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Ground states for the ising model with an external field on the Cayley tree

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Abstract. We consider the Ising model with non-zero external field on the Cayley tree of order k with $k \geq 2$. Described translation-invariant ground states for the Ising model with translation-invariant external field. Some periodic ground states for the Ising model with periodic external field are described.

Keywords: Cayley tree, Ising model, external field, translation-invariant external field, periodic external field, configuration, translation-invariant ground state, periodic ground state.

MSC (2010) Classification: 82B26; 60K35

1 Introduction

One of fundamental problems is to describe the extreme Gibbs measures corresponding to a given Hamiltonian. Each Gibbs measure is associated with a single phase of a physical system. Existence of two or more Gibbs measures means that phase transitions exist.

As is known, the phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure. Therefore the problem of description of periodic ground states naturally arises (see [1], [3],[5]-[8]).

In [5] and [6], [8] for the Ising model with competing interactions, periodic and weakly periodic ground states were described.

In [7] for the Potts model with competing interactions on the Cayley tree of order k with $k \geq 2$, periodic and weakly periodic ground states for normal subgroups of index 4 were studied.

In [3] for the $\lambda-$model on the Cayley tree of order two, periodic and weakly periodic ground states were studied.

In this paper we shall study translation-invariant and periodic ground states for the Ising model with external fields.

2 Main definitions and known facts

Let $\tau^k = (V, L)$ be a Cayley tree of order k, i.e, an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of τ^k.

Let G_k denote the free product of $k+1$ cyclic groups $\{e, a_i\}$ of order 2 wit generators $a_1, a_2, \ldots, a_{k+1}$, i.e., let $a_i^2 = e$ (see [4]).

There exists a one-to-one correspondence between the set V of vertices of th Cayley tree of order k and the group G_k, see [1], [2].

We show how to construct this correspondence. We choose an arbitrary verte $x_0 \in V$and associate it with the identity element e of the group G_k. Since we ma assume that the graph under consideration is planar, we associate each neighbo of x_0 (i.e., e) with a single generator $a_i, i = 1, 2, \ldots, k+1$, where the orde corresponds to the positive direction, see Figure 1.

