O‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS

TA’LIM VAZIRLIGI

URGANCH DAVLAT UNIVERSITETI

“FIZIKA-MATEMATIKA” FAKULTETI

141-matematika guruhi talabasi

Abdullayev Hamidjonning

5130100-Matematika ta’lim yo‘nalishi bo‘yicha

bakalavr darajasini olish uchun

BITIRUV MALAKAVIY ISHI

Mavzu: Kompleks Monj-Amper operatori.

Urganch 2018 yil

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OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

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“FIZIKA-MATEMATIKA” FAKULTETI

“MATEMATIKA” KAFEDRASI

Mavzu: Kompleks Monj-Amper operatori

Bajaruvchi: **Abdullayev H. A.**

Ilmiy rahbar: **Kamolov X.Q.**

Urganch 2018-yil

URGANCH DAVLAT UNIVERSITETI FIZIKA –MATEMATIKA FAKULTETI MATEMATIKA KAFEDRASI
BITIRUV MALAKAVIY ISHNI BAJARISH BO‘YICHA TOPSHIRIQLAR REJASI:

1. Talaba Abdullayev Hamidjon Universitet rektorining 2017 yil 1-noyabrdagi 187-T §4 buyrug‘i bilan bitiruv malakaviy ish bajarish uchun “Kompleks Monj-Amper operatori” mavzusi tasdiqlangan.

2. “Matematika” kafedrasining 201__ yil “___” _____ dagi “___”-sonli majlisining qaroriga binoan X.Q.Kamolov bitiruv malakaviy ishini bajarishda Abdullayev Hamidjonga rahbar qilib tayinlangan.

3. Bitiruv malakaviy ishining tarkibiy tuzilmasi: Ushbu bitiruv malakaviy ishi referativ xarakterda bo‘lib, Kompleks Monj-Amper operatoriga bag‘ishlangan. Bitiruv malakaviy ishi kirish, ikkita bobdan iborat bo‘lib, kirish qismida mavzuning dolzarbligi, tadqiqot maqsadi va vazifalari keltirilgan. Birinchi bobda plyurisubgarmonik funksiyalar, maksimal plyurisubgarmonik funksiyalar va musbat aniqlangan formalar va oqim o‘rganilgan. Ikkinchi bobda kompleks Monj-Amper operatori, plyurisubgarmonik funksiyalarning kvaziuzluksizligi va Monj-Amper operatorining uzluksizlik xossalari o‘rganilgan. Bitiruv malakaviy ishning so‘ngida xulosa, hamda foydalanilgan adabiyotlar ro‘yxati keltirilgan.

4. Bitiruv malakaviy ish uchun ma’lumotlar quyidagi adabiyotlardan olinadi:

1. Bedford, E. (1979). Extremal plurisubharmonic functions for certain domains in $\mathbb{C}^2$. *Indiana University Mathematical Journal*, **28,** 613-26.
2. Bedford, E. and Kalka, M. (1977). Foliations and complex Monge-Ampere equations. *Communications on Pure and Applied Mathematics*, **30,** 543-71.
3. Bedford, E. and Taylor, B.A. (1976). The Dirichlet problem for a complex Monge-Ampere equation. *Inventiones Mathematicae*, **37,** 1-44.
4. Bedford, E. and Taylor, B.A. (1982). A new capacity for plurisubharmo-nic functions. *Acta mathematica*, **149,** 1-40
5. Bremermann, H.J. (1959). On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains. Characterization of Silov boundaries. *Transactions of the American Mathematical Society*, **91,** 246-76.
6. Caffarelli, L., Kohn, J.J., Nirenberg, L., and Spruck, J. (1985). The Dirichlet problem for nonlinear second order elliptic equations, II: complex Monge-Ampere, and uniformly elliptic, equations. *Communications* in *Pure and Applied Mathematics*, **38,** 209-52.

7. Cegrell, U. *(1983b).* An estimate of the complex Monge-Ampere operator. In *Analytic functions. Proceedings, Blatejewsko* 1982, Lecture Notes in Mathematics **1039**, pp. 84-7. Springer, Berlin.
8. Cegrell, U. (1988). Capacities in complex analysis, Aspects of Mathematics. Vieweg, Wiesbaden.
9. Chern, S.S., Levine, H., and Nirenberg, L. (1969). Intrinsic norms on a complex manifold. In *Global analysis, papers in honour of K. Kodaira,* pp.119-39. University of Tokyo Press.
10. Hayman, W.K. and Kennedy, P.B. (1976). *Subharmonic functions,* Vol.1, London Mathematical Society Monographs. Academic Press, London.
11. Kerzman, N. (1977). A Monge-Ampere equation in complex analysis. In *Proceedings of symposia in pure mathematics of the American mathematical Society,* (ed. R.O. Wells, jr.), Vol. 30, Part 1, pp. 161-7. American Mathematical Society, Providence, RI.
12. Kiselman, C.O. (1983). Sur la definition de l'operateur de Monge-Ampere complexe. In *Analyse complexe, proceedings, Toulouse* 1*983,* (ed. E. Amar, R.Gay, and Nguyen Thanh Van), Lecture Notes in Mathematics 1094, pp. 139-50. Springer, Berlin.
13. Klimek, M (1991). Pluripotential Theory. *Published in the United States by Oxford University Press, New York* . pp 260.
14. Lempert, L. (1983). Solving the degenerate Monge-Ampere equation with one concentrated singularity. *Mathematische Annalen,* 263, 515-32.
15. Sadullaev, A. (1981). Plurisubharmonic measures and capacities on complex manifolds. *Russian Mathemaitcal Surveys,* 36:4, 61-119.

5. Bitiruv malakaviy ishga PowerPoint dasturida ishlangan multimediali taqdimot slaydlari ilova qilinadi.

Bitiruv malakaviy ishni bajarish jadvali

№	***Bajarilgan ishning mazmuni***	***Bajarish muddati***
1.	Bitiruv malakaviy ishni mavzusi bo‘yicha adabiyotlarni to‘plash, o‘rganish va tahlil qilish	10 noyabr – 12 yanvar 2017 - 2018 yy.
2.	Bitiruv malakaviy ishnining birinchi bobini tayyorlash	12 yanvar – 15 fevral 2018 y.
3.	Ikkinchi bob bo‘yicha ma’lumotlarni o‘rganish va tahlil qilish	15 fevral – 6 mart 2018 y.
4.	Bitiruv malakaviy ishnining ikkinchi bobini tayyorlash	6 mart – 6 aprel 2018 y.
5.	Bitiruv malakaviy ishning foydalanilgan adabiyotlar ro‘yxatini tuzish	6 aprel – 25 aprel 2018 y.

6.	Bitiruv malakaviy ishnining loyhasini tayyorlash va kafedra muxokamasidan o‘tkazish	25 aprel – 4-may 2018 y.

Bitiruv malakaviy ish rahbari: X.Q. Kamolov

Bajaruvchi talaba: H.A. Abdullayev

2018 yil “_____” _____________

Topshiriqlar rejasi va jadvali kafedra majlisida 2018 yil tasdiqlandi
(“____” - sonli bayonnoma)

Kafedra mudiri: dots. Atamuratov A.A.

Urganch davlat universiteti

Fizika-matematika fakulteti “Matematika” kafedrasi

5130100-matematika ta’lim yo‘nalishi

Tasdiqlayman

Fizika-matematika fakulteti dekani

___________dots. J.U.Xujamov

“______”_____________ 2018 y.

BITIRUV MALAKAVIY ISH BO‘YICHA TOPSHIRIQ

Talaba Abdullayev Hamidjon Abdikarim o‘g‘li

1. Ishning mavzusi: “Kompleks Monj-Amper operatori” 2017 yil 1-noyabrda universitet rektorining 187-T §4 sonli buyrug‘i bilan tasdiqlangan.

2. Ishni topshirish muddati: “___”__________2018 y.

3. Mavzu bo‘yicha dastlabki ma’lumotlar beruvchi adabiyotlar ro‘yxati

16. Bedford, E. (1979). Extremal plurisubharmonic functions for certain domains in $\mathbb{C}^2$. *Indiana University Mathematical Journal*, **28,** 613-26.
17. Bedford, E. and Kalka, M. (1977). Foliations and complex Monge-Ampere equations. *Communications on Pure and Applied Mathematics*, **30,** 543-71.
18. Bedford, E. and Taylor, B.A. (1976). The Dirichlet problem for a complex Monge-Ampere equation. *Inventiones Mathematicae*, **37,** 1-44.
19. Bedford, E. and Taylor, B.A. (1982). A new capacity for plurisubharmonic functions. *Acta mathematica*, **149,** 1-40
20. Bremermann, H.J. (1959). On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains. Characterization of Silov boundaries. *Transactions of the American Mathematical Society*, **91,** 246-76.
21. Caffarelli, L., Kohn, J.J., Nirenberg, L., and Spruck, J. (1985). The Dirichlet problem for nonlinear second order elliptic equations, II: complex Monge-Ampere, and uniformly elliptic, equations. *Communications* in *Pure and Applied Mathematics*, **38,** 209-52.

22. Cegrell, U. *(1983b).* An estimate of the complex Monge-Ampere operator. In *Analytic functions. Proceedings, Blatejewsko* 1982, Lecture Notes in Mathematics **1039**, pp. 84-7. Springer, Berlin.
23. Cegrell, U. (1988). Capacities in complex analysis, Aspects of Mathematics. Vieweg, Wiesbaden.
24. Chern, S.S., Levine, H., and Nirenberg, L. (1969). Intrinsic norms on a complex manifold. In *Global analysis, papers in honour of K. Kodaira,* pp.119-39. University of Tokyo Press.
25. Hayman, W.K. and Kennedy, P.B. (1976). *Subharmonic functions,* Vol.1, London Mathematical Society Monographs. Academic Press, London.
26. Kerzman, N. (1977). A Monge-Ampere equation in complex analysis. In *Proceedings of symposia in pure mathematics of the American mathematical Society,* (ed. R.O. Wells, jr.), Vol. 30, Part 1, pp. 161-7. American Mathematical Society, Providence, RI.
27. Kiselman, C.O. (1983). Sur la definition de l'operateur de Monge-Ampere complexe. In *Analyse complexe, proceedings, Toulouse* 1*983,* (ed. E. Amar, R.Gay, and Nguyen Thanh Van), Lecture Notes in Mathematics 1094, pp. 139-50. Springer, Berlin.
28. Klimek, M (1991). Pluripotential Theory. *Published in the United States by Oxford University Press, New York* . pp 260.
29. Lempert, L. (1983). Solving the degenerate Monge-Ampere equation with one concentrated singularity. *Mathematische Annalen,* 263, 515-32.
30. Sadullaev, A. (1981). Plurisubharmonic measures and capacities on complex manifolds. *Russian Mathemaitcal Surveys,* 36:4, 61-119.

4. Ishning maqsadi: Kompleks Monj-Amper operatori va uning muhim xossalarini oʻrganib, tahlil qilish.

5. Chizma materiallar roʻyxati: - - - - -

6. Maslahatchilar:

Boʻlimlar	**Maslahatchi F.I.Sh.**	**Imzo, sana**	
		Topshiriq berdi	**Topshiriq qabul qildi**
Plyurisubgarmonik funksiyalar	X.Q.Kamolov		
Maksimal plyurisubgarmonik funksiyalar	X.Q.Kamolov		
Musbat aniqlangan formalar va oqim	X.Q.Kamolov		

Kompleks Monj-Amper operatori	X.Q.Kamolov		
Plyurisubgarmonik funksiyalarning kvaziuzluksizligi	X.Q.Kamolov		
Monj-Amper operatorining uzluksizlik xossalari	X.Q.Kamolov		

Ishga taqrizchi **«Amaliy matematika» kafedrasi mudiri dots., f.-m.f.n., B. Babajanov**

Ilmiy rahbar: **X.Q. Kamolov**

BMI bajaruvchi talaba: **H. Abdullayev**

Kafedra mudiri: **A.A.Atamuratov**

Urganch davlat universiteti Fizika-matematika fakulteti
“Matematika” kafedrasi

Bitiruv malakaviy ish ____________ sonli tartib raqam bilan qayd qilindi.

Bitiruv malakaviy ishni bajaruvchining ismi-sharifi: *Abdullayev Hamidjon Abdikarim o‘g‘li*

Bitiruv malakaviy ishning mavzusi: *Kompleks Monj-Amper operatori*.

Ilmiy rahbar (maslahatchi)ning ismi-sharifi: *X.Q.Kamolov*

Bitiruv malakaviy ish kafedraning 2018 yil «___»________da o‘tkazilgan majlisi qaroriga muvofiq DAK majlisida himoya qilishga tavsiya qilindi.

Bitiruv malakaviy ishga taqrizchi qilib «Amaliy matematika» kafedrasi mudiri dotsent, f.-m.f.n. B. Babajonov tayinlandi.

Kafedra mudiri: **dots. A. A. Atamuratov**

Kafedraning bitiruv malakaviy ishni DAK majlisida himoya qilish bo‘yicha tavsiyasiga roziman.

Fakultet dekani: **dots. J.U.Xujamov**

MUNDARIJA

Kirish

Oliy ta’lim mamlakatning ertangi taraqqiyotini belgilab beradigan, jamiyat hayotining barcha sohasini isloh etishda hal qiluvchi vazifani bajaradigan omil hisoblanadi. Shuning uchun ham bugungi kunda mehnat bozorida raqobatdosh, yuqori malakali kadrlar tayyorlash, oliy ta’lim tizimi sifati va samaradorligini tubdan oshirish har bir mamlakat ijtimoiy siyosatining asosini tashkil qiladi.

O‘zbekiston Respublikasini keyingi besh yilda taraqqiy toptirish bo‘yicha Harakatlar strategiyasida Oliy ta’lim tizimini 2017-2021 yillarga mo‘ljallangan kompleks rivojlantirish dasturini ishlab chiqish belgilangan edi. Prezidentimizning “Oliy ta’lim tizimini yanada rivojlantirish chora-tadbirlari to‘g‘risida”gi qarori bu borada muhim qadam bo‘lib, tizmni yanada sifat bosqichiga ko‘tarishga qaratilganligi bilan ahamiyatlidir. Qaror Oliy ta’lim tizimining zamon talablari darajasida rivojlanishiga, sohaning yurtimizda kechayotgan islohotlar bilan hamohangligini ta’minlashga zamin yaratadi.

Shu sababli qaror bilan oliy ma’lumotli kadrlarni tayyorlashning maqsadli mezonlarini shakllantirish, oliy ta’lim muassasalaridagi ixtisoslik yo‘nalishlari va mutaxasisliklarni hududlar hamda, sohalar bo‘yicha joriy etilayotgan dasturlarning talab va ehtiyojlari, iqtisodiyot tarmoqlari hamda hududlarni kompleks rivojlantirish istiqbollarini inobatga olgan holda, isloh qilishdagi ustuvor vazifalar sifatida belgilandi.

Bu yondashuv oliy ta’lim tizimining mehnat bozoriga, ijtimoiy-iqtisodiy islohotlar jarayoniga moslashuvchanligini oshirib, tizimda innovatsion faoliyatni yuksaltirishning muhim sharti bo‘lib xizmat qiladi.

Albatta, qaror bilan oliy ta’lim tizimida o‘rta, uzoq muddatga mo‘ljallangan ijobiy o‘zgarishlar ro‘yobga chiqariladi. Bu sa’y-harakatlarning vaqtida samarali bajarilishi har birimizdan qat’iy tartib-intizom hamda shaxsiy ma’suliyat talab qiladi.

Mavzuning dolzarbligi. Yuqoridagi maqsad va g‘oyalarning ijobiy natijaga ega bo‘lishi, eng avvalo, biz yosh avlodga ilmiy bilimlar asoslarini puxta o‘rgatish, keng dunyoqarash hamda tafakkur ko‘lamini hosil qilish, ma’naviy axloqiy sifatlarni shakllantirish borasidagi ta’limiy va tarbiyaviy ishlarni samarali tashkil etishga bog‘liqdir.

Ana shu vazifalardan kelib chiqqan holda bugungi kunda matematika fanida erishilayotgan fundamental va amaliy natijalarni o‘rganish uchun uning asosi bo‘lgan tushunchalarni mukammal o‘zlashtirishimiz kerak. Shu maqsadda ushbu bitiruv malakaviy ishi kompleks Monj-Amper operatorini o‘rganishga bag‘ishlangan.

Tadqiqot maqsadi: plyurisubgarmonik funksiyalar sinfida kompleks Monj-Amper operatorning ba’zi xossalarini o‘rganish.

Tadqiqotning ob’ekti: plyurisubgarmonik funksiyalar, musbat aniqlangan forma va oqimlar.

Tadqiqot predmeti: plyurisubgarmonik funksiyalar sinfida aniqlangan kompleks Monj-Amper operatori.

Ish tuzilishi va tarkibi. Bitiruv malakaviy ishi kirish, ikki bob, xulosa hamda adabiyotlar ro‘yxatidan iborat. Har bir bob paragraflardan tuzilgan. Birinchi bob

Plyurisubgarmonik funksiyalar, musbat aniqlangan forma va oqimlar o‘rganilgan, o‘z navbatida uchta paragrafdan tarkib topadi. Ikkinchi bobda kompleks Monj-Amper operatori va uning uzluksizlik xossalari o‘rganilgan bo‘lib, uch paragrafdan tashkil topgan. Har bobning ta’rif, teoremalari va formulalar o‘z belgilashlar tartibiga ega, masalan (1.2.3) belgilash mos ravishda 1-bob, 2-paragraf 3-nomerli belgilanishni bildiradi.

Bitiruv malakaviy ishining mazmuni.

1.2.19-teorema. $u, v \in C^3(\Omega, \mathbb{C})$ funksiyalar bo‘lsin. Bu yerda $\Omega \subset \mathbb{C}^n$ ochiq to‘plam. Faraz qilaylik, $dd^c u$ va $dd^c v$ kamayuvchi operator. $p, q \in \mathbb{Z}_+$ $0 < p + q < n$ shartlarni qanoatlantirsin, u holda

$$(dd^c u)^p \wedge (dd^c v)^{q+1} = 0,$$

$$(dd^c u)^{p+1} = 0,$$

va

$$(dd^c u)^p \wedge (dd^c v)^q \neq 0$$

Ω dagi har bir nuqtada o‘rinli bo‘ladi.

1.2.20-natija(q. [2],[13]). $v \in C^3(\Omega)$ funksiya bo‘lsin. Bu yerda $\Omega \in \mathbb{C}^n$ to‘plam va Ω to‘plamning ixtiyoriy nuqtasida

$$(dd^c v)^n = 0,$$
$$(dd^c v)^{n-1} \neq 0$$

(Ω to‘plamda u funksiyaning Gessian matrisasining rankidir va $n-1$ ga teng **).**

1.3.11-natija. Agar $u \in \mathcal{PSH}(\Omega)$ bo‘lsa, u holda $dd^c u$ o‘lchovli koeffitsiyentlariga ega bo‘lgan musbat $(1,1)$ - oqim bo‘ladi.

2.1.1- ta’rif. 2-marta silliq $u \in C^2(\Omega)$ funksiya uchun uning ikkinchi tartibli differensiali

$$dd^c u = \frac{i}{2}\sum_{j,k} u_{j\bar{k}} dz_j \wedge d\overline{z_k} \quad \left(bunda \;\; u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} \right) \qquad (2.1.1)$$

(tayinlangan $z^0 \in \Omega$ nuqtada) Ermit kvadratik formasidan iborat bo‘ladi.

$\left(dd^c\right)^n = \underbrace{dd^c \wedge \ldots \wedge dd^c}_{n-marta}$ operator $\mathbb{C}^n$ fazodagi *Monj-Amper operatori* deyiladi.

Agar $u \in C^2(\Omega)$ bo‘lsa, u holda

$$(dd^c u)^n = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right] dV,$$

bu yerda,

$$dV = (\frac{i}{2})^n dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n$$

forma $\mathbb{C}^n$ fazodagi odatdagi hajm formasi.

