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"Riman integrali va uning tadbiqlari"mavzusidagi

BITIRUVMALAKAVIYISHI

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_______DSc.M.M.Rahmatullayev
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BMIrahbari:
PhD. Dots. R. xakimov

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MUNDARIJA

KIRISH... 3

ASOSIY QISM 6

1. Boshlang’ich funksiya va aniqmas integral 6

2. Aniq integral va integrallanuvchi funksiyalar sinfi 25

3. Aniq integral yordamida tekis shaklning yuzini hisoblash................................ 45

4. Aniq integral yordamida yoy uzunligi va aylanma jism yuzini hisoblash.......... 57

5. Aniq integralning fizika va mexanikaga tadbiqlari.. 71

6. Maple dasturi asosida misollar yechish 78

7. Mavzuga doir testlar.. 80

XULOSA 83

FOYDALANILGAN ADABIYOTLAR RO`YXATI.. 84

ILOVA. INTERNET MA’LUMOTLARI... 85

KIRISH.

Shuni unutmaslik kerakki, kelajagimiz poydevori bilimdargoxlarida yaratiladi, boshqacha aytganda, xalqimizning ertangikuni qanday bo'lishi farzandlarimizning bugun qanday ta'lim va tarbiya olishiga bog'liq.

Sh.M.Mirziyoyev.

Matematika fani insoniyat hayotida eng asosiy ahamiyat kasb etuvchi fan ekanini butun dunyo tan olgandir. Uning har bir kishi uchun har soniyada zarur ekani ravshan, undan hamma u yoki bu darajada foydalanadi, lekin ushbu jarayonni o'zi anglab yetmaydi. Bunga misol qilib, vaqt o'lchovini, kundalik xarajatlarni, o'qiladigan fanlar va darslar soni, mavzular, transport, yo'nalishlar nomeri va hokazolarni keltirish mumkin.

Matematikaning turli sohalarga: xalq xo'jaligiga, transportga, sanoatga,meditsinaga, biologiya, kimyo, fizika, genetika va boshqa o'nlab fanlarga tatbiqlari turmush darajasi va turli fanlarning shahdam qadamlar bilan olg'a ketishiga omil ekani ravshan.

Matematika, bir qarashda, matematikadan yiroq bo'lgan sohalarga, masalan adabiyotga, tilshunoslikka, sport sohasiga, psixologiyaga, tarixga, biologiyaga, meditsinaga va boshqa sohalarga kirib bormoqda. Matematikaning insoniyat tarixida va rivojlanish jarayonida nechog'lik ahamiyatga ega ekanini juda ko'p allomalar munosib baholaganlar. Masalan, ulug' shoh va shoir, astronom, matematik alloma - Mirzo Ulug'bek matematika haqida shunday yozgan: - "Matematika g'oyat bir yuksak fanki, unda bir olam mo'jiza yotadi".

Haqiqatan ham, matematika ilmi - insoniyat uchun bebaho ekanligini tan olmaydigan aqlli odamni topish amri mahol, chunki har bir fanning rivojlanish darajasi - bu fan matematik bilimlardan qanchalik foydalana olishi bilan baholanadi.

Hozirgi zamon matematikasining barcha yutuqlari haqida batafsil so`zlash imkoni kichik bir ish ichida beimkon muammodir, chunki matematika fani shunchalik rivojlanib ketganki, uning yuzlab tarmoqlarida minglab ilmiy ishlar qilinmoqda, chop etilgan ishlarning bir necha foizigina o’rganilib, kundalik hayot ehtiyojlariga tatbiq etiladi, xolos.

Mavzuning dolzarbligi va ahamiyati. Riman integrali va uning tadbiqlari nazaryasi matematika fanini katta qismini qamrab olgan, shuningdek, boshlang’ich funksiyalar, aniqmas integrallar, aniq integrallar, integrallanuvchi funsiyalar sinfi, aniq integral yordamida tekis shaklning yuzini hisoblash, aniq integral yordamida yoy uzunligi va aylanma jism yuzini hisoblash, aniq integralning fizika va mexanikaga tadbiqlari hozirgi kunda har bir soha ko’rishimiz mumkin, bularga misol qilib jadal darajada o’sib borayotgan muhandislik sohalarida yuzalarini, hajmlarini topishda bir muncha qiyinchilik tug’dirgan aylanma jismlar, egri chiziqli trapetsiyalar, egri chiziqli sektorlar va boshqa shakllarni yuzalarini hajmlarini hisoblashda katta hissa qo’shmoqda.

