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JORDAN MATRISASI VA UNI TOPISH USULLARI

uslubiy qo`llanma

«**5 110100 MATEMATIKA O`QITISH METODIKASI**»
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«Algebra va sonlar nazariyasi» fanidan uslubiy qo`llanma .

Ushbu uslubiy qo`llanma «Algebra va sonlar nazariyasi» fani bo‘yicha 5110100 – matematika o`qitish metodikasi ta’lim yo‘nalishi bakalavr talabalari uchun mo‘ljallangan bo‘lib, unda shu fanning namunaviy o‘quv dasturida keltirilgan matrisalar algebrasining usullariga qo`shimcha ravishda Jordan matrisasi va uni topish usullariga oid qisqacha nazariy ma’lumotlar, bu usullarning taqbiqiga oid namunaviy misollar yechimlari, mustaqil ish topshiriqlari va boshqa materiallar keltirilgan. Bular talabalarga shu fanni yanada chuqurroq o‘zlashtirishga yaqindan yordam beradi degan umiddaman.

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1-§. Matritsalar haqida umumiy ma`lumotlar

Sonlardan tuzilgan quyidagi to'g'ri burchakli jadvalga (tablisaga) *matritsa* deb aytiladi:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Matritsaning gorizontal qatoridagi sonlari uning *satrlari*, vertikal qatoridagi sonlari uning *ustunlari* deb aytiladi. a_{ij} sonlar *matritsaning elementlari* deb aytiladi. Matritsa *m* ta satrlarga va *n* ta ustunlarga ega bo'lsa, uni $m \times n$ *matritsa* deb aytiladi. Agar $m = n$ bo'lsa, bunday matritsa *n-tartibli kvadrat matritsa* deb aytiladi.

B matritsa *A* matritsa bilan α sonning ko'paytmasidan iborat deb aytiladi, agar ularning hamma elementlari uchun $b_{ij} = \alpha a_{ij}$ tenglik bajarilsa (*A* va *B* matritsalarning o'lchovlari bir xil) va $B = \alpha A$ deb belgilanadi.

Uchta *A, B, C* – matritsalar bir xil o'lchovli bo'lsin. Agar *i* va *j* indekslarning hamma qiymatlari uchun $c_{ij} = a_{ij} + b_{ij}$ tenglik bajarilsa, *C* matritsa *A* va *B* matritsalarning yig'indisi deb aytiladi va $C = A + B$ deb belgilanadi.

Faraz qilaylik, $m \times n$-o'lchovli $A = (a_{ij})$ va $n \times p$-o'lchovli $B = (b_{ij})$ matritsalar berilgan bo'lsin. Bu matritsalarning ko'paytmasi deb shunday $C = AB = (c_{ik})$ matritsaga aytiladiki, uning elementlari quyidagi formula bilan beriladi:

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}, \quad i = 1,2,\ldots,m; \quad k = 1,2,\ldots,p.$$

Agar *B* matritsaning ustunlari *A* matritsaning mos satrlari bo'lsa, ya'ni hamma *i, j* indekslar uchun $b_{ij} = a_{ji}$,bo`lsa *B* matritsa *A* matritsaga nisbatan *transponirlangan matritsa* deb aytiladi va $B = A^T$ deb belgilanadi, *A* matritsadan A^T matritsaga o'tish amali *A* matritsani *transponirlash* deb aytiladi. Agar *A* matritsa $m \times n$ o'lchovli bo'lsa, A^T matritsa $n \times m$ o'lchovli bo'ladi.

Agar hamma *i, j* indekslar uchun $b_{ij} = \overline{a}_{ij}$ tenglik bajarilsa, *B* matritsa *A* kompleks matritsaga nisbatan *qo'shma kompleks matritsa* deb ataladi va $B = \overline{A}$ deb belgilanadi. Agar hamma *i, j* lar uchun $b_{ij} = \overline{a}_{ji}$ tenglik bajarilsa, *B* matritsa *A* matritsaga nisbatan *qo'shma ermit matritsa* deb aytiladi va $B = A^H$ deb belgilanadi.

Agar hamma elementlari 0 ga teng bo'lsa, *A* matritsa *nol matritsa* deb aytiladi, va *A=0* deb belgalanadi. Agar $a_{i_0 j_0} = 1$ bo'lib, qolgan elementlari nolga teng bo'lsa, *A* matritsa i_0, j_0 *indeksli birlik matritsa* deb aytiladi,.

$a_{11}, a_{22}, \ldots, a_{nn}$ elementlar *n* tartibli $A = (a_{ij})$ kvadrat matritsaning bosh dioganalini tashkil qiladi va uning *diagonal elementlari* deb aytiladi. Matritsaning diagonal elementlari yig'indisi *A* matritsaning *izi* deb aytiladi va trA deb belgilanadi. Shunday qilib, $trA = \sum_{i=1}^{n} a_{ii}$.

Agar diagonalda bo'lmagan elementlari 0 ga teng bo'lsa, ya'ni $a_{ij}=0$, $i\neq j$, kvadrat matritsa diagonal matritsa deb aytiladi*n*-tartibli diagonal matritsa $diag(a_{11},...,a_{nn})$ deb belgilanadi. Diagonal elementlari 1 ga teng bo'lgan *n*-tartibli diagonal matritsa birlik matritsa deb aytiladi va E yoki E_n deb belgilanadi. Birlik matritsaning elementlari δ_{ij} deb belgilanadi: $E=(\delta_{ij})$,

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

Bizga $f(x)=a_0+a_1x+...+a_kx^k$ - ko'phad berilgan bo'lsin. $B=a_0E+a_1A+...+a_kA^k$ matritsa A matritsadan *ko'phad* deb aytiladi va $B=f(A)$ deb belgilanadi.

Quyidagi uchta almashtirishlar *matritsaning elementar almashtirishlari* deb aytiladi:

1) matritsaning biror satriga noldan farqli songa ko'paytirish;
2) matritsaning biror satrini songa ko'paytirib boshqa satriga qo'shish;
3) ikkita satrlarning o'rinlarini almashtirish.

Xuddi shunday uchta almashtirishlarni matritsaning ustunlari uchun ham ta'riflash mumkin.

A – matritsa *n*-chi tartibli kvadrat matritsa bo'lsin. *A* matritsa uchun *AB*=*BA*=*E* tenglikni qanoatlantiruvchi *B* matritsa *A* ga *teskari matritsa* deyiladi va u $B=A^{-1}$ ko'rinishda belgilanadi. Teskari matritsa elementlarini

$$b_{ij}=\frac{A_{ji}}{\det A}$$

formula yordamida topish mumkin, bunda A_{ji}lar a_{ji} elementlarning algebraik to'ldiruvchisidir. Agar $\det A\neq 0$, ya'ni A matritsa xosmas bo'lsa, A matritsa *teskarilanuvchi* deb aytiladi.

Har qanday xosmas A matritsani faqat satrlar (yoki faqat ustunlar) elementar almashtirishlari yordamida birlik matritsaga keltirish mumkin. Elementar almashtirishlarni xuddi shunday ketma-ketlikda E birlik matritsaga tadbiq qilsak, teskari matritsa A^{-1} ni hosil qilamiz. A va E matritsalarni chiziq yordamida qo'shni yozib ular ustida elementar almashtirishlarni bir vaqtda bijarish juda qulaydir.

Teskari matritsani hisoblash $AX=B$, $YA=B$ matritsaviy tenglamalarni yechish bir-biri bilan bog'langandir, bunda *A*, *B* – berilgan matritsalar, *X*, *Y* – izlanayotgan noma'lum matritsalar. Agar *A* matritsa to'g'ri burchakli matritsa yoki xosmas matritsa bo'lsa, matritsaviy tenglamalarni yechish *X* matritsaning har bir ustuni yoki *Y* matritsaning har bir satri elementlari uchun hosil bo'ladigan chiziqli tenglamalar sistemasini yechishga keltiriladi. Bu tenglamalarni hosil qilish uchun tenglamaning har ikkala tomonidagi matritsalarning mos elementlarini bir-biriga tenglashtirish lozim. Agar *A* matritsa xosmas bo'lsa, matritsaviy tenglamalarning yechimlari quyidagi formulalar yordamida topiladi:

$$X = A^{-1}B, \ \ Y = BA^{-1}.$$

2-§. Maxsus ko'rinishdagi matritsalar

Endi n–tartibli matritsaning ba'zi bir maxsus ko'rinishlarini qarab chiqamiz: $A=(a_{ij})$

skalyar matritsa: $A = diag(\lambda,...,\lambda)$, λ – son;

unimodulyar matritsa: detA=1;

xos (maxsus)matritsa: detA=0;

xosmas (maxsusmas) matritsa: $detA \neq 0$;

o'rin almashtirish matritsasi: A matritsa birlik matritsadan satrlarning o'rinlarini almashtirishdan hosil bo'ladi;

elementar matritsa: A matritsa birlik matritsadan elemetar almashtirishlar orqali hosil bo'ladi*;*

yuqori uchburchakli matritsa: $a_{ij} = 0$ agar $i > j$;

pastki uchburchakli matritsa: $a_{ij} = 0$ agar $i < j$;

simmetrik matritsa: $A^T=A$;

kososimmetrik matritsa: $A^T=-A$;

ermit matritsasi: $A^N=A$;

kosoermit matritsasi: $A^N=-A$;

ortogonal matritsa: $A^T=A^{-1}$;

unitar matritsa: $A^N=A^{-1}$;

manfiymas matritsa: $a_{ij} \geq 0$ hamma i, j lar uchun;

Markov (stoxastik) matritsasi: $a_{ij} \geq 0$ hamma i, j lar uchun va $\sum_{k=1}^{n} a_{ik} = 1$ $i=1, 2,...,n$;

nilpotent matritsa: *k natural sonning qandaydir qiymatida* $A^k=0$ (bunday k ning eng kichik qiymatiga *A matritsaning nilpotentlik ko'rsatkichi* deb aytiladi);

davriy matritsa: *k natural sonning qandaydir qiymatida* $A^k=E$ (bunday k son A matritsaning *davri* deb aytiladi).

