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MEXANIKA – MATEMATIKA FAKULTETI

“Ehtimollar nazariyasi va matematik statistika” kafedrasi magistranti

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“5130106-Ehtimollar nazariyasi va matematik statistika” ta’lim yo‘nalishi bo‘yicha magistrlik darajasini olish uchun

MAGISTRLIK DISSERTATSIYASI

Ilmiy raxbar: ________ k.o‘q. K.D. Kuliyev

2018 yil “____” ________________

“UMUMLASHGAN DISKRET HARDI TENGSIZLARI VA UNING BAZI TADBIQLARI”

Kafedraning 2018 yil “2”-maydagi majlisida muhokama qilindi va himoyaga tavsiya etildi (“ ”-bayonnoma)

Fakultet dekani: _______ prof. A.X. Begmatov

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Bitiruv malakaviy ishi YaDAKning 2018 yil“____”iyundagi majlisida himoya qilindi va ____ ball bilan baholandi (___ bayonnoma)

YaDAK raisi: ________________________

A’zolari: ____________________________

MUNDARIJA

Kirish

1. Masalaning dolzarbligi: Disstatsiyada umulashgan Hardi tengsizligi yadrosi $K(n,m)$ ko’phad korinishda bo’lgan hollar o’rganilgan. Umulashgan Hardi tengsizligiga oid juda ko’p masalalarda tengsizlik yadrosi no’manfiyligi talab etiladi. Agar yadro o’z ishorasini berilgan intervalda o’zgartirsa, u holda tengsizlik bajarilishini ta’minlaydigan shartlarni hosil qilish uchun qo’llaniladigan ko’p matematik metodlar ishlamay qoladi. Ikkinchi tomondan yadrosi o’z ishorasini almashtiradigan tengsizliklar matematikda juda ko’p uchraydi, masalan, katta sonlar qonuni haqidagi ta’rifda uchraydigan ifodalarda yadro qatnashadi va u o’z ishorasini o’zgartirib turadi. Shuning uchun ham bu ishda yadro o’z ishorasini almashtirishi juda muhim hisoblanadi. Hardi tengsizligidan foydalanib o’rta qiymat haqidagi natijalarni ishorasi o’zgaruvchan yadroli ifodalar uchun ham kengaytirish masalalarini ko’rish mumkin bo’ladi.

2. Masalaning qo‘yilishi: Tasodifiy miqdorlar ketma-ketligio‘rta qiymatdan ko‘ra umumiyroq ifodalar uchun katta sonlar qonuniga bo‘ysinishi shartlarini topish.

3. Ishning maqsad va vazifalari: Diskret Hardy tengsizligidan foydalanibkatta sonlar qonuni haqidagi natijalarni ma’lum ma’noda umulashtirish.

4. Ilmiy tadqiqot metodlari: Ushbu magistrlik dissertatsiyani bajarish jarayonida ehtimollar nazariyasi, funksional analiz va matematik analizning metodlaridan foydalanildi.

5. Ishning ilmiy ahamiyati: Magistrlik dissertatsiyasi ishida olingan natijalar ehtimollar nazariyasida bazi teoremalarni umumlashtirishda foydalanish mumkin.

6. Ishning amaliy ahamiyati: Magistrlik dissertatsiyasida jamlangan materiallardan, ehtimollar nazariyasidagi katta sonlar qonuni qo‘llaniladigan amaliy masalalarda foydalanish mumkin.

7. Ishning tuzilishi: Ushbu magistrlik dissertatsiyasi ishi kirish qismi, uchta bob, xulosa va foydanilgan adabiyotlar ro‘yxatidan iborat.

Birinchi bobda diskret Hilbert tengsizligi va Hardy tengsizligiga oid zaruriy ta’rif va teoremalar keltirilgan. Keltirilgan teoremalar ishning uchinchi bobida foydalanganligi sababli ularning isbotlari ham berilgan.Ikkinchi bobda katta sonlar qonuni va unga oid natijalar (Chebishev, Markov, Kolmogorov va Xinchin teoremalari) keltirilgan. Shuningdek, ba’zi ta’riflar misollar orqali tushuntirilgan. Uchinchi bobda Hardy tengsizligi va uning tadbiqlari o‘rganilgan.

8. Olingan natijalarning qisqacha bayoni:

Faraz qilaylik yadro quyidagi xossaga ega bo’lsin, ya’ni shunday b_0, b_1, b_2, b_3 *sonlar mavjudki*

$$a_{00} + a_{10}n + a_{01}m + a_{11}nm = (b_0 + b_1 n)(b_2 + b_3 m)$$

tenglik o’rinli. Ma’lumki, bu tenglik bajarilishi uchun ushbu shart bajarilish kerak bo’ladi:

$$a_{00}a_{11} = a_{10}a_{01}.$$

Faraz qilaylik

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00} + a_{10}n + a_{01}m + a_{11}nm)x_m\right|^p u_n\right)^{\frac{1}{p}} \leq C \cdot \left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}},$$

Tengsizlikdagi yadro uchun $a_{00}a_{11} = a_{10}a_{01}$ *shart qanoatlantirsin. U holda tengsizlik o’rinli bo’lishi uchun*

$$\sup_n \left(\sum_{m=1}^{n}|b_2 + b_3 m|^{p'} v_m^{1-p}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty}|b_0 + b_1 m| \cdot u_m\right) < \infty$$

shartning bajarilishi zarur va yetarlidir.

2-teorema.

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\,x_m\right|^p u_n\right)^{\frac{1}{p}} \le C\cdot\left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}}$$

tengsizlik o'rinli bo'lishi uchun

$$A=\sup_{n\in N}\left(\sum_{m=1}^{n}|a_{10}+a_{11}m|^{p'}v_m^{1-p'}\right)^{p-1}\cdot\left(\sum_{m=n}^{\infty}m^p u_m\right)<\infty$$

Shartning *bajarilishi zarur va yetarlidir.*

3-ta'rif. Agar $A=\left(c_{j,k}\right)_{j,k=1}^{\infty}$ matritsaning elementlari barcha $k>j$ uchun $c_{j,k}\ge 0, k\le j, \sum_{k=1}^{j}c_{j,k}=1$ shartlarni qanoatlantirsa A matritsani jamlanuvchi deb ataymiz. Agar A jamlanuvchi matrisaning elementlari ushbu ko'rinishga ega bo'lsa, ya'ni shunday $\{\lambda_i\}_{i=1}^{\infty}$ ketma-ketlik mavjudki, bunda $\lambda_1>0$ va $\lambda_i\ge 0$, $(i\ge 2)$

$$c_{j,k}=\frac{\lambda_k}{\Lambda_j}, 1\le k\le j, \qquad \Lambda_j=\sum_{i=1}^{j}\lambda_j,$$

bo'lsa u holda A matrisaga o'rta vaznli matritsa deyiladi.

4-teorema.

$$\sum_{j=1}^{\infty}\left|\sum_{k=1}^{n}c_{j,k}a_k\right|^p \le C\sum_{n=1}^{\infty}|a_k|^p$$

tengsizlik ixtiyoriy o'rta vaznli A matritsa uchun o'rinli.

5-teorema (asosiy natija). *Faraz qilaylik, $\{u_n\}$ va $\{v_n\}$ ketma-ketliklar quyidagi shartlarni qanoatlantirsin*

$$A=\sup_{n\in N}\left(\sum_{m=1}^{n}|a_{10}+a_{11}m|^{p'}v_m^{1-p'}\right)^{p-1}\cdot\left(\sum_{m=n}^{\infty}m^p u_m\right)<\infty.$$

Faraz qilaylik $\xi_1, \xi_2, \dots, \xi_n, \dots$*o'zaro bog'liq bo'lmagan va quyidagi shartni qanoatlantiruvchi tasodifiy miqdorlar*

$$\sum_{n=1}^{\infty} M|\xi_n - M\xi_n|^p \cdot v_n < \infty$$

ketma-ketligibo'lsa, u holda ixtiyoriy $\varepsilon > 0$ *uchun ushbu tenglik o'rinli*

$$\lim_{n\to\infty} P\{|\eta_n - M\eta_n| < \varepsilon\} = 1.$$

I BOB. DISKRET HARDY TENGSIZLIGI

1.1-§. Hilbert tengsizligi

Bu bo‘limda Hardy tengsizligiga juda yaqin bo‘lgan David Hilbert tomonidan 1900- yildan oldin topilgan tengsizlik to‘g‘risida ma’lumot keltiramiz (bu to‘g‘risida [18] ishda keltirilgan).Bundan tashqari bizlar ko‘rsatamizki, bu tengsizlik [18] dagi asosiy natijani olish uchun juda muhim rol o‘ynaydi.

Hilbert tengsizligining ko‘p uchraydigan ko‘rinishi quyidagichadir: agar

$$\sum_{m=1}^{\infty} a_m^2 < \infty \text{va} \sum_{n=1}^{\infty} b_n^2 < \infty$$

o‘rinli bo‘lsa, bunda $a_m > 0$ va $b_n > 0$, u holda quyidagi qo‘sh qatorlar

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n}$$

yaqinlashuvchidir. Aniq qilib aytganda, ushbu tengsizlik

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \le \pi \left(\sum_{m=1}^{\infty} a_m^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{1/2} \qquad (1.1.1)$$

o‘rinli, bunda π eng aniq konstanta. Bu konstantani [35] da Schur topgan, Hilbert isbotida bu konstanta o‘rnida 2π bo‘lgan. Ta’kidlash mumkinki, (1.1.1) tengsizlikning umumiyroq ko‘rinishi – adabiyotlarda Hilbert tengsizliklari nomi bilan ataluvchi – tengsizlik quyidagichadir:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \le \frac{\pi}{sin\left(\frac{\pi}{p}\right)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^p \right)^{1/p} \qquad (1.1.2)$$

bunda$p > 1$ va $p' = p/p - 1$.

[8]da Hardy, Hilbert teoremasi bilan bog‘liq teoremani keltiradi va shu asosida ba’zi natijalar olgan.

1.1.1-teorema. *Faraz qilamiz* $a_n \geq 0 va A_n = \sum_{k=1}^{n} a_k$ *lar o'rinli bo'lsin. U holda ushbu qatorlar*

$$(i) \sum_{n=1}^{\infty} \frac{a_n A_n}{n}, \qquad (ii) \sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^2, \qquad (iii) \sum\nolimits_{n=1}^{\infty} \sum\nolimits_{m=1}^{\infty} \frac{a_m b_n}{m+n}$$

yaqinlashuvchi bo'lishlari uchun ixtiyoriy birortasi yaqinlashuvchi bo'lishi yetarli va zarurdir.

1.2-§. Hardy tengsizligi

1.2.1-teorema. *Agar* $\sum_{n=1}^{\infty} a_n^2$ *yaqinlashuvchi bo'lsa, bunda* $a_n \geq 0$*, u holda* $\sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^2$ *ham yaqinlashuvchi bo'ladi, bunda* $A_n = \sum_{k=1}^{n} a_k$.

Isbot. Faraz qilamiz $\{a_n^*\}$, $\{a_n\}$ ketma-ketlikning monoton o'smovchi qayta tartiblovchisi bo'lsin, hamda $A_n^* = \sum_{k=1}^{n} a_k^*$. Malumki, $A_n^* \geq A_n$va $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} (a_n^*)^2$, bundan ko'rinib turibdiki teorema isboti uchun ushbu qatorning yaqinlashuvchiligini ko'rsatish yetarli. Bundan, umumiylikga zarar keltirmasdan ketma-ketlikni monoton o'smovchi deb faraz qilish mumkinligi kelib chiqadi. Hamda, 1.2.1-teoremaga ko'ra$\sum_{n=1}^{\infty} \frac{a_n A_n}{n}$ning yaqinlashuvchiligini ko'rsatish yetarli. Oxirgi qatorni ushbu $S = \sum_{n=1}^{\infty} \sum_{m=1}^{n} a_m b_n / n$ko'rinishda yozish mumkin va u yaqinlashuvchi bo'lishi uchun $\sum_{k=1}^{\infty} S_k < \infty$ bo'lishi yetarli, bunda

$$S_k = \sum_{k \leq \frac{n}{m}} \sum_{< k+1} \frac{a_m a_n}{n} \, (k = 1,2, \dots)$$

Bundan tashqari, oddiy tengsizliklar va Koshi-Schwartz tengsizligi yordamida ushbu bahoni

$$S_k \leq \frac{1}{k} \sum_{k \leq \frac{n}{m}} \sum_{< k+1} \frac{a_m a_n}{m} = \frac{1}{k} \sum_{m=1}^{\infty} \frac{a_m}{m} \sum_{k \leq \frac{n}{m} < k+1} a_n$$

$$\leq \frac{1}{k}\sum_{m=1}^{\infty}\frac{a_m}{m}ma_{km} \leq \frac{1}{k}\left(\sum_{m=1}^{\infty}a_m^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}a_{km}^2\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{k}\left(\sum_{m=1}^{\infty}a_m^2\right)^{1/2}\left(\frac{1}{k}\sum_{m=1}^{\infty}a_m^2\right)^{1/2} = \frac{1}{k^{3/2}}\sum_{m=1}^{\infty}a_m^2$$

hosil qilamiz. Shunday qilib,

$$S = \sum_{k=1}^{\infty}S_k \leq \sum_{k=1}^{\infty}\frac{1}{k^{3/2}}\sum_{m=1}^{\infty}a_m^2 < 3\sum_{m=1}^{\infty}a_m^2 < \infty,$$

ya'ni $S < \infty$ ga ega bo‘lamiz va bu teoremani isbotlaydi.

1.2.2-teorema (M.Riesz) [5]. *Agar* $p > 1, a_n \geq 0$ *va* $\sum_{n=1}^{\infty}a_n^p$ *bo‘lsa, u holda*

$\sum_{n=1}^{\infty}(A_n/n)^p$ *yaqinlashuvchi bo‘ladi. Bunda* $A_n = \sum_{k=1}^{n}a_k$.