Bitiruv malakaviy ishi mavzusining maqsad va vazifalar. Ushbu bitiruv malakaviy ishi mavzusini o`rganishdan asosiy maqsad, boshlang’ich funksiya, aniqmas integral,aniq integral, integrllanuvchi funksiyalar sinfi, aniq integral yordamida tekis shaklni yuzini hisoblash, aniq integral yordamida yoy uzunligi va aylanma jism yuzini hisoblash, aniq integralning fizik va mexanikaga tadbiqlarida kengroq, umumiyroq tasavvurga ega bo`lish, ularning o`ziga xos xususiyatlarini, ko`rinishlarini o`rganish, ular bilan bog`liq teoremalarni o’rganish, bu amallar qanday xossalarga ega, aniqmas va aniq integrallarni hisoblash usullarini o’rganish, bunga doir misollarni matematik dasturi asosida oson va qulay yechish, kabi masalalarini chuqur tahlil qilish va o`rganishdan iborat.

Bitiruv malakaviy ishi mavzusining ahamiyati va tadbiq sohalari. Ishning ahamiyati bu bir qator boshlang’ich funksiyalar, aniqmas va aniq integral, integrallanuvchi funksiyalar sinfini o‘rganishdan iborat. Aniq integralarning fizika va mexanika kabi sohalarda ko‘plab tadbiqqa ega.

Bitiruv malakaviy ishi mavzusining yangiligi. Mazkur bitiruv malakaviy ishda ilgari o‘rganilgan integral funksiyalarni o‘rganish bilan birga yangi misollarni analitik yechilish usullari, matematik dasturlardan biri bo’lgan Maple dasturida yordamida misollarni yechilishi ko’rsatildi va kelgusida ilmiy izlanishlar olib borish uchun tegishli yo‘nalishlar aniqlab olindi.

Ishni yozishda o‘zbek, rus va ingliz tilidagi adabiyotlardan va ilmiy maqolalardan foydalanildi.

Ishning tuzilishi. Bitiruv malakaviy ishi kirish qismi, asosiy qismi, xulosa, internet ma’lumotlari va foydalanilgan adabiyotlar ro‘yxatidan iborat.

ASOSIY QISM

1. BOSHLANG'ICH FUNKSIYA VA ANIQMAS INTEGRAL

1.1.Boshlang'ich funksiya tushunchasi

Faraz qilaylik, $f(x)$ funksiya (a,b) intervalda (bu interval chekli yoki cheksiz bo'lishi mumkin) aniqlangan bo'lib, $F(x)$ funksiya esa shu intervalda differensiallanuvchi bo'lsin.

1.1-ta'rif. Agar $\forall x \in (a,b)$ da $F'(x) = f(x)$ yoki $dF(x) = f(x)dx$ bo'lsa, $F(x)$funksiya (a,b) da $f(x)$ ning boshlang'ich funksiyasi deyiladi.

Masalan.

1) $f(x) = x^2$ funksiyaning $R = (-\infty, +\infty)$ dagi boshlang'ich funksiyasi $F(x) = \frac{1}{3}x^3$ bo'ladi, chunki

$$F'(x) = \left(\frac{x^3}{3}\right)' = x^2 = f(x).$$

2) $f(x) = -\frac{x}{\sqrt{1-x^2}}$ funksiyaning $(-1,1)$ intervaldagi boshlang'ich funksiyasi

$F(x) = \sqrt{1-x^2}$ bo'ladi, chunki

$$F'(x) = (\sqrt{1-x^2})' = -\frac{x}{\sqrt{1-x^2}} = f(x).$$

Aytaylik, $f(x)$ funksiya $[a,b]$ da aniqlangan bo'lib, $F(x)$ funksiya shu segmentga differensiyalanuvchi bo'lsin.

