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«*Oddiy Li algebralarida lokal avtomorfizmlar*»

mavzusidagi

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MUNDARIJA

KIRISH

So'nggi o'n yil mobaynida bir qator ilmiy ishlar assotsiativ algebralarning avtomorfizmlari va differensiallashlariga doir muammolarni o'rganishga bag'ishlandi. Masalan, lokal avomorfizmlar va 2-lokal avtomorfizmlarga doir muammolar shular jumlasidandir. Keyinroq bu muammolar assotsiativ bo'lmagan algebralar uchun, xususan, Li algebralari uchun o'rganildi. Bu bitiruv malakaviy ishida sl_n(izi nolga teng bo'lgan barcha matritsalar algebrasi)da lokal avtomorfizmlar o'rganilgan.

A biror assotsiativ algebra bo'lsin. $\Phi : A \to A$ chiziqli akslantirish lokal avtomorfizm deyiladi, agar ixtiyoriy $x \in A$ uchun x ga bog'liq $\Phi_x$avtomorfizm topilib, $\Phi(x) = \Phi_x(x)$ tenglik o'rinli bo'lsa. Bu tushunchani Keydison [14], Larson va Sourour [16] lar bir-biridan mustaqil ravishda o'rgana boshlashgan. Keyinroq 1997-yilda P.Shemrl [19] 2-lokal avtomorfizm va 2-lokal differensiallash tushunchalarini o'rgangan. Agar $\Phi : A \to A$ chiziqli (umumiy holda ixtiyoriy) akslantirishda ixtiyoriy $x, y \in A$ uchun $\Phi_{x,y}$ avtomorfizm (differensiallash) topilib, (x, y ga bog'liq), $\Phi(x) = \Phi_{x,y}(x)$, $\Phi(y) = \Phi_{x,y}(y)$ tengliklar o'rinli bo'lsa, u holda Φ akslantirish 2-lokal avtomorfizm (2-lokal differensiallash) deyiladi. P.Semrl H Gilbert fazosidagi barcha chegaralangan operatorlar algebrasidagi har bir 2-lokal avtomorfizm avtomorfizm bo'lishini ko'rsatgan.

Yuqoridagi tushunchalar assotsiativ algebralar uchun keltirilgan. Aytaylik L biror Li algebra bo'lsin. Agar $\Phi : L \to L$ chiziqli akslantirish uchun barcha $x, y \in L$ larda$\Phi[x, y] = [\Phi(x), \Phi(y)]$ tenglik o'rinli bo'lsa, u holda Φ avtomorfizm deyiladi. Li algebralari uchun local va 2-lokal avtomorfizm ta'riflari assotsiativ algebralardagi ta'riflari bilan bir xil keltiriladi.

Chen va Wang [11] da chekli o'lchamli Li algebralar uchun 2-lokal avtomorfizmlarni o’rganishni boshlaydi. Ular oddiy Li algebralarida har bir 2-lokal avtomorfizm avtomorfizm ekanligini ko’rsatishgan. Ayupov va Kudaybergenov [7] da [11] dagi natijani umumiy holda, ya’ni, chekli o’lchamli yarim oddiy Li algebrasidagi har bir 2-lokal avtomorfizm avtomorfizm bo’lishini isbotlashgan.

Bu ish kirish qismi, besh paragraf, xulosa va foydalanilgan adabiyotlar ro’yhatidan iborat.

Birinchi paragraf Li algabralarining umumiy xossalari deb nomlanadi va bu paragrafda Li algebrasining ta’rifi, misollar va bir nechta xossalari keltirilgan.

Ikkinchi paragrafda nilpotent Li algebralari ta’rifi, nilpotent va nilpotent bo’lmagan Li algebralariga misollar o’rganilgan.

Uchinchi paragraf oddiy Li algebralariga bag’ishlangan va oddiy Li algebralarga oid teoremalar qaralgan.

To’rtinchi paragraf oddiy Li algebralarida avtomorfizmlar deb nomlanadi va bu paragrafda avtomorfizm ta’rifi, misollar va teoremalar keltirilgan.

Bu ishning asosiy qismi bo’lgan beshinchi paragrafda sl_n dagi lokal avtomorfizmning avtomorfizm bo’lishining zaruriy va yetarli sharti o’rganilgan.

§1. Li algebralarining umumiy xossalari

F sonli maydon va L bu F maydon ustida biror chiziqli fazo bo'lsin. Agar

$$[\cdot,\cdot]: L \times L \to L \tag{1.1}$$

akslantirish quyidagi xossalarga ega bo'lsa:

$$[\alpha_1 x_1 + \alpha_2 x_2, y] = \alpha_1 [x_1, y] + \alpha_2 [x_2, y]$$

$$[x, \beta_1 y_1 + \beta_2 y_2] = \beta_1 [x, y_1] + \beta_2 [x_1, y_1]$$

u holda bu akslantirish bichiziqli forma deyiladi.

1.1-ta'rif. Bizga **F** maydon ustida Lchiziqli fazo va unda aniqlangan $[\cdot,\cdot]: L^2 \to L$ bichiziqli akslantirish berilgan bo'lib , u quyidagi shartlarni qanoatlantirsin:

$$\forall x \in L \quad \text{uchun} \quad [x, x] = 0 \tag{L1}$$

$$\forall x, y, z \in L \quad \text{uchun} \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \,. \tag{L2}$$

U holda L chiziqli fazo Li algebra deyiladi.

1.1-ta'rifdagi (L2) shart Yakobi ayniyati deyiladi. $[x, y]$ elementga x va y elementlarning ko'paytmasi yoki kommutatori deyiladi. (L1) dan Li algebrasi antikommutativ ekanligi kelib chiqadi. Haqiqatan ,

$$[x+y, x+y]=0$$

bundan

$$[x,x]+[x,y]+[y,x]+[y,y]=0$$

kelib chiqadi, (L1) ni hisobga olsak

$$[x,y]+[y,x]=0 \qquad \text{(L1')}$$

ekanligi kelib chiqadi. Agar $charF \neq 2$ bo'lsa , u holda (L1') dan (L1) kelib chiqadi.

1.1-*misol.* Aytaylik A assotsiativ algebra bo'lsin, ya'ni,

$$x(yz)=(xy)z, \forall x,y,z \in A \ .$$

U holda A algebra $[x,y]=xy-yx$ kabi aniqlangan ko'paytirishga nisbatan Li algebra bo'ladi .Dastlab bichiziqli ekanini tekshiramiz. Unda

$$[\alpha_1 x_1+\alpha_2 x_2, y]=-[y,\alpha_1 x_1+\alpha_2 x_2]=$$
$$=(\alpha_1 x_1+\alpha_2 x_2)y-y(\alpha_1 x_1+\alpha_2 x_2)=$$
$$=(\alpha_1 x_1)y+(\alpha_2 x_2)y-y(\alpha_1 x_1)-y(\alpha_2 x_2)=$$
$$=\alpha_1(x_1 y)+\alpha_2(x_2 y)-\alpha_1(yx_1)-\alpha_2(yx_2)=$$
$$=\alpha_1(x_1 y-yx_1)+\alpha_2(x_2 y-yx_2)=\alpha_1[x_1,y]+\alpha_2[x_2,y]$$

hamda

$$[x,\beta_1 y_1+\beta_2 y_2]=x(\beta_1 y_1+\beta_2 y_2)-(\beta_1 y_1+\beta_2 y_2)x=$$
$$=\beta_1 xy_1+\beta_2 xy_2-\beta_1 y_1 x-$$
$$-\beta_2 y_2 x=\beta_1(xy_1-y_1 x)+\beta_2(xy_2-y_2 x)=$$
$$=\beta_1[x,y_1]+\beta_2[x,y_2].$$

Endi Yakobi ayniyatini tekshiramiz.

$1^{\circ})$ $\left[x,[y,z]\right] = x(yz - zy) - (yz - zy)\cdot x = xyz - xzy - yzx + zyx$,

$2^{\circ})$ $\left[y,[z,x]\right] = y(zx - xz) - (zx - xz)\cdot y = yzx - yxz - zxy + xzy$,

$3^{\circ})$ $\left[z,[x,y]\right] = z(xy - yx) - (xy - yx)\cdot z = z\cdot xy - zyx - xyz + yxz$

tengliklarni qo’shsak talab etilgan ayniyat kelib chiqadi.

1.2-misol. $L = \mathbb{R}^3$ chiziqli fazoni qaraymiz. Bu fazo $[x, y]$ vektor ko’paytmaga nisbatan Li algebra bo’ladi. Bunda $x = (\mu_1, \mu_2, \mu_3)$ $y = (\nu_1, \nu_2, \nu_3)$ vektorlar ko’paytmasi

$$[x, y] = \left(\begin{vmatrix} \mu_2 & \mu_3 \\ \nu_2 & \nu_3 \end{vmatrix}; \begin{vmatrix} \mu_3 & \mu_1 \\ \nu_3 & \nu_1 \end{vmatrix}, \begin{vmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{vmatrix} \right). \tag{1.2}$$

(1.2) formula bilan aniqlangan Li ko’paytmasiga nisbatan R^3 fazo Li algebrasi ekanligini ko’rsatamiz. Uning uchun dastlab quyidagi tenglikni isbotlaymiz.

$$\left[x,[y,z]\right] = (x,z)\,y - (x,y)\,z \; . \tag{1.3}$$

Yakobi ayniyatini isbotlash uchun dastlab (1.2) tenglikni tekshiramiz.