2.1.2-teorema. Ω to‘plam $\mathbb{C}^n$ fazoning tekis chegaralangan ochiq to‘plami bo‘lsin. Agar $v^1,\ldots,v^k$ haqiqiy qiymatli $\mathcal{C}^2$-funksiyalar, $\overline{\Omega}$ to‘plamning atrofida va φ $(n-k,n-k)$ tartibli silliq formadir, shuningdek $\partial\Omega$ da $\varphi = 0$ bo‘lsa, u holda $j = 1,2,\ldots,k$ uchun quyidagi o‘rinli

$$\int\limits_{\Omega} dd^c v^1 \wedge \ldots \wedge dd^c v^k \wedge \varphi = \int\limits_{\Omega} v^j dd^c v^2 \wedge \ldots \wedge dd^c v^{j-1} \wedge dd^c v^{j+1} \wedge \ldots \wedge dd^c v^k \wedge dd^c \varphi$$

xuddi shu formula $v^1, \ldots, v^k \in \mathcal{C}^2(\Omega)$ da ixtiyoriy ochiq $\Omega \subset \mathbb{C}^n$ to‘plam uchun o‘rinli va φ Ω to‘plamdagi test formadir.

Agar $u \in \mathcal{PSH}(\Omega)$ funksiya bo‘lsa, u holda $dd^c u$ musbat $(1,1)$- oqim bo‘ladi. E.Bedford va B.A.Taylor 2.1.2-teoremani integrallashgan formula bilan ko‘rsatdi, u chegaralangan plyurisubgarmonik funksiyalar uchun Monj-Amper operatorining ta’rifini keltirishda xizmat qiladi.

$(dd^c)^n$ operator plyurisubgarmonik funksiyalar chegaralarida umumiy kompleks *Monj-Amper operatori* deyiladi. Ba’zan $(u_1, \ldots, u_k) \to dd^c u_1 \wedge \ldots \wedge dd^c u_k$ operator ***Monj-Amper operatori*** deyiladi.

2.1.4-teorema. Ω to‘plam $\mathbb{C}^n$ ning ochiq qism to‘plami bo‘lsin. $k \le n$ va Ω to‘plamda $\{v_j^1\}_{j \in N}, \ldots, \{v_j^k\}_{j \in N}$ plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketigi. $v^1, \ldots, v^k \in \mathcal{PSH}(\Omega) \bigcap L_{loc}^{\infty}(\Omega)$ nuqtalar bo‘lib, $i = 1, \ldots, k$ uchun Ω to‘plamga tegishli nuqtalarda $\lim_{j \to \infty} v_j^i = v^i$ ga teng. U holda

$$\lim_{j \to \infty} dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^k = dd^c v^1 \wedge \ldots \wedge dd^c v^k$$

nolinchi tartibli oqimlarining kuchsiz yaqinlashish ma’nosida.

2.1.10-natija. Agar $u, v \in \mathcal{PSH} \bigcap L_{loc}^{\infty}(\Omega)$ funksiya bo‘lsa, u holda

$$(dd^c(u+v)^n) \ge (dd^c u^n) + (dd^c v^n) \text{ bo‘ladi.}$$

2.1.12-teorema. Ω to‘plam $\mathbb{C}^n$ fazoning ochiq qism to‘plami, $k \le n$ va $\{v_j^0\}_{j \in N}, \ldots, \{v_j^k\}_{j \in N}$ plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketligi bo‘lsin. $v^1, \ldots, v^k \in \mathcal{PSH}(\Omega) \bigcap L_{loc}^{\infty}(\Omega)$ lar $i = 0, \ldots, k$ uchun Ω da $\lim\limits_{j \to \infty} v_j^i = v^i$ shartni qanoatlantirsin. U holda tartibi nolga teng bo‘lgan oqimlarning kuchsiz yaqinlashish ma’nosida

$$\lim_{j \to \infty} L_k(v_j^0, \ldots, v_j^k) = L_k(v^0, \ldots, v^k)$$

bo‘ladi.

Plyurisubgarmonik funksiyalarning eng muhim xossalaridan biri kvaziuzluksizlikdir: agar $u \in \mathcal{PSH}(\Omega)$ funksiya bo‘lsa, u holda u istalgancha kichik qilib olingan ochiq to‘plamdan tashqarida uzluksiz bo‘ladi.

Kvaziuzluksizlik haqidagi teorema E.Bedford va B.A.Teylor(q [4] 1982 yil) ga tegishli. Bu teoremaning boshqacha isboti 1984 yilda A.Sadullaev (q. [13] tomonidan berilgan. Biz keltiradigan natijalar E. Bedford va B.A.Teylor (q.[13] 1982 yil) va U.Segrel (q.[8] 1988) larning gʻoyalariga asoslangan.

2.2.2-lemma. Agar u funksiya $\Omega \subset \mathbb{C}^n$ ochiq toʻplamda haqiqiy C^2- sinfga tegishli funksiya boʻlsa, u holda $du \wedge d^c u$ forma qatʼiy musbat boʻladi.

2.2.3–natija. Agar $u, v \in C^2(\Omega)$ funksiyalar va χ musbat $(n-1, n-1)$-forma boʻlsa, u holda

$$\left| \int_\Omega du \wedge d^c v \wedge \chi \right|^2 \leq \int_\Omega du \wedge d^c u \wedge \chi \int_\Omega dv \wedge d^c v \wedge \chi$$

boʻladi.

2.2.4-lemma. $0 < r_1 < r_2$, $a \in \mathbb{C}^n$, $B_1 = B(a, r_1)$ va $B_2 = B(a, r_2)$ boʻlsin. Agar $\{u_j\}_{j \in N} \subset \mathcal{PSH} \cap C^2(B_2)$ kamayuvchi ketma-ketlik boʻlib, bu ketma-ketlik $u \in \mathcal{PSH} \cap L^\infty(B_2)$ funksiyaga yaqinlashsa, u holda

$$\lim_{j \to \infty} \left(\sup \left\{ \int_{\bar{B}_1} (u_j - u)(dd^c v)^n \; : \; v \in \mathcal{PSH}(B_2, (0,1)) \right\} \right) = 0 \qquad (2.3.1)$$

boʻladi.

Lokal chegaralangan plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketligida Monj-Amper operatorining uzluksizligini koʻrdik. Bundan tashqari, u yaqinlashuvchi ketma-ketliklarda ham oʻrinli. Bu natijalar 1982-yilda E.Bedford va B.A. Teylor tomonidan isbotlangan(q. [4]).

2.3.1-teorema. Ω toʻplam $\mathbb{C}^n$ dagi biror ochiq qism toʻplam va $\{U_j\}_{j \in N}$ ketma-ketlik esa $\mathcal{PSH} \cap L^\infty_{Loc}(\Omega)$ dagi ketma-ketlik boʻlib, bu ketma-ketlik Ω toʻplamning hamma yerida $u \in \mathcal{PSH} \cap L^\infty_{Loc}(\Omega)$ funksiyaga oʻsib yaqinlashuvchi boʻlsin, u holda $(dd^c u_j)^n$ Radon oʻlchovlari ketma-ketligi kuchsiz-topologiya maʼnosida $(dd^c u)^n$ oʻlchovga yaqinlashuvchi boʻladi.

2.3.2-teorema. Aytaylik, Ω, u, $\{u_j\}$ lar teorema shartlarini qanoatlantirsin va $v^1, \ldots, v^n \in \mathcal{PSH} \cap L^\infty_{Loc}(\Omega)$ boʻlsin. U holda $L_n(u_j, v^1, \ldots, v^n)$ oʻlchovlar ketma-ketligi $L_n(u, v^1, \ldots, v^n)$ ga yaqinlashuvchi boʻladi.

1.1. Plyurisubgarmonik funksiyalar va uning xossalari

1.1.1-ta’rif. Agar $z^0 \in \mathbb{C}^n$ nuqtaning biror atrofida aniqlangan haqiqiy $\varphi,\ -\infty \le \varphi \le \infty$ funksiya shu nuqtada yuqoridan yarim uzluksiz deyiladi, agar ixtiyoriy $\varepsilon > 0$ son uchun shunday, $\delta > 0$ topilsaki,

$$\left|z - z_0\right| < \delta \quad \Rightarrow \quad \begin{cases} \varphi\left(z\right) - \varphi\left(z^0\right) < \varepsilon, & agar\ \ \varphi\left(z^0\right) \ne -\infty \\ \varphi\left(z\right) < -\dfrac{1}{\varepsilon}, & agar\ \ \ \varphi\left(z^0\right) = -\infty \end{cases} \tag{1.1.1}$$

yoki

agar

$$\lim_{z \to z^0} \varphi\left(z\right) \le \varphi\left(z^0\right) \tag{1.1.2}$$

bo‘lsa, φ funksiya shu nuqtada yuqoridan yarim uzluksiz deyiladi. Agar φ funksiya har bir $z^0 \in \Omega$ nuqtada yuqoridan yarim uzluksiz bo‘lsa, $\Omega \subset \mathbb{C}^n$ sohada yuqoridan yarim uzluksiz deyiladi.

1.1.2-ta’rif. $\varphi : \Omega \to [-\infty;\ +\infty]$ funksiya $\Omega \subset \mathbb{C}^n$ sohada plyurisubgarmonik deyiladi, agarda u quyidagi shartlarni qanoatlantirsa:

1) φ funksiya Ω da yuqoridan yarim uzluksiz;

2) ixtiyoriy $z^0 \in \Omega$ nuqta uchun va ixtiyoriy $z = l\left(\xi\right) = z^0 + \omega\xi$, bunda $\omega \in \mathbb{C}^n$, $\xi \in \mathbb{C}$ analitik to‘g‘ri chiziq uchun, φ funksiyaning shu chiziqdagi kesim funksiyasi, ya’ni $\varphi\left(l\left(\xi\right)\right)$ ochiq $\{\xi \in \mathbb{C},\ l\left(\xi\right) \in \Omega\}$ to‘plamda subgarmonik bo‘lsa.

Plyurisubgarmonik funksiyalar to‘plam $\mathcal{PSH}(\Omega)$ kabi belgilanadi. Plyurisubgarmonik funksiyalar ko‘p o‘zgaruvchili golomorf funksiyalar bilan ham bog‘liq.

$\Omega \subset \mathbb{C}^n$ sohada golomorf bo‘lgan ixtiyoriy f funksiya uchun

$$\varphi\left(z\right)=\ln\left|f\left(z\right)\right| \qquad (1.1.3)$$

funksiya shu sohada plyurisubgarmonik bo‘ladi.

1.1.1. Plyurisubgarmonik funksiyalarning xossalari.

1^0. Agar Ω sohada plyurisubgarmonik bo‘lgan φ funksiya biror $z^0\in\Omega$ nuqtada lokal maksimumga erishsa, u holda u shu sohada o‘zgarmas bo‘ladi**.**

2^0. Har bir $z^0\in\Omega$ nuqtaning biror atrofida plyurisubgarmonik bo‘lgan funksiya, shu Ω sohada plyurisubgarmonik bo‘ladi.

3^0. Ω sohada yuqoridan yarim uzluksiz φ funksiya shu sohada plyurisubgarmonik bo‘lishi uchun, har bir $z\in\Omega$ nuqta uchun va har bir $\omega\in\mathbb{C}^n$ vektor uchun shunday $r_0=r_0(z,w)$, ixtiyoriy $r<r_0$ larda

$$\varphi\left(z\right)\le\frac{1}{2\pi}\int_0^{2\pi}\varphi\left(z+\omega re^{it}\right)dt \qquad (1.1.4)$$

tengsizlikni bajarilishi zarur va yetarli

C^2 sinfidan olingan funksiyalar uchun yengilroq tekshiriladigan kriteriyani aytish mumkin. Murakkab funksiyalarni differensiallash qoidasidan foydalanib, quyidagiga ega bo‘lamiz: $\varphi\in C^2\left(\Omega\right),\ \Omega\subset\mathbb{C}^n$ funksiyani $z=l\left(\xi\right)=z^0+\omega\xi$ to‘g‘ri chiziqdagi kesim funksiyasi uchun, ya’ni $u\left(\xi\right)=\varphi\left(l\left(\xi\right)\right)$ murakkab funksiya uchun,

$$\frac{\partial^2 u}{\partial\xi\partial\overline{\xi}}=\sum_{\mu,\nu=1}^{n}\frac{\partial^2\varphi}{\partial z_\mu\partial\overline{z}_\nu}\bigg|\omega_\mu\overline{\omega}_\nu$$

Bundan esa quyidagi kriteriya kelib chiqadi :

1.1.3-teorema. C^2 sinfidan olingan φ funksiya biror $\Omega\subset\mathbb{C}^n$ sohada plyurisubgarmonik bo‘lishi uchun, har bir $z^0\in\Omega$ nuqtadagi ixtiyoriy $\omega\in\mathbb{C}^n$ vektor uchun

$$H\left(\omega,\overline{\omega}\right)=\sum_{\mu,\nu=1}^{n}\frac{\partial^2\varphi}{\partial z_{\mu}\partial\overline{z}_{\nu}}\bigg|_{z^0}\omega_{\mu}\overline{\omega}_{\nu}\geq 0 \qquad (1.1.5)$$

tengsizlikni bajarilishi zarur va yetarli.

Bu yerda hosila z^0 nuqtada olinyapti.

1.1.4-teorema. φ funksiya $z^0\in\mathbb{C}^n$ nuqtaning biror atrofida plyurisubgarmonik funksiya bo‘lsin, u holda yetarlicha kichik r son uchun

$$\varphi\left(z^0\right)\leq\frac{1}{\sigma(r)}\int_{\left|\xi-z^0\right|=r}\varphi\left(z\right)d\sigma \qquad (1.1.6)$$

tengsizlik o‘rinli.

1.1.5-teorema. $z^0\in\mathbb{C}^n$ nuqtaning atrofida φ plyurisubgarmonik funksiyaning o‘rta qiymati

$$S\left(r\right)=\frac{1}{\sigma\left(r\right)}\int_{\left|\xi-z\right|=r}\varphi\left(\xi\right)d\sigma$$

r ning o‘suvchi funksiyasi bo‘ladi.

1.2. Maksimal plyurisubgarmonik funksiyalar

Ω to‘plam $\mathbb{C}^n$ fazodagi ochiq qism to‘plam va $u: \Omega \to \mathbb{R}$ plyurisubgarmonik funksiya bo‘lsin. Har bir $G \subset \Omega$ nisbiy kompakt qism to‘plam va $\overline{G}$ da yuqoridan yarim uzluksiz v funksiya uchun $v \in \mathcal{PSH}(G)$ va ∂G da $v \le u$ tengsizlik o‘rinli bo‘lib, G to‘plamda $v \le u$ tengsizlik bajarilsa, u funksiya maksimal plyurisubgarmonik funksiya (yoki ekstremal funksiya) deyiladi. Ω to‘plamda aniqlangan maksimal plyurisubgarmonik funksiyalar oilasi uchun $\mathcal{MPSH}(G)$ belgilash ishlatiladi. Maksimal plyurisubgarmonik funksiyalar va ularning xossalari taniqli olimlar H.J.Bremerman(1956), J.Sichiak(1962), A.Sadullaev(1981) ishlarida o‘rganilgan(q. [5],[13],[15]).

1.2.1- teorema(q. [15])**.** $\Omega \subset \mathbb{C}^n$ da ochiq to‘plam va $u: \Omega \to \mathbb{R}$ plyurisubgarmonik funksiya bo‘lsin. Quyidagi shartlar ekvivalent bo‘ladi:

(1) barcha $G \subset \Omega$ ochiq nisbiy kompakt qism to‘plam va har qanday $v \in \mathcal{PSH}(G)$ plyurisubgarmonik funksiya uchun, agar ixtiyoriy $\xi \in \partial G$ da $\liminf\limits_{z \to \xi} (u(z) - v(z)) \ge 0$ tengsizlik o‘rinli bo‘lsa, u holda G to‘plamda $v \le u$ tengsizlik bajariladi;

(2) agar $v \in \mathcal{PSH}(\Omega)$ va ixtiyoriy $\varepsilon > 0$ uchun, $K \subset \Omega$ kompakt to‘plam mavjud bo‘lib, $\Omega \setminus K$ to‘plamda $u - v \ge -\varepsilon$ tengsizlik bajarilsa, u holda Ω to‘plamda $v \le u$ tengsizlik o‘rinli;

(3) agar $v \in \mathcal{PSH}(\Omega)$ bo‘lib, $G \subset \Omega$ ochiq nisbiy kompakt to‘plam bo‘lsa va ∂G da $v \le u$ tengsizlik bajarilsa, u holda G to‘plamda $v \le u$ tengsizlik o‘rinli;

(4) agar $v \in \mathcal{PSH}(\Omega)$ bo‘lib, $G \subset \Omega$ ochiq nisbiy kompakt to‘plam va ixtiyoriy $\xi \in \partial G$ uchun

$$\liminf_{\substack{z\to\xi\\ z\in G}} (u(z)-v(z)) \geq 0$$

bo‘lsa, u holda G to‘plamda $v \leq u$ tengsizlik o‘rinli bo‘ladi;

(5) u maksimal plyurisubgarmonik funksiyadir;

Isbot. $(1) \Rightarrow (2)$. $\Omega \setminus K$ to‘plamda $K \subset \Omega$ kompakt to‘plam mavjud va $u - v \geq -\varepsilon$ tengsizlik bajarilganligidan, $\forall \varepsilon > 0$ uchun v plyurisubgarmonik funksiya bo‘ladi. Faraz qilamiz, biror $a \in \Omega$ nuqtada $u(a) - v(a) = \eta < 0$ o‘rinli bo‘lsin.

$E = \left\{ z \in \Omega : \ u(z) < v(z) + \eta/2 \right\}$ to‘plamning yopig‘i, Ω to‘plamning biror kompakt qism to‘plami bo‘ladi. Shuning uchun, E to‘plamni o‘z ichiga oluvchi G ochiq to‘plam mavjud va Ω fazoda bu to‘plam nisbiy kompaktdir. (1) xossada, to‘plamda $u(z) \geq v(z) + \eta/2$ tengsizlik bajariladi, bu esa $a \in E$ bo‘lishiga zid. Qolgan (3), (4), (5) natijalar ham shunday kelib chiqadi va Ω to‘plamda

$$\omega(z) = \begin{cases} \max\left\{u(z), v(z)\right\} & (z \in G) \\ u(z) & (z \in \Omega \setminus G) \end{cases}$$

funksiya plyurisubgarmonik funksiyadir.

Bir o‘lchamli holda, $\mathcal{PSH} = \mathcal{H}$ ($\mathcal{H}$ – biror to‘plamdagi garmonik funksiyalar sinfi) bo‘ladi. Jumladan, bir o‘zgaruvchili kompleks o‘zgaruvchining maksimal funksiyalari C^∞ dir. Bu yuqori o‘lchamlarda to‘g‘ri kelmaydi.

1.2.2-misol. $u(z_1, z_2, \ldots, z_n) = \ln(\max\{|z_1|, \ldots, |z_n|\})$ funksiya $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\}$ da berilgan bo‘lsin. Agar $n > 1$ bo‘lsa, u differensiallanuvchi funksiya bo‘lmaydi. Demak, u plyurisubgarmonik funksiyadir. u funksiyaning maksimal bo‘lishi ta’rifdan va $\mathbb{C}^n \setminus \{0\}$ to‘plamda ixtiyoriy $\omega \in \mathbb{C}^n \setminus \{0\}$ nuqta uchun, bir o'zgaruvchili $t \to u(t\omega)$ funksiya garmonik funksiya bo'lishidan kelib chiqadi.

1.2.3-misol(q. [13]). $\omega : \mathbb{C} \to \mathbb{R}$ biror uzilishli subgarmonik funksiya bo'lsin. $(z_1, z_2) \in \mathbb{C}^2$ nuqtalar uchun $u(z_1, z_2) = \omega(z_1)$ funksiyani aniqlaymiz. Ixtiyoriy $a \in \mathbb{C}$ nuqta uchun, $z_2 \to u(a, z_2)$ funksiya o'zgarmasdan iborat bo'lib, garmonik funksiya bo'ladi. Bu esa u plyurisubgarmonik funksiyaning maksimal ekanini bildiradi.

Endi uzluksiz maksimal funksiyalarning muhim sinfini qaraymiz.

