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**INTEGRAL MANBALI UMUMIY DAVRIY TODA TENGLAMASII
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KIRISH

Tadqiqot mavzusining dolzarbligi. Nochiziqli evalyutsion tenglamalarga qoyilgan Koshi masalalarining global yechimlarini teskari masalalar usulidan foydalanib topish masalasi matematika va fizikaning muhim masalalardan biridir. Bu masalalar hozirgi kunda ko`plab amaliy tatbiqlarga ega, ulardan kvant fizikasi, chiziqli va nochiziqli xususiy hosilali tenglamalar nazariyasi, kristallografiya, geologiya-razvedka kabi masalalarni hal qilishda matematik apparat sifatida umumli foydalanib kelinmoqda. Integral manbali umumiy davriy Toda tenglamasi yuqorida qayt qilingan Nochiziqli evalyutsion tenglamalardan biri bo`lib, qattiq jismlar mexanikasida muhim amaliy tadbiqlarga ega.

Hozirgi kunda moslangan manbali nochiziqli evalutsion tenglamalarning muhim amaliy tatbiqlari topilganligi bois ularni o`rganishga bo`lgan qiziqish yanada ortmoqda.

Ilmiy izlanishning maqsad va vazifalari. Ushbu ishda quyidagi

$$\left\{ \begin{array}{l} \dot{a}_n = P_m(a_n, b_n) + a_n \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_{n+1}^-(\lambda, t) \psi_{n+1}^+(\lambda, t) - \psi_n^-(\lambda, t) \psi_n^+(\lambda, t)] d\lambda \\ \dot{b}_n = Q_m(a_n, b_n) + a_n \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_n^-(\lambda, t) \psi_{n+1}^+(\lambda, t) + \psi_{n+1}^-(\lambda, t) \psi_n^+(\lambda, t)] d\lambda - \\ \quad - a_{n-1} \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_n^-(\lambda, t) \psi_{n-1}^+(\lambda, t) + \psi_{n-1}^-(\lambda, t) \psi_n^+(\lambda, t)] d\lambda \\ a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad n \in Z, \end{array} \right. \quad (1)$$

N -davrlı integral manbali umumiy davriy Toda tenglamasini ushbu

$$a_n(0) = a_n^0, \quad b_n(0) = b_n^0, \quad n \in Z, \quad (2)$$

boshlang`ich shartlar bilan qaraymiz. a_n^0 va b_n^0 , $n \in Z$ lar N -davrlı periodic xaqiqiy sonlar ketma-ketligi. (1) da

$$\begin{aligned} P_m(a_n, b_n) &= a_n[-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}], \\ Q_m(a_n, b_n) &= a_n^2\alpha_{n+1,m} - a_{n-1}^2\alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2\alpha_{n,m}, \quad m \in N, t \in R, \end{aligned}$$

hamda $\{\alpha_{n,j}(t)\}_{0 \leq j \leq m}$, $\{\beta_{n,j}(t)\}_{0 \leq j \leq m}$ lar ushbu

$$\beta_{n,0} = -1, \quad \alpha_{n,0} = 0, \quad \alpha_{n,1} = 2,$$

$$\beta_{n,j-1} - \beta_{n-1,j-1} = b_n(\beta_{n,j-2} - \beta_{n-1,j-2}) - a_n^2 \alpha_{n+1,j-2} + a_{n-1}^2 \alpha_{n-1,j-2}, \quad 2 \leq j \leq m,$$

$$\beta_{n,j} = \frac{a_{n-1}^2}{2} \alpha_{n-1,j-1} - \frac{a_n^2}{2} \alpha_{n+1,j-1} + \frac{b_n^2}{2} \alpha_{n,j-1} - b_n \beta_{n-1,j-1}, \quad 1 \leq j \leq m,$$

$$\alpha_{n,j} = b_n \alpha_{n,j-1} - \beta_{n-1,j-1} - \beta_{n,j-1}, \quad 2 \leq j \leq m,$$

requrent munosabatlarni qanoatlantiradi. (1) sistemada $m \in N$ ning o`zgarishi integral manbali umumiy davriy Toda tenglamasini beradi. $\{a_n(t)\}_{-\infty}^{\infty}$, $\{b_n(t)\}_{-\infty}^{\infty}$, $\{\psi_n^{\pm}(t)\}_{-\infty}^{\infty}$ – funksiyalar nomalum funksiyalar, hamda $\{\psi_n^{\pm}(t)\}_{-\infty}^{\infty}$ funksiyalar ushbu

$$(L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n \quad (3)$$

diskret Xill tenglamasini Floke-Blox yechimlari bo`lib, quyidagi

$$\psi_1^{\pm}(\lambda, t) = 1. \quad (4)$$

(1) sistemadagi E to`plam $L(0)$ operatorning spektridan iborat. $\tilde{\theta}_{N+1}(\lambda, t)$ ko`paytuvchi ushbu

$$\tilde{\theta}_{N+1}(\lambda, t) = \prod_{j=1}^{N-1} (\lambda - \mu_j(t))$$

tenglikdan aniqlanadi. Bu erda $\mu_1(t), \mu_2(t), \dots, \mu_{N-1}(t)$ lar quyidagi

$$\theta_{N+1}(\lambda, t) = 0$$

tenglamani ildizlari. $\theta_n(\lambda, t)$, $n \in Z$ funksiya (3) tenglamani ushbu

$$\theta_0(\lambda, t) = 1, \quad \theta_1(\lambda, t) = 0$$

Shartlarni qanoatlantiruvchi yechimlari.

Ushbu ishning asosiy maqsadi-(1)-(4) masalaning yechimini teskari masalalar usuludan foydalanib topish algoritmini berishdan iborat.

Mavzuning o`rganilish darajasining qiyosiy tahlili: Zarrachani to`g`ri chizikli o`zaro eksponentsial ta`sirini ifodalovchi ushbu

$$\frac{\partial^2 u_n}{\partial t^2} = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}), \quad n \in Z,$$

Toda zanjiri [1] Flashka o`zgaruvchilari [2] orqali quyidagicha

$$\begin{cases} \dot{a}_n = a_n(b_n - b_{n+1}), \\ \dot{b}_n = 2(a_{n-1}^2 - a_n^2), \quad n \in Z \end{cases}$$

ko`rinishga ega bo`ladi. Toda tenglamasi birinchi bor B.M. Toda tomonidan kiritilgan [1]. Shundan keyin, Flashka [2] diskret Xill tenglamasiga qo`yilgan teskar masala usuli yordamida Toda tenglamasiga qo`yilgan Koshi masalasining yechimi topishga muvassar bo`lgan. Xuddi shunday natija Flashkani natijasidan bexabar ravishda Manakov [3] tomonidan ham olingan. Davriy Toda tenglamasi [4-9] ishlarda o`rganilgan. Shundan keyin diskret Xill tenglamasiga qo`yilgan teskari masalalar usuli yordamida integrallanadigan barcha nochiqli evalutsion tenglamalarni topishga bo`lgan qiziqish ortgan. Bu tenglamalar odatda Umumiy Toda tenglamasi deb yuritiladi. Umumiy Toda tenglamasi birinchi bor K. Ueno va K. Takasakilar [10] tomonidan kiritilgan. Umumiy Toda tenglamasining diskret Xill tenglamasiga qo`yilgan teskari masalalar usulida integrallash mumkinligi [11-22] ishlarda ko`rsatilgan.

Tadqiqotning ilmiy yangiligi: Ushbu ishda diskret Xill tenglamasiga qo`yilgan teskari masalalar usuli yordamida integrallanadigan barcha nochiqli evalutsion tenglamalarni keltirib chiqarishning yangicha usuli berilgan. Diskret Xill tenglamasiga qo`yilgan teskari masalalar usulidan foydalanib (1)-(4) masalaning yechimi uchun aniq tasvir olingan.

Tadqiqot predmeti va ob`ekti. Bitiruv malakaviy ishda oddiy differensial tenglamalar, matematik fizika tenglamalari, funksional analiz, differensial operatorlarning spektral nazariyasi predmetlari usullaridan foydalaniladi. Tadqiqotning obyekt integral manbali umumiy davriy Toda tenglamasi.

Tadqiqotning ilmiy ahamiyati. O`ganilgan natijalardan amaliy ahamiyatga ega bo`lib, qattiq jismlar mexanikasi, kristallografiya va geologiya-razvedka masalalarida muhim tadbirlarga ega.

I-BOB. DAVRIY DISKRET XILL TENGLAMASI UCHUN TO`G`RI VA TESKARI SPEKTRAL MASALALAR

Ushbu bobda diskret Hill tenglamasiga qo`yilgan to`g`ri va teskari masalalar haqidagi zarur ma`lumotlar keltiriladi

1-§ Bazis yechimlar va ularning xossalari

Quyidagi tenglamalar sistemasini ko`rib chiqamiz:

$$(L\varphi)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in Z \quad (1.1.1)$$

bu yerda a_n, b_n haqiqiy N davrli koeffitsientlar:

$$a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad n \in Z \quad (1.1.2)$$

N biror natural son, λ kompleks parametr.

Misol 1. $a_n \equiv a \equiv const, b_n \equiv b \equiv const$ bo`lsin. Bu holda (1.1.1) sistema quyidagi ko`rinishda bo`ladi.

$$a y_{n-1} + b y_n + a y_{n+1} = \lambda y_n, \quad n \in Z \quad (1.1.3)$$

buning yechimini $y_n = \mu^n$ ko`rinishida izlaymiz.

$$a\mu^{n-1} + b\mu^n + a\mu^{n+1} = \lambda\mu^n,$$

$$a + b\mu + a\mu^2 = \lambda\mu, \quad a\mu^2 + (b - \lambda)\mu + a = 0.$$

$$D = (b - \lambda)^2 - 4a^2,$$

$$\mu_1 = \frac{-(b - \lambda) + \sqrt{(b - \lambda)^2 - 4a^2}}{2a}, \quad \mu_2 = \frac{-(b - \lambda) - \sqrt{(b - \lambda)^2 - 4a^2}}{2a}.$$

Agar $D \neq 0$ bo`lsa, u holda $y_n = c_1\mu_1^n + c_2\mu_2^n$ (1.1.3) tenglamaning umumiy yechimi bo`ladi. Agar $D = 0$ bo`lsa, u holda $y_n = c_1\mu_1^n + c_2\mu_2^n n$ (1.1.3) tenglamaning umumiy yechimi bo`ladi.

Misol 2. $a_n \equiv \frac{1}{2}, b_n \equiv 0$ bo'lsin. Bu holda (1.1.1) sistema quyidagi ko'rinishda bo'ladi:

$$\frac{1}{2}y_{n-1} + \frac{1}{2}y_n = \lambda y_n, \quad n \in Z \quad (1.1.4)$$

buning yechimini $y_n = \mu^n$ ko'rinishida izlaymiz.

$$\mu^{n-1} + \mu^n = 2\lambda\mu^n,$$

$$1 + \mu^2 = 2\lambda\mu, \quad \mu^2 - 2\lambda\mu + 1 = 0.$$

$$D = 4\lambda^2 - 4 = 4(\lambda^2 - 1),$$

$$\mu_1 = \lambda + \sqrt{\lambda^2 - 1},$$

Agar $D \neq 0$ bo'lsa, yani $\lambda \neq \pm 1$ bo'lsa $\mu_2 = \lambda - \sqrt{\lambda^2 - 1}$ u holda

$$y_n = c_1 \left(\lambda + \sqrt{\lambda^2 - 1} \right)^n + c_2 \left(\lambda - \sqrt{\lambda^2 - 1} \right)^n, \quad (1.1.5)$$

(1.1.4) tenglamaning umumiy yechimi bo'ladi. Agar $\lambda = \pm 1$ bo'lsa,

$$y_n = c_1 (\pm 1)^n + c_n (\pm 1)^n n,$$

(1.1.4) tenglamaning umumiy yechimi bo'ladi. (1.1.5) yechimda λ haqiqiy bo'lib $\lambda^2 - 1 < 0$ bo'lsa, $-1 < \lambda < 1$ bo'ladi. Bu holda $\lambda = \cos \alpha$ desak, (1.1.5) quyidagi tarzda yoziladi.

$$y_n = c_1 e^{in\alpha} + c_n e^{-in\alpha}$$

(1.1.1) tenglamaning quyidagi boshlang'ich shartlarni qanoatlantiruvchi yechimlarini $\theta_n(\lambda)$ va $\varphi_n(\lambda)$ orqali belgilaymiz.

$$\begin{cases} \theta_0(\lambda) = 1. \\ \theta_1(\lambda) = 0, \end{cases} \quad \begin{cases} \varphi_0(\lambda) = 0. \\ \varphi_1(\lambda) = 1, \end{cases} \quad (1.1.6)$$

$\theta_n(\lambda)$ va $\varphi_n(\lambda)$ yechimlar (1.1.1) tenglama yechimlarining fundamental sistemasini tashkil qiladi. Shuning uchun (1.1.1) tenglamaning ixtiyoriy $y_n(\lambda)$ yechimi ular orqali chiziqli ifodalanadi.

$$y_n(\lambda) = c_1 \cdot \theta_n(\lambda) + c_2 \varphi_n(\lambda) \quad n \in \mathbb{Z} \quad (1.1.7)$$

Teorema 1.1. $\theta_n(\lambda)$ va $\varphi_n(\lambda)$ yechim mavjud va yagona bo`lib, ular uchun
ushbu

$$\theta_n(\lambda) = -a_0 \left(\prod_{j=1}^{n-1} a_j \right)^{-1} \left\{ \lambda^{n-2} - \left(\sum_{j=2}^{n-1} b_j \right) \lambda^{n-3} + \left[\frac{1}{2} \left(\sum_{j=2}^{n-1} b_j \right)^2 - \frac{1}{2} \sum_{j=2}^{n-1} b_j^2 - \sum_{j=2}^{n-2} a_j^2 \right] \lambda^{n-4} + \dots \right\}$$

,

$$(1.1.8)$$

$$\varphi_n(\lambda) = \left(\prod_{j=1}^{n-1} a_j \right)^{-1} \left\{ \lambda^{n-1} - \left(\sum_{j=1}^{n-1} b_j \right) \lambda^{n-2} + \left[\frac{1}{2} \left(\sum_{j=1}^{n-1} b_j \right)^2 - \frac{1}{2} \sum_{j=1}^{n-1} b_j^2 - \sum_{j=1}^{n-2} a_j^2 \right] \lambda^{n-3} + \dots \right\}$$

$$(1.1.9)$$

tasvirlar o`rinli.

Isbot:

$$a_{n-1} \theta_{n-1} + b_n \theta_n + a_n \theta_{n+1} = \lambda \theta_n, \quad n \in \mathbb{Z} \quad (1.1.10)$$

tenglikda $n = 1$ desak, va (1.1.8) boshlang'ich shartlarni inobatga olsak

$$a_0 + a_1 \theta_2 = 0, \quad (1.1.11)$$

kelib chiqadi. Bunga ko'ra $\theta_2(\lambda) = -\frac{a_0}{a_1}$ bo'ladi (1.1.10) da $n = 2$ desak va

boshlang'ich shartlardan foydalansak

$$b_2 \theta_2 + a_2 \theta_3 = \lambda \theta_2, \quad (1.1.12)$$

kelib chiqadi (1.1.11) va (1.1.12) dan

$$\theta_3 = -\frac{\lambda - b_2}{a_2} \cdot \theta_2 = -\frac{(\lambda - b_2)a_0}{a_1 \cdot a_2} \quad (1.1.13)$$

kelib chiqadi. (1.1.10) da $n=3$ desak

$$a_2\theta_2 + b_3\theta_3 + a_3\theta_4 = \lambda\theta_3 \quad (1.1.14)$$

kelib chiqadi.(1.1.11), (1.1.13) va (1.1.14) dan

$$\begin{aligned} \theta_4 = -\frac{\lambda - b_3}{a_3} \cdot \theta_3 - \frac{a_2}{a_3} \cdot \theta_2 = -\frac{(\lambda - b_3)(\lambda - b_2) \cdot a_0}{a_1 \cdot a_2 \cdot a_3} + \frac{a_2 \cdot a_0}{a_3 \cdot a_1} = \\ -\frac{a_0}{a_1 \cdot a_2 \cdot a_3} [\lambda^2 - (b_2 + b_3)\lambda + b_2 \cdot b_3 - a_2^2]; \end{aligned} \quad (1.1.15)$$

kelib chiqadi. $a_n \neq 0$, $n \in \mathbb{Z}$ shartdan θ_n yechiminig mavjud va yagona bo'lishi ham shu tarzda ko'rsatiladi, umuman $n \geq 2$ bo'lganda shu jarayonni davom qildirish natijasida (1.1.8) va (1.1.9) tengliklar o'rinli bo'lishi kelib chiqadi.

