LINEAR DISCRETE DYNAMICAL SYSTEMS IN THE $\mathbb{C}^n$ SPACE

Dilmuradov N., Raximov A. (KarSU)

Keywords: *discrete dynamik system, Jordan matrix, eigenvalues of matrix.*

Discrete dynamic systems are met in mathematics. For example, we can use them in the iteration methods forsolving equations. In the article the n dimensional complex linear discrete dynamic systems $z^k = Az^{k-1}$ are explored.

Let $\mathbb{C}^n$ be, as usual, n dimensional complex linear space.

Its elements $z=(z_1,z_2,...,z_n)^T \in \mathbb{C}^n$ (T stands for transposition) are complex vectors. It consists of n components $z_j \in \mathbb{C}$, $j=\overline{1,n}$. In $\mathbb{C}^n$ space distance between z and w points (vectors) bi defined by the formula

$$|z-w| = \sqrt{\sum_{j=1}^{n} |z_j - w_j|^2}$$

.

By $\mathbb{M}_{n\times n}(\mathbb{C})$ we denote the linear space of $n\times n$ matrix with complex elements.

Let $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ and a $z^0 \in \mathbb{C}^n$ is fixed. Let's consider the following iterative sequence

$$z^k = Az^{k-1},\ k \in \mathbb{N} \tag{1}$$

We will explore the limit of this sequence.

From (1)

$$z^k = Az^{k-1} = A^2 z^{k-2} = ... = A^k z^0 . \tag{2}$$

equalitis are resulted.

To explore matrix A^k we transform the matrix A to Jordan normal form.

It is known $[1,2]$, that there is a matrix $S \in \mathbb{M}_{n\times n}(\mathbb{C})$, $\det S \neq 0$,which transforms matrix A to Jordan normal form:

$$A = SJS^{-1}\ ,\ J = S^{-1}AS\ , \tag{3}$$

where J is Jordan diagonal matrix that is J has Jordan blocks on diagonal and 0's elsewhere :

$$J = \operatorname{diag}(J_{\lambda_1,n_1},\dots,J_{\lambda_2,n_2},\dots,J_{\lambda_s,n_s}) = \begin{pmatrix} J_{\lambda_1,n_1} & & & \\ & \ddots & & \\ & & J_{\lambda_2,n_2} & \\ & & & \ddots\ J_{\lambda_s,n_s} \end{pmatrix}. \tag{4}$$

The Jordan diagonal matrix is constructed as following. Let $\lambda_1,\lambda_2,\dots,\lambda_s$ $(s \le n)$ be the eigenvalues of matrix A with the corresponding multiplicity $k_1,k_2,\dots,k_s$ such tehat $k_1+k_2+\dots+k_s=n$. Let p_q be the namber of lineary independent vectors corresponding to the eigenvalue λ_q , $q=\overline{1,s}$, that is $\dim\{\boldsymbol{x} \mid (A-\lambda_q E)\boldsymbol{x}=0\} = p_q$, $p_q = n - \operatorname{rank}(A-\lambda_q E)$.

The number of $d_{qj}\times d_{qj}$ Jordan blocks $J_{\lambda_q,d_{qj}}$ corresponding λ_q is p_q and

$$J_{\lambda_q,d_{qj}} = \begin{pmatrix} \lambda_q & 1 & & & \\ & \lambda_q & 1 & & \\ & & \lambda_q & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_q \end{pmatrix} \in \mathbb{M}_{d_{qj}\times d_{qj}}(\mathbb{C}),\ j=\overline{1,p_q},\ (d_{q1}+d_{q2}+\dots+d_{q p_q}=k_q). \tag{5}$$

