On the Phase Portrait Phase Portrait of the System $${{\dot x} = {Ax} + {\langle a, x \rangle x}}$$

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Qualitative Theory of Dynamical Systems

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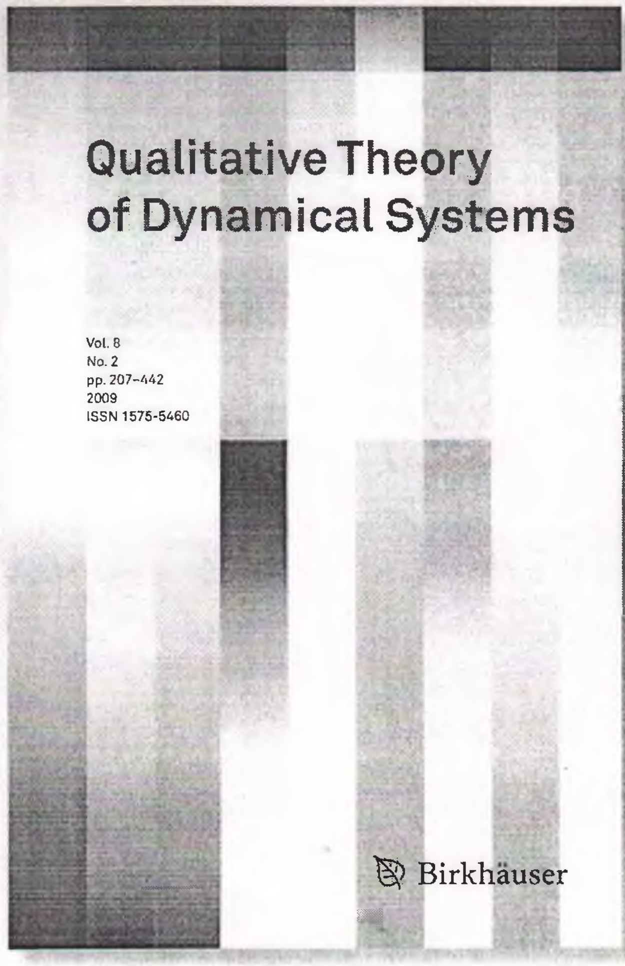

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Qualitative Theory of Dynamical Systems

On the Phase Portrait of the System $\dot{x} = Ax + \langle a, x\rangle x$

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Abstract Structure of the phase space of the nonlinear system $\dot{x} = Ax + \langle a, x\rangle x$ is clarified using saddle-node bifurcations ($x,\ a \in \mathbb{R}^d,\ A$ is a $d \times d$-matrix).

Keywords Quadratic nonlinearity · Phase portrait · Critical point · Invariant plane · Saddle-node bifurcation

1 Introduction

A phase portrait services for geometrical description of a dynamical system clarifying all characteristic specialities such as critical points, closed trajectories, separatrices, including polycycles, invariant manifolds and in some sense completes its consideration. The notion of a phase portrait is mainly connected with either linear systems $\dot{x} = dx/dt = Ax,\ x \in \mathbb{R}^d, A \in L(\mathbb{R}^d)$ [5,10] or with vector fields on two-dimensional manifolds [1,7–9].

What concerns with high order nonlinear systems, the task of construction of the phase portrait meets up serious difficulties even for local studies. For example, it is not known the classification of behaviour in a neighborhood of a critical point for such systems similar to the classification of sectors in a neighborhood of a critical point in two-dimensional systems, not to mention a classification problem of trajectories' behavior in the neighborhood of cycles or other singularities. We even may assume

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that, generally, it is impossible to realize such a classification for systems in $\mathbb{R}^d$, $d \geq 3$. From this point of view it may have some of interest (at least from the didactic point of view) that the system

$$\dot{x} = Ax + \langle a, x\rangle x, \tag{1}$$

with $a, x \in \mathbb{R}^d$, $\langle a, x\rangle$ is a scalar product, allows to build the phase portrait on the whole [2,4].

The Sect. 2 consists of the Preliminaries. First two properties of invariance for the systems of the form (1) are noted and two conditions of hyperbolicity are formulated. Then it is displayed the special principle of induction: restriction of the system (1) as a vector field to any invariant hyperplane would be vector field of the same type. This allows to restrict studying of behavior of trajectories for the situation when they lay outside of invariant hyperplanes. In the Sect. 3 it is considered the case when all eigenvalues of the matrix A are real. It is shown that in this case the system (1) has $n+1$ invariant hyperplanes dividing the phase space into $2^n - 1$ polyhedral regions including a simplex Σ with vertices those are saddles of types $(k, n-k)$, $k = 0, 1, \ldots, n$. Let (ω_-, ω_+) be the maximal interval of existence of a trajectory $x(t)$ of the system (1) not laying in any invariant hyperplane ($\omega_\pm$ may be as a finite number so the symbol $\pm\infty$ respectively). It is shown that every trajectory $x(t)$ belongs to one of the following types:

I. $|x(t)| \to \infty$ as $t \to \omega_\pm$ in both direction.
II. $|x(t)| \to \infty$ as $t \to \omega_-$ while $x(t)$ approaches at $t \to +\infty$ to the vertex of Σ being a stable critical point.
III. $|x(t)| \to \infty$ as $t \to \omega_+$ while $x(t)$ approaches at $t \to -\infty$ to the vertex of Σ being an unstable critical point.
IV. $x(t)$ approaches at $t \to -\infty$ to the vertex of Σ that is an unstable critical point and at $t \to +\infty$ to the vertex of Σ that is a stable critical point.