Teorema isbotlandi.

Teorema 1.2. Quyidagi tengliklar o'rinli

$$\begin{pmatrix} \theta_{n+N} \\ \varphi_{n+N} \end{pmatrix} = \begin{pmatrix} \theta_N & \theta_{N+1} \\ \varphi_N & \varphi_{N+1} \end{pmatrix} \cdot \begin{pmatrix} \theta_n \\ \varphi_n \end{pmatrix} \quad (1.1.16)$$

Isbot. Ushbu

$$a_{n-1}\theta_{n-1} + b_n\theta_n + a_n\theta_{n+1} = \lambda\theta_n, \quad (1.1.17)$$

$$a_{n-1}\varphi_{n-1} + b_n\varphi_n + a_n\varphi_{n+1} = \lambda\varphi_n, \quad (1.1.18)$$

tengliklarda uning o'rniga $n+N$ ni qo'yib, $a_{n+N} = a_n, b_{n+N} = b_n$ shartlardan foydalansak quyidagi

$$a_{n-1}\theta_{n+N-1} + b_n\theta_{n+N} + a_n\theta_{n+N+1} = \lambda\theta_{n+N}, \quad (1.1.19)$$

$$a_{n-1}\varphi_{n+N-1} + b_n\varphi_{n+N} + a_n\varphi_{n+N+1} = \lambda\varphi_{n+N}, \quad (1.1.20)$$

ayniyatlar kelib chiqadi. Bularga ko'ra,

$$\theta_{n+N} = c_1 \theta_n + c_2 \varphi_n, \quad (1.1.21)$$

$$\varphi_{n+N} = D_1 \theta_n + D_2 \varphi_n \quad (1.1.22)$$

bo'ladi. (1.1.21) va (1.1.22) larda $n = 0$ va $n = 1$ deb (1.1.8) boshlang'ich shartni xisobga olsak $c_1 = \theta_N$, $c_2 = \theta_{N+1}$, $D_1 = \varphi_N$, $D_2 = \varphi_{N+1}$, hosil bo'ladi. Bularni (1.1.21) va (1.1.22) ga qo'ysak

$$\theta_{n+N} = \theta_N \cdot \theta_n + \theta_{N+1} \cdot \varphi_n, \quad (1.1.23)$$

$$\varphi_{n+N} = \varphi_N \cdot \theta_n + \varphi_{N+1} \cdot \varphi_n \quad (1.1.24)$$

kelib chiqadi. Bularni matritsaviy tarzda yozsak (1.1.16) xosil bo'ladi. Teorema 2 isbotlandi.

Ta'rif 1.1. Ushbu

$$M = \begin{pmatrix} \theta_N(\lambda) & \theta_{N+1}(\lambda) \\ \varphi_N(\lambda) & \varphi_{N+1}(\lambda) \end{pmatrix} \quad (1.1.25)$$

matritsaga monodromiya matritsasi deyiladi.

Ta'rif 1.2. Ushbu

$$\Delta(\lambda) = \theta_N(\lambda) + \varphi_{N+1}(\lambda) \quad (1.1.26)$$

funksiyaga Lyapunov funktsiyasi yoki Xill diskriminanti deyiladi.

Teorema 1.3. Ushbu

$$\Delta^2(\lambda) - 4 = \left(\prod_{j=1}^N a_j \right)^{-2} \left\{ \lambda^{2N} - 2 \left(\sum_{j=1}^N b_j \right) \lambda^{2N-1} + \left[2 \left(\sum_{j=1}^N b_j \right)^2 - \sum_{j=1}^N b_j^2 - 2 \sum_{j=1}^N a_j^2 \right] \lambda^{2N-2} + \dots \right\} \quad (1.1.27)$$

tenglik o`rinli.

Isbot. Teoremani isboti Lyapunov funktsiyasini ko`rinishi (1.1.26) va bazis yechimlar uchun olingan (1.1.8)-(1.1.9) tasvirlardan to`g`ridan- to`g`ri hisoblashlar yordamida kelib chiqadi.

Ta`rif 1.3. Ushbu

$$W = a_n \cdot [\theta_n(\lambda) \cdot \varphi_{n+1}(\lambda) - \theta_{n+1}(\lambda) \cdot \varphi_n(\lambda)] \quad (1.1.28)$$

ifodaga $\theta_n(\lambda)$, $\varphi_n(\lambda)$ yechimlarning Vroniskiy determinanti deyiladi.

Teorema 1.3. $\theta_n(\lambda)$ va $\varphi_n(\lambda)$ larning W Vroniskiy determinanti n ga ham, λ ga ham bog`liq emas bo`lib, ushbu

$$W = a_n [\theta_n(\lambda) \varphi_{n+1}(\lambda) - \theta_{n+1}(\lambda) \varphi_n(\lambda)] = a_0 \quad (1.1.29)$$

bunga ko`ra $\det M = [\theta_N(\lambda) \varphi_{N+1}(\lambda) - \theta_{N+1}(\lambda) \varphi_N(\lambda)] = 1$ tenglik o`rinli.

Isbot. $\theta_n(\lambda)$ va $\varphi_n(\lambda)$ larni (1.1.1) ga qo`ysak

$$\begin{cases} a_{n-1} \theta_{n-1} + b_n \theta_n + a_n \theta_{n+1} = \lambda \theta_n \\ a_{n-1} \varphi_{n-1} + b_n \varphi_n + a_n \varphi_{n+1} = \lambda \varphi_n \end{cases} \quad (1.1.30)$$

tenglik o`rinli bo`ladi. Bundan esa

$$\begin{cases} a_{n-1} \theta_{n-1} \varphi_n + b_n \theta_n \varphi_n + a_n \theta_{n+1} \varphi_n = \lambda \theta_n \varphi_n \\ a_{n-1} \varphi_{n-1} \theta_n + b_n \varphi_n \theta_n + a_n \varphi_{n+1} \theta_n = \lambda \varphi_n \theta_n \end{cases},$$

$$\begin{cases} a_{n-1} \theta_{n-1} \varphi_n + a_n \theta_{n+1} \varphi_n = (\lambda - b_n) \theta_n \varphi_n \\ a_{n-1} \varphi_{n-1} \theta_n + a_n \varphi_{n+1} \theta_n = (\lambda - b_n) \varphi_n \theta_n \end{cases}$$

tengliklar kelib chiqadi. Pastdagi sistemada tengliklarni bir-biridan ayirib

$$a_n (\theta_{n+1} \varphi_n - \varphi_{n+1} \theta_n) = a_{n-1} (\varphi_{n-1} \theta_n - \varphi_n \theta_{n-1}) \quad (1.1.31)$$

tenglikni hosil qilamiz. Quyidagi bizga ma`lum munosabatlarni inobatga olib,

$$\begin{cases} \theta_{N+n}(\lambda) = \theta_n(\lambda) \\ \varphi_{N+n}(\lambda) = \varphi_n(\lambda) \end{cases}, \quad \begin{cases} \theta_N = \theta_0 \\ \varphi_N = \varphi_0 \end{cases} \quad (1.1.32)$$

va (1.1.8) tengliklardan

$$\varphi_1 \theta_0 - \varphi_0 \theta_1 = 1 \quad (1.1.33)$$

ekani kelib chiqadi, (1.1.2),(1.1.31) va (1.1.33) tengliklardan foydalansak

$$\begin{aligned} W &= a_n [\theta_n(\lambda) \varphi_{n+1}(\lambda) - \theta_{n+1}(\lambda) \varphi_n(\lambda)] = a_{n+N} [\theta_{n+N} \varphi_{n+N+1} - \theta_{n+N+1} \varphi_{n+N}] = \\ &= a_0 [\theta_0 \varphi_1 - \varphi_0 \theta_1] = a_0 \end{aligned}$$

tenglik o'rinli bo'lishi kelib chiqadi. Bunda $\det M = \theta_N \varphi_{N+1} - \theta_{N+1} \varphi_N = 1$ ekani ravshan. **Teorema 1.3** isbotlandi.

$\lambda_1, \lambda_2, \dots, \lambda_{2N}$ sonlar orqali ushbu

$$\Delta^2(\lambda) - 4 = 0$$

tenglamani, hamda $\mu_1, \mu_2, \dots, \mu_{N-1}$ sonlar orqali ushbu

$$\theta_{N+1}(\lambda) = 0$$

tenglamani ildizlarini belgilaymiz. Barcha $\lambda_i, i = 1, 2, \dots, 2N$ lar va $\mu_j, j = 1, 2, \dots, N-1$ lar haqiqiy bo'lib, μ_j ildizlar oddiydir. Quyidagi

$$\Delta^2(\lambda) - 4 = \left(\prod_{j=1}^N a_j \right)^{-2} \prod_{j=1}^{2N} (\lambda - \lambda_j),$$

$$\theta_{N+1}(\lambda) = -a_0 \left(\prod_{j=1}^N a_j \right)^{-1} \prod_{j=1}^{N-1} (\lambda - \mu_j)$$

tengliklarni bajarilishi ravshan. Ushbu

$$\sigma_j = \text{sign} \left[\theta_N(\mu_j) - \frac{1}{\theta_N(\mu_j)} \right], \quad j = 1, 2, \dots, N-1$$

belgilashni kiritamiz.

Ta`rif 1. $\mu_j, j=1, 2, \dots, N-1$ sonlar va $\sigma_j, j=1, 2, \dots, N-1$ ishoralar ketma-ketligiga diskret Xill tenglamasini spectral parametrlari deyiladi.

Ta`rif 2. $\{\mu_j, \sigma_j\}_{j=1}^{N-1}$ spectral parametrlar va $\lambda_i, i=1, 2, \dots, 2N$ sonlar ketma-ketligiga diskret Xill tenglamasini spectral berilganlari deyiladi.

Spectral berilganlarni topish va ularning xossalarini o`rganishga diskret Xill tenglamasi uchun qo`yilgan to`g`ri masala deyiladi. Aksincha, spectral berilganlar yordamida diskret Xill tenglamasini koeffitsientlari a_n, b_n larni topish masalasiga diskret Xill tenglamasini uchun qo`yilgan teskari masala deyiladi.

Ushbu

$$(L\varphi)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in Z \quad (1.1.34)$$

tenglamani koeffisientlarini $k \in Z$ butun songa siljitish natijasida hosil gilingan quyidagi tenglamani ko`rib chiqamiz:

$$a_{n-1+k}y_{n-1} + b_{n+k}y_n + a_{n+k}y_{n+1} = \lambda y_n, \quad n \in Z. \quad (1.1.35)$$

$\theta_{n+k}(\lambda)$ va $\varphi_{n+k}(\lambda)$ lar (1.1.35) ning yechimi bo`ladi. Haqiqatan: Ushbu

$$a_{n-1}\theta_{n-1} + b_n\theta_n + a_n\theta_{n+1} = \lambda\theta_n,$$

tenglama koeffisientlarini $k \in Z$ butun songa siljitsak

$$a_{n-1+k}\theta_{n-1+k} + b_{n+k}\theta_{n+k} + a_{n+k}\theta_{n+1+k} = \lambda\theta_{n+k}$$

bo`ladi. Oxirgi tenglikda quyidagi

$$y_n = \theta_{n+k},$$

belgilashni kiritsak, u holda

$$a_{n-1+k}y_{n-1} + b_{n+k}y_n + a_{n+k}y_{n+1} = \lambda y_n.$$

kelib chiqadi. (1.1.35) tenglamaning quyidagi boshlang'ich shartlarni qanoatlantiruvchi yechimlarini $u_{n,k}(\lambda)$ va $v_{n,k}(\lambda)$ orqali belgilaymiz:

$$\begin{cases} u_{0,k}(\lambda) = 1 \\ u_{1,k}(\lambda) = 0 \end{cases}, \quad \begin{cases} v_{0,k}(\lambda) = 0 \\ v_{1,k}(\lambda) = 1 \end{cases} \quad (1.1.36)$$

Teorema 3.1. (1.1.35) tenglamaning Lyapunov funksiyasi $\tilde{\Delta}(\lambda)$ bo'lsa, u holda $\tilde{\Delta}(\lambda) = \Delta(\lambda)$ bo'ladi.

Isbot: $\theta_{n+k}(\lambda)$ va $\varphi_{n+k}(\lambda)$ yechimlar chiziqli erkli bo'lgani uchun

$$u_{n,k}(\lambda) = A_1\theta_{n+k}(\lambda) + A_2\varphi_{n+k}(\lambda), \quad (1.1.37)$$

$$v_{n,k}(\lambda) = B_1\theta_{n+k}(\lambda) + B_2\varphi_{n+k}(\lambda), \quad (1.1.38)$$

(1.1.37) dan

$$\begin{cases} A_1\theta_k(\lambda) + A_2\varphi_k(\lambda) = 1 \\ A_1\theta_{k+1}(\lambda) + A_2\varphi_{k+1}(\lambda) = 0 \end{cases} \quad (1.1.39)$$

kelib chiqadi. (1.1.39) sistemani yechamiz:

$$\begin{vmatrix} \theta_k(\lambda) & \varphi_k(\lambda) \\ \theta_{k+1}(\lambda) & \varphi_{k+1}(\lambda) \end{vmatrix} = \frac{a_0}{a_k}, \quad \begin{vmatrix} 1 & \varphi_k(\lambda) \\ 0 & \varphi_{k+1}(\lambda) \end{vmatrix} = \varphi_{k+1}(\lambda), \quad \begin{vmatrix} \theta_k(\lambda) & 1 \\ \theta_{k+1}(\lambda) & 0 \end{vmatrix} = -\theta_{k+1}(\lambda)$$

bo'lgani uchun

$$A_1 = \frac{a_k}{a_0}\varphi_k(\lambda), \quad A_2 = -\frac{a_k}{a_0}\theta_k(\lambda)$$

bo'ladi. Bunga ko'ra

$$u_{n,k}(\lambda) = \frac{a_k}{a_0} \cdot [\varphi_{k+1}(\lambda)\theta_{n+k}(\lambda) - \theta_{k+1}(\lambda)\varphi_{n+k}(\lambda)] \quad (1.1.40)$$

bo'ladi. Xuddi shu tarzda quyidagi tenglik kelib chiqadi:

$$v_{n,k}(\lambda) = \frac{a_k}{a_0} \cdot [-\varphi_k(\lambda)\theta_{n+k}(\lambda) + \theta_k(\lambda)\varphi_{n+k}(\lambda)], \quad (1.1.41)$$

Ushbu

$$\theta_{N+k}(\lambda) = \frac{a_0}{a_k} \cdot [\theta_N(\lambda)\theta_k(\lambda) + \theta_{N+1}(\lambda)\varphi_k(\lambda)] \quad (1.1.42)$$

$$\varphi_{N+k}(\lambda) = \frac{a_0}{a_k} \cdot [\varphi_N(\lambda)\theta_k(\lambda) + \varphi_{N+1}(\lambda)\varphi_k(\lambda)] \quad (1.1.43)$$

tengliklar o'rinli bo'lishi ravshan. (1.1.39)-(1.1.43) larga muvofiq,

$$u_{N,k}(\lambda) = \varphi_{k+1}(\lambda)[\theta_N(\lambda)\theta_k(\lambda) + \theta_{N+1}(\lambda)\varphi_k(\lambda)] - \theta_{k+1}(\lambda)[\varphi_N(\lambda)\theta_k(\lambda) + \varphi_{N+1}(\lambda)\varphi_k(\lambda)] \quad (1.1.44)$$

$$v_{N,k}(\lambda) = -\varphi_k(\lambda)[\theta_N(\lambda)\theta_k(\lambda) + \theta_{N+1}(\lambda)\varphi_k(\lambda)] + \theta_k(\lambda)[\varphi_N(\lambda)\theta_k(\lambda) + \varphi_{N+1}(\lambda)\varphi_k(\lambda)] \quad (1.1.45)$$

(1.1.40) ga ko'ra

$$\begin{aligned} v_{N+1,k}(\lambda) &= -\varphi_k(\lambda)\theta_{N+k+1}(\lambda) + \theta_k(\lambda)\varphi_{N+k+1}(\lambda) = \\ &= -\varphi_k(\lambda)[\theta_N(\lambda)\theta_{k+1}(\lambda) + \theta_{N+1}(\lambda)\varphi_{k+1}(\lambda)] + \\ &+ \theta_k(\lambda)[\varphi_N(\lambda)\theta_{k+1}(\lambda) + \varphi_{N+1}(\lambda)\varphi_{k+1}(\lambda)] \end{aligned} \quad (1.1.46)$$

(1.1.44) va (1.1.46) ga ko'ra

$$\begin{aligned} \tilde{\Delta}(\lambda) &= u_{N,k}(\lambda) + v_{N+1,k}(\lambda) = \\ &= \theta_N(\lambda)[\theta_k(\lambda)\varphi_{k+1}(\lambda) - \theta_{k+1}(\lambda)\varphi_k(\lambda)] + \varphi_{N+1}(\lambda)[\theta_k(\lambda)\varphi_{k+1}(\lambda) - \\ &- \theta_{k+1}(\lambda)\varphi_k(\lambda)] = \theta_N(\lambda) + \varphi_{N+1}(\lambda) = \Delta(\lambda) \end{aligned}$$

teorema 3.1 isbot bo'ldi.