Ω to'plam $\mathbb{C}^n$ fazoda chegaralangan va $f \in C(\partial\Omega)$ bo'lsin. $(u|_{\Omega}) \in \mathcal{MPSH}(\Omega)$ va $(u|_{\partial\Omega}) \equiv f$ shartlarni qanoatlantiruvchi yuqoridan yarim uzluksiz $u : \overline{\Omega} \to \mathbb{R}$ funksiyani topish masalasi umumlashgan ***Direxle masalasi deyiladi.***

Ω to'plam $\mathbb{C}^n$ fazoda chegaralangan va $f \in C(\partial\Omega)$ bo'lsin. $\partial\Omega$ chegarada da $u^* \le f$ tengsizlikni qanoatlantiruvchi va ixtiyoriy $z \in \overline{\Omega}$ nuqta uchun

$$u^*(z) = \lim_{\substack{\omega \to z \\ \omega \in \Omega}} \sup u(\omega) \qquad (1.2.1)$$

shartni bajaruvchi $u \in \mathcal{PSH}(\Omega)$ funksiyalar oilasini $\mathcal{U}(\Omega, f)$ orqali belgilaymiz.

$$\Psi_{\Omega, f}(z) = \sup\{u(z) : u \in \mathcal{U}(\Omega, f)\} \quad z \in \Omega$$

$\Psi_{\Omega,f}$ funksiya Ω to'plam va f funksiya uchun *Perron-Bremermann funksiyasi* deyiladi. Endi Ω soha Evklid sharidan iborat bo'lgan holda $\Psi_{\Omega,f}$ funksiya asosiy Direxle masalasi yechimi bo'lishini isbotlaymiz. Quyida keltirilgan teoremaning birinchi ifodasi Bremermann, 2-ifodasi esa Valsh tomonidan berilgan(q. [5]).

1.2.4-teorema(q [13)]. Aytaylik $f \in C(\partial B)$ funksiya berilgan bo‘lib, $B = B\left(a, r\right)$ esa $\mathbb{C}^n$ fazoda berilgan ochiq shar bo‘lsin. Quyidagi o‘rinli:

$$\Psi(z) = \begin{cases} \psi_{B,f}(z)\ ,\ agar\ bo'lsa\ z \in B, \\ f(z)\ ,\ agar\ bo'lsa\ z \in \partial B. \end{cases}$$

Ψ funksiya B to‘plam va f funksiya uchun asosiy Direxle masalasining yechimidir. Bundan tashqari Ψ uzluksiz funksiyadir.

Isbot. Umumiy holda $a = 0$ deb faraz qilamiz. h funksiya B to‘plam va f funksiya uchun klassik Direxle masalasining yechimi bo‘lsin. Plyurisubgarmonik funksiyalar ham subgarmonik funksiya bo‘lgani uchun, subgarmonik funksiyalar uchun maksimum prinsipiga asosan B to‘plamda $\Psi_{\mathrm{B,f}} \leq h$ bo‘lishi kelib chiqadi. h funksiya $\bar{B}$ to‘plamda uzluksiz bo‘lgani uchun, $\bar{B}$ to‘plamda $(\Psi_{\mathrm{B,f}})^* \leq h$ tengsizlikka ega bo‘lamiz. Xususan, bundan $(\Psi_{\mathrm{B,f}})^* \in U(B, f)$ bo‘lishi kelib chiqadi, shuning uchun $(\Psi_{\mathrm{B,f}})^*$ funksiya Ψ funksiya bilan B to‘plamda ustma-ust tushadi. Demak, $\Psi \in \mathcal{PSH}(\mathrm{B})$ dir. Teoremaning birinchi qismini xulosalash uchun ∂B da $(\Psi_{\mathrm{B,f}}) \geq f$ tengsizlikni ko‘rsatish yetarli, ya’ni $z_0 \in \partial B$ da

$$\liminf_{\substack{z \to z_0 \\ z \in B}} \Psi_{\mathrm{B,f}}(z) \geq f(z_0)$$

$z_0 \in \partial B$ nuqta va $\forall \varepsilon > 0$ son olamiz. Agar $v : \bar{B} \to \mathbb{R}$ funksiyaning uzluksizligini ko‘rsatsak teorema isbot bo‘ladi, bu yerda $v\big|_B \in \mathcal{U}(B, f)$ va $v(z_0) = f(z_0) - \varepsilon$.

$$v(z) = c\left[\mathrm{Re}(z, z_0) - r^2\right] + f(z_0) - \varepsilon$$

bunda $c > 0$ o‘zgarmas son. Shuning uchun ∂B da $v \le f$ qilib tanlaymiz (bu kvadrat qavslardagi ifoda $\overline{B} \setminus \{z_0\}$ da manfiy bo‘ladi).

Natijada, har bir $z_0 \in \partial B$ nuqta uchun

$$\varliminf_{\substack{z \to z_0 \\ z \in B}} \Psi(z) \ge \Psi(z_0)$$

o‘rinliligi kelib chiqadi, ya’ni Ψ funksiya har bir chegaraviy nuqtada uzluksiz funksiyadir.

Ψ ning maksimumligi ma’lum. Haqiqatdan, agar G to‘plam B ning nisbiy kompakt ochiq to‘plami bo‘lsa, $v : \overline{G} \to \left[-\infty; \infty\right)$ funksiya ∂B da yuqoridan yarim uzluksiz, $v\big|_G \in \mathcal{PSH}(G)$ va $v \le \Psi$ bo‘lsa, u holda

$$V = \begin{cases} \max\left\{v, \Psi\right\} & G \ \ da \\ \Psi & B \setminus G \ \ da \end{cases}$$

funksiya $\mathcal{U}(B, f)$ to‘plamga tegishli bo‘ladi, jumladan G to‘plamda $V \le \Psi$ tengsizlik o‘rinli.

Ψ ning uzluksiz funksiya ekanini ko‘rsatishimiz uchun, uning quyidan yarim uzliksizligini ko‘rsatish yetarli. $\forall \varepsilon > 0$ son olamiz. Shuningdek, ∂B kompakt bo‘lgani uchun $\Psi\,|_{\partial B} = f$ funksiya tekis uzluksiz bo‘ladi. Bu shuni anglatadiki shunday $\delta \in (0, r/2)$ son mavjud bo‘lib, $z \in \overline{B}$, $w \in \partial B$ va $\left\| z - w \right\| < 3\delta$ bo‘lsa, u holda

$$\left|\Psi(z) - \Psi(w)\right| < \frac{\varepsilon}{2} \qquad (1.2.3)$$

bo‘ladi. Barcha $y \in B(0, \delta)$ nuqta uchun,

$$H_y(z)=\begin{cases}\max\left\{\Psi(z),\ \Psi(z+y)-\varepsilon\right\} & (z\in\bar{B}\cap(-y+\bar{B}))\\ \Psi(z) & (z\in\bar{B}\setminus(-y+\bar{B}))\end{cases}$$

o‘rinli.

H_y funksiyaning chetki nuqtalari $\mathcal{U}(B,f)$ funksiyaga tegishli bo‘ladi.

$$B(0,r-\delta)\subset B\cap(-y+B)=B(0,r)\cap B(-y,r)$$

bo‘lganidan $H_y\in\mathcal{PSH}(B(0,r-\delta))$ funksiya ikkita plyurisubgarmonik funksiyaning maksimumidir. Boshqa tomondan $\bar{B}\setminus B(0,r-2\delta)$ to‘plamda $H_y=\Psi$ bo‘ladi. Haqiqatdan H_y ning ta’rifiga ko‘ra bu hol $\bar{B}\setminus(-y+\bar{B})$ to‘plamda o‘rinli bo‘ladi. Agar $z\in(\bar{B}\cap(-y+\bar{B}))\setminus B(0,r-2\delta)$ bo‘lsa, shunday $z_0\in\partial B$ nuqta tanlaymizki, $\left\|z-z_0\right\|<2\delta$ bo‘ladi. $\left\|z+y-z_0\right\|<3\delta$ ga ega bo‘lamiz va shuning uchun (1.2.3) ga asosan

$$\left|\Psi(z)-\Psi(z_0)\right|<\frac{\varepsilon}{2}\ \text{ va }\left|\Psi(z+y)-\Psi(z_0)\right|<\frac{\varepsilon}{2}$$

o‘rinli.

Shunga ko‘ra $\Psi(z)\geq\Psi(z+y)-\varepsilon$ va bundan $H_y(z)=\Psi(z)$ kelib chiqadi. Shuning uchun ∂B chegarada $H_y\in\mathcal{PSH}(B)$ va $H_y=f$ bo‘ladi. Demak, talab qilinganidek $H_y\in\mathcal{U}(B,f)$.

Natijada, $H_y\leq\Psi$ tengsizlik bajariladi. Agar $z,w\in\bar{B}$ va $\left\|z-w\right\|<\delta$ bo‘lsa, u holda

$$\Psi(z)\geq H_{w-z}(z)-\varepsilon\geq\Psi(z+w-z)-\varepsilon=\Psi(w)-\varepsilon$$

bo‘ladi. Shuning uchun Ψ quyidan yarim uzluksiz funksiyadir.

1.2.5-natija. $\mathbb{C}^n$ fazoda Ω ochiq to‘plam va $u \in \mathcal{MPSH}(\Omega)$ funksiya berilgan bo‘lsin. Agar B to‘plam $\overline{B} \subset \Omega$ da ochiq shardan iborat bo‘lsa, u holda $u|_B$ funksiya uzluksiz maksimal plyurisubgarmonik funksiyalar kamayuvchi ketma-ketligining limiti bo‘ladi.

Yuqoridagilardan ko‘rinadiki $\mathbb{C}^n$ fazoda $n > 1$ bo‘lgan holda, differensial operator umumiy plyurisubgarmonik funksiyalar xususiyatlari bilan bir xil bo‘lsa, u umumiy ma’noda bo‘ladi. C^2 plyurisubgarmonik funksiyalarda Gessian matrisasi

$$\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right]$$

maksimallik haqida ba’zi tushunchalarni beradi. Agarda, $\Omega \subset \mathbb{C}^n$ bo‘lib, faqat va faqat Gessian matrisasi Ω to‘plamda nolga intilsa C^2 maksimal plyurisubgarmonik funksiya bo‘ladi.

1.2.6-teorema(q. [6],[11])**.** $u \in C^2(\Omega)$ funksiya berilgan bo‘lib, $\Omega \subset \mathbb{C}^n$ ochiq to‘plam bo‘lsin. Agar $u \in \mathcal{MPSH}(\Omega)$ bo‘lsa, u holda Ω to‘plamda

$$\det\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right] \equiv 0$$

bo‘ladi.

Isbot. Faraz qilaylik, Ω to‘plamda $\det\left[\partial^2 u / \partial z_j \partial \overline{z}_k\right] \neq 0$ bo‘lsin. Biror $a \in \Omega$ nuqta olamiz, ixtiyoriy $b \in \mathbb{C}^n \setminus \{0\}$ nuqta uchun $\left\langle Lu(a)b, b\right\rangle > 0$ o‘rinli. Uning ikkinchi tartibli hosilasi uzluksizligidan, shunday r musbat son mavjudki barcha $z \in \overline{B}(a,r)$, $b \in \mathbb{C}^n \setminus \{0\}$ nuqtalar uchun $\left\langle Lu(z)b, b\right\rangle > 0$ bo‘ladi. Bundan har bir $z \in \overline{B}(a,r)$, $b \in \mathbb{C}^n \setminus \{0\}$ va o‘zgarmas $c > 0$ son uchun $\left\langle Lu(z)b, b \right\rangle \geq c \left\| b \right\|^2$ tengsizlik kelib chiqadi. Demak,

$$v(z) = \begin{cases} u(z) & (z \in \Omega \setminus \overline{B}(a,r)) \\ u(z) + c(r^2 - \left\| z - a \right\|^2) & (z \in \overline{B}(a,r)). \end{cases}$$

U holda $\partial B(a,r)$ chegarada $v = u$ va $v(a) > u(a)$ tengsizlik bajariladi, bu esa u ning maksimal funksiyaligiga zid.

1.2.7-teorema. Ω to‘plam $\mathbb{C}^n$ ning chegaralangan ochiq qism to‘plami va v, u funksiyalar esa $\overline{\Omega}$ ning biror atrofidagi C^2-plyurisubgarmonik funksiyalar. Agar $\partial\Omega$ da $v \leq u$ va Ω to‘plamda

$$\det\left[\frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}\right] \geq \det\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right]$$

bo‘lsa, u holda Ω to‘plamda $v \leq u$ tengsizlik o‘rinli bo‘ladi.

Isbot. $\forall \varepsilon > 0$ son uchun

$$v_\varepsilon(z) = v(z) + \varepsilon(\left\| z \right\|^2 - \sup_{w \in \partial\Omega} \left\| w \right\|^2).$$

funksiya qurib olamiz. U holda

$$0 < \det\left[\frac{\partial^2 v_\varepsilon}{\partial z_j \partial \overline{z}_k}\right] - \det\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right] = \int_0^1 \frac{d}{dt} \det\left[\frac{\partial^2}{\partial z_j \partial \overline{z}_k}(tv_\varepsilon + (1-t)u)\right] dt =$$

$$= \int_0^1 \left(\sum_{j,k=1}^n A_t^{jk} \frac{\partial^2}{\partial z_j \partial \overline{z}_k}(v_\varepsilon - u) \right) dt = \sum_{j,k=1}^n B^{jk} \frac{\partial^2}{\partial z_j \partial \overline{z}_k}(v_\varepsilon - u),$$

bo‘ladi, bu yerda

$$\left[\frac{\partial^2}{\partial z_j \partial \overline{z}_k}(tv_\varepsilon + (1-t)u)\right]_{j,k=1,\ldots,n}$$

$\left[A_t^{jk}\right]$ ning kofaktor matrisasi va $\left[B^{jk}\right]$ uning t ga bog‘liq integrali bo‘ladi. Shuning uchun $\left[B^{jk}\right]$ musbat aniqlangan va Ω to‘plamda $v_\varepsilon - u$ lokal maksimumga ega bo‘lmaydi. Shuning uchun Ω to‘plamda $v_\varepsilon \leq u$ va $\varepsilon \to 0$ deb olib, kerakli natijaga erishamiz.

1.2.8-natija. Ω to‘plam $\mathbb{C}^n$ ning ochiq qism to‘plami va $u \in C^2 \bigcap \mathcal{PSH}(\Omega)$ bo‘lsin. Agar Ω to‘plamda

$$\det\left[\frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}\right] \equiv 0$$

bo‘lsa, u holda u maksimal funksiyadir.

Isbot (1). G to‘plam Ω ga tegishli ochiq nisbiy kompakt to‘plam va $u \in \mathcal{PSH}(\Omega)$ bo‘lsin. ∂G da $v \leq u$ tengsizlik bajarilsin. G to‘plamda v, u ning o‘rniga $(v-\delta)*\chi_\varepsilon$ ni qo‘ya olamiz va G to‘plamda $(v-\delta)*\chi_\varepsilon \leq u$ bo‘lishini topamiz. $\delta, \varepsilon \to 0$ deb olib, $v \leq u$ tengsizlikka ega bo‘lamiz.

Isbot (2). Faraz qilaylik, Ω to‘plam va u funksiya natijaning shartlarini qanoatlantirsin, ammo u maksimal funksiya bo‘lmasin. U holda, ∂G da biror $v \in \mathcal{PSH}(\Omega)$ funksiya va $G \subset \Omega$ ochiq nisbiy kompakt to‘plam uchun $v \leq u$ bajariladi, biroq $(v-u)$ funksiya $a \in G$ ichki nuqtada o‘zining maksimumiga erishadi. Faraz qilaylik, $\forall \varepsilon > 0$ son uchun $v_1 + \varepsilon(\left\|\cdot\right\|^2 + const)$ forma v funksiyaga tegishli bo‘lsin. $b \in \mathbb{C}^n \setminus \{0\}$ nuqtani shunday tanlaymizki, $\left\langle Lu(a)b, b\right\rangle = 0$ bo‘lsin. Nolning yetarlicha kichik atrofidagi $t \in \mathbb{C}$ nuqta uchun

$f(t)=(v-u)(a+tb)$ funksiyani aniqlaymiz. f funksiya nol nuqtada lokal maksimumga ega bo‘lgani uchun, $(\Delta f)(0)\le 0$ bo‘ladi. Boshqa tomondan

$$(\Delta f)(0)=4\left\langle L(v-u)(a)b,b\right\rangle=4\left\langle Lv(a)b,b\right\rangle>0,$$

shuning uchun ziddiyat kelib chiqadi.

1.2.9-natija(q. [13]). Ω to‘plam $\mathbb{C}^n$ ning ochiq qism to‘plami va $u\in C^2\bigcap \mathcal{PSH}(\Omega)$ bo‘lsin. U holda Ω to‘plamdan olingan u funksiya maksimal funksiya bo‘lishi uchun $(dd^c u)^n=0$ bo‘lishi zarur va yetarli.

$\Omega\in\mathbb{C}^n$ to‘plamdagi $\alpha=(i/2)\sum\alpha_{jk}dz_j\wedge d\bar{z}_k$ tenglik C^1 -forma bo‘lsin. Har bir $z\in\Omega$ nuqtada α forma nolga aylanadigan sohani aniqlaymiz:

$$N_z(\alpha)=\left\{x\in\mathbb{C}^n:\alpha_z(x,y)=0,\quad \forall y\in\mathbb{C}^n\right\}$$

$A:\mathbb{C}^n\to\mathbb{C}^n$- $\mathbb{C}$-chiziqli operator va E to‘plam $\mathbb{C}$ ning kompleks qism fazosi bo‘lsin. Agar $A(E)\subset E$ bo‘lsa, E to‘plam A operatorning *invariant fazosi* deyiladi. Agar E to‘plam va uning ortogonol kompanenti $E^{\perp}$ to‘plam A operatorning invariant fazolari bo‘lsa, u holda A operator E to‘plamning *mo‘tadil maydoni* deyiladi. $z\in\Omega$ nuqta uchun α $(1,1)$-forma Agar $\left[\alpha_{jk}(z)\right]$ matrisaga mos keladigan chiziqli operator , u holda α *keltirilgan* deyiladi. Har bir $\mathbb{C}$-chiziqli operator $A:\mathbb{C}^n\to\mathbb{C}^n$ ortogonal tarkibiy qismlarga ajraladi.

$$\mathbb{C}^n=RanA\otimes KerA^*=RanA^*\otimes KerA$$

Bu yerda A^*- A operatorning qo‘shma operatori. Bundan tashqari, agar P proyeksiyadan iborat va U unitar operator bo‘lsa, u holda UPU^* ham proyeksiyadan iborat bo‘ladi.

1.2.10-lemma. $p<n$ bo‘lib, $A:\mathbb{C}^n\to\mathbb{C}^n$ akslantirish $\mathbb{C}$-chiziqli bo‘lsin. Quyidagi shartlar ekvivalent.

(1) A operatorni $RangA$ keltiradi

(2) $RangA^* = RangA$ va $KerA^* = KerA$.

(3) A operator $RangA$ ga ortogonal proyeksiyasi bilan o‘rin almashadi

(4) $U:\mathbb{C}^n \to \mathbb{C}^n$ birlik operator va chiziqli $B:\mathbb{C}^p \to \mathbb{C}^p$ izomorfizm mavjud bo‘lib va ixtiyoriy $(z,w) \in \mathbb{C}^p \times \mathbb{C}^{n-p}$ nuqta uchun

$(B(z),0) = UPU^*(z,w)$

Agar A operator yuqoridagi shartlarni qanoatlantirsa, bunday operator *keltirilgan* deyiladi. Agar A keltirilgan operator bo‘lsa, u holda uning transpozitsiyasi ham kamayuvchi bo‘ladi. Agar A operatorni birlik transformatsiya bilan dioganallash mumkin bo‘lsa, u keltirilgan bo‘ladi. Misol uchun A normal operator bo‘lsa, A o‘zining qo‘shmasi bilan almashinadi.