Agar matrisaning elementlari $m_i \times n_j$ o'lchovli B_{ij} matritsalardan iborat bo'lsa B matritsa katakli matritsa deb aytiladi, bunda B matritsaning bitta satriga qarashli bo'lgan hamma B_{ij} matritsalar bir xil balandlikka ega bo'ladi, B matritsaning bitta ustuniga qarashli bo'lgan hamma B_{ij} matritsalar bir xil enga ega bo'ladi. Katakli matritsalar ustida bajariladigan amallar oddiy sonli matritsalar ustida bajariladigan amallardan iborat. Agar A sonli matritsa gorizontal va vertikal to'g'ri chiziqlar bilan B_{ij} kataklarga ajratilgan bo'lib, tabiiy holda nomerlangan bo'lsa va bu kataklardan $B=(B_{ij})$ katakli matritsa tuzilgan bo'lsa, V matritsa A matritsadan kataklarga bo'lish natijasida hosil bo'lgan deb aytiladi. B_{ij} matritsalarning elementlaridan tabiiy ravishda $\sum_i m_i \times \sum_j n_j$ o'lchovli sonli matritsani tuzish mumkin. Bu holda A matritsa B matritsa kataklarining birlashmasidan hosil bo'lgan deb aytiladi va $A=B^{\square}$ deb yoziladi. Agar tushunmovchilik uchun asos bo'lmasa, □ belgisini qoldirib sonli va katakli matritsalarni bir xil harflar bilan belgidash mumkin.

A=(a_{ij}) va B – matritsalar, $S=(s_{ij})$ – katakli matritsa $c_{ij}=a_{ij}B$ tenglik bilan hamma i, j lar uchun aniqlangan bo'lsin. S matritsa kataklarining birlashmasidan hosil bo'lgan sonli matritsa A va B matritsalarning *o'ng kroneker ko'paytmasi* deb aytiladi (yoki *o'ng to'g'ri ko'paytmasi deyiladi*) va $A\otimes B$ deb belgilanadi.

1-m i s o l. A diagonal matritsa bo'lib, uning hamma diagonal elementlari har xil va $AB=BA$ bo'lsin. U holda B matritsa ham diagonal ekanligini isbotlang.

Yechish. Bu tasdiqning to'g'riligini ikkinchi tartibli matritsa uchun isbotlaymiz.

$A=\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\ d_1\neq d_2,\ \ B=\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ bo'lsin. U holda

$$AB=\begin{pmatrix} d_1x & d_1y \\ d_2z & d_2t \end{pmatrix},\ BA=\begin{pmatrix} d_1x & d_2y \\ d_1z & d_2t \end{pmatrix}.$$

$AB=BA$ bo'lganligi uchun $d_1y=d_2y$ va $d_2z=d_1z$ bo'ladi. bundan esa $(d_1-d_2)y=0,(d_2-d_1)z=0$, ya'ni $y=0,\ z=0$ bo'ladi, chunki $d_1-d_2\neq 0$.

Xuddi shunday bu tasdiqni n-chi tartibli matritsalar uchun isbotlash mumkin.

2-m i s o l. Har qanday n chi tartibli maxsusmas matritsa bilan o'rin almashinuvchi matritsalarni toping.

Yechish. Dioganal matritsa *diag(1, 2, ..., n)* maxsusmasdir. Bu matritsadan foydalanib birinchi misolni qo'llasak, A matritsaning dioganalligi kelib chiqadi. endi faqat A matritsaning hamma dioganal elementlari teng ekanligini isbotlash qoladi. Agar A matritsa ikkinchi tartibli bo'lsa, $A=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, uni chapdan va o'ngdan $S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ matritsaga ko'paytiramiz. AS va SA matritsalarni tenglashtirib $\lambda_1=\lambda_2$ tenglikni hosil qilamiz. Xuddi shunday ixtiyoriy tartibli A dioganal matritsa uchun S ni tanlab A matritsaning har qanday ikkita diagonal elementlarining tengligini tekshiramiz.

3-m i s o l. $A=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ matritsa ermit matritsasidir, chunki

$$A^H=\overline{A}^T=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

4-m i s o l. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ matritsa o'rin almashtirish matritsasidir, chunki bu matritsa birlik matritsadan 1-,2- chi va 3-, 4- chi satrlarning o'rinlarini almashtirishdan hosil bo'ladi.

5-m i s o l. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ matritsa unitar matritsadir, chunki

$$A^{-1} = A^{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

6-m i s o l. Agar A matritsa yuqori uchburchakli bo'lsa va uning hamma dioganal elamentlari noldan farqli bo'lsa, A^{-1} matritsa mavjudligini va yuqori uchburchakli matritsa ekanligini isbotlang.

Yechish. Teskari matritsani $(A|E)$ matritsadan satrlarning elementar almashtirishlari yordamida izlaymiz. soddalashtirish prosessini oxirgi satrdan boshlaymiz. bunda A va E matritsalarning bosh dioganaldan pastda joylashgan elementlari o'zgarmaydi. natijada birlik matritsadan yuqori uchburchakli matritsa hosil bo'ladi.

7-m i s o l. $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ matritsa ortogonal matritsadir, chunki $A^{T} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = A^{-1}.$

8-m i s o l. $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ matritsa nilpotentdir va uning nilpotentlik ko'rsatkichi 2 ga teng, chunki bu matritsaning kvadrati nol matritsadir.

9-m i s o l. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ matritsa davriy matritsadir va uning davri 2 ga teng, chunki uning kvadrati birlik matritsaga tengdir.

10-m i s o l. $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ matritsa stoxastik matritsadir.

11-m i s o l. Œrinalmashtirish matritsasi davriy matritsa ekanligini isbotlang.

Yechish. Faraz qilaylik, A - o'rin almashtirish matritsasi bo'lsin. Mumkin bo'lgan barcha A^k matritsalarni qaraymiz. Tekshirish mumkinki, o'rinalmashtirishlar matritsalarning ko'paytmasi yana o'rin almashtirish matritsasi bo'ladi. Har xil o'rinalmashtirishlar matritsalarning bir xil tartibga ega bo'lgan matritsalari soni cheklidir. Shuning uchun shunday p, q natural sonlar mavjudki, $q>p$ va $A^q = A^p$ bo'ladi. Bundan esa $A^{q-p} = E$ kelib chiqadi.

12-m i s o l. Berilgan A va B matritsalarni bloklarga ajratib AB ko'paytmani toping.

$$A=\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}.$$

Yechish. Berilgan A va B matritslarani bloklarga quyidagicha ajratamiz. Bu matritsalarni blokli matritsalarni ko'paytirish qoidasiga asosan ko'paytirsak:

$$A=\left(\begin{array}{cc|cc}0&0&0&-1\\0&0&-1&0\\\hline 0&1&0&0\\1&0&0&0\end{array}\right),\quad B=\left(\begin{array}{cc|cc}0&0&-1&0\\0&0&0&1\\\hline 1&0&0&0\\0&-1&0&0\end{array}\right).$$

quyidagi matritsa hosil bo'ladi $AB=\left(\begin{array}{cc|cc}0&-1&0&0\\-1&0&0&0\\\hline 0&0&0&1\\0&0&-1&0\end{array}\right).$

13-m i s o l. Agar $N=\begin{pmatrix}E&A\\0&E\end{pmatrix}$ blokli matritsa bo'lsa, $(N\,\square)^{-1}$ matritsa topilsin.

Yechish. $\det H=E\neq 0$ bo'lganligi uchun teskari matritsa mavjud. algebraik to'ldiruvchilarni hisoblamaymiz.

$$A_{11}=E, A_{12}=0, A_{21}=-A, A_{22}=E.$$

bundan esa $(N\,\square)^{-1}=\begin{pmatrix}E&-A\\0&E\end{pmatrix}$ kelib chiqadi.