1.2-ta'rif. Agar $F'(x) = f(x)$ $\quad (x \in (a,b))$ bo'lib

$$F'(a+0) = f(a) \qquad F'(b-0) = f(b).$$

bo'lsa,$F(x)$ funksiya $[a,b]$ da $f(x)$ ning boshlang'ich funksiyasi deyiladi.

1.1-misol. Ushbu

$$f(x) = \begin{cases} 1, agar\ x > 0 \\ 0, agar\ x = 0 \\ -1, agar\ x < 0 \end{cases}$$

funksiya (-1,1) intervalda boshlang'ich funksiyaga ega emasligi ko'rsatilsin.

◄ Teskarisini faras qilaylik. Biror $F(x)$ funksiya (-1,1) da $f(x)$ ning boshlang'ich funksiyasi bo'lsin: (-1,1) da.

$$F'(x) = f(x).$$

Lagranj teoremasiga ko’ra $[0, x]$da ($0 < x < 1$)

$$F(x) - F(0) = F'(c)x = f(c)x = x$$

bo’ladi. Keyingi tenglikdan topamiz:

$$F'(+0) = \lim_{x \to +0} \frac{F(x) - F(0)}{x} = 1$$

Bu esa $F'(x) = F'(0) = f(0) = 0$ bo’lishiga zid. ►

1.1-eslatma. Keyinchalik, $F(x)$ funksiya boshlang’ich funksiyasi bo’ladigan oraliqni ko’rsatib o’tirmaymiz. Oraliq sifatida $f(x)$ ning aniqlanish oralig’i X ko’zda tutiladi va X sifatida

$$[a, b], (a, b), (a, b], [a, b), (-\infty, a], (-\infty, a), [b, +\infty), (b, +\infty), (-\infty, +\infty)$$

lar olinishi mumkin.

1.1-teorema. Agar $F(x)$ funksiya X oraliqda uzluksiz bo’lsa, $f(x)$ shu oraliqda har doim boshlang’ich funksiyaga ega bo’ladi.

$F(x)$ va $\Phi(x)$ funksiyalarni har biri $f(x)$ funksiya uchun boshlang’ich funksiya bo’lsin:

$$F'(x) = f(x), \qquad \Phi'(x) = f(x).$$

Demak, $F'(x) = \Phi'(x)$. Bunda bizga ma’lum bo’lgan natijaga ko’ra

$$F(x) = \Phi(x) + C \qquad\qquad (C = const)$$

tenglik kelib chiqadi.

Demak, $f(x)$ funksiyaning barcha boshlang’ich funksiyalari bir-biridan o’zgarmas songa farq qiladi va istalgan boshlang’ich funksiyasi ushbu ko’rinishda ifodalanadi: $F(x) + C \quad (C = const)$

1.3-ta’rif. $f(x)$ funksiya boshlang’ich funksiyaning umumiy ifodasi $F(x) + C \quad (C = const)$ shu $f(x)$ funksiyani aniqmas integrali deb ataladi va

$$\int f(x)dx$$

kabi belgilanadi. Bunda $\int$- integral belgisi, $f(x)$ integral ostidagi funksiya, $f(x)dx$ esa integral ostidagi ifoda deyiladi.

Demak,

$$\int f(x)dx = F(x) + C. \tag{1.1}$$

Masalan,

$$\int 2^x dx = \frac{2^x}{\ln 2} + C$$

bo'ladi, chunki hosila olish qoidaalariga ko'ra

$$\left(\frac{2^x}{\ln 2} + C\right)' = 2^x.$$

1.2.Integralni xossalari.

1°. $f(x)$ funksiya aniqmas integrali $\int f(x)dx$ ning differensiyali $f(x)dx$ ga teng:

$$d\left(\int f(x)dx\right) = f(x)dx.$$

◄ Haqiatdan ham $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi bo'lsin: $F'(x) = f(x)$. U holda

$$\int f(x)dx = F(x) + C$$

boladi. Keyingi tenglikdan topamiz:

$$d\left(\int f(x)dx\right) = d(F(x) + C) = dF(x) = F'(x)dx = f(x)dx$$

Bu xossa avval differensial belgisi d, so'ngra integral belgisi $\int$ kelib, ular yonma-yon turganda o'zaro bir-birini yo'qotishni ko'rsatadi.