Isbot. $\Phi_n = n^{-p} + (n+1)^{-p} + (n+2)^{-p} + \cdots$ bo‘lsin. U holda bizlar ixtiyoriy $N(A_0 = 0)$ uchun quyidagiga ega bo‘lamiz:

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p = \sum_{n=1}^{N}A_n^p(\Phi_n - \Phi_{n+1}) = \sum_{n=1}^{N}\left(A_n^p - A_{n-1}^p\right)\Phi_n - A_N^p\Phi_{N+1}$$

$$\leq \sum_{n=1}^{N}\left(A_n^p - A_{n-1}^p\right)\Phi_n \leq p\sum_{n=1}^{N}a_nA_{n-1}^p\,\Phi_n$$

Bundan tashqari

$$\Phi_n < n^{-p} + \int_n^{\infty}x^{-p}dx = n^{-p} + \frac{n^{-(p-1)}}{p-1} \leq \frac{p}{p-1}n^{-(p-1)}.$$

Bu tengsizlikdan, hamda Golder tengsizligidan ushbu

$$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}$$

$$\left(p > 1, \ \ \frac{1}{p} + \frac{1}{q} = 1\right) \tag{1.2.1}$$

bahoni hosil qilamiz. Shunday qilib,

$$\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p \leq \frac{p^2}{p-1} \sum_{n=1}^{N} a_n \left(\frac{A_n}{n}\right)^{p-1}$$

$$\leq \frac{p^2}{p-1} \left(\sum_{n=1}^{N} a_n^p\right)^{1/p} \left(\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p\right)^{1-(1/p)}.$$

Bundan quyidagi tengsizlik hosil bo‘ladi va teoremani isbotlaydi

$$\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p \leq \left(\frac{p^2}{p-1}\right)^p \sum_{n=1}^{N} a_n^p. \tag{1.2.2}$$

1.2.3-teorema.[6] *Agar* $\rho > 1$, $a > 0, f(x) \geq 0$ *va* $\int_a^{\infty} f(x)^p dx < \infty$ *bo‘lsa, u holda ushbu*

$$\int_a^{\infty} \left(\frac{\int_a^x f(t)dt}{x}\right)^p \leq \left(\frac{p}{p-1}\right)^p \int_a^{\infty} f(x)^p dx \tag{1.2.3}$$

Tengsizlik o‘rinli bo‘ladi va bunda $(p/(p-1))^p$ *konstanta eng kichikdir.*

1.2.4-teorema. *Faraz qilaylik,* $\rho > 1$, $a_n \geq 0$ *va* $\sum_{k=1}^{n} a_k$ *bo‘lsin. U holda quyidagi tengsizlik*

$$\sum_{n=1}^{N} \left(\frac{A_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{N} a_n^p \tag{1.2.4}$$

Barcha $N \in \boldsymbol{N}$ va $N = \infty$ larda o'rinli. Undan tashqari tengsizlikdagi konstanta $N = \infty$ uchun eng yaxshi konstanta bo'ladi.

(1.2.4) tengsizlik ba'zan Hardy-Landau tengsizligi ham deb ataladi ([27] ning 188 betiga qarang).

Isbot. Barcha $y \geq 0$lar uchun quyidagi tengsizlik o'rinli

$$y^p - py + p - 1 \geq 0.$$

Bu Bernulli tengsizligi deb ham aytiladi, agar uni ushbu ko'rinishda yozsak $y^p \geq 1 + p(y-1)$. Bu elementar tengsizlikda $y = y_1/y_2$ belgilash olsak quyidagiga ega bo'lamiz:

$$y_1^p - py_1y_2^{p-1} + (p-1)y_2^p \geq 0.$$

Endi $y_1 = b_n$ va $y_2 = (p-1)B_n/pn$ deb tanlasak, bunda $B_n = \sum_{k=1}^n b_k$, tengsizlik ushbu ko'rinishni oladi:

$$b_n^p - bp_n\left(\frac{p-1}{p}\frac{B_n}{n}\right)^{p-1} + (p-1)\left(\frac{p-1}{p}\frac{B_n}{n}\right)^p \geq 0.$$

Bundan tengsizlikning ikkala tomonini yig'ishdan

$$\sum_{n=1}^N b_n^p - \left(\frac{p-1}{p}\right)^{p-1}\sum_{n=1}^N pb_n\left(\frac{B_n}{n}\right)^{p-1} + (p-1)\left(\frac{p-1}{p}\right)^p\sum_{n=1}^N\left(\frac{B_n}{n}\right)^p \geq 0$$

ni hosil qilamiz. Bundan tashqari, $pb_nB_n^{p-1} = pB_n^{p-1}(B_n - B_{n-1}) \geq B_n^p - B_{n-1}^p$va bo'laklab yig'ish natijasida quyidagi bahoga ega bo'lamiz:

$$\sum_{n=1}^N pb_n\left(\frac{B_n}{n}\right)^{p-1} \geq \sum_{n=1}^N\left(B_n^p - B_{n-1}^p\right)\frac{1}{n^{p-1}}$$

$$\geq \sum_{n=1}^N B_n^p\left(\frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}}\right) \geq (p-1)\sum_{n=1}^N B_n^p\frac{1}{(n+1)^p}.$$

Oxirgi ikkala tengsizlikdan bizlar quyidagiga erishamiz:

$$\sum_{n=1}^{N} b_n^p \geq \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N} B_n^p \left(\frac{p}{(n+1)^p} - \frac{p-1}{n^p}\right) = \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N} c_n \left(\frac{B_n}{n}\right)^p \ ,$$

bu yerda $c_n = p(1+\frac{1}{n})^{-p} - p + 1 \to 1$, agar $n \to \infty$. Agar ushbu ko‘rinishda belgilashlar kiritsak

$$b_1 = b_2 = \cdots = b_m = a_1\ , \qquad b_{m+1} = b_{m+2} = \cdots = b_{2m} = a_2\ , \ldots$$

$$b_{(N-1)m+1} = b_{(N-1)m+2} = \cdots = b_{Nm} = a_N$$

Va N ni Nm bilan almashtirsak, u holda yuqoridagi tengsizlikdan

$$m\left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N} a_n^p \geq (c_1 + c_2 + \cdots + c_m)\left(\frac{A_1}{1}\right)^p + (c_{m+1} + c_{m+2} + \cdots +$$

$$+c_{2m})\left(\frac{A_2}{2}\right)^p + \cdots + (c_{(N-1)m+1} + c_{(N-1)m+2} + \cdots + c_{Nm})\left(\frac{A_N}{N}\right)^p$$

ga ega bo‘lamiz. Ikkala tomonini m ga bo‘lib, hamda $m \to \infty$, $(c_1 + c_2 + \cdots + c_m)/m \to 1$, $(c_{m+1} + c_{m+2} + \cdots + c_{2m})/m \to \infty$ e’tiborga olsak (12)ning barcha N larda bajarilishi kelib chiqadi (shunday qilib $N \to \infty$ dagi qismi qoldi).

$(p/(p-1))^p$ ning $N = \infty$ bo‘lganda eng yaxshi konstanta ekanligini ko‘rsatish uchun $a_n = n^{-1/p-\varepsilon}$ $(0 < \varepsilon > 1 - 1/p)$ ni qarash yetarli. $a_n$ning bunday tanlanilishidan, hamda

$$A_n = \sum_{k=1}^{n} k^{-\frac{1}{p}-\varepsilon} > \int_1^n x^{-\frac{1}{p}-\varepsilon} dx$$

$$= \frac{1}{1 - 1/p - \varepsilon}\left(n^{1-1/p-\varepsilon} - 1\right) > \frac{p}{p-1}\left(n^{1-1/p-\varepsilon} - 1\right)$$

quyidagi kelib chiqadi. Shunday qilib, ushbu bahoga ega bo‘lamiz:

$$\left(\frac{A_n}{n}\right)^p > \left(\frac{p}{p-1}\right)^p n^{-1-\varepsilon p}\left(1-\frac{1}{n^{1-1/p-\varepsilon}}\right)^p$$

$$\geq \left(\frac{p}{p-1}\right)^p n^{-1-\varepsilon p}\left(1-\frac{1}{n^{1-1/p-\varepsilon}}\right) = \left(\frac{p}{p-1}\right)^p \left(n^{-1-\varepsilon p} - pn^{-2+1/p-\varepsilon+\varepsilon p}\right) ,$$

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p > \left(\frac{p}{p-1}\right)^p \left(\sum_{n=1}^{N} a_n^P - p\sum_{n=1}^{N}\frac{1}{n^{2-\frac{1}{p}-\varepsilon+\varepsilon p}}\right)$$
$$= \left(\frac{p}{p-1}\right)^p \left(\sum_{n=1}^{N} a_n^P - pC_{N,\varepsilon}\right)$$

Bunda $C_{N,\varepsilon} \to C$ agar $N \to \infty$ barcha $\varepsilon > 0$ sababi $2 - \frac{1}{\varepsilon} + \varepsilon p > 1$. Va nihoyat

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p \Big/ \sum_{n=1}^{N} a_n^p > \left(\frac{p}{p-1}\right)^p \left(1 - pC_{N,\varepsilon} \Big/ \sum_{n=1}^{N} a_n^P\right) \to \left(\frac{p}{p-1}\right)^p ,$$

Chunki $\sum_{n=1}^{N} a_n^p = \sum_{n=1}^{N} n^{-1-\varepsilon p} \to \infty$ agar $N \to \infty$ va $\varepsilon \to 0^+$. Bu esa konstantaning eng yaxshi ekanligini ko‘rsatadi. Teorema isbot bo‘ldi.

1.2.5-teorema. *Faraz qilaylik,* $\rho > 1$ *hamda* $f(x) \geq 0$, $(0,\infty)$ *dap-integrallanuvchi bo‘lsin. Agar* $F(x) = \int_0^x f(t)dt < \infty$ *barcha* $x > 0$ *deb belgilasak u holda*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx$$

o‘rinli bo‘ladi.Bunda $(p/(p-1))^p$ *eng yaxshi konstantadir.*

Isbot. Bo‘laklab integralashni, hamda $d/dx(F(x)^p) = pF(x)^{p-1}f(x)$ ayniyatni, qaysiki deyarli barcha $x \in (0,\infty)$da o‘rinli qo‘llab barcha αva A, bunda $0 < \alpha < A < \infty$ uchun ushbu bahoni hosil qilamiz:

$$\left(\frac{F(x)}{x}\right)^p dx = -\frac{1}{p-1}\int_\alpha^A F(x)^p \frac{d}{dx}(x^{1-p})dx$$

$$= \int_\alpha^A \frac{\alpha^{1-p}}{p-1}F(\alpha)^p - \frac{A^{1-p}}{p-1}F(A)^p + \frac{1}{p-1}\int_\alpha^A x^{1-p}\frac{d}{dx}(F(x)^p)dx$$

$$\leq \frac{\alpha^{1-p}}{p-1} + \frac{p}{p-1}\int_\alpha^A \left(\frac{F(x)}{x}\right)^{p-1} f(x)dx\,.$$

Bunda Golder tengsizligini qo‘llab oxirgi integralni quyidagicha baholaymiz:

$$\int_\alpha^A \left(\frac{F(x)}{x}\right)^{p-1} f(x)dx \leq \left(\int_\alpha^A f(x)^p\right)^{1/p} \left(\int_\alpha^A \left(\frac{F(x)}{x}\right)^p\right)^{(p-1)/p}.$$

βni $\alpha \leq \beta \leq A$ shunday tanlab va yuqoridagi $F(x) - F(\alpha)$ning o‘rniga $F(x)$ qo‘yib huddi shunday takrorlasak ushbu ko‘rinishdagi bahoga ega bo‘lamiz

$$\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx \leq \frac{p}{p-1}\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^{p-1} f(x)dx$$

$$\leq \frac{p}{p-1}\left(\int_\alpha^A f(x)^p\right)^{\frac{1}{p}} \left(\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{(p-1)/p}.$$

Shunday qilib

$$\left(\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^{p-1} dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_\alpha^A f(x)^p dx\right)^{1/p}$$

va bundan

$$\left(\int_\beta^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^\infty f(x)^p dx\right)^{1/p}.$$

Bu tengsizlikda bizlar avval $\alpha \to 0^+$ va bunda $F(x) - F(\alpha)$ ushbu $F(x)$ ko‘rinishga kelishini etiborga olsak, va so‘ngra $A \to \infty$ va $\beta \to 0^+$ qarasak teorema isbotiga ega bo‘lamiz.

1.2.6-teorema. *Faraz qilaylik,* $a_n \geq 0$ *va* $\lambda_n > 0$ *bo‘lib* $A_n = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$ *va* $\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ *barcha* $n = 1,2,\ldots,$ *va hamda* $\sum_{n=1}^{\infty} \lambda_n a_n^p$. *U holda*

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p \quad .$$

II BOB. KATTA SONLAR QONUNI

2.1-§.Katta sonlar qonuni uchun yetarli shartlar

Biror o‘zaro bog‘liqmas tajribalar ketma-ketligi o‘tkazilgan bo‘lsin va har bir tajribada A hodisaning ro‘y berish ehtimoli p ga va ro‘y bermaslik ehtimoli $1-p$ ga teng bo‘lsin. Ahodisaning$k$ $-$tajribada ro‘y berish sonini $\xi_k$desak, u holda

$$\xi_k = \begin{cases} 0, & q = 1-p, \\ 1, & p. \end{cases}$$

Natijada $\xi_1, \xi_2, \ldots \ \xi_k, \ldots$o‘zaro bog‘liq bo‘lmagan bir xil taqsimlangan tasodifiy miqdorlar ketma-ketligi hosil bo‘ladi.Bu tasodifiy miqdorlar uchun $M\xi_k = p, D\xi_k = pq$ ekani ma’lum .Bu tasodifiy miqdorlar n tasining o‘rta arifmetigi A hodisa ro‘y berishlarining nisbiy chastotasi $\frac{S_n}{n}$ bo‘ladi, bu yerda $S_n = \xi_1 + \xi_2 + + \cdots + \xi_k$ n ta o‘zaro bog‘liq bo‘lmagan tajribada A hodisaning ro‘y berishlari soni. Ma’lumki, $MS_n = np,\ DS_n = npq$. Bu holat uchun Bernulli teoremasi o‘rinli.