14-m i s o l. A va B matritsalarning kroneker ko'paytmasini hisoblang.

$$A=\begin{pmatrix}3&5\\5&9\end{pmatrix},\quad B=\begin{pmatrix}1&2\\3&4\end{pmatrix}.$$

Yechish. Kroneker ko'paytmasining ta'rifiga asosan quyidagiga ega bo'lamiz:

$$A\otimes B=\begin{pmatrix}3\begin{pmatrix}1&2\\3&4\end{pmatrix}&5\begin{pmatrix}1&2\\3&4\end{pmatrix}\\5\begin{pmatrix}1&2\\3&4\end{pmatrix}&9\begin{pmatrix}1&2\\3&4\end{pmatrix}\end{pmatrix}=\begin{pmatrix}3&6&5&10\\9&12&15&20\\5&10&9&18\\15&20&27&36\end{pmatrix}.$$

M A S H Q L A R

1. *A, B* – bir xil tartibli diagonal matritsalar, α – son bo'lsin. $\alpha A, A+B, AB, BA$ matritsalar ham diagonal matritsalar va *AB*=*BA ekanligini* isbotlang.

2. $A=diag(\lambda_1,...,\lambda_n)$ bo'lsin.

1) BA matritsaning ustunlari B matritsaning ustunlarini $\lambda_1,...,\lambda_n$ sonlarga ko'paytirishdan hosil bo'lishini ;

2) AB matritsaning satrlari B matritsaning satrlarini $\lambda_1,...,\lambda_n$ sonlarga ko'paytrishdan hosil bo'lishini isbotlang.

3. A – diagonal matritsa, $f(t)$ – ko'phad bo'lsin. u holda $f(A)$ matritsa ham diagonal matritsa ekanligi isbotlang.

4. A matritsa diagonal matritsa bo'lib, uning hamma dioganal elementlari har xil va $AB=BA$ bo'lsin. U holda B matritsa ham diagonal matritsa ekanligi isbotlang.

5. A matritsa ixtiyoriy n-tartibli dioganal matritsa bilan o'rinalmashinuvchi bo'lsin. A matritsa n-tartibli dioganal matritsa ekanligini isbotlang.

6. A matritsa hamma n-tartibli matritsaviy birliklar bilan o'rinalmashinuvchi bo'lsin. A matritsa skalyar matritsa ekanligini isbotlang.

7. A matritsa n-tartibli har qanday matritsa bilan o'rinalmashinuvchi bo'lsin. A matritsa skalyar matritsa ekanligini isbotlang.

8. Berilgan matritsalarga qo'shma ermit matritsalar topilsin:

a) $\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$; b) $\begin{pmatrix} 5 & i \\ -i & 1 \end{pmatrix}$; c) $\begin{pmatrix} 1 & -i \\ 2+i & 1-2i \end{pmatrix}$; d) $\begin{pmatrix} 1+i\sqrt{2} & 3 \\ 1 & 1-i\sqrt{2} \end{pmatrix}$.

9. Ayniyatlarning to'g'riligini tekshiring:

a) $(A+B)^H = A^H + B^H$; b) $(\alpha A)^H = \overline{\alpha} A^H$;

c) $(A^H)^H = A$; d) $(AB)^H = B^H A^H$; e) $(A^H)^{-1} = (A^{-1})^H$.

10. Berilgan ikkinchi tartibli matritsalar dioganal, skalyar, uchburchakli, simmetrik, kososimmetrik, ermit, kosoermit, unitar, ortogonal yoki o'rinalmashtirish matritsalari ekanligini aniqlang:

a) $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$; b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}$; d) $\begin{pmatrix} 5 & i \\ -i & 1 \end{pmatrix}$;

e) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; f) $\begin{pmatrix} -4 & 0 \\ 1 & 4 \end{pmatrix}$; g) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$; h) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

11. a) kososimmetrik matritslarning hamma dioganal elementlari nolga teng;

b) ermit matritsasining dioganal elementlari haqiqiy;

c) kosoermit matritsaning dioganal elementlari mavhum son ekanligini isbot qiling.

12. a) agar A ermit matritsasi bo'lsa, iA – kosoermit matritsa ekanligini;

b) agar A kosoermit matritsasi bo'lsa, iA ermit matritsasi ekanligini isbot qiling.

13. a) ikkinchi tartibli ermit matritsalarning umumiy ko'rinishi topilsin;

b) ikkinchi tartibli kosoermit matritsalarning umumiy ko'rinishi topilsin;

c) hamma ikkinchi tartibli o'rinalmashtirishlarning matritsalarini ko'rsating.

14. Agar A diagonal matritsa bo'lib, uning hamma dioganal elementlari noldan farqli bo'lsa, teskari A^{-1} matritsa mavjud va dioganal matritsa ekanligini isbotlang.

15. Agar A – maxsusmas simmetrik matritsa bo'lsa, A^{-1} ham simmetrik matritsa ekanligini isbotlang.

16. Agar A – maxsusmas kososimmetrik matritsa bo'lsa, A^{-1} ham kososimmetrik matritsa ekanligini isbotlang..

17. Agar A – ortogona matritsa bo'lsa, A^{-1} matritsa mavjud va ortogonal ekanligini isbotlang.

18. Agar A – unitar matritsa bo'lsa, A^{-1} mavjud va unitar ekanligini isbotlang.

19. Agar A – o'rinalmashtirish matritsasi bo'lsa, A^{-1} matritsa mavjud va u ham o'rinalmashtirish matritsasi bo'lishini ekanligini isbotlang.

20. Quyidagi matritsalar ortogonal ekanligini isbotlang va ularga teskari matritsani toping:

$$\text{a) } \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{pmatrix}; \qquad \text{b) } \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{1}{2} & \frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix};$$

$$\text{c) } \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}; \quad \text{d) } \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{1}{2} & -\frac{1}{2} & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{10} \end{pmatrix}.$$

21. Quyidagi matritsalar unitar ekanlini isbotlang va ularga teskari matritsani toping:

$$\text{a) } \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}; \qquad \text{b) } \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

22. A va B – matritsalar yuqori uchburchakli matritsalar bo'lsin. AB matritsaning elementlarini A va B matritsalarning elementlari orqali ifodalang.

23. A va B – matritsalar yuqori uchburchakli matritsalar bo'lsin. U holda $A+B$ va AB – matritsalar ham yuqori uchburchakli matritsalar ekanligini isbotlang.

24. A va B matritsalar simmetrik matritsalar bo'lsin. Isbot qilish lozimki:

a) $A+B$ – simmetrik matritsadir;

b) har qanday k natural son uchun A^k – simmetrik matritsadir $k \in \boldsymbol{N}$;

c) AB matritsa simmetrik matritsa bo'lishi uchun A va B matritsalar o'rin almashinuvchi bo'lishi zarur va yetarlidir.

25. A va B – matritsalar kososimmetrik matritsalar bo’lsin. isbot qilish lozimki:

a) $A+B$ – kososimmetrik matritsa;

b) k toq son bo’lganda A^k – kososimmetrik matritsadir va k juft son bo’lganda

s) AB matritsa simmetrik bo’lishi uchun A va B matriyalar o’rinalmashuvchi bo’lishi zarur va yetarlidir;

d) A va B matritsalarning ko’paytmasi kososimmtrik matritsa bo’lishining yetarli va zarur shartlarini ifodalang va ularni isbotlang.

26. A – ixtiyoriy kvadrat matritsa bo’lsin. $A + A^T$ va AA^T matritsalar simmetrik, $A - A^T$ matritsa kososimmetrik ekanligini isbotlang.

27. Berilgan matritsalarni simmetrik va kososimmetrik matritsalar yig’indisiga yoying:

a) $\begin{pmatrix} 4 & -3 \\ 12 & -8 \end{pmatrix}$; b) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; c) $\begin{pmatrix} 0 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{pmatrix}$.

28. Har qanday haqiqiy ermit matritsasi simmetrik matritsa ekanligini isbotlang.

29. A – ermit matritsasi va $A+B=iC$ bo’lsin, bunda B va C matritsalar haqiqiydir. B – simmetrik matritsa, C – kososimmetrik matritsa ekanligini isbot qiling.

30. Haqiqiy unitar matritsa ortogonal matritsa ekanligini isbotlang.

31. Agar A va B matritsalar ortogonal bo’lsa, AB ortogonal ekanligini isbotlang.

32. Agar A va B matritsalar unitar bo’lsa, AB unitar ekanligini isbotlang.

33. O’rin almashtirish matritsasi ortogonal ekanligini isbotlang.

34. Berilgan matritslar davriy, nilpotent yoki stoxastik ekanligini tekshiring, ularning davri, nilpotentlik ko’rsatkichini toping:

a) $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; b) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$; c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$;

d) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{pmatrix}$; e) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{pmatrix}$; f) $\begin{pmatrix} -1 & 1 & -2 \\ 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$;

g) $\begin{pmatrix} -2 & 5 & 3 \\ -2 & 5 & 3 \\ 2 & -5 & -3 \end{pmatrix}$; h) $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$; i) $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$;

j) $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$; k) $\frac{1}{7}\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{pmatrix}$; l) $\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$.

35. Agar A va B matritsalar stoxastik bo'lsa, AB ham stoxastik matritsa ekanligini isbotlang.

36. A matritsa stoxastik bo'lsin. A^{-1} matritsa mavjudmi? Agar A^{-1} mavjud bo'lsa, stoxastik bo'ladimi?

37. Qaysi holda stoxastik matritsa ortogonal bo'ladi?

38. A – ixtiyoriy matritsa bo'lsa, quyidagilarni hisoblang:

a) $tr(A^T A)$;

b) $tr(A^H A)$;

c) agar $tr(A^H A) = 0$ bo'lsa, $A = 0$ ekanligini isbotlang.