2°. Funksiya differensialining aniqmas integrali shu funksiya bilan o'zgarmas son yig'indisiga teng:

$$\int f(x)dx = F(x) + C$$

◄ F(x) funksiya f(x) ning biror boshlang'ich funksiyasi bo'lsin: F′(x)=f(x) u holda

$$\int f(x)dx = F(x) + C$$

tenglik o'rinli bo'ladi. Ikkinchi tomondan,

$$\int f(x)dx = \int F'(x)\, dx = \int dF(x).$$

Oxirgi ikki tenglik 2)—xossani isbot etadi. ►

Yuqorida keltirilganlardan, differensiyallash (funksiyaning hosilasini hisoblash) hamda integrallash (funksiyani aniqmas integarlini hisoblash) amallari o'zaro teskari amallar ekanligi kelib chiqadi.

Ayni paytda funksiya hosilasi hisoblanganda natija bitta funksiya bo'lsa, uning aniqmas integrali hisoblanganda esa natija cheksiz ko'p funksiya (ular bir—

biridan o'zgarmas songa farq qiladi) bo'ladi. Aniqmas integral deb yuritilishining boisi ham shu.

3°. Agar $f(x)$ funksiya boshlang'ich funksiyaga ega bo'lsa, u holda $kf(x)$ (k o'zgarmas son) funksiya ham boshlang'ich funksiyaga ega va $k \neq 0$ da

$$\int kf(x)dx = k\int f(x)dx \tag{1.2}$$

formula o'rinli bo'ladi.

◀ f(x) funksiyaning boshlang'ich funksiyasi F(x) bo'lsin. U holda

$$F'(x) = f(x) \text{ va } \int f(x)dx = F(x) + C$$

bo'lib,

$$k\int f(x)dx = k(F(x) + C) = kF(x) + kC \tag{1.3}$$

bo'ladi, bunda C-ixtiyoriy o'zgarmas son. Ushbu

$$\big(kF(x)\big)' = kF'(x) = kf(x)$$

tenglik o'rinli bo'lishidan $kf(x)$ funksiyaning boshlang'ich funksiyasi $kF(x)$ ekanini topamiz. Demak,

$$\int kf(x)dx = kF(x) + C_1 \tag{1.4}$$

bunda C_1 ixtiyoriy o'zgarmas son. Endi (1.3) va (1.4) munosabatlardan C va C_1 o'zgarmas sonlarning ixtiyoriligi hamda $k \neq 0$ bo'lishidan (1.2) formulaning o'rinli ekani kelib chiqadi. ▶

Shunga o'xshash integralning quyidagi xossasi isbotlanadi:

4°. Agar $f(x)$ va $g(x)$ funksiyar boshlang'ich funksiyalarga ega bo'lsa, $f(x) + g(x)$ ham boshlang'ich funksiyaga ega va

$$\int\big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx. \tag{1.5}$$

formula o'rinli bo'ladi.

Odatda bu xossa integralning additivlik xossasi deyiladi.

1.2-eslatma. Yuqorida keltirilgan (1.2) va (1.5) tengliklarni hamda kelgusida uchraydigan shunga o'xshash o'ng va chap tomonlaridagi ifodalar orasidagi ayirma o'zgarmas songa barobarligi ma'nosidagi (o'zgarmas son aniqligidan) tengliklar deb qaraladi.

Asosiy aniqmas integrallar jadvali.