2.1.1-teorema. *Ixtiyoriy $\varepsilon > 0$ son uchun ushbu tenglik o‘rinli*

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) = 0.$$

Isbot. Chebishev tengsizligiga asosan

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) \leq \frac{np}{n\varepsilon^2}.$$

Bu tengsizlikdan $n \to \infty$ da limitga o‘tilsa, teoremaning isboti kelib chiqadi.Demak, tajribalar soni yetarlicha katta bo‘lsa, A hodisa ro‘y berishining nisbiy chastotasi A hodisaning ro‘y berish ehtimoliga yaqin bo‘lar ekan.

Faraz qilaylik, bizga $\xi_1, \xi_2, \ldots \ \xi_n, \ldots$ tasodifiy miqdorlar ketma-ketligi va

$\eta_n = f_n(\xi_1, \xi_2, \ldots \ \xi_n)$ funksiya berilgan bo‘lsin.

2.1.1-ta’rif. Agar shunday o‘zgarmas sonlar ketma-ketligi $\{a_n\},\ n = 1,2,...$ mavjud bo‘lsa va ixtiyoriy $\varepsilon > 0$ son uchun $\lim\limits_{n\to\infty} P\{|\eta_n - a_n| < \varepsilon\} = 1$ munosabat o‘rinli bo‘lsa, u holda $\xi_1, \xi_2, ...\ \xi_n, ...$ tasodifiy miqdorlar ketma-ketligi *katta sonlar qonuniga bo‘ysunadi* deyiladi.

2.1.2-teorema(Chebishev teoremasi). *Agar $\xi_1, \xi_2, ...$,o‘zaro bog‘liq bo‘lmagan tasodifiy miqdorlar bo‘lib, ularning dispersiyalari c son bilan tekis chegaralangan bo‘lsa, u holda ixtiyoriy $\varepsilon > 0$ son uchun quyidagi tenglik o‘rinli bo‘ladi:*

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j - \frac{1}{n}\sum_{j=1}^{n}M\xi_j\right| \geq \varepsilon\right) = 0,$$

$\xi_1, \xi_2, ...\ \xi_n, ...$*tasodifiy miqdorlar katta sonlar qonuniga bo‘ysunadi.*

Isbot. 2.1.1-ta’rifda $\eta_n = \frac{1}{n}\sum_{j=1}^{n}\xi_j$, $a_n = \frac{1}{n}\sum_{j=1}^{n}M\xi_j$ deb olish kerak. Chebishev tengsizligiga ko‘ra $\{\xi_j\}$ larni juft-jufti bilan bog‘liqsizligini hisobga olgan holda

$$P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j - \frac{1}{n}\sum_{j=1}^{n}M\xi_j\right| \geq \varepsilon\right\}$$

$$\leq \frac{D\left(\frac{1}{n}\sum_{j=1}^{n}\xi_j\right)}{\varepsilon^2} = \frac{\sum_{j=1}^{n}D\xi_j}{n^2\varepsilon^2}$$

tengsizlikni yozamiz. Agar $D\xi_j$ larning C bilan tekis chegaralanganligini e’tiborga olsak,

$$P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j - \frac{1}{n}\sum_{j=1}^{n}M\xi_j\right| \geq \varepsilon\right\} \leq \frac{C}{ns^2}$$

tengsizlikka ega bo‘lamiz. Bundan esa,

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j - \frac{1}{n}\sum_{j=1}^{n}M\xi_j\right| \geq \varepsilon\right\} = 0.$$

Demak, Chebishev teoremasiga ko‘ra $\xi_1, \xi_2, \dots \xi_n, \dots$ tasodifiy miqdorlar o‘zaro bog‘liqsiz va dispersiyalari chegaralangan bo‘lsa, u holda bu tasodifiy miqdorlarning o‘rta arifmetigini no‘sishi bilan bu tasodifiy miqdorlar o‘rta qiymatlarining o‘rta arifmetigiga istalgancha yaqin bo‘ladi.

Bernulli va Chebishev teoremalarining turli xil umumlashmalari yoki xususiy hollari mavjud, shulardan ba’zi birini keltiramiz. Bog‘liq bo‘lmagan tajribalar ketma-ketligida A hodisaning $k-$tajribada ro‘y berish ehtimoli p_k ga teng, ya’ni tajribaning soniga bog‘liq bo‘lsin. U holda ξ_k tasodifiy miqdorni quyidagicha kiritamiz:

$$\xi_k = \begin{cases} 0, & q_k = 1 - p_k \\ 1, & p_k. \end{cases}$$

$D\xi_k = p_k q_k \leq \frac{1}{4}$ligini e’tiborga olsak, bu $\xi_1, \xi_2, \dots \xi_n, \dots$ uchun katta sonlar qonuni o‘rinli bo‘lishini ko‘ramiz:

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{4}\sum_{j=1}^{n}\xi_j - \frac{p_1 + p_2 + \dots + p_n}{n}\right| < \varepsilon\right\} = 1.$$

2.1.3-teorema (Markov teoremasi). *Agar $\xi_1, \xi_2, \dots \xi_n, \dots$tasodifiy miqdorlar uchun $n \to \infty$ da*

$$\frac{1}{n^2} D\left(\sum_{k=1}^{n}\xi_k\right) \to 0$$

munosabat bajarilsa, $\xi_1, \xi_2, \dots \xi_n, \dots$ tasodifiy miqdorlar ketma-ketligi katta sonlar qonuniga bo‘ysunadi.

Xususan, $\xi_1, \xi_2, ...$ tasodifiy miqdorlar o'zaro bog'liq bo'lmasa va$D\xi_k < C$ bo'lsa, yuqoridagi shart bajariladi. Bu esa Chebishev teoremasi Markov teoremasining xususiy holi ekanligini ko'rsatadi.

2.1.4-teorema. *Aytaylik,* $\xi_1, \xi_2, ... -$*bog'liqsiz va chekli ikkinchi tartibli momentga ega ketma-ketlik bo'lsin. Aytaylik,* $m_k = M\xi_k$, $\sigma_k^2 = D\xi_k > 0$, $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, $D_n^2 = \sum_{k=1}^{n} \sigma_k^2$*va*$F_k = F_k(x) - \xi_k$ *tasodifiy miqdorning taqsimot funksiyasi bo'lsin.Faraz qilaylik, quyidagi hol uchun «Lindenberg sharti» bajarilsin: ixtiyoriy* $\varepsilon > 0$ *uchun*

$$\frac{1}{D_n^2}\sum_{k=1}^{n} \int_{\{x:\,|x-m_k|\geq\varepsilon D_n\}} (x-m_k)^2 dF_k(x) \to 0,\ \ n \to \infty. \quad (2.1.1)$$

U holda quyidagi o'rinli

$$\frac{S_n - MS_n}{\sqrt{DS_n}} \xrightarrow{d} N \quad (0{,}1). \qquad (2.1.2)$$

Isbot. Umumiylikni chegaralamaslik maqsadida $m_k = 0, k \geq 1$ deb hisoblashimiz mumkin. $\varphi(t) = Me^{it\xi_k}$, $T_n = \frac{S_n}{\sqrt{DS_n}} = \frac{S_n}{D_n}$, $\varphi_{S_n}(t) = Me^{itS_n}$, $\varphi_{T_n}(t) = Me^{itT_n}$.

U holda

$$\varphi_{T_n}(t) = Me^{itT_n} = Me^{i\frac{t}{D_n}S_n} = \varphi_{S_n}\left(\frac{t}{D_n}\right) = \prod_{k=1}^{n} \varphi_k\left(\frac{t}{D_n}\right) \qquad (2.1.3)$$

va (2.1.2) ni isbot qilish uchun har qanday $t \in R$ uchun

$$\varphi_{T_n} \to e^{-t^2/2}, \qquad n \to \infty \qquad (2.1.4)$$

ekanini ko'rsatish yetarli.

Ixtiyoriy $t \in R$ ni tanlaymiz va uni isbot davomida tayinlangan deb qabul qilamiz. Yoyish qoidasiga ko'ra

$$e^{iy} = 1 + iy + \frac{\theta_1 y^2}{2},$$

$$e^{iy} = 1 + iy - \frac{y^2}{2} + \frac{\theta_2 |y|^3}{3!},$$

$\theta_1 = \theta_1(y), \theta_2 = \theta_2(y)$ bilan har qanday haqiqiy shunaqa y uchunki, $|\theta_1| \leq 1$, $|\theta_2| \leq 1$ dan quyidagini topamiz.

$$\varphi_k(t) = Me^{it\xi_k} = \int_{-\infty}^{\infty} e^{itx} dF_k(x)$$

$$= \int_{|x| \geq \varepsilon D_n} \left(1 + itx + \frac{\theta_1 (tx)^2}{2}\right) dF_k(x)$$

$$+ \int_{|x| < \varepsilon D_n} \left(1 + itx - \frac{t^2 x^2}{2} + \frac{\theta_2 |tx|^3}{6}\right) dF_k(x)$$

$$= 1 + \frac{t^2}{2} \int_{|x| \geq \varepsilon D_n} \theta_1 x^2 dF_k(x)$$

$$- \frac{t^2}{2} \int_{|x| < \varepsilon D_n} x^2 dF_k(x) + \frac{|t|^3}{6} \int_{|x| < \varepsilon D_n} \theta_2 |x|^3 dF_k(x)$$

bu yerda faraz qilganimizdek $m_k = \int_{-\infty}^{\infty} x dF_k(x) = 0$ dan foydalandik.

Bundan kelib chiqadiki,

$$\varphi_k\left(\frac{t}{D_n}\right) = 1 - \frac{t^2}{2D_n^2} \int_{|x| < \varepsilon D_n} x^2 dF_k(x) + \frac{t^2}{2D_n^2} \int_{|x| \geq \varepsilon D_n} \theta_1 x^2 dF_k(x) +$$

$$+\frac{|t|^3}{6D_n^3}\int\limits_{|x|<\varepsilon D_n}\theta_2|x|^3dF_k(x). \qquad (2.1.5)$$

Quyidagiga ko‘ra,

$$\left|\frac{1}{2}\int\limits_{|x|\geq\varepsilon D_n}\theta_1 x^2 dF_k(x)\right|\leq\frac{1}{2}\int\limits_{|x|\geq\varepsilon D_n}x^2dF_k(x),$$

u holda

$$\frac{1}{2}\int\limits_{|x|\geq\varepsilon D_n}\theta_1 x^2dF_k(x)=\tilde{\theta}_1\int\limits_{|x|\geq\varepsilon D_n}x^2dF_k(x) \qquad (2.1.6)$$

Bunda $\tilde{\theta}_1=\tilde{\theta}_1(t,k,n)$ va $\left|\tilde{\theta}_1\right|\leq 1/2$.

Xuddi shunday

$$\left|\frac{1}{6}\int\limits_{|x|<\varepsilon D_n}\theta_2|x|^3dF_k(x)\right|\leq\frac{1}{6}\int\limits_{|x|<\varepsilon D_n}\frac{\varepsilon D_n}{|x|}\cdot|x|^3d\,F_k(x)$$

$$\leq\frac{1}{6}\int\limits_{|x|<\varepsilon D_n}\varepsilon D_n x^2dF_k(x),$$

demak,

$$\frac{1}{6}\int\limits_{|x|<\varepsilon D_n}\theta_2|x|^3dF_k(x)=\tilde{\theta}_2\int\limits_{|x|<\varepsilon D_n}\varepsilon D_n x^2dF_k(x) \qquad (2.1.7)$$

Bunda $\tilde{\theta}_2=\tilde{\theta}_2(t,k,n)$ va $\left|\tilde{\theta}_2\right|\leq 1/6$.

Endi quyidagilarni qo‘yamiz:

$$A_{kn} = \frac{1}{D_n^2} \int\limits_{|x|<\varepsilon D_n} x^2 dF_k(x)\,,$$

$$B_{kn} = \frac{1}{D_n^2} \int\limits_{|x|\geq\varepsilon D_n} x^2 dF_k(x).$$

U holda, (2.1.5) – (2.1.7) tenglikdan

$$\varphi_k\left(\frac{t}{D_n}\right) = 1 - \frac{t^2 A_{kn}}{2} + t^2\tilde{\theta}_1 B_{kn} + |t|^3 \varepsilon \tilde{\theta}_2 A_{kn} = 1 + C_{kn}. \qquad (2.1.8)$$

Bilamizki,

$$\sum_{k=1}^{n} (A_{kn} + B_{kn}) = 1$$

va (2.1.1) tenglikka ko‘ra

$$\sum_{k=1}^{n} B_{kn} \to 0, \;\; n \to \infty\,.$$

Shuning uchun barcha katta n lar uchun

$$\max_{1\leq k\leq n} |C_{kn}| \leq t^2\varepsilon^2 + \varepsilon|t|^3 \qquad (2.1.9)$$

va

$$\sum_{k=1}^{n} |C_{kn}| \leq t^2 + \varepsilon|t|^3 \qquad (2.1.10)$$

$|z| \geq 1/2$ ekanidan, ixtiyoriy z kompleks sonlar uchun quyidagidan foydalanamiz,

$$\ln(1 + z) = z + \theta|z|^2 \qquad (2.1.11)$$

Bunda $\theta = \theta(z), |\theta| \leq 1$ va ln logarifmning asosiy qiymatini bildiradi. Unda yetarlicha katta $n$lar uchun (2.1.8) va (2.1.11) dan yetarlicha kichik bo‘lgan $\varepsilon > 0$ dan quyidagi munosabat kelib chiqadi,

$$ln\varphi_k\left(\frac{t}{D_n}\right) = \ln(1 + C_{kn}) = C_{kn} + \theta_{kn}|C_{kn}|^2,$$

bunda $|\theta_{kn}| \leq 1$. Bundan kelib chiqadiki, (2.1.3) tenglikdan

$$\frac{t^2}{2} + ln\varphi_{T_n}(t) = \frac{t^2}{2} + \sum_{k=1}^{n} ln\varphi_k\left(\frac{t}{D_n}\right) = \frac{t^2}{2} + \sum_{k=1}^{n} C_{kn} + \sum_{k=1}^{n} \theta_{kn}|C_{kn}|^2.$$

Lekin

$$\frac{t^2}{2} + \sum_{k=1}^{n} C_{kn} = \frac{t^2}{2}\left(1 - \sum_{k=1}^{n} A_{kn}\right) +$$

$$t^2 \sum_{k=1}^{n} \tilde{\theta}_1(t,k,n)\, B_{kn} + +\varepsilon|t|^3 \sum_{k=1}^{n} \tilde{\theta}_2(t,k,n)\, A_{kn}$$

va (2.1.9) tenglikdan ixtiyoriy $\delta > 0$ uchun shunday katta $n_0$va $\varepsilon > 0$ topiladiki, barcha $n \geq n_0$ uchun

$$\left|\frac{t^2}{2} + \sum_{k=1}^{n} C_{kn}\right| \leq \frac{\delta}{2}$$

undan so‘ng, (2.1.11) va (2.1.12) dan

$$\left|\sum_{k=1}^{n} \theta_{kn}|C_{kn}|^2\right| \leq \max_{1\leq k\leq n}|C_{kn}| \sum_{k=1}^{n}|C_{kn}| \leq (t^2\varepsilon^2 + \varepsilon|t|^3)(t^2 + \varepsilon|t|^3)\ .$$

Shuning uchun yetarlicha katta n lar uchun $\varepsilon > 0$ tanlash hisobiga quyidagiga erishish mumkin

$$\left|\sum_{k=1}^{n}\theta_{kn}|C_{kn}|^2\right| \le \frac{\delta}{2}$$

va natijada

$$\left|\frac{t^2}{2} + in\varphi_{T_N}(t)\right| \le \delta.$$

Shunday qilib, har qanday haqiqiy t uchun

$$\varphi_{T_N}(t)e^{\frac{t^2}{2}} \to 1, \qquad n \to \infty$$

hosil qilamiz va natijada

$$\varphi_{T_N}(t) \to e^{\frac{-t^2}{2}}, \quad n \to \infty.$$

Teorema isbotlandi.