39. Agar A – ikkinchi tartibli nilpotent matritsa bo'lsa, $trA = 0$ bo'lishini isbotlang.

40. $AB - BA = E$ tenglikni qanoatlantiruvchi A va B matritsalar mavjud emasligini isbotlang.

41. Berilgan matritsalarni kataklarga ajratib matritsalar ko'paytmasini hisoblang:

a) $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$;

c) $\begin{pmatrix} 2 & -3 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ 6 & 3 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$;

d) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & -2 \\ -2 & -1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$;

e) $\begin{pmatrix} 1 & 2 & 5 & 5 & 5 \\ 3 & 4 & 6 & 6 & 6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$.

4**2**. Matritsalarning kroneker ko'paytmasi hisoblansin:

a) $\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}\otimes\begin{pmatrix} 4 \\ -2 \end{pmatrix}$; b) $\begin{pmatrix} 4 \\ -2 \end{pmatrix}\otimes\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\otimes\begin{pmatrix} -1 \\ 1 \end{pmatrix}$;

d) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; e) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\otimes\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$; f) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

3-§. Jordan matritsalari
Jordan matritsasi haqida tushuncha

Kompleks sonlar maydonidagi n-tartibli kvadrat matritsalarni tekshiramiz:

$$A=\begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix},$$

bunda a_{ik} – kompleks sonlar.

A matritsani λ-matritsa deb qarash mumkin. Haqiqatan, $a_{ik}\neq 0$ elementar λ ga nisbatan nolinchi darajali ko`phadlarni, $a_{ik}\neq 0$ elementar esa nolь ko`phadlarni ifodalaydi.

Sonlardangina tuzilgan A matritsani, ya’ni λ-matritsaning xususiy holini o`zgarmas matritsa deb ataymiz.

O`zgarmas matritsalarning eng sodda shakli – diagonal matritsa, ya’ni

$$\begin{Vmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{Vmatrix}$$

ekanligi ma’lum, bunda a_1, a_2, …, a_n – kompleks sonlar.

Bu bobda asosan diagonal matritsalarga qaraganda murakkabroq bo`lsa ham, lekin ko`pgina hollarda ko`rinishi sodda bo`lgan maxsus bir xil shakldagi o`zgarmas matritsalar tekshiriladi.

Bu avval Jordan katagi deb yuritiladigan o`zgarmas matritsani ko`rib o`tamiz.

1-ta’rif. Ushbu

$$J_0=\begin{Vmatrix} \lambda_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_0 & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_0 \end{Vmatrix}$$

shaklidagi n-tartibli o`zgarmas matritsa $\lambda_0$ ga oid n-tartibli Jordan katagi deyiladi. Bu matritsada: bosh diagonal bo`yicha joylashgan elementlar biror λ_0 o`zgarmas songa, bosh diagonal ustidagi elementlar 1 ga va qolgan hamma elementlar nolga teng.

Masalan, ushbu matritsa

$$\begin{Vmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{Vmatrix}$$

2 ga oid 5-tartibli Jordan katagi. SHunga o`xshash

$$\begin{Vmatrix} -3 & 1 \\ 0 & -3 \end{Vmatrix}$$

matritsa -3 ga oid 2-tartibli Jordan katagi. Biror songa oid 1-tartibli Jordan katagi ham bo`lishi mumkin. Masalan, $\|4-7i\|$ matritsa $4\text{-}7_i$ ga oid shunday katakdir.

Endi Jordan matritsasi tushunchasiga o`tamiz.

2-ta’rif. Bosh diagonali bo`ylab $\lambda_1$, $\lambda_2$, ..., $\lambda_s$ sonlarga oid $m_1$, $m_2$, …, $m_s$ – tartibli $J_1$, $J_2$, …, $J_s$ kataklar joylashgan va qolgan hamma elementlari nollardan iborat bo`lgan n-tartibli o`zgarmas matritsa Jordanning nolmal shakldagi matritsasi yoki qisqacha Jordan matritsasi deb ataladi.

Jondan matritsasini yozganda, odatda, kataklar tashqarisidagi nollarga teng elementlarni tushirib qoldirib, matritsani

$$J = \begin{Vmatrix} \boxed{}\, J_1 & & & & \\ & \boxed{}\, J_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \boxed{}\, J_s \end{Vmatrix}$$

ko`rinishda yozadigan, bunda $J_1$ bilan $\lambda_1$ ga oid $m_1$-tartibli katak, $J_2$ bilan $\lambda_2$ ga oid $m_2$-tartibli katak, ..., $J_s$ ga oid $m_s$-tartibli katak belgilangan bo`lib, λ_1, λ_2, ..., λ_s sonlarning hammasi bosh diagonal bo`yicha joylashgan.

λ_1, λ_2, ..., λ_s sonlar, shuningdek, m_1, m_2, ..., m_s sonlar ham har xil yoki bir-biriga teng bo`lishi mumkin. Bunda $m_1+m_2+\ldots+m_s=n$.

Masalan, 6-tartibli

$$J = \left\| \begin{array}{cccccc} 2 & 1 & & & & \\ 0 & 2 & & & & \\ & & -1 & 1 & 0 & \\ & & 0 & 1- & 1 & \\ & & 0 & 0 & -1 & \\ & & & & & 8 \end{array} \right\|$$

Jordan matritsasi: 2 ga oida 2-tartibli, -1 ga oid 3-tartibli va 8 ga oid 1-tartibli kataklardan tuzilgan.

J_0 Jordan katagini λ_0 ga oid bitta 2-tartibli katakdangina tuzilgan Jordan matritsasi deb atash mumkin. Masalan,

$$\left\| \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right\|$$

matritsa 1 ga oid bitta 3-tartibli katakdan iborat Jordan matritsasi.

Jordan katagining xarakteristik matritsasi

J_0 Jordan katagining xarakteristik matritsasini hosil qilish uchun uning boshqa diagonalidagi λ_0 sondar λ ni ayirish kifoya ekanligi ma'lum. Demak, bu xarakteristik matritsa

$$J_0 - \lambda E = \left\| \begin{array}{cccc} \lambda_0 - \lambda & 1 & \ldots & 0 \\ 0 & \lambda_0 - \lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_0 - \lambda \end{array} \right\|$$

ko`rinishda bo`lib, u λ-matritsadan iborat.

Bu λ-matritsani normal diagonal shaklga keltiraylik. J_0-λE-matritsaning tuzilishiga qarab,

$$D_n(\lambda) = (\lambda - \lambda_0)^{n^1}$$

ekanini topamiz.

Endi, J_0-λE da 1-ustun va n-yo`lni o`chirsak, ushbu (n-1)-tartibli minor hosil bo`ladi:

$$\begin{vmatrix} 1 & 0 & \ldots & 0 & 0 \\ \lambda_0-\lambda & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & \lambda_0-\lambda & 1 \end{vmatrix} = 1$$

Bundan, barcha (n-1)-tartibli minorlarning eng katta umumiy bo`luvchisi 1 ga teng, ya'ni:

$$D_{n-1}(\lambda)=1$$

degan natija kelib chiqadi.

$D_{n-1}(\lambda)$ ko`phad $D_{n-2}(\lambda)$ ko`phadga, $D_{n-1}(\lambda)$ esa $D_{n-3}(\lambda)$ ko`phadga va hakazo bo`linishi lozimligidan:

$$D_{n-1}(\lambda)=D_{n-2}(\lambda)=\ldots=D_2(\lambda)=D_1(\lambda)=1.$$

Demak,

$$E_1(\lambda)=E_2(\lambda)=\ldots=E_{n-1}(\lambda)=1, \quad E_n(\lambda)=(\lambda-\lambda_0)^n$$

bo`lib, $J_0-\lambda E$ matritsa ushbu normal diagonal shaklga keltiriladi:

$$\begin{Vmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & (\lambda-\lambda_0)^n \end{Vmatrix} \qquad (1)$$

Bunga quyidagini ta'kidlab o`tish lozim: $J_0-\lambda E$ matritsaning yo`llari va ustunlariga nisbatan chekli son marta elementar almashtirishlarni qo`llanish orqali ham bu matritsani normal diagonal shaklga, ya'ni (1) shaklga keltirishimiz mumkin.

Jordan matritsasining xarakteristik matritsasi

Masalan, ushbu Jordan matritsasini olaylik:

$$J=\begin{Vmatrix} \boxed{\begin{matrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{matrix}} & & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & & \boxed{5} \end{Vmatrix}$$

Bu matritsa quyidagi bitta 3-tartibli, bitta 2-tartibli va bitta 1-tartibli kataklardan tuzilgan:

$$J_1 = \begin{Vmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{Vmatrix}, \quad J_2 = \begin{Vmatrix} 2 & 1 \\ 0 & 2 \end{Vmatrix}, \quad J_3 = \|5\|.$$

J matritsaning xarakteristik matritsasi

$$J = \begin{Vmatrix} \boxed{\begin{matrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{matrix}} & & \\ & \boxed{\begin{matrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{matrix}} & \\ & & \boxed{5-\lambda} \end{Vmatrix}$$

ko`rinishga ega ekanligi ravshan. Bundagi kataklar ushbu matritsalarni ifodalaydi:

$$J_1 - \lambda E_1 = \begin{Vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{Vmatrix} = \begin{Vmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{Vmatrix} - \lambda \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix};$$

$$J_2 - \lambda E_2 = \begin{Vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{Vmatrix} = \begin{Vmatrix} 2 & 1 \\ 0 & 2 \end{Vmatrix} - \lambda \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix};$$

$$J_3 - \lambda E_3 = \|5-\lambda\| = \|5\| - \lambda\|1\|.$$

Demak, E_1 birlik matritsaning tartibil J_1 ning tartibiga E_2 ning tartibi J_2 ning tartibiga va E_3 ning tartibi J_3 ning tartibiga teng.