$$\left[x,[y,z]\right] = \left(\begin{vmatrix} \mu_2 & \mu_3 \\ \nu_2 & \nu_3 \end{vmatrix}; \begin{vmatrix} \mu_3 & \mu_1 \\ \nu_3 & \nu_1 \end{vmatrix}, \begin{vmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{vmatrix} \right) =$$

$$= \left| \begin{array}{cc} \mu_2 & \mu_3 \\ \begin{vmatrix} \nu_3 & \nu_1 \\ \gamma_3 & \gamma_1 \end{vmatrix} & \begin{vmatrix} \nu_1 & \nu_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} \end{array} \right|, \left| \begin{array}{cc} \mu_3 & \mu_1 \\ \begin{vmatrix} \nu_1 & \nu_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} & \begin{vmatrix} \nu_2 & \nu_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} \end{array} \right|, \left| \begin{array}{cc} \mu_1 & \mu_2 \\ \begin{vmatrix} \nu_2 & \nu_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} & \begin{vmatrix} \nu_1 & \nu_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} \end{array} \right|.$$

1-koordinatasi

$$\begin{vmatrix} \mu_2 & \mu_3 \\ \nu_3\gamma_1 - \nu_1\gamma_3 & \nu_1\gamma_2 - \nu_2\gamma_1 \end{vmatrix} =$$

$$= \mu_2\nu_1\gamma_2 - \mu_2\nu_2\gamma_1 - \mu_3\left(\nu_3\gamma_1 - \nu_1\gamma_3\right) =$$

$$= \mu_2\nu_1\gamma_2 - \mu_2\nu_2\gamma_1 - \mu_3\nu_3\gamma_1 + \mu_3\nu_1\gamma_3,$$

2-koordinatasi

$$\begin{vmatrix} \mu_3 & \mu_1 \\ \nu_1\gamma_2 - \nu_2\gamma_1 & \nu_2\gamma_3 - \nu_3\gamma_2 \end{vmatrix} =$$

$$= \mu_3\left(\nu_2\gamma_3 - \nu_3\gamma_2\right) - \mu_1\left(\nu_1\gamma_2 - \nu_2\gamma_1\right) =$$

$$= \mu_3\nu_2\gamma_3 - \mu_3\nu_3\gamma_2 - \mu_1\nu_1\gamma_2 + \mu_1\nu_2\gamma_1,$$

3-koordinatasi

$$\begin{vmatrix} \mu_1 & \mu_2 \\ \nu_2\gamma_3 - \nu_3\gamma_2 & \nu_1\gamma_2 - \nu_2\gamma_1 \end{vmatrix} =$$

$$= \mu_1\left(\nu_1\gamma_2 - \nu_2\gamma_1\right) - \mu_2\left(\nu_2\gamma_3 - \nu_3\gamma_2\right) =$$

$$= \mu_1\nu_1\gamma_2 - \mu_1\nu_2\gamma_1 - \mu_2\nu_2\gamma_3 + \mu_2\nu_3\gamma_2.$$

Yuqoridagidan foydalansak:

$$\left[y,\left[z,x\right]\right] = \left(y,x\right)z - \left(y,z\right)x ,$$

$$\left[z,\left[x,y\right]\right] = \left(z,y\right)z - \left(z,x\right)y ,$$

$$\left[x,\left[y,z\right]\right]=\left(x,z\right)y-\left(x,y\right)z\ .$$

U holda

$$\left[x\left[y,z\right]\right]+\left[y,\left[z,x\right]\right]+\left[z,\left[x,y\right]\right]=$$
$$=\left(x,z\right)y-\left(x,y\right)z+\left(y,x\right)z-\left(y,z\right)x+$$
$$+\left(z,y\right)x-\left(z,x\right)y=0.$$

$x=\left(\mu_1,\mu_2,\mu_3\right)$, $y=\left(v_1,v_2,v_3\right)$, $z=\left(\gamma_1,\gamma_2,\gamma_3\right)$ uchun $\left(x,z\right)y-\left(x,y\right)z$ vektor koordinatalari:

1-koordinatasi

$$\left(\mu_1\gamma_1+\mu_2\gamma_2+\mu_3\gamma_3\right)v_1-\left(\mu_1v_1+\mu_2v_2+\mu_3v_3\right)\gamma_1=$$
$$=\mu_1\gamma_1v_1+\mu_2\gamma_2v_1+\mu_3\gamma_3v_1-\mu_1\gamma_1v_1-\mu_2\gamma_2v_1-\mu_3\gamma_3v_1=$$
$$=\mu_2\left(\gamma_2v_1-v_2\gamma_1\right)+\mu_3\left(\gamma_3v_1-v_3\gamma_1\right),$$

2-koordinatasi

$$\left(\mu_1\gamma_1+\mu_2\gamma_2+\mu_3\gamma_3\right)v_2-\left(\mu_1v_1+\mu_2v_2+\mu_3v_3\right)\gamma_2=$$
$$=\mu_1\gamma_1v_2+\mu_2\gamma_2v_2+\mu_3\gamma_3v_2-\mu_1\gamma_1v_2-\mu_2\gamma_2v_2-\mu_3\gamma_3v_2=$$
$$=\mu_1\left(\gamma_1v_2-v_1\gamma_2\right)-\mu_3\left(v_3\gamma_2-\gamma_3v_2\right),$$

3-koordinatasi

$$\left(\mu_1\gamma_1+\mu_2\gamma_2+\mu_3\gamma_3\right)v_3-\left(\mu_1v_1+\mu_2v_2+\mu_3v_3\right)\gamma_3=$$
$$=\mu_1\gamma_1v_3+\mu_2\gamma_2v_3-\mu_1v_1\gamma_3-\mu_2v_2\gamma_3.$$

Mos koordinatalar tengligidan Yakobi ayniyati kelib chiqadi.

Bizga F maydon ustida chekli o'lchamli V vektor fazo berilgan bo'lsin. V dagi barcha chiziqli operatorlar to'plamini $EndV$ kabi belgilaymiz. Bu to'plamda algebraik amallarni quyidagicha kiritaylik: $x, y \in End(V)$ uchun

$$(x+y)(\xi) = x(\xi) + y(\xi), \quad \xi \in V,$$

$$(\lambda x)(\xi) = \lambda \cdot x(\xi), \quad \xi \in V,$$

$$(xy)(\xi) = y(x(\xi)), \quad \xi \in V.$$

Bu algebraik amallarga nisbatan $End\ (V)$ to'plam assotsiativ algebra boladi. Nol vektor: $\theta(\xi) = 0, \quad \xi \in V$.

Algebraning assotsiativlik shartini tekshiraylik:

$$x(yz) = (xy)z.$$

Unda

$$(x(yz))(\xi) = (yz)(x(\xi)) = z(y(x(\xi))),$$

$$((xy)z)(\xi) = z((xy)(\xi)) = z(y(x(\xi))),$$

$$(x(yz))(\xi) = ((xy)z)(\xi), \quad \forall \xi \in V.$$

Bundan

$$x(yz) = (xy)z.$$

Yuqoridagi 1.1-misoldan $End\ (V)$ fazo $[x, y] = xy - yx$ ko'paytmaga nisbatan Li algebrasi boladi. Bu algebra $gl(V)$ orqali belgilanadi.

1.1-teorema. Agar $\dim V = n$ bo'lsa, u holda $gl(V)$ algabra va $M_n(F)$ o'zaro izomorf bo'ladi.

Bu yerda $M_n(F)$ F maydon ustida $n \times n$ matritsalar algabrasi.

Isbot: V fazoda $e_1, e_2, \ldots, e_n$ bazisni olamiz. U holda $\forall \xi \in V$, vektorni $\xi = \sum_{i=1}^{n} \lambda_i e_i$ kabi ifodalash mumkin.

$$x(\xi) = x\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sum_{i=1}^{n} \lambda_i x(e_i) = \lambda_1 x(e_1) + \lambda_2 x(e_2) + \ldots + \lambda_n x(e_n)$$

bo'ladi, chunki x chiziqli operator.

Quyidagini hisobga olsak,

$$\begin{aligned} x(e_1) &= a_{11}e_1 + a_{21}e_2 + \ldots + a_{n1}e_n, \\ x(e_2) &= a_{12}e_1 + a_{22}e_2 + \ldots + a_{n2}e_n, \\ &\ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \\ x(e_n) &= a_{1n}e_1 + a_{2n}e_2 + \ldots + a_{nn}e_n. \end{aligned}$$

U holda

$$x \mapsto \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

moslikni o'rnatish mumkin.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} a_{1i} \ \lambda_i \\ \sum_{i=1}^{n} a_{2i} \ \lambda_i \\ \dots \dots \dots \\ \sum_{i=1}^{n} a_{ni} \ \lambda_i \end{pmatrix}$$

Bu moslik izomorfizm ekanligini ko'rsatamiz. Buning uchun $y \in gl(\mathrm{V})$ olamiz.

$$y \mapsto \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

U holda

$$x + y \mapsto \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots \ \dots \ \dots & \dots \dots \ \dots & \dots & \dots \ \dots \ \dots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nn} + b_{nn} \end{pmatrix}.$$

va

$$\lambda x \mapsto \begin{pmatrix} \lambda\, a_{11} & \lambda\, a_{12} & \dots & \lambda\, a_{1n} \\ \lambda\, a_{21} & \lambda\, a_{22} & \dots & \lambda\, a_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda\, a_{n1} & \lambda\, a_{n2} & \dots & \lambda\, a_{nn} \end{pmatrix}$$

bo'ladi. Endi operatorlar ko'paytmasi matritsalar ko'paytmasiga mos kelishini ko'rsatamiz , ya'ni

$$y\cdot x \mapsto \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

$$\begin{aligned}(xy)(e_j) &= y\left(x(e_j)\right) = y\left(\sum_{i=1}^{n} a_{ij}e_i\right) = \\ &= \sum_{i=1}^{n} a_{ij}y(e_i) = \\ &= a_{1j}y(e_1) + a_{2j}y(e_2) + \dots + a_{nj}y(e_n) = \\ &= a_{1j}\left(b_{11}e_1 + b_{21}e_2 + \dots + b_{n1}e_n\right) + \\ &+ a_{2j}\left(b_{12}e_1 + b_{22}e_2 + \dots + b_{n2}e_n\right) + \dots \\ &+ a_{nj}\left(b_{1n}e_1 + b_{2n}e_2 + \dots + b_{nn}e_n\right).\end{aligned}$$

$$\begin{aligned}(xy)(e_j) &= \left(a_{1j}b_{11} + a_{2j}b_{21} + \dots + a_{nj}b_{n1}\right)e_1 + \\ &+ \left(a_{1j}b_{12} + a_{2j}b_{22} + \dots + a_{nj}b_{n2}\right)e_2 + \dots \\ &+ \left(a_{1j}b_{1n} + a_{2j}b_{2n} + \dots + a_{nj}b_{nn}\right)\mathrm{e}_n .\end{aligned}$$

$j = \overline{1\dots n}$ ekanligini hisobga olsak kerakli natija kelib chiqadi.