2.2-§.Katta sonlar qonunining bajarilishi uchun zaruriy va yetarli shart

2.2.1-teorema. *Ixtiyoriy* $\{\xi_n\}$ *tasodifiy miqdorlar ketma-ketligi katta sonlar qonuniga bo'ysunishi uchun* $n \to \infty$ *da*

$$M\frac{(\sum_{k=1}^{n}\xi_k - M\xi_k)^2}{n^2 + (\sum_{k=1}^{n}\xi_k - M\xi_k)^2} \to 0 \qquad (2.1.1)$$

Munosabatning o'rinli bo'lishi zarur va yetarli.

Isbot. Faraz qilaylik, (2.1.1) shart bajarilsin.

$$\Phi_n(x) = P\{\eta_n < x\} = P\left\{\frac{1}{n}\sum_{k=1}^{n}(\xi_k - M\xi_k) < x\right\}$$

belgilashni kiritaylik. U holda

$$P\{|\eta_n| \geq \varepsilon\} = \int\limits_{|x|>\varepsilon} d\,\Phi_n(x) \leq \frac{1+\varepsilon^2}{\varepsilon^2} \int\limits_{|x|>\varepsilon} \frac{1+x^2}{x^2} d\Phi_n(x)$$

$$\leq \frac{1+\varepsilon^2}{\varepsilon^2} \int\limits_{-\infty}^{\infty} \frac{1+x^2}{x^2} d\Phi_n(x) = \frac{1+\varepsilon^2}{\varepsilon^2} M \frac{\eta_n^2}{1+\eta_n^2}.$$

Tengsizlikga ega bo‘lamizva bu tengsizlikdan katta sonlar qonunining o‘rinli ekani kelib chiqadi.

Aytaylik katta sonlar qonuni o‘rinli bo‘lsin. Quyidagi tengsizlikni

$$P\{|\eta_n| \geq \varepsilon\} = \int\limits_{|x|>\varepsilon} d\,\Phi_n(x) \geq \int\limits_{|x|>\varepsilon} \frac{1+x^2}{x^2} d\Phi_n(x) =$$

$$= \int\limits_{-\infty}^{\infty} \frac{1+x^2}{x^2} d\Phi_n(x) - \int\limits_{|x|<\varepsilon} \frac{1+x^2}{x^2} d\Phi_n(x) \geq$$

$$\geq \int\limits_{-\infty}^{\infty} \frac{1+x^2}{x^2} d\Phi_n(x) - \varepsilon^2 = M \frac{\eta_n^2}{1+\eta_n^2} - \varepsilon^2,$$

U holda

$$0 \leq M \frac{\eta_n^2}{1+\eta_n^2} \leq \varepsilon^2 + P\{|\eta_n| \geq \varepsilon\}.$$

Bu tengsizlikning o‘ng tomonini εva$n$ ni tanlash hisobiga kichraytirish mumkin.

2.2.2-teorema(Xinchin teoremasi). *Agar ξ_1, ξ_2, ... tasodifiy miqdorlar ketma-ketligi o‘zaro bog‘liq bo‘lmagan, bir xil taqsimlangan bo‘lib, matematik kutilmalari mavjud bo‘lsa, bu tasodifiy miqdorlar ketma-ketligi uchun katta sonlar qonuni o‘rinli bo‘ladi, $M\xi_1 = a$ bo‘lganda*

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j - a\right| < \varepsilon\right\} = 1. \quad (2.2.1)$$

Isbot. Teoremaning isbotini diskret tasodifiy miqdorlar uchun ko‘rsatamiz, umumiy holda ham teoremaning isboti shunday bajariladi.$\xi_i$tasodifiy miqdorlar yordamida yangi $\bar{\xi}_i$ va $\bar{\bar{\xi}}_i$ tasodifiy miqdorlarni quyidagicha kiritamiz.

$$\bar{\xi}_i = \xi_i, \bar{\bar{\xi}}_i = 0, \text{ agar } |\xi_i| < \delta \cdot n \text{ bo‘lsa}, \quad (2.2.2)$$

$$\bar{\xi}_i = 0, \bar{\bar{\xi}}_i = \xi_i, \text{ agar } |\xi_i| \geq \delta \cdot n \text{ bo‘lsa},$$

bu yerda $i = \overline{1,n}\,,\ \delta > 0$ −tayinlangan son. U holda ixtiyoriy $i(1 \leq i \leq n)$ uchun

$$\xi_i = \bar{\xi}_i + \bar{\bar{\xi}}_i (2.2.3)$$

teorema shartiga ko‘ra

$$M|\xi_i| = \sum_{j=1}^{\infty}|x_j|P(\xi_i = x_j) = A < \infty \quad (2.2.4)$$

buni e’tiborga olsak,

$$B_n = M\bar{\xi}_i = \sum_{|x_j|<\delta n} x_j\, P(\xi_i = x_j) (2.2.5)$$

ifoda$n \to \infty$ da a ga intiladi va ixtiyoriy $\varepsilon > 0$ uchun yetarlicha katta n larda $|B_n - M\xi_i| < \varepsilon$ tengsizlik o‘rinli bo‘ladi. U holda (2.2.4) va (2.2.5) ga ko‘ra

$$D\bar{\xi}_i \leq M\xi_i^2 = \sum_{|x_j|<\delta n} x_j^2\, P(\xi_i = x_j) \leq$$

$$\leq \delta n \sum_{|x_j|<\delta n} x_j\, P(\xi_i = x_j) \leq \delta n A. \quad (2.2.6)$$

Teorema shartlariga ko‘ra $\{\xi_i\}$lar o‘zaro bog‘liq bo‘lmagani uchun $\{\bar{\xi}_i\}$ lar ham o‘zaro bog‘liq bo‘lmaydi, u holda Chebishev tengsizligi va (2.2.6) ga ko‘ra

$$P\left\{\left|\frac{\bar{\xi}_1+\cdots+\bar{\xi}_n}{n}-B_n\right|\geq\varepsilon\right\}\leq\frac{D\bar{\xi}_i}{n\varepsilon^2}\leq\frac{A\delta}{\varepsilon^2}.$$

(2.2.6) ni e’tiborga olgan holda oxirgi tengsizlikdan quyidagi ifodani hosil qilamiz:

$$P\left\{\left|\frac{\bar{\xi}_1+\cdots+\bar{\xi}_n}{n}-M\xi\right|\geq 2\varepsilon\right\}\leq\frac{A\delta}{\varepsilon^2}. \quad (2.2.7)$$

Ikkinchi tomondan, (2.2.4) ni hisobga olgan holda

$$P(\bar{\bar{\xi}}_j\neq 0)=\sum_{|x_j|>\delta n}P(\xi_i=x_j)\leq\frac{1}{\delta n}\sum_{|x_j|>\delta n}|x_j|\,P(\xi_i=x_j)$$

tengsizlik hosil bo‘ladi va matematik kutilmasi mavjudligi sababli $n\to\infty$ da

$$P(\bar{\bar{\xi}}_1+\bar{\bar{\xi}}_2+\cdots+\bar{\bar{\xi}}_n\neq 0)\leq\sum_{k=1}^{n}P(\bar{\bar{\xi}}_j\neq 0)\leq\delta \quad (2.2.8)$$

hosil bo‘ladi. O‘z navbatida

$$\xi_1+\xi_2+\cdots+\xi_n=(\bar{\xi}_1+\bar{\xi}_2\cdots+\bar{\xi}_n)+(\bar{\bar{\xi}}_1+\bar{\bar{\xi}}_2+\cdots+\bar{\bar{\xi}}_n)$$

tenglikni e’tiborga olsak, (2.2.7) va (2.2.8) ga ko‘ra

$$P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j-a\right|\geq 2\varepsilon\right\}\leq P\left\{\left|\frac{\bar{\xi}_1+\cdots+\bar{\xi}_n}{n}-a\right|\geq 2\varepsilon\right\}+$$

$$+P(\bar{\bar{\xi}}_1+\bar{\bar{\xi}}_2+\cdots+\bar{\bar{\xi}}_n\neq 0)\leq\frac{A\delta}{\varepsilon^2}+\delta.$$

εvaδ larning ixtiyoriyligidan tenglama isbotlandi.

2.3-§.Katta sonlarning kuchaytirilgan qonuni

Ixtiyoriy $\varepsilon > 0$va$\eta > 0$ lar uchun shunday n_0 sonni topish mumkin bo‘lsa, ixtiyoriy $s > 0$ va ushbu $n_0 \leq n \leq n_0 + s$ tengsizlikni qanoatlantiruvchi barcha barcha n larda

$$P\left\{\max_{n_0 \leq n \leq n_0+s}\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k - \frac{1}{n}\sum_{k=1}^{n}M\xi_k\right| < \varepsilon\right\} > 1-\eta \qquad (2.3.1)$$

tengsizliko‘rinli bo‘lsa, $\{\xi_n\}$ tasodifiy miqdorlar ketma-ketligi *“katta sonlarning kuchaytirilgan qonuni”ga bo‘ysunadi* deyiladi.Katta sonlar qonuni bilan katta sonlarning kuchaytirilgan qonuni o‘zaro farq qiladi. Agar

$$\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k - \frac{1}{n}\sum_{k=1}^{n}M\xi_k\right| < \varepsilon$$

tengsizlikning$n \in [n_0, n_0 + s]$ ga nisbatan tekis bajarilishi talab qilinsa, katta sonlarning “kuchaytirilgan qonuni” ga aksincha,

$$\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k - \frac{1}{n}\sum_{k=1}^{n}M\xi_k\right| < \varepsilon$$

tengsizlikning yetarlicha katta n da bajarilishi talab qilinsa, oddiy “katta sonlar qonuni”ga ega bo‘lamiz.

Kolmogorov tengsizligi. *Agar o‘zaro bog‘liq bo‘lmagan*$\xi_1, \xi_2, \dots \xi_n, \dots$ *tasodifiy miqdorlar chekli* $D\xi_k$ $(k = 1, 2, \dots)$ *dispersiyalarga ega bo‘lsa, ushbu tengsizlik o‘rinli bo‘ladi:*

$$\mathrm{P}\left\{\max_{k=\overline{1,n}}\left|\sum_{k=1}^{n}(\xi_k - M\xi_k)\right| < \varepsilon\right\} \geq 1 - \frac{1}{\varepsilon^2}\sum_{k=1}^{n}D\xi_k$$

bu yerda $\varepsilon > 0$ $-$ixtiyoriy son.

Isbot. Endi$\eta_k = \xi_k - M\xi_k,\ S_k = \sum_{j=1}^{n}\eta_j,$

$$E_k = \{\omega: |S_i| < \varepsilon, \qquad i \leq k-1\} \cap \{\omega: |S_k| \geq \varepsilon\},$$

$$E_0 = \{\omega: |S_j| < \varepsilon, \qquad j < n\}$$

belgilashni kiritamiz, u holda

$$\left\{\omega: \max_{k=\overline{1,n}} |S_k| \geq \varepsilon\right\} = \bigcup_{k=1}^{n} E_k$$

bo‘ladi. Bu tenglikdan$E_i \cap E_j = V,\ i \neq j$ bo‘lganligidanquyidagiga ega bo‘lamiz.

$$P\left\{\max_{k=\overline{1,n}} |S_k| \geq \varepsilon\right\} = \sum_{k=1}^{n} P(E_k). \tag{2.3.2}$$

Endi $S_n$ning dispersiyasini hisoblaylik:

$$DS_n = \sum_{k=0}^{n} P(E_k) \cdot M(S_n^2/E_k) \geq \sum_{k=1}^{n} P(E_k) \cdot M(S_n^2/E_k),$$

bu yerda

$$M(S_n^2/E_k) = M\left\{S_k^2 + 2\sum_{i>k} S_k\eta_i + \sum_{i>k} \eta_i^2 + 2\sum_{i>n>k} \eta_i\eta_n/E_k\right\}$$
$$\geq M\left\{S_k^2 + 2\sum_{i>k} S_k\eta_i + 2\sum_{i>n>k} \eta_i\eta_n/E_k\right\}$$

$E_k$ning ro‘y berishi $\xi_i,\ i = \overline{1,k}$ largagina ta’sir etadi, lekin boshqa $\xi_i, i = \overline{k+1,n}$ tasodifiy miqdorlarga ta’sir qilmaydi, chunki $\xi_i, i = \overline{k+1,n}$ tasodifiy miqdorlar o‘zaro bog‘liq emasligicha qoladi.Shuning uchun

$$M(S_k \cdot \eta_j/E_k) = M(S_k/E_k) \cdot M(\eta_j/E_k) = 0$$

va

$$M(\eta_i\eta_h/E_k) = 0,$$

bu yerda $h \neq j,\ h > k,\ i \geq k \geq 1$. Bundan tashqari $k \geq 1$ bo‘lganda

$$M(S_n^2/E_k) = \frac{1}{P(E_k)} \int\limits_{E_k} x^2 dF(x) \geq \frac{\varepsilon^2}{P(E_k)} \int\limits_{E_k} dF(x) = \varepsilon^2.$$

Shu sabablarga ko‘ra

$$DS_n \geq \varepsilon^2 \sum_{k=0}^{n} P(E_k) \quad (2.3.3)$$

(2.3.2) va (2.3.3) dan

$$P\left\{\max_{k=\overline{1,n}} |S_k| \geq \varepsilon\right\} = \sum_{k=0}^{n} P(E_k) \leq \frac{DS_n}{\varepsilon^2}.$$

Shu bilan Kolmogorov tengsizligi ham isbotlandi.