SHu misolga asoslanib, Jordan matritsasining xarakteristik matritsasini umumiy ko`rinishda quyidagicha yozamiz:

$$J - \lambda E = \begin{Vmatrix} \boxed{J_1 - \lambda E_1} & & & \\ & \boxed{J_2 - \lambda E_2} & & \\ & & \ddots & \\ & & & \boxed{J_s - \lambda E_s} \end{Vmatrix} \qquad (2)$$

bunda E_i-birlik matritsaning tartibi J_i katakning tartibiga, ya’ni m_i ga teng.

J-λE matritsaning normal diagonal shaklini topish uchun, ushbu lemmani isbotlaymiz.

Lemma. $f_1(\lambda), f_2(\lambda), \dots f_n(\lambda)$ ko`phadlarning har ikkitasi o`zaro tub bo`lsa,

$$F(\lambda)=\begin{Vmatrix} f_1(\lambda) & 0 & \dots & 0 \\ 0 & f_2(\lambda) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f_n(\lambda) \end{Vmatrix} \quad ва \quad \Phi(\lambda)=\begin{Vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \varphi_n(\lambda) \end{Vmatrix}$$

matritsalar ekvivalent bo`ladi, bunda $\varphi_n(\lambda)=f_1(\lambda), f_2(\lambda)\dots f_3(\lambda)$.

Isbot. $f_1(\lambda)$ va $f_2(\lambda)$ o`zaro tub bo`lgani sababli shunday $q_1(\lambda)$ va $q_2(\lambda)$ ko`phadlar mavjud bo`lib,

$$f_1(\lambda)\, q_1(\lambda) + f_2(\lambda)\, q_2(\lambda) = 1 \qquad (3)$$

$F(\lambda)$ matritsaning 1-ustunini $q_1(\lambda)$ ga ko`paytirib 2-ustuneiga va 2-yo`lini $q_2(\lambda)$ ga ko`paytirib 1-yo`liga qo`shamiz. Bu vaqtda, (3) tenglikka asosan,

$$\begin{Vmatrix} f_1(\lambda) & 1 & \dots & 0 \\ 0 & f_2(\lambda) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f_n(\lambda) \end{Vmatrix} \qquad (4)$$

hosil bo`ladi. Bu (4) matritsaning 1- va 2-ustunlarini almashtiramiz va hosil bo`lgan yangi matritsada 1-ustunni $-f_2(\lambda)$ ga ko`paytirib 2-ustunga qo`shamiz va shuningdek 1-yo`lni $-f_2(\lambda)$ ga ko`paytirib 2-yo`lga qo`shamiz; undan keyin 2-ustunni -1 ga ko`paytiramiz va natijada ushbu matritsani hosil qilamiz:

$$\begin{Vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & \varphi_2(\lambda) & 0 & \dots & 0 \\ 0 & 0 & f_3(\lambda) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & f_n(\lambda) \end{Vmatrix}.$$

bunda $\varphi_2(\lambda)=f_1(\lambda)f_2(\lambda)$.

$\varphi_2(\lambda)$ va $f_3(\lambda)$ ham o`zaro tub bo`lganligi sababli yuqoridagi mulohazani 2- va 3-yo`llarga nisbatan takrorlab,

$$\left\| \begin{matrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & \varphi_3(\lambda) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & f_n(\lambda) \end{matrix} \right\|$$

matritsani hosil qilamiz, bunda $\varphi_2(\lambda)=\varphi_3(\lambda)f_3(\lambda)=f_1(\lambda)f_2(\lambda)f_3(\lambda)$.

YAna $\varphi_3(\lambda)$ va $f_4(\lambda)$ ning o`zaro tubligidan foydalanib, 3- va 4-ustunlar va yo`llar bilan ish ko`ramiz va hakazo. Bu prosessni oxirigacha davom ettirsak, xuddi $F(\lambda)$ matritsa hosil bo`ladi.

$F(\lambda)$ matritsadan $F(\lambda)$ matritsaga elementar almashtirishlar yordamida o`tganimiz uchun, bu ikkita λ-matritsa – ekvivalent matritsalardir.

Endi $J-\lambda E$ matritsani normal diagonal shaklga keltirish masalasiga o`tamiz.

Agar (2) matritsaning bitta $J_i-\lambda E_i$ katagidan o`tuvchi yo`llar va ustunlargina tegishli elementar almashtirishlarni bajarsak, bu katak (1) ga o`xgag ko`rinishni oladi va boshqa hamma kataklar o`zgarmay qoladi. Agar shu prosessni hamma kataklar uchun takrorlab chiqsak, $J-\lambda E$ matritsa ushbu

$$\left\| \begin{matrix} \boxed{\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (\lambda-\lambda_1)m_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (\lambda-\lambda_2)m_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (\lambda-\lambda_s)m_s \end{matrix}} \end{matrix} \right\| \quad (5)$$

diagonal matritsaga o`tadi, bunda yozilmagan elementlarning hammasi nolga teng.

$\lambda_1, \lambda_2, \dots, \lambda_s$ sonlar orasida bir – biriga tenglari mavjud bo`lishi mumkin.

(5) matritsada bosh diagonal elementlari bir – biri bilan shunday almashtirilgan bo`lib, buning natijasida quyidagilar bajarilgan deb faraz eta olamiz.

Birinchidan, o`zaro teng $\lambda_I$ lar ushbu qatorlar bo`yicha joylashgan:

$$\begin{array}{l} \lambda_1, \ \lambda_2, \ ..., \ \lambda_k, \\ \lambda_{k+1}, \ \lambda_{k+2}, \ ..., \ \lambda_i, \\ .... \\ \lambda_{r+1}, \ \lambda_{r+2}, \ ..., \ \ \lambda_s. \end{array} \qquad (6)$$

SHunday qilib, bitta qatordagi λ_I lar o`zaro teng bo`lib, turli qatorlardagi λ_i lar esa har xildir.

Ikkinchidan, k≥l-k≥…≥s-r, ya'ni har bir qatordagi λ_i larning soni keyingi qatordagi λ_i lar sonidan kichik emas.

Uchinchidan, (6) qatorlardagi λ_I lar shunday joylashganki, ularga oid kataklarning tartiblari ushbu tengsizliklarni qanoatlantiradi:

$$\begin{array}{l} m_1 \geq m_2 \geq ... \geq m_k, \\ m_{k+1} \geq m_{k+2} \geq ... \geq m_l, \\ .... \\ m_{r+1} \geq m_{r+2} \geq ... \geq m_s \end{array} \qquad (7)$$

(5) matritsaning bosh diagonalidagi 1 dan farqli hamma

$$(\lambda-\lambda_1)^m{}_1, (\lambda-\lambda_2)^m{}_2, \ldots,(\lambda-\lambda_k)^m{}_s$$

elementlarni quyidagi tartibda yozib chiqamiz:

$$\begin{array}{l} (\lambda-\lambda_k)^{m_1}, \ (\lambda-\lambda_k)^{m_2}, \ ..., \ (\lambda-\lambda_k)^{m_k}, \\ (\lambda-\lambda_l)^{m_{k+1}}, \ (\lambda-\lambda_l)^{m_{k+2}}, \ ..., \ (\lambda-\lambda_l)^{ml}. \\ .... \\ (\lambda-\lambda_s)^{m_{r+1}}, \ (\lambda-\lambda_s)^{m_{r+2}}, \ ..., \ (\lambda-\lambda_s)^{m_s} \end{array} \qquad (8)$$

chunki $\lambda_1=\lambda_2=\ldots=\lambda_k$, $\lambda_{k+1}=\lambda_{k+2}=\ldots=\lambda_l$ va hakazo.

(8) jadvalning 1-ustunidagi darajalarni o`zaro, 2-ustundagi darajalarni o`zaro va hakazo ko`paytirib chiqsak,

$$(\lambda-\lambda_k)^{m_1} \quad (\lambda-\lambda_l)^{m_{k+1}} \ ... \ (\lambda-\lambda_s)^{m_{r+1}}$$

ko`paytma

$$(\lambda-\lambda_k)^{m_2} \quad (\lambda-\lambda_l)^{m_{k+2}} \ ... \ (\lambda-\lambda_s)^{m_{r+2}}$$

ko`paytmaga bo`linadi [(7) ga asosan]. SHuningdek, ikkinchi ko`paytma uchinchi ustundagi darajalar ko`paytmasiga bo`linadi va hakazo.

Bu erda shuni aytib o`tish lozimki, agar, masalan, k>l-k bo`lsa, u holda (8) jadvalning (l-k) dan keyingi ustunlarida $\lambda-\lambda_l$ ning darajalari bo`lmaydi; biz o`sha joylarda $(\lambda-\lambda_l)^0=1$ turadi deb shartlashib olamiz.