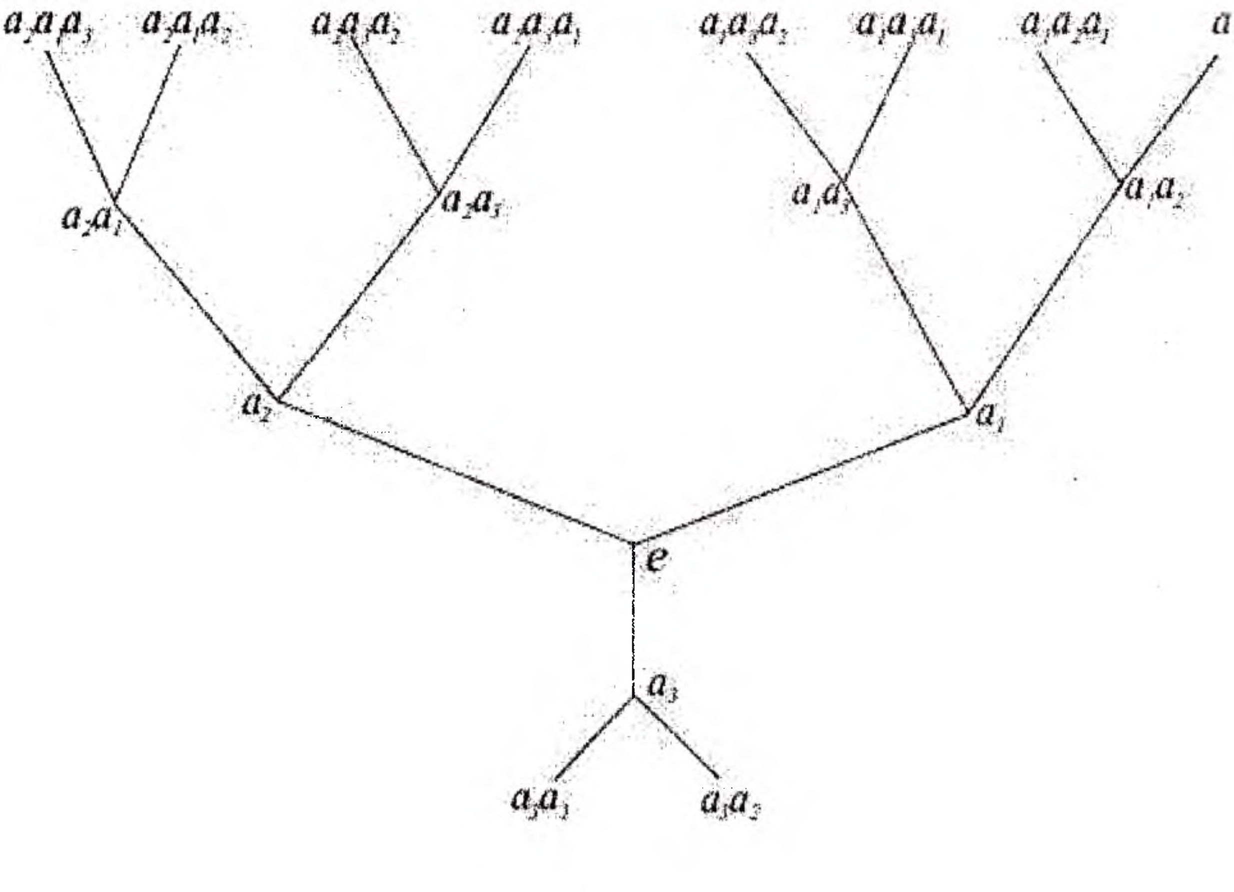

Рис. 1: The Cayley tree τ^2 and elements of the group representation of vertices

For every neighbor of a_i, we introduce words of the form $a_i a_j$. Since one of the neighbors of a_i is e, we put $a_i a_i = e$. The remaining neighbors of a_i are labeled according to the above order. For every neighbor of $a_i a_j$, we introduce words of length 3 in a similar way. Since one of the neighbors of $a_i a_j$ is a_i, we put $a_i a_j a_j = a_i$. The remaining neighbors of $a_i a_j$ are labeled by words of the form $a_i a_j a_l$, where $i, j, l = 1, 2, ..., k+1$, according to the above procedure. This agrees with the previous stage because $a_i a_j a_j = a_i a_j^2 = a_i$. Continuing this process, we obtain a one-to-one correspondence between the vertex set of the Cayley tree τ^k and the group G_k.

The representation constructed above is said to be *right* because, for all adjacent vertices x and y and the corresponding elements $g, h \in G_k$, we have either $g = ha_i$ or $h = ga_j$ for suitable i and j. The definition of the *left* representation is similar.

For the group G_k (or the corresponding Cayley tree), we consider the left (right) shifts. For $g \in G_k$, we put

$$T_g(h) = gh\ (T_g(h) = hg)\ for\ all\ h \in G^k$$

The group of all left (right) shifts on G_k is isomorphic to the group G_k.

Each transformation S on the group G_k induces a transformation S on the vertex set V of the Cayley tree τ^k. In the sequel, we identify V with G_k.

The following assertion is quite obvious (see [1], [2]).

Theorem 2.1. *The group of left (right) shifts on the right (left) representation of the Cayley tree is the group of translations.*

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x) = \{y \in G_k : \langle x, y\rangle\}$, where $\langle x, y\rangle$ means that the vertex x and y are nearest neighbor.

Assume that spin takes its values in the set $\Phi = \{-1, 1\}$. By a configuration σ on V we mean a function $\sigma : x \in V \to \sigma(x) \in \Phi$. The set of all configurations coincides with the set $\Omega = \Phi^V$.