The Sect. 4 is devoted to consider the case when just one couple of eigenvalues of the matrix A are complex. Here the system (1) has $n-1$ critical points and so many invariant hyperplanes. Using one-parameter family of systems of the form (1) that joins the given system and another one with real eigenvalues only and applying saddle-node bifurcation it is established that each trajectory belongs to one of the above mentioned types I–IV. In the Sect. 5 it is analyzed the case when m couple of eigenvalues are complex. Using one-parameter family of systems and applying saddle-node bifurcations consequently m times the same property of trajectories as above is derived.

2 Preliminaries

First of all note that by substitution of Hesse $x = y/\tau$, where $y \in \mathbb{R}^d$, τ is a scalar function, the linear and nonlinear parts of the system (1) can be separated. Namely, the initial value problem $x(0) = x_0$ for the system (1) is equivalent to that for the following $(d+1)$-dimensional system

$$\dot{y} = Ay, \; \dot{\tau} = -\langle a, y \rangle, \quad y(0) = x_0, \; \tau(0) = 1. \tag{2}$$

Thus, the general solution of the system (1) can be expressed in the explicit form

$$x(t) = \frac{e^{At}x_0}{1 - \int_0^t \langle e^{sA^*}a, x_0 \rangle \, ds}, \tag{3}$$

(A^* denotes the transposed matrix).

The formula (3) is useful, particularly, in studying of either continuability of solutions or stability of the solution $x = 0$ [2,3]. We'll turn to (3) below to find limit sets of trajectories. At the same time it should be noted that formula (3) itself is not suitable for construction of a phase portrait. (One can't even use it to answer the question: whether the maximal interval of existence of concrete solution is bounded or not?)

In the present paper, the problem of construction of a phase portrait will be solved based on specific properties of the system (1) and using inclusion it into one-parameter family with consequent handling to the saddle-node bifurcations [5,7].

The class of the systems (1) is invariant with respect to the linear transformation: if $x = Lz$, $\det L \neq 0$, then $\dot{z} = L^{-1}ALz + \langle L^*a, z \rangle z$. This property allows to reduce the general case to the situation when the matrix A has real Jordanian normal form. Therefore we can assume that such reduction has already been done: $A = \begin{bmatrix} B & O \\ O & C \end{bmatrix}$, where B is a block-diagonal matrix corresponding to complex eigenvalues of the matrix A consisting of blocks of the form $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $\beta \neq 0$, C is a diagonal submatrix consisting of real eigenvalues, O is a zero matrix of suitable sizes.

The class of systems of the form (1) has also the other property of invariance: if $\hat{x}$ is its critical point, then after shifting $x = \hat{x} + y$ the origin to the point $\hat{x}$ one obtains again a system of the form (1), namely $\dot{y} = \tilde{A}y + \langle a, y \rangle y$, where $\tilde{A} = A + a \otimes \hat{x} + \langle a, \hat{x} \rangle E$, ($\otimes$ denotes a tensor product, i.e. $a \otimes \hat{x} = (a_i x_j)$) [11].

Further we assume that the following conditions hold:

G1. All eigenvalues of the matrix A are different and have nonzero real part.

G2. All components of the vector a are different from zero. (If some component of a is equal to 0 then under the assumption G1 the order of the system (1) can be reduced.)

3 The Case, When All Eigenvalues are Real

In this case the system has the form

$$\dot{x}_i = \lambda_i x_i + x_i \sum_j a_j x_j, \quad i, j = 1, 2, \ldots, d. \tag{4}$$

Without loss of generality we can assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_d$. Further the origin of the coordinate system, denoted by $b^0 = (0, 0, \ldots, 0)$, is a critical point. Other critical points of (4) are $b^i = (0, 0, \ldots, -\lambda_i/a_i, \ldots, 0)$ where all coordinates equal 0 with exception of $ith, i = 1, 2, \ldots, d$.

Let us get sure that the system (4) has no other critical points. Indeed, let $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_2)$ be its critical point, i.e.

$$\lambda_i \hat{x}_i + \hat{x}_i \sum_j a_j \hat{x}_j = 0, \quad i = 1, 2, \ldots, d, \tag{5}$$

If $\hat{x}_i \neq 0, \hat{x}_j \neq 0$ then

$$\sum_j a_j \hat{x}_j = -\lambda_i, \quad \sum_j a_j \hat{x}_j = -\lambda_j \tag{6}$$

and hence $\lambda_i = \lambda_j$.

Assuming $\hat{x}_i \neq 0, i \neq 1$ by condition G1 we would obtain a contradiction that $\lambda_i \neq \lambda_j, i \neq j$, therefore (5) and condition G1 imply $\hat{x}_i = 0$, for $i = 2, 3, \ldots, d$.

Proposition 1 *For any $i, i = 0, 1, 2, \ldots, d$, one of the critical points is a saddle of the type $(i, d - i)$ (i.e. the number of positive eigenvalues equals i and the number of negative eigenvalues equals $d - i$.)*

Proof The eigenvalues $(\lambda_1^i, \lambda_2^i, \ldots, \lambda_d^i), i = 1, 2, \ldots, d$, of the system (4) at the critical point $b^i \in R^d$ can be easily calculated:

$$\lambda_i^j = \begin{cases} \lambda_j - \lambda_i & if\ i \neq j, \\ -\lambda_i & if\ i = j, \ j = 1, 2, \ldots, d. \end{cases} \tag{7}$$

If $\lambda_1 < 0$ then b^0 is a saddle of the type $(0, d)$ (i.e. a stable node). The formula (7) implies that b^d is a saddle of the type $(d, 0)$ (i.e. an unstable node) and b^i is a saddle of the type $(i, d - i), i = 1, 2, \ldots, d - 1$.