Natija: (1.1.34) tenglamani spektri k ga bog'liq emas.

Лемма 1.1. Ushbu tengliklar o`rinli:

$$b_1 = \frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2} \sum_{j=1}^{N-1} (\lambda_{2j} + \lambda_{2j+1} - 2\mu_j), \quad (1.1.47)$$

$$a_0^2 = \frac{\lambda_1^2 + \lambda_{2N}^2}{8} + \frac{1}{8} \sum_{j=1}^{N-1} (\lambda_{2j}^2 + \lambda_{2j+1}^2 - 2\mu_j^2) -$$

$$- \frac{1}{4} \left[\frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2} \sum_{j=1}^{N-1} (\lambda_{2j} + \lambda_{2j+1} - 2\mu_j) \right]^2 - \frac{1}{2} \sum_{j=1}^{N-1} \frac{\sigma_j \sqrt{\prod_{i=1}^{2N} (\mu_j - \lambda_i)}}{\prod_{\substack{i=1 \\ i \neq j}}^{N-1} (\mu_j - \mu_i)}. \quad (1.1.48)$$

Лемма 1.2. Ushbu tengliklar o`rinli:

$$b_{k+1} = \frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2} \sum_{j=1}^{N-1} (\lambda_{2j} + \lambda_{2j+1} - 2\mu_{j,k}), \quad (1.1.49)$$

$$a_k^2 = \frac{\lambda_1^2 + \lambda_{2N}^2}{8} + \frac{1}{8} \sum_{j=1}^{N-1} (\lambda_{2j}^2 + \lambda_{2j+1}^2 - 2\mu_{j,k}^2) -$$

$$- \frac{1}{4} \left[\frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2} \sum_{j=1}^{N-1} (\lambda_{2j} + \lambda_{2j+1} - 2\mu_{j,k}) \right]^2 - \frac{1}{2} \sum_{j=1}^{N-1} \frac{\sigma_{j,k} \sqrt{\prod_{i=1}^{2N} (\mu_{j,k} - \lambda_i)}}{\prod_{\substack{i=1 \\ i \neq j}}^{N-1} (\mu_{j,k} - \mu_{i,k})}, \quad (1.1.50)$$

Bu erda $\mu_{j,k}$, $j = \overline{1, N-1}$ sonlar $u_{N+1,k}(\lambda) = 0$ tenglamani ildizlari.

2-§ Floke teoremasi.

Teorema 2.1.(Floke).

a) Agar $\Delta(\lambda) - 2 \neq 0$ bo'lsa, (1.1.1) tenglama quyidagi chiziqli erkli ikkita yechimga ega:

$$\psi_n^+(\lambda) = (\rho_+(\lambda))^{\frac{n}{N}} \cdot P_n^+, \quad P_{n+N}^+(\lambda) \equiv P_n^+(\lambda), \quad (1.2.1)$$

$$\psi_n^-(\lambda) = (\rho_-(\lambda))^{\frac{n}{N}} \cdot P_n^-, \quad P_{n+N}^-(\lambda) \equiv P_n^-(\lambda) \quad (1.2.2)$$

b) Agar $\Delta(\lambda) - 2 = 0$ bo'lsa, (1.1.1) tenglamaning davri N bo'lgan $\psi_n(\lambda)$ yechimi mavjud:

$$\psi_{n+N}(\lambda) \equiv \psi_n(\lambda), \quad n \in Z$$

c) Agar $\Delta(\lambda) + 2 = 0$ bo'lsa, (1.1.1) tenglamaning davri $2N$ bo'lgan $\psi_n(\lambda)$ yechimi mavjud:

$$\psi_{n+2N}(\lambda) \equiv \psi_n(\lambda), \quad n \in Z ;$$

Isbot. (1.1.1) masalani biror ρ son uchun $y_{n+N}(\lambda) \equiv \rho y_n(\lambda), n \in Z$ shartni qanoatlantiradigan yechimlarini axtaramiz:

$y_n(\lambda) = c_1 \theta_n(\lambda) + c_2 \varphi_n(\lambda)$ - bu (1.1.1) ning umumiy yechimi. (1.1.2) shartga ko'ra $\theta_{n+N}(\lambda), \varphi_{n+N}(\lambda)$ larni chiziqli kombinatsiyasi ham (1.1.1) ni umumiy yechimi bo'ladi. (1.1.23) va (1.1.24) tengliklardan va

$$y_{n+N}(\lambda) = c_1 \cdot \theta_{n+N}(\lambda) + c_2 \varphi_{n+N}(\lambda) \quad (1.2.3)$$

munosabat o'rinli ekanidan hamda $y_{n+N} = \rho y_n$ shartda $n = 0$ va $n = 1$ desak, u holda

$$\begin{cases} C_1 \theta_N + C_2 \varphi_N = \rho C_1 \\ C_1 \theta_{N+1} + C_2 \varphi_{N+1} = \rho C_2 \end{cases}, \quad \begin{cases} C_1(\theta_N - \rho) + C_2 \varphi_N = 0 \\ C_1 \theta_{N+1} + C_2(\varphi_{N+1} - \rho) = 0 \end{cases} \quad (1.2.4)$$

tenglamalar sistemasini hosil qilamiz. Bu (1.2.4) bir jinsli tenglamalar sistemasi noldan farqli yechimga ega bo'lishi uchun

$$\begin{vmatrix} \theta_N - \rho \varphi_N & \\ \theta_{N+1} & \varphi_{N+1} - \rho \end{vmatrix} = 0 \quad (1.2.5)$$

tenglik bajarilishi kerak. Bundan teorema 3 ga ko'ra va (1.1.26) tenglikdan $\rho^2 - \Delta(\lambda)\rho + 1 = 0$ bo'lishi kelib chiqadi. Bu kvadrat tenglamani yechib

$$D = \Delta^2(\lambda) - 4,$$

$$\rho_+(\lambda) = \frac{\Delta(\lambda) + \sqrt{\Delta^2(\lambda) - 4}}{2},$$

$$\rho_-(\lambda) = \frac{\Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4}}{2} \quad (1.2.6)$$

tengliklarga ega bo'lamiz.

a) $\Delta^2(\lambda) - 4 \neq 0$ demak ushbu tenglamani $\rho_+(\lambda)$, $\rho_-(\lambda)$ ikkita yechimi mavjud. Bularni har biriga bittadan (1.1.1) tenglamaning yechimi mos keladi. Ularni $\psi_n^+(\lambda)$, $\psi_n^-(\lambda)$ deb belgilaymiz, $\psi_{n+N}^+(\lambda) = \rho_+ \psi_n^+$, $\psi_{n+N}^-(\lambda) = \rho_- \psi_n^-$.

Ushbu $P_n^+(\lambda)$, $P_n^-(\lambda)$ funktsiyalarni kiritamiz.

$$P_{n+N}^+(\lambda) = \rho_+^{-\frac{n}{N}} \psi_n^+, \quad P_n^-(\lambda) = \rho_-^{-\frac{n}{N}} \psi_n^- \quad (1.2.7)$$

(1.2.7) funktsiyalar N davrli ekanini ko'rsatamiz

$$\begin{aligned} P_{n+N}^+(\lambda) &= \rho_+^{-\frac{n+N}{N}} \psi_{n+N}^+ = \rho_+^{-\frac{n}{N}} \rho_+^{-1} \rho_+^1 \psi_n^+ = \rho_+^{-\frac{n}{N}} \psi_n^+ \\ P_{n+N}^-(\lambda) &= \rho_-^{-\frac{n+N}{N}} \psi_{n+N}^- = \rho_-^{-\frac{n}{N}} \rho_-^{-1} \rho_-^1 \psi_n^- = \rho_-^{-\frac{n}{N}} \psi_n^- \end{aligned}$$

Demak (1.2.7) tengliklardan (1.2.1) va (1.2.2) kelib chiqadi.

$C_1 = 1$ deylik, u holda $C_2 = \frac{\rho - \theta_N}{\varphi_N}$ ya'ni

$$C_2 = \frac{\Delta(\lambda) - 2\theta_N \pm \sqrt{\Delta^2(\lambda) - 4}}{2\varphi_N} = \frac{\varphi_{N+1} - \theta_N \pm \sqrt{\Delta^2(\lambda) - 4}}{2\varphi_N} \quad (1.2.8)$$

munosabat o'rinli. Demak

$$y_n^\pm(\lambda) = C_1\theta_n(\lambda) + C_2\varphi_n(\lambda) = \theta_n(\lambda) + \frac{\varphi_{N+1}(\lambda) - \theta_N(\lambda) \pm \sqrt{\Delta^2(\lambda) - 4}}{2\varphi_N(\lambda)}\varphi_n(\lambda) \quad (1.2.9)$$

(1.2.9) tenglik bilan aniqlanadigan yechimlar Floke yechimlari deyiladi. Bu yechimlar chiziqli erkli ekaninni tekshirib ko'rish qiyin emas. Bu tenglikdagi

$$m^\pm(\lambda) = \frac{\varphi_{N+1}(\lambda) - \theta_N(\lambda) \pm \sqrt{\Delta^2(\lambda) - 4}}{2\varphi_N(\lambda)}$$

funktsiyaga Veyl-Titchmarsh funktsiyasi deyiladi.

b) $\Delta(\lambda) = 2$ bo'lsa $\rho^2 - \Delta(\lambda)\rho + 1 = 0$ tenglama $\rho_+ = \rho_- = 1$ yechimga ega bo'ladi ya'ni (1.1.1) ni $y_{n+N} = 1 \cdot y_n$ shartni qanoatlantiruvchi yechimi mavjud.

c) $\Delta(\lambda) = -2$ bo'lsa, $\rho^2 - \Delta(\lambda)\rho + 1 = 0$ tenglama $\rho_+ = \rho_- = -1$ yechimga ega bo'ladi, ya'ni (1.1.1) tenglamaning $y_{n+N}(\lambda) = -1 \cdot y_n(\lambda)$ shartni qanoatlantiruvchi yechimi mavjud. Bu yechimning davri $2N$ ga teng

$$y_{n+2N}(\lambda) = y_{n+N+N}(\lambda) = -y_{n+N}(\lambda) = -(-y_n(\lambda)) = y_n(\lambda)$$

teorema 2.1 isbotlandi.

Ta'rif 2.1. λ o'qning (1.1.1) tenglamaning chegaralangan noldan farqli yechimi mavjud bo'ladigan qismiga spektr deyiladi.

Demak, spektr $E = \{\lambda \in R : -2 \leq \Delta\lambda \leq 2\}$, $R \setminus E$ - to'plamga lakunalar deyiladi.

Teorema 2.2. $\Delta^2(\lambda) - 4 = 0$ bo'lsa (1.1.1) tenglamani barcha xos qiymatlari haqiqiydir.

Isbot: $\Delta^2(\lambda) - 4 = 0$, bundan $\rho = \pm 1$ hamda $a_{n+N} = a_n$, $b_{n+N} = b_n$ va $y_{n+N} = \pm y_n$ shartlarga asosan (1.1.1) tenglamani matritsasini yozish mumkin.

$$L^\pm = \begin{pmatrix} b_1 & a_1 & \dots & \pm a_N \\ a_1 & b_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & b_{N-2} & a_{N-2} \\ \pm a_N & \dots & a_{N-2} & b_{N-1} \end{pmatrix} \quad (1.2.10)$$

Bu simmetrik matritsa bo'lib, elementlari haqiqiy. Xos qiymatlari haqiqiy ekanini ko'rsatamiz. λ xos qiymatga U xos funktsiya mos kelsin, agar xos qiymatni kompleks qo'shmasini λ^* desak unga U^* xos funktsiya mos keladi. $LU = \lambda U$, L - simmetrikligidan $(LU^*, U) = (U^*, LU)$ bo'ladi.

$$\begin{cases} \lambda U^* U = U^* (\lambda) U = U^* LU \\ \lambda^* U U^* = U \lambda^* U^* = U L U^* = U^* LU \end{cases} \quad (1.2.11)$$

(1.2.11) sistemadan $(\lambda - \lambda^*)U^* - U = 0$ bundan $\lambda = \lambda^*$. Demak λ - haqiqiy.

Teorema 2.2 isbot bo'ldi.

Teorema 2.3. Quyidagi tenglik o'rinli:

$$\frac{d\Delta}{d\lambda} = -\frac{\theta_{N+1}}{a_N} \cdot \sum_{n=1}^N \left\{ \left(\varphi_n + \frac{\theta_N - \varphi_{N+1}}{2\theta_{N+1}} \theta_n \right)^2 - \frac{\Gamma}{4\theta_{N+1}^2} \cdot \theta_n^2 \right\} \quad (1.2.12)$$

bunda

$$\Gamma = (\theta_N - \varphi_{N+1})^2 + 4\varphi_N \theta_{N+1} = (\theta_N + \varphi_{N+1})^2 - 4 = \Delta^2 - 4. \quad (1.2.13)$$

Isbot. (1.1.1) tenglamaning har bir λ ga mos ikkita $\theta_n(\lambda)$, $\psi_n(\lambda)$ chiziqli erkli yechimini qaraylik. Bularni (1.1.1) tenglamaga qo'ysak quyidagi tenglikka ega bo'lamiz.

$$\begin{cases} a_{n-1}\theta_{n-1} + b_n\theta_n + a_n\theta_{n+1} = \lambda\theta_n \\ a_{n-1}\psi_{n-1} + b_n\psi_n + a_n\psi_{n+1} = \tilde{\lambda}\psi_n \end{cases} \quad (1.2.14)$$

birinchi tenglikni ψ_n ga ikkinchini esa θ_n ga ko'paytirib birinchi tenglikdan ikkin-chisini ayirsak

$$\begin{aligned} (\lambda - \tilde{\lambda})\theta_n \cdot \psi_n &= \psi_n(a_n\theta_{n+1} + b_n\theta_n + a_{n-1}\theta_{n-1}) - \\ - \psi_n(a_n\psi_{n+1} + b_n\psi_n + a_{n-1}\psi_{n-1}) &= a_n(\theta_{n+1}\psi_n - \theta_n\psi_{n+1}) - a_{n-1}(\theta_n\psi_{n-1} - \theta_{n-1}\psi_n) \end{aligned} \quad (1.2.15)$$

tenglik hosil bo'ladi. Ikkala ta'rifdan $n=1$ dan N gacha yig'indi olsak quyidagiga ega bo'lamiz.

