{12}\right)=\frac{9-5}{60}a^5=\frac{1}{15}a^5;$$

$$M_{yz}=\iiint\limits_V xdV=\int_0^a dx\int_0^{a-x}dy\int_0^{x^2+y^2}xdz=$$

$$=\int_0^a xdx\int_0^{a-x}(x^2+y^2)dy=\int_0^a x\left(x^2y+\frac{y^3}{3}\right)\Bigg|_0^{a-x}dx=$$

$$= \int_0^a \left[ax^3 - x^4 + \frac{1}{3}(a^3x - 3a^2x^2 + 3ax^3 - x^4)\right] dx$$

$$= \left(a \cdot \frac{x^4}{4} - \frac{x^5}{5} + \frac{1}{3}a^3 \cdot \frac{x^2}{2} - \frac{1}{3}a^2x^3 + a \cdot \frac{x^4}{4} - \frac{x^5}{15}\right)\Bigg|_0^a =$$

$$= a^5\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \frac{1}{4} - \frac{1}{15}\right) = a^5\left(\frac{1}{2} - \frac{4}{15} - \frac{1}{6}\right) = a^5\left(\frac{1}{3} - \frac{4}{15}\right) = \frac{1}{15}a^5;$$

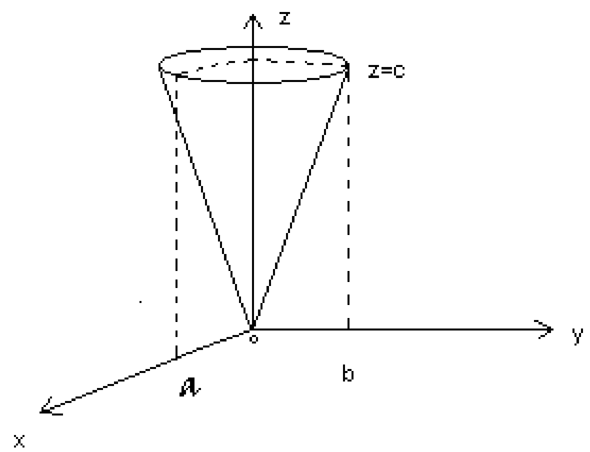
Demak, $M_{zx} = M_{yz} = \frac{1}{15}a^5$, $M_{xy} = \frac{7}{180}a^6$

Bularni (3.18) formulaga qo'ysak, og'irlik markazining koordinatalari kelib chiqadi:

$$x_0 = y_0 = \frac{a^5}{15} : \frac{a^4}{6} = \frac{2}{5}a; \qquad z_0 = \frac{7a^6}{180} : \frac{a^4}{6} = \frac{7}{30}a^2.$$

7-misol. $\left(\frac{z}{c}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$ konusning har bir $M(x, y, z)$ nuqtasidagi zichligi shu nuqtadan xOytekisligigacha bo'lgan masofaga proporsional bo'lib, proportsionallik koffitsienti k ga teng. Shu konusning $z = c$ $(c > 0)$ tekislik bilan kesilgan bo'lagining massaasi va og'irlik markazining koordinatalarini toping.

Yechish.



33-rasm

Shartga ko'ra $\rho \neq (x, y, z) = kz$

$$V = \left\{ x, y, z: \ -a \le x \le a; \ -b\sqrt{1 - \frac{x^2}{a^2}} \le y \right.$$

$$\left. \le b\sqrt{1 - \frac{x^2}{a^2}}; \quad c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} \le z \le c \right\}$$

Masalani yechishda umumlashgan silindrik koordinatalardan foydalanamiz: $x = ar\cos\varphi, \quad y = br\sin\varphi, \quad z = z$.

Bularni yuqoridagi tenglamaga qo'ysak, $z = cz$ kelib chiqadi. Natijada yangi koordinatalar uchun integrallash sohasi quyidagicha bo'ladi:

$$V' = \{(\varphi, r, z): \ 0 \le \varphi \le 2\pi; \ 0 \le r \le 1; \ cr \le z \le c\}, \quad I(\varphi, r, z) = abr$$

jism massasi

$$m = \iiint\limits_V kzdV = k\int_0^{2\pi} d\varphi \int_0^1 dr \int_{cr}^{c} z|I(\varphi, r, z)|dz = abk\int_0^{2\pi} d\varphi \cdot \int_0^1 dr \int_{cr}^{c} rzdz$$

$$= \frac{abk}{2}\int_0^{2\pi} d\varphi \int_0^1 r(c^2 - c^2r^2)dr =$$

$$= \frac{abc^2k}{2}\int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 d\varphi = \frac{abc^2k}{2 \cdot 4} \cdot 2\pi = \frac{\pi kabc^2}{4};$$

(3.17) formulalarga ko'ra quyidagini topamiz:

$$M_{xy} = \iiint\limits_V z\rho(\varphi, r, z)dV = k\iiint\limits_V z^2dV =$$

$$= k\int_0^{2\pi} d\varphi \int_0^1 dr \int_{cr}^{c} abcz^2dz = kab \cdot 2\pi\int_0^1 r\left(\frac{c^3}{3} - \frac{c^3r^3}{3}\right)dr =$$

$$= \frac{2\pi kabc^3}{3}\int_0^1 (r - r^4)dr = \frac{2\pi kabc^3}{3}\left(\frac{r^2}{2} - \frac{r^5}{5}\right)\Bigg|_0^1 = \frac{2\pi kabc^3}{3} \cdot \frac{3}{10}$$

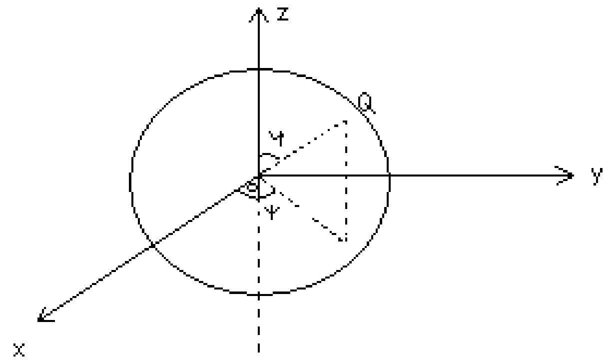
$$= \frac{1}{5}\pi kabc^2.$$

Berilgan jismning zichligi *x,y* larga bog'liq emas, demak *y* gorizontal kesimda o'zgarmasdir, bundan tashqari Oz jismning simmetriya o'qidir, shuning uchun og'rlik markazi Oz o'qida yotadi, demak $x_0 = y_0 = 0$ bo'ladi.

$$z_0 = \frac{M_{xy}}{m} = \frac{4}{5}c \ ; \qquad \left(0{,}0{,}\frac{4}{5}c\right)$$

8-misol. Radiusi R va massasi M ga teng bo'lgan sharning har bir nuqtasidagi zichligi shu nuqtadan shar markazigacha bo'lgan masofaga proporsional bo'lgan, shu sharning diametriga nisbatan inersiya momentini toping.

Yechish.



34-rasm

Bu yerda shu narsaga e'tibor qilish kerakki, sharning diametrlari cheksiz ko'p, lekin ularning hammasiga nisbatan inertsiya momentlari bir xil, ya'ni o'zgarmas- bo'ladi. Sharni koordinatalar sistemasida shunday joylashtiramizki, uning markazi koordinatalar boshi bo'lsin. Agar sharning koordinata o'qlaridan biriga (masalan Oz ga) nisbatan inertsiya momentini topsak, masala yechilgan bo'ladi. Shartga ko'ra ixtiyoriy $Q(x, y, z)$ nuqtasidagi zichligi$\rho(Q) = \rho(x, y, z) = k\sqrt{x^2 + y^2 + z^2}$ bo'ladi. Bu yerda proporsionallik koeffisienti k ni quydagicha shartdan aniqlaymiz:

$$M = \iiint\limits_V \rho(x, y, z)dV = k \iiint\limits_V \sqrt{x^2 + y^2 + z^2}\, dV$$

$$\left.\begin{aligned} x &= r\sin\psi \cos\varphi \\ y &= r\sin\psi \sin\varphi \\ z &= r\cos\psi \end{aligned}\right\} \text{sferik koordinatalarga o'tamiz}$$

Yakobian $|I(\varphi, \psi, r)| = r^2\sin\psi$ ekani bizga ma'lum. Bizning masalamizda

$$V' = \{(\varphi, \psi, r) : \ 0 \le \varphi \le \pi\,; \ \ 0 \le \varphi \le 2\pi\,; \ 0 \le r \le R\}$$

Bularni o‘rniga qo‘yib, yuqoridagi integralni hisoblaymiz.

$$M = k\int_0^{\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{R} r^3 \sin\psi dr$$

$$= k\cdot 2\pi \int_0^{\pi} \frac{R^4}{4}\sin\psi d\psi = \frac{k\pi R^4}{2}\cdot(-\cos\psi)\Big|_0^{\pi} = k\pi R^4.$$

Bundan $k = \frac{M}{\pi R^4}$ kelib chiqadi. (14) formulaga ko‘ra I_z quydagiga teng bo‘ladi:

$$I_z = \iiint\limits_V (x^2+y^2)\cdot\rho(x,y,z)dV = \iiint\limits_{V\prime} r^2\sin^2\psi\cdot k\cdot r|I(\varphi,\psi,r)|dV =$$

$$= k\cdot\int_0^{\pi} d\psi\int_0^{2\pi} d\varphi\int_0^{R} r^5\sin^3\psi\, dr =$$

$$= -2\pi k\cdot\frac{R^2}{6}\int_0^{\pi}(1-\cos^2\psi)d\cos\psi = -\frac{\pi k R^2}{3}\left(\cos\psi - \frac{\cos^3\psi}{3}\right)\Bigg|_0^{\pi}$$

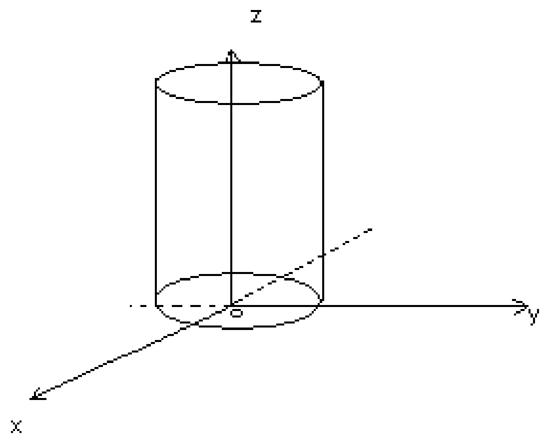
$$= -\frac{\pi k R^6}{3}\left(-2+\frac{2}{3}\right) = \frac{4\pi k R^6}{9};$$

Bu yerda k o‘rniga yuqorida topilgan qiymatini qo‘ysak, diametrga yoki baribir $O\ z$ o‘qiga nisbatan inertsiya momenti kelib chqadi.

$$I_z = \frac{4\pi R^6}{9}\cdot\frac{M}{\pi R^4} = \frac{4}{9}MR^2$$

9-misol. Zichligi o‘zgarmas ρ_0 ga, asosining radiusi a ga va balandligi h ga teng bo‘lgan bir jinsli silindir bilan asosining markazining massasi $m$ga teng.

Yechish. Silindrni koordinatalar sistemasida shunday joylashtiramizki, uning asosi $x\ O\ y$ tekisligida, o‘qi esa Oz o‘qi bilan ustma-ust tushsin. Natijada asosining markazi $O(0,0,0)$ nuqtada bo‘ladi.



35-rasm

V soha quyidagicha bo‘ladi:

$$V = \{(x, y, z) \in R^3 : \quad x^2 + y^2 \leq a^2\ ;\ \ 0 \leq z \leq h\ \}$$

Silindr $O\,z$ o‘qiga nisbatan simmetrik va bir jinsli bo‘lgani uchun O nuqtani tortish kuchi $\bar{F}$ OZ o‘qiga kollineyar vektor bo‘ladi. Demak, $F_x = F_y = 0$ bo‘ladi. (16) formulaga ko‘ra

$$F_z = k\rho_0 m \iiint\limits_V \frac{z}{r^3} dV = k\rho_0 m \iiint\limits_V \frac{zdV}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

$\left.\begin{matrix} x = p\cos\varphi \\ y = p\sin\varphi \\ z = z \end{matrix}\right\}$ silindrik koordinatalar kiritamiz. Yakobian $|I(\varphi, p, z)| =$

p

Bularni o‘rniga qo‘yib yuqoridagi integralni hisoblaymiz:

$$F_z = k\rho_0 m \int\limits_0^{2\pi} d\varphi \int\limits_0^a dp \int\limits_0^h \frac{pzdz}{(p^2 + z^2)^{\frac{3}{2}}} = \pi k\rho_0 m \int\limits_0^a \left.\frac{p(p^2 + z^2)^{-\frac{1}{2}}}{-\frac{1}{2}}\right|_0^h \cdot dp =$$

$$= -2\pi k\rho_0 m \int\limits_0^a \left[\frac{p}{\sqrt{p^2 + h^2}} - 1\right] dp =$$

$$= -2\pi k\rho_0 m \left[\sqrt{p^2 + h^2} - p\right]\Big|_0^a = 2\pi k\rho_0 m \left(a + h - \sqrt{c^2 + h^2}\right)$$

$\bar{F} = F_2 \cdot \bar{n}$ bo‘ladi, $\bar{n} - Oz$ o‘qidagi birlik vektor.