1. $\int 0dx = C$ =const;

2. $\int 1dx = \int dx = x + C;$

3. $\int x^{\mu}dx = \frac{x^{\mu}}{\mu+1} + C$ $(\mu \neq 0);$

4. $\int \frac{1}{x}dx = \int \frac{dx}{x} = ln|x| + C;$ (x$\neq$ 0)

5. $\int \frac{1}{1+x^2}dx = \int \frac{dx}{1+x^2} = arctgx + C;$

6. $\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} = arcsinx + c;$

7. $\int a^x dx = \frac{a^x}{lna} + C;$ (a>0)

8. $\int sinxdx = -cosx + C;$

9. $\int cosxdx = sinx + C;$

10. $\int \frac{1}{sinx^2}dx = \int \frac{dx}{sinx^2} = -ctgx + C;$

11. $\int \frac{1}{cosx^2}dx = \int \frac{dx}{cosx^2} = tgx + C;$

12. $\int shxdx = chx + C;$

13. $\int chxdx = shx + C;$

14. $\int \frac{dx}{shx^2} = -cthx + C;$

15. $\int \frac{dx}{chx^2} = thx + C;$

Sodda aniqmas integrallar asosan bevosita xossalardan hamda jadvaldan foydalanib hisoblanadi.

1.2– m i s o l . Ushbu

$$\int \frac{x^2 + x + 1}{\sqrt{x}}dx$$

integral hisoblansin.

◀Integral ostidagi funksiyani quydagicha

$$\frac{x^2+x+1}{\sqrt{x}} = x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

yozib olamiz. So‘ng integralni xossalar va jadvaldan foydalanib topamiz

$$\int \frac{x^2+x+1}{\sqrt{x}} dx = \int \left(x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx = \frac{x^{\frac{3}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + c =$$

$2\sqrt{x}\left(\frac{x^2}{5} + \frac{x}{3} + c\right)$.▶

1.3– m i s o l . Ushbu

$$I = \int \frac{\sqrt{4+x^2} - 3\sqrt{4-x^2}}{\sqrt{16-x^4}} dx$$

integral hisoblansin.

◀Aniqmas integrallarning xossalari va jalvaldan foydalanib topamiz:

$$\mathrm{I} = \int \frac{\sqrt{4+x^2}}{\sqrt{16-x^4}} dx - 3\int \frac{\sqrt{4-x^2}}{\sqrt{16-x^4}} dx = \int \frac{dx}{\sqrt{4-x^2}} - \int \frac{dx}{\sqrt{4+x^2}} = arcsin\frac{x}{2} +$$

$$3ln\left|x + \sqrt{4+x^2}\right| + c.\blacktriangleright$$

1.4 – m i s o l . Ushbu

$$I = \int \frac{dx}{3cos^2 + 4sin^2}$$

integral hisoblansin.

◀Integral ostidagi ifodani quyidagi ko‘rinishda yozib olamiz:

$$I = \int \frac{dtgx}{3 + 4tg^2x}$$

Natijada,

$$I = \int \frac{dtgx}{3 + 4tg^2x} = \frac{1}{4}\int \frac{dtgx}{(\frac{\sqrt{3}}{2})^2 + 4tg^2x} = \frac{1}{\sqrt{3}} arctg\frac{2tgx}{\sqrt{3}} + c$$

bo‘ladi.▶

1.3. Integrallash usullari.

1. O’zgaruvchilarni almashtirib integrallash usuli.

Ushbu $\int f(x)dx$ aniqmas integralni hisoblash talab etilgan bo’lsin. Bunda $f(x)$ funksiya biror $X = (a, b)$ intervalda aniqlangan va

$$f(x) = \varphi\big(g(x)\big)g(x)'$$

(1.6)

ko'rinishda yozilishi mumkin deylik.

Agar $\varphi(x)$ funksiya T=(t_1, t_2) intervalda boshlang'ich funksiya $\Phi(t)$ ga ega bo'lib, $g(x)$ funksiya $X = (a, b)$ intervalda (bunda $g(x) \complement T$ differensiyallanuvchi bo'lsa, u holda

$$\int f(x)dx = \int \varphi(g(x))g(x)'\,dx = \Phi(x) + C \qquad (1.7)$$

formula o'rinli .

◀Haqiqatan,$\big[\Phi\big(g(x)\big)\big]' = \Phi'\big(g(x)\big)g'(x) = \varphi\big(g(x)\big)g(x)'$.▶

Odatda integralni bunday usul bilan hisoblash o'zgaruvchini almashtirish usuli bilan integrallash deb ataladi.

O'zgaruvchilarni almashtirish usulining muhim tomoni o'zgaruvchilarni juda ko'p usul bilan almashtirish imkoniyati bo'lgan holda ular ichidan integralni sodda va hisoblash uchun qulay holga keltiradiganini tanlab olishdan iborat.