Analogously, if $\lambda_d > 0$ then b^0 is a saddle of the type $(d, 0)$ (i.e. an unstable node) while b^1 is a saddle of the type $(0, d)$ (i.e. a stable node), and b^i is a saddle of the type $(i - 1, d - i + 1), i = 2, \ldots, d$.

Now we expose the correspondence between a position of the spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ with respect to 0 and a type of saddles in the rest cases. Let $\lambda_1 > \cdots > \lambda_k > 0 > \lambda_{k+1} > \cdots > \lambda_d, 1 \leq k \leq d - 1$. Then (7) implies that the critical point b^i is a saddle of the type $(i - 1, d - i + 1)$ for $i \leq k$ and of the type $(i, d - i)$ for $i > k, i = 1, 2, \ldots, d$.

Thus, critical points make up a simplex Σ such that one of the vertices is b^0 while others lie on the points $b^i, i = 1, 2, \ldots, d$.

Proposition 2 *Affine planes spanned to bounds of any dimension of the simplex Σ are invariant.*

Proof It is enough to prove that hyperplanes containing $(d-1)$-bounds of the simplex Σ are invariant, because the others can be obtained from them by the operation of intersection that is preserves invariance.

Invariance of hyperplanes $x_i = 0, i = 1, 2, \ldots, d$ is obvious. Invariance of the hyperplane Π passing through the points b^i, $i = 1, 2, \ldots, d$, and given by the equality

$$\sum_{i=1}^{d} \frac{a_i}{\lambda_i} x_i + 1 = 0. \tag{8}$$

follows from the fact that the derivative of the left-part of the equality (8) by virtue of the system (4) equals zero.

Now we consider the system (4) as a vector field. It generates vector fields on all invariant planes. It turns out the following *reduction principle* holds.

Theorem 1 *The restriction of the system* (4) *to any invariant plane of dimension k has the form* (4) *again(in some Cartesian coordinate system) and satisfies the conditions G1 and G2.*

Proof It is sufficient to consider a restriction of the system (4) to the hyperplane Π. We define the affine coordinate system $(y_1, y_2, \ldots, y_{d-1})$ on Π as follows: a point $(x_1, x_2, \ldots, x_d) \in \Pi$ will have coordinates $y_i = -\frac{a_i}{\lambda_i} x_i$, $i = 1, 2, \ldots, d-1$. Then using the relation $x_n = \frac{\lambda_d}{a_d}(y_1 + y_2 + \cdots + y_{d-1} - 1)$ we will have the system of the form (4)

$$\dot{y}_i = (\lambda_i - \lambda_d) y_i + \left(\sum_{k=1}^{d-1} (\lambda_d - \lambda_i) y_k \right) y_i, \quad i = 1, 2, \ldots, d-1. \tag{9}$$

Because of the proposition $\lambda_i - \lambda_d \neq 0$, for the system (9) the conditions G1 and G2 hold.

According to the reduction principle one can restrict by studying behavior of trajectories not lying on invariant hyperplanes. Further we use the following property of dynamical systems, given on Euclidian spaces: either a solution can be continued to the right up to $+\infty$ (respectively to the left up to $-\infty$,) or $|x(t)| \to \infty$ as $t \to t^* - 0$ for some finite t^* (respectively as $t \to t_* + 0$ for some finite t_*) ([8], corollary 3.2; in our case $\partial E = \phi$.)

At first we consider the trajectory $x(t)$ with an initial condition $x(0) = x_0 \in Int\Sigma$.

Proposition 3 *If* $\lambda_1 > 0$ (*respectively* $\lambda_1 < 0$), *then* $\lim_{t \to +\infty} x(t) = b^1$ (*respectively* $\lim_{t \to +\infty} x(t) = b^0$).

Proof Since the boundary of the simplex Σ is invariant, its interior is also invariant and $x(t) \in Int\Sigma$ for all t. Consequently, the trajectory $x(t)$ is bounded and therefore is defined on the whole interval $(-\infty, \infty)$. In this case, the denominator of the fraction (3), denoted $F(t, \boldsymbol{x}_0)$, must stay positive for all t. (Note that $x_{0i} \neq 0, i = 1, 2, \dots, d$ for $\boldsymbol{x}_0 \in Int\Sigma$).

Further the cases $\lambda_1 > 0$ and $\lambda_1 < 0$ will be considered separately. In the former case $e^{\lambda_1 t}|x_{10}| \to +\infty$ as $t \to +\infty$. This implies that the function $F(t, \boldsymbol{x}_0)$ has to tend to infinity as $t \to +\infty$ (otherwise $\boldsymbol{x}(t)$ can't be bounded). Thus L'Hopital's rule can be applied to evaluate the limit of the fraction (3) which leads to the relation $x_1(t) \to -\lambda_1/a_1$ as $t \to +\infty$.

Besides, $x_i(t) \to 0$ as $t \to +\infty$ for $i = 2, \dots, d$. Thus, $\lim_{t\to+\infty} \boldsymbol{x}(t) = \boldsymbol{b}^1$.