Mustaqil ishlash uchun misollar

1. Yarim doira shakldagi bir jinsli plastinkaning diametriga nisbatan statik momentini toping.

2. Radiusi R va markaziy burchagining kattaligi α ga teng boʻlgan doiraviy sektor shaklidagi bir jinsli tekis figuraning ogʻirlik markazini toping.
Javob: ogʻirlik markazi α burchak bissektrissasida va doira markazidan $\frac{4}{3}R\,\frac{\sin(\alpha/2)}{\alpha}$ masofada yotadi.
3. Ikki karrali integral yordamida bir jinsli $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ ellipsoid shaklidagi jismning birinchi oktantda joylashgan boʻlagining ogirlik markazi koordinatlarini toping.

Quyidagi chiziqlar bilan chegaralangan, zichligi ρ ga teng plastinka massasini toping.
4. $y=x^2,\ x+y=2,\ y-x=2\ \ (x>0),$ agar $\rho=x+2$ boʻlsa.
5. $x=y,\ x-3y=1,\ y=1,\ y=3,$ agar $\rho=y$ boʻlsa.
6. $x^2+y^2=4x,\ x^2+y^2=4\ \ (xy\ge 0),$ agar $\rho=x$ boʻlsa.
7. $y^2=x+4,\ y^2=4-x,\ y=0\ (y\ge 0),$ agar $\rho=y$ boʻlsa.
8. $\frac{x^2}{9}+\frac{y^2}{4}=1,\ \frac{x^2}{18}+\frac{y^2}{8}=1,$ agar $\rho=4x^2+9y^2$ boʻlsa.
9. $x^2+y^2=1,\ x^2+y^2=9,\ x=0,\ y=0,\ (z\ge 0,\ y\ge 0),$ agar $\rho=x+y$ boʻlsa.
10. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{a^2}-\frac{y^2}{b^2}\ \ (x\ge 0),$ agar $\rho=x^2$ boʻlsa.
Quyidagi chiziqlar bilan chegaralangan, zichligi ρ ga teng bir jinsli plastinkaning massa markazi koordinatalarini toping:
11. $y=x^2,\ y=3x^2,\ y=3x.$
12. $y=2x-1,\ y^2=x,\ y=0.$
13. $y=4-x^2,\ y+2x=4.$
14. $y=\sqrt{2x-x^2},\ y=0.$
15. $y=4x+4,\ y^2=-2x+4.$
16. $y=x^2,\ y=2x^2,\ x=1,\ x=2.$
17. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ x=0,\ y=0\ \ (x\ge 0,\ y\ge 0).$
18. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ y=0,\ \ (y\ge 0).$
19. $\frac{x^2}{4}+\frac{y^2}{9}=1,\ 3x+2y=6,\ x\ge 0,\ y\ge 0.$
20. $\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{a}+\frac{y}{b}=2,\ \frac{x}{a}=\frac{y}{b},\ \frac{3x}{a}=\frac{y}{b}.$

21. Qirralari a, b, c larga teng bo‘lgan bir jinsli to‘g‘ri burchakli parallelepiped og‘irlik markazini berilgan. Uning yoqlariga nisbatan statik momentlarini toping $(\rho(x, y, z) = 1)$.
22. Quyidagi sirtlar bilan chegaralangan bir jinsli jismning og‘irlik markazining koordinatalarini toping:
$x^2 = 2pz,\ y^2 = 2px$ slindrlar, $x = \frac{p}{2},\ z = 0$ tekisliklar.
23. $x^2 + y^2 + z^2 \leq 2\alpha z$ sharning har bir $M(x, y, z)$ nuqtasidagi zichligi shu nuqtadan koordinatalar boshigacha bo‘lgan masofaga teskari proporsional bo‘lib, proporsionallik koeffisienti k ga teng. Shu sharning massasini va og‘irlik markazining koordinatalarini toping.
24. $x^2 + y^2 + z^2 = 2$ sfera va $x^2 + y^2 = z^2\ (z \geq 0)$ konus bilan chegaralangan bir jinsli jismning Oz o‘qiga nisbatan inersiya momentini toping.
Quyidagi sirtlar bilan chegaralangan zichligi ρ ga teng bo‘lgan jism massasini toping.
25. $\rho = x + y + z$. agar $\rho = z$ bo‘lsa.
26. $z = \frac{x^2}{4} + \frac{y^2}{9},\ z = 3$, agar $\rho = x^2$ bo‘lsa.
27. $z = x^2 + y^2,\ z = 2y,$ agar $\rho = y$ bo‘lsa.
28. $x + y + z = 2,\ x = 0,\ y = 0,\ z = 0,$ agar $\rho = x + y + z$ bo‘lsa.
29. $x = y^2,\ x = 4,\ z = 2,\ z = 5$ agar $\rho = |y|$ bo‘lsa.
30. $z = 6 - x^2 - y^2,\ z^2 = x^2 + y^2,\ z \geq 0,$ agar $\rho = z$ bo‘lsa.
31. $2x + z = 2a,\ x + z = a,\ y^2 = ax,\ y = 0\ (y \geq 0),$ agar $\rho = y$ bo‘lsa.
32. $2x^2 + 2y^2 - 4ax - 4ay + az = a^2,\ x^2 + y^2 - 2ax - 2ay + az = a^2,\ x^2 + y^2 - 2ax - 2ay + az = 0,\ z = 0,\ M(a, a, a) \in V,$
Quyidagi sirtlar bilan chegaralangan zichligi ρ ga teng bir jinsli jismning berilgan o‘qqa nisbatan inersiya momentini toping.
33. $x = 0,\ x = a,\ y = 0,\ y = b,\ z = 0,\ z = c$ koordinata o‘qlariga nisbatan.
34. $y = \sqrt{x},\ y = 2\sqrt{x},\ z = 0,\ z + x = 4$ koordinata o‘qlariga nisbatan.
35. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ koordinata o‘qlariga nisbatan.
36. $x^2 + y^2 - ax = 0,\ z^2 = 2ax,\ z = 0\ (z > 0)$ koordinata o‘qlariga nisbatan.

37. $z = \frac{1}{2}(y^2 + x^2),\ z = 1\ Ox$ o‘qiga nisbatan.
38. $x + y + z = 2,\ z = 0,\ x^2 + y^2 = 2\ \ (z > 0)\ Oz$ o‘qiga nisbatan.
39. $x^2 + y^2 = cz,\ z = c,\ \ Oz$ o‘qiga nisbatan.
40. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,\ x = 0,\ y = 0,\ z = 0,\ (x \geq 0,\ y \geq 0,\ z \geq 0)\ Oz$ o‘qiga nisbatan.

Javoblar:
Uch karrali integrallarning mexanikaga tadbiqlari

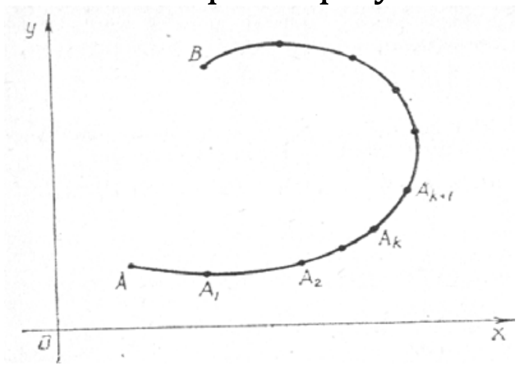
3. $x_0 = \frac{3}{8}a;\ \ y_0 = \frac{3}{8}b;\ \ z_0 = \frac{3}{8}c$ 4. $\frac{79}{12}$ 5. $21\frac{1}{3}$ 6. $2\pi - 4$ 7. 8 8. 1620π
9. $\frac{52}{3}$ 10. $\frac{a^3b}{8}\left(\frac{\pi}{4} + \frac{2}{3}\right)$ 11. $\left(\frac{13}{8};\ \frac{39}{10}\right)$ 12. $\left(\frac{23}{50};\ \frac{2}{5}\right)$ 13. $\left(1; \frac{12}{5}\right)$ 14. $\left(1; \frac{4}{3\pi}\right)$ 15. $\left(\frac{2}{5}; 0\right)$ 16. $\left(\frac{45}{28}; \frac{297}{70}\right)$ 17. $\left(\frac{49}{3}\pi\right);\ \frac{4b}{3\pi}$ 18. $\left(0;\ \frac{4b}{3\pi}\right)$
19. $\left(\frac{44}{(3\pi-6)}; \frac{22}{(\pi-2)}\right)$ 20. $\left(\frac{70}{12}; \frac{35b}{36}\right)$ 21. $\frac{a^2bc}{2};\ \frac{ab^2c}{2};\ \frac{abc^2}{2}$.
22. $x_0\frac{7}{18}p;\ \ y_0 = 0;\ \ z_0 = \frac{7}{176}p$ 23. $x_0 = y_0 = 0;\ \ z_0 = \frac{4}{5}a$. 24. $I_z = \frac{4\pi}{15}(4\sqrt{2} - 5)$ 25. $\frac{11\pi}{3}$ 26. 54π 27. $\frac{\pi}{2}$ 28. $\frac{4}{3}$ 29. $\frac{17^{\frac{3}{2}}-1}{2}$ 30. $\frac{92\pi}{3}$ 31. $\frac{a^4}{12}$
32. $\frac{31}{24}\pi a^5$ 33. $M = abc\rho;\ \mathfrak{I}_{OX} = \frac{M}{3}(b^2 + c^2);\ \mathfrak{I}_{OY} = \frac{M}{3}(a^2 + c^2);\ \mathfrak{I}_{OZ} = \frac{M}{3}(a^2 + b^2)$.
34. $\mathfrak{I}_{OX} = \frac{2^8 \cdot 101}{63}\rho;\ \mathfrak{I}_{OY} = \frac{2^{10} \cdot 41}{315}\rho, \mathfrak{I}_{OZ} = \frac{2^9 \cdot 41}{315}\rho$.
35. $M = \frac{4}{3}\pi abc\rho;\ \ \mathfrak{I}_{OX} = \frac{1}{5}M(b^2 + c^2);\ \ \mathfrak{I}_{OY} = \frac{1}{5}M(a^2 + c^2);\ \ \mathfrak{I}_{OZ} = \frac{1}{5}M(a^2 + b^2)$. 36. $M = \frac{8\sqrt{2}a^3}{15}\rho;\ \ \mathfrak{I}_{OX} = \frac{8a^2}{21}M;\ \ \mathfrak{I}_{OY} = \frac{80}{3}a^2M;\ \ \mathfrak{I}_{OZ} = \frac{4}{9}a^2M$. 37. $\mathfrak{I}_{O\mathrm{X}} = \frac{5\pi\rho}{6}$. 38. $\mathfrak{I}_{OZ} = 4\pi\rho$. 39. $\mathfrak{I}_{OZ} = \frac{\pi c^5\rho}{6}$. 40. $\frac{abc\rho(a^2+b^2)}{60}$.

IV- BOB.

EGRI CHIZIQLI INTEGRALLAR

4.1-§. BIRINChI TUR EGRI ChIZIQLI INTEGRALLAR.

1. **Birinchi tur egri chiziqli integral ta'rifi**. Tekislikda biror $\breve{AB}(A = (a_1, a_2) \in R^2, B = (b_1, b_2) \in R^2)$ egri chiziqni (yoyni) olaylik. Bu egri chiziqda ikki yo'nalishdan birini musbat yo'nalish, ikkinchisini manfiy yo'nalish deb qabul qilaylik.



36-rasm

AB egri chiziqni A dan B ga qarab

$$A_0 = A, A_1, A_2, \dots \dots, A_n = B\ (A_k = (x_k, y_k) \in AB\ k = 0,1,2, \dots \dots, n)$$
$$A_0 = (x_0, y_0) = (a_1, a_3), \qquad A_n = (x_n, y_n) = (b_1, b_2)$$

nuqtalar yordamida n ta bo'lakka bo'lamiz. Bu $A_0A, A_1, A_2, \dots \dots, A_n$nuqtalar sistemasi $\breve{AB}$ yoyning bo'linishi deb ataladi va u $P = \{A_0A, A_1, A_2, \dots \dots, A_n\}$ kabi belgilanadi: A_kA_{k+1} yoy uzunliklari $\Delta s_k\ (k = 0,1,2, \dots \dots, n)$ ning eng kattasi P bo'linishning diametri deyiladi va u λ_p bilan belgilanadi: $\lambda_p = max\{\Delta s_k\}$. Ma'lumki, AB egri chiziqni turli usullar bilan istalgan sonda bo'linishlarini tuzish mumkin.

AB egri chiziqda $f(x, y)$ funksiya berilgan bo'lsin. Bu egri chiziqning $P = \{A_0, A_1, A_2, \dots \dots, A_n\}$ bo'linishini va uning har bir A_kA_{k+1}yoyida ixtiyoriy

$$Q_k = (\xi_k, \eta_k),$$
$$(Q_k = (\xi_k, \eta_k) \in A_kA_{k+1}, \quad k = 0,1,2,3, \dots, n)$$

nuqta olamiz. Berilgan funksiyaning $Q_k = (\xi_k, \eta_k)$ nuqtadagi $f(\xi_k, \eta_k)$ qiymatini har bir mos A_kA_{k+1} ning Δs_k uzunligiga ko'paytirib quyidagi yig'indini tuzamiz:

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta s_k \tag{4.1}$$

Endi AB egri chiziqning shunday

$$P_1, P_2, \dots, P_m, \dots \tag{4.2}$$

bo‘linishlari ketma – ketligini qaraymizki, ularning mos diametrlaridan tashkil topgan

$$\lambda_{p_1}, \lambda_{p_2} \dots \dots . \lambda_{p_m}, \dots,.$$

ketma – ketlik nolga intilsin: $\lambda_{p_m} \to 0$. Bunday bo‘linishlarga nisbatan (4.1) kabi yig‘indilarni tuzib, ushbu $\sigma_1, \sigma_2, \dots \dots, \sigma_m$ ....ketma – ketlikni hosil qilamiz. Ma’lumki, bu ketma – ketlikning har bir hadi $Q_k = (\xi_k, \eta_k)$ nuqtalarga bog‘liq.