Consider the quotient group $G_k \setminus G_k^* = \{H_1, ..., H_r\}$, where G_k^* is a normal subgroup of index r with $r \geq 1$.

Definition 2.2. A configuration $\sigma(x)$ is said to be G_k^*−periodic if $\sigma(x) = \sigma_i$ for all $x \in G_k$ with $x \in H_i$. A G_k−periodic configuration is said to be translation invariant.

By *period* of a periodic configuration we mean the index of the corresponding normal subgroup.

Ising model with an external field is given by Hamiltonian:

$$H(\sigma) = J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) + \sum_{x \in V} \alpha_x \sigma(x), \tag{2.1}$$

where $J, \alpha_x \in \mathbb{R}$, α_x is an external field and $\sigma \in \Omega$.

3 Model with a external field

Let M be the set of all unit balls with vertices in V, i.e.

$$M = \{\{x\} \cup S_1(x) : \forall x \in V\}.$$

By the *restricted configuration* σ_b we mean the restriction of a configuration σ to a ball $b \in M$. Let c_b denote the center of a unit ball b. The energy of a configuration σ_b on b is defined by the formula

$$U(\sigma_b) = \frac{1}{2}J \sum_{\substack{\langle x,y\rangle,\\ x,y\in b}} \sigma(x)\sigma(y) + \alpha_{c_b}\sigma(c_b). \tag{3.1}$$

Note that $U(\sigma_b)$ has finitely many values for arbitrary configuration σ. Denote by $U_{\min}$ the minimal value of the function $U(\sigma_b)$.

Definition 3.1. A configuration φ is called a ground state for the Hamiltonian (2.1), if

$$U(\varphi_b) = U_{\min}$$

for any $\psi \in \Omega$ and $b \in M$.

Ising model with a translation-invariant external field, i.e. $\alpha_x = \alpha, \forall x \in V$, is defined by the following Hamiltonian:

$$H(\sigma) = J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) + \alpha \sum_{x\in V} \sigma(x), \tag{3.2}$$

where $J, \alpha \in \mathbb{R}$, and $\sigma \in \Omega$.

The energy of a configuration σ_b on b is defined by the formula

$$U(\sigma_b) = \frac{1}{2}J \sum_{\substack{\langle x,y\rangle,\\ x,y\in b}} \sigma(x)\sigma(y) + \alpha\sigma(c_b). \tag{3.3}$$

It is not difficult to prove the following lemma.

Lemma 3.2. *For each configuration σ_b, we have the following*

$$U(\sigma_b) \in \{U_{-,(k+1)}, ..., U_{-,0}, U_{+,0}, ..., U_{+,(k+1)}\},$$

where $U_{\pm,i} = \pm(\alpha + (i - \frac{k+1}{2})J), i = \overline{0, k+1}$.

Definition 3.3. A configuration φ is called a ground state for the Hamiltonian (3.1), if

$$U(\varphi_b) = \min\{U_{\pm,i} : i = 0, 1, ..., k+1\}$$

for any $b \in M$.

For every $m = 0, 1, ..., (k+1)$, we put

$$A_{\pm,m} = \{(J, \alpha) : U_{\pm,m}(\sigma_b) \leq U_{\pm,i}, i = 0, 1, ..., k+1\}.$$

Quite cumbersome but not difficult calculations show that

$$A_{\pm,(k+1)} = \{(J, \alpha) : \pm J \leq 0, \pm\alpha \leq 0\},$$

$$A_{\pm,j} = \{(J,\alpha) : J = 0, \pm\alpha \le 0\}, \text{where } j = \overline{1,k},$$
$$A_{\pm,0} = \{(J,\alpha) : \pm J \ge 0, \pm\alpha \le 0\}.$$

The following theorem is described the necessary condition to the configurations to be a ground states for the Ising model with random non-zero external fields.