In the case $\lambda_1 < 0$, formula (3) immediately gives

$$\lim_{t\to+\infty} \boldsymbol{x}(t) = \boldsymbol{b}^0.$$

When $t \to -\infty$ we have the inverse portrait: $\boldsymbol{x}(t) \to \boldsymbol{b}^0$ in the case $\lambda_1 > 0$ and $\boldsymbol{x}(t) \to \boldsymbol{b}^1$ in the case $\lambda_1 < 0$. Thus, the following is true.

Proposition 4 *In the case $\lambda_1 > 0$ (respectively $\lambda_1 < 0$) the vertex $\boldsymbol{b}^1$ (the vertex $\boldsymbol{b}^0$) of the simplex Σ will be ω-limit set, and the vertex $\boldsymbol{b}^0$ (the vertex $\boldsymbol{b}^1$) will be α-limit set for all trajectories lying inside of Σ.*

Further, invariant hyperplanes divide R^d into $2^{d+1} - 2$ unbounded open polyhedral domains. Each of them is invariant and has a common piece of boundary with the simplex Σ. There is a one-to-one correspondence between such domains and r-bounds of Σ, $0 \le r \le d - 1$. For example, to each vertex of Σ corresponds to d-hedral cone with that vertex.

We denote by Δ one of these domains and consider the trajectory $\boldsymbol{x}(t)$ passing through the point $\boldsymbol{x}_0 \in Int\Delta$ at t. It is clear that $\boldsymbol{x}(t)$ can't leave Δ. However, here as distinct from the case $\Delta = \Sigma$, being considered above the trajectory may not exist on the whole interval $(-\infty, \infty)$. Let (α, β) be the maximal interval of existence of $\boldsymbol{x}(t)$. If both α and β are finite, then as had been mentioned above

$$\lim_{t\to\alpha+0} |\boldsymbol{x}(t)| = \lim_{t\to\beta-0} |\boldsymbol{x}(t)| = \infty. \tag{10}$$

Thus, in this case $\boldsymbol{x}(t)$ has not any limit point.

Situation (10) and its consequent may be held when $\alpha = -\infty$ or $\beta = +\infty$ as well. So it is remained to consider the cases when trajectory $\boldsymbol{x}(t)$ has limit points when $t \to +\infty$, or $t \to -\infty$. In such cases repeating with insignificant modifications of the arguments conducted for the case of the simplex Σ, we come to the following conclusions:

(a) let $\boldsymbol{x}(t)$ have a limit point when $t \to +\infty$. Then $\boldsymbol{x}(t) \to \boldsymbol{b}^1$ in the case $\lambda_1 > 0$ and $\boldsymbol{x}(t) \to \boldsymbol{b}^0$ in the case $\lambda_1 < 0$;
(b) let $\boldsymbol{x}(t)$ have a limit point when $t \to -\infty$. Then $\boldsymbol{x}(t) \to \boldsymbol{b}^0$ in the case $\lambda_1 > 0$ and $\boldsymbol{x}(t) \to \boldsymbol{b}^1$ in the case $\lambda_1 < 0$.

Thus, for the trajectories in Δ there are possible the following types of behaviour:

Type I. Trajectory comes from infinity and leaves to infinity.

Type II. Trajectory comes from infinity and tends to one of the critical points $\boldsymbol{b}^0$, $\boldsymbol{b}^1$ (depending on the conditions $\lambda_1 < 0, \lambda_1 > 0$ respectively).

Type III. Trajectory goes out from one of critical points $\boldsymbol{b}^0$, $\boldsymbol{b}^d$ (depending on the conditions $\lambda_d > 0, \lambda_d < 0$ respectively) and goes off to infinity.

Type IV. Trajectory goes out from an critical point and tends to another. In order to concretize this type of behavior let us depict the situation $\boldsymbol{b}^\alpha = \lim_{t\to-\infty} \boldsymbol{x}(t), \boldsymbol{b}^\omega = \lim_{t\to+\infty} \boldsymbol{x}(t)$, symbolically as $\boldsymbol{b}^\alpha \to \boldsymbol{b}^\omega$.

Then

IV^a. $\boldsymbol{b}^0 \rightarrowtail \boldsymbol{b}^1$ if $\lambda_d > 0$;
IV^b. $\boldsymbol{b}^d \rightarrowtail \boldsymbol{b}^0$ if $\lambda_1 < 0$;
IV^c. $\boldsymbol{b}^d \rightarrowtail \boldsymbol{b}^1$ if $\lambda_1 > 0$ and $\lambda_d < 0$.

Theorem 2 *Every trajectory lying in one of the domains* Δ *and defined on the interval* $(-\infty, \infty)$ *belongs to one of the types* $\mathrm{IV}^a - \mathrm{IV}^c$ *only.*

Proof The statement is followed from the fact that existence of the finite limits $\lim_{t\to+\infty} \boldsymbol{x}(t)$, $\lim_{t\to-\infty} \boldsymbol{x}(t)$ and their values for the trajectories defined on the interval $(-\infty, \infty)$ depend on a situation of the spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_d\}$ relatively 0 only.

One can conclude observing upper considerations that each domain Δ including Σ contains only trajectories one of the types I–IV, for all t, $t \in (-\infty, \infty)$.

The following example shows there may be either continuable or incontinuable trajectories in some of the domains Δ.

What concerns trajectories not necessarily defined on $(-\infty, \infty)$, here types of trajectories may be mixed.