1–ta’rif. Agar AB egri chiziqning har qanday (4.2) ko‘rinishdagi bo‘linishlari ketma – ketligi $\{P_m\}$ olinganda ham, unga mos yig‘indilardan iborat $\{\sigma_m\}$ ketma – ketlik (ξ_k, η_k) nuqtalarning tanlab olinishiga bog‘liq bo‘lmagan holda hamma vaqt bitta I songa intilsa, bu son σ yig‘indining limiti deb ataladi va

$$\lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta s_k = I \tag{4.3}$$

kabi belgilanadi.

(4.1) yig‘indining limitini quyidagicha ham ta’riflash mumkin.

2–ta’rif. Agar $\forall \varepsilon > 0$ soni olinganda ham shunday $\delta > 0$ topilsaki, AB egri chiziqning diametri $\lambda_p < \delta$ bo‘lgan har qanday P bo‘linishi uchun tuzilgan σ yig‘indi ixtiyoriy $(\xi_k, \eta_k) \in A_k, A_{k+1}$ nuqtalarda $|\sigma - I| < \varepsilon$ tengsizlik bajarilsa, I son σ yig‘indining $\lambda_p \to 0$ dagi limiti deb ataladi va (4.3) kabi belgilanadi.

(4.1) yig‘indi limitining bu ta’riflari ekvivalent ta’riflardir.

3- ta’rif. Agar $\lambda_p \to 0$ da σ yig‘indi chetli limitga ega bo‘lsa, u holda $f(x, y)$ funksiya AB egri chiziq bo‘yicha integrallanuvchi deyiladi. Bu limit $f(x, y)$ funksiyaning egri chiziq bo‘yicha birinchi tur egri chiziqli integrali deb ataladi va u

$$\int_{AB} f(x, y) ds = \sigma$$

kabi belgilanadi.

Shunday qilib, kiritilgan egri chiziqli integral tushunchasining o‘ziga xosligi qaralayotgan ikki argumentli funksiyaning berilish sohasi tekislidagi biror AB egri chiziq ekanligidir. Qolgan boshqa mulohazalar yuqorida kiritilgan integral tushunchalari singaridir.

2. Uzluksiz funksiya birinchi tur egri chiziqli integrali. Endi birinchi tur egri chiziqli integralning mavjud bo‘lishini ta’minlaydigan shartni topish bilan shug‘ullanamiz. Yuqorida keltirilgan 3- ta’rifdan ko‘rinadiki, birinchi tur egri chiziqli integral AB egri chiziqqa hamda unda berilgan $f(x\,,y)$ funksiyaga bog‘liq bo‘ladi. Demak, integralning mavjud bo‘lishi shartini AB egri chiziq hamda $f(x\,,y)$ funksiyaga qo‘yiladigan shartlar orqali topish kerak bo‘ladi.

Faraz qilaylik, AB egri chiziq ushbu

$$\left.\begin{matrix} x = x(s) \\ y = y(s) \end{matrix}\right\} 0 \leq s \leq \text{ S} \qquad (4.4)$$

sistema bilan berilgan bo‘lsin. Bunda $s - AQ$ yoyning uzunligi $(Q = (x, y) \in AB), S$ esa AB ning uzunligi. $f(x\,,y)$ funksiya shu AB egri chiziqda berilgan bo‘lsin, modomiki, $x = x(s), y = y(s), (0 \leq s \leq\ S\ )$ ekan, unda $(x, y) = f(x(s), y(s))$ bo‘lib, natijada ushbu

$$f(x(s), y(S)) = F(S)\ (0 \leq s \leq S)$$

murakkab funksiyaga ega bo‘lamiz.

AB egri chiziqning $P = \{A_0, A_1\,, A_2, \dots \dots, A_n\}$ bo‘linishini va har bir $A_k A_{k+1}$ dan ixtiyoriy $Q_k(\xi_k\,, \eta_k)$ nuqtani olaylik. Har bir A_k nuqtaga mos keladigan AA_k ning uzunligi s_k, har bir Q_k nuqtaga mos keladigan AQ_k ning uzunligi s_k deylik. Ma‘lumki, mos ravishda $A_k A_{k+1}$ning uzunligiga

$$s_{k-1} - \ s_k = \Delta s_k$$

bo‘ladi.

Natijada P bo‘linishga nisbatan tuzilgan

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k\,, \eta_k) \cdot \Delta s_k$$

yig‘indi ushbu

$$\sum_{k=0}^{n-1} f(\xi_k\,, \eta_k) \cdot \Delta s_k = \sum_{k=0}^{n-1} f(x(s_k\,, y(s_k) \cdot \Delta s_k = \sum_{k=0}^{n-1} F(s_k) \cdot \Delta s_k$$

ko‘rinishga keladi. Demak,

$$\sum_{k=0}^{n-1} F(s_k)\cdot\Delta s_k \qquad (4.5)$$

Bu yig'indini $[0, S]$ oraliqdagi *F(s)* funksiyaning integral yig'indisi ekanligini payqash qiyin emas.

Agar $f(x, y)$ funksiya AB egri chiziqda uzluksiz bo'lsa, u holda $F(x)$ funksiya $[0, S]$ da uzluksiz bo'ladi. Demak, bu holda $F(s)$ funksiya $[0, S]$ da integrallanuvchi:

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1} F(s_k)\Delta s_k = \int_0^S F(s)ds \qquad (4.6)$$

Shunday qilib, (4.5), (4.6) munosabatlardan $\lambda_p \to 0$ da σ yig'indining limiti mavjud bo'lishi va

$$\lim_{\lambda_p\to 0}\int_0^S F(s)ds$$

ekanligini topamiz. Natijada quyidagi teoremaga kelamiz.

1 – teorema. Agar $f(x, y)$ funksiya AB egri chiziqda uzluksiz bo'lsa, u holda bu funksiyaning ABegri chiziq bo'yicha birinchi tur egri chiziqli integrali mavjud bo'ladi va

$$\int_{Ab} f(x,y)ds = \int_0^S f(x(s), y(s))ds$$

bo'ladi.

Bu teorema, bir tomondan uzluksiz funksiya birinchi tur egri chiziqli integralining mavjudligini aniqlab bersa, ikkinchi tomondan bu integralning aniq integralga (Riman integraliga) kelishini ko'rsatadi.

1 – eslatma. Egri chiziqli integral tushunchasi bilan Riman integrali tushunchasini solishtirib, ularning har ikkalasi yig'indining limiti sifatida ta'riflanishini ko'rdik. Ayni paytda bu tushunchalarning farqli tomoni ham bor. Ushbu

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\cdot\Delta s_k \qquad (4.5)$$

yig'indidagi Δs_k har doim musbat bo'lib, AB egri chiziqning yo'nalishiga bog'liq emas. Demak,

$$\int_{AB} f(x,y)ds = \int_{BA} f(x,y)ds.$$

3. Birinchi tur egri chiziqli integralning xossalari. Yuqorida ko‘rdikki, uzluksiz funksiyalarning birinchi tur egri chiziqli integrallari Riman integrallariga keladi. Binobarin, egri chiziqli integrallar ham Riman integrallari xossalari kabi xossalarga ega bo‘ladi. Shuni e’tiborga olib, egri chiziqli integrallarning asosiy xossalarini sanab o‘tish bilan kifoyalanamiz.

(4.4) sistema bilan aniqlangan AB egri chiziqda $f(x,y)$ funksiya berilgan va uzluksiz.

1^0. Agar $AB = AC + CB$ bo‘lsa, u holda

$$\int_{AB} f(x,y)ds = \int_{AC} f(x,y)ds + \int_{CB} f(x,y)ds$$

bo‘ladi.

2^0. Ushbu

$$\int_{AB} f(x,y)ds = c\int_{AB} f(x,y)ds \ (c = const)$$

tenglik o‘rinli.

AB egri chiziqda $f(x,y)$ funksiya bilanbirga $g(x,y)$ funksiya ham berilgan va u uzluksiz bo‘lsin.

3^0. Quyidagi

$$\int_{AB} [f(x,y) \pm g(x,y)]ds = \int_{AB} f(x,y)ds \pm \int_{AB} g(x,y)ds$$

formula o‘rinli bo‘ladi.

4^0. Agar $\forall(x\ ,y) \in AB$ da $f(x,y) \geq 0$ bo‘lsa, u holda

$$\int_{AB} f(x,y)ds \geq 0$$

bo‘ladi.

5^0. $|f(x,y)|$ funksiya shu AB da integrallanuvchi bo‘ladi:

$$\left|\int_{AB} f(x,y)ds\right| = \int_{AB} |f(x,y)|ds$$

bo‘ladi.

6^0. Shunday $(c_1, c_2) \in AB$ nuqta topiladiki,

$$\int_{AB} f(x,y)ds = f(c_1\,, c_2) \cdot S$$

bo‘ladi, bunda $S - AB$ ning uzunligi.

6^0 ***xossa o‘rta qiymat haqidagi teorema deb ataladi.***

4. Birinchi tur egri chiziqli integrallarni hisoblash. Birinchi tur egri chiziqli integrallar, asosan Riman integrallariga keltirilib hisoblanadi. Yuqorida keltirilgan 1 – teoremaga ko‘ra AB egri chiziq ushbu

$$\left.\begin{aligned} x &= x(s) \\ y &= y(s) \end{aligned}\right\} (\,0 \le s \le \,\mathrm{S})$$

sistema bilan berilganda (bunda S −yoy uzunligi)$f(x\,, y)$ funksiya shu AB da uzluksiz bo‘lganda egri chiziqli integral Riman integraliga keldi. Demak, bu Riman integralini hisoblash natijasida egri chiziqli integral topiladi.

Endi AB egri chiziq ushbu

$$\left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\} (\alpha \le t \le \beta) \qquad (4.7)$$

sistema bilan berilgan bo‘lsin. Bunda $\varphi(t)\,, \psi(t)$ funksiyalar $[\alpha\,, \beta]$ da $\varphi'(t)\,, \psi'(t)$ hosilalarga ega va hosilalar shu oraliqda uzluksiz hamda $\big(\varphi(a), \psi(\alpha)\big) = A$ va $\big(\varphi(\beta), \psi(\beta)\big) = B$ bo‘lsin.

Ma’lumki, (4.7) sistema $t \in [\alpha\,, \beta]$ oraliqni ABegri chiziqqa akslantiradi. Bunda $[\gamma\,, \delta\,] \subset [\alpha\,, \beta]$ ning ABchiziqdagi $A_\gamma A_\delta$ aksining uzunligi

$$\int_\gamma^\delta \sqrt{\varphi'^2(t) + \psi'^2(t)}\,dt$$

bo‘ladi.

2- teorema. Agar $f(x, y)$ funksiya ABda berilgan va uzluksiz bo‘lsa, u holda

$$\int_{AB} f(x,y)ds = \int_\alpha^\beta f(\varphi(t)\psi(t)\sqrt{\varphi'^2(t) + \psi'^2(t)}dt \qquad (4.8)$$

bo‘ladi.

Isbot .$[\alpha\,, \beta]$ oraliqning

$$P = \{t_{0\,,}t_1\,, t_{2,}\,, \dots, t_n\}\,(\alpha = t_0\, < t_1 < t_2 < \,\dots < t_n\, = \beta$$

bo‘linishini olaylik. Bu bo‘linishning bo‘luvchi nuqtalari t_k, $(k = 0,1,2,3, \dots, n)$ ning AB dagi mos akslari A_k $(k = 0,1,2,3, \dots n)$ deylik. Ma’lumki, bu

A_k $(k = 0,1,2,3, \dots n)$ nuqtalar AB egri chiziqning

$$\{A_0, A_1, A_2, \dots, A_n\}$$

bo‘linishini hosil qiladi. Bunda

$$A_k = \left(\varphi\left(t_{k\,,}\right), \psi(t_k)\right) (k = 0,1,2, \dots., n)$$

va bir $A_k A_{k+1}$ ning uzunligi

$$\Delta s_k = \int_{t_{k\,,}}^{t_{k+1\,,}} \sqrt{\varphi'^2(t) + \psi'^2(t)} dt$$

bo‘ladi. O‘rta qiymat haqidagi teoremadan foydalanib quyidagini topamiz:

$$\Delta s_k = \sqrt{\varphi'^2(\tau_k) + \psi'^2(\tau)} \cdot \left(t_{k+1,} - t_k\right) = \sqrt{\varphi'^2\left(\tau_{k\,,}\right) + \psi'^2(\tau)} \cdot \Delta t_{k\,,}$$

bunda $t_k < \tau_k < t_{k+1}$. Endi $\varphi(\tau_k) = \xi_k$, $\psi(\tau_k) = \eta_k$ deb olamiz. Ma’lumki,

$$f(\xi_k, \eta_k) \in A_k A_{k+1} (k = 0,1,2,3, \dots, n-1)$$

bo‘ladi. ABegri chiziqning yuqorida aytilgan $\{A_0, A_1, A_2, \dots \dots, A_n\}$ bo‘linishini va har bir $A_k A_{k+1}$ da (ξ_k, η_k) nuqtani olib,

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta s_k$$

yig‘indini tuzamiz. Uni quyidagicha ham yozish mumkin:

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta s_k$$

$$= \sum_{k=0}^{n-1} f\left(\varphi(\tau_k), \psi(\tau_k)\right) \sqrt{\varphi'^2(\tau_k) + \psi'^2(\tau_k)}\, \Delta t_k \ (4.8).$$

Bu tengliknig o‘ng tomondagi yig‘indi

$$f(\varphi(t), \psi(t)) \sqrt{\varphi'^2(t) + \psi'^2(t)}$$

funksiyaning $[\alpha, \beta]$ oraliqdagi Riman yig‘indisidir.

