ҚОРАҚАЛПОҒИСТОНДА ФАН ВА ТАЪЛИМ

ҚАРАҚАЛПАҚСТАНДА ИЛИМ ҲӘМ ТӘЛИМ

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NATURAL SCIENCES

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SOLVABILITY OF THE BOUNDARY VALUE PROBLEM FOR A MIXED-TYPE EQUATION OF THE FOURTH ORDER

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Summary. *In the rectangular domain, a criterion for the uniqueness and existence of a solution of the problem of a fourth-order mixed-type equation was established. A continuous dependence of the solution of the problem on the initial data was proved.*

Key words. *Mixed-type equation, boundary value problem, spectral method, uniqueness, existence, continuous dependence of solutions on the data.*

1. Introduction. Boundary value problems for second-order and third-order equations have been thoroughly studied in rectangular domains. In recent years, interest has arisen in the study of boundary value problems for fourth-order equations. In [1] a complete classification of fourth-order equations with two independent variables was given, as well as the application of such equations, and some boundary-value problems for one equation of a certain type and for mixed-type equations were investigated. In [5-9], direct and inverse boundary value problems for a fourth-order equation were studied.

In this paper, using the method of spectral expansions, necessary and sufficient conditions for the uniqueness of the solution of a boundary-value problem for a fourth-order equation were given. In this case, the solution of the problem is constructed in the form of a series in the system of eigenfunctions of the problem under consideration. A continuous dependence of the solution on the initial data was proved.

2. Statement of the problem. In a domain $\Omega=\{(x,t):0<x<1,-\alpha<t<\beta\}$ where α, β was given positive numbers, we consider the equation

$$Lu\equiv u_{tt}-u_t+\operatorname{sgn}t\cdot u_{xxxx}=0. \qquad (1)$$

Let $\Omega_+=\Omega\cap\{t>0\}$, $\Omega_-=\Omega\cap\{t<0\}$.

Problem. Find a function $u(x,t)$ in the domain Ω, satisfying the following conditions:

$$u(x,t)\in C_{x,t}^{2,1}(\bar{\Omega})\cap C_{x,t}^{4,2}(\Omega_+\cup\Omega_-) \qquad (2)$$

$$Lu(x,t)\equiv 0,\ (x,t)\in\Omega_+\cup\Omega_- \qquad (3)$$

$$\begin{gathered} u(0,t)=u(1,t)=0,\ -\alpha\le t\le\beta, \\ u_{xx}(0,t)=u_{xx}(1,t)=0,\ -\alpha\le t\le\beta, \end{gathered} \tag{4}$$

$$u(x,\beta)=\varphi(x),\ 0\le x\le 1, \tag{5}$$

$$u(x,-\alpha)=\psi(x),\ 0\le x\le 1, \tag{6}$$

where $\varphi(x)$, $\psi(x)$ are given smooth functions, $\varphi^{(2i)}(0)=\varphi^{(2i)}(1)=0$ and $\psi^{(2i)}(0)=\psi^{(2i)}(1)=0,\ i=0,1.$

3. Construction of partial solutions of the equation. Non-zero particular solutions of equation (1) in the domain Ω, will be sought in the form of a product.

$$u(x,t)=X(x)T(t). \tag{7}$$

Substituting (7) into equation (1) and separating the variables, we have

$$\frac{X^{IV}(x)}{X(x)}=\operatorname{sgn}t\cdot\left(-\frac{T''(t)}{T(t)}+\frac{T'(t)}{T(t)}\right)=\lambda, \tag{8}$$

where λ is the separation constant. From (8), in view of the boundary conditions (4), the following problems follow:

$$X^{IV}(x)-\lambda X(x)=0,\ 0\le x\le 1,\ X(0)=X(1)=X''(0)=X''(1)=0, \tag{9}$$

$$\operatorname{sgn}t\cdot\left(T''(t)-T'(t)\right)+\lambda T(t)=0,\ t\in(-\alpha,0)\cup(0,\beta). \tag{10}$$

The solution of the spectral problem (9) has the form

$$X_k(x)=\sqrt{2}\sin\xi_k x,\ \xi_k=\sqrt[4]{\lambda_k}=k\pi,\ k\in N. \tag{11}$$

and forms a complete orthonormal system of functions in $L_2(0,1)$.