(8) jadvalning 1-ustunidagi darajalar ko`paytmasini $E_n(\lambda)$ bilan, 2-ustundagi darajalar ko`paytmasini $E_{n-1}(\lambda)$ bilan, ..., k-ustunidagi darajalar ko`paytmasini $E_{n-k+1}(\lambda)$ bilan belgilaylik. YUqoridagi shartga muvofiq, masalan, k-ustunda $\lambda-\lambda_l$ ayirmaning darajasi bo`lmaydi, biz u joyda $(\lambda-\lambda_l)^0=1$ turadi deb hisoblaymiz.

YUqorida tushuntirganimizdek,

$$E_n(\lambda),\ E_{n-1}(\lambda),\ \ldots,\ E_{n-k+1}(\lambda) \qquad (9)$$

Ko`paytmalarning har qaysisi o`zidan keyingisiga bo`linadi.

Endi (5) matritsada bosh diagonal elementlarini o`zaro shunday almashtiraylikki, buning natijasida hosil bo`ladigan matritsaning bosh diagonalida avval hamma 1 lar joylashsin, undan keyin (8) jadvalning k-ustunidagi darajalar, so`ngra (k-1)-ustunidagi darajalar va hakazo, eng keyin 1-ustunidagi darajalar joylashsin.

Bunday matritsada (8) jadvalning k-ustunidagi darajalarini o`z ichiga olgan katak bilan ish ko`raylik.

λ_k, λ_l, …, λ_s sonlar har xil bo`lgani uchun, k-ustunidagi hamma darajalar o`zaro tub hadlardan iborat. SHu sababli mana shu aytilgan katakdan o`tuvchi satr va ustunlarga nisbatan yuqoridagi lemmani qo`llansak, bu katak

$$\begin{Vmatrix} 1 & 0\ldots & 0 \\ 0 & 1\ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & 0\ldots & E_{n-k+1}(\lambda) \end{Vmatrix}$$

ko`rinishni oladi.

Ana shu mulohazani ko`rilayotgan matritsada (8) jadvalning har bir ustunidagi darajalarnigina o`z ichiga olgan katak uchun takrorlab chiqamiz.

Bu almashtirishlardan keyin hosil qilingan matritsaning bosh diagonalidagi elementlarini o`zaro almashtirib, eng oxirida quyidagi matritsaga kelamiz:

$$\left\|\begin{array}{cccccccccc}
1 \\
& 1 \\
& & 1 \\
& & & \cdot \\
& & & & \cdot \\
& & & & & \cdot \\
& & & & & & 1 \\
& & & & & & & E_{n=k+1}(\lambda) \\
& & & & & & & & \cdot \\
& & & & & & & & & \cdot \\
& & & & & & & & & & \cdot \\
& & & & & & & & & & & E_{n-1}(\lambda) \\
& & & & & & & & & & & & E_n(\lambda)
\end{array}\right\|$$

Bu – xuddi (2) Jordan matritsasining normal diagonal shakli.

Misol.

Ushbu 15-tartibli xarakteristik matritsani normal diagonal shaklga keltiraylik:

$$\left\|\begin{array}{cccccc}
\boxed{\begin{array}{ccc} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{array}} \\
& \boxed{2-\lambda} \\
& & \boxed{\begin{array}{cc} 5-\lambda & 1 \\ 0 & 5-\lambda \end{array}} \\
& & & \boxed{\begin{array}{ccc} 5-\lambda & 1 & 0 \\ 0 & 5-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{array}} \\
& & & & \boxed{\begin{array}{cc} 2-\lambda & 1 \\ 0 & 2-\lambda \end{array}} \\
& & & & & \boxed{\begin{array}{cccc} -3-\lambda & 1 & 0 & 0 \\ 0 & -3-\lambda & 1 & 0 \\ 0 & 0 & -3-\lambda & 0 \\ 0 & 0 & 0 & -3-\lambda \end{array}}
\end{array}\right\|$$

Normal diagonal matritsani hosil qilish uchun (8) jadvalni tuzish kerak. Berilgan matritsada kataklarning eng ko`pchiligi 2 ga oiddir; bu kataklar 3-, 2- va 1-tartibli. Undan keyingi o`rnida 5 ga oid kataklar turadi, ular 3- va 2-tartibli. Nihoyat (-3) ga oid bitta 4-tartibli katak bor. Demak, bu matritsa uchun (8) jadval ushbu ko`rinishda bo`ladi:

$(\lambda-2)^3$, $(\lambda-2)^2$, $\lambda-2$,
$(\lambda-5)^3$, $(\lambda-5)^2$,

$(\lambda+3)^4$.

Ustunlar bo`yicha ko`paytirib, (9) ko`paytmalarni tuzamiz:

$$E_n(\lambda)=E_{15}(\lambda)=(\lambda-2)^3(\lambda-5)^3(\lambda+3)^4,$$
$$E_{n-1}(\lambda)=E_{14}(\lambda)=(\lambda-2)^2(\lambda-5)^2,$$
$$E_{n-2}(\lambda)=E_{13}(\lambda)=\lambda-2.$$

(Bunda n=15 va k=3 bo`lgani uchun oxirgi ko`paytma $E_{n-k+1}(\lambda)=E_{n-2}(\lambda)=E_{13}(\lambda)$ dir).

SHunday qilib, 15-tartibli normal diagonal matritsa quyidagidan iborat:

$$\begin{Vmatrix} 1 &&&&&&&&&&&&&& \\ & 1 &&&&&&&&&&&&& \\ && 1 &&&&&&&&&&&& \\ &&& 1 &&&&&&&&&&& \\ &&&& 1 &&&&&&&&&& \\ &&&&& 1 &&&&&&&&& \\ &&&&&& 1 &&&&&&&& \\ &&&&&&& 1 &&&&&&& \\ &&&&&&&& 1 &&&&&& \\ &&&&&&&&& 1 &&&&& \\ &&&&&&&&&& 1 &&&& \\ &&&&&&&&&&& 1 &&& \\ &&&&&&&&&&&& E_{13}(\lambda) && \\ &&&&&&&&&&&&& E_{14}(\lambda) & \\ &&&&&&&&&&&&&& E_{15}(\lambda) \end{Vmatrix}$$

Ixtiyoriy matritsani Jordan matritsasiga keltirish

Bu paragrafda kompleks sonlar maydonidagi n-tartibli ixtiyoriy A matritsa bilan o`xshash Jordan matritsasini topish masalasini ko`ramiz. Buning uchun avval λ-matritsalarga doir quyidagi lemmalarni isbotlaymiz.

1-lemma. N-darajali

$$F(\lambda)=A_0\lambda^n+A_1\lambda^{n-1}+\ldots+A_{n-1}\lambda+A_n$$

λ-matritsa va birinchi darajali

$$A-\lambda E$$

λ-matritsa uchun

$$F(\lambda)=(A-\lambda E)Q(\lambda)+R \qquad (10)$$

tenglikni qanoatlantiruvchi $Q(\lambda)$ va o`zgarmas R matritsalar mavjuddir.

Isbot. Ushbu

$$F(\lambda)+(A-\lambda E)A_0\lambda^{n-1}=F(\lambda)-A_0\lambda^n+AA_0\lambda^{n-1}$$

matritsaning darajasi (n-1) dan yuqori emas.

Agar

$$F(\lambda)+(A-\lambda E)A_0`\lambda^{n-1}+A_1`\lambda^{n-2}=$$
$$=F(\lambda)+(A-\lambda E)(A_0\lambda^{n-1}+A_0`\lambda^{n-2})$$

matritsaning darajasi (n-2) dan yuqori bo`lmaydi. Bu prosessni davom ettirib,

$$F(\lambda)+(A-\lambda E)(A_0\lambda^{n-1}+A_0`\lambda^{n-2}+\ldots+A_0^{(n-2)}\lambda+A_0^{(n-1)}=R$$

ko`rinishdagi R o`zgarmas matritsaga kelamiz. Buni

$$F(\lambda)=(A-\lambda E)(-A_0\lambda^{n-1}-A_0`\lambda^{n-2}-\ldots-A_0^{(n-2)}+R$$

ko`rinishda yozib, qavs ichidagi (n-1)-darajali matritsani $Q(\lambda)$ bilan belgilasak, xuddi (10) tenglik hosil bo`ladi.

SHunga o`xshash

$$F(\lambda)=Q_1(\lambda)(A-\lambda E)+R_1 \qquad (11)$$

tenglikni qanoatlantiruvchi $Q_1(\lambda)$ va o`zgarmas R_1 matritsalarning mavjudligi isbotlanadi.

2-lemma. Ikkita n-tartibli A va B o`zgarmas matritsalar bir – biriga o`xshash bo`lishi uchun, ularning $A-\lambda E$ va $B-\lambda E$ ‘arakteristik matritsalari ekvivalent bo`lishi zarur va etarli.