$$\begin{aligned} (\lambda - \tilde{\lambda})\sum_{n=1}^N \theta_n\psi_n &= a_1(\theta_2\psi_1 - \theta_1\psi_2) + a_0(\theta_1\psi_0 - \theta_0\psi_1) + a_2(\theta_3\psi_2 - \theta_2\psi_3) - \\ &- a_1(\theta_2\psi_1 - \theta_1\psi_2) + a_3a_1(\theta_4\psi_3 - \theta_3\psi_4) - a_2(\theta_3\psi_2 - \theta_2\psi_3) + \\ &+ \dots + \\ &+ a_N(\theta_{N+1}\psi_N - \theta_N\psi_{N+1}) - a_{N-1}(\theta_N\psi_{N-1} - \theta_{N-1}\psi_N) = \\ &= a_N(\theta_{N+1}\psi_N - \theta_N\psi_{N+1}) - a_0(\theta_1\psi_0 - \theta_0\psi_1), \end{aligned} \quad (1.2.16)$$

(1.2.16) tenglikda $\theta_0(\lambda)=1, \theta_1(\lambda)=0, \psi_0(\tilde{\lambda})=0, \psi_1(\tilde{\lambda})=1$ shartlarni inobatga olib, hamda $\psi_n(\tilde{\lambda})=\theta_n(\tilde{\lambda})$ deb olsak u holda quyidagi tenglikni hosil qilamiz.

$$(\lambda - \tilde{\lambda})\sum_{n=1}^N \theta_n(\lambda)\psi_n(\lambda) = a_N(\theta_{N+1}(\lambda)\psi_N(\lambda) - \theta_N(\lambda)\psi_{N+1}(\lambda)) \quad (1.2.17)$$

(1.2.17) tenglikda $\Delta\lambda$ nolga intilganda limitga o'tib $\theta'_n = \frac{d\theta_n}{d\lambda}$ belgilash kiritsak u holda

$$\begin{aligned}
\|\theta_n(\lambda)\|^2 &= \sum_{n=1}^N \theta_n^2(\lambda) = \\
&= a_N \left[\theta_N(\lambda) \lim_{\lambda \rightarrow \tilde{\lambda}} \frac{\theta_{N+1}(\lambda) - \theta_{N+1}(\tilde{\lambda})}{\lambda - \tilde{\lambda}} - \theta_{N+1}(\lambda) \lim_{\lambda \rightarrow \tilde{\lambda}} \frac{\theta_N(\lambda) - \theta_N(\tilde{\lambda})}{\lambda - \tilde{\lambda}} \right] = \\
&= a_N (\theta_N \theta'_{N+1} - \theta_{N+1} \theta'_N)
\end{aligned} \tag{1.2.18}$$

Xuddi shunga o'xshash $\varphi_n(\lambda)$ va $\varphi_n(\lambda) \cdot \theta_n(\lambda)$ ifodalar uchun quyidagilarni hosil qilish mumkin.

$$\sum_{n=1}^N \varphi_n^2(\lambda) = a_N [\varphi_N(\lambda) \varphi'_{N+1}(\lambda) - \varphi_{N+1}(\lambda) \varphi'_N(\lambda)], \tag{1.2.19}$$

$$\begin{aligned}
\sum_{n=1}^N \theta_n(\lambda) \varphi_n(\lambda) &= \frac{a_N}{2} (\varphi_N \theta'_{N+1} - \varphi_{N+1} \theta'_N + \theta_N \varphi'_{N+1} - \theta_{N+1} \varphi'_N) = \\
&= a_N (\theta_N \varphi'_{N+1} - \theta_{N+1} \varphi'_N) = a_N (\varphi_N \theta'_{N+1} - \varphi_{N+1} \theta'_N)
\end{aligned} \tag{1.2.20}$$

(2.18-20) va (1.1.26) tengliklardan foydalanib quyidagini keltirib chiqaramiz:

$$\begin{aligned}
&\frac{1}{a_N} \sum_{n=1}^N \{ \varphi_N \theta_n^2 - [\theta_N - \varphi_{N+1}] \theta_n \varphi_n - \theta_{N+1} \varphi_n^2 \} = \\
&\{ \theta_N \varphi_N \theta'_{N+1} - \varphi_N \theta_{N+1} \theta'_N + \theta_N \varphi_{N+1} \theta'_N - \theta'_N \theta_N \varphi_{N+1} + \theta_{N+1} \varphi_{N+1} \varphi'_N - \\
&- \theta_{N+1} \varphi_N \varphi'_{N+1} + \theta_N \varphi_{N+1} \varphi'_{N+1} - \varphi_{N+1} \theta_N \varphi'_{N+1} - (\theta_N - \varphi_{N+1}) (\theta_N \varphi'_{N+1} - \varphi'_N \theta_{N+1}) \} = \\
&= \{ \Delta'(\lambda) + (\theta_N - \varphi_{N+1}) (\theta_N \varphi'_{N+1} - \varphi'_N \theta_{N+1}) - (\theta_N - \varphi_{N+1}) (\theta_N \varphi'_{N+1} - \varphi'_N \theta_{N+1}) \} = \Delta'(\lambda)
\end{aligned} \tag{1.2.21}$$

(1.2.21) tenglikda (1.2.13) belgilashni kiritib uni quyidagicha yozish mumkin.

$$\frac{d\Delta}{d\lambda} = -\frac{\theta_{N+1}}{a_N} \cdot \sum_{n=1}^N \left\{ \left(\varphi_n + \frac{\theta_N - \varphi_{N+1}}{2\theta_{N+1}} \theta_n \right)^2 - \frac{\Gamma}{4\theta_{N+1}^2} \cdot \theta_n^2 \right\}$$

Teorema 2.3 isbot bo'ldi.

Natija : Teorema 2.3 dagi (1.2.12) tenglikda θ_{N+1} funktsiya $\Delta^2 - 4 < 0$ da nolga aylanmaydi va $sign(\Delta'(\lambda)) = sign(-\theta_{N+1})$ tenglik o'rinli bo'ladi.

Isbot: $\Delta^2 - 4 < 0$ bo'lsin.

Agar $\theta_{N+1} = 0$ bo'lsa u holda **Teorema 3** dan $\theta_N \cdot \varphi_{N+1} = 1$ va $\varphi_{N+1} = \frac{1}{\theta_N}$

ekani kelib chiqadi. Demak, $|\Delta(\lambda)| = |\theta_N + \varphi_{N+1}| = \left| \theta_N + \frac{1}{\theta_N} \right| \geq 2$ bo'lib bu esa

$\Delta^2 - 4 < 0$ ga ziddir. (1.2.12) tenglikdan esa $\Gamma = \Delta^2 - 4 < 0$ bo'lganda $sign(\Delta'(\lambda)) = sign(-\theta_{N+1})$ ekani bizga ravshan natija isbot bo'ldi.

Izoh 2.1: $\Delta^2 - 4 > 0$ bo'lganda ham (1.2.12) tenglikni shaklini o'zgartirib $\Delta'(\lambda)$ ifodani ishorasini aniqlash mumkin.

3 - § Teskari masala yechish algoritmini keltirib chiqarish.

Ushbu

$$(L\varphi)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in Z \quad (1.3.1)$$

tenglamani quyidagi

$$\begin{cases} \theta_0(\lambda) = 1, & \varphi_0(\lambda) = 0, \\ \theta_1(\lambda) = 0, & \varphi_1(\lambda) = 1, \end{cases} \quad (1.3.2)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini $\theta_n(\lambda)$ va $\varphi_n(\lambda)$ orqali belgilaymiz. Endi (1.3.1) tenglamani koefitsientlarini $k \in Z$ butun songa siljitish natijasida hosil gilingan quyidagi tenglamani ko'rib chiqamiz:

$$a_{n-1+k}y_{n-1} + b_{n+k}y_n + a_{n+k}y_{n+1} = \lambda y_n, \quad n \in Z. \quad (1.3.3)$$

$\theta_{n+k}(\lambda)$ va $\varphi_{n+k}(\lambda)$ lar (1.3.3) ning yechimi bo'ladi. Haqiqatan: Ushbu

$$a_{n-1}\theta_{n-1} + b_n\theta_n + a_n\theta_{n+1} = \lambda\theta_n,$$

tenglama koefitsientlarini $k \in Z$ butun songa siljitsak

$$a_{n-1+k}\theta_{n-1+k} + b_{n+k}\theta_{n+k} + a_{n+k}\theta_{n+1+k} = \lambda\theta_{n+k}$$

bo'ladi. Oxirgi tenglikda quyidagi

$$y_n = \theta_{n+k},$$

belgilashni kiritsak, u holda

$$a_{n-1+k}y_{n-1} + b_{n+k}y_n + a_{n+k}y_{n+1} = \lambda y_n.$$

kelib chiqadi. (1.3.3) tenglamaning quyidagi

$$\begin{cases} \theta_{0,k}(\lambda) = 1 \\ \theta_{1,k}(\lambda) = 0 \end{cases}, \quad \begin{cases} \varphi_{0,k}(\lambda) = 0 \\ \varphi_{1,k}(\lambda) = 1 \end{cases} \quad (1.3.4)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini $\theta_{n,k}(\lambda)$ va $\varphi_{n,k}(\lambda)$ orqali belgilaymiz.

Lemma 3.1. $\theta_{n,k}(\lambda)$ va $\varphi_{n,k}(\lambda)$ yechimlar uchun ushbu tengliklar bajariladi:

$$\theta_{n,k}(\lambda) = \frac{a_k}{a_0} \cdot [\varphi_{k+1}(\lambda)\theta_{n+k}(\lambda) - \theta_{k+1}(\lambda)\varphi_{n+k}(\lambda)] \quad (1.3.5)$$

$$\varphi_{n,k}(\lambda) = \frac{a_k}{a_0} \cdot [-\varphi_k(\lambda)\theta_{n+k}(\lambda) + \theta_k(\lambda)\varphi_{n+k}(\lambda)], \quad (1.3.6)$$

Isbot. $\theta_{n+k}(\lambda)$ va $\varphi_{n+k}(\lambda)$ yechimlar chiziqli erkli bo'lgani uchun

$$\theta_{n,k}(\lambda) = A_1\theta_{n+k}(\lambda) + A_2\varphi_{n+k}(\lambda), \quad (1.3.7)$$

$$\varphi_{n,k}(\lambda) = B_1\theta_{n+k}(\lambda) + B_2\varphi_{n+k}(\lambda), \quad (1.3.8)$$

(1.3.7) dan

$$\begin{cases} A_1\theta_k(\lambda) + A_2\varphi_k(\lambda) = 1 \\ A_1\theta_{k+1}(\lambda) + A_2\varphi_{k+1}(\lambda) = 0 \end{cases} \quad (1.3.9)$$

kelib chiqadi. (1.3.9) sistemani yechamiz:

$$\begin{vmatrix} \theta_k(\lambda) & \varphi_k(\lambda) \\ \theta_{k+1}(\lambda) & \varphi_{k+1}(\lambda) \end{vmatrix} = \frac{a_0}{a_k}, \quad \begin{vmatrix} 1 & \varphi_k(\lambda) \\ 0 & \varphi_{k+1}(\lambda) \end{vmatrix} = \varphi_{k+1}(\lambda), \quad \begin{vmatrix} \theta_k(\lambda) & 1 \\ \theta_{k+1}(\lambda) & 0 \end{vmatrix} = -\theta_{k+1}(\lambda)$$

bo'lgani uchun

$$A_1 = \frac{a_k}{a_0}\varphi_k(\lambda), \quad A_2 = -\frac{a_k}{a_0}\theta_k(\lambda)$$

bo'ladi. Bundan esa (1.3.7) va (1.3.8) ga ko'ra (1.3.5) va (1.3.6) kelib chiqadi.

Lemma isbotlandi.

$\theta_{N+1,k}(\lambda) = 0$ tenglamani ildizlarini $\mu_j(k)$, $j = \overline{1, N-1}$ orqali belgilaymiz. (1.3.5) va (1.3.6) dan

$$\theta_{n+1,k-1} = \frac{a_{k-1}}{a_0} [\varphi_k \theta_{n+k} - \theta_n \varphi_{n+k}] = \frac{a_{k-1}}{a_0} \cdot \frac{a_0}{a_k} (-\varphi_{n,k}) = -\frac{a_{k-1}}{a_k} \varphi_{n,k} \quad (1.3.10)$$

bo`ladi. (1.3.10) dan quyidagilar kelib chiqadi:

$$\theta_{n+1,k-1} = -\frac{a_{k-1}}{a_k} \varphi_{n,k}, \quad (1.3.11)$$

$$\varphi_{n-1,k+1} = -\frac{a_{k+1}}{a_k} \theta_{n,k} \quad (1.3.12)$$

Ushbu

$$a_{n-1} \theta_{n-1} + b_n \theta_n + a_n \theta_{n+1} = \lambda \theta_n$$

tenglikda n ni $n+N$ ga siljitamiz, natijada quyidagi

$$a_{n-1+N} \theta_{n-1+N} + b_{n+N} \theta_{n+N} + a_{n+N} \theta_{n+1+N} = \lambda \theta_{n+N},$$

$$a_{n-1} \theta_{n-1+N} + b_n \theta_{n+N} + a_n \theta_{n+1+N} = \lambda \theta_{n+N}$$

tengliklarga ega bo`lamiz. Bulardan esa θ_{n+N} ham (1.3.1) ni yechimi ekan

kelib chiqadi. U holda θ_{n+N} yechim θ_n va φ_n yechimlar orqali chiziqli

ifodalanadi, ya`ni

$$\theta_{n+N} = c_1 \theta_n + c_2 \varphi_n \quad (1.3.13)$$

(1.3.13) da c_1 va c_2 larni topamiz:

$$\begin{aligned} n=0, & \quad \theta_N = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = \theta_N \\ n=1, & \quad \theta_{N+1} = c_1 \cdot 0 + c_2 \cdot 1 \Rightarrow c_2 = \theta_{N+1} \end{aligned}$$

Topilganlarga ko`ra, quyidagi

$$\theta_{n+N} = \theta_N \theta_n + \theta_{N+1} \varphi_n \quad (1.3.14)$$

$$\varphi_{n+N} = \varphi_N \theta_n + \varphi_{N+1} \varphi_n \quad (1.3.15)$$

(1.3.14) va (1.3.15) larni (1.3.5) va (1.3.6) larga qo`yamiz:

$$\begin{aligned} \theta_{N+1,k}(\lambda) &= \frac{a_k}{a_0} [\varphi_{k+1} \theta_{N+1+k} - \theta_{k+1} \varphi_{N+1+k}] = \\ &= \frac{a_k}{a_0} \varphi_{k+1} [\theta_N \theta_{k+1} + \theta_{N+1} \varphi_{k+1}] - \frac{a_k}{a_0} \theta_{k+1} [\varphi_N \theta_{N+1} + \varphi_{N+1} \varphi_{k+1}], \end{aligned}$$

$$\theta_{N+1,k}(\lambda) = \frac{a_k}{a_0} \{ \theta_{N+1} \varphi_{n+1}^2 + [\theta_N - \varphi_{N+1}] \theta_{k+1} \varphi_{k+1} - \varphi_N \theta_{n+1}^2 \}. \quad (1.3.16)$$

(1.3.16) dan

$$\theta_{N+1,k+1}(\lambda) = \frac{a_{k+1}}{a_0} \{ \theta_{N+1} \varphi_{n+2}^2 + [\theta_N - \varphi_{N+1}] \theta_{k+2} \varphi_{k+2} - \varphi_N \theta_{n+2}^2 \} \quad (1.3.17)$$

kelib chiqadi. $\theta_{N+1,k+1}$ ni $\theta_{N+1,k}$, $\theta_{N,k} - \varphi_{N+1,k}$, $\varphi_{N,k}$ lar orqali chiziqli ifodalash kerak. Shu maqsadda quyidagilarni hisoblaymiz:

$$\begin{aligned} \theta_{N,k} &= \frac{a_k}{a_0} [\varphi_{k+1} \theta_{N+k} - \theta_{k+1} \varphi_{N+k}] = \frac{a_k}{a_0} [\varphi_{k+1} (\theta_N \theta_k + \theta_{N+1} \varphi_k) - \theta_{k+1} (\varphi_N \theta_k + \varphi_{N+1} \varphi_k)] = \\ &= \frac{a_k}{a_0} \{ \theta_N \theta_k \varphi_{k+1} + \theta_{N+1} \varphi_k \varphi_{k+1} - \varphi_N \theta_k \theta_{k+1} - \varphi_{N+1} \varphi_k \theta_{k+1} \}, \quad (1.3.18) \end{aligned}$$

$$\begin{aligned} \varphi_{N+1,k} &= \frac{a_k}{a_0} \{ -\varphi_k \theta_{N+1+k} + \theta_k \varphi_{N+1+k} \} = \\ &= -\frac{a_k}{a_0} \varphi_k (\theta_N \theta_{k+1} + \theta_{N+1} \varphi_{k+1}) + \frac{a_k}{a_0} \theta_k (\varphi_N \theta_{k+1} + \varphi_{N+1} \varphi_{k+1}), \end{aligned}$$

$$\varphi_{N+1,k} = \frac{a_k}{a_0} \{-\theta_N \theta_{k+1} \varphi_k - \theta_{N+1} \varphi_k \varphi_{k+1} + \varphi_N \theta_k \theta_{k+1} + \varphi_{N+1} \theta_k \varphi_{k+1}\}. \quad (1.3.19)$$