$\dim(\mathrm{End}V) = n^2$ bo’ladi. Bu Li algebra umumiy chiziqli algebra deyiladi. $gl(V)$ algebraning ixtiyoriy qismalgebrasi chiziqli Li algebra deyiladi. Qaralayotgan bazisga nisbatan har bir chiziqli operatorga biror matritsa mos keladi. Demak , $gl(V)$ sifatida barcha $n\times n$ matritsalar algebrasini qarashimiz mumkin ekan. Bu algebrani $gl(n, F)$ kabi belgilaymiz. $gl(n, F)$ uchun e_{ij} standart bazisni tanlab olamiz. Bu yerda, $e_{ij} - (i, j)$ o'rinda 1 qolgan o'rinlarda esa 0 bo'lgan $n \times n$ matritsa. Bazis elementlarni bir-biriga ko'paytirishni quyidagicha yozish mumkin:

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$

bu yerda , δ_{jk} Kroniker simvoli deyiladi va quyidagicha aniqlanadi:

$$\delta_{jk} = \begin{cases} 1 & agar\ j = k \\ 0 & agar\ j \neq k \end{cases} .$$

Yuqoridagi ko'paytirish qoidasidan foydalanib bazis elementlarning Li ko'paytmasini quyidagicha yozamiz:

$$\left[e_{ij}, e_{kl}\right] = e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{jk}e_{il} - \delta_{li}e_{kj} \tag{1.4}$$

$[L, L] = \left\{ \sum_{i=1}^{n} \alpha_i [x_i, y_i] \mid x_i, y_i \in L, \alpha_i \in F, i \in \mathbb{N} \right\}$ to'plam L – Li algebraning kvadrati deyiladi.

1.2-ta'rif. Kvadrati nolga teng bo'lgan Li algebra Abel Li algebra deyiladi.

Bu ta'rif $\forall x, y \in L$ uchun $[x, y] = 0$ ifodaga teng kuchli bo'ladi.

1.3-ta'rif. K fazo L (Li algebra) ning qismfazosi bo'lsin. K qismalgebra deyiladi, agar $\forall x, y \in K$ uchun $[x, y] \in K$ bo'lsa.

Aytaylik L biror Li algebra va I bu algebraning qism fazosi bo'lsin. Agar I qism fazodan olingan ixtiyoriy vektor bilan L algebradan olingan ixtiyoriy vektorning ko'paytmasi I qism fazoga tegishli bo'lsa, u holda I ideal deyiladi, ya'ni,

$$\forall x \in I, \quad \forall y \in L, \quad [x, y] \in I .$$

Ko’paytirish amalining antikomutativlik xossasidan quyidagiga ega bo’lamiz:

$$[y,x]=-[x,y]\in I$$

Ta’rifdan ko’rinadiki, ixtiyoriy ideal qism algebra bo’ladi. Endi matritsalar algebrasining idealni qaraymiz.

$gl(n,F)$ bu F maydon ustidagi n -tartibli kvadrat matritsalar Li algebrasi:

$$[x,y]=xy-yx\,.$$

1.4-ta’rif. Kvadrat matritsaning barcha diagonal elementlari yig’indisiga matritsaning izi deyiladi va $tr(x)$ kabi belgilanadi.

$$tr(x)=\sum_{i=1}^{n}x_{ii}=x_{11}+x_{22}+x_{33}+\ldots+x_{nn}\;.$$

Matritsaning izi quyidagi xossalarga ega:

1º. $tr(x+y)=tr(x)+tr(y)$;

2º. $tr(\alpha x)=\alpha tr(x)$;

3º. $tr(xy)=tr(yx)$.

Isbot:

1º. $tr(x+y)=\sum_{i=1}^{n}(x_{ii}+y_{ii})=\sum_{i=1}^{n}x_{ii}+\sum_{i=1}^{n}y_{ii}=tr(x)+tr(y)$.

2º. $tr(\alpha x)=\sum_{i=1}^{n}\alpha x_{ii}=\alpha\sum_{i=1}^{n}x_{ii}=\alpha tr(x)$.

3º. xy matritsaning diagonal elementlari:

$$x_{11}y_{11} + x_{12}y_{21} + \ldots + x_{1n}y_{n1}$$

$$x_{21}y_{12} + x_{22}y_{22} + \ldots + x_{2n}y_{n2}$$

$$\ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots$$

$$x_{n1}y_{1n} + x_{n2}y_{2n} + \ldots + x_{n}y_{n}$$

bo'ladi. Endi $\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}y_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} y_{ji}x_{ij}$ yig'indilarni solishtirsak, talab etilgan tenglikka ega bo'lamiz.

$sl(n,F)$ orqali izi nolga teng bo'lgan barcha n-ta'rtibli kvadrat matritsalar to'plamini belgilaymiz.

$$sl(n,F) = \{x \in yl(n,F) : tr(x) = 0\}.$$

1.2-lemma. $sl(n,F)$ to'plam $gl(n,F)$ algebraning ideali bo'ladi.

Isbot: $sl(n,F)$ to'plamning qism fazo ekanligi matritsa izining additivlik va birjinslilik xossalaridan kelib chiqadi , ya'ni $\forall x, y \in sl(n,F), \quad \alpha \in F$ uchun

$$tr(x+y) = tr(x) + tr(y) = 0 + 0 = 0.$$

Bundan

$$x + y \in sl(n,F).$$

Endi $x, y \in sl(n,F), \quad \alpha \in F$ uchun

$$tr(\alpha x) = \alpha tr(x) = 0$$

tengligidan $\alpha x \in sl(n,F)$ ekanligi kelib chiqadi.

Ko'paytma uchun

$$tr([x,y]) = tr(xy - yx) = tr(xy) - tr(yx) = 0 .$$

Lemma isbotlandi.

Aytaylik L li algebra I va J uning ideallari bo'lsin. Quyidagi to'plamlarni qaraymiz:

$$I \cap J = \{x \in L: \ x \in I, \quad x \in J\}$$

$$I + J = \{x + y: \ x \in I, \quad y \in J\}$$

$$[I, J] = span\{[x, y]: \ x \in I, \ y \in J\} =$$

$$= \left\{ \sum_{i=1}^{n} \alpha_i [x_i, y_i]; \quad x_i \in I, \quad y_i \in J, \ \alpha_i \in F, \quad i \in \overline{1, n}, \ n \in N \right\}$$

1.3-teorema. Aytaylik L Li algebra bo\`lsin. Agar $I$, $J$ bu algebraning ideallari bo'lsa, u holda $I \cap J$, $I + J$, $[I, J]$ to'plamlar ham bu algebraning ideallari bo\`ladi.

Isbot: $I \cap J$ ideal ekanligini isbotlaymiz. $x, y \in I \cap J$, $\alpha \in F$ olamiz. I va J lar ideal bo\`lgani uchun quyidagiga ega bo\`lamiz:

$$x + y \in I, \quad x + y \in J \qquad \Rightarrow \quad x + y \in I \cap J$$
$$\alpha x \in I, \qquad \alpha x \in J \qquad \Rightarrow \quad \alpha x \in I \cap J$$
$$[x, y] \in I \quad [x, y] \in J \qquad \Rightarrow \quad [x, y] \in I \cap J, \ \ y \in L$$

Endi $I + J$ ning ideal ekanligini tekshiraylik ,

$$\forall x, y \in I + J, \quad \alpha \in L.$$

Ta'rifdan quyidagiga ega bo'lamiz:

$$x = a + b, \quad a \in I, \ b \in J,$$

$$y = u + \vartheta, \ u \in I, \ \vartheta \in J$$

bu munosabatlardan foydalansak

$$x + y = (a + u) + (b + \vartheta) \in I + J$$

va

$$\alpha x = \alpha a + \alpha b \in I + J \ .$$

yana

$$[x, y] = [a + b,\ u + \vartheta] = [a,u] + [a,\vartheta] + [b,u] + [b,\vartheta] \in I + J \ ,$$

$$[x, y] = [a + b, y] = [a, y] + [b, y] \in I + J \ .$$

$[I, J]$ kommutator qism fazo ekanligi aniq. Chunki, bu to'plamning har bir elementi biror elementning chiziqli qobig'dan tashkil topgan.

$\forall x \in [I, J],\ y \in L \Rightarrow [x, y] \in [I, J]$ ni ko'rsatamiz.

$x = [u, \vartheta],\ u \in I,\ \vartheta \in J,\ \ y \in L$ bo'lsin. U holda

$$[[u, \vartheta], y] = -[y, [u, \vartheta]] \ .$$

Yakobi ayniyatidan quyidagilarga ega bo'lamiz:

$$[y, [u, \vartheta]] = [u, [y, \vartheta]] + [[y, u], \vartheta] \in [I, J]$$

bundan $[[u, \vartheta], y] = -[y, [u, \vartheta]] \in [I, J]$ bo'lishi kelib chiqadi.

Endi $x = \sum_{i=1}^{n} \alpha_i [u_i, \vartheta_i]$ ko\`rinishdagi element olsak, u holda

$$[x, y] = \sum_{i=1}^{n} \alpha_i [[u_i, \vartheta_i], y] \in [I, J]$$

bo'ladi.Teorema isbotlandi.

1.4-natija. $[L, L]$ to'plam L algebraning ideali bo\`ladi.