1.5-misol. $\int \frac{xdx}{x^2+a^2}$ (a=const) ni hisoblansin.

◀Berilgan integralda o'zgaruvchi x ni $x^2 + a^2 = t$ kabi almashtiramiz. Bunda $2xdx = dt$ bo'lib ((1.6) va (1.7)) larga qarang).

$$\int \frac{xdx}{x^2+a^2} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}ln|t| + c = \frac{1}{2}ln|x^2 + a^2| + c \qquad ▶$$

1.6-misol. $\int e^{cosx} sinxdx$ ni hisoblansin.

◀ Bu integralda $\cos x = t$ almashtirishni bajaramiz. Natijada –sinxdx=dt bo'lib,

$$\int e^{cosx} sinxdx = -\int e^t dt = -e^t + c = -e^{cosx} + c.$$

bo'ladi. ▶

2. Bo'laklab integrallash usulli**.**

Ikki $u = u(x)$ va $v = v(x)$ funksiya (a,b) intervalda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. Ma'lumki,

$$d\big(u(x)v(x)\big) = u(x)dv(x) + v(x)du(x)$$

Bu tenglikdan

$$u(x)dv(x) = d(u(x)v(x)) - v(x)du(x) \quad (1.8)$$

bo'lishi kelib chiqadi.

Endi (1.8) tenglikni integrallab topamiz:

$$\int u(x)dv(x) = \int d\left(u(x)v(x)\right) - \int v\,(x)du(x) = u(x)v(x) - \int v(x)dx.$$

Shunday qilib, quyidagi

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)dx. \quad (1.9)$$

formulaga kelamiz. Bu (1.9) formula bo'laklab integrallash formulasi deyiladi.

Bo'laklab integrallash formulasidan foydalanish uchun integral ostidagi ifodani u(x) hamda dv(x) lar ko'paytmasi ko'rinishida yozib olinadi, bunda albatta dv(x) hamda v(x)dx ifodalarning integrallarini oson hisoblana olinishi lozimligini e'tiborda tutish kerak.

1.7-misol. $\int \boldsymbol{lnxdx}$ ni hisoblansin.

◀Integral ostidagi $\ln xdx$ ifodani $u = lnx$, $dv = dx$ lar ko'paytmasi deb olamiz. U holda $du = \frac{1}{2}dx, v = x$ bo'ladi. Bo'laklab integrallash formulasidan foydalanib topamiz:

$$\int ln\,xdx = xlnx - \int x\frac{1}{x}dx = xlnx - x + c = xln\frac{x}{e} + c\;.$$ ▶

1.8 – m i s o l . Ushbu

$$I = \int \arcsin x dx$$

integral hisoblansin.

◀Bu integralni bo'laklab integralash formulasidan foydalani hisoblaymiz:

$$I = \int \arcsin x dx = \begin{bmatrix} u = \arcsin x, & du = \dfrac{1}{\sqrt{1-x^2}}dx \\ dv = dx, & \mathrm{v} = \mathrm{x} \end{bmatrix} = x\arcsin x - \int \frac{xdx}{\sqrt{1-\mathrm{x}^2}}$$

Ravshanki,

$$\int \frac{xdx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \left(1-x^2\right)^{-\frac{1}{2}} d\left(1-x^2\right) = -\left(1-x^2\right)^{\frac{1}{2}}$$

Demak,

$$I = x\arcsin x + \sqrt{1-x^2} + C\,.$$ ▶

1.9– m i s o l . Ushbu

$$I = \int \sqrt{a^2 - x^2}dx \qquad (-\mathrm{a} \le \mathrm{x} \le \mathrm{a})$$

integral hisoblansin.