Theorem 3.4. *For the Ising model with non-zero external field, a translation-invariant configuration is a ground state, if only if the external field is translation-invariant.*

Proof. Necessity. First we shall prove that if translation-invariant configuration is a ground state, then the external field is translation-invariant. Assume, $\alpha_x \in \{\alpha_0, \alpha_1, ..., \alpha_n, ...\}, \forall x \in V$.

Let $\sigma(x) = 1, \forall x \in V$ be a ground state. Then the energy of the unit balls $b \in M$ may be one of the following:

$$\frac{(k+1)J}{2} + \alpha_0, \frac{(k+1)J}{2} + \alpha_1, \frac{(k+1)J}{2} + \alpha_2, ..., \frac{(k+1)J}{2} + \alpha_n, ...$$

Since $\sigma(x) = 1, \forall x \in V$ is a ground state the energy $\frac{(k+1)J}{2} + \alpha_0$ must be minimal. From minimality of this energy for variable of external field we take the following set $\{(\alpha_0, \alpha_1, ..., \alpha_n, ...) : \alpha_0 \le 0, \alpha_0 \le \alpha_1, ..., \alpha_0 \le \alpha_n, ...\}$. From minimality of $\frac{(k+1)J}{2} + \alpha_1$ we take the set $\{(\alpha_0, \alpha_1, ..., \alpha_n, ...) : \alpha_1 \le \alpha_0, \alpha_1 \le 0, ..., \alpha_1 \le \alpha_n, ...\}$ etc. Consequently we take the following:

$$\{(\alpha_0, \alpha_1, ..., \alpha_n, ...) : \alpha_0 \le 0, \alpha_0 \le \alpha_1, ..., \alpha_0 \le \alpha_n..\} \cap$$
$$\cap \{(\alpha_0, \alpha_1, ..., \alpha_n, ...) : \alpha_1 \le \alpha_0, \alpha_1 \le 0, ..., \alpha_1 \le \alpha_n, ...\} \cap ...$$
$$\cap \{(\alpha_0, \alpha_1, ..., \alpha_n, ...) : \alpha_n \le \alpha_0, \alpha_n \le \alpha_1, ..., \alpha_n \le 0...\} \equiv$$
$$\equiv \{(\alpha_0, \alpha_1, ..., \alpha_n ...) : \alpha_0 = \alpha_1 = ... = \alpha_n = ...\},$$

i.e. external field must be a translation-invariant.

If of the translation invariant configuration $\sigma(x) = -1, \forall x \in V$, the energy of the unit balls $b \in M$ may be one of the following form:

$$\frac{(k+1)J}{2} - \alpha_0, \frac{(k+1)J}{2} - \alpha_1, \frac{(k+1)J}{2} - \alpha_2, ..., \frac{(k+1)J}{2} - \alpha_n, ...$$

By similar way it easy to prove that in this case also the external field must be translation-invariant.

Sufficiency. Now we shall prove that, if external field is translation-invariant (i.e. for the model (3.2)), then any ground state is translation-invariant.

Let σ be a ground state. For any $b \in M$ we consider the following sets:

$$\{\sigma_b : \sigma_b(c_b) = 1\}; \quad \{\sigma_b : \sigma_b(c_b) = -1\}.$$

May be the following cases:

1) Both sets are nonempty, then σ configuration is no translation-invariant and it is ground state on the set $A_{+i} \cap A_{-j} = \{(J, \alpha) : \alpha = 0\}, i, j \in \{0, 1, ..., k+1\}$;

2) one of the sets is empty, then σ is translation-invariant and it is ground state on the set $\{(J, \alpha) : \alpha \neq 0\}$.

This finishes the proof of Theorem 3.4 □

We let $GS(H)$ denote the set of all ground states of the Hamiltonian H (see (3.2)).