The differential equation (10) has the general solution

$$T_k(t)=\begin{cases} e^{\frac{1}{2}t}\left(a_k\cos v_k t+b_k\sin v_k t\right), & t>0, \\ e^{\frac{1}{2}t}\left(c_k e^{\mu_k t}+d_k e^{-\mu_k t}\right), & t<0, \end{cases} \tag{12}$$

where a_k, b_k, c_k and d_k are arbitrary constants, and $v_k=\frac{1}{2}\sqrt{4\xi_k^4-1}$, $\mu_k=\frac{1}{2}\sqrt{4\xi_k^4+1}$.

The connection between these arbitrary constants is obtained from condition (2) in the form

$$T_k(0-0)=T_k(0+0),\ T_k'(0-0)=T_k'(0+0). \tag{13}$$

Then (12) and (13) it follows that

$$c_k=\frac{1}{2\mu_k}\left[\mu_k a_k+v_k b_k\right],\ d_k=\frac{1}{2\mu_k}\left[\mu_k a_k-v_k b_k\right].$$

Then the function (12) is represented in the form

$$T_k(t)=\begin{cases} e^{\frac{1}{2}t}\left(a_k\cos v_k t+b_k\sin v_k t\right), & t>0, \\ \frac{1}{\mu_k}e^{\frac{1}{2}t}\left(\mu_k a_k ch\mu_k t+v_k b_k sh\mu_k t\right), & t<0. \end{cases} \tag{14}$$

4. Uniqueness and existence of the solution of the problem. Let $u(x,t)$ be the solution of problem (2) - (6). Following [2-4], we consider the functions

$$u_k(t)=\int_0^1 u(x,t)X_k(x)dx,\ k=1,2,\dots . \tag{15}$$

Differentiating (15), by virtue of equation (1) and the boundary conditions (4), we have

$$u_k''(t)-u_k'(t)+\operatorname{sgn}t\cdot\xi_k^4 u_k(t)=0.$$

The equation obtained coincides with equation (10) for at $\lambda=\xi_k^4$.

Consequently, $u_k(t)\equiv T_k(t)$ and $u_k(t)$ are determined according to (14).

From (5), (6) and (15) we have

$$u_k(\beta)=\sqrt{2}\int_0^1 u(x,\beta)\sin\xi_k x dx=\sqrt{2}\int_0^1\varphi(x)\sin\xi_k x dx=\varphi_k, \tag{16}$$

$$u_k(-\alpha)=\sqrt{2}\int_0^1 u(x,-\alpha)\sin\xi_k x dx=\sqrt{2}\int_0^1\psi(x)\sin\xi_k x dx=\psi_k. \tag{17}$$

Using conditions (16) and (17), we obtain a system of equations for finding the coefficients of a_k and b_k in the expression (14):

$$\begin{cases}\mu_k a_k ch\mu_k\alpha-v_k b_k sh\mu_k\alpha=\mu_k\psi_k e^{\frac{1}{2}\alpha},\\ a_k\cos v_k\beta+b_k\sin v_k\beta=\varphi_k e^{-\frac{1}{2}\beta}.\end{cases} \tag{18}$$

If the determinant of the system (18) is not equal to zero for all $k\in N$, i.e.

$$\Delta_k(\alpha,\beta)=v_k sh\mu_k\alpha\cos v_k\beta+\mu_k ch\mu_k\alpha\sin v_k\beta\neq 0, \tag{19}$$

then the system has a unique solution

$$a_k=\frac{v_k\varphi_k e^{-\frac{1}{2}\beta}sh\mu_k\alpha+\mu_k\psi_k e^{\frac{1}{2}\alpha}\sin v_k\beta}{\Delta_k(\alpha,\beta)},\quad b_k=\frac{\mu_k\varphi_k e^{-\frac{1}{2}\beta}ch\mu_k\alpha-\mu_k\psi_k e^{\frac{1}{2}\alpha}\cos v_k\beta}{\Delta_k(\alpha,\beta)}.$$