§ 2.Nilpotent Li algebralari

Endi Li algebrasi nazariyasining muhim tushunchalaridan biri bo’lgan nilpotent Li algebralari bilan tanishamiz. Aytaylik L Li algebra bo`lsin. Quyidagi ketma-ketlikni qaraymiz:

$$L^1 = [L, L], \dots, L^k = \left[L^{k-1}, L\right], \ k \geq 2 \tag{2.1}$$

ketma-ketlik aniqlanishidan quyidagi munosabat kelib chiqadi:

$$L^1 \supseteq L^2 \supseteq \dots \supseteq L^n \supseteq \dots \tag{2.2}$$

2.1-lemma. Ixtiyoriy natural k son uchun L^k to’plam L algebraning ideali bo`ladi.

Isbot: Matematik induksiya bo`yicha isbotlaymiz.

$k = 1$, $L^1 = [L, L]$ ideallarning ko’paytmasi ideal bo`lgani uchun L^1 ham ideal bo’ladi.

$k = n-1$, $L^n = \left[L^{n-1}, L\right]$ ideal ekanligi yuqoridagiday kelib chiqadi. Lemma isbotlandi.

**2.1-ta`rif.** Agar biror $n \in N$ topilib, $L^n = \{0\}$ bo`lsa, unda L nilpotent algebra deyiladi.

Abel Li algebra nilpotent bo`ladi, chunki $L^1 = [L, L] = \{0\}$.

Avval nilpotent bo`lmagan algebraga misol keltiramiz.

2.1-misol.

$L:[e_1,e_2]=e_2$.

Bu algebra nilpotent emas , chunki

$L^1=[L,L]=\{\alpha e_2:\alpha\in F\}$,

$L^2=\left[L^1,L\right]=L^1$,

$L^k=L^1,\quad k\geq 2,\quad L^k\neq\{0\}$.

2.2-misol. Agar 5 o`lchamli Li algebraning basis vekto`rlari uchun ko`paytirishni quydagicha kiritsak ,

$$[e_1,e_2]=e_3$$

$$[e_1,e_3]=e_4$$

$$[e_1,e_4]=e_5 \quad .$$

Bu algebra nilpotent bo`ladi , haqiqatan, $x,y\in L$ uchun $x=\sum_{i=1}^{5}\alpha_i e_i$, $y=\sum_{j=1}^{5}\beta_j e_j$ bo'lsin , u holda

$$[x,y]=\left[\sum_{i=1}^{5}\alpha_i e_i,\sum_{j=1}^{5}\beta_j e_j\right]=\alpha_1\beta_1 e_3+\alpha_1\beta_3 e_4+\alpha_1\beta_4 e_5 \quad .$$

Bundan

$$L^1=[L,L]=span\{e_3,e_{4,}e_5\}$$

$x\in L$, $y\in[L,L]$ uchun $x=\sum_{i=1}^{5}\alpha_i e_i$, $y=\sum_{j=3}^{5}\beta_j e_j$ bo'lsin, u holda

$$[x,y]=\left[\sum_{i=1}^{5}\alpha_i e_i,\sum_{j=3}^{5}\beta_j e_j\right]=\alpha_1\beta_3 e_4+\alpha_1\beta_4 e_5$$

bundan

$$L^2 = \left[L^1, L\right] = span\{e_4, e_5\}$$

$x \in L$, $y \in L^2$ uchun

$$[x, y] = \left[\sum_{i=1}^{5} \alpha_i e_i, \sum_{j=4}^{5} \beta_j e_j\right] = \alpha_1 \beta_4 e_5$$

bo'ladi. Bundan

$$L^3 = \left[L^2, L\right] = span\{e_5\} .$$

$[e_i, e_5] = 0 \ \left(i = \overline{1,5}\right)$ bo`lgani uchun $L^4 = \left[L^3, L\right] = 0$ bo`ladi . Demak L-nilpotent .

2.2-teorema. Aytaylik L - n o'lchamli nilpotent Li algebra bo`lsin. U holda

$$\dim L^1 \leq n - 2 \tag{2.3}$$

Isbot. Teskarisini faraz qilib isbotlaymiz. $\dim L^1 > n - 2$ bo'lsin. Agar $\dim L^1 = n$ bo`lsa , $L^1 \leq L$ bo`lgani uchun , $L = L^1$. Bundan

$$L^k = \left[L^{k-1}, L\right] = L^1 = L ,$$

$$L^2 = \left[L^1, L\right] = L ,$$

$$L^k = \left[L^{k-1}, L\right] = [L, L] = L .$$

$L^k = L \neq \{0\}$ munosabat L ning nilpotent ekanligiga zid.

$\dim L^1 = n - 1$ bo`lsin. U holda $L^1$ fazoning biror bazisini qaraymiz. $L^1 : e_2, ..., e_n$ bu bazisni $L$ ning bazisigacha to`ldiramiz.

$$L : e_1, e_2, ..., e_n$$

$$L^2 : x = \sum_{i=1}^{n} \alpha_i e_i \in L, \quad y = \sum_{j=2}^{n} \beta_j e_j \in L^1 .$$

Bundan

$$[x, y] = \sum_{i=1}^{n} \sum_{j=2}^{n} \alpha_i \beta_j \left[e_i, e_j \right] = \sum_{s=2}^{n} * e_s .$$

Demak, $L^2 : e_2, ..., e_n$ va $L^2 = L^3 = ... = L^k = ...$ Bundan $L^k = L^1 \neq \{0\}$ $k \geq 2$. Oxirgi munosabat algebraning nilpotent ekanligiga zid. Teorema isbotlandi.

Endi Li algebrasida differensiallashlarni qaraymiz. Aytaylik A birorta assotsiativ algebra va $D : A \to A$ chiziqli operator bo`lsin

2.2-ta'rif. D chiziqli differensiallash deyiladi, agar $\forall x, y \in A$ elementlar uchun

$$D(xy) = D(x) y + xD(y) \tag{2.4}$$

tenglik o'rinli bo`lsa. (4) Leybnits ayniyati deyiladi

2.3-misol. $$A = P[t, s] = \left\{ x = x(t, s) = \sum_{i=j} a_{ij} t^i s^j ; a_{ij} \in R \right\}$$

$P[t, s]$ - t va s o`zgaruvchilarning ko`phadi. $P[t, s]$ -kommutativ, assotsiativ algebra $p, q \in A$, $p = p(t, s)$, $q = q(t, s)$ tayinlab olamiz

$$\partial(x) = p \frac{\partial x}{\partial t} + q \frac{\partial x}{\partial s}, \; x \in A .$$

∂ -differensiallash boladi, haqiqatan, $\forall x, y \in A$ uchun

$$\partial(xy) = p \frac{\partial(xy)}{\partial t} + q \frac{\partial(xy)}{\partial s} = p\left(\frac{\partial x}{\partial t} y + \frac{\partial y}{\partial t} x \right) + q\left(\frac{\partial x}{\partial s} y + x \frac{\partial y}{\partial s} \right) =$$
$$= \left(p \frac{\partial x}{\partial t} + q \frac{\partial y}{\partial t} \right) y + x \left(p \frac{\partial x}{\partial s} + q \frac{\partial y}{\partial s} \right) = \partial(x) y + x \partial(y) .$$

2.4-misol. A -biror assotsiativ algebrava $a \in A$ bo`lsin.

$$D_a(x) = [a, x] = ax - xa \ , \ \forall x \in A$$

D chiziqli operator differensiallash bo`ladi. Haqiqtan,

$$\begin{aligned} D(x)y + xD(y) &= (ax - xa)y + x(ay - ya) = \\ &= axy - xay + xay - xya = \\ &= axy - xya = D_a(xy). \end{aligned}$$

A biror assotsiativ algebra bo`lsa uni $[x, y] = xy - yx$ ko`paytma amalini kiritib Li algebrasiga aylantirish mumkin.

2.3-teorema. D assotsiativ algebraning differensiali bo’lsa, u holda Li algebrasi ham differensiali bo`ladi.

Isbot:

$$D([x, y]) = [D(x)y] + [x, D(y)] \tag{2.5}$$

ekanini ko`rsatishimiz kerak

$$\begin{aligned} D([x, y]) &= D(xy - yx) = \\ &= D(xy) - D(yx) = D(x)y + xD(y) - D(y)x - yD(x) = \\ &= D(x)y - yD(x) + xD(y) - D(y)x = \\ &= [D(x), y] + [x, D(y)] \ . \end{aligned}$$

Bu tenglikdan D differensiallash Li algebraning ham differensiallashi ekanligi kelib chiqadi.

§ 3. Oddiy Li algebralari

Biz oldingi bobda Li algebralari Li algebralarining ideallari va nilpotent Li algebralari kabi tushunchalarni o`rgangan edik. Bu bobda esa Li algebralarining muhim sinfi bo'lgan oddiy Li algebralarini o'ganamiz.

Bizga L Li algebra berilgan bo'lsin.

3.1-ta'rif. Agar L algebra 0 va L dan boshqa ideallarga ega bo'lmasa, u holda L oddiy Li algebradeyiladi.

Oddiy Li algebrada $[L,L]=L$ bo'ladi. Haqiqatan, $[L,L]$ ideal bo'lishi oldingi bobdan ma`lum. Agar $[L,L]=0$ bo'lsa L Abel Li algebra bo'ladi. Abel Li algebrasining ixtiyoriy qismfazosi uning ideali bo'lishini hisobga olsak L algebraning oddiy ekanligiga zid fikrga kelamiz.

3.1-misol. L Li algebraning $\{e_1,e_2,e_3\}$ bazis elementlarini ko'paytirish quydagicha aniqlansin:

$[e_1,e_3]=2e_3$

$[e_1,e_2]=2e_2$

$[e_2,e_3]=e_1$.

a) L- Li algebrasi ekanligini ko'rsating.

b) L - oddiy Li algebrasi ekanligini ko`rsating.