◀Bu integralni bo’laklab integralash formuladan foydalanib hisoblaymiz:

$$I=\int\sqrt{a^2-x^2}dx=\begin{bmatrix} x=a\sin t & \left(-\frac{\pi}{2}\le t\le\frac{\pi}{2}\right) \\ dx=a\cos tdt & \end{bmatrix}=\int\sqrt{a^2-a^2\sin^2 t}\,a\cos tdt=$$

$$=a^2\int\cos^2 tdt=\frac{a^2}{2}\int(1+\cos 2t)dt=\frac{a^2}{2}t+\frac{a^2}{4}\cdot\sin 2t+C\,.$$

Agar $\sin t=\frac{x}{a}$, $t=\arcsin\frac{x}{a}$ bo`lishini e`tiborga olsak, u holda

$$\mathrm{I}=\int\sqrt{\mathrm{a}^2-\mathrm{x}^2}\,\mathrm{dx}=\frac{\mathrm{a}^2}{2}\arcsin\frac{\mathrm{x}}{\mathrm{a}}+\frac{\mathrm{x}}{2}\sqrt{\mathrm{a}^2-\mathrm{x}^2}+\mathrm{C}$$

bo`lishi kelib chiqadi.▶

1.10-misol. $c=\int\frac{dx}{(x^2+a^2)^n}$ (n=1,2,3...) ni hisoblansin.

◀Bu integralda

u=$\frac{1}{(x^2+a^2)^n}$ dv=dx, du=$\frac{2nxdx}{(x^2+a^2)^{n+1}}$ v=x

bo’ladi. (1.9) formuladan foydalanib topamiz:

$$J_n=\int\frac{dx}{(x^2+a^2)^n}=\frac{x}{(x^2+a^2)^n}+2n\int\frac{x^2dx}{(x^2+a^2)^n}\,. \qquad (1.10)$$

Bu tenglikning o’ng tomonidagi $\int\frac{x^2dx}{(x^2+a^2)^{n+1}}$ ni

$$\int\frac{x^2dx}{(x^2+a^2)^{n+1}}=\int\frac{dx}{(x^2+a^2)^n}-a^2\int\frac{dx}{(x^2+a^2)^{n+1}}$$

ko’rinishda yozsak, unda (1.10) munosabat ushbu

$$J_n=\frac{x}{(x^2+a^2)^n}+2n\ \ J_n-2n\,a^2\ \ J_{n+1}$$

ko’rinishni oladi. Keyingi tenglikdan esa quyidagi

$$J_{n+1}=\frac{1}{2a^2x}\frac{x}{(x^2+a^2)^n}+\frac{2n-1}{2n}\ \ J_n. \qquad (1.11)$$

rekurrent formula kelib chiqadi.

Ravshanki, n=1 bo’lganda

$$J_1=\int\frac{dx}{x^2+a^2}=\frac{1}{a}\int\frac{d\left(\frac{x}{a}\right)}{1+(\frac{x}{a})^2}=\frac{1}{a}arctg\frac{x}{a}+c$$

bo'ladi.
n≥2 bo'lganda, mos J_n integrallar (1.11) rekurrent formula yordamida topiladi. Masalan:

$$J_2=\int\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^2}\frac{x}{x^2+a^2}+\frac{1}{2a^2}\quad J_1=\frac{1}{2a^2}\frac{x}{x^2+a^2}+\frac{1}{2a^2}arctg\frac{x}{a}+c. \blacktriangleright$$

1.4. Sodda kasrlarni integrallash.

Sodda kasrlarning aniqmas integrallarini hisoblaymiz.

1) $\frac{A}{x-a}$ sodda kasrning aniqmas integrali .

$$\int\frac{A}{x-a}dx=A\int\frac{d(x-a)}{x-a}=A\ln|x-a|+C.$$

2) $\frac{A}{(x-a)^m}$ $(m>1)$ sodda kasrning aniqmas integrali ham tez hisoblanadi:

$$\int\frac{Adx}{(x-a)^m}=A\int\frac{d(x-a)}{(x-a)^m}=A\int(x-a)^{-m}d(x-a)=\frac{A}{1-m}\cdot\frac{1}{(x-a)^{m-1}}+C.$$