Tasodifiy miqdorlar ketma-ketligi qanday shartlar qo‘yilganda katta sonlarning kuchaytirilgan qonuni o‘rinli bo‘ladidegan savolga Kolmogorov teoremalari javob beradi.

2.3.1-teorema. *O‘zaro bog‘liq bo‘lmagan tasodifiy miqdorlar ketma-ketligi* $\xi_1, \xi_2, \dots \xi_n, \dots (M\xi_k = 0,\ k = \overline{1,\ \infty})$ *katta sonlarning kuchaytirilgan qonuniga bo‘ysunishi uchun ushbu*

$$\sum_{n=1}^{\infty} \frac{D\xi_n}{n^2} < +\infty$$

munosabatning bajarilishi yetarli.

Isbot. Faraz qilaylik,

$$S_n = \sum_{k=1}^{n} \xi_k, \quad D\xi_k = \sigma^2$$

bo‘lsin. U holda ixtiyoriy $\varepsilon > 0$ uchun $n \to \infty$ da

$$P\left(\sup_{k\geq n}\left|\frac{S_k}{k}\right| > \varepsilon\right) \to 0$$

ekanini ko‘rsatsak, teoremani isbot qilgan bo‘lamiz.

Agar

$$A_n = \left\{\omega\colon \max_{2^{n-1}\leq k<2^n}\left|\left|\frac{S_k(\omega)}{k}\right| > \varepsilon\right|\right\} \subset \bigcup_{m=l}^{\infty} A_m$$

bo‘ladi. Bundan

$$P\left(\sup_{k\geq n}\left|\frac{S_k}{k}\right| > \varepsilon\right) \leq P\left(\bigcup_{m=l}^{\infty} A_m\right).$$

Bu yerda $l$ni$2^l \leq n < 2^{k+1}$ dan aniqlanadi. Ko‘rinib turibdiki $l \to \infty$ dan $n \to \infty$ kelib chiqadi. Endi$l \to \infty$ da $P(\bigcup_{m=l}^{\infty} A_m) \to 0$ekanini ko‘rsatamiz. Kolmogorov tengsizligidan foydalanib,

$$P(A_n) \leq P\left(\max_{2^{n-1}\leq k<2^n}|S_k| > \varepsilon\, 2^{n-1}\right) \leq$$

$$\leq P\left(\left|\tilde{S}_2^n\right| > \varepsilon\, 2^{n-1}\right) \leq 4\frac{DS_n}{\varepsilon^2 2^{2n}}$$

ga ega bo‘lamiz. Bu ifodada $\tilde{S}_n = \max(|S_1|, |S_2|, \dots, |S_n|)$. Oldingi tengsizlikning ixtiyoriy $n \in N$ da o‘rinli ekanidan ushbu tengsizlikning o‘rinli ekanini ko‘ramiz:

$$\sum_{k=1}^{\infty} P(A_k) \leq 4\varepsilon^{-2}\sum_{k=1}^{\infty} 2^{-2k}\sum_{j=1}^{2^k}\sigma_j^2 \leq$$

$$\leq 4\varepsilon^{-2}\sum_{j=1}^{2^k}\sigma_j^2 \sum_{\{k:2^k\geq j\}} 2^{-2k} \leq 4\varepsilon^{-2}\sum_{j=1}^{\infty}\sigma_j^{2\frac{2}{j^2}} < +\infty.$$

Bu esa $\sum_{k=1}^{\infty} P(A_k)$ qatorning yaqinlashuvchi ekanligini ko‘rsatadi. Demak $l \to \infty$ da

$$P\left(\bigcup_{m=l}^{\infty} A_m\right) \leq \bigcup_{m=l}^{\infty} P(A_m) \to 0.$$

Teorema isbotlandi.

2.3.1-natija. Agar o‘zaro bog‘liq bo‘lmaganξ_k tasodifiy miqdorlarning dispersiyalari bitta Co‘zgarmas son bilan chegaralangan bo‘lsa, $\xi_1, \xi_2, \dots \xi_n, \dots$ tasodifiy miqdorlar ketma-ketligi katta sonlarning kuchaytirilgan qonuniga bo‘ysunadi.

Bir xil taqsimlangan tasodifiy miqdorlar ketma-ketligi uchun katta sonlarning kuchaytirigan qonuni o‘rinli bo‘lishi uchun zarur va yetarli shartni Kolmogorov quyidagi teoremada keltirgan.

2.3.2-teorema. *O‘zaro bog‘liq bo‘lmagan bir xil taqsimlangan tasodifiy miqdorlar ketma-ketligi katta sonlarning kuchaytirilgan qonuniga bo‘ysunishi uchun ular matematik kutilmasining mavjud bo‘lishi zarur va yetarlidir*.

Faraz qilaylik, biror $\xi_1, \xi_2, \dots \xi_n, \dots$tasodifiy miqdorlar ketma-ketligi berilgan bo‘lsin.Katta sonlarning kuchaytirilgan qonuniga oid Markov teoremasini keltiramiz.Bu teorema Kolmogorov teoremasining xususiy holi hisoblanadi.Avval quyidagi lemmani isbotlaylik.

2.3.1-lemma. *Agar ixtiyoriy butun musbat r da*

$$\sum_{n=1}^{\infty} P\left\{|\xi_n - \xi| \geq \frac{1}{r}\right\} < \infty \qquad (2.3.4)$$

bajarilsa, u holda

$$P\{\xi_n \to \xi\} = 1 \qquad (2.3.5)$$

yoki

$$P\{\xi_n \nrightarrow \xi\} = 0$$

bo'ladi.

Isbot. Quyidagi belgilashni kiritamiz:

$$E_n^r = \left\{|\xi_n - \xi| \geq \frac{1}{r}\right\}$$

va

$$S_n^r = \sum_{k=1}^{\infty} E_{n+k}^r.$$

Ushbu

$$P(S_n^r) \leq \sum_{k=1}^{\infty} P\{E_{n+k}^r\} = \sum_{c=n+1}^{\infty} P\left(|\xi_n - \xi| \geq \frac{1}{r}\right)$$

munosabat va (2.3.4) ga asosan

$$\lim_{n\to\infty} P(S_n^r) = 0 \tag{2.3.6}$$

kelib chiqadi. Endi faraz qilaylik,

$$S^r = S_1^r \cdot S_2^r \cdot S_3^r \cdot \ldots$$

Biroq S_n^r hodisa ixtiyoriy S^r hodisani o'z ichiga olgani sababli (2.3.3) ga ko'ra quyidagini hosil qilamiz:

$$P(S^r) = 0 \tag{2.3.7}$$

Ushbu $S = S^1 + S^2 + \cdots$ hodisani ko'raylik. Bu hodisa quyidagi tenglikning bajarilishini bildiradi:

$$|\xi_{n+k} - \xi| \geq \frac{1}{r}, \quad P(S) \leq \sum_{r}^{\infty} P(S^r)$$

bo'lgani uchun (2.3.7) ga ko'ra $P(S) = 0$.

2.3.3-teorema(Borel teoremasi). *n ta o'zaro bog'liq bo'lmagan tajribalarda A hodisaning ro'y berishlari soni μ, bu tajribalarning har birida A hodisaning ro'y berish ehtimoli p ga, $P(\bar{A}) = q = 1 - p$ bo'lsin, u holda $n \to \infty$ da*

$$P\left(\frac{\mu}{n} - p \to 0\right) = 1.$$

Isbot. A hodisaning i −tajribada ro'y berishlari sonini μ_i bilan belgilaymiz. U holda

$$M\left(\frac{\mu}{n} - p\right)^4 = \frac{1}{n^4}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M\,(\mu_i - p)(\mu_j - p)(\mu_k - p)(\mu_l - p).$$

Elementar hisoblar

$$M\left(\frac{\mu}{n} - p\right)^4 = \frac{np}{n^4}[n(p^3 + q^3) + 3pq(n^2 - n)] < \frac{1}{4n^2}$$

munosabatning to'g'riligini ko'rsatadi. Bu tengsizlik va Chebishev tengsizligiga ko'ra

$$P\left\{\left|\frac{\mu}{n} - p\right| \geq \frac{1}{r}\right\} \leq r^4 M\left(\frac{\mu}{n} - p\right)^4 < \frac{r^4}{4n^2}.$$

Bundan,

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{\mu}{n} - p\right| \geq \frac{1}{r}\right\}.$$

Qatorning yaqinlashishi kelib chiqadi.Bu esa 1-lemmaga asosan, Borel teoremasining to'g'riligini ko'rsatadi.

2.4-§.Yaqinlashish turlari

Biz tasodifiy miqdorlarni bitta $< \Omega, F, P >$ ehtimollik fazosida berilgan deb faraz qilamiz. O'lchovli funksiyalar nazariyasidan ma'lumki, o'lchovli funksiyalar ustida qo'shish, ayirish, ko'paytirish va(maxraji 0 dan farqli bo'lsa) bo'lish amali bajarish natijasida hosil bo‘ladigan funksiya yana o‘lchovli, shu bilan birga o‘lchovli funksiyalar ketma-ketligininglimiti (agar mavjud bo‘lsa) yana o‘lchovli bo‘ladi. Shunga o‘xshash natijalar tasodifiy miqdorlar uchun ham o‘rinli.Tasodifiy miqdorlar ketma-ketligining yaqinlashishi masala talabiga qarab turlicha bo‘lishi mumkin.

2.4.1-ta’rif. Agar ixtiyoriy musbat $\varepsilon > 0$ son uchun $n \to \infty$ da $P\{\xi_n - \xi\} \to 0$ bo‘lsa, u holda $\xi_1, \xi_2, \dots\ \xi_n, \dots$ tasodifiy miqdorlar ketma-ketligi p *ehtimol bo‘yicha* ξ *tasodifiy miqdorga yaqinlashadi* deyiladi va

$$\xi_n \xrightarrow{p} \xi$$

kabi belgilanadi.

Aytaylik, g ixtiyoriy uzluksiz, chegaralangan funksiya bo‘lsin. Agar $\xi_n \xrightarrow{p} \xi$bo‘lsa, u holda

$$Mg(\xi_n) \to Mg\xi. \tag{2.4.1}$$

Agar $\xi_n$vaξ larning taqsimot funksiyalarini mos ravishda $F_n(x)$ va $F(x)$ deb belgilasak, u holda (2.4.1) ni quyidagicha yozamiz.

$$\int_{-\infty}^{\infty} g(x)dF_n(x) \to \int_{-\infty}^{\infty} g(x)dF(x). \tag{2.4.2}$$

2.4.2-ta’rif. Agar $\xi_1, \xi_2, \dots\ \xi_n, \dots$ tasodifiy miqdorlar ketma-ketligi uchun

$$P\left\{\omega: \lim_{n\to\infty} \xi_n(\omega) = \xi(\omega)\right\} = 1$$

tengliko‘rinli bo‘lsa, u holda $\xi_1, \xi_2, \dots\ \xi_n, \dots$ tasodifiy miqdorlar ketma-ketligi ξ *tasodifiy miqdorga* 1 *ehtimol bilan yaqinlashadi* deyiladi, ya’ni bunday yaqinlashish uchun $\lim\limits_{n\to\infty} \xi_n(\omega) = \xi(\omega)$ munosabatni qanoatlantirmaydigan ω nuqtalarning o‘lchovi nolga teng bo‘ladi.

Bir ehtimol bilan yaqinlashishni $\xi_n \xrightarrow{p(1)} \xi$ kabi belgilaymiz. Bir ehtimol bo‘yicha yaqinlashish

$$\lim_{n\to\infty} P\left\{\omega: \left(\sup_{m\geq n}|\xi_n(\omega) - \xi(\omega)| > \varepsilon\right)\right\} = 0$$

ga teng kuchlidir.

2.4.3-ta’rif. Agar $n \to \infty$ da $M\{\xi_n - \xi\}^r \to 0$ shart bajarilsa, $\{\xi_n\}$ tasodifiy miqdorlar ketma-ketligi ξ ga *o‘rtacha* $r-$ *tartibda yaqinlashadi* deyiladi.Bu yaqinlashishni $\xi_n \xrightarrow{r} \xi$ kabi belgilanadi.Xususan $r = 2$ da bu yaqinlashish *o‘rta kvadratik yaqinlashish* deyiladi.

Bir ehtimol bo‘yicha yaqinlashishdan ehtimol bo‘yicha yaqinlashish kelib chiqadi, lekin teskarisi har doim ham o‘rinli emasligini quyidagi misoldan ko‘rish mumkin.