1.2.11-misol. Faraz qilaylik, $A = S + iT$ bo‘lsin, bu yerda S va T Ermit operatorlari va ularning har biri yarim aniqlangan. U holda A keltirilgan operator bo‘ladi. Haqiqatdan ham, agar $z \in KerA$ nuqta bo‘lsa, u holda $\left\langle Az, z \right\rangle = 0$ va bundan $\left\langle Sz, z \right\rangle = \left\langle Tz, z \right\rangle = 0$ ekani kelib chiqadi. Misol uchun, $T \geq 0$ bo‘lsa , u holda T operator $T^{\frac{1}{2}}$ kvadrat ildizga ega va bundan tashqari $\left\langle T^{\frac{1}{2}}z, T^{\frac{1}{2}}z \right\rangle = 0$. Shuning uchun $z \in KerT$ va $z \in KerS$ bo‘ladi. Demak, $z \in KerA^*$

1.2.12–teorema(q. [13]). Aytaylik, $\alpha = (i/2)\sum \alpha_{jk} dd_{zj} \wedge d\overline{z}_k$ va $\beta = (i/2)\sum \beta_{jk} d_{zj} \wedge d\overline{z}_k$ lar ochiq $\Omega \in \mathbb{C}^n$ qism to‘plamda aniqlangan keltirilgan C^1-formalar bo‘lsin. $A = \left[\alpha_{jk}\right]^{tr}$ va $A = \left[\beta_{jk}\right]^{tr}$. Faraz qilaylik, $p, q \in Z_+$, $0 < p + q < n$ bo‘lib, Ω to‘plamda $\alpha^p \wedge \beta^q \neq 0$, $\alpha^p \wedge \beta^{q+1} = 0$, $\alpha^{p+1} = 0$ shartlar o‘rinli bo‘lsin. U holda $\Delta(z) = N_z(\alpha) \bigcap N_z(\beta)$ ifoda

$2(n-p-q)$ haqiqiy o‘lchovli Ω to‘plamda C^1- taqsimot bo‘ladi va barcha $z \in \Omega$ nuqtalar uchun $\Delta(z) = KerA(z) \bigcap KerB(z)$ o‘rinli.

1.2.13-lemma. Aytaylik $A : \mathbb{C}^n \to \mathbb{C}^n$ -$\mathbb{C}$- chiziqli operator va

$$N = \left\{ z \in \mathbb{C}^n : \left\langle Az, w \right\rangle = \left\langle Aw, z \right\rangle, \ \forall w \in \mathbb{C}^n \right\}$$

bo‘lsin. U holda $N = KerA \bigcap KerA^*$ bo‘ladi.

Isbot. $S = (A + A^*)/2$ va $T = (A - A^*)/2i$ deb olamiz. S va T o‘z-o‘ziga qo‘shma operatorlar va $A = S + iT$ bo‘lsin. Ravshanki, $KerA \bigcap KerA^* \subset N$ teskari ifodani isbotlash uchun $N \subset KerS \bigcap KerT$ ni ko‘rsatish yetarli. $z \in \mathbb{N}$ nuqta olamiz. Har bir $\forall w \in \mathbb{C}^n$ nuqtalar uchun,

$$\left\langle Sz, w \right\rangle - \left\langle Sw, z \right\rangle + i\left(\left\langle Tz, w \right\rangle - \left\langle Tw, z \right\rangle \right) = 0$$

bo‘ladi. Bundan tashqari $\left\langle Sz, w \right\rangle - \left\langle Sw, z \right\rangle = 2i \operatorname{Im} \left\langle Sz, w \right\rangle$ va $\left\langle Tz, w \right\rangle - \left\langle Tw, z \right\rangle = 2i \operatorname{Im} \left\langle Tz, w \right\rangle$. $\forall x, y \in \mathbb{C}^n$ nuqtalar uchun

$$\left\langle x, y \right\rangle = \operatorname{Im} \left\langle x, -iy \right\rangle + i \operatorname{Im} \left\langle x, y \right\rangle$$

bo‘lishidan $\forall w \in \mathbb{C}^n$ nuqtada quyidagiga ega bo‘lamiz:

$$\left\langle Sz, w \right\rangle = \left\langle Tz, w \right\rangle = 0$$

1.2.14-natija. $N = (Ran\, A + Ran\, A^*)^{\perp}$.

1.2.15-lemma. Agar $\alpha = (i/2) \sum \alpha_{jk}\, dz_j \wedge d\bar{z}_j$ - o‘zgarmas koeffitsiyentli (1,1)-forma va $A = \left[\alpha_{jk} \right]^{tr}$ bo‘lsa, u holda

$$\alpha(x, y) = \frac{i}{2} \left(\left\langle Ax, y \right\rangle - \left\langle Ay, x \right\rangle \right)$$

bo‘ladi.

Isbot $x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in\mathbb{C}^n$ nuqtalar uchun quyidagini ko‘rsatish yetarli

$$dz_j\wedge d\overline{z}_k(x,y)=x_j\overline{y}_k-y_j\overline{x}_k$$

$$c_p=\begin{cases}2^{-p} & (p\in\mathbb{Z}_+\ toq)\\ i2^{-p} & (p\in\mathbb{Z}_+\ juft\ va\ toq)\end{cases}$$

ni aniqlab olamiz.

Har bir $J=(j_1,\dots,j_p)$ va $K=(k_1,\dots,k_p)$ multiindekslar uchun

$$\left(\frac{i}{2}\right)^p dz_{j_1}\wedge d\overline{z}_{k_1}\wedge\ldots\wedge dz_{j_p}\wedge d\overline{z}_{k_p}=c_p dz^J\wedge d\overline{z}^K$$

ga ega bo‘lamiz.

Natijada, (1,1)-forma uchun quyidagi natijalar kelib chiqadi. Agar $\alpha=(i/2)\sum\alpha_{jk}\,dz_j\wedge d\overline{z}_k$ - o‘zgarmas koeffitsiyentli (1,1)- forma bo‘lsa, u holda

$$\alpha^p=\underbrace{(\alpha\wedge\ldots\wedge\alpha)}_{p\quad marta}=p!\,c_p\sum_{J,K}\alpha_{J,K}dz^J\wedge d\overline{z}^K \tag{1.2.4}$$

bo‘ladi, bu yerda yig‘indi barcha $J=(j_1,\dots,j_p)$ va $K=(k_1,\dots,k_p)$ multiindekslar bo‘yicha olingan bo‘lib, $1\le j_1\le\ldots\le j_p\le n$, $1\le k_1\le\ldots\le k_p\le n$ va $\alpha_{J,K}=\det\,[\alpha_{j,k}]_{j\in J,k\in K}$.

Yuqoridagi mulohazalardan quyidagi natijalar kelib chiqadi.

1.2.16-natija. Agar Ω to‘plamda α -Ω to‘plamdagi keltirilgan C^1-forma bo‘lsa, u holda $N_z(\alpha)=(Rang\,A(z))^{\perp}=Ker\,A(z)$ bo‘ladi.

Xususan, $z \in \Omega$ nuqtadagi a ning bo‘sh qiymati, $\mathbb{C}^n$ fazoning kompleks qism sohasini keltirib chiqaradi.

1.2.17-natija. C^1 $-$fazoda $a(1,1)$ dagi α forma bo‘lsin, bu yerda $\Omega \subset \mathbb{C}^n$ ochiq qism to‘plam va $1 \le p < n$. Agar Ω to‘plamdagi biror nuqtada $\alpha^{p+1} = 0$, $\alpha^p \neq 0$ bo‘lib, faqat va faqat α forma kamayuvchi bo‘lsa $z \to N_z(\alpha)$ akslantirish $2(n-p)$ o‘lchovli C^1 $-$fazo bo‘ladi.

1.2.18–misol. $\forall z_1, z_2, z_3 \in \mathbb{C}^3$ nuqtalar uchun u funksiya

$$u(z_1,z_2,z_3) = z_1\overline{z}_2 + z_1\overline{z}_3 + (1+i)(z_2\overline{z}_1 + z_3\overline{z}_1)$$

shu ko‘rinishda aniqlanadi. Quyidagi kelib chiqadi:

$$dd^c u = 2i(dz_1 \wedge d\overline{z}_2 + d\,z_1 \wedge d\overline{z}_3 + (1+i)(dz_2 \wedge d\overline{z}_1 + dz_3 \wedge d\overline{z}_1))$$

Shuning uchun $dd^c u$ kamayuvchi operatordir. Bundan tashqari, $(dd^c u)^2 \neq 0$ va $(dd^c u)^3 = 0$.

Ixtiyoriy $t \in \mathbb{C}$ nuqta uchun $\mathrm{Re}\; u(t, \pm t, 0) = \pm 2|\,t\,|^2$, $\mathrm{Im}\; u(t, \pm t, 0) = \pm 2|\,t\,|^2$, bundan ko‘rinadiki, $\pm \mathrm{Re}\; u, \pm \mathrm{Im}\; u$ funksiyalar plyurisubgarmonik funksiyalar emas.

1.2.12-teoremaning isboti. $z \in \Omega$ nuqta olamiz. 1.2.13 va 1.2.15 lemmadan $\Delta(z) = Ker\,A(z) \cap Ker\,B(z)$ kelib chiqadi**.** α va β formalar kamayuvchi bo‘lgani uchun,

$$Ker\,A(z) \cap Ker\,B(z) = (\,Ran\,A(z) + Ran\,B(z))^{\perp}$$

ga ega bo‘lamiz.

1.2.10-lemmaga asosan, koordinatalarni birlik almashtirish va

$$\alpha(z)=\sum_{j,k=1}^{p}\alpha_{j,k}(z)\,dz_j\wedge d\overline{z}_k .$$

deb olish mumkin.

$\alpha(z)^p\wedge\beta(z)^q\neq 0$ va $\alpha(z)^p\wedge\beta(z)^{q+1}=0$ bo‘lgani uchun 1.2.4 formulaga asosan $\left[\beta_{j,k}(z)\right]_{j,k\in\{p+1,\dots,n\}}$ matrisaning rangi q ga teng bo‘ladi. Natijada, $rank\left[A(z)\mid B(z)\right]=p+q$, ya’ni $Ran\,A(z)+Ran\,B(z)$ ning kompleks o‘lchami $p+q$ ga teng bo‘ladi. Teoremaning 1-qismi isbotlandi.

Faraz qilaylik, $\Delta\times\Delta\times T\Omega$ da $d\alpha$ va $d\beta$ formalar nolga intilsin. Agar X,Y,Z lar Ω to‘plamdagi vektor maydonlar va $X,Y\in\Delta$ bo‘lsa, u holda

$$\begin{aligned}&0=d\alpha(X,Y,Z)=X\alpha(Y,Z)-Y\alpha(X,Z)+Z\alpha(X,Y)\\&-\alpha([X,Y],Z)-\alpha(Y,[X,Z])+\alpha(X,[Y,Z])\end{aligned}$$

Shuning uchun

$$\alpha([X,Y],Z)=0$$

va shunga o‘xshash

$$\beta([X,Y],Z)=0$$

Demak, $[X,Y]\in\Delta$ bo‘ladi.

1.2.19-teorema. $u,v\in C^3(\Omega,\mathbb{C})$ funksiyalar bo‘lsin. Bu yerda $\Omega\subset\mathbb{C}^n$ ochiq to‘plam. Faraz qilaylik, dd^cu va dd^cv kamayuvchi operator. $p,q\in\mathbb{Z}_+$ $0<p+q<n$ shartlarni qanoatlantirsin, u holda

$$(dd^cu)^p\wedge(dd^cv)^{q+1}=0,$$
$$(dd^cu)^{p+1}=0,$$

va

$$(dd^c u)^p \wedge (dd^c v)^q \neq 0$$

Ω dagi har bir nuqtada o‘rinli bo‘ladi.

$p+q$ larning kompleks qism ko‘pxilliklari aniqlangan Ω to‘plamning $\mathcal{F}$ kompleks birikmalari mavjud. Natijada $\mathcal{F}$ ning biror kichik M atrofi uchun $\operatorname{Re} u, \operatorname{Im} u, \operatorname{Re} v, \operatorname{Im} v$ mavjud bo‘ladi va M da plyurisubgarmonik.

Natijada

$$\frac{\partial \operatorname{Re} u}{\partial z_j}, \quad \frac{\partial \operatorname{Im} u}{\partial z_j}, \quad \frac{\partial \operatorname{Re} v}{\partial z_j}, \quad \frac{\partial \operatorname{Im} v}{\partial z_j}$$

funksiyalar M da golomorf, $j = 1, \dots, n$

Isbot. Oldingi teoremaga asosan $\alpha = dd^c u$ va $\beta = dd^c v$ bo‘lganidan $F(\alpha, \beta)$ foliatsiya bo‘ladi. $F(\alpha, \beta)$ qatlamlarning har bir M qismi kompleks qism ko‘pxillikdir, $\partial = \partial_M$, $\bar{\partial} = \bar{\partial}_M$ operatorlar M to‘plamda aniqlangan. Shuningdek, agar $a \in M$ nuqta bo‘lsa, u holda

$$T_a M = N_a(dd^c u) \bigcap N_a(dd^c v)$$

bo‘ladi. Agar $I: \ M \to \ \Omega$ akslantirish bo‘lsa, u holda

$$\partial_M \bar{\partial}_M \ (u \mid M) = \partial_M \bar{\partial}_M (I * U) = I * \partial \bar{\partial} u = 0$$

bo‘ladi va shunga o‘xshash

$$\partial_M \bar{\partial}_M (v \mid_M) = 0$$

Demak, $\operatorname{Re} u \mid_M, \ \operatorname{Im} u \mid_M, \ \operatorname{Re} v \mid_M, \ \operatorname{Im} v \mid_M \subset \mathcal{PSH}(M)$.

$a \in M$ nuqta bo‘lsin. M to‘plamda (φ, U) funksiya berilgan, shuningdek $a \in U$ nuqta va $\varphi(a) = 0$ bo‘lsin. $\psi = (\psi_1, \psi_2, \ldots, \psi_n) = \varphi^{-1}$ belgilash kiritamiz. U holda, quyidagi tenglikka ega bo‘lamiz

$$T_a M = \partial_0 \psi(\mathbb{C}^{n-p-q}) \subset Ker \left[\frac{\partial^2 u(a)}{\partial z_j \partial \overline{z}_k} \right]^{tr} = Ker \overline{\left[\frac{\partial^2 u(a)}{\partial z_j \partial \overline{z}_k} \right]}$$

Natijada $l \in \{1, \ldots, n-p-q\}$ va $j \in \{1, \ldots, n\}$ lar uchun

$$\frac{\partial}{\partial \overline{\varphi}_l} \left(\frac{\partial u}{\partial z_j} \right)(a) = \sum_{k=1}^{n} \frac{\partial^2 u(a)}{\partial z_j \partial \overline{z}_k} \overline{\frac{\partial \psi_k}{\partial \omega_l}}(0) = 0$$

Bundan esa

$$\frac{\partial u}{\partial z_j} |_M \in \mathcal{O}(M)$$

$$\frac{\partial v}{\partial z_j} |_M \in \mathcal{O}(M)$$

bo‘lishi kelib chiqadi.

1.2.20-natija(q. [2],[13])**.** $v \in C^3(\Omega)$ funksiya bo‘lsin. Bu yerda $\Omega \in \mathbb{C}^n$ to‘plam va Ω to‘plamning ixtiyoriy nuqtasida

$$(dd^c v)^n = 0,$$
$$(dd^c v)^{n-1} \neq 0$$

(Ω to‘plamda u funksiyaning Gessian matrisasining rankidir va $n-1$ ga teng **).**

Bu matrisaning rangi

$$\left[\frac{\partial^2 v}{\partial z_j \partial \overline{z}_k} \right] \qquad (1.2.5)$$

1.2.21-misol. $(z_1, z_2, z_3) \in \mathbb{C}^3$ nuqtalar uchun

$$v(z_1, z_2, z_3) = |z_1 + z_3|^4 + |z_2|^4$$

bo‘lsin. (1.2.5) matrisaning ranki quyidagi to‘plam uchun 2 ga teng

$$A = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_2(z_1 + z_3) \neq 0\}$$

$\{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus A : z_2 \neq z_1 + z_3\}$ to‘plamda ranki 1 ga va $z_2 = z_1 + z_3 = 0$ bo‘lsa esa ranki 0 ga teng bo‘ladi.

Hisoblang

$$u(z_1, z_2, z_3) = |z_1 - 1 + z_3|^2 + |z_2|^2$$

$p = 2$, $q = 0$ bo‘lganda u va v funksiyalar 1.2.19- teoremadan ko‘rsatiladi.

Shuningdek, bir o‘lchamli $\mathbb{C}^3$ fazoning komleks qatlamlari bor. Natijada bu qatlamning biror kichik M atrofida $v|_M$ garmonik bo‘ladi.

1.3. Musbat aniqlangan formalar va oqim

1.3.1. Musbat aniqlangan formalar

1.3.1-teorema(q. [13]). $\omega \in \Lambda^{1,1}(\mathbb{C}^n, \mathbb{C})$ haqiqiy forma, Ermit matrissasi bo‘lishi uchun $\omega = (i \setminus 2)\sum_{k,j=1}^{n} a_{k,j} dz_k \wedge d\overline{z}_j$ bo‘lishi zarur va yetarli.

Bunda $[a_{k,j}]$ $(n \times n)$ o‘lchamli Ermit matrisasidir.

$\mathbb{C}^n$ fazoda standart Keyler formasi quyidagicha aniqlanadi:

$$\beta = \beta_n = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j \tag{1.3.1}$$

$\mathbb{C}^n$ fazoning unitar izomorfizmi β bilan invariant bo‘lsa, u holda $U * \beta = \beta$ bo‘ladi.

$\mathbb{C}^n$ fazoning standart hajmli elementi $dV = dV_n$ bilan belgilanadi va quyidagicha aniqlanadi:

$$dV_n = (n!)^{-1}(\beta_n)^n = (n!)^{-1} \underbrace{\beta_n \wedge \ldots \wedge \beta_n}_{n \quad marta} \tag{1.3.2}$$

Agar $\omega = \tau dV$ bo‘lsa, ω musbat deyiladi, bunda $\tau \geq 0$.

$\omega \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ forma qat’iy musbat forma deyiladi, agar chiziqli erkli $\mathbb{C}$ fazoning aksi $\eta_j : \mathbb{C}^n \to \mathbb{C}$, $j = 1,\ldots,p$ uchun quyidagi o‘rinli bo‘lsa

$$\omega = \frac{i}{2}\eta_1 \wedge \overline{\eta}_1 \wedge \ldots \wedge \frac{i}{2}\eta_p \wedge \overline{\eta}_p \tag{1.3.3}$$

Agar ω forma $\Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ da $SP^{p,p}(\mathbb{C}^n, \mathbb{R})$ qavariq konusga tegishli bo‘lsa, $\omega \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ qat’iy musbat deyiladi, bunda $\omega = \sum_j \lambda_j \omega_j$, $\lambda_1,\ldots,\lambda_m$ manfiy bo‘lmagan sonlar va ba’zi elementar qat’iy musbat forma $\omega_1,\ldots,\omega_m$ $(m \geq 1)$ formalari uchun.

1.3.2-natija. $\Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ haqiqiy vektor fazo qat’iy musbat formali bazisga ega bo‘ladi.

Isbot. $p = 1$ xususiy holni qaraymiz. Agar $\omega \in \Lambda^{1,1}(\mathbb{C}^n, \mathbb{R})$ bo‘lsa, u holda 1.3.1-teoremadan Ermit matrisasiga ko‘ra $[a_{kj}]$ mavjud, unda

$$\omega = \frac{i}{2}\sum_{k,j=1}^{n} a_{kj} dz_k \wedge d\overline{z}_j$$

$k < j$ bo‘lishi yetarli

$$a_{kj}dz_k \wedge d\overline{z}_j + a_{jk}dz_j \wedge d\overline{z}_k =$$
$$(a_{kj}dz_k + dz_j) \wedge (\overline{a}_{kj}d\overline{z}_k + d\overline{z}_j) - | a_{kj} |^2 \ dz_k \wedge d\overline{z}_k - dz_j \wedge dz_j$$

1.3.3-natija. $\Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ fazoda bo‘sh bo‘lmagan $SP^{p,p}(\mathbb{C}^n)$ konus mavjud.

Agar $\eta \in SP^{n-p,n-p}(\mathbb{C}^n)$ bo‘lsa, $\omega \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ musbat forma deyiladi, $\omega \wedge \eta$ forma esa musbatdir. $\mathbb{C}^n$ fazodagi barcha $(p,p)-$formalarning oilasini $\lambda_+^{p,p}(\mathbb{C}^n)$ deb belgilaymiz. Bunda $\lambda_+^{p,p}(\mathbb{C}^n) \subset \Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$.

Shubhasiz, $SP^{p,p}(\mathbb{C}^n) \subset \lambda_+^{p,p}(\mathbb{C}^n)$. Bu ikki oilaning $p = 1$ va $p = n-1$ hol uchun mos kelishini tekshirish qiyin emas. $2 \le p \le n-2$ da to‘g‘ri. Kuchli musbat formalardan farqli o‘laroq, 2 ta musbat formaning tashqi ko‘paytmasi musbat bo‘lmaydi.