Example $\dot{x} = x + (x + 2y)x, \quad \dot{y} = 2y + (x + 2y)y.$

The straights $x = 0$, $y = 0$ and $x + y + 1 = 0$ divides the plane into 7 domains as depicted in Fig. 1 (not continuability of trajectories up to $+\infty$ or $-\infty$ showed by dashed lines). Each invariant domain is filled by trajectories one of the types I–IV only except for the domain $x > 0$, $-(1 + x) < y < 0$, containing three types trajectories defined on intervals of the types $(-\infty, +\infty)$, $(-\infty, \beta)$, $(\alpha, +\infty)$.

4 The Case With One Couple of Complex Eigenvalues

Let $\alpha \pm i\beta$ be a unique couple of complex eigenvalues, $\alpha \neq 0$, $\beta > 0$ [6]. In this case, the system (1) can be written in the form

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - \beta x_2 + (\boldsymbol{a} \cdot \boldsymbol{x})x_1, \\ \dot{x}_2 &= \beta x_1 + \alpha x_2 + (\boldsymbol{a} \cdot \boldsymbol{x})x_2, \\ \dot{y} &= Cy + (\boldsymbol{a} \cdot \boldsymbol{x})y, \end{aligned} \tag{11}$$

where $\boldsymbol{x} = (x_1, x_2, y)$, $\boldsymbol{x} \in \mathbb{R}^d$, $x_1, x_2 \in \mathbb{R}$, $y \in \mathbb{R}^{d-2}$, $\boldsymbol{a}$ is a constant vector, $\boldsymbol{a} \in \mathbb{R}^d$. (As before assumed that conditions G1, G2 hold. We remind that C is a diagonal matrix composed of $\lambda_3, \lambda_4, \dots, \lambda_n$.)

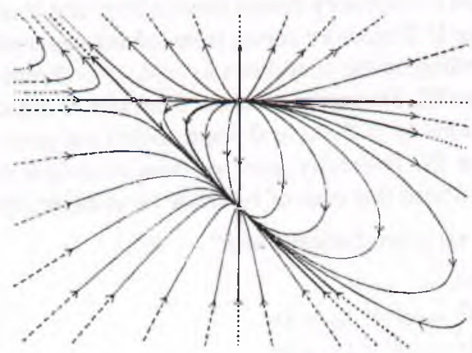

Fig. 1 The portrait for the example

The system (11) has $d-1$ critical points $\boldsymbol{b}^0 = (0, 0, \ldots, 0)$, $\boldsymbol{b}^j = (0, 0, \ldots, -\lambda_j/a_j, 0, \ldots, 0)$, $j = 3, \ldots, d$ and, as a result, the coordinate hyperplanes $y_i = 0$, $i = 1, \ldots, d-2$ keep their invariance only, while $x_1 = 0$ and $x_2 =$ will not be invariant. Here the inclined invariant hyperplane $\frac{a_1\alpha - a_2\beta}{\alpha^2+\beta^2}x_1 + \frac{a_2\alpha + a_1\beta}{\alpha^2+\beta^2}x_2 + \sum_{i=3}^{d} \frac{a_i}{\lambda_i} y_i + 1 = 0$ presents also.

For clarifying of the structure of the phase space we use bifurcation method [5,8]. For that let us join the system (11) with system that all eigenvalues are real. A simple way of that is to include system (11) to the one-parameter family

$$\begin{aligned} \dot{x}_1 &= a_{11}(\mu)x_1 - a_{12}(\mu)x_2 + (\boldsymbol{a}\cdot\boldsymbol{x})x_1, \\ \dot{x}_2 &= a_{21}(\mu)x_1 + a_{22}(\mu)x_2 + (\boldsymbol{a}\cdot\boldsymbol{x})x_2, \\ \dot{y}_j &= \lambda_j y_j + (\boldsymbol{a}\cdot\boldsymbol{x})y_j, \end{aligned} \tag{12}$$

where $a_{11}(\mu) = \lambda_1 + (\alpha - \lambda_1)\mu$, $a_{12}(\mu) = -\mu\beta$, $a_{21}(\mu) = \mu\beta$, $a_{22}(\mu) = \lambda_2 + (\alpha - \lambda_2)\mu$, $\boldsymbol{a} \in \mathbb{R}^d$, $j = 3, \ldots, d$, λ_1, λ_2 are arbitrarily selected real numbers not equal to zero, $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_d$, μ is a real parameter of bifurcation, $\mu \in [0, 1]$.

When $\mu = 0$ the matrix $A(0)$ is diagonal and satisfies the conditions G1 and G2. Hence, in this case phase portrait has been depicted already. Now we study changing of phase portrait when the parameter μ increases from 0 to 1. Let

$$\Delta = \begin{vmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a_1 & a_2 \\ a_{21}(\mu) & a_{22}(\mu) \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{12}(\mu) & a_{22}(\mu) \\ a_1 & a_2 \end{vmatrix},$$

$$a_{11}(\mu) = \lambda_1 + (\alpha - \lambda_1)\mu, \quad a_{12} = -\mu\beta, \quad a_{21} = \mu\beta, \quad a_{22} = \lambda_2 + (\alpha - \lambda_2)\mu.$$

Proposition 5 *The hyperplane*

$$\frac{\Delta_1}{\Delta}x_1 + \frac{\Delta_2}{\Delta}x_2 + \sum_{i=3}^{d} \frac{a_i}{\lambda_i}x_i + 1 = 0 \tag{13}$$

is invariant for every $\mu \in [0, 1]$.

Proof is analogously with one for the Proposition 2.