(1.3.18) va (1.3.19) dan

$$\begin{aligned} \theta_{N,k} - \varphi_{N+1,k} &= \frac{a_k}{a_0} \{\theta_N (\theta_k \varphi_{k+1} + \theta_{k+1} \varphi_k) + 2\theta_{N+1} \varphi_k \varphi_{k+1} - \\ &\quad - 2\varphi_N \theta_k \theta_{k+1} - \varphi_{N+1} (\theta_k \varphi_{k+1} + \theta_{k+1} \varphi_k)\} = \\ &= \frac{a_k}{a_0} \{2\theta_{N+1} \varphi_k \varphi_{k+1} + (\theta_N - \varphi_{N+1}) (\theta_k \varphi_{k+1} + \theta_{k+1} \varphi_k) - 2\varphi_N \theta_k \theta_{k+1}\}. \end{aligned} \quad (1.3.20)$$

(1.3.6) ga ko`ra

$$\varphi_{N,k} = \frac{a_k}{a_0} [-\varphi_k \theta_{N+k} + \theta_N \varphi_{N+k}] = -\frac{a_k}{a_0} \varphi_k (\theta_N \theta_k + \theta_{N+1} \varphi_k) + \frac{a_k}{a_0} \theta_k (\varphi_N \theta_k + \varphi_{N+1} \varphi_k),$$

$$\varphi_{N,k} = \frac{a_k}{a_0} \{-\theta_{N+1} \varphi_k^2 - (\theta_N - \varphi_{N+1}) \theta_k \varphi_k + \varphi_N \theta_k^2\}. \quad (1.3.21)$$

(1.3.19)-(1.3.21) ko`rinishlaridan foydalanib, ushbu

$$\theta_{N+1,k+1} = \alpha \theta_{N+1,k} + \beta (\theta_{N,k} - \varphi_{N+1,k}) + \gamma \varphi_{N,k} \quad (1.3.22)$$

tenglikdagi α, β, γ larni aniqlaymiz:

$$\frac{a_{k+1}}{a_0}(\varphi_{k+2}^2) = \frac{a_k}{a_0}\varphi_{k+1}^2\alpha + \frac{2a_k}{a_0}\varphi_k\varphi_{k+1}\beta - \frac{a_k}{a_0}\varphi_k^2\gamma, \quad (1.3.23)$$

$$\frac{a_{k+1}}{a_0}\theta_{k+2}\varphi_{k+2} = \frac{a_k}{a_0}\theta_{k+1}\varphi_{k+1}\alpha + \frac{a_k}{a_0}(\theta_k\varphi_{k+1} + \theta_{k+1}\varphi_k)\beta - \frac{a_k}{a_0}\theta_k\varphi_k\gamma, \quad (1.3.24)$$

$$-\frac{a_{k+1}}{a_0}\theta_{k+2}^2 = -\frac{a_k}{a_0}\theta_{k+1}^2\alpha - \frac{2a_k}{a_0}\theta_k\theta_{k+1}\beta + \frac{a_k}{a_0}\theta_k^2\gamma. \quad (1.3.25)$$

(1.3.25) dan,

$$a_{k+1}^2\theta_{k+2}^2 = a_k a_{k+1}\theta_{k+1}^2\alpha + 2a_k a_{k+1}\theta_k\theta_{k+1}\beta - a_k a_{k+1}\theta_k^2\gamma. \quad (1.3.26)$$

(1.3.26) ni chap tarafidagi ifodani, tenglamadan foydalanib, quyidagicha yozib olamiz:

$$((\lambda - b_{k+1})\theta_{k+1} - a_k\theta_k)^2 = (\lambda - b_{k+1})^2\theta_{k+1}^2 - 2(\lambda - b_{k+1})a_k\theta_k\theta_{k+1} + a_k^2\theta_k^2. \quad (1.3.27)$$

(1.3.27) ni (1.3.26) ga qo'yib, mos koeffitsientlarni tenglashtiramiz, natijada ushbu

$$a_n a_{n+1}\alpha = (\lambda - b_{k+1})^2 \Rightarrow \alpha = \frac{(\lambda - b_{k+1})^2}{a_k a_{k+1}}, \quad (1.3.28)$$

$$2a_k a_{k+1} \beta = -2(\lambda - b_{k+1}) a_k \Rightarrow \beta = -\frac{(\lambda - b_{k+1})}{a_{k+1}}, \quad (1.3.29)$$

$$\gamma = -\frac{a_k}{a_{k+1}}. \quad (1.3.30)$$

(1.3.28)-(1.3.30) ni (1.3.22) ga qo'yib, quyidagiga ega bo'lamiz:

$$\theta_{N+1,k+1}(\lambda) = \frac{(\lambda - b_{k+1})^2}{a_k a_{k+1}} \theta_{N+1,k}(\lambda) - \frac{(\lambda - b_{k+1})}{a_{k+1}} (\theta_{N,k}(\lambda) - \varphi_{N+1,k}(\lambda)) - \frac{a_k}{a_{k+1}} \varphi_{N,k}(\lambda). \quad (1.3.31)$$

Xuddi shuningdek

$$\theta_{N,k+1}(\lambda) - \varphi_{N+1,k+1}(\lambda) = \frac{2(\lambda - b_{k+1})}{a_k} \theta_{N+1,k}(\lambda) - (\theta_{N,k}(\lambda) - \varphi_{N+1,k}(\lambda)), \quad (1.3.32)$$

$$\varphi_{N,k+1}(\lambda) = -\frac{a_{k+1}}{a_k} \theta_{N+1,k}(\lambda). \quad (1.3.33)$$

Shu bilan birgalikda quyidagi

$$(\theta_{N,k}(\lambda) - \varphi_{N+1,k}(\lambda))^2 = \Delta^2(\lambda) - 4 - 4\varphi_{N,k}(\lambda)\theta_{N+1,k}(\lambda), \quad (1.3.34)$$

$$\theta_{N,k}(\lambda)\varphi_{N+1,k}(\lambda) = 1 + \varphi_{N,k}(\lambda)\theta_{N+1,k}(\lambda).$$

tengliklarni bajarilishi ravshan.

Quyidagi belgilashlarni kiritamiz:

$$R(\lambda) = \prod_{j=1}^{2N} (\lambda - \lambda_j), \quad (1.3.35)$$

$$S_k(\lambda) = \prod_{j=1}^{N-1} (\lambda - \mu_{j,k}), \quad (1.3.36)$$

$$Q_k(\lambda) = \frac{\varphi_{N+1,k}(\lambda) - \theta_{N,k}(\lambda)}{\prod_{i=1}^N a_i}. \quad (1.3.37)$$

Ushbu munosabat bajariladi:

$$Q_k(\lambda) = (\lambda - b_{k+1})S_k(\lambda) + S_k(\lambda) \sum_{j=1}^{N-1} \frac{Q_k(\mu_{j,k})}{S'_k(\mu_{j,k})(\lambda - \mu_{j,k})}, \quad (1.3.38)$$

bunda

$$Q_k(\mu_{j,k}) = \tau_{j,k} \sqrt{R(\mu_{j,k})}, \quad (1.3.39)$$

va $\tau_{j,k} = \text{sign } Q_k(\mu_{j,k})$.

(1.3.35)-(1.3.37) larga ko`ra (1.3.31)-(1.3.34) tengliklardan quyidagilarga ega bo`lamiz:

$$Q_k^2(\lambda) = R(\lambda) + 4a_k^2 S_{k-1}(\lambda) S_k(\lambda), \quad (1.3.40)$$

$$S_{k+1}(\lambda) = \frac{a_k^2}{a_{k+1}} S_{k-1}(\lambda) - \frac{(\lambda - b_{k+1})}{a_{k+1}^2} Q_k(\lambda) + \frac{(\lambda - b_{k+1})^2}{a_{k+1}^2} S_k(\lambda), \quad (1.3.41)$$

$$Q_{k+1}(\lambda) = 2(\lambda - b_{k+1})S_k(\lambda) - Q_k(\lambda). \quad (1.3.42)$$

(1.3.38)-(1.3.42) tengliklar diskret Xill tenglamasi uchun teskari masala yechish algoritmini beradi:

Bizga $\{\mu_j, \sigma_j\}_{j=1}^{N-1}$ spektral parametrlar va $\lambda_1, \lambda_2, \dots, \lambda_{2N}$ sonlar berilgan.

1. (1.1.47) va (1.1.48) formulalardan b_1, a_0, a_1 larni topamiz.
2. $\sigma_{j,0} = \sigma_j, \mu_{j,0} = \mu_j, j = 1, 2, \dots, N-1$ bo`lsin.
3. (1.3.35), (1.3.36) va (1.3.39) munosabatlardan $R(\lambda), S_0(\lambda)$ va $Q_0(\mu_{j,0})$ larni aniqlaymiz.
4. (1.3.8) tenglikdan $Q_0(\lambda)$ ni topamiz.
5. (1.3.40) dan $S_{-1}(\lambda)$ ga ega bo`lamiz.

6. $S_1(\lambda)$ ko`phadni (1.3.11) orqali aniqlaymiz va uning ildizlari $\{\mu_{j,1}\}_{j=1}^{N-1}$ larni topamiz.
7. (1.3.12) yordamida $Q_1(\lambda)$ ifodani tiklaymiz va uning ishorasi $\tau_{j,1} = \text{sign } Q_1(\mu_{j,1})$ ni aniqlaymiz.
8. (1.1.49) va (1.1.50) tengliklardan b_2 va a_2 larni topamiz.
9. Ushbu jarayonni davom qildirish natijasida b_k va a_k , $k \in \mathbb{Z}$ ketma-ketliklarni aniqlaymiz.

Izox: Ushbu algoritmnning har bir qadamida Floke yechimlarini ham quramiz.

Buning uchun bazis yechimlar $\theta_n(\lambda)$ va $\varphi_n(\lambda)$ larni quyidagi

$$\theta_2(\lambda) = -\frac{a_0}{a_1},$$

$$\theta_3(\lambda) = \frac{(\lambda - b_2)}{a_2} \theta_2(\lambda),$$

$$\theta_{k+1}(\lambda) = \frac{(\lambda - b_k) \theta_k(\lambda) - a_{k-1} \theta_{k-1}(\lambda)}{a_k}, \quad k \in \mathbb{Z}.$$

$$\varphi_2(\lambda) = \frac{(\lambda - b_1)}{a_1},$$

$$\varphi_3(\lambda) = \frac{(\lambda - b_2) \varphi_2(\lambda) - a_1}{a_2},$$

$$\varphi_{k+1}(\lambda) = \frac{(\lambda - b_k) \varphi_k(\lambda) - a_{k-1} \varphi_{k-1}(\lambda)}{a_k}, \quad k \in \mathbb{Z}.$$

munosabatlardan aniqlaymiz. Ushbu belgilashni kiritamiz:

$$c(\lambda) = \frac{Q_0(\lambda)}{2a_0 S_0(\lambda)}.$$

Floke yechimini quyidagi

$$f_n^i = c(\lambda_i)\theta_n(\lambda_i) + \varphi_n(\lambda_i), \quad i = \overline{1, 2N}, \quad n \in Z.$$

tengliklardan aniqlaymiz.

Xulosa

Mazkur bobda diskret Xill tenglamasi uchun qo`yilgan to`g`ri va teskari spectral masalalar o`rganilgan. Xususan, diskret Xill tenglamasi uchun qo`yilgan teskari masalani yechinning yangicha usuli keltirib chiqarilgan bo`lib, bu usul qaralayotgan masalani sonly usullar yordamida yechishda juda qo`l keladi. Diskret Xill tenglamasining bazis yechimlari uchun yoyilmalar keltirib chiqarilgan. Yagonalik teoremasi isbotlangan. Teskari masala yechishning sxematik algoritmi ko`rsatilgan.

II-BOB. INTEGRAL MANBALI UMUMIY DAVRIY TODA TENGLAMASNI INTEGRALLASH

Ushbu bob integral manbali umumiy davriy Toda tenglamasni integrallashga bag`ishlanga.

1-§ Integral manbali umumiy davriy Toda tenglamasni keltirib chiqarish

Ushbu paragrafda biz Hill tenglamasiga qo`yilgan teskari masala yordamida integral manbali umumiy davriy Toda tenglamasini keltirib chiqarishning effektiv usulini bayon qilamiz.

Quyidagi

$$\begin{cases} \dot{a}_n = P_m(a_n, b_n) + a_n \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_{n+1}^-(\lambda, t) \psi_{n+1}^+(\lambda, t) - \psi_n^-(\lambda, t) \psi_n^+(\lambda, t)] d\lambda, \\ \dot{b}_n = Q_m(a_n, b_n) + a_n \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_n^-(\lambda, t) \psi_{n+1}^+(\lambda, t) + \psi_{n+1}^-(\lambda, t) \psi_n^+(\lambda, t)] d\lambda - \\ - a_{n-1} \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_n^-(\lambda, t) \psi_{n-1}^+(\lambda, t) + \psi_{n-1}^-(\lambda, t) \psi_n^+(\lambda, t)] d\lambda, \\ a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad a_n > 0, \quad n \in Z, \quad t \in R, \end{cases} \quad (2.1.1)$$

sistemani ushbu

$$a_n(0) = a_n^0, \quad b_n(0) = b_n^0, \quad n \in Z, \quad (2.1.2)$$

boshlang`ich shart bilan qaraymiz. Bu yerda a_n^0 va b_n^0 , $n \in Z$ lar N davrli davriy ketma-ketliklar. P_m va Q_m lar a_n va b_n larning ko`padi. Ushbu paragrafning asosiy maqsadi quyidagi

$$(L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in Z. \quad (2.1.3)$$

diskret Hill tenglamasi yordamida integrallanuvchi barcha P_m va Q_m larni topishdan iborat.

Ushbu $y^j(t) = (y_0^j(t), y_1^j(t), \dots, y_N^j(t))^T$, $j = 1, 2, \dots, N-1$ funksiyalar orqali quyidagi

$$\begin{cases} (L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, & 1 \leq n \leq N \\ y_1 = 0, \quad y_{N+1} = 0. \end{cases}$$

chegaraviy masalani $\lambda = \mu_j(t)$, $j = 1, 2, \dots, N-1$ xos qiymatlariga mos keluvchi ortonormallangan xos funksiyalarni begilaymiz. Ushbu

$$L(t)y^j = \mu_j(t)y^j, \quad j = 1, 2, \dots, N-1$$

ayniyatni t bo'yicha differentsiallab skalyar ko'payitmadan foydalansak quyidagiga ega bo'lamiz

$$\dot{\mu}_j(t) = \sum_{n=1}^N (2\dot{a}_n(t)y_n^j y_{n+1}^j + \dot{b}_n(t)(y_n^j)^2), \quad j = 1, 2, \dots, N-1. \quad (2.1.4)$$

(2.1.1) ni etiborga olib oxirgi tenglik quyidagicha yozamiz

$$\dot{\mu}_j(t) = \sum_{n=1}^N [2P_m(a_n, b_n)y_n^j y_{n+1}^j + Q_m(a_n, b_n)(y_n^j)^2] + \sum_{n=1}^N [\Phi_n(t)y_n^j y_{n+1}^j + F_n(t)(y_n^j)^2]. \quad (2.1.5)$$

Bunda,

$$\begin{aligned} \Phi_n(t) &= 2a_n \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_{n+1}^-(\lambda, t)\psi_{n+1}^+(\lambda, t) - \psi_n^-(\lambda, t)\psi_n^+(\lambda, t)] d\lambda, \\ F_n(t) &= a_n \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_n^-(\lambda, t)\psi_{n+1}^+(\lambda, t) + \psi_{n+1}^-(\lambda, t)\psi_n^+(\lambda, t)] d\lambda - \\ &\quad - a_{n-1} \int_E \tilde{\theta}_{N+1}(\lambda, t) [\psi_n^-(\lambda, t)\psi_{n-1}^+(\lambda, t) + \psi_{n-1}^-(\lambda, t)\psi_n^+(\lambda, t)] d\lambda. \end{aligned}$$

Qulaylik uchun quyidagi belgilashni kiritamiz

$$H_n = 2P_m(a_n, b_n)y_n^j y_{n+1}^j + Q_m(a_n, b_n)(y_n^j)^2. \quad (2.1.6)$$

Biz ushbu

$$u_{n+1} - u_n = H_n. \quad (2.1.7)$$

Shartni qanoatlantiruvchi u_n funksiyani topamiz. Buning uchun u_n ni quyidagi ko'rinishda izaymiz

$$u_n = A_n (y_n^j)^2 + 2a_n(t) B_n y_n^j y_{n+1}^j + a_n^2(t) C_n (y_{n+1}^j)^2, \quad (2.1.8)$$

Bu yerda $A_n = A_n(\mu_j(t))$, $B_n = B_n(\mu_j(t))$ va $C_n = C_n(\mu_j(t))$ lar xozircha nomalum koefitsiyentlar.