Then the functions $u_k(t)$ take the following form:

$$u_k(t)=\begin{cases}\dfrac{1}{\Delta_k(\alpha,\beta)}\left[\varphi_k e^{-\frac{1}{2}(\beta-t)}\Delta_k(\alpha,t)+\mu_k\psi_k e^{\frac{1}{2}(t+\alpha)}\sin v_k(\beta-t)\right],\ t>0,\\ \dfrac{1}{\Delta_k(\alpha,\beta)}\left[v_k\varphi_k e^{-\frac{1}{2}(\beta-t)}\operatorname{sh}\mu_k(t+\alpha)+\psi_k e^{\frac{1}{2}(t+\alpha)}\Delta_k(-t,\beta)\right],\ t<0.\end{cases} \tag{20}$$

Now, let $\varphi(x)\equiv 0$ and $\psi(x)\equiv 0$ in $[0,1]$. Then on the basis of (16), (17) and (20) we find that $u_k(t)\equiv 0$ in the segment $[-\alpha,\beta]$, $k=1,2,\dots$. Then from (15) we have

$$\int_0^1 u(x,t)X_k(x)dx=0,\ \ k=1,2,\dots .$$

Hence, in view of the completeness of the system $\{X_k(x)\}_{k=1}^{\infty}$ in space $L_2[0,1]$, it follows that the function $u(x,t)=0$ is almost everywhere in $[0,1]$ for any $t\in[-\alpha,\beta]$. Since by (2) the function $u(x,t)\in C(\bar{\Omega})$, it follows that $u(x,t)\equiv 0$ in $\bar{\Omega}$.

Suppose that for some α, β and $k=k_0\in N$ the following $\Delta_{k_0}(\alpha,\beta)=0$ holds. Then the problem (2) - (6) (where $\varphi(x)\equiv 0,\ \psi(x)\equiv 0$) has a nontrivial solution of the form

$$u_{k_0}(x,t)=\begin{cases}\sqrt{2}e^{\frac{1}{2}t}\sin v_{k_0}(\beta-t)\sin\xi_{k_0}x, & t>0,\\ \sqrt{2}\dfrac{1}{\mu_{k_0}}e^{\frac{1}{2}t}\Delta_{k_0}(-t,\beta)\sin\xi_{k_0}x, & t<0.\end{cases}$$

In this case, the solution of problem (2)-(6) is not unique.

Now, we will consider the question of the zeros of the expression $\Delta_k(\alpha,\beta)$. We transform (19) to the form:

$$\Delta_k(\alpha,\beta)=A_k\sin(v_k\beta+\gamma_k) \tag{21}$$

Where $A_k=\sqrt{\xi_k^4 ch2\mu_k\alpha+\frac{1}{4}}$, $\gamma_k=\arcsin\frac{v_k sh\mu_k\alpha}{A_k}$.

It follows from (21) that only in the case $\Delta_k(\alpha,\beta)=0$ when

$$\beta=\frac{n\pi}{v_k}-\frac{\gamma_k}{v_k},\quad k,n\in N.$$

For such values β, the uniqueness of the solution of problem (2) - (6) is violated.

So, the following is valid.

Theorem 1. In order that problem (2) - (6) have a unique solution, it is necessary and sufficient that the inequality $\Delta_k(\alpha,\beta)\neq 0$, $\forall k\in N$

A direct verification shows that there exist positive constants c_0 and k_0, and depending on the β, such that for the estimate at $k>k_0$

$$|\Delta_k(\alpha,\beta)|\geq c_0\xi_k^2 e^{\xi_k^2\alpha}>0. \tag{22}$$

Lemma 1. Let the estimate (22) hold. Then for sufficiently large k the estimates are valid

$$u_k(t)=\begin{cases}c_1(|\varphi_k|+|\psi_k|), & t>0,\\ c_2(|\varphi_k|+|\psi_k|), & t<0,\end{cases}\qquad u_k'(t)=\begin{cases}c_3k^2(|\varphi_k|+|\psi_k|), & t>0,\\ c_4k^2(|\varphi_k|+|\psi_k|), & t<0,\end{cases}$$

$$u_k''(t)=\begin{cases}c_5k^4(|\varphi_k|+|\psi_k|), & t>0,\\ c_6k^4(|\varphi_k|+|\psi_k|), & t<0.\end{cases}$$

The validity of these estimates for the function (20) and its derivatives can be shown on the bases (20) and (22).