Let us put $D(\mu) = (\mu - 1)^2(\lambda_2 - \lambda_1)^2 - 4\mu^2\beta^2$ (the discriminant of the characteristic equation of the matrix $A(\mu)$). Equation $D(\mu) = 0$ has the unique positive solution $\mu^* = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + 2\beta}$ laying strongly between 0 and 1 that will serve as a bifurcation value of μ.

For $0 < \mu < \mu^*$ the system (12) has $d + 1$ critical points as before. Two of them:

$$b^i(\mu) = (b_1^i(\mu), b_2^i(\mu), 0, \dots, 0), \quad i = 1, 2,$$

where

$$b_1^i(\mu) = -\frac{(\lambda_1 + \mu(\alpha - \lambda_1))t^i - \mu\beta}{a_1 t^i + a_2}, \quad b_2^i(\mu) = \frac{b_1^i(\mu)}{t^i}, \quad i = 1, 2, \tag{14}$$

$$t^{1,2}(\mu) = \frac{(\mu - 1)(\lambda_2 - \lambda_1) \pm \sqrt{D}}{2\mu\beta}, \tag{15}$$

while the other critical points $b^0 = (0, 0, \dots, 0)$, $b^j = (0, 0, \dots, -\lambda_j/a_j, 0, \dots, 0)$, $j = 3, \dots, d$ do not depend on μ. Similarly there are invariant hyperplanes $b_2^i(\mu)x_1 - b_1^i(\mu)x_2 = 0$ $(i = 1, 2)$, $x_j = 0$ $(j = 3, \dots, d)$ and (13).

Thus the phase portrait for $\mu \in (0, \mu^*)$ coincides with the one for the case $\mu = 0$.

Formulas (14)–(15) imply that as $\mu \to \mu^* - 0$ the following properties hold:

1. the critical points $b^1(\mu)$ and $b^2(\mu)$, having come closer, tend to common limit $b^*(\mu^*)$ being a saddle-node critical point: $b^*(\mu^*) = (b_1^*, b_2^*, 0, \dots, 0)$, $b_1^* = b_2^* = -\frac{\lambda_1 + \mu^*(\alpha - \lambda_1 - \beta)}{a_1 + a_2}$ (other critical points do not change);
2. invariant hyperplanes $b_2^i(\mu)x_1 - b_1^i(\mu)x_2 = 0$, $i = 1, 2$, passing through critical points b^0, $b^i(\mu)$ $(i = 1, 2)$ and b^j $(j = 3, \dots, d)$, also stick together to common limit, that would be the invariant hyperplane $x_1 - x_2 = 0$, while hyperplanes $x_j = 0$, $j = 3, \dots, d$, don't change.

Therefore, when $\mu = \mu^*$ the number of critical points and invariant hyperplanes decreases exactly by one.

When $\mu^* < \mu < 1$ "saddle-node" critical point disappears, happening "breakdown of saddle-node equilibrium" bifurcation [5–7]. As a result there remained critical points b^0, b^j, $j = 3, \dots, d$, only. The number of invariant hyperplanes also decreases to $d - 1$.

We point out to essential property of the system (12): in spite of to disappearing of two critical points, inclined invariant hyperplane having passed through them until the bifurcation value $\mu = \mu^*$, keeps its track as (13) for $\mu^* < \mu \le 1$ too.

Now as in the Sect. 2 one can establish that any trajectory belongs to one of the types I–IV, that completes the construction of the phase portrait.

In Fig. 2, the corresponding portrait for $d = 2$ is shown.

In Fig. 3 it is shown decreasing of the number of critical points and invariant hyperplanes (the later are marked by "a local coordinate system") for the case $d = 3$.