(2.1.8) ni (2.1.7) ga qo'yib ushbu

$$A_{n+1}(y_{n+1}^j)^2 + 2a_{n+1}(t)B_{n+1}y_{n+1}^j y_{n+2}^j + a_{n+1}^2(t)C_{n+1}(y_{n+2}^j)^2 - A_n(y_n^j)^2 - 2a_n(t)B_n y_n^j y_{n+1}^j - a_n^2(t)C_n(y_{n+1}^j)^2 = H_n. \quad (2.1.9)$$

Tenglikka ega bo'lamiz. Quyidagi

$$y_{n+2}^j = \frac{1}{a_{n+1}(t)}[(\mu_j(t) - b_{n+1}(t))y_{n+1}^j - a_n(t)y_n^j]$$

Tenglikni etiborga olsak (2.1.9) ni quyidagicha yozishimiz mumkin

$$(A_{n+1} - a_n^2 C_n)(y_{n+1}^j)^2 - A_n(y_n^j)^2 - 2B_n y_n^j y_{n+1}^j + 2B_{n+1} y_{n+1}^j [(\mu_j - b_{n+1})y_{n+1}^j - a_n y_n^j] + C_{n+1}(\mu_j - b_{n+1})^2 (y_{n+1}^j)^2 - 2a_n C_{n+1}(\mu_j - b_{n+1})y_n^j y_{n+1}^j + C_{n+1}a_n^2 (y_n^j)^2 = H_n. \quad (2.1.10)$$

(2.1.6) ni etiborga olgan holda (2.1.10) ning o'ng va chap taraflarini solishtirib quyidagiga ega bo'lamiz

$$-A_n + a_n^2 C_{n+1} = Q_m,$$

$$A_{n+1} - a_n^2 C_n + 2B_{n+1}(\mu_j - b_{n+1}) + C_{n+1}(\mu_j - b_{n+1})^2 = 0,$$

$$-a_n B_n - a_n B_{n+1} - a_n C_{n+1}(\mu_j - b_{n+1}) = P_m.$$

Bundan esa,

$$P_m = -a_n B_n - a_n B_{n+1} - a_n C_{n+1}(\mu_j - b_{n+1}), \quad (2.1.11)$$

$$Q_m = a_n^2 C_{n+1} - a_n^2 C_n + 2B_n(\mu_j - b_n) + C_n(\mu_j - b_n)^2. \quad (2.1.12)$$

(2.1.11) va (2.1.12) ning chap tarafi μ_j ga bog'lik b'lmagani uchun B_n va C_n

larni quidagi ko'rinishda izlamiz

$$B_n = \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k}, \quad C_n = \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k}. \quad (2.1.13)$$

(2.1.13) ni (2.1.11) ga qo'yib ushbu

$$P_m = -a_n \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} - a_n \sum_{k=0}^m \beta_{n+1,k} \mu_j^{m-k} - a_n(\mu_j - b_{n+1}) \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k} =$$

$$\begin{aligned}
&= -a_n \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} - a_n \sum_{k=0}^m \beta_{n+1,k} \mu_j^{m-k} - a_n \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k+1} + a_n b_{n+1} \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k} = \\
&= -a_n \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} - a_n \sum_{k=0}^m \beta_{n+1,k} \mu_j^{m-k} - a_n \sum_{k=-1}^{m-1} \alpha_{n+1,k+1} \mu_j^{m-k} + a_n b_{n+1} \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k} = \\
&\quad = -a_n \beta_{n,m} - a_n \beta_{n+1,m} + a_n b_{n+1} \alpha_{n+1,m} - a_n \alpha_{n+1,0} \mu_j^{m+1} + \\
&\quad + \sum_{k=0}^{m-1} (-a_n \beta_{n,k} - a_n \beta_{n+1,k} - a_n \alpha_{n+1,k+1} + a_n b_{n+1} \alpha_{n+1,k}) \mu_j^{m-k}.
\end{aligned}$$

Tenglikka ega bo`lamiz. Buning o`ng va chap taraflarni solishtirish natijasida quyidagi tenglikni hosil qilamiz

$$\alpha_{n+1,0} = 0, \quad (2.1.14)$$

$$P_m = -a_n \beta_{n,m} - a_n \beta_{n+1,m} + a_n b_{n+1} \alpha_{n+1,m}, \quad (2.1.15)$$

va

$$-\beta_{n,k} - \beta_{n+1,k} - \alpha_{n+1,k+1} + b_{n+1} \alpha_{n+1,k} = 0, \quad k = 0, 1, \dots, m-1. \quad (2.1.16)$$

Xuddi shunga o`xshash (2.1.13) ni (2.1.12) ga qo`yib quyidagi tenglikni topamiz

$$\begin{aligned}
Q_m &= a_n^2 \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k} - a_{n-1}^2 \sum_{k=0}^m \alpha_{n-1,k} \mu_j^{m-k} + 2(\mu_j - b_n) \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} + \\
&+ (\mu_j - b_n)^2 \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k} = a_n^2 \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k} - a_{n-1}^2 \sum_{k=0}^m \alpha_{n-1,k} \mu_j^{m-k} - 2b_n \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} + \\
&+ 2\mu_j \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} + \mu_j^2 \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k} - 2b_n \mu_j \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k} + b_n^2 \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k} = \\
&= a_n^2 \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k} - a_{n-1}^2 \sum_{k=0}^m \alpha_{n-1,k} \mu_j^{m-k} - 2b_n \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} + b_n^2 \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k} + \\
&\quad + 2 \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k+1} + \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k+2} - 2b_n \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k+1} = \\
&= a_n^2 \sum_{k=0}^m \alpha_{n+1,k} \mu_j^{m-k} - a_{n-1}^2 \sum_{k=0}^m \alpha_{n-1,k} \mu_j^{m-k} - 2b_n \sum_{k=0}^m \beta_{n,k} \mu_j^{m-k} + b_n^2 \sum_{k=0}^m \alpha_{n,k} \mu_j^{m-k} + \\
&\quad + 2 \sum_{k=-1}^{m-1} \beta_{n,k+1} \mu_j^{m-k} + \sum_{k=-2}^{m-2} \alpha_{n,k+2} \mu_j^{m-k} - 2b_n \sum_{k=-1}^{m-1} \alpha_{n,k+1} \mu_j^{m-k} =
\end{aligned}$$

$$\begin{aligned}
&= a_n^2 \alpha_{n+1,m} - a_{n-1}^2 \alpha_{n-1,m} - 2b_n \beta_{n,m} + b_n^2 \alpha_{n,m} + \\
&+ a_n^2 \alpha_{n+1,m-1} \mu_j - a_{n-1}^2 \alpha_{n-1,m-1} \mu_j - 2b_n \beta_{n,m-1} \mu_j + b_n^2 \alpha_{n,m-1} \mu_j + 2\beta_{n,m} \mu_j - 2b_n \alpha_{n,m} \mu_j + \\
&+ 2\beta_{n,0} \mu_j^{m+1} + \alpha_{n,0} \mu_j^{m+2} + \alpha_{n,1} \mu_j^{m+1} - 2b_n \alpha_{n,0} \mu_j^{m+1} + \\
&+ \sum_{k=0}^{m-2} (a_n^2 \alpha_{n+1,k} - a_{n-1}^2 \alpha_{n-1,k} - 2b_n \beta_{n,k} + b_n^2 \alpha_{n,k} + 2\beta_{n,k+1} + \alpha_{n,k+2} - 2b_n \alpha_{n,k+1}) \mu_j^{m-k} = \\
&= a_n^2 \alpha_{n+1,m} - a_{n-1}^2 \alpha_{n-1,m} - 2b_n \beta_{n,m} + b_n^2 \alpha_{n,m} + \\
&+ (a_n^2 \alpha_{n+1,m-1} - a_{n-1}^2 \alpha_{n-1,m-1} - 2b_n \beta_{n,m-1} + b_n^2 \alpha_{n,m-1} + 2\beta_{n,m} - 2b_n \alpha_{n,m}) \mu_j + \\
&+ \sum_{k=0}^{m-2} (a_n^2 \alpha_{n+1,k} - a_{n-1}^2 \alpha_{n-1,k} - 2b_n \beta_{n,k} + b_n^2 \alpha_{n,k} + 2\beta_{n,k+1} + \alpha_{n,k+2} - 2b_n \alpha_{n,k+1}) \mu_j^{m-k} + \\
&+ (2\beta_{n,0} + \alpha_{n,1} - 2b_n \alpha_{n,0}) \mu_j^{m+1} + \alpha_{n,0} \mu_j^{m+2}.
\end{aligned}$$

Buning o'ng va chap taraflarni solishtirish natijasida quyisdagi tenglikarni hosil qilamiz

$$\alpha_{n,0} = 0, \quad (2.1.17)$$

$$2\beta_{n,0} + \alpha_{n,1} = 0, \quad (2.1.18)$$

$$a_n^2 \alpha_{n+1,m-1} - a_{n-1}^2 \alpha_{n-1,m-1} - 2b_n \beta_{n,m-1} + b_n^2 \alpha_{n,m-1} + 2\beta_{n,m} - 2b_n \alpha_{n,m} = 0, \quad (2.1.19)$$

$$Q_m = a_n^2 \alpha_{n+1,m} - a_{n-1}^2 \alpha_{n-1,m} + b_n^2 \alpha_{n,m} - 2b_n \beta_{n,m}, \quad (2.1.20)$$

va

$$a_n^2 \alpha_{n+1,k} - a_{n-1}^2 \alpha_{n-1,k} - 2b_n \beta_{n,k} + b_n^2 \alpha_{n,k} + 2\beta_{n,k+1} + \alpha_{n,k+2} - 2b_n \alpha_{n,k+1} = 0, \quad (2.1.21)$$

$$k = 0, 1, \dots, m-2.$$

(2.1.17), (2.1.18) va (2.1.16) ning birinchi munsabatidan ushbu

$$\alpha_{n,0} = 0, \quad \beta_{n,0} = c_0 = const, \quad \alpha_{n,1} = 2c_0, \quad (2.1.22)$$

Tenglikni hosil qilamiz. Qulaylik uchun (2.1.16), (2.1.19) va (2.1.21) formulalarni quyidagicha yozib olamiz

$$-\beta_{n,k-1} - \beta_{n+1,k-1} - \alpha_{n+1,k} + b_{n+1}\alpha_{n+1,k-1} = 0, \quad k = 2, 3, \dots, m, \quad (2.1.23)$$

$$a_n^2\alpha_{n+1,k-2} - a_{n-1}^2\alpha_{n-1,k-2} - 2b_n\beta_{n,k-2} + b_n^2\alpha_{n,k-2} + 2\beta_{n,k-1} + \alpha_{n,k} - 2b_n\alpha_{n,k-1} = 0, \quad (2.1.24)$$

$k = 2, 3, \dots, m,$

$$a_n^2\alpha_{n+1,m-1} - a_{n-1}^2\alpha_{n-1,m-1} - 2b_n\beta_{n,m-1} + b_n^2\alpha_{n,m-1} + 2\beta_{n,m} - 2b_n\alpha_{n,m} = 0, \quad (2.1.25)$$

(2.1.23)-(2.1.25) ifodalarda oddiy soddalashtirishlarni bajarib quyidagiga ega bo`lamiz

$$a_n^2\alpha_{n+1,k-2} - a_{n-1}^2\alpha_{n-1,k-2} - 2b_n\beta_{n,k-2} + b_n^2\alpha_{n,k-2} + 2\beta_{n,k-1} +$$

$$+ (-\beta_{n-1,k-1} - \beta_{n,k-1} + b_n\alpha_{n,k-1}) - 2b_n\alpha_{n,k-1} = 0,$$

$$a_n^2\alpha_{n+1,k-2} - a_{n-1}^2\alpha_{n-1,k-2} - 2b_n\beta_{n,k-2} + b_n^2\alpha_{n,k-2} +$$

$$+ \beta_{n,k-1} - \beta_{n-1,k-1} - b_n\alpha_{n,k-1} = 0,$$

$$a_n^2\alpha_{n+1,k-2} - a_{n-1}^2\alpha_{n-1,k-2} - 2b_n\beta_{n,k-2} + b_n^2\alpha_{n,k-2} +$$

$$+ \beta_{n,k-1} - \beta_{n-1,k-1} - b_n(-\beta_{n-1,k-2} - \beta_{n,k-2} + b_n\alpha_{n,k-2}) = 0,$$

$$\beta_{n,k-1} - \beta_{n-1,k-1} = b_n(\beta_{n,k-2} - \beta_{n-1,k-2}) + a_{n-1}^2\alpha_{n-1,k-2} - a_n^2\alpha_{n+1,k-2}, \quad 2 \leq k \leq m. \quad (2.1.26)$$

(2.1.23) ni (2.1.25) ga qo`yib ushbu munosabatni hosil qilamiz

$$\beta_{n,m} = -\frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} + b_n\beta_{n,m-1} - \frac{b_n^2}{2}\alpha_{n,m-1} + b_n\alpha_{n,m},$$

$$\begin{aligned} \beta_{n,m} = & -\frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} + b_n\beta_{n,m-1} - \frac{b_n^2}{2}\alpha_{n,m-1} + \\ & + b_n(-\beta_{n-1,m-1} - \beta_{n,m-1} + b_n\alpha_{n,m-1}), \end{aligned}$$

$$\beta_{n,m} = \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} - \frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{b_n^2}{2}\alpha_{n,m-1} - b_n\beta_{n-1,m-1}. \quad (2.1.27)$$

(2.1.22), (2.1.23), (2.1.26) va (2.1.27) rekurent munosabatlarni juda sodda

ko`rinishda quyidagicha yozamiz

$$\alpha_{n,0} = 0, \quad \beta_{n,0} = c_0 = const, \quad \alpha_{n,1} = 2c_0, \quad (2.1.28)$$

$$\beta_{n,k-1} - \beta_{n-1,k-1} = b_n(\beta_{n,k-2} - \beta_{n-1,k-2}) - a_n^2\alpha_{n+1,k-2} + a_{n-1}^2\alpha_{n-1,k-2}, \quad 2 \leq k \leq m, \quad (2.1.29)$$

$$\alpha_{n,k} = b_n\alpha_{n,k-1} - \beta_{n-1,k-1} - \beta_{n,k-1}, \quad 2 \leq k \leq m, \quad (2.1.30)$$

$$\beta_{n,m} = \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} - \frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{b_n^2}{2}\alpha_{n,m-1} - b_n\beta_{n-1,m-1}, \quad (2.1.31)$$