Lemma 2. Let $\varphi(x), \psi(x) \in C^5[0,1]$, $\varphi^{(2i)}(0) = \varphi^{(2i)}(1) = 0$, $\psi^{(2i)}(0) = \psi^{(2i)}(1) = 0$, $i = 0,1,2$. Then we have

$$\varphi_k = -\frac{1}{\pi^5}\frac{\varphi_k^{(5)}}{k^5},\ \psi_k = -\frac{1}{\pi^5}\frac{\psi_k^{(5)}}{k^5},$$

where $\varphi_k^{(5)} = \sqrt{2}\int_0^1 \varphi^{(5)}(x)\cos\pi kx dx$, $\psi_k^{(5)} = \sqrt{2}\int_0^1 \psi^{(5)}(x)\cos\pi kx dx$ and

$$\sum_{k=1}^{\infty}\left|\varphi_k^{(5)}\right|^2 \le \int_0^1 \left|\varphi^{(5)}(x)\right|^2 dx,\ \sum_{k=1}^{\infty}\left|\psi_k^{(5)}\right|^2 \le \int_0^1 \left|\psi^{(5)}(x)\right|^2 dx.$$

The proof of the lemma follows by integrating by parts five times the integrals from (16) - (17), taking into account the conditions of the lemma and the Bessel inequality in the trigonometric system.

If conditions (19) and (22) are satisfied, then on the basis of particular solutions (20), the solution of problem (2) - (6) can be represented as a Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x) \tag{23}$$

where the functions $u_k(t)$ and $X_k(x)$ are respectively determined by the formulas (20) and (11).

We consider the problem of the convergence of the Fourier series, that on the right-hand side of (23) and

$$u_t(x,t) = \sum_{k=1}^{\infty} u_k'(t) X_k(x),$$
$$u_{tt}(x,t) = \sum_{k=1}^{\infty} u_k''(t) X_k(x), \tag{24}$$
$$u_{xxxx}(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k^{(4)}(x).$$

The series (24) for any $(x,t) \in \bar{\Omega}$ by virtue of (22) and the conditions of the lemma are dominated by the following numerical series

$$C\sum_{k=1}^{\infty}\frac{1}{k}\left(\left|\varphi_k^{(5)}\right| + \left|\psi_k^{(5)}\right|\right). \tag{25}$$

Then, by virtue of the convergence of (25), on the basis of the Weierstrass test, the first series (24) converges absolutely and uniformly in the closed domain $\bar{\Omega}$, and the second and third series in the domains $\bar{\Omega}_+$ and $\bar{\Omega}_-$. Substituting the series (24) into equations (1), we see that the function (23) satisfies equations (1) on the $\bar{\Omega}_+ \cup \bar{\Omega}_-$ domains.

Thus, the following assertion is true.

Theorem 2. Let the functions $\varphi(x), \psi(x)$ satisfy the conditions of Lemma 2 and let (22) be satisfied. Then there exists a unique solution of problem (2) - (6) and is determined by the series (23).

5. The continuous dependence of solutions on the initial data.

Theorem 3. Let the conditions of Theorem 2 be satisfied. Then for the solution of problem (2) - (6) we have the estimates

$$\|u(x,t)\|_{L_2} \le c_{11}\left(\|\varphi\|_{L_2} + \|\psi\|_{L_2}\right),\ t>0,$$

$$\|u(x,t)\|_{L_2} \le c_{22}\left(\|\varphi\|_{L_2} + \|\psi\|_{L_2}\right),\ t<0,$$

$$\|u(x,t)\|_{C(\bar{\Omega}_+)} \le c_{33}\left(\|\varphi\|_{W_2^1} + \|\psi\|_{W_2^1}\right),\ t>0,$$

$$\|u(x,t)\|_{C(\bar{\Omega}_-)} \le c_{44}\left(\|\varphi\|_{W_2^1} + \|\psi\|_{W_2^1}\right),\ t<0$$

where c_{ii} does not depend on the functions $\varphi(x)$ and $\psi(x)$, and the norms are defined by formulas $\|u(x,t)\|_{L_2(0,1)} = \|u(x,t)\|_{L_2} = \left(\int_0^1 |u(x,t)|^2 dx\right)^{\frac{1}{2}}$, $\|u(x,t)\|_{C(\bar{\Omega}_\pm)} = \max_{\bar{\Omega}_\pm} |u(x,t)|$.