Shartga ko‘ra $f(x, y)$ va $\varphi'(t), \psi'(t)$ funksiyalar uzluksiz. Demak, murakkab funksiyaning uzluksizligi haqidagi teoremaga ko‘ra$(\varphi(t), \psi(t))$ va demak, $f(\varphi(t), \psi(t)) \sqrt{\varphi'^2(t) + \psi'^2(t)}$ funksiya

$[\alpha\,,\beta]$ oraliqda uzluksiz. Demak, bu funksiya $[\alpha\,,\beta]$ da integrallanuvchi bo‘ladi. Ya’ni

$$\lim_{\max(\Delta t_k)\to 0}\sum_{k=1}^{n-1} f(\varphi(\tau_k),(\psi(\tau_k))\sqrt{\varphi'^2(\tau_k)+\psi'^2(\tau_k)}\cdot\Delta t_k\,,$$
$$=\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}\;dt$$

Demak,$x=\varphi(t), y=\psi(t)$ funksiyalar $[\alpha\,,\beta]$ da uzluksiz ekan, unda $\max_k\{\Delta\, t_k\}\to 0$ da $\Delta x_k\ \to 0, \Delta y_k\ \to 0$ va demak, $\Delta s_k\ \ \to 0$. Bundan esa $\lambda_p\to 0$ bo‘lishi kelib chiqadi. (4.8) munosabatdan foydalanib

$$\lim_{\lambda_{p\to 0}}\sigma\ =\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{{\varphi'}^2(t)+{\psi'}^2(t)}\,dt$$

bo‘lishini topamiz. Bu esa

$$\int\limits_{AB} f(x,y)ds=\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$$

ekanini bildiradi. **Teorema isbot bo‘ldi.**

Bu teoremadan quyidagi natijalar kelib chiqadi.

1 – natija. AB egri chiziq ushbu

$$y=y(x)\ (a\le x\le b, y(a)=A, y(b)=B)$$

tenglama bilan aniqlanga bo‘lib, $y(x)$ funksiya $[a\,,b]$ da hosilaga ega va u uzluksiz bo‘lsin. Agar $f(x,y)$ funksiya shu AB da berilgan va uzluksiz bo‘lsa, u holda

$$\int\limits_{AB} f(x,y)ds=\int\limits_{AB} f(x,y(y))\sqrt{1+y'^2(x)}dx\ \ (4.9)$$

bo‘ladi.

2 – natija. ABegri chiziq ushbu $\rho=\rho(\theta)\ (\theta_0\le\theta\le\theta_1)$ tenglama bilan berilgan bo‘lib, $\rho(\theta)$ funksiya $[\sigma_0\,,\sigma_1]$ da hosilaga ega va u uzluksiz bo‘lsin. Agar $f(x,y)$ funksiya shu AB da berilgan va uzluksiz bo‘lsa, u holda

$$S=\int\limits_{AB} ds=\int\limits_{\sigma_1}^{\sigma_2} f(\rho cos\theta,\rho sin\theta)\sqrt{\rho^2+\rho'^2(x)}d\theta \qquad (4.10)$$

bo‘ladi.

4.1-misol. Ushbu

$$S = \int\limits_{AB} \sqrt{x^2 + y^2} ds$$

egri chiziqli integral hisoblansin, bunda ABmarkazi kordinata boshida, radiusi $r > 0$ ga teng boʻlgan aylananing yuqori yarim tekislikdagi qismi.

Ma'lumki, bu ABegri chiziq quyidagi

$$\left.\begin{aligned} x = r\,cost \\ y = r\,cost \end{aligned}\right\} \; (0 \le t \le \pi)$$

sistema bilan aniqlanadi. AB da

$$f(\,x, y) = \sqrt{x^2 + y^2} = \sqrt{(r\,cost)^2 + (r\,sint)^2}$$

funksiya uzluksiz. Demak,

$$\int\limits_{AB} \sqrt{x^2 + y^2} ds$$

$$= \int_0^{\pi} \sqrt{(r\,cost)^2 + (r\,sint)^2}$$

$$\cdot \sqrt{\left((r\,cost)'\right)^2 + \left((r\,sint)'\right)^2} dt = r^2 \int_0^{\pi} dt = \pi r^3$$

boʻladi.

5. Birinchi tur egri chiziqli integrallarning ba'zi bir tadbiqlari. Birinchi tur egri chiziqli integrallar yordamida yoy uzunligi, jismning massasini, ogʻirlik markazlarini topish mumkin. Quyida biz birinchi tur egri chiziqli integrallar yordamida yoy uzunligi qanday hisoblanishini koʻrsatamiz.

Tekislikda sodda AB egri chiziq berilgan boʻlsin. Bu chiziqda $f(x, y) = 1$ funksiyani qaraylik. Maʻlumki, bu funksiya AB da uzluksiz, $f(x, y)$ funksiyaning birinchi tur egri chiziqli integrali ta'rifidan quyidagini topamiz:

$$\int\limits_{AB} f(x\,, y) ds = \lim_{\lambda_p \to 0} \sum_{k=0}^{n=1} f(\xi_k\,, \eta_k) \cdot \Delta s_k = \lim_{\lambda_p \to 0} \sum_{\lambda_p \to 0}^{n-1} \Delta s_k = S.$$

Demak,

$$S = \int\limits_{AB} ds \qquad (4.11)$$

4.2-misol. Ushbu

$$\left.\begin{aligned} x = x(t) = acos^3 t \\ y = y(t) = asin^3 t \end{aligned}\right\}$$

sistema bilan berilgan AB chiziqning uzunligi topilsin. Bu chiziq astroidani ifodalaydi.

(4.11) formulaga koʻra astroidaning uzunligi

$$S = \int_{AB} ds$$

boʻladi. Astroida koordinata oʻqlariga nisbatan simmetrik boʻlishini e'tiborga olib, yuqorida keltirilgan (4.8) formuladan fordalanib quyidagini topamiz:

$$\int_{AB} ds = 4\int_0^{\frac{\pi}{2}} \sqrt{x'^2(t) + y'^2(t)}\, dt$$

$$= 4\int_0^{\frac{\pi}{r}} \sqrt{(-3acos^2 t \cdot sint)^2 + (3asin^2 t \cdot cost)^2}\, dt$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{\frac{9a^2}{4} sin^2 2t}\, dt$$

$$= 6a\int_0^{\frac{\pi}{2}} sin2t\, dt = 6a - \frac{\cos 2t}{2}\Big|_0^{\frac{\pi}{2}} = 6a$$

4.2-§. IKKINChI TUR EGRI ChIZIQLI INTEGRALLAR.

1. Ikkinchi tur egri chiziqli integral tarifi. Tekislikda biror sodda AB egri chiziqni qaraylik. Bu egri chiziqda $f(x, y)$ funksiya berilgan boʻlsin. $\overset{\frown}{AB}$ egri chiziqning

$$P = \{A_0, A_1, A_2, \dots, A_n\}$$

boʻlishni va uning har bir (k=0,1,…,n-1) yoyida ixtiyoriy

$Q_k = \left(\xi_k, \eta_k\right)$ nuqtani

$$Q_k = (\xi_k, \eta_k) \in \overset{\frown}{A_k A_{k+1}} (k = 0,1,2,3, \dots, n-1)$$

olaylik. Berilgan funksiyaning $Q_k = (\xi_k, \eta_k)$ nuqta $f(\xi_k, \eta_k)$ qiymatini $\overset{\frown}{A_k A_{k+1}}$ ning $Ox(Oy)$ o'qidagi Δx_k (Δy_k) nuqtadagi proyeksiyasiga ko'paytirib quyidagi yig'indini tuzamiz.

$$\sigma' = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k \sigma'' = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta y_k \qquad (4.12)$$

endi, $\overset{\frown}{AB}$ egri chiziqning shunday $P_0, P_1, P_2, \dots, P_m \dots$ bo'linishlari ketma-ketligini qaraymizki, ularning diametrlaridan tashkil topgan mos $\lambda_{p_1}, \lambda_{p_2}, \dots, \lambda_{p_m}, \dots$ketma-ketlik nolga intilsin: $\lambda_{p_m} \to 0$. Bunday bo'linishlarga nisbatan (5.1) kabi yig'indilarni tuzib, ushbu $\sigma_1', \sigma_2', \dots, \sigma_m'(\sigma_1'', \sigma_2'', \dots, \sigma_m'')$ ketma-ketliklarni hosil qilamiz. Ma'lumki, bu ketma-ketliklarning har bir hadi, xususan, (ξ_k, η_k) nuqtalarga ham bog'liq.

3- ta'rif: Agar $\overset{\frown}{AB}$ egri chiziqning har qanday bo'linishlari ketma-ketligi egri $\{P_m\}$ olinganda ham, uning mos yig'indilaridan iborat.$\{\sigma_m'\}\{\sigma_m''\}$ ketma–ketlik (ξ_k, η_k) nuqtalarining $(\xi_k, \eta_k) \in \overset{\frown}{A_k A_{k+1}}$ tanlab olinishiga bog'liq bo'lmagan ravishda hamma vaqt bitta I' songa $(I''\ songa)$ intilsa,bu son σ_m' (σ_m'') yig'indining limiti deb ataladi va

$$\lim_{\lambda_p \to 0} \sigma' = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k = I'$$

$$\lim_{\lambda_p \to 0} \sigma'' = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k = I'' \qquad (4.13)$$

kabi belgilanadi.

σ_m' (σ_m'') yig'indining bu limitini quyidagicha ham ta'riflash mumkin.

4-ta'rif.Agar $\forall \varepsilon > 0$olinganda ham, shunday $\delta > 0$ topilsaki, $\overset{\frown}{AB}$ egri chiziqning diametric $\lambda_p < \delta$ bo'lganda har qanday P bo'lishi uchun tuzilgan. σ_m' (σ_m'') yig'indi uchun ixtiyoriy (ξ_k, η_k) nuqtalarda

$$(\xi_k, \eta_k) \in A_k A_{k+1}\ k = 0,1,2,3, \dots \dots n-1)$$

$$|\sigma' - I'| < \varepsilon\ (|\sigma'' - I''| < \varepsilon)$$

tengsizlik bajarilsa, I' son $(I''\ son)$, σ' yig'indining (σ'' yig'indining) $\lambda_p \to 0$ dagi limiti deb ataladi va (5.1) kabi belgilanadi.

Yig'indi limitining bu ta'riflari ekvivalent ta'riflaridir.

5-ta'rif. Agar $\lambda_p \to 0$ da, σ' yig'indi (σ''yig'indi) chekli limitga ega bo'lsa, u holda $f(x,y)$ funksiya$\int_{AB} f(x,y)ds$ egri chiziq bo'yicha integrallanuvchi deyiladi. Bu limit $f(x,y)$ funksiyaning $\breve{AB}$ egri chiziq bo'yicha ikkinchi tur egri chiziqli integrali deb ataladi va u

$$\int\limits_{\breve{AB}} f(x,y)dx = \int\limits_{\breve{AB}} f(x,y)dy$$

kabi belgilanadi. Demak,

$$\int\limits_{\breve{AB}} f(x,y)dx = \lim_{\lambda_p \to 0} \sigma' = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k$$

$$\int\limits_{\breve{AB}} f(x,y)dy = \lim_{\lambda_p \to 0} \sigma'' = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k$$

Shunday qilib, $\breve{AB}$ egri chiziqda berilgan $f(x,y)$ funksiyadan ikkita – Ox o'qidagi proyeksiyalar vositasida va Oy o'qidagi proyeksiyalar vositasida olingan ikkinchi tur egri chiziqli integral tushunchalari kiritildi.

Faraz qilaylik, $\breve{AB}$ egri chiziqda ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo'lib,

$$\int\limits_{\breve{AB}} P(x,y)dx, \int\limits_{\breve{AB}} Q(x,y)dx$$

lar esa ularning ikkinchi tur egri chiziqli integrallari bo'lsin.Ushbu

$$\int\limits_{\breve{AB}} P(x,y)dx + \int\limits_{\breve{AB}} Q(x,y)dx$$

yig'indi ikkinchi tur egri chiziqli integralning umumiy ko'rinishi deb ataladi va

$$\int\limits_{\breve{AB}} P(x,y)dx + (x,y)dy$$

kabi yoziladi.
Demak,

$$\int\limits_{\breve{AB}} P(x,y)dx + Q(x,y)dx.$$

Ikkinchi tur egri chiziqli integral ta'rifidan quyidagi natijalar kelib chiqadi.