Isbot. 1.A va B matritsalar o`xshash bo`lsa,xosmas C matritsa mavjud bo`lib,

$$B=C^{-1}AC$$

tenglik bajariladi. Bunga asosan:

$$B-\lambda E=C^{-1}AC-\lambda E=C^{-1}(A-\lambda E)C.$$

Xosmas C matritsa teskarilanuvchi matritsani ifodalaganligi sababli $A-\lambda E$ va $B-\lambda E$ matritsalar uchun shart bajarilganini ko`ramiz. Demak, bu matritsalar ekvivalent ekan.

2.Endi $A-\lambda E$ va $B-\lambda E$ matritsalar ekvivalent desak, $P(\lambda)$ va $Q(\lambda)$ teskarilanuvchi matritsalar mavjud bo`lib,

$$B-\lambda E=P(\lambda)(A-\lambda E)Q(\lambda) \qquad (12)$$

tenglik bajariladi.

(10) va (11) tengliklarga asosan:

$$\begin{aligned} &P(\lambda)=(B-\lambda E)P_1(\lambda)+P_0,\\ &Q(\lambda)=Q_1(\lambda)(B-\lambda E)+Q_0, \end{aligned} \qquad (13)$$

bunda P_0 va Q_0 – o`zgarmas matritsalar.

(13) ifodalarni (12) tenglikka qo`yamiz:

$$\begin{aligned} B-\lambda E&=[(B-\lambda E)P_1(\lambda)+P_0]\,(A-\lambda E)\,[Q_1(\lambda)(B-\lambda E]+Q_0]=\\ &=(B-\lambda E)P_1\lambda)(A-\lambda E)Q_1(\lambda)(B-\lambda E)+\\ &+(B-\lambda E)P_1(\lambda)(A-\lambda E)Q_0+\\ &+P_0(A-\lambda E)\,Q_1(\lambda)(B-\lambda E)+P_0(A-\lambda E)Q_0. \end{aligned} \qquad (14)$$

Bu tenglikda $P_0(A-\lambda E)Q_0$ ifodani chap tomonga o`tkazib, o`ng tomonda qolgan yiғindini $S(\lambda)$ bilan belgilaymiz. Bu vaqtda (14) tenglik

$$B-\lambda E-P_0(A-\lambda E)Q_0=S(\lambda) \qquad (15)$$

ko`rinishni oladi, bunda

$$\begin{aligned} S(\lambda)&=(B-\lambda E)P_1(\lambda)(A-\lambda E)Q_1(\lambda)(B-\lambda E)+\\ &+(B-\lambda E)P_1(\lambda)(A-\lambda E)Q_0+\\ &+P_0(A-\lambda E)Q_1(\lambda)(B-\lambda E) \end{aligned} \qquad (16)$$

(13) tengliklarning ikkinchisini nazarda tutib, (16) yiғindining avvalgi ikkita qo`shiluvchisini quyidagicha yozish mumkin.

$$\begin{aligned} &(B-\lambda E)P_1(\lambda)(A-\lambda E)Q_1(\lambda)(B-\lambda E)+\\ &+(B-\lambda E)P_1(\lambda)(A-\lambda E)\cdot Q_0=\\ &=(B-\lambda E)P_1(\lambda)(A-\lambda E)[Q_1(\lambda)(B-\lambda E)+Q_0]=\\ &=(B-\lambda E)P_1(\lambda)(B-\lambda E)Q(\lambda). \end{aligned}$$

Endi (16) yiғindining uchinchi qo`shiluvchisiga

$$(B-\lambda E)P_1(\lambda)(A-\lambda E)Q_1(\lambda)(A-\lambda E)$$

ifodani qo`shamiz va ayiramiz [(13) tengliklarning birinchisidan foydalanib]. Buning natijasida $S(\lambda)$ yiғindining ko`rinishi bunday bo`ladi:

$$S(\lambda)=(B-\lambda E)P_1(\lambda)(A-\lambda E)\,Q(\lambda)+$$
$$+P(\lambda)(A-\lambda E)\,Q_1(\lambda)(B-\lambda E)-$$
$$-(B-\lambda E)P_1(\lambda)(A-\lambda E)Q_1(\lambda)(B-\lambda E) \qquad (17)$$

(12) tenglikdan hosil qilingan

$$(A-\lambda E)Q(\lambda)=P^{-1}(\lambda)(B-\lambda E),$$
$$P(\lambda)\,(A-\lambda E)=(B-\lambda E)Q^{-1}(\lambda)$$

ifodalarni (17) ga qo`yib, ushbuga ega bo`lamiz:

$$S(\lambda)=(B-\lambda E)[P_1(\lambda)P^{-1}(\lambda)+Q^{-1}(\lambda)Q_1(\lambda)+$$
$$P_1(\lambda)(A-\lambda E)Q_1(\lambda)]\,(B-\lambda E).$$

Kvadrat qavs ichidagi yiғindi λ ga nisbatan $T(\lambda)$ ko`phad. Bu ko`phadning darajasini m deylik. U holda $S(\lambda)=(B-\lambda E)T(\lambda)\,(B-\lambda E)$ ko`phadning darajasi m+2 bo`ladi. Demak, $m\geq 0$ ekanini nazarda tutsak, $S(\lambda)$ ning darajasi 2 dan kam bo`lmasligi kerak. Ammo, (15) tenglikka asosan, $S(\lambda)$ ning darajasi 1 dan yuqori bo`la olmaydi. Bu ziddiyat faqat $T(\lambda)=0$ shartdagina yo`qotilishi mumkin. Bu vaqtda $S(\lambda)=0$ bo`lib, (15) tenglikdan

$$B-\lambda E=P_0(A-\lambda E)Q_0$$

yoki

$$B-\lambda E=P_0AQ_0-\lambda P_0Q_0$$

tenglik hosil bo`ladi. Demak, matritsali ko`phadlarning teng bo`lishi shartiga asosan:

$$P_0Q_0=E \qquad (18)$$

va

$$B=P_0AQ_0 \qquad (19)$$

(18) tenglikdan P_0 va Q_0 matritsalarning xosmas matritsalar bo`lishi ko`rinadi va $P_0=Q^{-1}_0$ ekani hosil bo`ladi. SHu sababli, (19) tenglik

$$B=Q^{-1}_0AQ_0.$$

ko`rinishni oladi. Bu esa A va B matritsalarning o`xshashligini ifodalaydi.

Endi bu paragrafning asosiy masalasiga kelamiz.

Dastlab quyidagi eslatmaga e’tibor beraylik: (8) jadvaldagi har bir $(\lambda-\lambda_t)^{mi}$ daraja J-λE matritsaning λ_t soniga oid m_i – tartibli katagini xarakterlaydi. SHunday qilib, (8) jadvaldagi hamma darajalar J-λE matritsani xarakterlaydi. (8) jadval bilan aniqlangan kataklarni J-λE matritsada joylashtirish tartibi yoxtiyoriydir, chunki bu kataklarni turlicha tartibda joylashtirish bilan hosil qilingan hamma J-λE ko`rinishdagi matritsalar bir – biriga ekvivalent.

(8) jadvalga asoslanib, J-λE matritsani tuzgandan keyin J matritsani topish engil.

Masalan,

$$\begin{gathered}(\lambda-8)^2,\ \lambda-8\\ (\lambda+1),\\ (\lambda-11)\end{gathered}$$

jadval bo`yicha avval

$$\begin{Vmatrix} 8-\lambda & 1 & & & \\ 0 & 8-\lambda & & & \\ & & 8-\lambda & & \\ & & & -1-\lambda & \\ & & & & 11-\lambda \end{Vmatrix}$$

matritsani tuzib, so`ngra ushbu Jordan matritsasini hosil qilamiz:

$$\begin{Vmatrix} 8 & 1 & & & \\ 0 & 8 & & & \\ & & 8 & & \\ & & & -1 & \\ & & & & 11 \end{Vmatrix}$$

1-teorema. Kompleks sonlar mayjonidagi har bir n-tartibli o`zgarmas A matritsaga o`xshash bo`lgan J Jordan matritsasi mavjud.

Isbot. A matritsaning A-λE xarakteristik matritsasini olib, ma’lum qoida bilan uning $D_n(\lambda)$, D_{n-1}, …, $D_1(\lambda)$ ko`phadlarini tuzamiz va ularga asoslanib,

$$E_1(\lambda),\ \ E_2(\lambda),\ \ ...,\ \ E_n(\lambda) \qquad (20)$$

ko`paytuvchilarni hosil qilamiz.