Yechish: a) $\forall x\in L$, $x=\alpha e_1+\beta e_2+\gamma e_3$ bo`lsin. U holda

$$\begin{aligned}[x,x]&=[\alpha e_1+\beta e_2+\gamma e_3,\alpha e_1+\beta e_2+\gamma e_3]=\\&=\alpha^2[e_1,e_1]+\beta^2[e_2,e_2]+\gamma^2[e_3,e_3]+\\&+2(\alpha\gamma-\gamma\alpha)e_2-2(\alpha\gamma-\gamma\alpha)e_3+(\beta\gamma-\beta\gamma)e_1=0\quad.\end{aligned}$$

Agar L Li algebra chekli o`lchamli bo`lsa, u holda Yakobi ayniyatini faqat bazis vektorlar uchun tekshirish yetarli bo'ladi, ya'ni, $x, y, z \in \{e_1, e_2, e_3\}$ $x = e$, $y = e_2 , z = e_3$ desak , u holda

$$\left[x,[y,z]\right] + \left[y,[z,x]\right] + \left[z,[x,y]\right] = \left[e_1,[e_2,e_3]\right] + \left[e_2\,[e_3,e_1]\right] + \left[e_3,[e_1,e_2]\right] =$$
$$= \left[e_1, e_1\right] + \left[e_2 + 2e_3\right] + \left[e_3, 2e_2\right] =$$
$$= 2e_1 - 2e_1 = 0 \qquad .$$

Bu algebraning oddiy Li algebrasi ekanligini ko`rsatish uchun unga izomorf bo`lgan boshqa algebraning oddiy Li algebra ekanligini ko`satamiz .

Shu maqsadda sl_2 matritsalar algebrasini qaraymiz.

$$sl_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} : \alpha, \beta, \gamma \in \square \right\}$$

Bu algebra uchun $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ matritsalar bazis elementlar bo'ladi. Dastlab sl_2ning Li algebra bo'lishini ko'rsatamiz

$sl_2 \subset M_2(\square)$ ekanligi aniq, shuning uchun $\forall x, y \in sl_2$, $\forall \alpha \in \square$ uchun

$x + y \in sl_2$, $\alpha x \in sl_2$, $[x, y] \in sl_2$ bo'lishini ko'rsatish yetarli.

$$\forall x, y \in sl_2 \ , \quad x = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \ y = \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix}$$

Bo'lsin, u holda

$$x + y = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -(a_1 + a_2) \end{pmatrix} \in sl_2$$

$$\alpha x = \alpha \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & -\alpha a_1 \end{pmatrix} \in sl_2$$

$$[x,y] = xy - yx = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 - b_1a_2 \\ c_1a_2 - a_1c_2 & c_1b_2 + a_1a_2 \end{pmatrix} - \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 - a_1b_2 \\ a_1c_2 - c_1a_2 & c_1b_2 + a_1a_2 \end{pmatrix} =$$

$$= \begin{pmatrix} b_1c_2 - b_2c_1 & 2(a_1b_2 - b_1a_2) \\ 2(c_1a_2 - a_1c_2) & -(b_1c_2 - b_2c_1) \end{pmatrix} \in sl_2 \qquad .$$

Demak, sl_2 Li algebra bo'lar ekan.

$L \to sl_2$ moslikni quydagicha o'rnatamiz

$$e_1 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e_2 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$e_3 \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\alpha e_1 + \beta e_2 + \gamma e_3 \to \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} .$$

Bu moslikni izomorfizm bo'lishini osongina tekshirish mumkin.

$$[e_1, e_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \equiv 2e_2$$

$$[e_1, e_3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \equiv -2e_3$$

$$[e_2, e_3] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv e_1 \qquad .$$

Endi sl_2 ning oddiy Li algebra ekanini ko`rsatamiz. Buning uchun L algebraning $I \neq 0$ idealini olamiz.

$I \subseteq L$ munosabatdan $I = L$ kelib chiqishini ko`rsatish kerak. $\forall x \in I$, $x \neq 0$ element olamiz, u holda $x = \alpha e_1 + \beta e_2 + \gamma e_3$.

1-hol. $\alpha \neq 0$ bo'lsin, u holda

$$[x, e_3] = [\alpha e_1 + \beta e_2 + \gamma e_3, e_3] = -2\alpha e_3 + \beta e_1 .$$

Demak, $-2\alpha e_3 + \beta e_1 \in I$ bo'lar ekan. $[x, e_3] \in I$, $-\frac{1}{4\alpha} e_1 \in L$ element olamiz, u holda

$$\left[[x, e_3], -\frac{1}{4\alpha} e_1\right] = \left[-2\alpha e_3 + \beta e_1, -\frac{1}{4\alpha} e_1\right] = \frac{1}{2}[e_3, e_1] - \frac{\beta}{4\alpha}[e_1, e_1] = e_3 .$$

Bundan $e_3 \in I$. $e_3 \in I$, $e_2 \in L$ olamiz, u holda $[e_2, e_3] = e_1 \in I$

$e_1 \in I$, $e_2 \in L$ uchun $\frac{1}{2}[e_1, e_2] = e_2 \in I$ bo'ladi.

Demak, $L = span\{e_1, e_2, e_3\} \subseteq I$ boshqa tomondan $I \subseteq L$ ya`ni $L \subseteq I \subseteq L$ bundan $I = L$.

2-hol. $\beta \neq 0$ bo'lsin, u holda

$$[x, e_1] = [\alpha e_1 + \beta e_2 + \gamma e_3, e_1] = -2\beta e_2 + 2\gamma e_3 \in I .$$

$-2\beta e_2 + 2\gamma e_3 \in I$ va $-\frac{1}{2\beta} e_3 \in L$ olamiz.

$$\left[[x, e_2], -\frac{1}{2\beta} e_3\right] = \left[-2\beta e_2 + 2\gamma e_3, -\frac{1}{2\beta} e_3\right] = [e_1, e_3] - \frac{\gamma}{\beta}[e_3, e_3] = e_1 \in I$$

$$e_1 \in I , \quad e_3 \in L \Rightarrow -\frac{1}{2}[e_1, e_3] = e_3 \in I$$

$$e_1 \in I , \quad e_2 \in L \Rightarrow \frac{1}{2}[e_1, e_2] = e_2 \in I$$

Demak, $L = span\{e_1, e_2, e_3\} \subseteq I$ boshqa tomondan $I \subseteq L$ bundan $I = L$.

3-hol. $\gamma \neq 0$ bo'lsin, u holda

$$[x, e_2] = [\alpha e_1 + \beta e_2 + \gamma e_3, e_2] = 2\alpha e_2 - \gamma e_1 \in I$$

$2\alpha e_2 - \gamma e_1 \in I$ va $-\dfrac{1}{2\gamma}e_2 \in L$ olamiz, u holda

$$\left[[x,e_2],-\frac{1}{2\gamma}e_2\right] = \left[2\gamma e_2 - \gamma e_1, -\frac{1}{2\gamma}e_2\right] = -\frac{\alpha}{\gamma}[e_2,e_2] + \frac{1}{2}[e_3,e_3] = e_2 \in I \ .$$

$e_2 \in I$, $e_3 \in L$ olamiz, bundan $e_1 = [e_1,e_3] \in I$

$e_1 \in I$, $e_3 \in L$ olamiz, u holda $e_3 = -\dfrac{1}{2}[e_1,e_2] \in I$

Demak, $L = span\{e_1,e_2,e_3\} \subseteq I$ boshqa tomondan $I \subseteq L$ bundan $I = L$ bo`lishi kelib chiqadi.

Endi yuqoridagi misolning umumiy holi bo'lgan sl_n matritsalar algebrasini qaraymiz. sl_n izi nolga teng bo`lgan barcha $(n \times n)$ matritsalar to`plamidir.

$$sl_n = \{x \in M_n(\mathbb{C}) : tr(x) = 0\}.$$

Dastlab sl_n ning Li algebra ekanini ko'rsatamiz. Buning uchun quydagilarni ko'rsatish yetarli:

$\forall x, y \in sl_n$, $\forall \lambda \in \mathbb{C}$ uchun

1) $x + y \in sl_n$

2) $\forall \lambda \in sl_n$

3) $[x, y] \in sl_n$.

Chunki $M_n(\mathbb{C})$ Li ko'paytmasiga nisbatan Li algebra bo'ladi

1) $tr(x+y) = tr(x) + tr(y) = 0 \Rightarrow x + y \in sl_n$

2) $tr(\lambda x) = \lambda tr(x) = 0 \Rightarrow \lambda x \in sl_n$

3) $tr([x,y]) = tr(xy - yx) = tr(xy) - tr(yx) = 0 \Rightarrow [x,y] \in sl_n$.

Demak sl_n Li algebra bo'lar ekan.

3.1-teorema. sl_n - oddiy Li algebra bo'ladi .

Isbot: $B=\left\{h_i=e_{ii}-e_{i+1i-1}:i=\overline{1,n-1}\right\}\cup\left\{e_{ij}:i\neq j\right\}$ to`plam $sl_n$ uchun bazis elementlari to`plami bo'ladi.

sl_n ning oddiy Li algebra ekanini ko'rsatish uchun $I\neq 0$ ideal bo'lib $I\subseteq sl_n$ ekanligidan $I=sl_n$ bo'lishini keltirib chiqarishimiz kerak .