It is clear that the biggest dimension of Jordan blocks $J_{\lambda_q, d_{qj}}$ is $\tilde{k}_q \overset{def}{=}$ $= \max\{d_{q1}, d_{q2}, \dots, d_{q p_q}\} \le k_q$. In (4) we have

$$J_{\lambda_1, d_{11}} = J_{\lambda_1, n_1}, \dots, J_{\lambda_1, d_{1p_1}}, \; J_{\lambda_2, d_{21}} = J_{\lambda_2, n_2}, \dots, \; J_{\lambda_s, d_{s\, p_s}} = J_{\lambda_s, n_s} \quad .$$

It is known [2], in formula (3) the transformation matrix S has the form $S = [\boldsymbol{s}^1 \vdots \boldsymbol{s}^2 \vdots \cdots \vdots \boldsymbol{s}^n]$, where $\boldsymbol{s}^1, \boldsymbol{s}^2, \dots, \boldsymbol{s}^n$ are Jordan basis.

From the equation (3) we have

$$A^2 = SJS^{-1}SJS^{-1} = SJEJS^{-1} = SJ^2S^{-1}, \dots, A^k = SJ^kS^{-1} . \tag{6}$$

Letôs consider the cases $\det A = 0$ and $\det A \neq 0$ separetly.

If $\det A = 0$ the rank of matrix A is $r, r < n$. Therefore the matrix A maps $\mathbb{C}^n$ on a r dimensional hiperplane of $\mathbb{C}^n$. Thus $z^1 = Az^0$ is in the r dimensional hiperplane. It is known $[2]$, the product of several matrices with the same ranks has the same rank. From this statement it follows that the points $z^2, z^3, \dots, z^k$ are in the r dimensional hiperplane. Therefore in this case $A-$ mapping is ekvivalent to the mapping of the r dimensional hiperplane to itself by some nonesingular matrix.

Let $\det A \neq 0$. In this case $\operatorname{rank} A = n$ and the matrix A bijectively maps $\mathbb{C}^n$ onto $\mathbb{C}^n$. This statement shows that, the case $\det A \neq 0$ is enough to consider.

Letôs return to the forrmula (2) giving the solution to the equation (1). From the formula (6) we have

$$A^k = SJ^kS^{-1}.$$

It is clear that

$$J^n = \operatorname{diag}(J^n_{\lambda_1, n_1}, \dots, J^n_{\lambda_2, n_2}, \dots, J^n_{\lambda_s, n_s}) = \begin{pmatrix} J^n_{\lambda_1, n_1} & & & \\ & \ddots & & \\ & & J^n_{\lambda_2, n_2} & \\ & & & \ddots \; J^n_{\lambda_s, n_s} \end{pmatrix} . \tag{7}$$

(7) shows that, we need calculate the n^{th} degree of Jordan block for to calculate J^n. The typial Jordan block is

$$J_{\lambda, m} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \in M_{m \times m}(\mathbb{C}). \tag{8}$$

Given $J_{\lambda, m}$. we introduce the denotations

$$E_m = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, N_m = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} , \tag{9}$$

where $E_m, N_m -$are $m \times m$ matrices, and wright

$$J_{\lambda, m} = \lambda E_m + N_m \; . \tag{10}$$

Since $N_m^m, N_m^{m+1} \dots$ are null matrices from (10) and (9) we get

$$J_{\lambda,m}^{k}=\left(\lambda E_m+N_m\right)^k=\lambda^k E_m+\frac{n\lambda^{k-1}N_m}{1!}+\ldots+\frac{k!}{(m-1)!(k-m+1)!}\lambda^{k-m+1}N_m^{m-1}\ . \qquad (11)$$

From (11) we have

$$J_{\lambda,m}^{k}=\begin{pmatrix} \lambda^k & \frac{k\lambda^{k-1}}{1!} & \frac{k(k-1)\lambda^{k-2}}{2!} & \cdots & \frac{k!\lambda^{k-m+1}}{(k-1)!(k-m+1)!} \\ 0 & \lambda^k & \frac{k\lambda^{k-1}}{1!} & \cdots & \frac{k!\lambda^{k-m+1}}{(k-2)!(k-m+2)!} \\ \vdots & \vdots & \lambda^k & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{k\lambda^{k-1}}{1!} \\ 0 & 0 & 0 & \cdots & \lambda^k \end{pmatrix} \qquad (12)$$

Letôs explore the limit

$$\lim_{k\to\infty} J_{\lambda,m}^{k}\,. \qquad (13)$$

There are three cases $|\lambda|<1$, $|\lambda|>1$ and $|\lambda|=1$.