1.3.4-natija. $\omega = \frac{i}{2}\sum_{k,j=1}^{n} a_{k,j}\, dz_k \wedge d\overline{z}_j$ $(1,1)-$ haqiqiy forma, musbat aniqlangan bo‘lishi uchun har bir $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n$ nuqtalar uchun

$$\sum_{k,j=1}^{n} a_{k,j}\omega_k\overline{\omega}_j \ge 0$$

bo‘lishi zarur va yetarli.

1.3.5-natija. β Keyler standart formasi $SP^{1,1}(\mathbb{C}^n)$ konusning ichki qismiga tegishlidir.

1.3.6-teorema. Har bir $\omega \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ forma uchun $C\left\|\omega\right\|\beta^p + \omega$ musbat forma bo‘ladigan, o‘zgarmas $C = C(n,p)$ mavjud bo‘ladi.

1.3.2. Oqimlar.

Ω to‘plam $\mathbb{C}^n$ fazoning ochiq qism to‘plami va φ esa $C_0(\Omega, \mathbb{C})$ fazodagi barcha uzluksiz funksiyalarning oilasi bo‘lsin, shuningdek, $\operatorname{supp} \varphi$ Ω to‘plamning kompakt to‘plamidir. Ω to‘plamning biror K kompakt to‘plami uchun quyidagi o‘rinli:

$$C_0(K, \mathbb{C}) = \left\{\varphi \in C_0(\Omega, \mathbb{C}) \ : \ \operatorname{supp} \ \varphi \subset K\right\}$$

C_0 to‘plamni C_0^∞ bilan almashtirib, o‘xshashliklarga ega bo‘lamiz. $\left\{\Omega_j\right\}_{j\in N}$ ketma-ketlik Ω to‘plamning nisbiy kompakt ochiq to‘plamlar ketma-ketligi bo‘lib, barcha j lar uchun $\overline{\Omega}_j \subset \Omega_{j+1}$ va $\bigcup \Omega_j = \Omega$ bo‘lsin. Har bir $C_0(\overline{\Omega}_j, \mathbb{C})$ fazolarning topologiyasi bilan ustma-ust tushishini ko‘ramiz. Unda barcha $C_0(\overline{\Omega}_j, \mathbb{C})$ fazolar Banax fazosi bo‘ladi. Endi $C_0(\overline{\Omega}_j, \mathbb{C})$ fazo chegarasining topologiyasi bilan $C_0(\Omega, \mathbb{C})$ fazoni to‘ldiramiz. Ushbu topologiyada, faqat va faqat $\varphi_m \to \varphi_0$ va $\bigcup_{m\geq 0} \operatorname{supp} \varphi_m$ Ω to‘plamning nisbiy ochiq kompakt to‘plami bo‘lsa $\left\{\varphi_m\right\}_{m\in N} \subset C_0(\Omega, \mathbb{C})$ ketma-ketlik $\varphi_0 \subset C_0(\Omega, \mathbb{C})$ ga yaqinlashadi. Ω to‘plamda Radon o‘lchovi $C_0(\Omega, \mathbb{C})$ fazodagi uzluksiz $\mathbb{C}-$chiziqli funksionaldir. Riss teoremasiga ko‘ra, Ω to‘plamda har bir Radon μ o‘lchoviga mos Borel μ o‘lchovi mos keladi(q. [13]).

$$\mu(\varphi) = \int_\Omega \varphi d\mu \quad (\varphi \in \mathbb{C}_0(\Omega, \mathbb{C})) \tag{1.3.4}$$

Bundan tashqari, μ Radon o‘lchovi, faqat va faqat $C_0(\Omega, \mathbb{C})$ fazoda Borel o‘lchoviga mos μ musbat bo‘lsa, musbat funksionaldir ($\mu(\varphi) \geq 0$ biror manfiy bo‘lmagan $\varphi \in C_0(\Omega)$ uchun). Musbat chiziqli funksional o‘z-o‘zidan uzluksizdir, ya’ni u Radon o‘lchovidir.

(1.3.4) dan Radon o‘lchovlari Borel o‘lchoviga mosligi kelib chiqadi va shuning uchun odatda ular *o‘lchovlar* deyiladi.

μ musbat Radon o‘lchovi bo‘lganida Borel o‘lchoviga o‘xshashligi doim kelib chiqadi. Agar V to‘plam Ω to‘plamning ochiq qism to‘plami bo‘lsa, u holda

$$\mu(V) = \sup\left\{\mu(\varphi) \;:\; \varphi \in C_0(V, [0,1])\right\}$$

bo‘ladi, agar E to‘plam Ω to‘plamdagi Borel to‘plami bo‘lsa, u holda

$$\mu(E) = \inf\{\mu(V) \;:\; V \quad ochiq,\, E \subset V \subset \Omega\}$$

Bundan tashqari, har bir $K \subset \Omega$ to‘plam uchun

$$\mu(K) = \inf\{\mu(\varphi) \;:\; \varphi \in \mathbb{C}_0(\Omega, [0,1]),\, K \subset \varphi^{-1}(1)\}$$

$(C_0(\Omega, \mathbb{C}))'$ Radon o‘lchovining fazosi topologiya bilan quriladi. Har bir $\varphi \in C_0(\Omega, \mathbb{C})$ nuqta uchun $\mu_j(\varphi) \to \mu(\varphi)$ bo‘lsa, bu topologiyada $\mu_j \to \mu$, $j \to \infty$ bo‘ladi.

Agar μ Radon o‘lchovi bo‘lsa, $|\mu|$ μ ning to‘la variatsiyasi bo‘ladi.

1.3.7-natija. Agar $\mu_j, \mu \in (C_0(\Omega, \mathbb{C}))'$ bo‘lsa, u holda $\mu_j \to \mu$ bo‘lishi uchun har bir $\varphi \in C_0^\infty(\Omega)$ uchun $\mu_j(\varphi) \to \mu(\varphi)$ va har bir kompakt $K \subset \Omega$ to‘plam uchun

$$\sup_{j \geq 1} \left\{ |\mu_j|(K) \right\} < \infty \qquad (1.3.5)$$

zarur va yetarli, agar barcha o‘lchovlar musbat bo‘lsa (1.3.5) shart ortiqcha bo‘ladi.

1.3.8-natija. Agar T - Ω to‘plamdagi bo‘laklash va T - musbat bo‘lsa (ya’ni har bir manfiy bo‘lmagan funksiya uchun $T(\varphi) \geq 0$), u holda T bo‘laklash o‘lchovdan iborat bo‘ladi.

$\mathbb{C}^n$ fazoda Ω ochiq to‘plam berilgan bo‘lsin. Barcha (p,q) bidarajali differensial formalarining oilasining $C_0(\Omega)$ fazoga tegishli koeffitsiyentlari $D_0^{p,q}(\Omega)$ bilan belgilanadi (mos ravishda $D^{p,q}(\Omega)$). Bunda

$$D_0^{p,q}(\Omega) = C_0(\Omega, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C}))$$

va

$$D^{p,q}(\Omega) = C_0^\infty(\Omega, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C}))$$

ularning elementlari odatda *test formalar* deyiladi.

Ω to‘plamning ochiq nisbiy kompakt qismto‘plamlari ketma-ketligi $\{\bar{\Omega}_j\}_{j \in N}$ bo‘lib, unda barcha j lar uchun $\bar{\Omega}_j \subset \Omega_{j+1}$ va $\bigcup \Omega_j = \Omega$ bo‘lsin. Uning har birini

$$C_0(\bar{\Omega}_j, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C})) \qquad (1.3.6)$$

fazolarning koeffitsiyentlarining yagona konvergentsiyasi bilan belgilaymiz va u holda (1.3.6) fazoning induktiv limitining topologiyasi bilan $D_0^{p,q}(\Omega)$ ni belgilaymiz. Xuddi shunday, har birini

$$C_0^\infty(\bar{\Omega}_j, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C})) \qquad (1.3.7)$$

koeffitsiyentlarining yagona konvergentsiyasi va barcha hosilalarning topologiyasi bilan belgilaymiz. Bu topologiyada, (1.3.7) fazo *Freshe fazosi* deyiladi. Keyin $D^{p,q}(\Omega)$ ni (1.3.7) fazoning qat’iy induktiv limitining topologiyasi bilan belgilaymiz.

Qolaversa, $p=q=0$ uchun, $D_0^{p,q}(\Omega)=C_0(\Omega,\mathbb{C})$ va $D^{p,q}(\Omega)=C_0^{\infty}(\Omega)=D(\Omega)$ funksiyalar fazosi bo‘ladi.

Shuningdek, agar $\{\varphi^j\}_{j\geq o}\subset D^{p,q}(\Omega)$ va

$$\varphi^j=\sum_{I,J}\varphi_{I,J}^j dz^I\wedge d\overline{z}^J$$

bo‘lsa (bu yerda I,J o‘suvchi, $\# J=q,\# I=p$), u holda $j\to\infty$ da $\varphi^j\to\varphi^0$ bo‘lishi uchun quyidagi shartlar bajarilishi zarur va yetarli:

(1) $K\subset\Omega$ kompakt to‘plam mavjud bo‘lib, barcha j,I,J lar uchun $\operatorname{supp}\ \varphi_{I,J}^j\subset K$ bo‘ladi;

(2) barcha I,J, va $\alpha\in(\mathbb{Z}_+)^{2n}$ uchun $j\to\infty$ da $D^{\alpha}(\varphi_{I,J}^j)\to D^{\alpha}(\varphi_{I,J}^0)$;yaqinlashish tekis bo‘ladi,

$\left(D^{n-p,n-p}(\Omega)\right)'$ dual fazoning elementlari (p,q) bidarajali oqimlar deyiladi. $\left(D^{n-p,n-p}(\Omega)\right)'$ dual fazoning elementlari (p,q) bidarajali va nolinchi tartibli oqimlar deyiladi. Darhaqiqat, $(n,n)-$oqimlar Ω to‘plamdagi bo‘laklashlar bo‘laib, nolinchi tartibli $(0,0)-$oqimlari Ω to‘plamlardagi Radon o‘lchovlari bo‘ladi.

Ω to‘plam (p,q)forma bo‘lib, uning koeffitsiyentlari Ω to‘plamda lokal integrallanuvchi. bo‘ladi ψ ga mos T_{ψ} oqim Ω to‘plamdagi har qanday $(n-p,n-q)-$forma uchun.

$$T_{\psi}(\varphi)=\int_{\Omega}\psi\wedge\varphi \tag{1.3.8}$$

formula bilan aniqlanadi.

Aytaylik T- Ω to‘plamdagi $(p,q)-$oqim bo‘lsin uchun $I=(i_1,\ldots,i_p)$ va $J=(j_1,\ldots,j_q)$ multiindekslar bo‘lib. . $1\leq i_1<\ldots<i_p\leq n$ va $1\leq j_1<\ldots<j_q\leq n$ bo‘lsin. $K=(k_1,\ldots,k_{n-p})$ va $L=(l_1,\ldots,l_{n-q})$ lar orqali mos ravishda I va J larning $\{1,\ldots,n\}$ to‘plamga yaqinlashuvchi to‘ldiruvchilarini belgilaymiz.

$$T_{I,J}(\varphi)=\varepsilon_{I,J}\,T\,(\varphi dz^K\wedge d\overline{z}^L)$$

qilib olamiz, bu yerda $\varphi\in C_0^\infty(\Omega,C)$ va $\varepsilon_{I,J}$ ni shunday tanlaymizki, bunda

$$\varepsilon_{I,J}\,dz^I\wedge d\overline{z}^J\wedge dz^K\wedge d\overline{z}^L=dV_n$$

bo‘ladi.

U holda $T_{I,J}$ bo‘laklash bo‘ladi, T oqim esa $T_{I,J}$ bo‘laklash koeffitsiyentlariga mos differensial forma sifatida qaralishi mumkin. Shuning uchun

$$T=\sum_{I,J}T_{I,J}\,dz^I\wedge d\overline{z}^J$$

o‘rinli bo‘ladi, bu yerda yig‘indi p va q o‘suvchi multiindekslar bo‘ycha olingan.

Demak,

$$T(\varphi)=\sum_{I,J}(\varepsilon_{I,J})^{-1}T_{I,J}\,(\varphi_{K,L})$$

bu yerda, I,J,K,L multiindekslar o‘suvchi bo‘lib, $K-I$ multiindeksning , $L-J$ multiindeksning komponentasi va $\varphi=\sum_{K,L}\varphi_{K,L}dz^K\wedge d\overline{z}^L$ test formasidir.

Agar T nolinchi tartibli oqimi bo‘lsa, u holda $T_{I,J}$ bo‘laklashlar Radon o‘lchovi bo‘ladi. Agar T_ψ oqim ψ differensial formaning oqimi bo‘lsa, T mavjud bo‘ladi. Ya’ni $T(\varphi)$ ning o‘rniga

$$\int_\Omega T\wedge\varphi$$

ni yozish mumkin. Agar $T=T_\psi$ va

$$\psi=\sum_{I,J}\psi_{I,J}\,dz^I\wedge d\overline{z}^J$$

bo‘lsa, u holda

$$T_{I,J}(\varphi)=\int_\Omega\psi_{I,J}\,\varphi d\lambda$$

bo‘ladi.

T oqim Ω to'plamdagi (p,q)-oqim, ψ oqim esa (k,l)-oqim bo'lib koeffitsiyentlari $C^{\infty}(\Omega,\mathbb{C})$ fazoda bo'lib va $\max\{p+k,q+l\}\le n$ bo'lsa, u holda

$$(T\wedge\psi)(\varphi)=T(\psi\wedge\varphi) \qquad (\varphi\in D^{n-p-k,n-q-l}(\Omega))$$

$(p+k,q+l)-$oqimni aniqlaydi.

(p,q) bidarajali oqimlar fazosini kuchsiz topologiya bilan ta'minlaymiz. Agar (p,q) oqimlar ketma-ketligi $D_0^{n-p,n-q}(\Omega)$ to'plamda nuqtaviy yaqinlashchi bo'lsa, u holda bu ketma-ketlik ushbu topologiyada yaqinlashuvchi bo'ladi.

1.3.9-teorema. Aytaylik T_j,T lar $\Omega\subset\mathbb{C}^n$ ochiq to'plamda (p,q)- oqimlar bo'lsin, $j=1,2,\ldots$. T_j ketma-ketlik T ga kuchsiz yaqinlashishi uchun T_j ketma-ketlik $C_0^{\infty}(\Omega,\Lambda^{n-p,n-q}(\mathbb{C}^n,\mathbb{C}))$ fazoda T ga nuqtaviy yaqinlashishi va har qanday $K\subset\Omega$ kompakt to'plam uchun

$$\sup_{j\ge 0}\left\{\sum_{I,J}|\,(T_j)_{I,J}\,|(K)\right\}<\infty$$

bo'lishi zarur va yetarli ,

Agar har bir ketma-ketlik T to'plamda yaqinlashuvchi ketma-ketlik bo'lsa, $\Omega\subset\mathbb{C}^n$ to'plamda nolinchi tartibli (p,q)- oqimlarning T to'plami deb kuchsiz topologiyadagi kompakt to'plamga aytiladi,. T to'plam yaqinlashuvchi qism ketma-ketlikni o'z ichiga oladi, T to'plam kuchsiz topologiyadagi kompakt to'plam bo'lishi uchun yetarlilik sharti quyidagicha bo'ladi: har bir nisbiy kompakt $G\subset\Omega$ qismsoha uchun shunday shunday $C>0$ mavjudki barcha $T\in \mathrm{T}$ lar va $D^{n-p,n-q}(G)$ uchun

$$|\,T(\varphi)\,|\le C\left\|\varphi\right\|_G$$

o'rinli bo'ladi.

Agar T oqim $(p,p)-$oqim bo'lib, har bir $\omega\in C_0^{\infty}(\Omega,SP^{n-p,n-q}(\mathbb{C}^n))$, uchun $T(\omega)\ge 0$, u holda ning T oqim p darajali musbat oqim deyiladi,. Boshqacha aytganda, agar $\omega\in SP^{n-p,n-q}(\mathbb{C}^n)$ lar uchun $T\wedge\omega$ musbat bo'laklash bo'lsa, u holda T oqim musbat oqim deyiladi.

Agar ψ- Ω to‘plamdagi musbat $(p,p)-$ bo‘lsa, u holda (1.3.8) munosabat bilan anilangan T_ψ oqim musbat bo‘ladi.

Agar T oqim musbat bo‘lsa, har bir ω uchun

$$T(\omega)=\overline{T(\overline{\omega})}$$

o‘rinli.

1.3.10-natija. $\Omega\subset\mathbb{C}^n$ to‘plamda barcha (p,p) bidarajali oqim o‘lchovli koeffitsiyentlarga ega. Bo‘ladi.

Isbot. 3.2.2 ga asosan, $\Lambda^{n-p,n-p}(\mathbb{C}^n,\mathbb{R})$ uchun $\beta=\{\varphi_j\}\subset SP^{n-p,n-p}(\mathbb{C}^n)$ bazisni aniqlash mumkin. Aytaylik $\overline{\beta}=\{\tilde{\varphi}_j\}$-$\beta$ ga dual bo‘lgan $\Lambda^{p,p}(\mathbb{C}^n,\mathbb{R})$ uchun basis bo‘lsin , ya’ni β ning qo‘shmasi bo‘lsin. Toqim Ω to‘plamdagi musbat (p,p)- oqimi bo‘lsin. U holda $T=\sum_J T_J\tilde{\varphi}_j$. Agar ψmanfiy bo‘lmagan funksiya bo‘lsa, u holda $T_J(\psi)=T(\psi\varphi_j)\geq 0$. Shuning uchun, 1.3.8-natijaga ko‘ra, T_J musbat o‘lchov bo‘ladi. Natijada, agar T oqim ixtiyoriy bazisning elementlari orqali ifodalansa, uning koeffitsiyentlari o‘lchovlardan iborat bo‘ladi.

1.3.11-natija. Agar $u\in\mathcal{PSH}(\Omega)$ bo‘lsa, u holda dd^cu o‘lchovli koeffitsiyentlariga ega bo‘lgan musbat $(1,1)$- oqim bo‘ladi.

Isbot. dd^cu operatorning musbatligini ko‘rsatish yetarli. Agar $u\in\mathbb{C}^2(\Omega)$ funksiya bo‘lsa, u holda 1.3.4-natijaga ko‘ra dd^cu musbat operator bo‘ladi. Umumiy holda, $\varepsilon\to 0$ da $dd^c(u*\chi_\varepsilon)$ operator dd^cu operatorga yaqinlashishini ko‘rsatish yetarli. Agar $\varphi\in C_0^\infty(\Omega)$ bo‘lsa, u holda $\varepsilon\to 0$ da yaqinlashuvchilik teoremasiga asosan

$$\int\frac{\partial^2(u*\chi_\varepsilon)}{\partial z_j\partial\overline{z}_k}\varphi d\lambda=\int(u*\chi_\varepsilon)\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}d\lambda \;\to\; \int u\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}d\lambda=\frac{\partial^2 u}{\partial z_j\partial\overline{z}_k}(\varphi)\;.$$

bo‘ladi.

II bob. Monj-Amper operatori

2.1. Kompleks Monj-Amper operatori ta'rifi

2.1.1- ta'rif. 2-marta silliq $u \in C^2(\Omega)$ funksiya uchun uning ikkinchi tartibli differensiali

$$dd^c u = \frac{i}{2}\sum_{j,k} u_{j\bar{k}} dz_j \wedge d\overline{z_k} \qquad \left(bunda \;\; u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} \right) \tag{2.1.1}$$

(tayinlangan $z^0 \in \Omega$ nuqtada) Ermit kvadratik formasidan iborat bo'ladi.

$\left(dd^c\right)^n = \underbrace{dd^c \wedge \ldots \wedge dd^c}_{n-marta}$ operator $\mathbb{C}^n$ fazodagi *Monj-Amper operatori* deyiladi.