Theorem 3.5. *For the Ising model with non-zero translation-invariant external field*

a) If $(J, \alpha) \in A_{+(k+1)}$ *then* $GS(H) = \{\sigma(x) = 1, \forall x \in V\}$,

b) If $(J, \alpha) \in A_{-0}$ *then* $GS(H) = \{\sigma(x) = -1, \forall x \in V\}$.

Proof.a) Consider the configuration $\sigma(x) = 1, \forall x \in V$. For any $b \in M$ by (3.3) we have $U(\sigma_b) = U_{+(k+1)}$. Thus the configuration $\sigma(x) = 1, \forall x \in V$ is ground state on the set $A_{+(k+1)}$.

b) Consider the configuration $\sigma(x) = -1, \forall x \in V$. For any $b \in M$ by (3.3) we have $U(\sigma_b) = U_{-0}$. Thus the configuration $\sigma(x) = -1, \forall x \in V$ is ground state on the set A_{-0}.

This finishes the proof of Theorem 3.5 □

Remark 3.6. Note that in [5] periodic ground states for the Ising model with two step interactions on Cayley tree and with zero external fields are described. In [6] weakly periodic ground states for the Ising model with competing interactions and with zero external fields are described.

So, obviously seen from Theorem ??, when an external field is non-zero translation-invariant, all ground states for the Ising model are translation-invariant.

Let $G_k^{(2)} = \{x \in G_k : |x| \text{ is even}\}$, where $|x|$ means length of the word x. Now we shall study $G_k^{(2)}$–periodic ground states for the Ising model with $G_k^{(2)}$–periodic external field .

4 Model with a periodic external field

Ising model with $G_k^{(2)}$–periodic external field is defined according to the following Hamiltonian:

$$H(\sigma) = J \sum_{\langle x,y \rangle \in L} \sigma(x)\sigma(y) + \sum_{x \in V} \alpha_x \sigma(x), \tag{4.1}$$

where $J, \alpha_x \in \mathbb{R}$ and

$$\alpha_x = \begin{cases} \alpha_0, \text{ if } x \in G_k^{(2)}, \\ \alpha_1, \text{ if } x \in G_k \setminus G_k^{(2)}, \end{cases}$$

where $\alpha_0 \neq \alpha_1$.

The energy of a configuration σ_b on b is defined by the formula

$$U(\sigma_b) = \frac{1}{2}J \sum_{x:\langle x,c_b\rangle \in L} \sigma(x)\sigma(c_b) + \alpha_{(c_b)}\sigma(c_b). \tag{4.2}$$

It is not difficult to prove the following

Lemma 4.1. *We have*

$$U(\sigma_b) \in \{U^{(0)}_{+,0}, U^{(0)}_{-,0}, U^{(1)}_{+,0}, U^{(1)}_{-,0}, ..., U^{(0)}_{+,(k+1)}, U^{(0)}_{-,(k+1)}, U^{(1)}_{+,(k+1)}, U^{(1)}_{-,(k+1)}\}$$

for all σ_b, *where* $U^{(j)}_{\pm,i} = \pm(\alpha_j + (i - \frac{k+1}{2})J), i = 0, 1, ..., k+1, j = 0, 1.$

Definition 4.2. A configuration φ is called a ground state of the Hamiltonian (4.1), if

$$U(\varphi_b) = \min\{U^{(0)}_{+,0}, U^{(0)}_{-,0}, U^{(1)}_{+,0}, U^{(1)}_{-,0}, ..., U^{(0)}_{+,(k+1)}, U^{(0)}_{-,(k+1)}, U^{(1)}_{+,(k+1)}, U^{(1)}_{-,(k+1)}\}$$

for all $b \in M$.