Let $|\lambda|<1$. In this case we have obviously

$$\lim_{k\to\infty} J_{\lambda,m}^{k}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

If $|\lambda|>1$, the (13) limit is infinity. If $|\lambda|=1$, one of two conditions is fitting.

If the dimesion of (8) Jordan block is 1×1, the limit (13) is finite, else the limit (13) is equal to infinity.

We have obtained the following theorem from these remarks .

Theorem 1. $A^k \xrightarrow[k\to\infty]{} \theta$, iff the conditions $\left\{|\lambda_j|<1,\ j=\overline{1,n}\right\}$ hold, (θ – null matrix). Thus, if for all eigenvalues λ_j of the A matrix $\left\{|\lambda_j|<1,\ j=\overline{1,n}\right\}$ hold, then given every $z^0\in\mathbb{C}^n$, the dynamical system $z^k \xrightarrow[k\to\infty]{} o$ (o – null vector) and the point $z=o$ is attracting.

For many theoritical and applicational problems it is very important to know if all eigenvalues of a given matrix have eigenvalues with negative real parts.

Suppose, all eigenvalues of a matrix A belong to unit disk in the complex plane $\mathbb{C}$. Then obvi-ously $(E-A)^{-1}$ exists. Letôs construckt eigenequation of following matrix $B=(A-E)(E+A)^{-1}$:

$$\det\left((A-E)(E+A)^{-1}-\mu E\right)=0\,. \qquad (14)$$

This equation is ekvivalent to the following equation:

$$\det\left(A-\frac{1+\mu}{1-\mu}E\right)=0\,. \qquad (15)$$

It is follows from (15) that the eigenvalues of B belong to the left half-plane $\mathrm{Re}\,\mu<0$ of $\mathbb{C}$ unit circle because of the properties of the map $\mu\to\lambda=\frac{1+\mu}{1-\mu}$.

Conversly, it follows from (14) and (15) that if all eigenvalues of the matrix $B=(A-E)(E+A)^{-1}$ belong to the left half-plane, then all eigenvalues of the matrix $A=(E+B)(E-B)^{-1}$ are in the unit disk.

We can formulate the following theorem.

Teorema 2. For all eigenvalues μ_j of the matrix $B=(A-E)(E+A)^{-1}$ the conditions $\{\operatorname{Re}\mu_j<0,\ j=\overline{1,n}\}$ hold, iff all eigenvalues of the matrix A belong to the unit disk $\{|\lambda_j|<1,\ j=\overline{1,n}\}$.

Leturature

1. Dilmurodov N. Differensial tenglamalar kursi. I jild. Qarshi, 2013, ï 306 b.
2. [illegible], 2010. ï 576 [illegible]
3. Robert L. Devaney An introduction to chaotic dynmical system. Copyright 1989 by Addison-Wesley Publition Company. ï 331 p.
4. Alan F. Beardon Iteration Of Rational Functions. Springer-Werlag,1990. ï 283 p.

REZYUME

Maqolada kompleks chiziqli Discrete dinamik sistemaning tabiati o`rganilgan.Matritsa xarakteristik sonlarining chap yarimkompleks tekislikda yotishining yetarli va zaruriy sharti topilgan.

[illegible]

[illegible]

SUMMARY

The behavior of the complex linear discrete dynamic system is explored. A necessary and sufficient condition for eigenvalues of a matrix to belong to the left complex half-plane is obtained.

Recommended for publication by prof. Yu.Eshkabilov