Bunda

$$d = \partial + \bar{\partial} \qquad d^c = \frac{1}{4i}\left(\partial - \bar{\partial}\right)$$

$$du = \partial u + \bar{\partial} u = \sum_{k=1}^{n} \frac{\partial u}{\partial z_k} dz_k + \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{z_k}} d\overline{z_k}$$
$$d^c u = \frac{1}{4i}\left(\sum_{k=1}^{n} \frac{\partial u}{\partial z_k} dz_k - \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{z_k}} d\overline{z_k} \right).$$

(2.1.1) tenglikni $n = 1$ uchun ko'rib chiqamiz:

$$du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}$$
$$d^c u = \frac{1}{4i}\left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial \bar{z}} d\bar{z} \right)$$

$$dd^c u = \frac{1}{4i}\left(\partial\left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial \bar{z}} d\bar{z} \right) + \bar{\partial}\left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial \bar{z}} d\bar{z} \right) \right) =$$
$$\frac{1}{4i}\left(\frac{\partial^2 u}{\partial z^2} dz \wedge dz - \frac{\partial^2 u}{\partial z \partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial^2 u}{\partial \bar{z} \partial z} dz \wedge d\bar{z} - \frac{\partial^2 u}{\partial^2 \bar{z}} d\bar{z} \wedge d\bar{z} \right)$$

tashqi ko'paytmada

$dz \wedge dz = 0, \quad d\bar{z} \wedge d\bar{z} = 0, \quad d\bar{z} \wedge dz = -dz \wedge d\bar{z}$ bo'lishidan foydalansak,

$$dd^c u = \frac{1}{2i}\frac{\partial^2 u}{\partial \bar{z}\partial z} dz \wedge d\bar{z}$$

ifodaga ega bo‘lamiz. Demak, $u \in C^2(D), D \subset \mathbb{C}$ uchun $dd^c u = \frac{i}{2}\partial\bar{\partial}u$ bo‘ladi.

(2.1.1) formulani $n = 3$ uchun ham ko‘rib chiqamiz.

$$du = \partial u + \bar{\partial} u = \frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2}\partial z_2 + \frac{\partial u}{\partial z_3}\partial z_3 + \frac{\partial u}{\partial \bar{z}_1}\partial \bar{z}_1 + \frac{\partial u}{\partial \bar{z}_2}\partial \bar{z}_2 + \frac{\partial u}{\partial \bar{z}_3}\partial \bar{z}_3$$

$$d^c u = \frac{1}{4i}\left(\frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2} dz_2 + \frac{\partial u}{\partial z_3} dz_3 - \frac{\partial u}{\partial \bar{z}1} d\bar{z}1 - \frac{\partial u}{\partial z_2} d\bar{z}_2 - \frac{\partial u}{\partial z_3} d\bar{z}_3\right)$$

$$dd^c u = \frac{1}{4i}\left[\partial\left(\frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2}\partial z_2 + \frac{\partial u}{\partial z_3}\partial z_3 - \frac{\partial u}{\partial \bar{z}_1}\partial \bar{z}_1 - \frac{\partial u}{\partial \bar{z}_2}\partial \bar{z}_2 - \frac{\partial u}{\partial \bar{z}_3}\partial \bar{z}_3\right) + \right.$$

$$\left. + \bar{\partial}\left(\frac{\partial u}{\partial z_1} dz_1 + \frac{\partial u}{\partial z_2}\partial z_2 + \frac{\partial u}{\partial z_3}\partial z_3 - \frac{\partial u}{\partial \bar{z}_1}\partial \bar{z}_1 - \frac{\partial u}{\partial \bar{z}_2}\partial \bar{z}_2 - \frac{\partial u}{\partial \bar{z}_3}\partial \bar{z}_3\right) = \right.$$

$$= \frac{1}{4i}\left[\left(\frac{\partial^2 u}{\partial z_2 \partial z_1} dz_1 \wedge dz_2 + \frac{\partial^2 u}{\partial z_3 \partial z_1} dz_1 \wedge dz_3 + \frac{\partial^2 u}{\partial z_1 \partial z_2} dz_2 \wedge dz_1 + \frac{\partial^2 u}{\partial z_3 \partial z_2} dz_2 \wedge dz_3 + \right.\right.$$

$$+ \frac{\partial^2 u}{\partial z_1 \partial z_3} dz_3 \wedge dz_1 + \frac{\partial^2 u}{\partial z_2 \partial z_3} dz_3 \wedge dz_2 - \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} d\bar{z}_1 \wedge dz_1 - \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} d\bar{z}_1 \wedge dz_2 -$$

$$- \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_1} d\bar{z}_1 \wedge dz_3 - \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} d\bar{z}_2 \wedge dz_2 - \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} d\bar{z}_2 \wedge dz_2 - \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_2} d\bar{z}_2 \wedge dz_3 -$$

$$\left. - \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_3} d\bar{z}_3 \wedge dz_1 - \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_3} d\bar{z}_3 \wedge dz_2 - \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_3} d\bar{z}_3 \wedge dz_3 \right) + \frac{\partial^2 u}{\partial \bar{z}_1 \partial z_1} dz_1 \wedge d\bar{z}_1 +$$

$$+ \frac{\partial^2 u}{\partial \bar{z}_2 \partial z_1} dz_1 \wedge d\bar{z}_2 + \frac{\partial^2 u}{\partial \bar{z}_3 \partial z_1} dz_1 \wedge d\bar{z}_3 + \frac{\partial^2 u}{\partial \bar{z}_1 \partial z_2} dz_2 \wedge d\bar{z}_1 + \frac{\partial^2 u}{\partial \bar{z}_2 \partial z_2} dz_2 \wedge d\bar{z}_2 +$$

$$+ \frac{\partial^2 u}{\partial \bar{z}_3 \partial z_2} dz_2 \wedge d\bar{z}_3 + \frac{\partial^2 u}{\partial \bar{z}_1 \partial z_3} dz_3 \wedge d\bar{z}_1 + \frac{\partial^2 u}{\partial \bar{z}_2 \partial z_3} dz_3 \wedge d\bar{z}_2 + \frac{\partial^2 u}{\partial \bar{z}_3 \partial z_3} dz_3 \wedge d\bar{z}_3 -$$

$$- \frac{\partial^2 u}{\partial \bar{z}_2 \partial \bar{z}_1} d\bar{z}_1 \wedge d\bar{z}_2 - \frac{\partial^2 u}{\partial \bar{z}_3 \partial \bar{z}_1} d\bar{z}_1 \wedge d\bar{z}_3 - \frac{\partial^2 u}{\partial \bar{z}_1 \partial \bar{z}_1} d\bar{z}_2 \wedge d\bar{z}_1 - \frac{\partial^2 u}{\partial \bar{z}_3 \partial \bar{z}_2} d\bar{z}_2 \wedge d\bar{z}_3 -$$

$$\left.\left. - \frac{\partial^2 u}{\partial \bar{z}_1 \partial \bar{z}_3} d\bar{z}_3 \wedge d\bar{z}_1 - \frac{\partial^2 u}{\partial \bar{z}_2 \partial \bar{z}_3} d\bar{z}_3 \wedge d\bar{z}_2 \right)\right] =$$

$$= \frac{1}{4i}\left(2\frac{\partial^2 u}{\partial \overline{z}_1 \partial z_1} dz_1 \wedge d\overline{z_1} + 2\frac{\partial^2 u}{\partial \overline{z}_2 \partial z_1} dz_1 \wedge d\overline{z_2} + 2\frac{\partial^2 u}{\partial \overline{z}_3 \partial z_1} dz_1 \wedge d\overline{z_3} + \right.$$

$$+2\frac{\partial^2 u}{\partial \overline{z}_2 \partial z_1} dz_1 \wedge d\overline{z_2} + 2\frac{\partial^2 u}{\partial \overline{z}_2 \partial z_3} dz_3 \wedge d\overline{z_2} + 2\frac{\partial^2 u}{\partial \overline{z}_3 \partial z_3} dz_3 \wedge d\overline{z_3} +$$

$$\left. 2\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_1} d\overline{z_1} \wedge dz_2 + 2\frac{\partial^2 u}{\partial \overline{z}_3 \partial z_1} dz_2 \wedge d\overline{z_3} + 2\frac{\partial^2 u}{\partial \overline{z}_1 \partial z_3} dz_3 \wedge d\overline{z_1}\right) =$$

$$= \frac{1}{2i}\Big(u_{1\overline{1}} dz_1 \wedge d\overline{z}_1 + u_{1\overline{2}} dz_1 \wedge d\overline{z}_2 + u_{1\overline{3}} dz_1 \wedge d\overline{z}_3 + u_{2\overline{1}} dz_2 \wedge d\overline{z}_1 +$$

$$+u_{2\overline{2}} dz_2 \wedge d\overline{z}_2 + u_{2\overline{3}} dz_2 \wedge d\overline{z}_3 + u_{3\overline{1}} dz_3 \wedge d\overline{z}_1 + u_{3\overline{2}} dz_3 \wedge d\overline{z}_2 +$$

$$+u_{3\overline{3}} dz_3 \wedge d\overline{z}_3 = \frac{1}{2i}\sum_{j,k=1}^{3} u_{j\overline{k}} dz_j \wedge d\overline{z}_k.$$

Demak, $n = 3$ uchun ham $dd^c u = \frac{i}{2}\partial\overline{\partial} u$ tenglik o‘rinli ekan.

Agar $u \in C^2(\Omega)$ bo‘lsa, u holda

$$(dd^c u)^n = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right] dV,$$

bu yerda,

$$dV = (\frac{i}{2})^n dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n$$

forma $\mathbb{C}^n$ fazodagi odatdagi hajm formasi.

Endi Monj-Amper operatori ta’rifini kengaytiramiz, biz bu holda differensiallanuvchi bo‘lmagan plyurisubgarmonik funksiyalarni ko‘rib chiqamiz(q. [13]).

Isbotlashlarimizda Stoks teoremasi muhim ro‘l o‘ynaydi.

2.1.2-teorema. Ω to‘plam $\mathbb{C}^n$ fazoning tekis chegaralangan ochiq to‘plami bo‘lsin. Agar $v^1, \ldots, v^k$ haqiqiy qiymatli C^2-funksiyalar, $\overline{\Omega}$ to‘plamning atrofida va φ $(n-k, n-k)$ tartibli silliq formadir, shuningdek $\partial\Omega$ da $\varphi = 0$ bo‘lsa, u holda $j = 1, 2, \ldots, k$ uchun quyidagi o‘rinli

$$\int_\Omega dd^c v^1 \wedge \ldots \wedge dd^c v^k \wedge \varphi = \int_\Omega v^j dd^c v^2 \wedge \ldots \wedge dd^c v^{j-1} \wedge dd^c v^{j+1} \wedge \ldots \wedge dd^c v^k \wedge dd^c \varphi$$

xuddi shu formula $v^1,\ldots,v^k \in \mathcal{C}^2(\Omega)$ da ixtiyoriy ochiq $\Omega \subset \mathbb{C}^n$ to‘plam uchun o‘rinli va φ Ω to‘plamdagi test formadir.

Isbot. Dastlab oxirgi qismni isbotlaymiz. Faraz qilaylik, $v^1,\ldots,v^k$ va φ natijaning 1-qismini qanoatlantirsin. $j=1$ bo‘lsin.

Quyidagiga ega bo‘lamiz:

$$\int_\Omega dd^c v^1 \wedge \ldots \wedge dd^c v^k \wedge \varphi = \int_\Omega \varphi \wedge dd^c v^1 \ldots \wedge dd^c v^k =$$

$$= -\int_\Omega d\varphi \wedge d^c v^1 \wedge dd^c v^2 \wedge \ldots \wedge dd^c v^k =$$

$$= -\int_\Omega dv^1 \wedge d^c\varphi \wedge dd^c v^2 \wedge \ldots \wedge dd^c v^k =$$

$$= \int_\Omega v^1 dd^c\varphi \wedge dd^c v^2 \wedge \ldots \wedge dd^c v^k =$$

$$= \int_\Omega v^1 dd^c v^2 \wedge \ldots \wedge dd^c v^k \wedge dd^c\varphi .$$

Ikkinchi va to‘rtinchi tenglik Stoks teoremasidan kelib chiqadi;

uchinchi esa $(n-k+1, n-k+1)$ qismlarga $d\varphi \wedge d^c v^1$ va $dv^1 \wedge d^c\varphi$ to‘g‘ri keladi.

Yuqoridagilar, $\mathbb{C}^n$ fazoda $SP^{1,1}(\mathbb{C}^n)$ konusning ichki qismiga tegishli bo‘lib, undan quyidagi tengsizlikni keltirib chiqarishda foydalanish mumkin.

2.1.3-teorema. Ω to‘plam kompakt ochiq $K \subset \mathbb{C}^n$ to‘plamning biror atrofi bo‘lsin. O‘zgarmas K va Ω to‘plamga bog‘liq, kompakt $L \subset \Omega \setminus K$ to‘plam mavjud, u holda barcha $u_1,\ldots,u_n \in \mathcal{PSH} \cap \mathcal{C}^2(\Omega)$ funksiyalar uchun quyidagi tengsizlik o‘rinli

$$\int_K dd^c u_1 v^1 \ldots v^1 dd^c u_n \le C \left\|u_1\right\|_L \cdot \ . \ . \ . \ \cdot \left\|u_n\right\|_L \qquad (2.1.2)$$

Isbot. 1.3.5-natijadan, o‘zgarmas $R>0$ son mavjud, shuningdek, Ω to‘plamda har bir haqiqiy $(1,1)-$forma $\omega \neq 0$ koeffitsiyentlari chegaralangan,

$$(R\left\|\omega\right\|_\Omega \beta - \omega(z)) \in \operatorname{int} SP^{1,1}(\mathbb{C}^n) \qquad (z \in \Omega)$$

bu yerda,

$$\omega = \frac{1}{2}\sum_{j,k}^{n} \omega_{j,k} dz_j \wedge d\overline{z}_k$$

va har bir Ω to‘plamning S qism to‘plami uchun

$$\|\omega\|_S = \max_{j,k}\{\|\omega_{j,k}\|_S\}$$

o‘rinli.

Demak, agar Ω to‘plam yuqoridagidek bo‘lsa, η musbat $(n-1,n-1)$-forma va agar ikkala forma koeffitsiyentlari bilan o‘lchovli bo‘lsa, u holda

$$\int_{\Omega} \eta \wedge \omega \le R\|\omega\|_{\Omega} \int_{\Omega} \eta \wedge \beta \qquad (2.1.3)$$

$K_i \subset \operatorname{int} K_{i+1}$, $(i = 0,\dots,n-1)$ to‘plamlar uchun Ω to‘plamga tegishli $K_0 = K, K_1,\dots,K_n$ kompakt to‘plam tanlaymiz. $\chi_i \in C_0^{\infty}(\Omega)$ bo‘lsin, shuningdek, $i = 0,\dots,n-1$ son uchun $0 \le \chi_i \le 1$, $\operatorname{supp} \chi_i \subset \operatorname{int} K_i$ va $K_{i-1} \subset \operatorname{int} \chi_i^{-1}(1)$. $L = K_n \setminus \operatorname{int} \chi_1^{-1}(1)$ deb belgilaymiz.

$dd^c u_j$ ning qat’iy musbatligini va shuning uchun $dd^c u_1 \wedge \dots \wedge dd^c u_n$ musbat. 2.1.2-teoremadan quyidagiga ega bo‘lamiz:

$$\begin{aligned} 0 &\le \int_K dd^c u_1 \wedge \dots \wedge dd^c u_n \le \int_{\operatorname{int} K_1} \chi_1 dd^c u_1 \wedge \dots \wedge dd^c u_n \\ &= \int_{\operatorname{int} K_1} u_1 dd^c \chi_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_n \\ &\le \|u_1\|_L \|dd^c \chi_1\|_L R \int_{\operatorname{int} K_1} dd^c u_2 \wedge \dots \wedge dd^c u_n \wedge \beta \end{aligned}$$

chunki, bu yerda $\operatorname{supp} dd^c \chi_1 \subset L$.

$dd^c(\|z\|^2/4) = \beta$ dan boshlab, bu jarayonni $n-1$ marta takrorlash mumkin; $u_2,\dots,u_n, \|z\|^2/4$ va K_1; $u_3,\dots,u_n, \|z\|^2/4, \|z\|^2/4$ va K_1; va hokazo. Demak,

$$\int_K dd^c u_1 \wedge \dots \wedge dd^c u_n \le \|u_1\|_L \cdot \ldots \cdot \|u_n\|_L C$$

bu yerda, $C = \|dd^c \chi_1\|_L \cdot \ldots \cdot \|dd^c \chi_n\|_L \mathbb{R}^n \int_{K_n} \beta^n$.

Yuqoridagi isbotni Chern va boshqalar ko‘rsatgan. (2.1.2) tengsizlik ***Chern-Levin- Nirenberg tengsizligi*** deyiladi(q. [9],[13]).

Agar $u \in \mathcal{PSH}(\Omega)$ funksiya bo‘lsa, u holda $dd^c u$ musbat $\left(1,1\right)$- oqim bo‘ladi. E.Bedford va B.A.Taylor 2.1.2-teoremani integrallashgan formula bilan ko‘rsatdi, u chegaralangan plyurisubgarmonik funksiyalar uchun Monj-Amper operatorining ta’rifini keltirishda xizmat qiladi.

Bu ta’rifni o‘rganib chiqamiz, $u^1,\ldots,u^n \in L^\infty_{loc}(\Omega)\bigcap\mathcal{PSH}(\Omega)$ funksiyalar bo‘lsin. Agar $1 \le k \le n$ bo‘lsa, u holda $dd^c u^1 \wedge \ldots \wedge dd^c u^k$ nolning (k,k)- oqimi bo‘lishi mumkin, demak

$$\int\limits_\Omega dd^c u^1 \wedge \ldots \wedge dd^c u^k \wedge \chi = \int\limits_\Omega u^k dd^c u^1 \wedge \ldots \wedge dd^c u^{k-1} \wedge dd^c \chi \qquad (2.1.4)$$

bu yerda, χ Ω to‘plamdagi $(n-k,n-k)$ bidarajali test formasi.

Faraz qilaylik, $dd^c u^1 \wedge \ldots \wedge dd^c u^{k-1}$ allaqachon musbat deb ta’riflangan $(k-1,k-1)$ oqimlar, koeffitsiyentlari o‘lchovli.

u^k yuqoridan yarim uzluksiz va chegaralangan funksiya, u $u^k dd^c u^1 \wedge \ldots \wedge dd^c u^{k-1}$ ning koeffitsiyentlarining har biriga nisbatan L^1 fazo bo‘ladi va $u^k dd^c u^1 \wedge \ldots \wedge dd^c u^{k-1}$ o‘lchovli koeffitsiyentlarga ega. Shuning uchun (2.1.4) bilan aniqlangan $dd^c u^1 \wedge \ldots \wedge dd^c u^k$, (k,k)- oqimdir.

Musbatlik quyidagi tarzda ko‘rsatilishi mumkin. Ω da qat’iy musbat $(n-k,n-k)$ bidarajali test forma χ qaraylik. G to‘plam $\operatorname{supp}\chi$ ni o‘z ichiga oluvchi Ω to‘plamning nisbiy kompakt to‘plami bo‘lsin. Teorema shartiga ko‘ra, G to‘plamda kamayuvchi u^k funksiyaga yaqinlashuvchi $\{u^k_j\}_{j\in N} \subset \mathcal{PSH}\bigcap C^\infty(G)$ ketma-ketlikni topamiz.

Har bir j uchun, $dd^c u^k_j \wedge \chi$ qat’iy musbatdir. Shuning uchun bu gipotezadan, quyidagi kelib chiqadi:

$$\int\limits_\Omega (dd^c u^1 \wedge \ldots \wedge dd^c u^{k-1}) \wedge (dd^c u^k_j \wedge \chi) \ge 0$$

Yuqoridagi teoremadan

$$\int_{\Omega} dd^c u^1 \wedge \ldots \wedge dd^c u^k \wedge \chi = \int_{\Omega} u^k dd^c u^1 \wedge \ldots \wedge dd^c u^{k-1} \wedge dd^c \chi =$$
$$= \lim_{j\to\infty} \int_{\Omega} u_j^k dd^c u^1 \wedge \ldots \wedge dd^c u^{k-1} \wedge dd^c \chi =$$
$$= \lim_{j\to\infty} \int_{\Omega} dd^c u^1 \wedge \ldots \wedge dd^c u_j^k \wedge \chi \geq 0$$

ni topamiz.

1.3.10-natijaga ko‘ra, $dd^c u^1 \wedge \ldots \wedge dd^c u^k$ nolinchi tartibli oqimidir, ya’ni u o‘lchovli koeffitsiyentlidir.

$(dd^c)^n$ operator plyurisubgarmonik funksiyalar chegaralarida umumiy kompleks *Monj-Amper operatori* deyiladi. Ba’zan $(u_1,\ldots,u_k) \to dd^c u_1 \wedge \ldots \wedge dd^c u_k$ operator ***Monj-Amper operatori*** deyiladi.