2.4.1-misol. Faraz qilaylik, $\Omega = [0,1]$, $\mathcal{F}$ esa Borel to‘plamlarining σ algebrasi, $P -$ Lebeg o‘lchovi bo‘lsin va

$$A_n^k = \left[\frac{k-1}{n}, \frac{k}{n}\right],$$

$$\chi_n^k = I_{A_n^k}(\omega) = \begin{cases} 1, & agar\ \ \omega \in A_n^k\ \ bo'lsa \\ 0, & agar\ \ \omega \notin A_n^k\ \ bo'lsa \end{cases}$$

(bu yerdagi $I_{A_n^k}(\omega)$ ni A_n^k to‘plamning indikatori deymiz). U holda

$$\{\chi_2^1,\ \chi_2^2,\ \chi_3^1,\ \chi_3^2,\ \chi_3^3, \dots\}.$$

Tasodifiy miqdorlar ketma-ketligi ehtimol bo‘yicha nolga intiladi, biroq hech bir $\omega \in [0,1]$ nuqtada limitga ega emas. Endi o‘rtacha $r-$ tartibda yaqinlashishga to‘xtalib o‘tamiz. Chebishev tengsizligiga asosan

$$P\{|\xi_n - \xi| \geq \varepsilon\} \leq \frac{M|\xi_n - \xi|^r}{\varepsilon^2}.$$

Bu tengsizlik esa $r-$ tartibda yaqinlashishdan ehtimol bo‘yicha yaqinlashish kelib chiqishini ko‘rsatadi.

Bizga $\{\xi_n\}$ tasodifiy miqdorlar ketma-ketligi berilgan bo‘lib, $F_n(x)=P\{\xi_n<x\}$ bo‘lsin.

2.4.4-ta’rif. Agar $\{F_n(x)\}$ taqsimot funksiyalar ketma-ketligi $n\to\infty$ da $F(x)$ taqsimot funksiyaning har bir uzluksiz nuqtasida $F(x)=P\{\xi<x\}$ ga yaqinlashsa, u holda $\{\xi_n\}$ tasodifiy miqdorlar ketma-ketligi ξ ga *taqsimot bo‘yicha yaqinlashadi* deyiladi va$\xi_n\xrightarrow{D}\xi$ kabi belgilanadi (bu yerda D inglizcha “distribution” – taqsimot so‘zining bosh harfidan olingan).

Aytaylik $\mathcal{H}=\{H\}$ to‘plam $H=H(x)$ funksiyalardan iborat sinf bo‘lib, bu to‘plamdagi funksiyalar quyidagi shartlarni qanoatlantirsin:

a) $H(x)-$kamaymaydigan funksiya;
b) $H(-\infty)=0,\ \mathrm{H}(+\infty)\le 1$;
c) $H(x)$chapdan uzluksiz funksiya.

Biz $\mathcal{F}=\{F\}$ deb H sinfning shunday qism to‘plamini olamizki, bunda $F(+\infty)=1$ ya’ni $F(x)\xi$ tasodifiy miqdor taqsimot funksiyaning xuddi o‘zi.

2.4.5-ta’rif. Agar ixtiyoriy uzluksiz va chegaralangan $h(x)$ funksiya uchun

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}h(x)dH_n(x)=\int_{-\infty}^{+\infty}h(x)dH(x)$$

tenglik o‘rinli bo‘lsa, $\mathcal{H}\ni H_n$ funksiyalar ketma-ketligi $\mathcal{H}\ni H$ funksiyaga *sust yaqinlashadi* deyiladi va qisqacha $H_n\xrightarrow{W}H^*$ kabi belgilanadi,bu yerda W harfi inglizcha “Weak” – “sust” so‘zining bosh harfidan olingan.

2.4.1-teorema. *Faraz qilaylik,* $F_n,\ F\in\mathcal{F}$, *u holda*

$$\left(F_n\xrightarrow{W}F\right)\Leftrightarrow\left(F_n\xrightarrow{D}F\right).$$

Bu teoremani $\mathcal{H}$ sinfdagi funksiyalar uchun quyidagicha bayon etish mumkin.

2.4.2-teorema.*Faraz qilaylik,* $H_n,\ H\in\mathcal{H}$, *u holda*

1) $\left(H_n \xrightarrow{W} H\right) \Leftrightarrow \left(H_n \xrightarrow{D} H\right)$;

2) $H_n \xrightarrow{D} H,\ H_n(+\infty) \to H(+\infty) \Longrightarrow H_n \xrightarrow{W} H$.

III BOB. HARDY TENGSIZLIKLARI VA KATTA SONLAR QONUNI

Ushbu bobda dissertatsiyaning asosiy olingan natijalarni keltiramiz.

3.1-§. Umumlashgan Hardi tengsizligiga oid ba'zi natijalar

Umulashgan Hardi tengsizligining umumiy ko'rinishi quyidagi ko'rinishga ega:

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n} K(n,m)\, x_m\right|^{q} u_n\right)^{\frac{1}{q}} \le C\cdot\left(\sum_{n=1}^{\infty}|x_n|^{p} v_n\right)^{\frac{1}{p}}. \qquad (3.1.1)$$

Bu tengsizlikni o'rganishda asosiy masala, tengsizlikda qatnashuvchi yadro $\{K(n,m)\}$va$\{u_n\},\{v_n\}$ ketma-ketliklar, hamda, p,q haqiqiy sonlar uchun shunday shart topish kerakki u (3.1.1) tengsizlikning bajarishini to'liq taminlasin, ya'ni, shart bajarilganda (3.1.1) o'rinli bo'lsin va teskarisi. (3.1.1) tengsizlik shu vaqtgacha juda ko'p olimlar tomonidan o'rganilgan bo'lib, deyarli barcha ilmiy ishlar yadro musbatligi talab qilinadi. Ushbu ishda bizlar bu yadro o'z ishorasini almashtirishi mumkin bo'lgan holda tengsizlik bajarilishi uchun zaruriy va yetarli shart topdik.

Faraz qilaylik, yadro $p=q$ va $\{K(n,m)\}$ quyidagi ko'rinishga ega bo'lsin:

$$K(n,m)=\sum_{i=0}^{l}\sum_{j=0}^{r} a_{ij} n^{i} m^{j}.$$

- Agar bu yerda $l=r=0$ bo'lsa, u holda yadro $K(n,m)=a_{00}$ bo'ladi. Bu holda (3.1.1) tengsizlik ushbu ko'rinishga ega bo'lib

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n} x_m\right|^{p} u_n\right)^{\frac{1}{p}} \le \frac{C}{|a_{00}|}\cdot\left(\sum_{n=1}^{\infty}|x_n|^{p} v_n\right)^{\frac{1}{p}}, \qquad (3.1.2)$$

bu tengsizlik o'rinli bo'lishi uchun quyidagi shartning bajarilishi yetarli va zarurdir:

$$\sup_{n\in N}\left(\sum_{m=1}^{n} v_m^{1-p'}\right)^{p-1}\cdot\left(\sum_{m=n}^{\infty} u_m\right)<\infty.$$

- Agar $l=1$ va $r=1$ bo'lsa yadroning ko'rinishi

$$K(n,m)=a_{00}+a_{10}n+a_{01}m+a_{11}nm$$

kabi bo'ladi. U holda (1) ushbu ko'rinishga ega bo'ladi:

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\,x_m\right|^p u_n\right)^{\frac{1}{p}}$$

$$\leq C\cdot\left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}}. \qquad (3.1.3)$$

Quyida bizlar shu tengsizlikning bajarishilkishi bilan shugullanamiz:

1-hol. Faraz qilaylik yadro quyidagi xossaga ega bo'lsin, ya'ni shunday b_0, b_1, b_2, b_3 sonlar mavjudki

$$a_{00}+a_{10}n+a_{01}m+a_{11}nm=(b_0+b_1n)(b_2+b_3m)$$

tenglik o'rinli. Ma'lumki, bu tenglik bajarilishi uchun ushbu shart bajarilish kerak bo'ladi:

$$a_{00}a_{11}=a_{10}a_{01}.$$

Demak, bu holda (3.1.3) tengsizlik ushbu ko'rinishga ega bo'ladi

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(b_0+b_1n)(b_2+b_3m)\,x_m\right|^p u_n\right)^{\frac{1}{p}}$$

$$\leq C\cdot\left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}}. \qquad (3.1.4)$$

Endi bu yerda $\mathrm{y_n} = (\mathrm{b_2} + \mathrm{b_3 m}) \cdot \mathrm{x_m}, \bar{\mathrm{u}}_\mathrm{n} = (\mathrm{b_0} + \mathrm{b_1 n}) \cdot \mathrm{v_n}$ va $\bar{\mathrm{v}}_\mathrm{n} = \mathrm{v_n} \cdot |\mathrm{b_2} + \mathrm{b_3 m}|^{-\mathrm{p}}$ deb belgilash kiritsak (3.1.4) tengsizlik quyidagi ko'rinishni oladi:

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n} y_m\right|^p \bar{u}_n\right)^{\frac{1}{p}} \leq C \cdot \left(\sum_{n=1}^{\infty}|y_n|^p \bar{v}_n\right)^{\frac{1}{p}}. \qquad (3.1.5)$$

Demak bu holda (3.1.3) tengsizligimiz (3.1.2) ko'rinishni olar ekan. (3.1.2) tengsizligimizning bajarilish uchun keltirilgan shartdan foydalansak, quyidagi natijaga ega bolamiz:

3.1.1-teorema. *Faraz qilaylik (*3.1.*2) tengsizlikdagi yadro uchun* $a_{00}a_{11} = a_{10}a_{01}$ *shart qanoatlantirsin. U holda bu tengsizlik o'rinli bo'lishi uchun*

$$\sup_n \left(\sum_{m=1}^{n}|b_2 + b_3 m|^{p'} v_m^{1-p}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty}|b_0 + b_1 m| \cdot u_m\right) < \infty$$

shartning bajarilishi zarur va yetarlidir.

Bu ishda bizlar (3.1.3) dagi yadro koeffisientlari ixtiyoriy bo'lgan holda qaraymiz.

Izoh. Yadroning bunday ko'rinishda qaralganining sababi, bu yadro Green yadrosi bo'lib Hardi tengsizligining uzluksiz holda juda muhim yadrolardan hisoblanadi. Bunda $a_{11} \neq 0$ deb faraz qilinadi.

Quyidagi teorema o'rinli.

3.1.2-teorema. (3.1.3) tengsizlik o'rinli bo'lishi uchun

$$A = \sup_{n \in N}\left(\sum_{m=1}^{n}|a_{10} + a_{11} m|^{p'} v_m^{1-p'}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m\right) < \infty$$

shartning bajarilishi zarur va yetarlidir.

Isbot. [46] (Yetarliligi) (3.1.3)ning chap tomonini normaning uchburchak tengsizligidan foydalanib quyidagicha

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\,x_m\right|^p u_n\right)^{\frac{1}{p}}$$

$$\leq\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00}+a_{01}m)\,x_m\right|^p u_n\right)^{\frac{1}{p}}$$

$$+\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}n(a_{10}+a_{11}m)\,x_m\right|^p u_n\right)^{\frac{1}{p}}$$

baholash mumkin. Bu ko'rsatadiki tengsizlikning o'rinli bo'lishi uchun ushbu tengsizliklarning har biri o'rinli bo'lishi yetarli ekan

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00}+a_{01}m)\,x_m\right|^p u_n\right)^{\frac{1}{p}}\leq C\cdot\left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}} \tag{3.1.6}$$

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}n(a_{10}+a_{11}m)\,x_m\right|^p u_n\right)^{\frac{1}{p}}\leq C\cdot\left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}}. \tag{3.1.7}$$

(3.1.6) va (3.1.7) tengsizliklarda mos ravishda $y_m=(a_{00}+a_{01}m)x_m$ va $z_m=(a_{10}+a_{11}m)x_m$ almashtirishlar olsak u holda quyidagi tengsizliklarni hosil qilamiz

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}y_m\right|^p u_n\right)^{\frac{1}{p}}\leq C\cdot\left(\sum_{n=1}^{\infty}|y_n|^p\frac{v_n}{|a_{00}+a_{01}n|^p}\right)^{\frac{1}{p}} \tag{3.1.8}$$

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n} z_m\right|^p n^p u_n\right)^{\frac{1}{p}} \le C \cdot \left(\sum_{n=1}^{\infty} |z_n|^p \frac{v_n}{|a_{10} + a_{11}n|^p}\right)^{\frac{1}{p}}. \qquad (3.1.9)$$

(3.1.8) va (3.1.9) lar Hardining klassik tengsizligi ko'rinishida bo'lib bu tengsizlik uchun yetarlishart Muckenhoupt tomonidan topilgan ([44-43] qarang). O'sha shartlardan foydalanadigan bo'lsak (3.1.6) va (3.1.7) ular uchun yetarlishartlar mos ravishda quyidagi ko'rinishni oladi

$$A^1 = \sup_{n \in N}\left(\sum_{m=1}^{n} |a_{00} + a_{01}m|^{p'} v_m^{1-p'}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} u_m\right) < \infty$$

$$A^2 = \sup_{n \in N}\left(\sum_{m=1}^{n} |a_{10} + a_{11}m|^{p'} v_m^{1-p'}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m\right) < \infty.$$

Bu yerda ko'rinadiki, agar A^2 chekli bo'lsa u holda A^1 ham chekli bo'ladi, sababi, A^2 ning ikkinchi yig'indi qatnashgan ifodasida qo'shimcha m^p had uchraydi, bu esa m ning yetarlicha katta qiymatlarida yig'indi A^1 uchraydigan huddi shunday yig'indidan ancha katta bo'lishini ko'rsatadi. Demak, (3.2.3) tengsizlik bajarilishi uchun $A^2 < \infty$ shartning bajarilishi yetarli ekan. Bundan osongina teorema shartining yetarli ekanligi kelib chiqadi.