Xuddi shuningdek

$$\begin{cases} P_m(a_n, b_n) = a_n[-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}], \\ Q_m(a_n, b_n) = a_n^2\alpha_{n+1,m} - a_{n-1}^2\alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2\alpha_{n,m}, \end{cases} \quad m \in N, t \in R. \quad (2.1.32)$$

bo`ladi. Oxirgi tenglikda $m \in N$ ni o`zgartirish hisobiga integral manbali davriy umumiy Toda tengslamasining barcha o`rinishi kelib chqadi

$$\begin{cases} \dot{a}_n = a_n[-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}] + \Phi_n(t), \\ \dot{b}_n = a_n^2\alpha_{n+1,m} - a_{n-1}^2\alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2\alpha_{n,m} + F_n(t), \end{cases} \quad m \in N, t \in R. \quad (2.1.33)$$

Explicitly, one obtains from (2.1.28)-(2.1.31)

$$\alpha_{n,1} = 2c_0,$$

$$\beta_{n,1} = -c_0b_n,$$

$$\alpha_{n,2} = 2c_0b_n - 2c_1,$$

$$\begin{aligned}
\beta_{n,2} &= c_0(a_{n-1}^2 - a_n^2 + b_n^2) - c_1 b_n, \\
\alpha_{n,3} &= 2c_0(b_n^2 + a_{n-1}^2 + a_n^2) - 2c_1 b_n - 2c_2, \\
\beta_{n,3} &= c_0(a_{n-1}^2 b_{n-1} - a_n^2 b_{n+1} + b_n^3 + 2a_{n-1}^2 b_n) + c_1(a_n^2 - a_{n-1}^2 - b_n^2) - c_2 b_n \quad (2.1.31)
\end{aligned}$$

etc.

and hence from (2.1.33), we find few equations of the periodic Toda lattice hierarchy an integral source,

$$m = 1, c_0 = -1,$$

$$\begin{cases} \dot{a}_n = a_n(b_{n+1} - b_n) + \Phi_n(t), \\ \dot{b}_n = 2(a_n^2 - a_{n-1}^2) + F_n(t), \end{cases}$$

$$m = 2,$$

$$\begin{cases} \dot{a}_n = c_1 a_n(b_n - b_{n+1}) + a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2) + \Phi_n(t), \\ \dot{b}_n = 2c_1(a_{n-1}^2 - a_n^2) - 2a_{n-1}^2(b_n + b_{n-1}) + 2a_n^2(b_n + b_{n+1}) + F_n(t), \end{cases}$$

$$m = 3,$$

$$\begin{cases} \dot{a}_n = c_2 a_n(b_n - b_{n+1}) - c_1 a_n(a_{n+1}^2 - a_{n-1}^2 + b_{n+1}^2 - b_n^2) + a_n(b_{n+1}^3 - b_n^3) + \\ \quad + a_n[a_{n+1}^2(b_{n+2} + 2b_{n+1}) + a_n^2(b_{n+1} - b_n) - a_{n-1}^2(b_{n-1} + 2b_n)] + \Phi_n(t), \\ \dot{b}_n = 2c_2(a_{n-1}^2 - a_n^2) + 2c_1[a_{n-1}^2(b_n + b_{n-1}) - a_n^2(b_n + b_{n+1})] + 2a_n^2(a_n^2 + a_{n+1}^2) + \\ \quad + 2a_n^2(b_n^2 + b_n b_{n+1} + b_{n+1}^2) - 2a_{n-1}^2(b_n^2 + b_n b_{n-1} + b_{n-1}^2) - 2a_{n-1}^2 a_{n-2}^2 - 2a_{n-1}^4 + F_n(t), \end{cases}$$

etc.

2-§ Spektral parametrlarning o`zgarish dinamikasini aniqlash

Ushbu paragrafda biz spektral parametрни vaqt bo`yicha o`zgarish qonuniyatini keltirib chiqaramiz.

Teorema. Agar $a_n(t)$, $b_n(t)$, $\psi_n^\pm(\lambda, t)$, $n \in Z$ funksiyalar (1.1.1)-(1.1.4) masalani yechimlari bo`lsa, u holda (1.1.3) diskret Hill operatorining spektri t o`zgaruvchiga bog`liq bo`lmaydi, $\mu_j(t)$, $j=1, 2, \dots, N-1$, spektral parametrlar ushbu

$$\mu_j(t) = 2 \frac{\sigma_j(t) \cdot \sqrt{\prod_{k=1}^{2N} (\mu_j(t) - \lambda_k)}}{\prod_{\substack{k=1 \\ k \neq j}}^{N-1} (\mu_j(t) - \mu_k(t))} \cdot \left\{ C_1(\mu_j(t)) - \int_E \tilde{\theta}_{N+1}(\lambda, t) \frac{1}{\lambda - \mu_j(t)} d\lambda \right\}, \quad (2.2.1)$$

Tenglamalar sistemasini qanoatlantiradi. Bunda $C_1(\mu_j(t)) = \sum_{k=0}^m \alpha_{1,k} \mu_j^{m-k}(t)$.

Isbot. (2.1.7) formulani (2.1.8) ga qo`yib n bo`yicha yig`indi olsak quyidagiga ega bo`lamiz

$$\begin{aligned} \sum_{n=1}^N H_n = u_{N+1} - u_1 = A_{N+1} (y_{N+1}^j)^2 + 2a_{N+1}(t) B_{N+1} y_{N+1}^j y_{N+2}^j + a_N^2(t) C_N (y_{N+2}^j)^2 - \\ - A_1 (y_1^j)^2 - 2a_1(t) B_1 y_1^j y_2^j - a_1^2(t) C_1 (y_2^j)^2. \end{aligned} \quad (2.2.2)$$

Ushbu

$$a_{N+1}(t) y_{N+2}^j = (\mu_j(t) - b_{N+1}(t)) y_{N+1}^j - a_N(t) y_N^j,$$

tenglik va $y_1^j = 0$, $y_{N+1}^j = 0$ shartlarni etiborga olsak, (2.2.2) dan quyidagi

$$\sum_{n=1}^N H_n = a_0^2(t) C_1 (y_0^j)^2 - a_N^2(t) C_{N+1} (y_N^j)^2 = a_0^2(t) C_1 [(y_0^j)^2 - (y_N^j)^2], \quad (2.2.3)$$

tenglikni olamiz. (2.2.3) ni (2.1.5) ga qo`yib quyidagiga ega bo`lamiz

$$\mu_j(t) = a_0^2(t) C_1 [(y_0^j)^2 - (y_N^j)^2] + \sum_{n=1}^N [\Phi_n(t) y_n^j y_{n+1}^j + F_n(t) (y_n^j)^2]. \quad (2.2.4)$$

Endi biz (2.2.4) tenglikni o`ng tarafidagi ikkinchi tenglikni soddalashtiramiz

$$\sum_{n=1}^N a_n \psi_{n+1}^-(\lambda, t) \psi_{n+1}^+(\lambda, t) y_n^j y_{n+1}^j - \sum_{n=1}^N a_n \psi_n^-(\lambda, t) \psi_{n+1}^+(\lambda, t) (y_{n+1}^j)^2 +$$

$$\begin{aligned}
& + \sum_{n=1}^N a_n \psi_{n+1}^-(\lambda, t) \psi_{n+1}^+(\lambda, t) y_n^j y_{n+1}^j - \sum_{n=1}^N a_n \psi_{n+1}^-(\lambda, t) \psi_n^+(\lambda, t) (y_{n+1}^j)^2 + \\
& + \sum_{n=1}^N a_n \psi_n^-(\lambda, t) \psi_{n+1}^+(\lambda, t) (y_n^j)^2 - \sum_{n=1}^N a_n \psi_n^-(\lambda, t) \psi_n^+(\lambda, t) y_{n+1}^j y_n^j + \\
& + \sum_{n=1}^N a_n \psi_{n+1}^-(\lambda, t) \psi_n^+(\lambda, t) (y_n^j)^2 - \sum_{n=1}^N a_n \psi_n^-(\lambda, t) \psi_n^+(\lambda, t) y_{n+1}^j y_n^j = \\
& = \sum_{n=1}^N a_n (\psi_{n+1}^+(\lambda, t) y_n^j - \psi_n^+(\lambda, t) y_{n+1}^j) \psi_n^-(\lambda, t) y_n^j + \sum_{n=1}^N a_n (\psi_{n+1}^+(\lambda, t) y_n^j - \psi_n^+(\lambda, t) y_{n+1}^j) \psi_{n+1}^-(\lambda, t) y_{n+1}^j + \\
& + \sum_{n=1}^N a_n (\psi_{n+1}^-(\lambda, t) y_n^j - \psi_n^-(\lambda, t) y_{n+1}^j) \psi_n^+(\lambda, t) y_n^j + \sum_{n=1}^N a_n (\psi_{n+1}^-(\lambda, t) y_n^j - \psi_n^-(\lambda, t) y_{n+1}^j) \psi_{n+1}^+(\lambda, t) y_{n+1}^j = \\
& = \sum_{n=1}^N T_n \psi_n^-(\lambda, t) y_n^j + \sum_{n=1}^N T_n \psi_{n+1}^-(\lambda, t) y_{n+1}^j + \sum_{n=1}^N W_n \psi_n^+(\lambda, t) y_n^j + \sum_{n=1}^N W_n \psi_{n+1}^+(\lambda, t) y_{n+1}^j = \\
& = \sum_{n=1}^N \psi_{n+1}^-(\lambda, t) y_{n+1}^j (T_{n+1} + T_n) + \sum_{n=1}^N \psi_{n+1}^+(\lambda, t) y_{n+1}^j (W_{n+1} + W_n) = \\
& = \sum_{n=1}^N \frac{1}{\lambda - \mu_j(t)} (W_{n+1} - W_n) (T_{n+1} + T_n) + \sum_{n=1}^N \frac{1}{\lambda - \mu_j(t)} (T_{n+1} - T_n) (W_{n+1} + W_n) = \\
& = \frac{2}{\lambda - \mu_j(t)} \sum_{n=1}^N (W_{n+1} T_{n+1} - W_n T_n) = \frac{2}{\lambda - \mu_j(t)} (W_{N+1} T_{N+1} - W_1 T_1), \tag{2.2.5}
\end{aligned}$$

bu yerda $T_n = a_n (\psi_{n+1}^+(\lambda, t) y_n^j - \psi_n^+(\lambda, t) y_{n+1}^j)$ va $W_n = a_n (\psi_{n+1}^-(\lambda, t) y_n^j - \psi_n^-(\lambda, t) y_{n+1}^j)$.

Oddiy hisoblashlardan quyidagilarga ega bo`lamiz

$$\begin{aligned}
W_{N+1} T_{N+1} & = a_{N+1} (\psi_{N+2}^-(\lambda, t) y_{N+1}^j - \psi_{N+1}^-(\lambda, t) y_{N+2}^j) a_{N+1} (\psi_{N+2}^+(\lambda, t) y_{N+1}^j - \psi_{N+1}^+(\lambda, t) y_{N+2}^j) = \\
& = (-\psi_{N+1}^-(\lambda, t) a_{N+1} y_{N+2}^j) (-\psi_{N+1}^+(\lambda, t) a_{N+1} y_{N+2}^j) = \\
& = (-\psi_{N+1}^-(\lambda, t) a_{N+1} y_{N+2}^j) (-\psi_{N+1}^+(\lambda, t) a_{N+1} y_{N+2}^j) = \psi_{N+1}^-(\lambda, t) a_N y_N^j \psi_{N+1}^+(\lambda, t) a_N y_N^j = \\
& = a_N (y_N^j)^2 \psi_{N+1}^-(\lambda, t) \psi_{N+1}^+(\lambda, t) = a_N (y_N^j)^2. \tag{2.2.6}
\end{aligned}$$

$$W_1 T_1 = a_0^2 (y_0^j)^2. \tag{2.2.7}$$

(2.2.5) formuladan foyalanib ushbu

$$\sum_{n=1}^N [\Phi_n(t) y_n^j y_{n+1}^j + F_n(t) (y_n^j)^2] = \int_E \tilde{\theta}_{N+1}(\lambda, t) \frac{2a_0^2}{\lambda - \mu_j(t)} [(y_N^j)^2 - (y_0^j)^2] d\lambda. \quad (2.2.8)$$

tenglikni keltirib chiqaramiz. (2.2.8) ni (2.2.4) ga qo'yib quyidagiga ega bo'lamiz

$$\begin{aligned} \dot{\mu}_j(t) &= 2a_0^2 C_1(\mu_j(t)) [(y_0^j)^2 - (y_N^j)^2] + \int_E \tilde{\theta}_{N+1}(\lambda, t) \frac{2a_0^2}{\lambda - \mu_j(t)} [(y_N^j)^2 - (y_0^j)^2] d\lambda = \\ &= -2a_0^2 [(y_N^j)^2 - (y_0^j)^2] \left\{ C_1(\mu_j(t)) - \int_E \tilde{\theta}_{N+1}(\lambda, t) \frac{1}{\lambda - \mu_j(t)} d\lambda \right\}, \end{aligned} \quad (2.2.9)$$

Bu yerda $C_1(\mu_j(t)) = \sum_{k=0}^m \alpha_{1,k} \mu_j^{m-k}(t) \cdot \alpha_{1,k}$, $k = 0, 1, \dots, m$ factorlar (2.1.28)-(2.1.31)

rekurent formulalardan aniqlanadi.

Ushbu

$$\|\theta^j\|^2 = \sum_{n=1}^N (\theta_n^j)^2 = a_N \theta_N^j (\theta_{N+1}^j)' \Big|_{\lambda=\mu_j}, \quad (\theta^j)' = \frac{d\theta^j}{d\lambda},$$

$$(y_0^j)^2 = \frac{(\theta_0^j)^2}{\|\theta^j\|^2}, \quad (y_N^j)^2 = \frac{(\theta_N^j)^2}{\|\theta^j\|^2},$$

tengliklarni etiborga olisak (2.2.9) quyidagicha yoziladi

$$\dot{\mu}_j(t) = - \frac{2a_0 \left(\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)} \right)}{(\theta_{N+1}^j)' \Big|_{\lambda=\mu_j(t)}} \cdot \left\{ C_1(\mu_j(t)) - \int_E \frac{\tilde{\theta}_{N+1}(\lambda, t)}{\lambda - \mu_j(t)} d\lambda \right\}. \quad (2.2.10)$$

Quyidagi

$$\theta_N(\lambda, t) \varphi_{N+1}(\lambda, t) - \theta_{N+1}(\lambda, t) \varphi_N(\lambda, t) = 1,$$

tengliklardan ushbu

$$\Delta^2(\mu_j(t)) - 4 = [\theta_N^j(\mu_j(t), t) - \varphi_{N+1}^j(\mu_j(t), t)]^2 + 4\theta_N^j(\mu_j(t), t) \varphi_{N+1}^j(\mu_j(t), t) - 4 =$$

$$=[\theta_N^j(\mu_j(t), t) - \varphi_{N+1}^j(\mu_j(t), t)]^2 = \left(\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)} \right)^2.$$

munosabatni hosil qilamiz, bunda esa quyidagiga ega bo`lamiz

$$\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)} = \sigma_j(t) \sqrt{\Delta^2(\mu_j(t)) - 4}, \quad (2.2.11)$$

Bu yerda

$$\sigma_j(t) = \text{sign} \left(\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)} \right), \quad j=1, 2, \dots, N-1.$$

(1.2.2) va (1.2.3) formulardan ushbu

$$\Delta^2(\lambda) - 4 = \left(\prod_{k=1}^N a_k \right)^{-2} \prod_{k=1}^{2N} (\lambda - \lambda_k), \quad (2.2.12)$$

$$\theta_{N+1}(\lambda, t) = -a_0 \left(\prod_{j=1}^N a_j \right)^{-1} \prod_{k=1}^{N-1} (\lambda - \mu_k(t)). \quad (2.2.13)$$

tengliklarga ega bo`lamiz. (1.3.12) ifodani λ bo`yicha differentsiallab $\lambda = \mu_j(t)$ bo`lsin desak quyidagi tenglikka ega bo`lamiz

$$\theta'_{N+1}(\lambda) \Big|_{\lambda=\mu_j(t)} = -a_0 \left(\prod_{k=1}^N a_k \right)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^{N-1} (\mu_j(t) - \mu_k(t)) \quad (2.2.14)$$

(2.2.11), (2.2.12) va (2.2.14) formulalarni (2.2.10) ga qo`yib (2.2.1) formulani hosil qilamiz.