Biror $I\subseteq sl_n$ ideal olamiz. $x\in I$, $x\neq 0$ bo'lsin, u holda

$x=\sum\limits_{i=1}^{h-1}\alpha_i h_i+\sum\limits_{i\neq j}\alpha_{ij}e_{ij}$ kabi yozish mumkin. Bu yerda, $h_i=e_{ii}-e_{i+1i+1}$, $i=\overline{1,n-1}$

1) $\alpha_{ks}\neq 0$ $k<s$ bo'lsin, unda $h_k\in sl_n$ uchun $[x,h_k]\in I$ bo'ladi

$$[x,h_k]=\sum_{i=1}^{h-1}\alpha_i[h_i,h_k]+\sum_{i\neq j}\alpha_{ij}\left[e_{ij},h_k\right]=\sum_{i\neq j}\alpha_{ij}\left[e_{ij},e_{kk}-e_{k+1k+1}\right]=$$

$$=\sum_{i\neq j}\alpha_{ij}\left(e_{ij}\left(e_{kk}-e_{k+1k+1}\right)-\left(e_{kk}-e_{k+1k+1}\right)e_{ij}\right)=$$

$$=\sum_{i\neq j}\alpha_{ik}e_{ik}-\sum_{i\neq j}\alpha_{kj}e_{kj}-\left(\sum_{i\neq k+1}\alpha_{ik+1}e_{ik+1}-\sum_{j\neq k+1}\alpha_{n+1j}e_{k+1j}\right) .$$

$[x,h_k]\in I$, $e_{1k}\in sl_n$ olamiz, u holda

$$\left[[x,h_k],e_{1k}\right]=\sum_{i\neq k}\alpha_{ik}\left[e_{ik},e_{1k}\right]-\sum_{j\neq k}\alpha_{kj}\left[e_{kj},e_{1k}\right]-$$

$$-\sum_{i\neq k+1}\alpha_{ik+1}\left[e_{ik+1},e_{1k}\right]+\sum_{j\neq k+1}\alpha_{k+1j}\left[e_{k+1j},e_{1k}\right]=$$

$$=\alpha_{k1}e_{11}-\alpha_{k1}e_{kk}+\alpha_{kk+1}e_{k+11}+\alpha_{k+11}e_{k+1k}\in I .$$

$\left[[x,h_k]e_{1k}\right]\in I$, $e_{kk}\in L$ bo'lsin, u holda

$$\left[\left[[x,h_k]e_{1k}\right]e_{kk}\right]=\alpha_{k+11}\left[e_{k+1k},e_{kk}\right]=\alpha_{k+11}e_{k+1k}\in I .$$

$e_{k+1k}\in I$, $e_{k1}\in L$ bo'lsin, u holda $\left[e_{k+1k},e_{k1}\right]=e_{k+11}\in I$ bo'ladi.

$e_{k+11}\in I$, $e_{ik+1}\in L$ bo'lsin, u holda $\left[e_{ik+1},e_{k+11}\right]=e_{i1}\in I$ bo'ladi.

Nihoyat $e_{i1}\in I$, $e_{1j}\in L$ bo'lsin, u holda $\left[e_{i1},e_{1j}\right]=e_{ij}\in I$ bo'ladi.

Demak, yuqoridagilarga asoslanib $I = L$ bo'lishi kelib chiqadi. Qolgan hollar ham shunga o'xshash ko'rsatiladi.

§ 4. Li algebralarida avtomorfizmlar

Bizga F maydon ustida L va L' Li algebralar va

$$\phi : L \to L'$$

chiziqli akslantirish berilgan bo'lsin.

4.1-ta'rif. Agar barcha $x, y \in L$ elementlar uchun

$$\phi([x,y]) = [\phi(x), \phi(y)] \tag{4.1}$$

tenglik o'rinli bo'lsa, u holda $\phi : L \to L'$ akslantirish gomomorfizm deyiladi.

Agar $\ker\phi = 0$ bo`lsa, ϕ monomorfizm, $\operatorname{Im}\phi = L'$ bo'lsa, ϕ epimorfizm deyiladi.

Agar ϕ akslantirish ham epimorfizm ham monomorfizm bo'lsa, unda ϕ izomorfizm deyiladi. L va L' algebralar o`zaro izomorf deyiladi va $L \cong L'$ kabi yoziladi.

$\phi : L \to gl(V)$ ko'rinishdagi gomorfizmga tasvirlash deyiladi. Bu yerda V F maydon ustida chiziqli fazo va chekli o`lchamli.

4.1-misol. Quydagi akslantirishni qaraymiz:

$$ad : L \to gl(L) \ .$$

Bu yerda adx akslantirish har bir y elementni x da chapdan ko'paytirish orqali aniqlanadi, ya'ni,

$$adx(y) = [x, y] \ .$$

Bu akslantirish gomomorfizm bo'ladi. Haqiqatan,

$$
\begin{aligned}
\left[ad\,x, ad\,y\right](z) &= ad\,x\,ad\,y(z) - ad\,y\,ad\,x(z) = \\
&= ad\,x([y,z]) - ad\,y([x,z]) = \\
&= \left[x,[y,z]\right] - \left[y,[x,z]\right] = \\
&= \left[x,[y,z]\right] + \left[[x,z],y\right] = \qquad (L\,2') \\
&= \left[[x,y],z\right] = ad\left[x,y\right](\mathrm{z}) \qquad (L\,3)
\end{aligned}
$$

Demak, $ad\left[x,y\right] = \left[ad\,x, ad\,y\right]$ bo’ladi. Bundan $ad\,x$ gomomorfizm ekanligi kelib chiqadi. Bu yerda $(L2')$ va $(L3)$ mos ravishda antikomutativlik va Yakobi ayniyatini bildiradi.

Endi ad gomorfizmning yadrosini o`rganamiz.
$Ker(ad) = \left\{x \in L : \left[x,y\right] = 0, \forall y \in L\right\}$

$$ad\,x = 0 \text{ yoki } \forall y \in L \text{ uchun } \left[x,y\right] = 0 \;.$$

Demak, $Ker(ad) = Z(L)$, ya’ni , $ad\,x$ ning yadrosi algebraning markazi bilan ustma-ust tushar ekan. Agar L oddiy Li algebrasi bo’lsa, u holda $Z(L) = 0$ bo’ladi, ya’ni

$$ad : L \to gl(L)$$

monomorfizm bo`ladi. Bu akslantirish epimorfizm ham bo’ladi. Shunga ko’ra quydagicha xulosaga kelish mumkin: har qanday oddiy Li algebra biror chiziqli Li algebraga izomorf bo’ladi.

Bizga L Li algebra va

$$\phi : L \to L$$

akslantirish berilgan bo’lsin.

4.2-ta’rif. Agar ϕ akslantirish izomorfizm bo’lsa, u holda ϕ avtomorfizm deyiladi. L algebradagi barcha avtomorfizmlar to`plami gruppa tashkil qiladi va bu gruppani $AutL$ kabi belgilaymiz . Haqiqatan, $f, g, h \in AutL$ uchun

$$(f + g) + h = f + (g + h)$$

tenglik o’rinli b’ladi.

$\forall f \in AutL$ uchun $\exists I \in AutL$ topilib ,

$$f \circ I = I \circ f = f$$

tenglik o`rinli bo`ladi. Bu yerda I ayniy almashtirish

$\forall f \in AutL$ uchun $\exists f^{-1} \in AutL$ topilib ,

$$f \circ f^{-1} = f^{-1} \circ f = I$$

tenglik o'rinli bo'ladi. Bu tenglik f aftomorfizimga teskari akslantirish ham aftomorfizm bo'lishidan kelib chiqadi. Demak, $AutL$ gruppa ekan.

Endi shunday $x \in L$ olamizki, $ad\ x$ nilpotent, ya'ni, biror $k > 0$ uchun $(ad\ x)^k = 0$ bo'lsin. U holda

$$\exp(ad\ x) = 1 + ad\ x + \frac{(ad\ x)^2}{2!} + \frac{(ad\ x)^3}{3!} + \ldots + \frac{(ad\ x)^{k-1}}{(k-1)!} \qquad (4.2)$$

bo'ladi.

Endi $\exp(ad\ x) \in AutL$, ya'ni $\exp(ad\ x)$ avtomorfizm ekanligini ko'rsatamiz. Bunda adx o'rniga ixtiyoriy nilpotent δ differensiallashni qarash mumkun .

Bizga ma'lum Leybnits formulasini quydagicha yozib olamiz:

$$\frac{\delta^n}{n!}(xy) = \sum_{i=0}^{n} \left(\frac{\delta^i x}{i!} \right) \left(\frac{\delta^{n-i} y}{(n-i)!} \right). \qquad (4.3)$$

4.1-teorema. Agar δ biror nilpotent differensiallash bo'lsa, u holda $\exp \delta \in AutL$ bo'ladi.

Isbot: $\exp \delta(xy) = \exp \delta(x) \exp \delta(y)$ tenglikni ko'rsatishimiz kerak.

$$\exp\delta(x)\exp\delta(y)=\left(\sum_{i=0}^{k-1}\left(\frac{\delta^i x}{i!}\right)\right)\left(\sum_{j=0}^{k-1}\left(\frac{\delta^j y}{j!}\right)\right)=$$

$$=\sum_{n=0}^{2k-2}\left(\sum_{i=0}^{n}\left(\frac{\delta^i x}{i!}\right)\left(\frac{\delta^{n-1} y}{(n-1)!}\right)\right)=$$

$$=\sum_{n=0}^{2k-2}\frac{\delta^n(xy)}{n!}=$$

$$=\sum_{n=0}^{k-1}\frac{\delta^n(xy)}{n!}= \qquad (\delta^k=0)$$

$$=\exp\delta(xy)\ .$$

Demak, $\exp\delta(x,y)=\exp\delta(x)\exp\delta(y)$ tenglik o'rinli ekan. δ differensiallash bo'lgani uchun Leybnits formulasi Li ko`paytmasi uchun ham o'rinli bo'ladi, ya'ni,

$$\frac{\delta^n[x,y]}{n!}=\sum_{i=0}^{n}\left[\frac{\delta^i}{i!},\frac{\delta^{n-i}y}{(n-i)!}\right]$$

shunga ko'ra

$$\exp\delta[x,y]=\left[\exp\delta(x),\exp\delta(y)\right] \tag{4.4}$$

bo`lishi kelib chiqadi. Bu esa $\exp\delta$ ning avtomorfizm ekanligini bildiradi. Isbot tugadi. Bunday ko'rinishdagi avtomorfizmlar ichki avtomorfizmlar deyiladi. Ichki bo'lmagan avtomorfizmga misol keltiramiz.

$$\phi: L\to Z(L)$$

akslantirishda

$$\phi\big|_{[L,L]}=0$$

munosabat o'rinli bo'lsin, ya'ni, algebraning kvadratini nolga o'tkazsin. U holda

$$\Phi: L\to L\ ,\ \Phi(x)=x+\phi(x)$$

kabi aniqlangan akslantirish avtomorfizm bo'ladi. Haqiqatan,

$$\left[\Phi(x),\Phi(y)\right]=\left[x+\phi(x),y+\phi(y)\right]=$$

$$=\left[x,y\right]+\left[x,\phi(y)\right]+\left[\phi(x),y\right]+\left[\phi(x),\phi(y)\right]=$$

$$=\left[x,y\right]=\left[x,y\right]+\phi\left(\left[x,y\right]\right)=$$

$$=\Phi\left(\left[x,y\right]\right).$$

Bu avtomorfizm ichki bo'lishi shart emas. Bunday avtomorfizmlar tashqi avtomorfizm deyiladi.