(Zaruriyligi) [46] Endi bizlar teorema shartining zarur ekanligini ko'rsatamiz. Faraz qilaylik, (3.1.3) tengsizlik bajarilgan bo'lsin. Umumiylikka zarar keltirmasdan $a_{11} > 0$ deyishimiz mumkin. U holda shunday $n_0 \in N$ son mavjudki

$$a_{00} + a_{10}n + a_{01}m + a_{11}nm \ge \frac{a_{11}nm}{2}$$

baho barcha $n \ge n_0$ va $1 \le m \le n$ larda o'rinli bo'ladi. (3.1.3) ning o'rinli ekanligidan ushbu tengsizlikning ham bajarilishi kelib chiqadi

$$\left(\sum_{n=n_0}^{\infty}\left|\sum_{m=1}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\,x_m\right|^p u_n\right)^{\frac{1}{p}} \le C\cdot\left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}}.$$

Faraz qilaylik $\{x_m\}_{m=0}^{\infty}$ ketma-ketlikning boshlang'ich n_0 ta hadi nolga teng bo'lsin. U holda yuqoridagi tengsizlik quyidagi ko'rinishni oladi

$$\left(\sum_{n=n_0}^{\infty}\left|\sum_{m=n_0}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\,x_m\right|^p u_n\right)^{\frac{1}{p}} \le C\cdot\left(\sum_{n=n_0}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}}.$$

Demak, (3.1.3) tengsizlik o'rinli bo'lishidan shunday n_0 son mavjudligi va yuqoridagi tengsizlikning o'rinli bo'lishi kelib chiqar ekan. Endi tengsizlikdagi ketma-ketlikni xususiy holda musbat deb qaraylik. U holda yuqoridagi tengsizlikning chap tomonini quyidagicha baholash mumkin

$$\left(\sum_{n=n_0}^{\infty}\left|\sum_{m=n_0}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\,x_m\right|^p u_n\right)^{\frac{1}{p}}$$

$$\ge \frac{a_{11}}{2}\left(\sum_{n=n_0}^{\infty}\left(\sum_{m=n_0}^{n} nm\,x_m\right)^p u_n\right)^{\frac{1}{p}}.$$

Demak, (3.1.3) o'rinli bo'lsa u holda

$$\left(\sum_{n=n_0}^{\infty}\left(\sum_{m=n_0}^{n} nm\,x_m\right)^p u_n\right)^{\frac{1}{p}} \le \frac{2C}{a_{11}}\cdot\left(\sum_{n=n_0}^{\infty} x_n^p v_n\right)^{\frac{1}{p}}$$

ham musbat ketma-ketliklar uchun o'rinli bo'lishi kelib chiqar ekan. Bu yerda $y_m = mx_m$ deb almashtirish olsak, u holda

$$\left(\sum_{n=n_0}^{\infty}\left(\sum_{m=n_0}^{n} y_m\right)^p n^p u_n\right)^{\frac{1}{p}} \leq \frac{2C}{a_{11}}\cdot\left(\sum_{n=n_0}^{\infty} y_n^p \frac{v_n}{n^p}\right)^{\frac{1}{p}}$$

tengsizlikka ega bo'lamiz. Bunda, $y_n = \chi_{[n_0,n_1]}(n)n^{p'}v_n^{1-p'}$ deb tanlab yuqoridagi tengsizlikning chap va o'ng tomonlariga qo'ysak ushbularni hosil qilamiz

$$\frac{2C}{a_{11}}\cdot\left(\sum_{n=n_0}^{\infty} y_n^p \frac{v_n}{m^p}\right)^{\frac{1}{p}} = \frac{2C}{a_{11}}\cdot\left(\sum_{n=n_0}^{n_1}\left(n^{p'}v_n^{1-p'}\right)^p \frac{v_n}{n^p}\right)^{\frac{1}{p}} = \frac{2C}{a_{11}}\cdot\left(\sum_{n=n_0}^{n_1} n^{p'}v_n^{1-p'}\right)^{\frac{1}{p}},$$

$$\left(\sum_{n=n_0}^{\infty}\left(\sum_{m=n_0}^{n} y_m\right)^p n^p u_n\right)^{\frac{1}{p}} = \left(\sum_{n=n_0}^{\infty}\left(\sum_{m=n_0}^{n} \chi_{[n_0,n_1]}(m)m^{p'}v_m^{1-p'}\right)^p n^p u_n\right)^{\frac{1}{p}} \geq$$

$$\geq \left(\sum_{n=n_1}^{\infty}\left(\sum_{m=n_0}^{n} \chi_{[n_0,n_1]}(m)m^{p'}v_m^{1-p'}\right)^p n^p u_n\right)^{\frac{1}{p}} \geq$$

$$\geq \left(\sum_{n=n_1}^{\infty}\left(\sum_{m=n_0}^{n_1} m^{p'}v_m^{1-p'}\right)^p n^p u_n\right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=n_1}^{\infty} n^p u_n\right)^{\frac{1}{p}}\left(\sum_{m=n_0}^{n_1} m^{p'}v_m^{1-p'}\right).$$

Demak, yuqoridagi tengsizlikdan quyidagi baho kelib chiqdi

$$\left(\sum_{n=n_1}^{\infty} n^p u_n\right)^{\frac{1}{p}}\left(\sum_{m=n_0}^{n_1} m^{p'}v_m^{1-p'}\right) \leq \frac{2C}{a_{11}}\cdot\left(\sum_{n=n_0}^{n_1} n^{p'}v_n^{1-p'}\right)^{\frac{1}{p}}$$

va

$$\left(\sum_{n=n_1}^{\infty} n^p u_n\right)^{\frac{1}{p}} \left(\sum_{m=n_0}^{n_1} m^{p'} v_m^{1-p'}\right)^{\frac{1}{p'}} \leq \frac{2C}{a_{11}}.$$

Bunda n_1 ning ixtiyoriyligidan

$$\mathrm{A}^3 = \sup_{\mathrm{n}\geq \mathrm{n}_0} \left(\sum_{\mathrm{m}=\mathrm{n}_0}^{\mathrm{n}} \mathrm{m}^{\mathrm{p}'} \mathrm{v}_{\mathrm{m}}^{1-\mathrm{p}'}\right)^{\mathrm{p}-1} \cdot \left(\sum_{\mathrm{m}=\mathrm{n}}^{\infty} \mathrm{m}^{\mathrm{p}} \mathrm{u}_{\mathrm{m}}\right) < \infty$$

ni hosil qilamiz. Endi $A \leq C_1 + C_2 A^3$ ekanligini ko'rsatsak zaruriylik isbot bo'ladi. $A^3 < \infty$ekanligidan, ixtiyoriy $n \in N$ uchun

$$\sum_{m=n}^{\infty} m^p u_m < \infty$$

bo'lishi kelib chiqadi. Bundan tashqari Ani ushbu ko'rinishda yozamiz

$$A^2 = \sup_{n\geq 1} \left(\sum_{m=1}^{n} m^{p'} v_m^{1-p'}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m\right) \leq$$

$$\leq \sup_{n\geq n_0} \left(\sum_{m=1}^{n} m^{p'} v_m^{1-p'}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m\right)$$

$$+ \sup_{n_0\geq n\geq 1} \left(\sum_{m=1}^{n} m^{p'} v_m^{1-p'}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m\right).$$

Bulardan ko'rinadiki ikkinchi yig'indi chekli ekan, chunki ikkinchi yig'indidagi birinchi qavslarda cheklita hadlarning yig'indisidan iborat. Shuning uchun birinchi yig'indi bilan bahoni davom ettiramiz:

$$\sup_{n\geq n_0} \left(\sum_{m=1}^{n} m^{p'} v_m^{1-p'}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m\right) \leq$$

$$\le \sup_{n\ge n_0}\left(\sum_{m=1}^{n_0}\mathrm{m}^{\mathrm{p}'}v_m^{1-p'}+\sum_{m=n_0}^{n}\mathrm{m}^{\mathrm{p}'}v_m^{1-p'}\right)^{p-1}\cdot\left(\sum_{m=n}^{\infty}m^p u_m\right)$$

$$\le 2^{p-1}\sup_{n\ge n_0}\left(\left(\sum_{m=1}^{n_0}\mathrm{m}^{\mathrm{p}'}v_m^{1-p'}\right)^{p-1}+\left(\sum_{m=n_0}^{n}\mathrm{m}^{\mathrm{p}'}v_m^{1-p'}\right)^{p-1}\right)\cdot\left(\sum_{m=n}^{\infty}m^p u_m\right)$$

$$\le 2^{p-1}\left(\sup_{n\ge n_0}\left(\sum_{m=1}^{n_0}\mathrm{m}^{\mathrm{p}'}v_m^{1-p'}\right)^{p-1}\cdot\left(\sum_{m=n}^{\infty}m^p u_m\right)+\sup_{n\ge n_0}\left(\sum_{m=n_0}^{n}\mathrm{m}^{\mathrm{p}'}v_m^{1-p'}\right)^{p-1}\cdot\left(\sum_{m=n}^{\infty}m^p u_m\right)\right).$$

Ohirgi ifodalarning ikkalasi ham chekli, sababi, bunda ham birinchi yig'indining birinchi qavsdagi hadlar cheklita, ikkinchi yig'indining chekliligi ko'rsatilgan edi. Teorema to'liq isbot bo'ldi.

Endi shu teoremaning ehtimollar nazariyasiga qo‘llanishi to‘g‘risidagi ushbu asosiy teoremani keltiramiz:

Bizga$\xi_1, \xi_2, \dots, \xi_n$, ...tasodifiy miqdorlar ketma-ketligi, hamda, shu ketma-ketlikka mos$\eta_n = \eta_n(\xi_1, \xi_2, \dots, \xi_n)$ ifodalar ketma-ketligi berilgan bo‘lsin.

3.1.3-ta'rif. Agar, shunday$\{b_n\}$sonlar ketma-ketligi mavjud bo‘lib, ushbu

$$\lim_{n\to\infty} P\{|\eta_n - b_n| < \varepsilon\} = 1$$

Tenglik o‘rinli bo‘lsa, $\{\xi_n\}$ tasodifiy miqdorlari ketma-ketligi katta sonlar qonuniga bo‘ysunadi deyiladi.

Berilgan $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ tasodifiy miqdorlar ketma-ketligi uchun ushbu ifodani qaraylik

$$\eta_n = \eta_n(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{k=1}^{n} (a_{00} + a_{10}n + a_{01}m + a_{11}nm)\xi_k.$$

3.1.4-teorema (asosiy natija). *Faraz qilaylik,* $\{u_n\}$ *va* $\{v_n\}$ *ketma-ketliklar quyidagi shartlarni qanoatlantirsin*

$$A = \sup_{n \in N} \left(\sum_{m=1}^{n} |a_{10} + a_{11}m|^{p'} v_m^{1-p'} \right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m \right) < \infty.$$

Faraz qilaylik $\xi_1, \xi_2, \ldots, \xi_n, \ldots$*o'zaro bog'liq bo'lmagan va quyidagi shartni qanoatlantiruvchi tasodifiy miqdorlar*

$$\sum_{n=1}^{\infty} M|\xi_n - M\xi_n|^p \cdot v_n < \infty$$

ketma-ketligi bo'lsa, u holda ixtiyoriy $\varepsilon > 0$ *uchun ushbu tenglik o'rinli*

$$\lim_{n \to \infty} P\{|\eta_n - M\eta_n| < \varepsilon\} = 1.$$

Isbot. Teoremani isbotlash uchun Hardi tengsizligidan, yani, 3.2.1-teoremadan foydalanamiz. Umumiylikga zarar keltirmasdan $M\xi_k = 0,\ k = 1,2, \ldots$ deb faraz qilaylik. Bundan,

$$M\eta_n = M\left(\sum_{k=1}^{n} (a_{00} + a_{10}n + a_{01}m + a_{11}nm)\xi_k \right)$$
$$= \sum_{k=1}^{n} (a_{00} + a_{10}n + a_{01}m + a_{11}nm)M\xi_k = 0$$

kelib chiqadi. U holda (3.1.3) ga ko'ra ixtiyoriy $l \in N$ uchun

$$\sum_{n=1}^{l}|\eta_n - M\eta_n|^p = \sum_{n=1}^{l}\left|\sum_{k=1}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\xi_k\right|^p u_n$$

$$\leq C\cdot\sum_{n=1}^{l}|\xi_n|^p\cdot v_n$$

ga ega bo‘lamiz, ya’ni,

$$\sum_{n=1}^{l}|\eta_n - M\eta_n|^p \leq C\cdot\sum_{n=1}^{l}|\xi_n|^p\cdot v_n.$$

Tasodifiy miqdorlarning matematik kutilmalari uchun ham tengsizlik saqlanadi

$$\sum_{n=1}^{l}M|\eta_n - M\eta_n|^p \leq C\cdot\sum_{n=1}^{l}M|\xi_n|^p\cdot v_n.$$

Bunda, $\sum_{n=1}^{\infty}M|\xi_n|^p\cdot v_n<\infty$ ni etiborga olsak, $l\to\infty$ da quyidagini hosil qilamiz:

$$\sum_{n=1}^{\infty}M|\eta_n - M\eta_n|^p \leq C\cdot\sum_{n=1}^{\infty}M|\xi_n|^p\cdot v_n.$$

Demak,

$$\sum_{n=1}^{\infty}M|\eta_n - M\eta_n|^p < \infty$$

o‘rinli. Bundan

$$\lim_{n\to\infty}M|\eta_n - M\eta_n|^p = 0$$

ga ega bo‘lamiz, yani, ixtiyoriy ε > 0 uchun umulashgan Chebishev tengsizligi, ya’ni

$$P(|\eta_n - M\eta_n| < \varepsilon) \geq 1 - \frac{M|\eta_n - M\eta_n|^p}{\varepsilon^p}$$

ga ko’raushbu tenglik o‘rinli

$$\lim_{n\to\infty} P\{|\eta_n - M\eta_n| < \varepsilon\} = 1.$$

Teorema isbot bo‘ldi.