Proof. Since $\{X_k(x)\}$ is an orthonormal system in $L_2[0,1]$, from (23) and Lemma 1, with $t>0$ we have

$$\|u(x,t)\|_{L_2}^2 \le 2c_1^2 \sum_{k=1}^{\infty}\left(|\varphi_k| + |\psi_k|\right)^2 \le 4c_1^2 \sum_{k=1}^{\infty}\left(|\varphi_k|^2 + |\psi_k|^2\right) \le$$
$$\le 4c_1^2\left(\|\varphi\|_{L_2}^2 + \|\psi\|_{L_2}^2\right) \le c_{11}^2\left(\|\varphi\|_{L_2}^2 + \|\psi\|_{L_2}^2\right) \le c_{11}^2\left(\|\varphi\|_{L_2} + \|\psi\|_{L_2}\right)^2. \tag{26}$$

Similarly, when $t<0$ we get

$$\|u(x,t)\|_{L_2}^2 \le c_{22}^2\left(\|\varphi\|_{L_2} + \|\psi\|_{L_2}\right)^2. \tag{27}$$

From (26) and (27) follows the first two estimates of the theorem.

Let (x,t) be an arbitrary point of the domain $\bar{\Omega}_+$. Then, using formulas (23), (20), and Lemma 1 at $t>0$, we have

$$u(x,t) \le \sqrt{2}c_1 \sum_{k=1}^{\infty}\left(|\varphi_k| + |\psi_k|\right). \tag{28}$$

Since

$$\varphi_k = \frac{\varphi_k^{(1)}}{k\pi},\ \psi_k = \frac{\psi_k^{(1)}}{k\pi},\ \varphi_k^{(1)} = \sqrt{2}\int_0^1 \varphi'(x)\cos\pi kx dx,\ \psi_k^{(1)} = \sqrt{2}\int_0^1 \psi'(x)\cos\pi kx dx,$$

then using the Cauchy-Bunyakovskii inequality from (28), we obtain

$$|u(x,t)| \le \frac{\sqrt{2}c_1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(\left|\varphi_k^{(1)}\right| + \left|\psi_k^{(1)}\right|\right) \le \frac{\sqrt{2}c_1}{\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}\left|\varphi_k^{(1)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\psi_k^{(1)}\right|^2\right)^{\frac{1}{2}}\right] \le$$
$$\le \frac{c_1}{\sqrt{3}}\left[\|\varphi'\|_{L_2} + \|\psi'\|_{L_2}\right] \le c_{12}\left[\|\varphi\|_{W_2^1} + \|\psi\|_{W_2^1}\right].$$

Here equality $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ is used.

In the same way, at $t<0$ the fourth inequality of the theorem is shown.

The theorem was proved.

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Rezyume. *To`rtburchakli sohada to`rtinchi tartibli aralash turdagi tenglama echimining mavjudligi va yagonaligi ko`rsatilgan. Masala echimining berilganlardan uzliksiz bog`likligi isbotlangan.*

Резюме. *В прямоугольной области установлен критерий единственности и существования решения задачи уравнения смешанного типа четвертого порядка. Доказана непрерывная зависимость решения задачи от начальных данных.*

***Kalit so`zlar.*** *Aralash turdagi tenglama, chegaraviy masala, spektral metod, echimning yagonaligi, mavjudligi va berilganlardan uzluksiz bog`likligi.*

Ключевые слова. *Уравнение смешанного типа, краевая задача, спектральный метод, единственность, существование, непрерывно зависимость решения от данных.*