2 – natija. Ikkinchi tur egri chiziqli integral egri chiziqning yo‘nalishiga bog‘liq bo‘ladi. Shuni isbotlaylik. Ma’lumki, $\breve{AB}$ egri chiziqda ikkita yo‘nalish (A nuqtadan B nuqtaga va B nuqtadan A nuqtaga) olish mumkin $(\breve{AB}, \breve{BA}, A \neq B)$ $\breve{AB}$ egri chiziqning yuqoridagi P bo‘lishini olib, bu bo‘linishga nisbatan (4.12) yig‘indini tuzamiz:

$$\sigma' = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k (\Delta x_k = x_{k+1} - x_k)$$

Aytaylik, $\lambda_p \to 0$ da bu yig‘indi chekli limitga ega bo‘lsin. Demak,

$$\lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k = \int_{\breve{AB}} f(x, y) dx \qquad (4.14)$$

Endi, $\breve{AB}$ ning o‘sha R bo‘linishini hamda har bir $\overbrace{A_k, A_{k+1}}$dagi o‘sha (ξ_k, η_k) nuqtalarni olib, $\breve{AB}$ egri chiziqning yo‘nalishini esa B dan A ga qarab deb ushbu yig‘indini tuzamiz:

$$\overline{\sigma'} = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot (x_k - x_{k+1})$$

$\lambda_p \to 0$ da bu yig‘indi chekli limitga ega bo‘lsa, u ta’rifga binoan ushbu

$$\int_{\breve{AB}} f(x, y) dx$$

integral bo‘ladi:

$$\lim_{\lambda_p \to 0} \overline{\sigma'} = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot (x_k - x_{k+1}) = \int_{\breve{AB}} f(x, y) dx$$

Agar

$$\overline{\sigma'} = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k = -\sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot (x_k - x_{k+1}) = -\overline{\sigma'}$$

ekanligini e’tiborga olsak, u holda $\lambda_p \to 0$ da σ' yig‘indining chekli limitga ega bo‘lishidan $\overline{\sigma'}$ yig‘indining ham chekli limitga ega bo‘lishi va

$$\lim_{\lambda_p \to 0} \overline{\sigma'} = -\lim_{\lambda_p \to 0} \overline{\sigma'}$$

tenglikning bajarilishini topamiz. Demak,

$$\int\limits_{\breve{AB}} f(x,y)dx = -\int\limits_{\breve{AB}} f(x,y)dx.$$

Xuddi shunga o‘xshash

$$\int\limits_{\breve{AB}} f(x,y)dy = -\int\limits_{\breve{AB}} f(x,y)dy$$

bo‘ladi.

3-natija. $\breve{AB}$ egri chiziq Ox o‘qiga ((Oy o‘qiga) perpendikulyar bo‘lgan to‘g‘ri chiziq kesmasidan iborat bo‘lsin. $f(x,y)$ funksiya shu chiziqda berilgan bo‘lsin.

U holda

$$\int\limits_{\breve{AB}} f(x,y)dx \left(\int\limits_{\breve{AB}} f(x,y)dy \right)$$

mavjud bo‘ladi va

$$\int\limits_{\breve{AB}} f(x,y)dx = 0 \qquad \left(\int\limits_{\breve{AB}} f(x,y)dy \right) = 0$$

Bu tenglik bevosita ikkinchi tur egri chiziqli integral ta’rifidan kelib chiqadi.

Endi$\breve{AB}$- sodda yopiq egri chiziq bo‘lsin, ya’ni A va B nuqtalar ustma-ust tushsin. Bu yopiq chiziqni K deb belgilaylik. Bu sodda yopiq chiziqda ham ikki yo‘nalish bo‘ladi.Ularning birini musbat yo‘nalish, ikkinchisini manfiy yo‘nalish deb qabul qilaylik. Shunday yo‘nalishni musbat deb qabul qilamizki, kuzatuvchi yopiq chiziq bo‘ylab harakat qilganda, yopiq chiziq bilan chegaralangan soha unga nisbatan har doim chap tomonda yotsin.Faraz qilaylik, K sodda yopiq chiziqda $f(x,y)$funksiya berilgan bo‘lsin. Bu K chiziqda ixtiyoriy ikkita turli nuqtalarni olib, ularni AvaB bilan belgilaylik. Natijada K yopiq chiziq ikkita $\breve{AB}$ va $\overparen{BbA}$ chiziqlarga ajratiladi.

Ushbu

$$\int\limits_{\overparen{AaB}} f(x,y)dx + \int\limits_{\overparen{BbA}} f(x,y)dx.$$

Integral (agar u mavjud bo‘lsa) $f(x,y)$ funksiya K yopiq chiziq bo‘yicha ikkinchi tur egri chiziq integrali deb ataladi va

$$\int_{\breve{K}} f(x,y)dx \text{ yoki } \int_{K} f(x,y)dy$$

kabi belgilanadi. Bunda K yopiq chiziqning musbat yo‘nalishi olingan (bundan buyon yopiq chiziq bo‘yicha olingan integrallarda, yopiq chiziq bo‘yicha olingan integrallarda, yopiq chiziq musbat yo‘nalishda deb qaraymiz). Demak,

$$\int_{\breve{K}} f(x,y)dx = \int_{\overset{\frown}{AaB}} f(x,y)dx + \int_{\overset{\frown}{BbA}} f(x,y)dx.$$

Xuddi shunga o‘xshash

$$\int_{\breve{K}} f(x,y)dx$$

hamda, umumiy hol

$$\int_{\breve{K}} f(x,y)dx + Q(x,y)dy$$

integrallar ta’riflanadi.

$\overset{\smile}{AB}$ fazoviy egri chiziq bo‘lib , bu chiziq $f(x,y,z)$ funksiya berilgan bo‘lsin. Yuqoridagidek, $f(x,y,z)$ funksiyaning $\overset{\smile}{AB}$ bo‘yicha ikkinchi tur egri chiziq integrallari tariflanadi va ular

$$\int_{\overset{\smile}{AB}} f(x,y,z)dx\,, \int_{\overset{\smile}{AB}} f(x,y,z)dy\,, \int_{\overset{\smile}{AB}} f(x,y,z)dz$$

kabi belgilanadi. Umumiy holda $\overset{\smile}{AB}$ da

$$P(x,y,z), Q(x,y,z)\ R(x,y,z),$$

funksiyalar berilgan bo‘lib, ushbu

$$\int_{\overset{\smile}{AB}} P(x,y,z)dx + \int_{\overset{\smile}{AB}} Q(x,y,z)dy + \int_{\overset{\smile}{AB}} R(x,y,z)dz\,,$$

integrallar mavjud bo‘lsa, u holda

$$\int_{\overset{\smile}{AB}} P(x,y,z)dx + \int_{\overset{\smile}{AB}} Q(x,y,z)dy + \int_{\overset{\smile}{AB}} R(x,y,z)dz$$

yig‘indini ikkinchi tur egri chiziqli integralning umumiy ko‘rinishi deb ataladi va u

$$\int\limits_{\breve{AB}} f(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

kabi belgilanadi. Demak,

$$\int\limits_{\breve{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

$$= \int\limits_{\breve{AB}} P(x,y,z)dx + \int\limits_{\breve{AB}} Q(x,y,z)dy + \int\limits_{\breve{AB}} R(x,y,z)dz\,.$$

2. Uzluksiz funksiya ikkinchi tur egri chiziqli integrali. Endi ikkinchi tur egri chiziqli integralning mavjud bo‘lishini ta’minlaydigan shartni topish bilan shug‘ullanamiz.

Faraz qilaylik, $\breve{AB}$ egri chiziq ushbu

$$\left.\begin{matrix} x = \varphi(t) \\ y = \psi(t) \end{matrix}\right\} \quad \alpha \le t \le \beta \tag{4.15}$$

sistema bilan (parametrik formada) berilgan bo‘lsin. Bunda $\varphi(t)$ funksiya $[\alpha\,,\beta]$ da $\varphi'(t)$ hosilaga ega va bu hosila shu oraliqda uzliksiz, $\psi(t)$ funksiya ham $[\alpha\,,\beta]$ da uzluksiz hamda $\big(\varphi(\alpha),\psi(\alpha)\big) = A$ va $\big(\varphi(\beta),\psi(\beta)\big) = B$ bo‘lsin.

T parametrga α dan β ga qadar o‘zgarganda $(x,y) = \big(\varphi(t),\psi(t)\big)$ A dan B ga qarab $\breve{AB}$ ni chiza borsin.

2 – teorema. Agar $f(x,y)$ funksiya $\breve{AB}$ da berilgan va uzluksiz bo‘lsa, u holda bu funksiyaning $\breve{AB}$ egri chiziq bo‘yicha ikkinchi tur egri chiziqli integrali

$$\int\limits_{\breve{AB}} f(x,y)dx$$

mavjud va

$$\int\limits_{\breve{AB}} f(x,y)dx = \int\limits_{\alpha}^{\beta} f(\varphi(t),\psi(t))\cdot\varphi'(t)dt$$

bo‘ladi.

Isbot. $[\alpha\,,\beta]$ oraliqning

$$P = \{t_0,t_1,\dots,t_n\}(\alpha = t_0 < t_1 < \cdots < t_n = \beta)$$

bo‘linishini olaylik. Bu bo‘linishning bo‘luvchi nuqtalari t_n $(k = 0,1, \ldots \ldots, n)$ ning $\overset{\smile}{AB}$ dagi mos akslarini A_k deylik $(k = 0,1, \ldots, n)$.

Ma’lumki, bu A_k nuqtalar AB egri chiziqning $\{A_0, A_1, A_2, \ldots, A_n\}$ bo‘lishini hosil qiladi. Bundan $A_k = \big(\varphi(t_k), \psi(t_k)\big)(k = 0,1,2, \ldots, n)$ bo‘ladi. Bu bo‘linishga nisbatan yig‘indi

$$\sigma' = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k \tag{4.16}$$

tuzamiz. Keyingi tenglikda

$$\Delta x_k - \overset{\frown}{A_k A_{k+1}}$$

ning Ox o‘qidagi proeksiyasi

$$(\Delta x_k = x_{k+1} - x_k = x = \varphi(t_{k+1}) - \varphi(t_k)$$

ga tengdir

Lagranj teoremasidan foydalanib topamiz:

$$\varphi(t_{k+1}) - \varphi(t_k) = \varphi'(\theta_k)(t_{k+1} - t_k)$$
$$= \varphi'(\theta_k)\Delta t_k(\theta_k \in [t_k, t_{k+1}])$$

Ma’lumki,

$$(\xi_k, \eta_k) \in \overset{\frown}{A_k A_{k+1}}(k = 0,1,2,3, \ldots, n-1).$$

Agar bu (ξ_k, η_k) nuqtaga akslanuvchi nuqtani

$$\tau_k \in [t_k, t_{k+1}]$$

deyilsa, unda

$$\big(\xi_k = \varphi(\tau_k), \eta_k = \psi(\tau_k)\big)$$

bo‘ladi. Natijada σ' yig‘indi quyidagi ko‘rinishga keladi:

$$\sigma' = \sum_{k=0}^{n-1} f\big(\varphi(\tau_k), \psi(\tau_k)\big)\varphi'(\theta_k) \cdot \Delta t_k.$$

Endi $\lambda'_p = \max_k\{\Delta t_k\} \to 0$ da (bu holda $\lambda_p$ham nolga intiladi σ' yig‘indining limitini topish maqsadida uning ifodasini o‘zgartirib quyidagicha yozamiz:

$$\sigma' = \sum_{k=0}^{n-1} f\big(\varphi(\tau_k), \psi(\tau_k)\big) \cdot \varphi'(\tau_k) \cdot \Delta t_k$$
$$+ \sum_{k=0}^{n-1} f\big(\varphi(\tau_k), \psi(\tau_k)\big)$$
$$\cdot \left[\varphi'^{(\theta_k)} - \varphi'^{(\tau_k)}\right] \cdot \Delta t_k \tag{4.17}$$

bu tenglikning o‘ng tomonidagi ikkinchi qo‘shiluvchini baholaymiz:

$$\left|\sum_{k=0}^{n-1} f\big(\varphi(\tau_k),\psi(\tau_k)\big)\cdot\ [\varphi'(\theta_k)-\varphi'(\tau_k)]\cdot\Delta t_k\right|$$

$$\leq \sum_{k=0}^{n-1}|f(\varphi(\tau_k),\psi(\tau_k))\ |\cdot\big|[\varphi'(\theta_k)-\varphi'(\tau_{k,})]\cdot\Delta t_k\big|$$

$$\leq M\sum_{k=0}^{n-1}|\varphi'(\theta_k)-\varphi'(\tau_k)|\cdot\Delta t_k,$$

bunda

$$M=\max_{\alpha\leq t\leq\beta}|f(\varphi(t),\psi(t))|$$

$\varphi'(t)$ funksiya $[\alpha\,,\beta]$ da uzliksiz bo‘ladi. U holda Kantor natijasiga ko‘ra,$\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ topiladiki, $[\alpha\,,\beta]$ oraliqning diametri $\lambda'_p<\delta$ bo‘lgan har qanday P bo‘linish uchun

$$\left|\varphi'^{(\theta_k)}-\varphi'^{(\tau_k)}\right|<\frac{\varepsilon}{M\cdot(\beta-\alpha)}\,(\theta_k,\tau_k\in[t_{\ k},t_{\ k+1}])$$

bo‘ladi. Unda

$$\left|\sum_{k=0}^{n-1} f\big(\varphi(\tau_k),\psi(\tau_k)\big)\cdot\ \left[\varphi'^{(\theta_k)}-\varphi'^{(\tau_k)}\right]\cdot\Delta t_k\right|$$

$$<M\sum_{k=0}^{n-1}\frac{\varepsilon}{M\cdot(\beta-\alpha)}\cdot\Delta t_{\ k}=\ \frac{\varepsilon}{(\beta-\alpha)}\sum_{k=0}^{n-1}\Delta t_k=\varepsilon$$

demak ,

$$\lim_{\lambda_{p\to0}}\sum_{k=0}^{n-1} f\big(\varphi(\tau_k),\psi(\tau_k)\big)\cdot\ \left[\varphi'^{(\theta_k)}-\varphi'^{(\tau_k)}\right]\cdot\Delta t_{\ k}=0$$

bo‘ladi. Bu munosabatni e’tiborga olib, (4.17) tenglikda, $\lambda_p\to0$ da, limitga o‘tib quyidagini topamiz :

$$\lim_{\lambda_p\to0}\sigma'=\ \lim_{\lambda_p\to0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\cdot\Delta t_k=\ \int_\alpha^\beta f\big(\varphi(t),\psi(t)\big)\varphi'(t)dt$$

Demak,

$$\int_{\overset{\frown}{AB}} f(x,y)dx=\int_\alpha^\beta f(\varphi(t),\psi(t))\cdot\ \varphi'(t)dt$$

Teorema isbot bo‘ldi.