U holda A-λE matritsaning normal diagonal shakli quyidagicha bo`lishi ma’lum:

$$\begin{Vmatrix} E_1(\lambda) & 0 & \ldots & 0 \\ 0 & E_2(\lambda) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & E_n(\lambda) \end{Vmatrix} \qquad (21)$$

(20) ko`phadlarning 1 dan farqlilari

$$E_{n-k+1}(\lambda),\ E_{n-k+2}(\lambda),\ \ldots,\ E_n(\lambda) \qquad (22)$$

bo`lsin. Agar (22) ko`phadlarni $(\lambda-\lambda_i)$ larning darajalari bo`yicha ko`paytuvchilarga yoysak:

$$E_n(\lambda)=(\lambda-\lambda_1)^{\alpha_1}(\lambda-\lambda_2)^{\alpha_2}\ldots(\lambda-\lambda_l)^{\alpha_l}$$
$$E_{n-1}(\lambda)=(\lambda-\lambda_1)^{\beta_1}(\lambda-\lambda_2)^{\beta_2}\ldots(\lambda-\lambda_l)^{\beta_l}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$E_{n-k+1}(\lambda)=(\lambda-\lambda_1)^{\gamma_1}(\lambda-\lambda_2)^{\gamma_2}\ldots(\lambda-\lambda_l)^{\gamma_l} \qquad (23)$$

bo`ladi, bunda $\lambda_1$, $\lambda_2$, ..., $\lambda_l$ ildizlar har xil va (22) ko`phadlarning har biri o`zidan avvalgisiga bo`lingani uchun:

$$\alpha_i \geq \beta_i \geq \ldots \geq \gamma_i \geq 0.$$

(23) yoyilmalardagi darajalarni ustunlar bo`yicha yozib chiqsak, quyidagi jadval hosil bo`ladi:

$$(\lambda-\lambda_1)^{\alpha_1},\ (\lambda-\lambda_1)^{\beta_1}, \ldots, (\lambda-\lambda_1)^{\gamma_1},$$
$$(\lambda-\lambda_2)^{\alpha_2},\ (\lambda-\lambda_2)^{\beta_2}, \ldots, (\lambda-\lambda_2)^{\gamma_2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(\lambda-\lambda_l)^{\alpha_l},\ (\lambda-\lambda_l)^{\beta_l}, \ldots, (\lambda-\lambda_l)^{\gamma_l} \qquad (24)$$

Bu jadvalga asosan, $J-\lambda E$ matritsani tuzamiz. Uning normal diagonal formasi xuddi (21) matritsadan iborat.

SHunday qilib, $A-\lambda E$ va $J-\lambda E$ matritsaning har qaysisi (21) matritsa bilan ekvivalent bo`lgani uchun, ular o`zaro ham ekvivalent bo`ladi. Demak, 2-lemmaga asosan, A bilan J o`xshash matritsalardir.

Berilgan A matritsaga o`xshash J Jordan matritsasini topish – bu A matritsani Jordan matritsasiga keltirish deyiladi.

λ_1, λ_2, ..., λ_l sonlar $|A-\lambda E|$ xarakteristik ko`phadning ildizlarini ifodalaydi. Umuman, bu ko`phadning ildizlari karrali.

Isbot qilinganlardan quyidagi xulosa kelib chiqadi: har bir A o`zgarmas matritsa kataklarning joylashish tartibi aniqligida Jordan matritsasiga birdan – bir usul bilan keltiriladi, ya'ni A ning turlicha Jordan matritsalari bir – biridan faqat kataklarning joylanish tartibi bilangina farq qiladi.

2-teorema. A matritsaning J Jordan matritsasi diagonal matritsadan iborat bo`lishi uchun (24) jadvaldagi ayirmalarning hammasi 1-darajali bo`lish zarur va etarli.

Isbot. I. (24) jadvaldagi hamma ayirmalar birinchi darajali bo`lsin:

$$\begin{gathered}\lambda-\lambda_1,\ \lambda-\lambda_1,\ \ldots,\\ \lambda-\lambda_2,\ \lambda-\lambda_2,\ \ldots,\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \lambda-\lambda_i,\ \lambda-\lambda_i,\ \ldots\end{gathered}$$

U holda

$$J-\lambda E=\begin{Vmatrix}\boxed{\lambda_1-\lambda} & & & & \\ & \boxed{\lambda_2-\lambda} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \boxed{\lambda_i-\lambda}\end{Vmatrix} \qquad (25)$$

va, demak,

$$J=\begin{Vmatrix}\lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \lambda_l\end{Vmatrix}$$

II.Aksincha, J matritsaning shakli (26) dan iborat bo`lsa, J-$\lambda$E matritsaning shakli (25) dan iborat bo`ladi. Bu esa (24) jadvaldagi ayirmalarning 1-darajali ekanini ko`rsatadi.

Misollar.

Quyidagi matritsalarni Jordan matritsalariga keeltiraylik.

Quyidagi matritsalarni Jordan matritsalariga keltiraylik.

1.

$$\begin{Vmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ 1 & 0 & 2\end{Vmatrix}.$$

Xarakteristik ko`phad tuzib, uni birinchi darajali ko`paytuvchilarga ajratamiz:

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)+1+(1-\lambda)-(2-\lambda)=$$

$$=(1-\lambda)^2(2-\lambda).$$

Demak, $D_3(\lambda)=(\lambda-1)^2(2-\lambda)$.

Ushbu

$$\begin{vmatrix} 1 & -1 \\ 1-\lambda & 1 \end{vmatrix} = 2-\lambda, \quad \begin{vmatrix} 1 & 1-\lambda \\ 1 & 0 \end{vmatrix} = \lambda-1.$$

2-tartibli minorlar o`zaro tub bo`lgani sababli $D_2(\lambda)=1$. U holda $D_1(\lambda)=1$. Endi $E_3(\lambda)=\dfrac{D_3(\lambda)}{D_2(\lambda)}=(\lambda-1)^2(\lambda-2)$; $E_2(\lambda)=\dfrac{D_2(\lambda)}{D_1(\lambda)}=1$; $E_1(\lambda)=D_1(\lambda)=1$. (24) jadval ushbu ko`rinishga ega bo`ladi:

$(\lambda-1)^2$,

$(\lambda-2)$.

SHunday qilib, I-λE matritsa 1 ga oid 2-tartibli va 2 ga oid 1-tartibli kataklardan tuziladi. SHuning uchun J matritsaning shakli bunday bo`ladi:

$$\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{Vmatrix}$$

2. $\begin{Vmatrix} 2 & -2 & 1 \\ 3 & -5 & 3 \\ 6 & -12 & 7 \end{Vmatrix}$

Bunga asosan:

$$\begin{Vmatrix} 2-\lambda & -2 & 1 \\ 3 & -5-\lambda & 3 \\ 6 & -12 & 7-\lambda \end{Vmatrix} = -(\lambda-1)^2(\lambda-2).$$

Demak, $D_3(\lambda)=(\lambda-2)^2(\lambda-2)$.

2-tartibli minorlar: $(\lambda-1)(\lambda+4)$; $12(\lambda-1)$; $6(\lambda-1)$; $3(\lambda-1)$; $(\lambda-1)(\lambda-8)$; $-3(\lambda-1)$; $(\lambda-1)$; $2(\lambda-1)$; $(\lambda-1)^2$ bo`lib, bundan $D_2(\lambda)=\lambda-1$.

Birinchi tartibli minorlar uchun: $D_1=(\lambda)=1$. SHunday qilib, $E_3(\lambda)=\dfrac{D_3(\lambda)}{D_2(\lambda)}=(\lambda-1)(\lambda-2)$;

$$E_3(\lambda)=\frac{D_2(\lambda)}{D_1(\lambda)}=(\lambda-1); \qquad E_1(\lambda)=D_1(\lambda)=1.$$

(24) jadvalning ko`rinishi:

$(\lambda\text{-}1), (\lambda\text{-}1)$

$(\lambda\text{-}2).$

SHu sababli

$$J = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{Vmatrix}.$$

3-teorema. R unitar fazoda har bir A chiziqli almashtirishning istalgan

$$\bar{e}_1,\ \bar{e}_2, ...,\ \bar{e}_n \qquad (27)$$

bazisdagi A matritsasi uchun shunday

$$\bar{f}_1,\ \bar{f}_2, ...,\ \bar{f}_n \qquad (28)$$

bazis mavjudki, bu bazisda A almashtirishning matritsasi Jordan matritsasidan iborat.

Isbot. 1-teoremaga asosan A almashtirishning (27) bazisdagi A matritsasi bilan o`xshash J matritsa mavjud. Demak xosmas C matritsa uchun

$$J=C^{-1}AC$$

tenglik bajariladi, bunda

$$C = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{Vmatrix}$$

deylik. Agar

$$\begin{gathered} \bar{f}_1 = c_{11}\bar{e}_1 + c_{21}\bar{e}_2 + ... + c_{n1}\bar{e}_n, \\ \bar{f}_2 = c_{12}\bar{e}_1 + c_{22}\bar{e}_2 + ... + c_{n2}\bar{e}_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{f}_n = c_{1n}\bar{e}_1 + c_{2n}\bar{e}_2 + ... + c_{nn}\bar{e}_n \end{gathered}$$

bazisni olsak, u holda J matritsa A almashtirishning shu (28) bazisdagi matritsasi bo`ladi.

Mashqlar

Ushbu matrisalar Jordan matritsalariga keltirilsin:

1. $\begin{Vmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 7 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & -1 & -4 & -4 \end{Vmatrix};$

2. $\begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -1 & 0 \end{Vmatrix}$;

3. $\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 19 \\ 0 & 1 & 1 & 1 \end{Vmatrix}$;

4. $\begin{Vmatrix} 1 & 2 & 5 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$;

5. $\begin{Vmatrix} 2 & 1 & 1 \\ 1 & 6 & 5 \\ -1 & -3 & -2 \end{Vmatrix}$;

6. $\begin{Vmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$;

7. $\begin{Vmatrix} 2 & 1 & 1 \\ -1 & 4 & 2 \\ 1 & -2 & 0 \end{Vmatrix}$;

8. $\begin{Vmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{Vmatrix}$.

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