Ma'lumki δ chiziqli akslantirish. $\forall x,y\in L$ uchun

$$\delta\left[x,y\right]=\left[\delta x,y\right]+\left[x,\delta y\right]$$

tenglikni qanoatlantirsa, δ differensiallash deyiladi. Barcha differensiallashlar to\`plamini $autL$ kabi belgilaymiz. $L$ dagi barcha ichki differensiallashlar (chapdan ko\`paytirishlar) to'plamini adL kabi belgilaymiz.

4.2-teorema. L yarim oddiy Li algebrasidagi barcha differensiallashlar ichki differensiallash bo\`ladi, ya'ni, $autL=adL$.

Isbot: Aytaylik δ akslantirish L Li algebrada differensiallash bo'lsin. U holda

$$L_1=\left\{(x,\lambda\delta)\,\middle|\,x\in L,\lambda\in\square\right\}$$

to'plamda amallarni quyidagicha aniqlaymiz:

$$(x,\lambda\delta)+(y,\mu\delta)=\left(x+y,(\lambda+\mu)\delta\right),$$

$$\left[(x,\lambda\delta)+(y,\mu\delta)\right]=\left(\left[x,y\right]+\lambda\delta(y)-\mu\delta(x),0\right).$$

L_1 - Li algebra bo'ladi. Haqiqatan, $\forall\,(x,\lambda\delta)\in L_1$ uchun

$$\left[(x,\lambda\delta),(x,\lambda\delta)\right]=\left[\left[x,x\right]+\lambda\delta(x)-\lambda\delta(x),0\right]=\left[0,0\right]=0$$

Endi Yakobi ayniyati o'rinli ekanligini ko'rsatamiz.

$$\left[\left[(x,\lambda\delta),(y,\mu\delta)\right],(z,\upsilon\delta)\right]=$$
$$=\left[\left(\left[x,y\right]+\lambda\delta(y)-\mu\delta(x),0\right),(z,\upsilon\delta)\right]=$$
$$=\left(\left[\left[x,y\right]+\lambda\delta(y)-\mu\delta(x),z\right]-\upsilon\delta\left(\left[x,y\right]+\lambda\delta(y)-\mu\delta(x),0\right)\right)=$$
$$=\left(\left[\left[x,y\right],z\right]+\lambda\left[\delta(y),z\right]-\mu\left[\delta(x),z\right]-\upsilon\delta\left[x,y\right]-\lambda\upsilon\delta^2(y)+\mu\upsilon\delta^2(x)\right)$$

$$\left[\left[(y,\mu\delta),(z,\nu\delta)\right],(x,\lambda\delta)\right]=$$
$$=\left(\left[\left[y,z\right],x\right]+\mu\left[\delta(z),x\right]-\nu\left[\delta(y),x\right]-\lambda\delta\left[y,z\right]-\mu\lambda\delta^2(z)+\lambda\nu\delta^2(x),0\right)$$

$$\left[\left[(z,\nu\delta),(x,\lambda\delta)\right],(y,\mu\delta)\right]=$$
$$=\left(\left[\left[z,x\right],x\right]+\nu\left[\delta(x),y\right]-\lambda\left[\delta(z),y\right]-\mu\delta\left[z,x\right]-\nu\mu\delta^2(x)+\mu\lambda\delta^2(z),0\right)$$

Hosil bo`lgan uchala tenglikni qo’shsak Yakobi ayniyati hosil bo’ladi.

$L_0=\left\{(x,0)\,\middle|\,x\in L\right\}$ belgilash kiritamiz. Ravshanki, $L_0\cong L$, ya’ni, L ga izomorf bo`ladi. Shunga ko’ra L_0 ham yarim oddiy bo’ladi. Demak L_0 to’plam L_1 Li algebraning yarimoddiy ideali bo’ladi. U holda

$$L_1=L_0+I$$

munsabatni yozish mumkin. Bu yerda I maximal ideal. Tabiiyki I bir o`lchamli.

Aytaylik $(0,\delta)\in L_1$ uchun

$$(0,\delta)=(a,0)+\upsilon\quad,\ \upsilon\in I$$

bo`lsin. U holda

$$\left[(0,\delta),(x,0)\right]=\left[(a,0),(x,0)\right],$$

ya`ni,

$$\left(\delta(x),0\right)=\left(\left[a,x\right],0\right)$$

bundan

$$\delta(x)=\left[a,x\right]$$

yoki

$$\delta = ada .$$

Isbot tugadi.

$L \subset gl(V)$ ixtiyoriy chiziqli Li algebra bo`lsin . Bu algebrada avtomorfizmlarning ko`rinishini aniqlaymiz. Buning uchun bizga ma`lum adx ichki differensiallashni quydagicha yozib olamiz:

$$ad\ x = \lambda_x + \rho_{-x} \tag{4.5}$$

bu yerda, λ_x va ρ_{-x} lar mos ravishda x ga chapdan va o'ngdan ko'paytirish. 4.1-teoremaga ko'ra $\exp ad\ x \in AutL$, u holda

$$\exp adx = \exp\left(\lambda_x + \rho_{-x}\right) = \exp\lambda_x \exp\rho_{-x} = \lambda_{\exp x}\rho_{\exp(-x)} .$$

Demak, quydagiga ega bo'lamiz.

$$\exp adx(y) = (\exp x)\, y\, (\exp x)^{-1} \quad , \quad \forall y \in L \tag{4.6}$$

$\exp ad\ x = \phi$ va $\exp x = a$ belgilashlarni kiritsak, quydagiga ega bo'lamiz:

$$\phi(y) = aya^{-1} \qquad , \qquad \forall y \in L$$

ϕ - avtomorfizm bo'ladi. Haqiqatan,

$$\begin{aligned}\phi([x, y]) &= a[x, y]a^{-1} = a(xy - yx)a^{-1} = \\ &= axya^{-1} - ayxa^{-1} = \\ &= axa^{-1}aya^{-1} - aya^{-1}axa^{-1} = \\ &= \phi(x)\phi(y) - \phi(y)\phi(x) = \\ &= [\phi(x), \phi(y)]\end{aligned}$$

tenglik o'rinli bo`ladi. $\phi : x \to -x^t$ akslantirish ham umumiy holda

$$\phi : x \to -ax^t a^{-1}, \forall x \in L$$

akslantirish ham avtomorfizm bo`ladi. Haqiqatan,

$$\phi([x,y]) = a([x,y])^t a^{-1} = -a(xy-yx)^t a^{-1} =$$
$$= -a\left(y^t x^t - x^t y^t\right)^t a^{-1} =$$
$$= ax^t y^t a^{-1} - ay^t x^t a^{-1} =$$
$$= ax^t a^{-1} a y^t a^{-1} - a y^t a^{-1} a x^t a^{-1} =$$
$$= \phi(x)\phi(y) - \phi(y)\phi(x) =$$
$$= [\phi(x), \phi(y)] \ .$$

Bundan ,$\phi([x,y]) = [\phi(x), \phi(y)]$ tenglik o`rinli bo`ladi. Yuqoridagilarga asoslanib quydagi teoremani keltirish mumkin:

4.3-teorema. Izi nolga teng 2×2 matritsalar Li algebrasida avtomorfizmlar gruppasi barcha $\phi(x) = a\,xa^{-1}$ ko`rinishdagi akslantirishlar to`plami bo`ladi. Izi nolga teng $n\times n$ matritsalar Li algebrasidagi barcha $\phi(x) = a\,xa^{-1}$ va $\phi(x) = -a\,x^t a^{-1}$ ko`rinishdagi akslantirishlar to`plami avtomorfizmlar gruppasi bo`ladi.

§ 5. sl_n da lokal avtomorfizmlar

Bu mavzuda biz oddiy Li algebra sl_n (izi nolga teng bo`lgan barcha $n\times n$ matritsalar ) da lokal avtomarfizmlarni o`rganamiz. A biror Li algebra bo'lsin. $\Phi : A \to A$ chiziqli akslantirish lokal avtomorfizm deyiladi, agar ixtiyoriy $x\in A$ uchun x ga bog'liq Φ_x avtomorfizm topilib, $\Phi(x) = \Phi_x(x)$ tenglik o'rinli bo'lsa. Quyidagi ikki teoremani isbotsiz keltiramiz.

5.1-teorama. $M_n(\mathbb{C})$ kompleks sonlar ustida barcha $n\times n$ - matritsalar algebrasi bo'lsin. $\Delta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ akslantirish lokal avtomorfizm bo`lishi uchun u avtomorfizm yoki anti-avtomorfizm bo`lishi zarur va yetarli, ya`ni, $\forall x \in M_n(\mathbb{C})$ uchun teskarilanuvchi $a \in M(\mathbb{C})$ topilib, $\Delta(x) = axa^{-1}$ yoki $\Delta(x) = ax^t a^{-1}$ ko`rinishda bo`ladi.

Bu yerda x^t matritsa x matritsaning transponirlangan matritsasi.