Endi (4.15) sistemada ($\varphi(t)$ funktsiya $[\alpha\,,\beta]$ da uzluksiz, $\psi(t)$ funksiya esa $[\alpha\,,\beta]$ da $\varphi'(t)\psi'(t)$ hosilaga ega va bu hosila shu oraliqda uzluksiz bo‘lsin.

3-Teorema. Agar $f(x,y)$ funksiya $\breve{AB}$ da berilgan va uzliksiz bo‘lsa, u holda funksiyaning $\breve{AB}$ egri chiziq bo‘yicha olingan ikkinchi tur egri chiziqli integrali

$$\int_{\breve{AB}} f(x,y)dy$$

mavjud va

$$\int_{\breve{AB}} f(x,y)dx = \int_{\alpha}^{\beta} f\big(\varphi(t),\psi(t)\big)\cdot\varphi'(t)dt$$

bo‘ladi.

Bu teorema yuqoridagi 2- teoremaga kabi isbotlanadi.

Bu teoremalar, bir tomonidan, uzluksiz funksiya ikkinchi tur egri chiziqli integralning mavjudligini aniqlab bersa, ikkinchi tomonidan, bu integral (Riman integrali) orqali ifodalanishini ko‘rsatadi.

$\breve{AB}$ egri chiziq (4.15) sistema bilan berilgan bo‘lib, $\big(\varphi(t),\psi(t)\big)$ funksiyalar $[\alpha\,,\beta]$ da $(\varphi'(t),\psi'(t))$ hosilalarga ega va bu hosilalar uzliksiz bo‘lsin.

$\breve{AB}$ egri chiziq ikkita $P(x,y)$, va $Q(x,y)$ funksiyalar berilgan bo‘lib, ular shu chiziqda uzluksiz bo‘lsa, u holda

$$\int_{\breve{AB}} P(x,y)d$$

$$+\int_{\breve{AB}} Q(x,y)dy$$

$$=\int_{\breve{AB}}\big[P\big(\varphi(t),\psi(t)\big)\varphi'(t)dt + Q\big(\varphi(t),\psi(t)\big)\varphi'(t)\big]dt$$

bo‘ladi.

3. Ikkinchi tur egri chiziqli integralarning xossalari. Yuqorida keltirilgan teoremalar uzluksiz funksiyalarning ikkinchi tur egri chiziqli integrallarni bizga malum bo‘lgan aniq integral- Riman integraliga kelishini ko‘rsatadi.Binobarin , bu egri chiziqli integrallar ham Riman integrali xossalari kabi xossalarga ega bo‘ladi . O‘tgan paragrafda xuddi shunday mulohaza 1-tur egri chiziqli integrallarga nisbatan bo‘lgan edi. Shularni e’tiborga olib, 2- tur egri chiziqli integrallarning xossalarni keltirishni va tegishli xulosalar chiqarishni o‘quvchiga xovola etamiz.

4. Ikkinchi tur egri chiziqli integrallarni hisoblash. Yuqorida keltirilgan teoremalar funksiyaning ikkinchi tur egri chiziqli integrallarning mavjudligini tasdiqlabgina qolmasdan ularni hissoblash yo‘lini ko‘rsatadi . Demak , ikkinchi tur egri chiziqli integrallar ham, asosan Riman integraliga keltirib hisoblanadi:

$$\int_{\breve{AB}} f(x,y)dx = \int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\cdot\varphi'(t)dt \qquad (4.18)$$

$$\int_{\breve{AB}} f(x,y)dy = \int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\cdot\psi'(t)dt \qquad (4.19)$$

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy$$
$$= \int_{\breve{AB}} \big[P\big(\varphi(t),\psi(t)\big)\varphi'(t)dt$$
$$+ Q\big(\varphi(t),\psi(t)\big)\psi'(t)\big]\,dt \qquad (4.20)$$

Xussusan , $\breve{AB}$ egri chiziq $y = y(x)\ a \le y \le b$ tenglama bilan aniqlangan bo‘lib, $y(x)$ funksiya $[a,b]$ da hosilaga ega va u uzluksiz bo‘lsa, (4.19), (2.20) formulalar quyidagi

$$\int_{\breve{AB}} f(x\ ,y)dy = \int_{a}^{b} f\big(x,y(x)\big)dx \qquad (4.21)$$

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_{a}^{b} [P(x,y(x)) + Q(x,y(x))y'(x)]dx$$

ko‘rinishiga keladi.

Shunikdek, $\breve{AB}$ egri chiziq $x = x(x)\ c \le y \le d$ tenglama bilan aniqlangan bo‘lib, (5.7) va (5.8) formulalar quyidagi

$$\int_{\breve{AB}} P(x,y)dy = \int_{\breve{c}}^{d} f(x(y),y)dy\,,$$

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy$$

$$= \int_{c}^{d} [P(x(y),y)x'(y) + Q(x(y),y)]dy \qquad (4.22)$$

ko‘rinishga keladi.

1- misol. Ushbu

$$\int_{\breve{AB}} y^2dx + x^2dy$$

integralni qaraylik. Bunda

$$\breve{AB} - \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Ellipsning yuqori yarim tekislikdagi qismidan iborat.

Ellipsning parametrik tenglamasi quyidagicha bo‘ladi:

$$\left.\begin{array}{l} x = acost \\ y = bsint \end{array}\right\} \qquad (4.23)$$

$A = (a,0)$ nuqtaga parameter t ning $t = 0$ qiymati, B$(-a,0)$ nuqtaga esa $t = \pi$ qiymati mos kelib, 0 dan π gacha o‘zgarganda (x,y) nuqta A dan B ga qarab ellipsning yuqori yarim tekislikdagi qismini chizadi.

$P(x,y) = y^2 Q(x,y) = x^2$ funksiyalar esa $\breve{AB}$ da uzluksiz. (4.23) formuladan foydalanib quyidagini topamiz:

$$\int_{\breve{AB}} y^2dx + x^2dy = \int_{0}^{\pi} [b^2sin^2t(-asint) + a^2cos^2t(bcost)]dt$$

$$= ab\int_{0}^{\pi} (acos^3t - bsin^3t)dt = -\frac{4}{3}ab^2$$

Ushbu

$$\int_{\breve{AB}} 3x^2ydx + (x^3+1)dy$$

integralni qaraylik. Bunda $\breve{AB}$ egri chiziq:

a) (0,0) nuqtadan chiqqan (0,0) va (1,1) nuqtalarni birlashtiruvchi to‘g‘ri chiziq kesmasi,

b) (0,0) dan chiqqan (0,0) va (1,1) nuqtalarni birlashtiruvchi $y = x^2$ parabolaning yoyi,

v) (0,0) nuqtadan chiqan (0,0), (1,0), (1,1) nuqtalarni birlashtiruvchi siniq chiziqdan iborat.

Yuqoridagi (4.20), (4.21) va (4.22) formulalardan foydalanib quyidagilarni topamiz: a) holda

$$\int_{\breve{AB}} 3x^2ydx + (x^3+1)dy$$

$$= \int_0^1 [3x^2x + (x^3+1)dx] = \int_0^1 (4x^3+1)dx = 2.$$

b) holda

$$\int_{\breve{AB}} 3x^2ydx + (x^3+1)dy$$

$$= \int_0^1 [3x^2x^2 + (x^3+1)2x]dx = \int_0^1 (5x^4+2x)dx = 2$$

v) holda

$$\int_{\breve{AB}} 3x^2ydx + (x^3+1)dy = \int_{\breve{AC}} 3x^2ydx + (x^3+1)dy$$

$$= \int_{\breve{CB}} 3x^2ydx + (x^3+1)dy$$

Bunda $\breve{AC}$ - (0, 0) va (1,0) nuqtalarni, $\breve{CB}$ - (1,0) va (1,1) nuqtalarni birlashtiruvchi to‘g‘ri chiziq kesmalaridan iborat.

Ma’lumki,

$$\int_{\breve{AC}} 3x^2ydx + (x^3+1)dy = 0, \int_{\breve{CB}} 3x^2ydx + (x^3+1)dy =$$

$$= \int_0^1 2dy = 2.$$

Demak,

$$\int\limits_{\breve{AB}} 3x^2y dx + (x^3+1)dy = 2$$

Ikkinchi tur egri chiziqning yoy uzunligini hisoblashga doir misol keltiramiz.

2-misol. Ushbu

$$f(x) = x^{\frac{3}{2}} \quad (0 \le x \le 4)$$

funksiya tasvirlangan egri chiziq yoyining uzunligini toping. Avvalo berilgan funksiyaning hosilasini topamiz.

$$f'(x) = \left(x^{\frac{3}{2}}\right)' = \frac{3}{2}x^{\frac{1}{2}}.$$

Demak, AB yoyning uzunligi formulasidan foydalanib, quyidagiga ega bo'lamiz.

$$l = \int\limits_a^b \sqrt{1 + f'(x)^2 dx} \qquad (*)$$

formuladan

$$1 + f'(x)^2 = 1 + (\frac{3}{2}x^{\frac{1}{2}})^2 = 1 + \frac{9}{4}x$$

bo'lib , (*) formulaga binoan

$$l = \int\limits_0^4 \sqrt{1 + \frac{9}{4}x\, dx}$$

natijaga ega bo'lamiz . Keyingi integraldan $1 + \frac{9}{4}x = t$ almashtirishni bajaramiz unda $dx = \frac{4}{9}dt \quad 1 \le t \le 10$ bo'lib,

$$\int\limits_0^4 \sqrt{1 + \frac{9}{4}x\, dx}$$

$$= \frac{4}{9}\int\limits_1^{10} t^{\frac{1}{2}}dt$$

$$= \frac{8}{27}t^{\frac{3}{2}}\Big|_1^{10} = \frac{8}{27}\left(\sqrt{1000} - 1\right) = \frac{8}{27}\left(10\sqrt{10} - 1\right)$$

bo'ladi. Demak$l = \frac{8}{27}(10\sqrt{10} - 1)$.

V- BOB.

GRIN FORMULASI

5.1-§. GRIN FORMULASI VA UNING TADBIQLARI

Ma'lumki, Nyuton – Leybnits formulasi $f(x)$ funksiyaning $[a, b]$ oraliq bo'yicha olingan aniq integralini shu funksiya boshlang'ich funksiyasining oraliq chetlari (chegaralari) dagi qiymatlari orqali ifodalanar edi.

Biror $(D)((D) \subset R^2)$sohada berilgan $f(x, y)$uzluksiz funksiyaning ikki karrali

$$\iint\limits_{(D)} f(x, y)dxdy$$

integralini tegishli funksiyaning shu soha chegarasidagi qiymatlari orqali ifodalaydigan formulalar ham mavjud. Quyida bu formulani keltiramiz.

1. Grin formulasi. Yuqoridan $y = \varphi_2(x)\ (a \leq x \leq b)$ funksiya grafigi, yon tomonlardan$x = a,\ x = b$ vertikal chiziqlar hamda pastdan $y_1 = \varphi_1(x)(a \leq x \leq b)$ funksiya grafigi bilan chegaralangan soha egri chiziqli trapetsiyai qaraylik. Bu sohani (D) bilan, uning chegarasi – yopiq chiziqni $\partial(D)$ bilan belgilaylik.

Ma'lumki, $AB - \varphi_2(x)$ funksiya grafigi $EC - \varphi_1(x)$ funksiya grafigi hamda

$$\partial(D) = EC + CB + BA + AE$$

$P(x, y)$funksiyashu (D)sohada berilgan va uzluksiz bo'lib,

$$\frac{\partial P(x, y)}{\partial y}$$

xususiy hosilaga ega va u ham (D) da uzluksiz bo'lsin. U holda ushbu

$$\iint\limits_{(D)} \frac{\partial P(x, y)}{\partial y} dxdy$$

integral mavjud bo'ladi.

$$\iint\limits_{(D)} \frac{\partial P(x, y)}{\partial y} dxdy = \int\limits_a^b \left(\int\limits_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P(x, y)}{\partial y} \right) dx$$

bo‘ladi. Endi

$$\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P(x,y)}{\partial y} dy = P(x,y) \Big|_{y=\varphi_1(x)}^{y=\varphi_2(x)} = P\big(x,\varphi_2(x)\big) - P(x,\varphi_1(x))$$

bo‘lishini e’tiborga olib quyidagini topamiz:

$$\iint\limits_{(D)} \frac{\partial P(x,y)}{\partial y} dxdy = \int_a^b P\big(x,\varphi_2(x)\big)dx - \int_a^b P(x,\varphi_1(x))dx$$

$$\int_a^b P\big(x,\varphi_2(x)\big)dx = \int\limits_{AB} P(x,y)dx, \int_a^b P(x,\varphi_1(x))dx = \int\limits_{EC} P(x,y)dx$$

bo‘ladi. Demak,

$$\iint\limits_{(D)} \frac{\partial P(x,y)}{\partial y} dxdy = \int\limits_{AB} P(x,y)dx - \int\limits_{EC} P(x,y)dx$$
$$= -\int\limits_{BA} P(x,y)dx - \int\limits_{EC} P(x,y)dx$$

Ma’lumki,

$$\int\limits_{CB} P(x,y)dx = 0, \int\limits_{EA} P(x,y)dx = 0$$

bu tengliklarni hisobga olib quyidagini topamiz:

$$\iint\limits_{(D)} \frac{\partial P(x,y)}{\partial y} dxdy$$
$$= -\int\limits_{EC} P(x,y)dx - \int\limits_{CB} P(x,y)dx - \int\limits_{BA} P(x,y)dx$$
$$- \int\limits_{AE} P(x,y)dx$$
$$= -\Bigg(\int\limits_{EC} P(x,y)dx + \int\limits_{CB} P(x,y)dx + \int\limits_{BA} P(x,y)dx$$
$$+ \int\limits_{AE} P(x,y)dx\Bigg) = -\int\limits_{\partial(D)} P(x,y)dx$$

Demak,

$$\iint_{(D)} \frac{\partial P(x,y)}{\partial y} dxdy = - \int_{\partial(D)} P(x,y)\, dx \quad (5.1)$$

Endi yuqoridan $y = c$, pastdan $y = d$chiziqlar, yon tomondan esa $x = \psi_1(y), y = \psi_2(y)$ funksiyalar grafiklari bilan chegaralangan soha – egri chiziqli trapetsiyani qaraylik. Bu sohani (D) bilan, uning chegarasi – yopiq $\partial(D)$ bilan belgilaylik.