Umumiy Monj-Amper operatori uchun u aniq ta’rif deb qaralmaydi. Yuqoridagi ta’rifdan ko‘rinadiki, agar $u_1,\ldots,u_n$ chegaralangan plyurisubgarmonik funksiyalar ochiq $\mathbb{C}^n$ fazoda berilgan bo‘lsa, $dd^c u_1 \wedge \ldots \wedge dd^c u_n$ aniqlangan bo‘ladi. Shiffmann va Teylor barcha $n-$ tipli plyurisubgarmonik funksiyalarda misol bilan ko‘rsatib bergan(q. [13])

Endi Monj-Amper operatorining lokal chegaralangan plyurisubgarmonik funksiyalarga ta’sir ko‘rsatadigan ba’zi xususiyatlarini ko‘rib chiqamiz. Ulardan eng muhimi Monj-Amper operatori kamayuvchi ketma-ketligining uzluksizligidir. Har bir plyurisubgarmonik funksiyalar, silliq plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketligi bilan ustma-ust tushganligi uchun, Monj-Amper operatorining silliq funksiyalardagi xususiyatlari saqlanadi.

2.1.4-teorema. Ω to‘plam $\mathbb{C}^n$ ning ochiq qism to‘plami bo‘lsin. $k \leq n$ va Ω to‘plamda $\{v_j^1\}_{j\in N},\ldots,\{v_j^k\}_{j\in N}$ plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketigi. $v^1,\ldots,v^k \in \mathcal{PSH}(\Omega)\bigcap L_{loc}^{\infty}(\Omega)$ nuqtalar bo‘lib, $i=1,\ldots,k$ uchun Ω to‘plamga tegishli nuqtalarda $\lim_{j\to\infty} v_j^i = v^i$ ga teng. U holda

$$\lim_{j\to\infty} dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^k = dd^c v^1 \wedge \ldots \wedge dd^c v^k$$

nolinchi tartibli oqimlarining kuchsiz yaqinlashish ma’nosida.

Teoremani isbotlash uchun bir necha lemma kerak bo‘ladi.

2.1.5-lemma. $\{v_j\}_{j\in N} \subset C(\Omega)$ kamayuvchi ketma-ketlik, yuqoridan yarim uzluksiz $v \in L_{loc}^{\infty}(\Omega)$ funksiyaga yaqinlashuvchi bo‘lsin. $\{\mu_j\}_{j\in N}$ ketma-ketlik μ

o‘lchovga ega manfiy bo‘lmagan o‘lchovlarga ega ketma-ketlik bo‘lsin. Agar barcha o‘lchovlar $K \subset \Omega$ kompakt to‘plamga tegishli bo‘lsa, u holda

$$\lim_{j\to\infty} \sup \int_K v_j \, d\mu_j \leq \int_K v d\mu .$$

Isbot. Faraz qilaylik, $\varepsilon > 0$ son va $j_1 < j_2 < \ ...$ ketma-ketlik berilgan bo‘lib, $\forall p$ uchun $\int_K v_{j_p} d\mu_{j_p} > L + \varepsilon$ o‘rinli, bunda $L = \int_K v d\mu$. U holda

$$\int_K v_{j_q} d\mu_{j_p} > L + \varepsilon \qquad (p > q)$$

$p \to \infty$ bo‘lsa, quyidagiga ega bo‘lamiz:

$$\int_K v_{j_q} d\mu \geq L + \varepsilon$$

$q \to \infty$ da, chap tomoni L ga yaqinlashadi va ziddiyatga ega bo‘lamiz. Lemma isbotlandi.

2.1.6-lemma. Qo‘shimcha gipotezaga ko‘ra 2.1.4-teorema o‘rinli, bunda Ω to‘plam ochiq shar va v_j^m funksiya Ω to‘plamda tekis va Ω kompakt qism to‘plamiga mos keladi.

Isbot Chern-Levin-Nirenberg tengsizligi va 1.3.9-teoremaga ko‘ra, Ω to‘plamda barcha ψ test formalar uchun ko‘rsatish yetarli,

$$\lim_{j\to\infty} \int_\Omega dd^c v_j^1 \wedge ... \wedge dd^c v_j^k \wedge \psi = \int_\Omega dd^c v^1 \wedge ... \wedge dd^c v^k \wedge \psi$$

Teoremani induksiyadan foydalanib isbotlaymiz. $k = 1$ to‘g‘ri. Natijani $k = 1, ..., k-1$ uchun to‘g‘ri deb faraz qilamiz. 1.3.9-teorema va Monj-Amper operatori ta’rifidan, shuni ko‘rsatamiz

$$\lim_{j\to\infty} \int_\Omega v_j^k \, dd^c v_j^1 \wedge ... \wedge dd^c v_j^{k-1} \wedge dd^c \psi = \int_\Omega v^k dd^c v^1 \wedge ... \wedge dd^c v^{k-1} \wedge dd^c \psi \quad (2.1.5)$$

Ω to‘plamda, barcha ψ lar uchun, bunda ψ $(n-k, n-k)$ bidarajali formadir. ψ ni $B = B(a,r) \subset \overline{B} \subset \Omega$ to‘plamga tegishli ekanini ko‘rsatish yetarli, bunda B to‘plam ∂B ning atrofida v_j^m funksiyalarni o‘z ichiga oladi. Bundan tashqari, (2.1.5) ni $(n-k, n-k)$ haqiqiy formalar uchun ψ ning tegishli ekanini quyidagidan ko‘rsatiladi: $dd^c\psi$ operator B to‘plamda musbat, ψ $\overline{B}$ to‘plamning atrofida C^∞ dir va ∂B da $\psi = 0$. Haqiqatdan ham, agar $\tilde{\psi}$ $(n-k, n-k)$ tekis, B to‘plamning qism kompakt to‘plamiga mos bo‘lsa, u holda

$\psi = \rho(dd^c\rho)^{n-k} + \varepsilon\tilde{\psi}$ o‘rinli, bundan $\varepsilon > 0$ son va $z \in \mathbb{C}^n$ nuqta uchun $\rho(z) = \left\|z - a\right\|^2 - r^2$. Agar (2.1.5) ga ψ va $\rho(dd^c\rho)^{n-k}$ mos bo‘lsa, u holda $\tilde{\psi} = \varepsilon^{-1}(\psi - \rho(dd^c\rho)^{n-k})$ o‘rinli.

$B_1 = B(a, r_1)$, $0 < r < r_1$ bo‘lsin va barcha v_j^m funksiyalar $B \setminus \tilde{B}_1$ to‘plamga to‘g‘ri keladi. 2.1.5-lemmadan $v_j = v_j^k$, $v = v^k$, $\mu_j = dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^{k-1} \wedge dd^c\psi$ va $\mu = dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c\psi$ va induksion gipotezadan quyidagiga ega bo‘lamiz:

$$\limsup_{j\to\infty} \int_B v_j^k dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^{k-1} \wedge dd^c\psi \le \int_B v^k dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c\psi$$

Isbotni tugatish uchun quyidagini ko‘rsatishimiz kerak

$$\liminf_{j\to\infty} \int_B v_j^k dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^{k-1} \wedge dd^c\psi \ge$$

$$\ge \int_B v^k dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c\psi \qquad (2.1.6)$$

Bundan quyidagiga ega bo‘lamiz

$$\int_B v^k dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c\psi =$$

$$= \int_{\bar{B}_1} + \int_{B\setminus\bar{B}_1} \le \int_{\bar{B}_1} v_j^k dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^{k-1} \wedge dd^c\psi +$$

$$+ \int_{B\setminus\bar{B}_1} v^k dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c\psi$$

$$= \int_B v_j^k dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c\psi +$$

$$+ \int_{B\setminus\bar{B}_1} (v^k - v_j^k) dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c\psi$$

Oxirgi integral $v^k = v_j^k$ da nolga intiladi. $\bar{B}_1$ to‘plamning tashqarisida birinchi integral quyidagiga teng:

$$\int_B v^{k-1} dd^c v^1 \wedge \ldots \wedge dd^c v^{k-2} \wedge dd^c v_j^k \wedge dd^c\psi$$

$$\le \int_B v_j^{k-1} dd^c v^1 \wedge \ldots \wedge dd^c v^{k-2} \wedge dd^c v_j^k \wedge dd^c\psi$$

$$= \int_B v_j^k dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c \psi \quad .$$

Xuddi shuni $v^{k-1}, \ldots, v^1$ nuqtalarga qo‘llaymiz

$$\int_B v^k dd^c v^1 \wedge \ldots \wedge dd^c v^{k-1} \wedge dd^c \psi \le \int_B v_j^k dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^{k-1} \wedge dd^c \psi$$

j cheksizligidan (2.1.6) kelib chiqadi.

2.1.7-lemma. 2.1.4- teorema to‘g‘ri bo‘lgani uchun qo‘shimcha xulosaga kelamiz. Ω to‘plam ochiq shar bo‘lsa, v_j^m funksiya Ω to‘plam bilan ustma-ust tushadi.

Isbot. $\forall \varepsilon > 0$ uchun, plyurisubgarmonik asosiy yaqinlashish teoremasida tasvirlangani uchun $(v_j^m)_\varepsilon$; v_j^m standart **regularizatsiyani** bildiradi. Ω to‘plamda ψ $(n-k, n-k)$- test forma bo‘lsin.

Quyidagiga ega bo‘lamiz

$$\lim_{\varepsilon \to 0} \int_\Omega dd^c (v_j^1)_\varepsilon \wedge \ldots \wedge dd^c (v_j^k)_\varepsilon \wedge \psi = \int_\Omega dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^k \wedge \psi$$

Shuning uchun, dioganal ko‘rinishga keltirish jarayonidan foydalanib, $\varepsilon_j \to 0$ ketma-ketlik topa olamiz, bunda $\tilde{v}_j^m = (v_j^m)_\varepsilon$ tenglik v^m gacha kamayadi va

$$\lim_{j \to \infty} \int_\Omega dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^k \wedge \psi = \lim_{j \to \infty} \int_\Omega dd^c \tilde{v}_j^1 \wedge \ldots \wedge dd^c \tilde{v}_j^k \wedge \psi$$

2.1.6-lemmadan, oxirgi limit $\int_\Omega dd^c v^1 \wedge \ldots \wedge dd^c v^k \wedge \psi$ ga teng.

2.1.4-teoremani isbotlaymiz. Masala lokal bo‘lgani uchun biz umumiylikga ziyon qilmagan holda. $B = B(a,r) \subset \bar{B} \subset \Omega$ deb olib. $z \in \mathbb{C}^n$ nuqta uchun $\rho(z) = \|z-a\|^2 - r^2$ aniqlangan. r_1, r_2 ni $0 < r_1 < r_2 < r$ shartlarni qanoatlantiradigan qilib olamiz. Teoremadagi $\varphi(t) = at + b$, $t \in \mathbb{R}$ o‘suvchi funksiyani inobatga olgan holda, faraz qilaylik v_j^i funksiya $\left[r_1^2 - r^2, r_2^2 - r^2\right]$ intervalda joylashgan. B to‘plamda $u_j^i = \max\{\rho, v_j^i\}$, $u^i = \max\{\rho, v^i\}$ funksiyalar uchun teoremani isbotlasak, natijada B to‘plamda v_j^i, v^i funksiyalar uchun to‘g‘riligi kelib chiqadi, chunki kichik sharlarda $u_j^i = v_j^i$, $u^i = v^i$ o‘rinli. $B \setminus B(a, r_2)$ to‘plamda $u_j^i = u^i = \rho$ teng.

2.1.8-natija. Agar $1 \le k \le n$ uchun $u^1, \dots, u^k \in L^\infty_{loc}(\Omega) \bigcap \mathcal{PSH}(\Omega)$ funksiya bo‘lsa, u holda $j = 1, \dots, k$ lar uchun

$$\int_\Omega dd^c u^1 \wedge \dots \wedge dd^c u^k \wedge \chi =$$
$$= \int_\Omega u^j dd^c u^1 \wedge \dots \wedge dd^c u^{j-1} \wedge dd^c u^{j+1} \wedge \dots \wedge dd^c u^k \wedge dd^c \chi$$

Ω to‘plamdagi har bir $(n-k, n-k)$ bidarajali χ test forma $(n-k, n-k)$ uchun.

$$(u^1, \dots, u^k) \;\to\; dd^c u^1 \wedge \dots \wedge dd^c u^k$$

akslantirish simmetrik va chiziqlidir.

2.1.9-natija. Agar $u_1, \dots, u_n \in \mathcal{PSH} \bigcap L^\infty_{loc}(\Omega)$ funksiya bo‘lsa, (2.1.2) Chern-Levin-Nirenberg tengsizligi o‘rinli bo‘ladi.

2.1.10-natija. Agar $u, v \in \mathcal{PSH} \bigcap L^\infty_{loc}(\Omega)$ funksiya bo‘lsa, u holda

$$(dd^c(u+v)^n) \ge (dd^c u^n) + (dd^c v^n) \text{ bo‘ladi.}$$

Isbot. 2.1.4-teoremada u va v C^2-sinf funksiyalari uchun bu farazni isbotlash yetarli. U holda

$$(dd^c(u+v))^n = (dd^c u)^n + (dd^c v) + \sum_{j=1}^{n-1} \begin{pmatrix} n \\ j \end{pmatrix} (dd^c u)^j \wedge (dd^c v)^{n-j}$$

o‘rinli.

$v^0, v^1, \dots, v^k \in \mathcal{PSH} \bigcap L^\infty_{loc}(\Omega)$ funksiyalar uchun

$$L_k(v^0, v^1, \dots, v^k) = v^0 dd^c v^1 \wedge \dots \wedge dd^c v^k$$

aniqlangan.

Ravshanki, $L_k(v^0, v^1, \dots, v^k)$ nolinchi tartibli oqimidir. Demak, L_k operator ham Monj-Amper operatori kabi muhim ahamiyatga ega(q.[13]).

2.1.4-teoremaning isbotini osongina ko‘rsata olamiz, shuning uchun L_k operator uchun shunga o‘xshash natijaga olib kelamiz. Zaruriy o‘zgartirishlardan keyin quyidagi lemma o‘rinli bo‘ladi.

2.1.11-lemma. Faraz qilaylik $T-$ $\Omega \subset \mathbb{C}^n$ ochiq to‘plamdagi musbat (k,k) -oqim va Ω ning K kompakt qism to‘plamidan tashqarisida $T = 0$ va barcha $\psi \in D^{n-k-1, n-k-1}(\Omega)$ lar uchun $T(dd^c \psi) = 0$ bo‘lsin. U holda $T = 0$ bo‘ladi.

Isbot. 1.3.6-teoremaga ko‘ra, barcha $\chi \in C_0^\infty$ son uchun $\chi \geq 0$ va $\chi = 1$. K to‘plamning atrofida, $T(\chi\beta^{n-k}) = 0$ bo‘lishini ko‘rsatish yetarli.

$\omega = \frac{\|z\|^2}{4}\beta^{n-k-1} = \frac{\|z\|^2}{4}\left(dd^c \frac{\|z\|^2}{4}\right)^{n-k-1}$ deb olsak, u holda $dd^c\omega = \beta^{n-k}$ va

$$T(\chi\beta^{n-k}) = T(\chi dd^c\omega) = T(dd^c(\chi\omega) - d\chi \wedge d^c\omega - d\omega \wedge d^c\chi - \omega dd^c\chi).$$

Gipotezaga ko‘ra, $T(dd^c(\chi\omega)) = 0$. $d\chi \wedge d^c\omega$, $d\omega \wedge d^c\chi$, $\omega dd^c\chi$ ifodalar χ funksiyaning hosilalarini o‘z ichiga oladi va shuning uchun ular K to‘plamning biror atrofida nolga yaqinlashadi. Shunday qilib, $T = 0$ da, ularning havzalari $\Omega \setminus K$ to‘plamda yotadi.

Keyingi teorema Bedford va Teylor tomonidan keltirilgan(q. []).

2.1.12-teorema. Ω to‘plam $\mathbb{C}^n$ fazoning ochiq qism to‘plami, $k \leq n$ va $\{v_j^0\}_{j\in N}, \dots, \{v_j^k\}_{j\in N}$ plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketligi bo‘lsin. $v^1, \dots, v^k \in \mathcal{PSH}(\Omega) \bigcap L_{loc}^\infty(\Omega)$ lar $i = 0, \dots, k$ uchun Ω da $\lim\limits_{j\to\infty} v_j^i = v^i$ shartni qanoatlantirsin. U holda tartibi nolga teng bo‘lgan oqimlarning kuchsiz yaqinlashish ma’nosida

$$\lim_{j\to\infty} L_k(v_j^0, \dots, v_j^k) = L_k(v^0, \dots, v^k)$$

bo‘ladi.

Isbot. Birinchidan, $n > k$ holni qaraymiz. ν nuqta $\lim\limits_{j\to\infty}\{L_k(v_j^0, \dots, v_j^k)\}_{j\in N}$ ketma-ketlikning ixtiyoriy kuchsiz limit nuqtasi bo‘lsin va $\mu = L_k(v^0, \dots, v^k)$ bo‘lsin. $\nu \leq \mu$ tengsizlik 2.1.4-teoremaning isbotida ko‘rsatilgan. 2.1.4-teoremaning isbotidagidek Ω biror ochiq shar va biror $K \subset \Omega$ kompakt to‘plam va i, j lar uchun va $\Omega \setminus K$ to‘plamda $v_j^i = v^i$ o‘rinli bo‘lsin deb faraz qilamiz. $T = \mu - \nu$ deb olamiz. (2.1.4) formuladan va 2.1.4-teoremaga ko‘ra, T 2.1.11-lemmaning shartlarini qanoatlantiradi va shuning uchun $T = 0$. Demak, teorema $n > k$ uchun o‘rinli.

Faraz qilaylik, $n = k$ bo‘lsin. Biz v_j^i, v^i funksiyalarni $\Omega \times \mathbb{C} \subset \mathbb{C}^{n+1}$ da aniqlangan deb hisoblashimiz mumkin. So‘ngra biz faqatgina isbotlagan ishni qo‘llashimiz mumkin va bu bizga bu holda u $\Omega \times \mathbb{C}$ to‘plamda $\{L_n(v_j^0, \dots, v_j^n)\}_{j\in N}$ oqimlarning kuchsiz yaqinlashuvchi ekanini beradi. Nihoyat, biz ushbu

$$\varphi(z_1, \dots, z_n)\,\psi(z_{n+1})dz_{n+1} \wedge d\overline{z}_{n+1}$$

ko‘rinishdagi test formalarga yaqinlashishni Ω to‘plamdagi $\{L_n(v_j^0,\dots,v_j^n)\}_{j\in N}$ oqimlar bo‘yicha kuchsiz yaqinlashish sifatida olishimiz mumkin.

2.2. Plyurisubgarmonik funksiyalarning kvaziuzluksizligi

Plyurisubgarmonik funksiyalarning eng muhim xossalaridan biri kvaziuzluksizlikdir: agar $u \in \mathcal{PSH}(\Omega)$ funksiya bo‘lsa, u holda u istalgancha kichik qilib olingan ochiq to‘plamdan tashqarida uzluksiz bo‘ladi.

Kvaziuzluksizlik haqidagi teorema E.Bedford va B.A.Teylor(q [4] 1982 yil) ga tegishli. Bu teoremaning boshqacha isboti 1984 yilda A.Sadullaev (q. [13] tomonidan berilgan. Biz keltiradigan natijalar E. Bedford va B.A.Teylor (q.[13] 1982 yil) va U.Segrel (q.[8] 1988) larning g‘oyalariga asoslangan.

Ω to‘plam $\mathbb{C}^n$ dagi ochiq qism to‘plam va K to‘plam Ω to‘plamga tegishli kompakt to‘plam bo‘lsin. K ning (Ω dagi) sig‘imini quyidagi formula orqali aniqlaymiz:

$$C(K,\Omega) = \sup\left\{ \int_K (dd^c u)^n \quad : \quad u \in \mathcal{PSH}(\Omega,(0,1)) \right\}$$

Eslatib o‘tamiz, Chern-Levin-Nirenberg tengsizligiga ko‘ra

$$C(K,\Omega) < \infty .$$

Agar $E \subset \Omega$ bo‘lsa, sig‘im sifatida

$$C(K,\Omega) = \sup\{C(K,\Omega) : K \quad E \text{ ning} \quad \text{kompakt} \quad \text{qism} \quad \text{to‘plami}\}$$

ni olamiz. Biz quyidagi teoremaga ega bo‘lamiz.