Agar x matritsa uchun $x^2 = 0$ bo`lsa, u holda x kvadrati nol matritsa deyiladi.

Agar $\Phi : sl_n \to sl_n$ akslatirish $x \in sl_n$, $x^2 = 0$ uchun $\Phi(x)^2 = 0$ tenglikni qanoatlantirsa, u holda bu akslantirishni kvadrati nol matritsani saqlaydi deymiz. Ravshanki, kvadrati nol matritsani saqlovchi chiziqli akslantirishlar quydagicha aniqlanadi:

$$\Phi(x) = \lambda axa^{-1}, \qquad x \in sl_n \tag{5.1}$$

va

$$\Phi(x) = \lambda ax^t a^{-1}, \qquad x \in sl_n \tag{5.2}$$

Bu natija P. Semrlning chiziqli kvadrati nol akslantirishlar haqidagi maqolasida keltirilgan.

5.2-teorema. $\Phi : sl_n \to sl_n$ biektiv, kvadrati nol matritsani saqlovchi chiziqli akslantirish bo`lsin. U holda $\Phi$ yo (5.1) yoki (5.2) ko`rinishda bo`ladi.

Bizga ma'lumki, sl_n dagi har qanday Φ avtomorfizm uchun teskarilanuvchi $a \in M_n(\mathbb{C})$ matritsa topilib, $\Phi(x) = axa^{-1}$ yoki $\Phi(x) = -ax^t a^{-1}$ bo`ladi.

5.3-teorema. $\Delta : sl_n \to sl_n$ chiziqli akslantirish lokal avtomorfizm bo'lishi uchun bu akslantirish avtomorfizm yoki anitavtomorfizm bo'lishi zarur va yetarli, ya'ni , Δ akslantirish

$$\Delta(x) = \pm axa^{-1}, \qquad x \in sl_n \tag{5.3}$$

yoki

$$\Delta(x) = \pm a\, x^t a^{-1}\ , \qquad x \in sl_n \tag{5.4}$$

ko'rinishda bo`ladi.

Isbot: *zarurligi.* Aytaylik Δ - sl_n da lokal avtomorfizm bo`lsin. Biz Δ chiziqli akslantirish biektiv va kvadrati nol matritsani saqlaydigan akslantirish ekanligini ko'rsatamiz.

$x \in sl_n$ nolmas element bo'lsin. Tarifga ko'ra sl_n da shunday Φ^x avtomorfizm topilib, $\Delta(x) = \Phi^x(x)$ bo`ladi. $\Phi^x$ avtomorfizm bo'lgani uchun $\Phi^x(x) \neq 0$ bo`ladi. Bundan $\Delta(x) \neq 0$ va Δ ning yadrosi faqat nol elementdan iborat ekanligi kelib chiqadi. Demak, Δ - biektiv akslantirish .

$x \in sl_n$ kvadrati nol matritsa bo`lsin, ya`ni $x^2 = 0$. Lokal avtomorfizmning ta'rifiga ko'ra teskarilanuvchi $a_x \in M_n(\mathbb{C})$ topilib, $\Delta(x) = a_x x a_x^{-1}$ yoki $\Delta(x) = -a_x x^t a_x^{-1}$ kabi bo`ladi. U holda

$$\Delta(x)^2 = \left(a_x x a_x^{-1}\right)^2 = a_x x^2 a_x^{-1} = 0$$

yoki

$$\Delta(x)^2 = \left(-a_x x^t a^{-1}\right)^2 = a_x \left(x^2\right)^t a_x^{-1} = 0 \quad .$$

Demak, Δ akslantirish biektiv va kvadrati nol matritsalarni saqlovchi akslantirish. 5.2-teoremaga ko'ra teskarilanuvchi $a \in M_n(\mathbb{C})$ va $\lambda \neq 0$ skalyar topilib, barcha $x \in sl_n$ uchun

$$\Delta(x) = \lambda a x a^{-1}$$

yoki (5.5)

$$\Delta(x) = \lambda a x^t a^{-1}$$

bo`ladi.

$$y = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & \dots & & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$ matritsa olamiz. Lokal avtomorfizm ta'rifiga

ko'ra $y \in M_n(\mathbb{C})$ uchun shunday a_y teskarilanuvchi matritsa mavjud bo`lib ,

$$\Delta(y) = a_y y a_y^{-1}$$

yoki (5.6)

$$\Delta(y) = -a_y y^t a_y^{-1}$$

bo'ladi.

$P_x(t) = \det(t\mathbf{1} - x)$ ko`phad x matritsaning xarakteristik ko'phadi bo'lsin. Bu yerda , $\mathbf{1} \in M_n(\mathbb{C})$ - birlik matritsa.

Biz (5.5) va (5.6) tengliklardan foydalanib $\Delta(y)$ ning xarakteristik ko'phadini topishimiz mumkin. y elementning tanlanishiga ko`ra $y = y^t$ bo`ladi. Shunga ko'ra (5.5) va (5.6) tengliklardan

$$\Delta(y) = \lambda a_y y a_y^{-1} = \pm a_y y a_y^{-1}$$

bo'lishi kelib chiqadi. U holda

$$P_{\Delta(y)}(t) = P_{\lambda a y a^{-1}}(t) = \det\left(t\mathbf{1} - \lambda a y a^{-1}\right) = \det\left(a\left(t\mathbf{1} - \lambda y\right)a^{-1}\right) =$$
$$= \det\left(t\mathbf{1} - \lambda y\right) = (t-\lambda)(t+\lambda)t^{n-2}$$

va

$$P_{\Delta(y)}(t) = P_{\pm a_y y a_y^{-1}}(t) = \det\left(t\mathbf{1} \mp a_y y a_y^{-1}\right) = \det\left(a_y\left(t\mathbf{1} \mp y\right)a_y^{-1}\right) =$$
$$= \det\left(t\mathbf{1} \mp y\right) = (t-1)(t+1)t^{n-2} .$$

Bundan $(t-\lambda)(t+\lambda)t^{n-2}=(t-1)(t+1)t^{n-2}$ bo`ladi. Shunga ko`ra $\lambda=\pm 1$ ekanligi kelib chiqadi. Demak,

$$\Delta(x)=\pm axa^{-1}$$

yoki

$$\Delta(x)=\pm ax^{t}a^{-1}$$

bo`lishi kelib chiqadi.

Yetarliligi: Har qanday avtomorfizm lokal avtomorfizm bo`lishi aniq. Shuning uchun biz $sl_n$ dagi har bir anti-avtomorfizm lokal avtomorfizm bo`lishini ko`rsatishimiz yetarli.

Avval quyidagi ko`rinishdagi anti-avtomorfizmlar olib ularning lokal avtomorfizm ekanligini ko`rsatamiz.

$$\Delta(x)=x^{t}\quad,\qquad x\in sl_n \tag{5.7}$$

va

$$\Delta(x)=-x\quad,\qquad x\in sl_n. \tag{5.8}$$

5.1-teoremaga ko’ra $M_n(\mathbb{C})$ da aniqlangan $z\to z^{t}$ kabi akslantirish assotsiativ lokal avtomorfizm bo`ladi. Shunga ko’ra teskarilanuvchi $a_x\in M_n(\mathbb{C})$ matritsa mavjud bo’lib, $x^{t}=a_x x a_x^{-1}$ bo`ladi . U holda

$$x^{t}=a_x x a_x^{-1}\text{ va }-x=-a_x^{t}x\left(a_x^{t}\right)^{-1}.$$

Bundan ko`rinadiki, (5.7) va (5.8) ko`rinishda aniqlangan akslantirish sl_n da lokal avtomorfizm bo`ladi. Nihoyat $sl_n$ dagi har bir anti-avtomorfizm (5.7) yoki (5.8) ko`rinishdagi anti-avtomorfizm va avtomorfizmning superpozitsiyadan iborat ekanligi kelib chiqadi, ya`ni , sl_n da ixtiyoriy anti-avtomorfizm lokal avtomorfizm ekan. Isbot tugadi.

XULOSA

Mazkur bitiruv malakaviy ishi oddiy Li algebralarida avtomorfizmlar, lokal avtomorfizmlar va ularning ko'rinishlarini o'rganishga bag'ishlangan. Bir qancha adabiyotlar va maqolalardan foydalangan holda Li algebralari, ulardagi differensiallashlar, lokal avtomorfizmlar va bir qancha teoremalar o'rganilgan. Ushbu bitiruv malakaviy ishida nilpotent Li algebralari, Li algebralarining ideallari, ichki differensiallashlar o'rganilgan va nilpotent Li algebrasi kvadratining o'lchami haqidagi teorema isboti bilan keltirilgan. Shuningdek, yarim oddiy Li algebrasidagi har bir ichki differensiallash differensiallash bo'lishi va biror nilpotent differensiallashning eksponentasi avtomorfizm ekanligi isbotlangan. Olimlar ([16] da) Larson, D. R., Sourour, A. R. larning kompleks matritsalar algebrasida har bir lokal avtomorfizm avtomorfizm bo'lishining zaruriy va yetarli sharti va ([18] da) P.Semrlning kvadrati nol matritsalarni saqlovchi akslantirishlarning ko'rinishini aniqlovchi teorema ham keltirilgan.

Bundan tashqari chekli o'lchamli chiziqli fazodagi barcha endomorfizmlar to'plami barcha matritsalar to'plami bilan izomorf ekanligi, har qanday assotsiativ algebrani yangi ko'paytirish amali kiritish orqali Li algebrasiga aylantirish mumkinligi va assotsiativ algebraning differensiali Li algebrasining ham differensiali bo'lishi isbotlangan.

Ushbu bitiruv malakaviy ishining oxirgi mavzusi oddiy Li algebrasi sl_n(izi nolga teng bo'lgan barcha $n \times n$ matritsalar algebrasi)da har bir $\Phi : sl_n \to sl_n$ lokal avtomorfizm avtomorfizm bo'lishining zaruriy va yetarli sharti isbotlangan.

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