$Q(x,y)$ funksiya shu (D) sohada berilgan, uzluksiz bo‘lib,

$$\frac{Q(x,y)}{\partial x}$$

xususiy hosilaga ega va bu hosila (D) da uzluksiz bo‘lsin. U holda

$$\iint_{(D)} \frac{\partial Q(x,y)}{\partial x} dxdy = \int_{\partial(D)} \partial Q(x,y)\, dy \quad (5.2)$$

bo‘ladi.

Bu formulaning to‘g‘riligi yuqoridagidek mulohaza yuritish bilan isbotlanadi.

Endi R^2 fazoda qaralayotgan (D) soha yuqoridagi ikki holda qaralgan sohaning har birining xarakteriga ega bo‘lgan soha bo‘lsin, $\partial(D)$ esa uning chegarasi bo‘lsin. Bu (D)sohada ikkita $P(x,y)$ va $Q(x,y)$funksiyalar berilgan, uzluksiz bo‘lib, ular $\frac{\partial P(x,y)}{\partial y}, \frac{\partial Q(x,y)}{\partial x}$ xususiy hosilalarga ega hamda bu hosilalar ham (D) da uzluksiz bo‘lsin. Ma’lumki, bu holda (5.1) va (5.2) formulalar o‘rinli bo‘ladi. Ularni hadlab qo‘shib ushbuni topamiz:

$$\int_{\partial(D)} P(x,y)\, dx + Q(x,y) dx$$

$$= \iint_{(D)} \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy \quad (5.3)$$

bu Grin formulasi deb ataladi.

Demak, Grin frmulasi soha bo‘yicha olingan ikki karrali integralni shu soha chegarasi bo‘yicha olingan egri chiziqli integral bilan bog‘laydigan formula ekan.

Biz yuqorida Grin formulasini maxsus ko‘rinishdagi (D) sohalar uchun keltirdik. Aslida bu formula ancha keng sinfdagi sohalar uchun ham to‘g‘ri bo‘lib, bu faqat u sohalarni chekli sondagi egri chiziqli trapetsiyalar yig‘indisi sifatida tasvirlash bilan isbot qilinadi.

2. Grin formulasining ba’zi bir tatbiqlari.

1^0 . Shaklning yuzini topish. Grin formulasidan foydalanib, yassi shaklning yuzini sodda funksiyalarning egri chiziqli integrallari yordamida hisoblanishini ko‘rsatish qiyin emas. Haqiqatan ham, (5.3) formulada $P(x,y) - y, Q(x,y) = 0$ deyilsa, u holda

$$\int_{\partial(D)} (-y)\, dx = \iint_{(D)} dxdy = D$$

bo‘ladi. Demak,

$$D = - \int_{\partial(D)} ydx$$

Agar (6.3) formulada $P(x,y) = 0, Q(x,y) = x$ deyilsa, u holda

$$D = \int_{\partial(D)} xdy \qquad (5.4)$$

bo‘ladi.

(6.3) formulada $P(x,y) = -\frac{1}{2}y, Q(x,y) = \frac{1}{2}x$ deb olinsa, (D) sohaning yuzi

$$D = \frac{1}{2} \int_{\partial(D)} xdy - ydx \qquad (5.5)$$

bo‘ladi.

1-misol.

$$\left.\begin{array}{l} x = acost \\ y = bsint \end{array}\right\} (0 \leq t \leq 2\pi.$$

ellips bilan chegaralangan shaklning yuzi topilsiz. (6.4) formulaga ko‘ra

$$D=\frac{1}{2}\int\limits_{\partial(D)} xdy-ydx$$
$$=\frac{1}{2}\int\limits_{0}^{2\pi}(a\cos t\ \ b\cos t+b\ \ \sin t\ a\ \ \sin t)dt$$
$$=\frac{1}{2}ab\int\limits_{0}^{2\pi}(cos^2t+sin^2t)dt=\pi ab$$

bo‘ladi.

2^0. Ikki karrali integrallarni o‘garuvchilarni almashtirib hisoblash. Sohani (D) sohaga akslantiruvchi

$$\left.\begin{aligned} x&=\varphi(u\,,v\,)\\ y&=\psi(u\,,v\,)\end{aligned}\right\}\ (6.6)$$

sistema 1-3 shartlar bajarilganda (D)sohaning yuzi

$$D=\iint\limits_{(\Delta)}|I(u,v)|dudv=\iint\limits_{(\Delta)}\left|\frac{D(x,y)}{D(u,v)}\right|dudv \qquad (5.7)$$

bo‘lishi aytilgan edi. Grin formulasidan foydalanib shu formulaning to‘g‘riligini isbotlaymiz. Avvalo (6.4) formuladan foydalanib, (D)sohaning yuzi

$$D=\int\limits_{\partial(D)} xdy \qquad (5.8)$$

bo‘lishini topamiz. Faraz qilaylik, $\partial(\Delta)$ parametrik formada ushbu

$$\begin{aligned} u&=u(t)\\ v&=v(t)\end{aligned}\ (\alpha\le t\le\beta\ \ yoki\ \ \alpha\ge t\ge\beta)$$

sistema bilan ifodalansin. U holda quyidagi

$$\left.\begin{aligned} x&=\varphi(u\,,v\,)=\varphi(u(t),v(t),\\ y&=\psi(u\,,v\,)=\psi(u(t)\,,v(t\,)\end{aligned}\right\}$$

Sistema (D) sohaning $\partial(D)$ chegarasini ifodalaydi. Bunda parametrning o‘zgarish chegarasini shunday tanlab olamizki, t parameter α dan β ga qarab o‘zgarganda $\partial(D)$ egri chiziq musbat yo‘nalishda bo‘lsin. U holda (5.8) tenglik ushbu

$$D = \int\limits_{\partial(D)} xdy = \int\limits_{\partial(D)} \varphi(u\,,v\,)d\psi(u\,,v\,) =$$

$$= \int\limits_{\alpha}^{\beta} \varphi\big(u(t),v(t)\big)\left[\frac{\partial(\psi)}{\partial u}u'(t) + \frac{\partial(\psi)}{\partial v}v'(t)\right]dt \qquad (5.9)$$

ko'rinishga keladi.

Agar

$$\int\limits_{\partial(D)} \varphi(u\,,v\,)\left[\frac{\partial\psi}{\partial u}du + \frac{\partial\psi}{\partial v}dv\right]$$

$$= \int\limits_{\alpha}^{\beta} \varphi(u(t)\,,v(t)\,)\left[\frac{\partial\psi}{\partial u}u'(t) + \frac{\partial\psi}{\partial v}v'(t)\right]dt \quad (6.10)$$

bo'lishini e'tiborga olsak, u holda

$$D = \pm \int\limits_{\partial(D)} x\frac{\partial y}{\partial u}du + x\frac{\partial y}{\partial v}dv \qquad (5.11)$$

bo'lishini topamiz. Bu tenglikdagi integral belgisi oldiga qo'yilgan ishorani tushuntiramiz. Yuqorida, t parametr αdan β ga qarab o'zgarganda $\partial(D)$egri chiziqni musbat yo'nalishda bo'lishini aytdik. Bu holda , $\partial(\Delta)$egri chiziqning yo'nalishi musbat ham bo'lishi mumkin, manfiy ham bo'lishi mumkin. Shuning uchun (5.9) va (5.10) munosabatlar bir – biridan ishora bilan farq qiladi. Agar $\partial(D)$egri chiziqning musbat yo'nalishiga, $\partial(\Delta)$egri chiziqning ham musbat yo'nalishi mos kelsa, unda "+" ishora olinadi, aks holda esa "-" ishora olinadi.

Endi ushbu

$$\int\limits_{\partial(D)} P\,(u,v)du + Q(u,v)dv$$

$$= \iint\limits_{(\Delta)} \left(\frac{\partial Q(u,v)}{\partial u} - \frac{\partial P(u,v)}{\partial v}\right)dudv \qquad (5.12)$$

Grin formulasida

$$P(u,v) = x\frac{\partial y}{\partial u}\ ,Q(u,v) = x\frac{\partial y}{\partial v}$$

deb olsak, u holda bu formula quyidagi ko'rinishga keladi:

$$\int\limits_{(\Delta)} x\frac{\partial y}{\partial u}du + x\frac{\partial y}{\partial v}dv = \iint\limits_{(\Delta)} \left[\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(x\frac{\partial y}{\partial u}\right)\right]dudv \qquad (5.13)$$

Agar

$$\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + x\frac{\partial^2 y}{\partial v\partial u}, \qquad \frac{\partial}{\partial v}\left(x\frac{\partial y}{\partial u}\right) = \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} + x\frac{\partial^2 y}{\partial u\partial v}$$

va

$$\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{D(x,y)}{D(u,v)}$$

ekanini e’tiborga olsak, unda (5.11), (5.12) va (5.13) munosabatlardan

$$D = \pm\iint\limits_{(\Delta)} \frac{D(x,y)}{D(u,v)}dudv$$

bo‘lishi kelib chiqadi.

Ma’lumki,

$$I(u,v) = \frac{D(x,y)}{D(u,v)}$$

Yakobian aniq ishorali, D esa ma’nosiga ko‘ra musbat bo‘lishi kerak. Demak, integral belgisi oldidagi ishora Yakobianning ishorasi bilan bir xil bo‘lishi kerak. Shuning uchun

$$D = \iint\limits_{(\Delta)} \left|\frac{\partial Q(x,y)}{D(u,v)}\right| dudv$$

bo‘ladi. Shuni isbotlash lozim edi.

3^0. Egri chiziqli integral qiymatining integrallash yo‘liga bog‘liq bo‘lmasligi. Chegaralangan yopiq bog‘lamli $(D)((D) \subset R^2)$ sohada ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo‘lsin. Bu funksiyalar (D) sohada uzluksiz va

$$\frac{\partial P(x,y)}{\partial y}, \frac{Q(x,y)}{dx}$$

xususiy hosilalarga ega bu hosilalar ham shu sohada uzluksiz bo‘lsin.

1) Agar (D) sohada

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{dx} \qquad (5.14)$$

bo‘lsa, u holda (D) sohaga tegishli har qanday K yopiq chiziq bo‘yicha olingan ushbu

$$\int_K P\,(x,y)dx + Q(x,y)dy$$

integral nolga teng bo‘ladi:

$$\int_K P\,(x,y)dx + Q(x,y)dy = 0$$

Isbot. K yopiq chiziq chegaralangan sohani (G) deylik. Ma’lumki,

$(G) \subset (D)$. Grin formulasiga ko‘ra

$$\int_K P\,(x,y)dx + Q(x,y)dy = \iint_{(G)} \left(\frac{\partial Q(x,y)}{dx} - \frac{\partial P(x,y)}{dy}\right) dxdy$$

bo‘ladi. Shartga ko‘ra (D) da, demak, (G) da

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

U holda (5.14) munosabatdan

$$\int_{(G)} \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right) dxdy = 0$$

bo‘ladi. Demak,

$$\int_K P\,(x,y)dx + Q(x,y)dy = 0$$

2) Agar (D) sohaga tegishli bo‘lgan har qanday K yopiq chiziq bo‘yicha olingan ushbu integral

$$\int_K P\,(x,y)dx + Q(x,y)dy = 0$$

bo‘lsa, u holda quyidagi

$$\int_{AB} P(x,y)dx + Q(x,y)dy \quad \big(AB \subset (D)\big) \quad (5.15)$$

integral A va Bnuqtalarni birlashtiruvchi egri chiziqqa bog‘liq bo‘lmaydi, ya’ni (5.15) integral qiymati integrallash yo‘liga bog‘liq bo‘lmaydi.