For a fixed $m = 0, 1, ..., (k+1), j = 0, 1$, we set

$$A^{(j)}_{\pm,m} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : U^{(j)}_{\pm,m} = \min\{U^{(0)}_{+,0}, U^{(0)}_{-,0}, ..., U^{(1)}_{+,(k+1)}, U^{(1)}_{-,(k+1)}\}.$$

Quite cumbersome but not difficult calculations show that

$$A^{(0)}_{\pm,0} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : \pm J \geq 0, \pm\alpha_0 \leq 0, |\alpha_1| \leq \mp\alpha_0\},$$

$$A^{(1)}_{\pm,0} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : \pm J \geq 0, \pm\alpha_1 \leq 0, |\alpha_0| \leq \mp\alpha_1\},$$

$$A^{(0)}_{\pm,m} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : J = 0, \pm\alpha_0 \leq 0, |\alpha_1| \leq \mp\alpha_0\}, m = 1, 2, ..., k,$$

$$A^{(1)}_{\pm,m} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : J = 0, \pm\alpha_1 \leq 0, |\alpha_0| \leq \mp\alpha_1\}, m = 1, 2, ..., k,$$

$$A^{(0)}_{\pm,(k+1)} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : \pm J \leq 0, \pm\alpha_0 \leq 0, |\alpha_1| \leq \mp\alpha_0\},$$

$$A^{(1)}_{\pm,(k+1)} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : \pm J \leq 0, \pm\alpha_1 \leq 0, |\alpha_0| \leq \mp\alpha_1\},$$

and

$$\mathbb{R}^3 = \bigcup_{m=0}^{k+1} (A^{(0)}_{\pm m} \cup A^{(1)}_{\pm m}).$$

Theorem 4.3. *a) If* $(J, \alpha_0, \alpha_1) \in A^{(0)}_{+,0} \cap A^{(1)}_{-,(k+1)}$, *then*

$$\sigma(x) = \begin{cases} 1, \ if x \in G_k^{(2)}, \\ -1, \ if x \in G_k \setminus G_k^{(2)}, \end{cases} \tag{4.3}$$

$G_k^{(2)}$*−periodic configuration is* $G_k^{(2)}$*−periodic ground state for the (4.1) model;*

b)*If* $(J, \alpha_0, \alpha_1) \in A^{(0)}_{-,(k+1)} \cap A^{(1)}_{+,0}$, *then*

$$\sigma(x) = \begin{cases} -1, \ if x \in G_k^{(2)}, \\ 1, \ if x \in G_k \setminus G_k^{(2)}, \end{cases} \tag{4.4}$$

$G_k^{(2)}$*−periodic configuration is* $G_k^{(2)}$*−periodic ground state for the (4.1) model.*

Proof. a) When we define the configuration for the model of (4.1) on the Cayley tree in the form of (4.2) then $\forall b \in M$ we have $\sigma(c_b) = 1$ or $\sigma(c_b) = -1$.

If $\sigma(c_b) = 1$ then $\forall x \in S_1(c_b)$ we have $\sigma(x) = -1$. In this case by (4.2) we take $U(\sigma_b) = U^{(0)}_{+,0} = \frac{-(k+1)}{2} J + \alpha_0$.

If $\sigma(c_b) = -1$ then $\forall x \in S_1(c_b)$ $\sigma(x) = 1$. Then we have $U(\sigma_b) = U^{(1)}_{-,(k+1)} = \frac{-(k+1)}{2} J - \alpha_1$.

From these cases, $G_k^{(2)}$−periodic configuration (see (4.3)) for the model of (4.1) is ground state on the set of $A^{(0)}_{+,0} \cap A^{(1)}_{-,(k+1)} = \{(J, \alpha_0, \alpha_1) \in \mathbb{R}^3 : J \geq 0, \alpha_0 \leq 0, \alpha_1 = -\alpha_0\}$.

b)Simularly proved.

This finishes the proof of Theorem 4.3.

Remark 4.4. Note that for the Ising model with $G_k^{(2)}$−periodic external field not exist translation-invariant ground states.

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