3.2-§. O‘rta vaznli matritsa yadroli Hardy tengsizligiga oid natijalar

Mashhur Hardy tengsizligi quyidagi ko‘rinishga ega:

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|^p \le \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty}|a_k|^p, \qquad (3.2.1)$$

bunda$p>1$, $\{a_k\}$ $-$ixtiyoriy sonli ketma-ketlik. [46] da bu tengsizlik va uning ba’zi umumlashmalari to‘g’risida batafsil natijalar keltirilgan. Bundan tashqari xozirgi zamon matematikasida ushbu ko‘rinishdagi tengsizlikni

$$\sum_{j=1}^{\infty}\left|\sum_{k=1}^{n}c_{j,k}a_k\right|^p \le C\sum_{n=1}^{\infty}|a_k|^p \qquad (3.2.2)$$

o‘rganish zaruriyati paydo bo‘lmoqda. Bunda $\left(c_{j,k}\right)_{j,k=1}^{\infty}$ -haqiqiy cheksiz o‘lchovli matritsadir. (3.2.2) tengsizlik ba’zi operatorlarning spektral xossalari haqida ham muhim xulosalar chiqarishda ishlatilishi mumkin. Biz bu ishda (3.2.2) tengsizlikning xususiy hollarda o‘rinli bo‘lish masalasi bilan shug’ullanamiz.

3.2.1-ta’rif. Agar $A=\left(c_{j,k}\right)_{j,k=1}^{\infty}$ matritsaning elementlari barcha $k>j$ uchun $c_{j,k}\ge 0, k\le j, \sum_{k=1}^{j}c_{j,k}=1$ shartlarni qanoatlantirsa A matritsani jamlanuvchi deb ataymiz. Agar A jamlanuvchi matrisaning elementlari ushbu ko‘rinishga ega bo‘lsa, ya’ni shunday $\{\lambda_i\}_{i=1}^{\infty}$ ketma-ketlik mavjudki, bunda $\lambda_1>0$ va $\lambda_i\ge 0$, $(i\ge 2)$

$$c_{j,k}=\frac{\lambda_k}{\Lambda_j}, 1\le k\le j, \qquad \Lambda_j=\sum_{i=1}^{j}\lambda_j,$$

bo‘lsa u holda$A$ matrisaga o‘rta vaznli matritsa deyiladi.

3.2.2-Teorema. *(*3.2.*2) tengsizlik ixtiyoriy o‘rta vaznli A matritsa uchun o‘rinli.*

Izoh. (3.2.2) tengsizlikni$\left(c_{j,k}\right)_{j,k=1}^{\infty}$o‘rta vaznli matritsa bilan ehtimollar nazariyasidagi katta sonlar qonuni uchun bazi natijalarni olishda foydalanish mumkin.

Teoremaning isboti. Faraz qilaylik,(3.2.2) tengsizlik o‘rinli bo‘lsin va faraz qilaylijk$\{w_i\}_{i=1}^{\infty}$ ixtiyoriy musbat ketma-ketlik bo‘lsin.U holda ushbu Hölder tengsizligi o‘rinli

$$\left(\sum_{k=1}^{n}\lambda_k|a_k|\right)^p=\left(\sum_{k=1}^{n}\lambda_k|a_k|\,{w_k}^{-\frac{1}{p^*}}\cdot{w_k}^{\frac{1}{p^*}}\right)^p$$

$$\leq\left(\sum_{k=1}^{n}{\lambda_k}^p|a_k|^p\,{w_k}^{-(p-1)}\right)\left(\sum_{j=1}^{n}w_j\right)^{p-1}.$$

Bundan va (3.2.2) dan foydalanib quyidagini hosil qilamiz

$$\sum_{n=1}^{\infty}\left|\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k a_k\right|^p\leq\sum_{n=1}^{\infty}\frac{1}{\Lambda_n^p}\left(\sum_{k=1}^{n}{\lambda_k}^p|a_k|^p\,{w_k}^{-(p-1)}\right)\left(\sum_{j=1}^{n}w_j\right)^{p-1}$$

$$=\sum_{k=1}^{\infty}{w_k}^{-(p-1)}{\lambda_k}^p\left(\sum_{n=k}^{\infty}\frac{1}{\Lambda_n^p}\left(\sum_{j=1}^{n}w_j\right)^{p-1}\right)|a_k|^p.$$

Farazqilaylik, harbir$p>1$uchunmusbato‘zgarmas$C$mavjudbo‘lsin va$w$ketma-ketlikw_n^{p-1}/λ_n^phadlari 0 gaintilsin. U holda barcha$n\geq 1$butunsonlaruchun

$$(w_1+\cdots+w_n)^{p-1}<C\Lambda_n^p\left(\frac{w_n^{p-1}}{\lambda_n^p}-\frac{w_{n+1}^{p-1}}{\lambda_{n+1}^p}\right),$$

o‘rinli bo‘ladi. Bundanoxirgi bahoni quyidagicha davom ettirish mumkin

$$\sum_{n=1}^{\infty}\left|\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k a_k\right|^p\leq\sum_{k=1}^{\infty}{w_k}^{-(p-1)}{\lambda_k}^p\left(\sum_{n=k}^{\infty}\frac{1}{\Lambda_n^p}C\Lambda_n^p\left(\frac{w_n^{p-1}}{\lambda_n^p}-\frac{w_{n+1}^{p-1}}{\lambda_{n+1}^p}\right)\right)|a_k|^p$$

$$= C\sum_{k=1}^{\infty} w_k{}^{-(p-1)}\lambda_k{}^{p}\frac{w_k^{p-1}}{\lambda_k^{p}}|a_k|^p = C\sum_{k=1}^{\infty}|a_k|^p .$$

Teorema isbot bo‘ldi.

Endi shu teoremaning ehtimollar nazariyasiga qo‘llanishi to‘g‘risidagi ushbu ikkinchi asosiy teoremani keltiramiz:

Bizga $\xi_1, \xi_2, \dots, \xi_n,$ ...tasodifiy miqdorlar ketma-ketligi, hamda, shu ketma-ketlikka mos $\eta_n = \eta_n(\xi_1, \xi_2, \dots, \xi_n) = \frac{\sum_{i=1}^{n}\lambda_i\xi_i}{\sum_{j=1}^{n}\lambda_j}$ ifodalar ketma-ketligini qaraymiz.

3.2.2-teorema (2-asosiy natija). *Faraz qilaylik,*$\xi_1, \xi_2, \dots, \xi_n,$ *...o‘zaro bog‘liq bo‘lmagantasodifiy miqdorlar ketma-ketligi bo‘lib, quyidagi shartni qanoatlantirsin:*

$$\sum_{n=1}^{\infty} M|\xi_n - M\xi_n|^p < \infty.$$

U holda ixtiyoriy $\varepsilon > 0$ *uchun ushbu tenglik*

$$\lim_{n\to\infty} P\{|\eta_n - M\eta_n| < \varepsilon\} = 1$$

o‘rinli bo‘ladi.

Isbot. Teoremani isbotlash uchun Hardi tengsizligi (3.3.1-teorema)dan foydalanamiz. U holda (3.3.1) ga ko‘ra ixtiyoriy $l \in N$ uchun

$$\sum_{n=1}^{l}|\eta_n - M\eta_n|^p = \sum_{n=1}^{l}\left|\frac{\sum_{i=1}^{n}\lambda_i\xi_i}{\sum_{j=1}^{n}\lambda_j} - \frac{\sum_{i=1}^{n}\lambda_i M\xi_i}{\sum_{j=1}^{n}\lambda_j}\right|^p$$

$$= \sum_{n=1}^{l}\left|\sum_{i=1}^{n}\frac{\lambda_i}{\sum_{j=1}^{n}\lambda_j}(\xi_i - M\xi_i)\right|^p \le C\cdot\sum_{n=1}^{l}|\xi_n - M\xi_n|^p$$

ga ega bo‘lamiz, ya’ni,

$$\sum_{n=1}^{l} |\eta_n - M\eta_n|^p \leq C \cdot \sum_{n=1}^{l} |\xi_n - M\xi_n|^p.$$

Tasodifiy miqdorlarning matematik kutilmalari uchun ham tengsizlik saqlanadi

$$\sum_{n=1}^{l} M|\eta_n - M\eta_n|^p \leq C \cdot M \sum_{n=1}^{l} |\xi_n - M\xi_n|^p = C \cdot \sum_{n=1}^{l} M|\xi_n - M\xi_n|^p.$$

Bunda, $\sum_{n=1}^{\infty} M|\xi_n - M\xi_n|^p < \infty$ ni etiborga olsak, $l \to \infty$ da quyidagini hosil qilamiz:

$$\sum_{n=1}^{\infty} M|\eta_n - M\eta_n|^p \leq C \cdot \sum_{n=1}^{\infty} M|\xi_n - M\xi_n|^p.$$

Demak,

$$\sum_{n=1}^{\infty} M|\eta_n - M\eta_n|^p < \infty$$

o‘rinli. Bundan

$$\lim_{n\to\infty} M|\eta_n - M\eta_n|^p = 0$$

ga ega bo‘lamiz. Bundan umulashgan Chebishev tengsizligiga asosan quyidagiga ega bo’lamiz, yani, ixtiyoriy $\varepsilon > 0$ uchun ushbu tenglik o‘rinli

$$\lim_{n\to\infty} P\{|\eta_n - M\eta_n| < \varepsilon\} = 1.$$

Teorema isbot bo‘ldi.

Xulosa

Ushbu magistrlik dissertatsiyasi “Umumlarshgan Hardy tengsizligi va uning ehtimolga tadbiqlari”ni o‘rganishga bag‘ishlangan. Odatda katta sonlar qonuni haqidagi asosiy teoremalar tasodifiy miqdorlarning o‘rta qiymati uchun keltirilgan. Lekin ularning isbotlarini Hardy tengsizligidan yordamida ham qilish mumkin. Ikkinchi tomondan Hardy tengsizliklari ancha rivojlangan soha hisoblanadi. Bu esa katta sonlar qonuni nafaqat o‘rta qiymat uchun, balki mumkin qadar murakkabroq ifodalar uchun ham keltirish imkoniyatini beradi.

Dissertatsiyaning uchinchi bobida avval Hardy tengsizligiga oid olingan natijalar keltirilgan.

Faraz qilaylik yadro quyidagi xossaga ega bo’lsin, ya’ni shunday b_0, b_1, b_2, b_3 *sonlar mavjudki*

$$a_{00} + a_{10}n + a_{01}m + a_{11}nm = (b_0 + b_1 n)(b_2 + b_3 m)$$

tenglik o’rinli. Ma’lumki, bu tenglik bajarilishi uchun ushbu shart bajarilish kerak bo’ladi:

$$a_{00}a_{11} = a_{10}a_{01}.$$

1-teorema.*Faraz qilaylik*

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00} + a_{10}n + a_{01}m + a_{11}nm)x_m\right|^p u_n\right)^{\frac{1}{p}} \leq C \cdot \left(\sum_{n=1}^{\infty}|x_n|^p v_n\right)^{\frac{1}{p}},$$

Tengsizlikdagi yadro uchun $a_{00}a_{11} = a_{10}a_{01}$ *shart qanoatlantirsin. U holda tengsizlik o’rnli bo’lishi uchun*

$$\sup_n \left(\sum_{m=1}^{n}|b_2 + b_3 m|^{p'} v_m^{1-p}\right)^{p-1} \cdot \left(\sum_{m=n}^{\infty}|b_0 + b_1 m| \cdot u_m\right) < \infty$$

shartning bajarilishi zarur va yetarlidir.

2-teorema.

$$\left(\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n}(a_{00}+a_{10}n+a_{01}m+a_{11}nm)\,x_m\right|^{p}u_n\right)^{\frac{1}{p}}\leq C\cdot\left(\sum_{n=1}^{\infty}|x_n|^{p}v_n\right)^{\frac{1}{p}}$$

tengsizlik o'rinli bo'lishi uchun

$$A=\sup_{n\in N}\left(\sum_{m=1}^{n}|a_{10}+a_{11}m|^{p'}v_m^{1-p'}\right)^{p-1}\cdot\left(\sum_{m=n}^{\infty}m^{p}u_m\right)<\infty$$

Shartning *bajarilishi zarur va yetarlidir.*

3-ta'rif. Agar $A=\left(c_{j,k}\right)_{j,k=1}^{\infty}$ matritsaning elementlari barcha $k>j$ uchun $c_{j,k}\geq 0, k\leq j, \sum_{k=1}^{j}c_{j,k}=1$ shartlarni qanoatlantirsa A matritsani jamlanuvchi deb ataymiz. Agar A jamlanuvchi matrisaning elementlari ushbu ko'rinishga ega bo'lsa, ya'ni shunday $\{\lambda_i\}_{i=1}^{\infty}$ ketma-ketlik mavjudki, bunda $\lambda_1>0$ va $\lambda_i\geq 0$, $(i\geq 2)$

$$c_{j,k}=\frac{\lambda_k}{\Lambda_j}, 1\leq k\leq j, \qquad \Lambda_j=\sum_{i=1}^{j}\lambda_j,$$

bo'lsa u holda A matrisaga o'rta vaznli matritsa deyiladi.

4-teorema.

$$\sum_{j=1}^{\infty}\left|\sum_{k=1}^{n}c_{j,k}a_k\right|^{p}\leq C\sum_{n=1}^{\infty}|a_k|^{p}$$

tengsizlik ixtiyoriy o'rtavaznli A matritsa uchun o'rinli.

5-teorema (asosiy natija). *Faraz qilaylik,* $\{u_n\}$ *va* $\{v_n\}$ *ketma-ketliklar quyidagi shartlarni qanoatlantirsin*

$$A = \sup_{n \in N} \left(\sum_{m=1}^{n} |a_{10} + a_{11}m|^{p'} v_m^{1-p'} \right)^{p-1} \cdot \left(\sum_{m=n}^{\infty} m^p u_m \right) < \infty.$$

Faraz qilaylik $\xi_1, \xi_2, \dots, \xi_n, \dots$*o'zaro bog'liq bo'lmagan va quyidagi shartni qanoatlantiruvchi tasodifiy miqdorlar*

$$\sum_{n=1}^{\infty} M|\xi_n - M\xi_n|^p \cdot v_n < \infty$$

*ketma-*ketligi *bo'lsa, u holda ixtiyoriy* $\varepsilon > 0$ *uchun ushbu tenglik o'rinli*

$$\lim_{n \to \infty} P\{|\eta_n - M\eta_n| < \varepsilon\} = 1.$$

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