2.2.1-teorema. Ω to‘plam $\mathbb{C}^n$ fazoning ochiq qism to‘plami bo‘lsin.

(1) Agar E to‘plam Ω to‘plamning Borel qism to‘plami bo‘lsa, u holda

$$C(E,\Omega) = \sup\left\{ \int_E (dd^c u)^n \quad : \quad u \in \mathcal{PSH}(\Omega,(0,1)) \right\}$$

(2) Agar $E_1 \subset E_2 \subset \Omega$ bo‘lsa, u holda $C(E_1,\Omega) \le C(E_2,\Omega)$

(3) Agar $E \subset \Omega \subset \Omega'$ bo‘lsa, u holda $C(E,\Omega) \ge C(E,\Omega')$

(4) Agar $E_1, E_2, \ldots$ to‘plamlar Ω to‘plamning qism to‘plamlari bo‘lsa, u holda

$$C\left(\bigcup_{j=1}^{\infty} E_j, \Omega\right) \leq \sum_{j=1}^{\infty} C(E_j, \Omega)$$

(5) Agar $E_1 \subset E_2 \subset \ldots$ to‘plamlar Ω to‘plamning Borel qism to‘plamlari bo‘lsa, u holda

$$C\left(\bigcup_{j=1}^{\infty} E_j, \Omega\right) = \lim_{j\to\infty} C(E_j, \Omega).$$

Isbot. (1)-(4) xossalar ta’rifdan to‘g‘ridan-to‘g‘ri kelib chiqadi. (5)-xossani isbotlash uchun $\varepsilon > 0$ sonni tayinlab, shunday $u \in \mathcal{PSH}(\Omega,(0,1))$ plyurisubgarmonik funksiya olamizki, u ushbu

$$C(E,\Omega) \leq \int_E (dd^c u)^n + \varepsilon,$$

tengsizlikni qanoatlantirsin, bu yerda $E = \bigcup E_j$. Quyidagiga ega bo‘lamiz

$$C(E,\Omega) \geq \lim_{j\to\infty} C(E_j,\Omega) \geq \lim_{j\to\infty} \int_{E_j} (dd^c u)^n = \int_E (dd^c u)^n \geq C(E,\Omega) - \varepsilon.$$

2.2.2-lemma. Agar u funksiya $\Omega \subset \mathbb{C}^n$ ochiq to‘plamda haqiqiy C^2- sinfga tegishli funksiya bo‘lsa, u holda $du \wedge d^c u$ forma qat’iy musbat bo‘ladi.

Isbot. Quyidagiga ega bo‘lamiz:

$$\begin{aligned} & du \wedge d^c u = ((\partial + \overline{\partial})u) \wedge (i(\overline{\partial} - \partial)u) \\ & = 2i\partial u \wedge \overline{\partial} u - i\partial u \wedge \partial u + i\overline{\partial} u \wedge \overline{\partial} u = 2i\partial u \wedge \overline{\partial} u = 2i\partial u \wedge \overline{\partial u} \end{aligned}$$

2.2.3–natija. Agar $u, v \in C^2(\Omega)$ funksiyalar va χ musbat $(n-1, n-1)$- forma bo‘lsa, u holda

$$\left|\int_\Omega du \wedge d^c v \wedge \chi\right|^2 \leq \int_\Omega du \wedge d^c u \wedge \chi \int_\Omega dv \wedge d^c v \wedge \chi$$

bo‘ladi.

Isbot. $(u,v) \to \int_\Omega du \wedge d^c v \wedge \chi$ bichiziqli va simmetrik akslantirish qaraymiz. Yuqoridagi lemmadan ko‘rinadiki, bu akslantirish musbat aniqlangan va Koshi-Shvars tengsizligini qanoatlantiradi.

2.2.4-lemma. $0 < r_1 < r_2$, $a \in \mathbb{C}^n$, $B_1 = B(a, r_1)$ va $B_2 = B(a, r_2)$ bo‘lsin. Agar $\{u_j\}_{j\in N} \subset \mathcal{PSH} \cap C^2(B_2)$ kamayuvchi ketma-ketlik bo‘lib, bu ketma-ketlik $u \in \mathcal{PSH} \cap L^\infty(B_2)$ funksiyaga yaqinlashsa, u holda

$$\lim_{j\to\infty}\left(\sup\left\{\int_{\bar{B}_1}(u_j - u)(dd^c v)^n \,:\, v \in \mathcal{PSH}(B_2,(0,1))\right\}\right) = 0 \qquad (2.3.1)$$

bo‘ladi.

Isbot. Faraz qilaylik $B_2 \setminus B_1$ da, barcha j, k lar uchun $0 < u_j, u < 1$ va $u_j = u_k$ tengliklar o‘rinli bo‘lsin. $v \in \mathcal{PSH}(B_2,(0,1))$ funksiya olamiz va $r_1 < s_1 < s_2 < r_2$ tengsizliklarni qanoatlantiruvchi s_1, s_2 larni tanlaymiz. $\forall z \in \mathbb{C}^n$ uchun $\rho(z) = (\|z - a\| / s_1)^\gamma$ ni aniqlaymiz, bu yerda $\bar{B}_1$ to‘plamda $\gamma > 0$ son tanlaymizki $\rho \le 1/2$ bo‘ladi, ya’ni

$$\omega = \begin{cases} \max\{\rho, (v+1)/2\} & B_2 \ da \\ \rho & \mathbb{C}^n \setminus B_2 \ da \end{cases}$$
$$\tilde{\omega} = \max\{\rho, 1\}$$

Ravshanki, $\mathbb{C}^n \setminus B(a, s_1)$ da $\omega, \overline{\omega} \in \mathcal{PSH}(\mathbb{C}^n)$ va $\omega = \overline{\omega} = \rho$ bo‘ladi. Agar χ_ε standart silliq yadro va $0 < \varepsilon < \min\{s_2 - s_1, s_1 - r_1\}$ bo‘lsa, u holda $\mathbb{C}^n \setminus B(a, s_2)$ da $\omega_\varepsilon = \omega * \chi_\varepsilon = \tilde{\omega} * \chi_\varepsilon = \tilde{\omega}_\varepsilon$ va B_1 da $\tilde{\omega}_\varepsilon = 1$ bo‘ladi.

Stoks teoremasidan,

$$\int_{\bar{B}_1}(u_j - u_k)(dd^c\omega_\varepsilon)^n = -\int_{\bar{B}_1} d(u_j - u_k)\wedge d^c\omega_\varepsilon \wedge (dd^c\omega_\varepsilon)^{n-1}$$
$$= -\int_{\bar{B}_1} d(u_j - u_k)\wedge d^c(\omega_\varepsilon - \tilde{\omega}_\varepsilon)\wedge(dd^c\omega_\varepsilon)^{n-1}$$

chunki, $\tilde{\omega}_\varepsilon$ funksiya $\bar{B}_1$ to‘plamning biror atrofida o‘zgarmasdir. Faraz qilaylik, $k \ge j$ bo‘lsin. 2.2.3-natijaga ko‘ra va Chern-Levin-Nirenberg tengsizligidan, quyidagiga ega bo‘lamiz:

$$\left|\int_{\bar{B}_1} d(u_j - u_k)\wedge d^c(\omega_\varepsilon - \tilde{\omega}_\varepsilon)\wedge(dd^c\omega_\varepsilon)^{n-1}\right| \le$$

$$\left[\int_{\bar B_1} d(u_j-u_k)\wedge d^c(\omega_\varepsilon-\tilde\omega_\varepsilon)\wedge(dd^c\omega_\varepsilon)^{n-1}\right]^{1/2}\times$$

$$\times\left[\int_{\bar B_1} d(\omega_\varepsilon-\tilde\omega_\varepsilon)\wedge d^c(\omega_\varepsilon-\tilde\omega_\varepsilon)\wedge(dd^c\omega_\varepsilon)^{n-1}\right]^{1/2}\le$$

$$=\left[\int_{\bar B_1} (u_j-u_k)\wedge d^c(u_j-u_k)\wedge(dd^c\omega_\varepsilon)^{n-1}\right]^{1/2}\times$$

$$\times\left[\int_{\bar B(a,s_2)} (\omega_\varepsilon-\tilde\omega_\varepsilon)\wedge d^c(\omega_\varepsilon-\tilde\omega_\varepsilon)\wedge(dd^c\omega_\varepsilon)^{n-1}\right]^{1/2}\le$$

$$\le\left[\int_{\bar B_1} (u_j-u_k)\wedge d^c(u_j+u_k)\wedge(dd^c\omega_\varepsilon)^{n-1}\right]^{1/2} 2(\sup\rho(\bar B_2))^{(n+1)/2}.$$

Bu jarayonni $n-1$ marta takrorlab, quyidagicha baholashga ega bo‘lamiz.

$$\int_{\bar B_1}(u_j-u_k)\wedge(dd^c\omega_\varepsilon)^n\le A\left[\int_{\bar B_1}(u_j-u_k)\wedge(dd^c(u_j+u_k))^n\right]^B$$

bu yerda $A,B>0$ lar $u_j,u_k,\omega_\varepsilon$ larga bog‘liq bo‘lmagan o‘zgarmaslar.

Agar $\eta\in C_0^\infty(B_2)$ nomanfiy va $\bar B_1$ to‘plamning biror atrofida $\eta\equiv 1$ bo‘lsa, u holda 2.1.12-teoremadan, quyidagiga ega bo‘lamiz:

$$\int_{\bar B_1}(u_j-u_k)\wedge(dd^c\omega_\varepsilon)^n=\int_{B_2}\eta(u_j-u_k)\wedge(dd^c\omega_\varepsilon)^n$$

$\varepsilon\to 0$ da limitga o‘tsak

$$\int_{B_2}\eta(u_j-u_k)\wedge(dd^c\omega)^n=2^{-n}\int_{\bar B_1}(u_j-u_k)\wedge(dd^c v)^n.$$

Shuning uchun,

$$\int_{\bar B_1}(u_j-u_k)(dd^c v)^n\le 2^nA\left[\int_{\bar B_1}(u_j-u_k)(dd^c(u_j+u_k))^n\right]^B$$

$k \to \infty$ va 2.1.12-teoremadan, ifodaning chap tomonidan ($v \in \mathcal{PSH}(B_2,(0,1))$ bo‘yicha) supremum olsak quyidagiga ega bo‘lamiz:

$$\sup_v \int_{\overline{B}_1} (u_j - u)(dd^c)^n \le 2^n A \left[\int_{\overline{B}_1} (u_j - u)(dd^c(u_j + u_k))^n \right]^B$$

Endi j ni ∞ ga intiltirib, 2.1.12-teoremani qo‘llash kifoya.

2.2.5-teorema. $u \in \mathcal{PSH} \bigcap L^\infty_{loc}(\Omega)$ funksiya bo‘lsin, bu yerda Ω to‘plam $\mathbb{C}^n$ fazoning ochiq qism to‘plami. Har qanday $\varepsilon > 0$ son uchun, Ω to‘plamning shunday ω ochiq qism to‘plami mavjudki, $C(\omega,\Omega) < \varepsilon$ tengsizlik bajariladi va u funksiya $\Omega \setminus \omega$ to‘plamda uzluksizdir.

Isbot. 2.2.1-teoremada, ba’zi $a \in \mathbb{C}^n$ lar va $r > 0$ son uchun $\Omega = B(a,r)$ deb faraz qilishimiz mumkin. Aytaylik $\{r_j\}$ musbat va $\lim_{j\to\infty} r_j = r$ bo‘lgan nomerlar uchun o‘suvchi ketma-ketlik va $B_j = B(a,r_j)$ bo‘lsin. $\varepsilon > 0$ fiksirlaymiz. Berilishiga ko‘ra $\{u_j\}_{j\in N}$ ketma-ketlikni quyidagicha tanlash mumkin:

(1) u_j funksiya $\tilde{B}_j$ to‘plamning biror atrofida plyurisubgarmonik va silliq;

(2) $\{u_j\}$ ketma-ketlik kamayuvchi va u ga yaqinlashadi;

(3) 2.2.4-lemmadan quyidagi baholashga ega bo‘lamiz

$$\sup\left\{ \int_{\overline{B}_j} (u_j - u)(dd^c v)^n : v \in \mathcal{PSH}(\Omega,(0,1)) \right\} < 2^{-j} \qquad (j \in N)$$

$\omega_j = \{z \in B_j : (u_j - u)(z) > 1/j\}, j \in N$ deymiz. Ravshanki,

$$C(\omega_j,\Omega) < j2^{-j} .$$

$k \in N$ ni shunday tanlaymizki, natijada $\sum_{j=k}^{\infty}(j2^{-j}) < \varepsilon$ bo‘lsin va $\omega = \bigcup_{j\ge k}\omega_j$.

U holda ω ochiq to‘plam va

$$C(\omega,\Omega) \le \sum_{j=k}^{\infty} C(\omega_j,\Omega) \le \sum_{j=k}^{\infty}(j2^{-j}) < \varepsilon .$$

Bundan tashqari, $\{u_j\}$ ketma-ketlik u ga $\Omega \setminus \omega$ to‘plamda lokal tekis yaqinlashadi va shuning uchun $u|_{\Omega\setminus\omega}$ uzluksiz funksiyadir.

Shuni ta’kidlash kerakki, subgarmonik funksiyalar ham kvaziuzluksizlik xossalariga ega. Bu natijalar 1945 –yilda H. Kartan tomonidan isbotlangan. Biz bu ishga to‘xtalmaymiz(q. [13]).

2.3. Monj-Amper operatorining uzluksizlik xossalari.

Lokal chegaralangan plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketligida Monj-Amper operatorining uzluksizligini ko‘rdik. Bundan tashqari, u yaqinlashuvchi ketma-ketliklarda ham o‘rinli. Bu natijalar 1982-yilda E.Bedford va B.A. Teylor tomonidan isbotlangan(q. [4]).

2.3.1-teorema. Ω to‘plam $\mathbb{C}^n$ dagi biror ochiq qism to‘plam va $\{U_j\}_{j\in N}$ ketma-ketlik esa $\mathcal{PSH}\cap L^{\infty}_{Loc}(\Omega)$ dagi ketma-ketlik bo‘lib, bu ketma-ketlik Ω to‘plamning hamma yerida $u\in\mathcal{PSH}\cap L^{\infty}_{Loc}(\Omega)$ funksiyaga o‘sib yaqinlashuvchi bo‘lsin, u holda $(dd^c u_j)^n$ Radon o‘lchovlari ketma-ketligi kuchsiz-topologiya ma’nosida $(dd^c u)^n$ o‘lchovga yaqinlashuvchi bo‘ladi.

Bu teoremaning isboti U.Segrelning 1988-yildagi maqolasida keltirilgan (q. [8]). Dastlab, biz L_n operatorning birinchi o‘zgaruvchisi bo‘yicha uzluksizlik xossasini kvaziuzluksizlikdan foydalanamiz.

2.3.2-teorema. Aytaylik, Ω, u, $\{u_j\}$ lar teorema shartlarini qanoatlantirsin va $v^1,\ldots,v^n\in\mathcal{PSH}\cap L^{\infty}_{Loc}(\Omega)$ bo‘lsin. U holda $L_n(u_j,v^1,\ldots,v^n)$ o‘lchovlar ketma-ketligi $L_n(u,v^1,\ldots,v^n)$ ga yaqinlashuvchi bo‘ladi.

Isbot. Masala lokal bo‘lgani uchun, barcha funksiyalarning qiymatlarini $(0,1)$.da bo‘lsin deb faraz qilamiz va $v=\lim u_j$ bo‘lsin.

Lebegning monoton yaqinlashish teoremasiga ko‘ra $L_n(v^1,\ldots,v^n)$ funksiya $L_n(u_j,v^1,\ldots,v^n)$ ning kuchsiz limiti bo‘ladi. Biz $\{v<u\}$ to‘plamning Lebeg o‘lchovi nolga tengligini bilamiz. Ammo $dd^c v^1\wedge\ldots\wedge dd^c v^n$ Lebeg o‘lchovi bo‘yicha absolyut uzluksiz emas (masalan, bir kompleks o‘zgaruvchili $dd^c(\max\{0,\operatorname{Re} z\})$ ni qaraymiz).

$L_n(u,v^1,\ldots,v^n)\geq L_n(v,v^1,\ldots,v^n)$ tengsizlikdan, bu ikki o‘lchovning lokal bir xil ma’noga ga ega ekanligini ko‘rsatish yetarli. Ω to‘plamda joylashgan biror K yopiq sharni olaylik. $B\setminus K$ sohada $v^i=v^i*\chi_\varepsilon$ deb faraz qilamiz, bu yerda B esa K ni o‘z ichiga oluvchi ochiq shar. v^i_ε orqali $v^i*\chi_\varepsilon$ ifodani belgilaymiz.

$\eta\in C_0^{\infty}(\Omega)$ bo‘lib, K ning biror atrofida $\eta=1$ va $\operatorname{supp}\eta\subset B$ bo‘lsin. U holda

$$\int_\Omega \eta u_j\, dd^c v^1 \wedge \ldots \wedge dd^c (v^n - v^n_\varepsilon)$$

$$= \int_\Omega u_j\, dd^c v^1 \wedge \ldots \wedge dd^c (v^n - v^n_\varepsilon)$$

$$= \int_\Omega (v^n - v^n_\varepsilon)\, dd^c v^1 \wedge \ldots \wedge dd^c v^{n-1} \wedge dd^c u_j$$

v^n funksiyaning kvaziuzluksizligidan va Chern-Levin-Nirenberg tengsizligiga ko‘ra oxirgi integral $\varepsilon \to 0$ da j bo‘yicha nolga intiladi.

$(n-1)$- marta takrorlashdan keyin, agar $\delta > 0$ va $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ lar yetarlicha kichik bo‘lsa, quyidagiga ega bo‘lamiz:

$$\int_\Omega \eta u_j dd^c v^1 \wedge \ldots \wedge dd^c v^n \geq -\delta + \int_\Omega \eta u_j dd^c v^1 \wedge \ldots \wedge dd^c v^{n-1} \wedge dd^c v^n_{\varepsilon_n}$$

$$\geq -n\delta + \int_\Omega \eta u_j dd^c v^1_{\varepsilon_1} \wedge \ldots \wedge dd^c v^n_{\varepsilon_n}$$

Natijada, $\delta, \varepsilon_1, \ldots, \varepsilon_n \to 0$ da (va 2.1.12-teoremadan foydalanib),

$$\liminf_{j\to\infty} \int_\Omega \eta u_j dd^c v^1 \wedge \ldots \wedge dd^c v^n \geq \int_\Omega \eta u_j dd^c v^1_{\varepsilon_1} \wedge \ldots \wedge dd^c v^n_{\varepsilon_n} - \delta n$$

kerakli natijaga erishamiz. Teorema isbot bo‘ldi.

Xulosa

Biz yuqorida ko‘rib chiqqan masalalardan shuni xulosa qilishimiz mumkinki, lokal chegaralangan plyurisubgarmonik funksiyalarning kamayuvchi ketma-ketligida Monj-Amper operatorining uzluksizligini, bu natijalar 1982-yilda E.Bedford va B.A. Teylor tomonidan isbotlangan.

Ya’na shuni aytish mumkinki, berilga Ω to‘plam $\mathbb{C}^n$ dagi biror ochiq qism to‘plam va $\{u_j\}_{j\in N}$ ketma-ketlik esa $\mathcal{PSH}\bigcap L^{\infty}_{Loc}(\Omega)$ dagi ketma-ketlik bo‘lib, bu Ω to‘plamning hamma yerida $u\in\mathcal{PSH}\bigcap L^{\infty}_{Loc}(\Omega)$ funksiyaga o‘sib yaqinlashuvchi bo‘lsin, u holda $(dd^c u_j)^n$ Radon o‘lchovlari ketma-ketligi kuchsiz-topologiya ma’nosida $(dd^c u)^n$ o‘lchovga yaqinlashuvchi bo‘ladi. Bu teoremaning isboti U.Segrelning 1988-yildagi maqolasida keltirilgan isboti murakkab.

Yuqoridagi masala ixtiyoriy plyurisubgarmonik funksiyalar sinfida $(dd^c u_j)^n$ Radon o‘lchovlari ketma-ketligi kuchsiz-topologiya ma’nosida $(dd^c u)^n$ o‘lchovga yaqinlashuvchi bo‘lishi umumiy ma’noda kelib chiqmaydi.

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