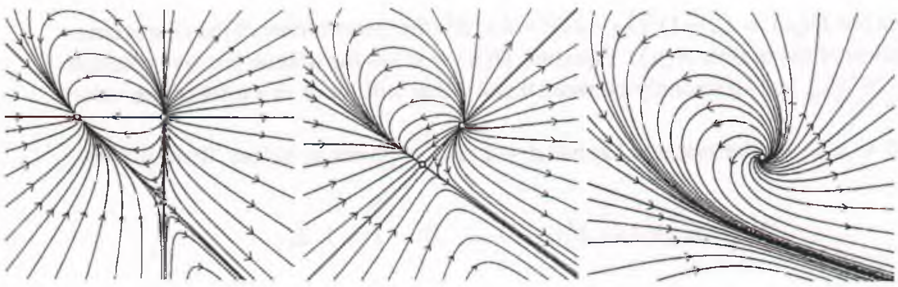

Fig. 2 A saddle-node bifurcation, $d = 2$

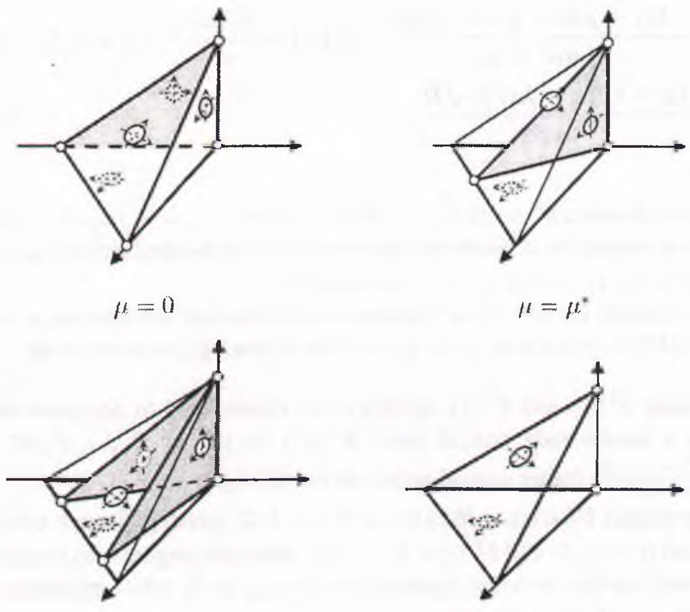

Fig. 3 A saddle-node bifurcation, $d = 3$

5 The Case With M Couples of Complex Eigenvalues

Now let us suppose the matrix A has m pairs of complex eigenvalues $\alpha_i \pm i\beta_i$, $\beta_i > 0$, $i = 1, \dots, m$, while $n - 2m$ eigenvalues are real. Then system (1) can be written in the form

$$\begin{aligned} \dot{x}_{2i-1} &= \alpha_i x_{2i-1} - \beta_i x_{2i} + (a \cdot x)x_{2i-1}, \\ \dot{x}_{2i} &= \beta_i x_{2i-1} + \alpha_i x_{2i} + (a \cdot x)x_{2i}, \quad i = 1, 2, \dots, m, \\ \dot{y} &= Cy + (a \cdot x)y, \end{aligned} \tag{16}$$

where $y \in \mathbb{R}^{d-2m}$, so that $x = (x_1, x_2, \dots, x_{2m}, y) \in \mathbb{R}^d$.

The system (16) has $d+1-2m$ critical points $\boldsymbol{b}^0 = (0, 0, \dots, 0)$, $\boldsymbol{b}^j = (0, 0, \dots, -\lambda_j/a_j, 0, \dots, 0)$, $i = 2m+1, \dots, d$, [in particular for $d = 2m$ there is only unique critical point $\boldsymbol{b}^0 = (0, 0, \dots, 0)$] and the same number of invariant coordinate planes $y_i = 0$, $(i = 1, \dots, d-2m)$ and the inclined invariant hyperplane

$$\sum_{i=1}^{m} \left(\frac{a_{2i-1}\alpha_i - a_{2i}\beta_i}{\alpha_i^2 + \beta_i^2} x_{2i-1} + \frac{a_{2i}\alpha_i - a_{2i-1}\beta_i}{\alpha_i^2 + \beta_i^2} x_{2i} \right) + \sum_{j=2m+1}^{d} \frac{a_j}{\lambda_j} x_j + 1 = 0.$$

In order to clear up the phase portrait we include the system (16) to one-parameter family

$$\begin{aligned} \dot{x}_{2i-1} &= a_{2i-1,2i-1}(\mu)x_{2i-1} - a_{2i-1,2i}(\mu)x_{2i} + (\boldsymbol{a}\cdot\boldsymbol{x})x_{2i-1}, \\ \dot{x}_{2i} &= a_{2i,2i-1}(\mu)x_{2i-1} + a_{2i,2i}(\mu)x_{2i} + (\boldsymbol{a}\cdot\boldsymbol{x})x_{2i}, \\ \dot{y}_j &= \lambda_j y_j + (\boldsymbol{a}\cdot\boldsymbol{x})y_j, \end{aligned} \tag{17}$$

where $a_{2i-1,2i-1}(\mu) = \lambda_{2i-1} + (\alpha_i - \lambda_{2i-1})\mu$, $a_{2i-1,2i}(\mu) = -\mu\beta_i$, $a_{2i,2i-1}(\mu) = \mu\beta_i$, $a_{2i,2i}(\mu) = \lambda_{2i} + (\alpha_i - \lambda_{2i})\mu$, $i = 1, 2, \dots, m$, $j = 2m+1, \dots, d$, $\boldsymbol{a} \in \mathbb{R}^d$. What concerns the real numbers $\lambda_1, \lambda_2, \dots, \lambda_{2m}$ they are arbitrarily selected, but should be different, differ from $\lambda_{2m+1}, \lambda_{2m+2}, \dots, \lambda_n$, and values

$$\frac{\lambda_{2i-1} - \lambda_{2i}}{\beta_i}, \quad i = 1, 2, \dots, m, \tag{18}$$

are different as well (μ is a bifurcation parameter as above, $\mu \in [0, 1]$).

When $\mu = 0$ the matrix $A(0)$ is diagonal and so the constructions of the Sect. 3 are valid.

If μ increases from 0 to 1 saddle-node type bifurcations will occur m times.

Note that for all values of parameter μ invariant hyperplane is kept in the form

$$\sum_{i=1}^{m} \left(\frac{\Delta_i^1}{\Delta_i} x_{2i-1} + \frac{\Delta_i^2}{\Delta_i} x_{2i} \right) + \sum_{j=2m+1}^{d} \frac{a_j}{\lambda_j} x_j + 1 = 0, \tag{19}$$

where

$$\Delta_i^1 = \begin{vmatrix} a_{2i-1} & a_{2i} \\ a_{2i,2i-1}(\mu) & a_{2i,2i}(\mu) \end{vmatrix}, \quad \Delta_i^2 = \begin{vmatrix} a_{2i-1,2i-1}(\mu) & a_{2i-1,2i}(\mu) \\ a_{2i-1} & a_{2i} \end{vmatrix},$$

$$\Delta_i = \begin{vmatrix} a_{2i-1,2i-1}(\mu) & a_{2i-1,2i}(\mu) \\ a_{2i,2i-1}(\mu) & a_{2i,2i}(\mu) \end{vmatrix}, \quad i = 1, 2, \dots, m.$$

Let us put $D_i(\mu) = (\mu-1)^2(\lambda_{2i} - \lambda_{2i-1})^2 - 4\mu^2\beta_i^2$, $i = 1, 2, \dots, m$. Each equation $D_i(\mu) = 0$ has the unique positive solution $\mu_i^* = \frac{\lambda_{2i-1}-\lambda_{2i}}{\lambda_{2i-1}-\lambda_{2i}+2\beta_i}$, $\mu_i^* \in (0, 1)$, serving as a bifurcation value of μ. The condition (17) implies the values μ_i^* are different, $i = 1, 2, \dots, m$.