Endi biz $\lambda_k(t)$ xos qiymatlarni t parametrغا bog`liq emasligini ko`rsatamiz. $\{g_n^k(t)\}$ funktsiyalar $L(t)$ operatorning $\lambda_k(t)$, $k=1, 2, \dots, 2N$ xos qiymatlarga mos keluvchi ortonormallangan xos funktsiyalari bo`lsin, yani

$$a_{n-1} g_{n-1}^k + b_n g_n^k + a_n g_{n+1}^k = \lambda_k g_n^k.$$

Oxirgi tenglikni t parameter bo'yicha differensiallaymiz va natijani g_n^k ga ko'paytirib n bo'yicha yig'indi olsak quyidagiga ega bo'lamiz

$$\frac{d\lambda_k}{dt} = \sum_{n=1}^N \left(2\dot{a}_n(t) g_n^k g_{n+1}^k + \dot{b}_n(t) (g_n^k)^2 \right). \quad (2.2.15)$$

(1.1) tenglikni etiborga olsak (2.2.15) ni ushbu

$$\begin{aligned} \frac{d\lambda_k}{dt} = & \sum_{n=1}^N (2a_n b_n g_n^k g_{n+1}^k - 2a_n b_{n+1} g_n^k g_{n+1}^k) + \sum_{n=1}^N 2(a_{n-1}^2 - a_n^2) (g_n^k)^2 + \\ & + \int_E \tilde{\theta}_{N+1}(\lambda, t) G^k(\lambda, t) d\lambda, \end{aligned} \quad (2.2.16)$$

ko'rinishda yozish mumkin. Bu yerda

$$\begin{aligned} G^k(\lambda, t) = & \sum_{n=1}^N \left\{ 2a_n [\psi_{n+1}^-(\lambda, t) \psi_{n+1}^+(\lambda, t) - \psi_n^-(\lambda, t) \psi_n^+(\lambda, t)] \right\} g_n^k g_{n+1}^k + \\ & + \sum_{n=1}^N \left\{ a_n [\psi_n^-(\lambda, t) \psi_{n+1}^+(\lambda, t) + \psi_{n+1}^-(\lambda, t) \psi_n^+(\lambda, t)] \right\} (g_n^k)^2 - \\ & - \sum_{n=1}^N \left\{ a_{n-1} [\psi_n^-(\lambda, t) \psi_{n-1}^+(\lambda, t) + \psi_{n-1}^-(\lambda, t) \psi_n^+(\lambda, t)] \right\} (g_n^k)^2. \end{aligned}$$

(1.2.1) tenglikdan foydalanib (2.2.16) tenglikni o'ng tarafini quyiagicha yozish mumkin

$$\begin{aligned} \frac{d\lambda_k}{dt} = & \sum_{n=1}^N \left\{ 2a_n [\lambda_k(t) g_n^k - a_{n-1} g_{n-1}^k - a_n g_{n+1}^k] g_{n+1}^k - 2a_n [\lambda_k(t) g_{n+1}^k - a_n g_n^k - a_{n+1} g_{n+2}^k] g_n^k \right\} + \\ & + \sum_{n=1}^N 2(a_{n-1}^2 - a_n^2) (g_n^k)^2 + \int_E \tilde{\theta}_{N+1}(\lambda, t) G^k(\lambda, t) d\lambda = \sum_{n=1}^N 2 \left[-a_{n-1} a_n g_{n-1}^k g_{n+1}^k - a_n^2 (g_{n+1}^k)^2 \right] + \\ & + \sum_{n=1}^N 2 \left[a_n^2 (g_n^k)^2 + a_n a_{n+1} g_{n+2}^k g_n^k \right] + \sum_{n=1}^N 2(a_{n-1}^2 - a_n^2) (g_n^k)^2 + \int_E \tilde{\theta}_{N+1}(\lambda, t) G^k(\lambda, t) d\lambda = \\ & = - \sum_{n=1}^N 2a_{n-1} a_n g_{n-1}^k g_{n+1}^k + \sum_{n=2}^{N+2} 2a_{n-1} a_n g_{n-1}^k g_{n+1}^k - \end{aligned}$$

$$-\sum_{n=2}^{N+1} 2a_{n-1}^2 (g_n^k)^2 + \sum_{n=1}^N 2a_{n-1}^2 (g_n^k)^2 + \int_E \tilde{\theta}_{N+1}(\lambda, t) G^k(\lambda, t) d\lambda = \int_E \tilde{\theta}_{N+1}(\lambda, t) G^k(\lambda, t) d\lambda. \quad (2.2.17)$$

$\lambda \neq \lambda_k$ bo`lsin, u holda

$$G^k(\lambda, t) = \frac{2a_0^2}{\lambda - \lambda_k} [(g_N^k)^2 - (g_0^k)^2] = 0. \quad (2.2.18)$$

bo`ladi. Agar $\lambda = \lambda_k$ bo`lsa u holda $G^k(\lambda, t) = 0$ bo`lishi oson ko`rsatiladi.

(2.2.17) va (2.2.18) formulalardan $\dot{\lambda}_k(t) = 0$ ekani kelib chiqadi. **Teorema isbotlandi.**

Izoh 2. $\alpha_{1,k}$, $k = 0, 1, \dots, m$ faktorlar $a_n(t)$, $b_n(t)$ larga bog`lik bolib (2.1.28)-(2.1.31) rekurent formulalardan aniqlanadi. Ma`lumki $a_n(t)$, $b_n(t)$ lar izlar formulasi yordamida λ_k va $\mu_j(t)$ lar orqali yoziladi.

Izoh 3. Teorema 1 qo`yilgan (1.1.1)-(1.1.4) masalani yechish usulini beradi.

1. Koeffitsienti $\{a_n^0\}$ va $\{b_n^0\}$ ketma-ketlikdan iborat diskret Hill tenglamasi uchun qo`yilgan to`g`ri masalani yechib λ_i , $i = 1, 2, \dots, 2N$ va $\mu_j(0)$, $\sigma_j(0)$, $i = 1, 2, \dots, N-1$ spectral berilganlar topiladi.

2. Teorema 1 ni qo`llab $\mu_j(t)$, $\sigma_j(t)$, $i = 1, 2, \dots, N-1$ spectral parametrlar topiladi.

3. Teskari masala yechish algoritmini qo`llab $a_k(t)$, $b_k(t)$ va $\psi_k^\pm(\lambda, t)$ lar topiladi.

Natija 1. Agar $N = 2p$ bo`lib p son $\{a_n^0\}$ va $\{b_n^0\}$ ketma-ketlikning davri bo`lsa u holda $\Delta(\lambda) + 2 = 0$ tengamani barcha ildizlari ikki karrali bo`ladi. Borg teskari teoremasining diskret analogiga ko`ra p son $a_n(t)$, $b_n(t)$ yechimlarning ham n o`zgaruvchi bo`yica davri bo`ladi.

3-§ Misollar

Ushbu paragrafda yuqorida olingan natijalarni aniq misollar yordamida ko`rib chiqamiz.

(1.1.1), (1.1.2) masalada $m = 2$ bo`lsin. Ya`ni ushbu

$$\begin{cases} \dot{a}_n = a_n(b_{n+1} - b_n) + a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2) + \Phi_n(t), \\ \dot{b}_n = 2(a_n^2 - a_{n-1}^2) - 2a_{n-1}^2(b_n + b_{n-1}) + 2a_n^2(b_n + b_{n+1}) + F_n(t). \end{cases}$$

ikkinchi davriy Toda tenglamasini quyidagi

$$(a_n^0)^2 = \frac{\alpha_1^2 + \alpha_2^2}{4} - (-1)^n \frac{\alpha_1 \alpha_2}{2}, \quad b_n^0 = 0, \quad n \in Z,$$

boshlang`ich shart bilan qaraymiz. Bunda α_1 va α_2 ushbu $0 < \alpha_1 < \alpha_2$ shartlarni qanoatlantiruvchi musba sonlar va $\Phi_n(t)$, $F_n(t)$ lar (2.1.5) formula bilan berilgan. By holda

$$N = 2, \quad \lambda_1 = -\alpha_2, \quad \lambda_2 = -\alpha_1, \quad \lambda_3 = \alpha_1, \quad \lambda_4 = \alpha_2, \quad \mu_1(0) = 0, \quad \sigma_1(0) = 1.$$

Izoh 3 ni qo`llab quyidagiga ega bo`lamiz

$$a_n^2(t) = \frac{\alpha_1^2 + \alpha_2^2}{4} - \frac{1}{2} \mu^2(t) - (-1)^n \frac{\sigma(t)}{2} \sqrt{(\mu^2(t) - \alpha_1^2)(\mu^2(t) - \alpha_2^2)},$$

$$b_n(t) = (-1)^n \mu(t), \quad n \in Z,$$

$$\psi_0^\pm(\lambda, t) = \frac{\lambda^2 - \mu^2(t) - \sigma(t) \sqrt{(\mu^2(t) - \alpha_1^2)(\mu^2(t) - \alpha_2^2)} \pm \sqrt{(\lambda^2 - \alpha_1^2)(\lambda^2 - \alpha_2^2)}}{2a_0(t)(\lambda - \mu(t))},$$

$$\psi_1^\pm(\lambda, t) = 1,$$

Bu yerda $\mu(t)$ funksiya ushbu

$$\frac{d\mu(t)}{dt} = 4(1 + \alpha_2 - \alpha_1) \sigma(t) \sqrt{(\mu^2(t) - \alpha_1^2)(\mu^2(t) - \alpha_2^2)},$$

tenglamani $\mu(0) = 0$, $\sigma(0) = 1$ boshlang`ich shartni qanoatlantiruvchi yechimi.

$\sigma(t)$ ishora $\mu(t)$ o`z lakunasini $[-\alpha_1, \alpha_1]$ ning chekka nuqtalari bilan to`qnashganda o`z ishorasini o`zgartiradi. Ushbu $\mu(t) = \alpha_1 \sin x(t)$ almashtirish va quyidagi

$$\text{sign } \sigma(t) \cdot \text{sign } (\cos x(t)) = \sigma(0),$$

tenglikdan ushbu

$$a_n^2(t) = \frac{\alpha_1^2 + \alpha_2^2}{4} - \frac{\alpha_1^2}{2} \sin^2 x(t) - (-1)^n \frac{\alpha_1 \alpha_2}{2} \cos x(t) \sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2 \sin^2 x(t)},$$

$$b_n(t) = (-1)^n \alpha_1 \sin x(t),$$

$$\psi_n^\pm(\lambda, t) = \frac{\lambda^2 - \alpha_1^2 \sin^2 x(t) - \alpha_1 \alpha_2 \cos x(t) \sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2 \sin^2 x(t)} \pm \sqrt{(\lambda^2 - \alpha_1^2)(\lambda^2 - \alpha_1^2)}}{2a_0(t)(\lambda - \alpha_1 \sin x(t))}$$

,

natijaga ega bo`lamiz. Bu yerda $x(t)$ funksia quyidagi

$$\dot{x}(t) = 4\alpha_1 \alpha_2 (1 + \alpha_2 - \alpha_1) \sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2 \sin^2 x(t)},$$

$$x(0) = 0.$$

Koshi masalasining yechimi ([24]). Oxirgi ifodani sodda o`rinishda quyidagicha yozib olish mumkin:

$$x(t) = am\left(\beta t, \frac{\alpha_1}{\alpha_2}\right),$$

Bunda $\beta = 4\alpha_1 \alpha_2 (1 + \alpha_2 - \alpha_1)$ va am Yakobini amplitude funktsiyasi. Oxirgi ifodani etiborga olsak qaralayotgan masalaning yechimi ushbu

$$a_n(t) = \sqrt{\frac{\alpha_1^2 + \alpha_2^2}{4} - \frac{\alpha_1^2}{2} \text{sn}^2\left(\beta t, \frac{\alpha_1}{\alpha_2}\right) - (-1)^n \frac{\alpha_1 \alpha_2}{2} \text{cn}\left(\beta t, \frac{\alpha_1}{\alpha_2}\right) \text{dn}\left(\beta t, \frac{\alpha_1}{\alpha_2}\right)}$$

,

$$b_n(t) = (-1)^n \alpha_1 \text{sn}\left(\beta t, \frac{\alpha_1}{\alpha_2}\right),$$

$$\psi_n^\pm(\lambda, t) = \frac{\lambda^2 - \alpha_1^2 \operatorname{sn}^2\left(\beta t, \frac{\alpha_1}{\alpha_2}\right) - \alpha_1 \alpha_2 \operatorname{cn}\left(\beta t, \frac{\alpha_1}{\alpha_2}\right) \sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2 \operatorname{sn}^2\left(\beta t, \frac{\alpha_1}{\alpha_2}\right)}}{2a_0(t) \left(\lambda - \alpha_1 \operatorname{sn}\left(\beta t, \frac{\alpha_1}{\alpha_2}\right) \right)} \pm \frac{\sqrt{(\lambda^2 - \alpha_1^2)(\lambda^2 - \alpha_2^2)}}{2a_0(t) \left(\lambda - \alpha_1 \operatorname{sn}\left(\beta t, \frac{\alpha_1}{\alpha_2}\right) \right)},$$

Ifodalardan iborat bo`ladi. Bu yerda sn , cn va dn lar Yakobi elliptic funksiyalari.

Boshlang`ich shartlarni aniq sonlar yordamida quyidagicha olsak

$$\alpha_1 = 1, \alpha_2 = 3, (a_n^0)^2 = \sqrt{\frac{5}{2} - (-1)^n \frac{3}{2}}, b_n^0 = 0, n \in Z,$$

U holda qaralayotgan masala yechimi grafiklarini ham topish mumkin. Ular quyidagicha bo`ad:

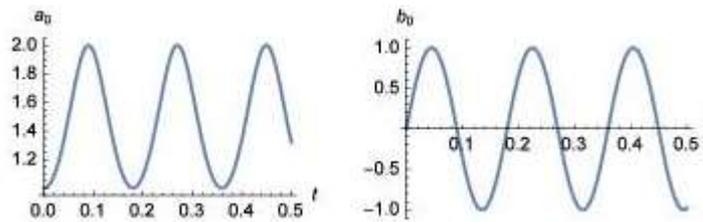


Figura 1: $a_0(t)$ va $b_0(t)$ larni $[0, 0.5]$ kesmadagi grafqi.

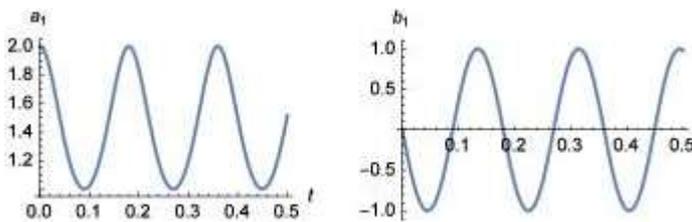


Figura 2: $a_1(t)$ va $b_1(t)$ larni $[0, 0.5]$ kesmadagi grafqi.

Xulosa

Ushbu bobda diskret Hill tenglamasiga qo'yilgan teskari masala usulida integrallanuvchi barcha noxiziqli tenglamalar keltirib chiqarilgan. Spectral parametrlarning vaqt bo'yicha o'zgarish dinamikasi keltirib chiqarilgan. Aniq boshlang'ich shartlarda ikkinchi Toda tenglamasini yechimini aniq ko'rinishi keltirib chiqarilga.

XULOSA

Ushbu magistrlik dissertatsiyasi integral manbali umumiy davriy Toda tenglamasini teskari masalalar usulidan foydalanib, integrallashga bag'ishlangan bo'lib, quyidagi natijalar olindi:

1. Diskret Xill tenglamasi uchun qo'yilgan to'g'ri va teskari masalalar o'rganildi.
2. Diskret Xill tenglamasi uchun izlar formulasi topildi.
3. Teskari masala yechish algoritmi keltirib chiqarildi.
4. Diskret Hill tenglamasi yordamida integrallanuvchi nochiziqli tenglamalarni keltirib chiqarish algoritmi keltirildi
5. Spektral parametrlarning o'zgarish dinamikasi o'rganildi.
6. Integral manbali II-tur Toda tenglamasi $m=2$ holida integrallandi.
7. $m=2$ holida Toda tenglamasi yechimlari grafiklari keltirildi.

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