Isbot. (D) sohaning A va Bnuqtalarini birlashtiruvchi va shu sohaga tegishli bo‘lgan ixtiyoriy ikkita AaB va AbB egri chiziqni olaylik. Bu holda AaB va AbB egri chiziqlar birgalikda (D) sohaga tegishli bo‘lgan yopiq chiziqni tashkil etadi. Uni K bilan belgilaylik:

$$K = AaBbA.$$

Shartga ko‘ra

$$\int_{K} P\,(x,y)dx + Q(x,y)dy = \int_{AaBbA} P(x,y)\,dx + Q(x,y)dy = 0$$

bo‘ladi. Integralning xossasidan foydalanib ushbuni topamiz:

$$\int_{AaBbA} P(x,y)\,dx + Q(x,y)dy$$

$$= \int_{AaB} P(x,y)dx + Q(x,y)dy + \int_{BbA} P(x,y)dx + Q(x,y)dy$$

$$= \int_{AaB} P(x,y)dx + Q(x,y)dy - \int_{AbB} P(x,y)dx + Q(x,y)dy$$

Demak,

$$\int_{AaB} P(x,y)dx + Q(x,y)dy - \int_{AaB} P(x,y)dx + Q(x,y)dy = 0$$

Bundan esa

$$\int_{AaB} P(x,y)dx + Q(x,y)dy = \int_{AbB} P(x,y)dx + Q(x,y)dy$$

ekanligi kelib chiqadi.

1 – eslatma. Yuqoridagi tasdiq, isbot jarayonida ko‘rinadiki, AB egri chiziq sodda egri chiziqlar to‘plamidan ixtiyoriy olinganda o‘rinlidir.

3) Agar ushbu

$$\int_{AB} P(x,y)dx + Q(x,y)dy \quad \big(AB \subset (D)\big) \qquad (5.15).$$

Integral A va B nuqtalarini birlashtiruvchi egri chiziqqa bog‘liq bo‘lmasa, ya’ni integral integrallash yo‘liga bog‘liq bo‘lmasa, u holda

$$P(x,y)dx + Q(x,y)dy$$

ifoda D sohada berilgan biror funksiyaning to‘liq differensiali bo‘ladi.

Isbot. Modomiki, (5.15) integrallash yo‘liga bog‘liq emas ekan, u holda integral A=$(x_0 y_0)$, va A=$(B = x_1 y_1)$, nuqtalar bilan bir qiymatli aniqlanadi. Shuning uchun bu holda (5.5) integral quyidagicha ham yoziladi:

$$\int_{(x_0,y_0)}^{(x_1,y_1)} P(x,y)dx + Q(x,y)dy$$

Endi A nuqtani tayinlab, B nuqta sifatida D sohaning ixtiyoriy (x,y) nuqtasini olib, ushbu:

$$\int_{(x_0,y_0)}^{(x_1,y_1)} P(x,y)dx + Q(x,y)dy$$

integralni qaraymiz. Ravshani, bu integral (x,y) ga bog‘liq bo‘ladi:

$$F(x,y) = \int_{(x_0,y_0)}^{(x_1,y_1)} P(x,y)dx + Q(x,y)dy$$

Bu funksiyaning xususiy hosilalarini hisoblaymiz. (x,y) nuqtaning x koordinatasiga shunday orttirma beraylikki, $(x+\Delta x, y)$ nuqta va (x,y),
$(x+\Delta x, y)$ nuqtalarni birlashtiruvchi to‘g‘ri chiziq kesmasi ham (D) sohaga tegishli bo‘lsin. Natijada $F(x,y)$ funksiya ham xususiy orttirmaga ega bo‘ladi:

$$F(x+\Delta x, y) - F(x,y)$$
$$= \int_{(x_0,y_0)}^{(x+\Delta x,y)} P(x,y)dx + Q(x,y)dy - \int_{(x_0,y_0)}^{(x,y)} P(x,y)dx$$
$$+ Q(x,y)dy = \int_{(x_0,y_0)}^{(x+\Delta x,y)} P(x,y)dx + Q(x,y)dy$$
$$= \int_{(x_0,y_0)}^{(x+\Delta x,y)} P(x,y)dx.$$

O‘rta qiymat haqidagi teoremadan foydalanib quyidagini topamiz:

$$\int_{(x_0,y_0)}^{(x+\Delta x,y)} P(x,y)dx = P(x+\theta\cdot\Delta x, y) \quad (0<\theta<1)$$

Natijada
$$\frac{F(x+\Delta x,y)-F(x,y)}{\Delta x}=P(x+\theta\cdot\Delta x,y)$$
bo'ladi. Bundan
$$\lim_{\Delta x\to 0}\frac{F(x+\Delta x,y)-F(x,y)}{\Delta x}=\lim_{\Delta x\to 0}P(x+\theta\cdot\Delta x,y)=P(x,y)$$
bo'ladi. Demak,
$$\frac{\partial F(x,y)}{\partial x}=P(x,y)$$
Xuddi shunga o'xshash
$$\frac{\partial F(x,y)}{\partial y}=Q(x,y)$$
bo'lishi ko'rsatiladi.
Shunday qilib,
$$P(x,y)dx+Q(x,y)dy=\frac{\partial F(x,y)}{\partial y}\,dx+\frac{\partial F(x,y)}{\partial y}dy=dF(x,y)$$
bo'ladi.
4) Agar
$$P(x,y)dx+Q(x,y)dy \text{ (5.16)}$$
ifoda (D) sohada berilgan biror funksiyaning to'liq differensiali bo'lsa, u holda
$$\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$$
bo'ladi.
Isbot. Aytaylik, (5.16) ifoda (D) sohada berilgan $F(x,y)$ funksiyaning to'liq differensiali bo'lsin:
$$P(x,y)dx+Q(x,y)dy=dF(x,y)$$
Ma'lumki,
$$P(x,y)=\frac{\partial F(x,y)}{\partial x},Q(x,y)=\frac{\partial F(x,y)}{\partial y}.$$
Keyingi tengliklardan ushbuni topamiz:
$$\frac{\partial P(x,y)}{\partial y}=\frac{\partial^2 P(x,y)}{\partial x\partial y},\quad \frac{\partial Q(x,y)}{\partial x}=\frac{\partial^2 F(x,y)}{\partial x\partial y}.$$
Shartga ko'ra:
$$\frac{\partial P(x,y)}{\partial y},\quad \frac{\partial Q(x,y)}{\partial x}$$

lar (D) sohada uzluksiz. Aralash hosilalarning tengligi haqidagi teoremaga binoan:

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

bo‘ladi.

Shunday qilib, Grin formulasidan foydalangan holda, yuqoridagi 1) – 4) tasdiqlar orasida

$$1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 1)$$

Munosabat borligi ko‘rsatildi.

2- misol. $\oint_\gamma \frac{xdy+ydx}{x^2+y^2}$ integralni hisoblang, bunda γ koordinatalar boshidan o‘tmaydigan musbat yo‘nalishda harakatlanuvchi oddiy yopiq kontur.

Agar γ kontur koordinata boshini o‘rab olmasa, u holda Grin formulasini qo‘llab quyidagiga ega bo‘lamiz.

$$I = \iint_D \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right) dxdy$$

$$= \iint_D \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2} dxdy = 0.$$

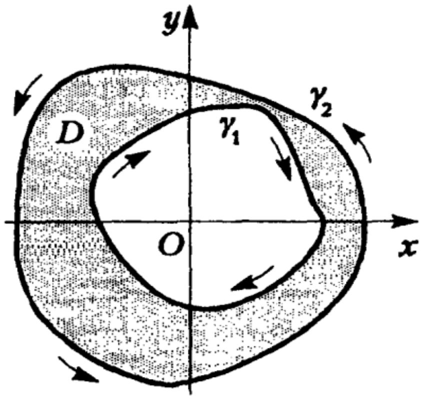
Agar γ kontur koordinata boshini o‘rab olsa, u holda Grin formulasini qo‘llab bo‘lmydi, chunki D soha bir tomonlama bog‘liq emas. Bu holda I integralni to‘g‘ridan-to‘g‘ri hisoblaymiz.

Integral ostidagi differensialni ω orqali belgilaymiz.

$$I = \oint_\gamma \omega$$

integral koordinata boshini o‘rab turuvchi γ egri chiziqni tanlashga bo‘liq bo‘lmasiligini ko‘rsatamiz. γ_1 va γ_2 koordinata boshini o‘rab olgan va

$D \subset \mathbb{R}^2 \backslash \{(0,0)\}$ oddiy soha bilan chegaralangan ixtiyoriy kesishmaydigan yopiq silliq yoki silliq bo‘lakli konturlar bo‘lsin.



$\gamma = \gamma_1 \cup \gamma_2$ chegaraning musbat yo‘nalishni aniqlashda D soha γ_1 va $\gamma_2$chiziqlarning aylanib chiqish yo‘nalishi qarama-qarshi bo‘ladi.

Ikki bog‘lamli D oddiy soha o‘zida integral ostidagi ω ifodaning maxsus nuqtalarini saqlamaydi, shu sababli Grin formulasiga ko‘ra quyidagiga ega bo‘lamiz:

$$I = \iint\limits_D \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right) dxdy = \int\limits_{\gamma_1} \omega - \int\limits_{\gamma_2} \omega = 0.$$

Oxirgi integraldan

$$\int\limits_{\gamma_1} \omega = \int\limits_{\gamma_2} \omega$$

tenglikka ega bo‘lamiz. I integral γ yopiq egri chiziqni tanlanishiga bo‘liq emas.

$\gamma = \{(x, y) \in \mathbb{R}^2 : x = \varepsilon \cos\varphi, y = \sin\varphi, 0 \leq \varphi \leq 2\pi\}$ aylanani tanlab, quyidagiga ega bo‘lamiz:

$$I = \int\limits_0^{2\pi} \frac{\varepsilon^2 \cos^2\varphi + \varepsilon^2 \sin^2\varphi}{\varepsilon^2} d\varphi = \int\limits_0^{2\pi} d\varphi = 2\pi.$$

FOYDALANILGAN ADABIYOTLAR

1. Sadullaev A.S., Mansurov X., Xuydayberganov G., Vorisov A., G‘ulomov R. Matematik analiz kursidan misol va masalalar to‘plami. 1, 2- tomlar. Toshkent. “O‘zbekiston”, 1996.
2. Xudayberganov G., Vorisov A., Mansurov X., Shoimqulov B. Matematik analizdan ma’ruzalar. Toshkent. “Voris nashriyoti”, 2010.
3. Alimov Sh.A., Ashurov R.R. Matematik Tahlil.1-qism.“Kamalak” nashriyoti. Toshkent 2012
4. Alimov Sh.A., Ashurov R.R. Matematik Tahlil. 2-qism.“Kamalak” nashriyoti. Toshkent 2016
5. Демидович Б.П. Сборник задач и упражнений по математическому анализу. Изд. Наука, Москва, 1977
6. Пискунов Н. С. Дифференциал ва интеграл ҳисоб. 1-Том. Ўқитувчи нашриёти. Тошкент, 1972
7. Arfken G. Matematical Methods for physicits. Academic Press New York and London, 1970.

Mundarija

SO'ZBOSHI	3
I BOB. ANIQMAS INTEGRAL	4
1.1-§. Sodda funksiyalarning aniqmas integrallari jadvali	5
1.2-§. O'zgaruvchini almashtirish usuli	7
1.3-§. Bo'laklab integrallash usuli	9
1.4-§. Ratsional funksiyalar. Eng sodda ratsional funksiyalar va ularni integrallash	12
1.5-§. Ratsional funksiyalarni eng sodda ratsional funksiyalarga ajratish	16
1.6-§. Ratsional funksiyalarni integrallash	18
1.7-§. Trigonometrik funksiyalarning ba'zi sinflarini integrallash	20
1.8-§. Ba'zi bir irratsional funksiyalarni integrallash. Eyler almashtirishlari	23
1.9-§. Binomial formula bilan berilgan funksiyani integrallash	28
1.10-§. Har xil funksiyalarni integrallashga misollar	30
II BOB. ANIQ INTEGRAL	35
2.1-§. Aniq integralning ta'rifi	35
2.2-§. Aniq integralning asosiy xossalari	37
2.3-§. Aniq integralni hisoblash	41
III BOB. KARRALI INTEGRAL	49
3.1-§. Ikki karrali integral	49
3.2-§. Uch karrali integral	64
3.3-§.Ikki karrali va uch karrali itegrallarning mexanikaga tatbiqi	72
IV BOB. EGRI CHIZIQLI INTEGRALLAR	93
4.1-§. Birinchi tur egri chiziqli entegrallar	93
4.2-§. Ikkinchi tur egri chiziqli entegrallar	102
V BOB.GRIN FORMULASI	116
5.1-§. Grin formulasi va uning tatbiqlari	116
Foydalanilgan adabiyotlar	129

S.A.Imomkulov, Z.Sh.Ibragimov R.A.Sharipov

"INTEGRAL"

(uslubiy qo'llanma)

Muharrir Ro'zimboy Yo'ldoshev
Texnik muharrir Sherali Yo'ldoshev
Musahhih Tamara Turumova

UrDU noshirlik bo'limi O'zbekiston matbuot va axborot agentligining 2009-yil 19-avgustdagi №148 raqamli buyrug'i bilan qayta ro'yxatdan o'tkazilgan.
Terishga berildi: 23.10.2018
Bosishga ruxsat etildi: 6.11.2018
Ofset qog'ozi. Qog'oz bichimi 60x84 $^1/_{16}$.
Tayms garniturasi. Adadi 50. Buyurtma №.83.
Hisob-nashriyot tabag'i 6.
Shartli bosma tabag'i 5,6.
UrDU noshirlik bo'limida tayyorlandi.
Manzil: 220110. Urganch shahri,
H. Olimjon ko'chasi, 14-uy.
Telefon: (0-362)-224-66-01.

UrDU bosmaxonasida chop etildi.
Manzil: 220110. Urganch shahri,
H. Olimjon ko'chasi, 14-uy.
Telefon: (0-362)-224-66-01.