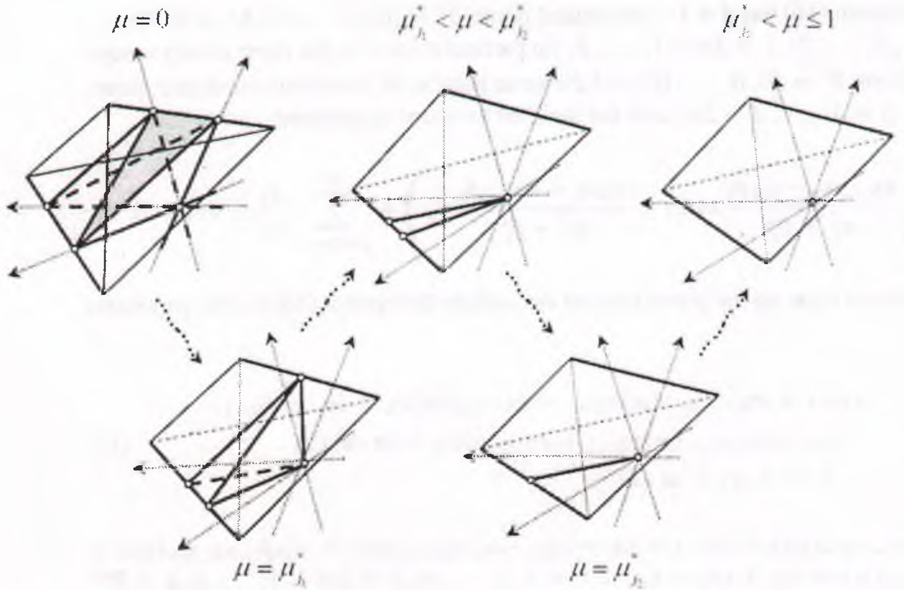

Fig. 4 Two consequent saddle-node bifurcation, $d = 4$

Let us rearrange the sequnce $\mu_i^*, i = 1, 2, \dots, m$ in the increasing order $\mu_{j_1}^*, \mu_{j_2}^*, \mu_{j_3}^*, \mu_{j_4}^*, \dots, \mu_{j_{m-1}}^*, \mu_{j_m}^*$ the least of the bifurcation values $\mu_i^* = \frac{\lambda_{2i-1}-\lambda_{2i}}{\lambda_{2i-1}-\lambda_{2i}+2\beta_i}, i = 1, 2, \dots, m$, denote by $\mu_{j_1}^*$, $j_1 \in \{1, 2, \dots, m\}$, the least of the residuary denote $\mu_{j_2}^*$. In the same way bifurcation values are defined.

Now notice that when μ changes from 0 to $\mu_{j_1}^*$ and passes through $\mu_{j_1}^*$ behavior of the system (17) coincides with the case considered in the Sect. 4.

Analogous bifurcations will happen when μ passes through values $\mu_{j_l}^*, l = 2, \dots, m$, also. Namely, when $\mu \to \mu_{j_l}^* - 0$ two of the residuary critical points after previous bifurcations tend to common limit being a saddle-node (other critical points save hiperpolicity). Just as two of residuary invariant hyperplanes tend as a sloping sheet of paper to common limit (the other are kept).

Thus, when $\mu_{j_l}^*, l = 2, \dots, m$, the system has $d+2-2l$ critical points and $d+2-2l$ invariant hyperplanes.

Then, at $\mu_{j_{l-1}}^* < \mu < \mu_{j_l}^*, l = 2, \dots, m$, a pair of eigenvalues turns to a pair of complex eigenvalues. Here "saddle-node" critical point $b^*(\mu_{j_l}^*)$ disappears and $d + 1 - 2l$ critical points and as many invariant hyperplanes reminded.

When $l = m$ we obtain the last bifurcation after that the system will have $d+2-2m$ critical points and $d + 2 - 2m$ invariant hyperplanes. Trajectories belong to one of types I-IV as before. These properties are held naturally for $\mu = 1$ too, i.e. for the given system (16).

In particular, if an order of system is even and all eigenvalues are complex then system will have unique critical point $b^0 = (0, 0, \dots, 0)$ and unique invariant hyperplane $\sum_{i=1}^{n/2} \left(\frac{a_{2i-1}\alpha_i - a_{2i}\beta_i}{\alpha_i^2+\beta_i^2} x_{2i-1} + \frac{a_{2i}\alpha_i - a_{2i-1}\beta_i}{\alpha_i^2+\beta_i^2} x_{2i} \right) + 1 = 0$ for the trajectories of system it remind only three possibilities of the types I, II and III.

In Fig. 4 it is shown decreasing of the number of critical points and invariant hyperplanes (the later are marked by "a local coordinate system") for the case $d = 4$.

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