3) $\frac{Bx+C}{x^2+px+q}$ sodda kasrning (bunda x^2+px+q kvadrat uchhad haqiqiy ildizga ega emas) integrali $\int\frac{Bx+C}{x^2+px+q}dx$ ni hisoblash uchun avval kasrning maxrajida turgan x^2+px+q kvadrat uchhadni ushbu

$$x^2+px+q=\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}$$

ko'rinishda yozib olamiz. U holda

$$\int\frac{Bx+C}{x^2+px+q}dx=\int\frac{Bx+C}{\left(x+\frac{p}{2}\right)^2+a^2}dx$$

bo'ladi, bunda $a^2=q-\frac{p^2}{4}$. Bu integralda $x+\frac{p}{2}=t$ almashtirishni bajaramiz:

$$\int\frac{Bx+C}{x^2+px+q}dx=B\int\frac{tdt}{t^2+a^2}+(C-\frac{Bp}{2})\int\frac{dt}{t^2+a^2}=\frac{B}{2}\int\frac{d(t^2+a^2)}{t^2+a^2}+$$

$$+\left(C-\frac{Bp}{2}\right)\frac{1}{a}\int\frac{d(\frac{t}{a})}{1+(\frac{t}{a})^2}=\frac{B}{2}\ln(t^2+a^2)+\left(C-\frac{Bp}{2}\right)\frac{1}{a}arctg\frac{t}{a}+C_1=$$

$$=\frac{B}{2}\ln(x^2+px+q)+\frac{2C-Bp}{2\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C_1\ .$$

Demak,

$$\int\frac{Bx+C}{x^2+px+q}dx=\frac{B}{2}\ln(x^2+px+q)+\frac{2C-Bp}{2\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C_1$$

bunda C_1 ixtiyoriy o'zgarmas.

4) $\frac{Bx+C}{\left(x^2+px+q\right)^m}$ $(m>1)$ sodda kasrning integrali

$$J_m=\int\frac{Bx+C}{\left(x^2+px+q\right)^m}dx$$

ni hisoblash uchun 3)-holdagidek o'zgaruvchini almashtiramiz: $x+\frac{p}{2}=t$. Natijada quyidagiga ega bo'lamiz:

$$J_m=\int\frac{Bx+C}{\left(x^2+px+q\right)^m}dx=\int\frac{Bx+C}{\left(\left(x+\frac{p}{2}\right)^2+p-\frac{p^2}{4}\right)^m}dx=\int\frac{Bt+\left(C-\frac{Bp}{t}\right)}{\left(t^2+a^2\right)^m}\mathrm{dt}=$$

$$=\frac{B}{2}\int\frac{d\left(t^2+a^2\right)}{\left(t^2+a^2\right)^m}+\left(C-\frac{Bp}{2}\right)\int\frac{dt}{\left(t^2+a^2\right)^m}=\frac{B}{2}\cdot\frac{1}{1-m}\cdot\frac{1}{\left(t^2+a^2\right)^{m-1}}+\left(C-\frac{Bp}{2}\right)\int\frac{dt}{\left(t^2+a^2\right)^m}.$$

Bu munosabatdagi $\int\frac{dt}{(t^2+a^2)^m}$ integral quyda keltirilgan bo'lib, u rekurrent formula orqali hisoblanadi.

1.11-misol. $\int\frac{xdx}{x^2+a^2}$ $(a=\text{const})$ ni hisoblansin.

Berilgan integralda o'zgaruvchi x ni $x^2+a^2=t$ kabi almashtiramiz. Bunda $2xdx=dt$ bo'lib ((1.6) va (1.7)) larga qarang)

$$\int\frac{xdx}{x^2+a^2}=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\ln|t|+C=\frac{1}{2}\ln(x^2+a^2)+C$$

1.2-teorema. Har qanday to'g'ri kasr soda kasrlar yig'indisi orqali ifodalanadi.

1.12-misol. $\frac{2x-1}{x^2-5x+6}$ to'g'ri kasrni soda kasrlarga ajratilsin.

◀ Bu kasrning maxraji $x^2-5x+6=(x-3)(x-2)$ bo'lgani uchun teoremaga ko'ra

$$\frac{2x-1}{x^2-5x+6}=\frac{A}{x-3}+\frac{B}{x-2}$$

bo'ladi. Uni

$$\frac{2x-1}{x^2-5x+6}=\frac{A}{x-3}+\frac{B}{x-2}=\frac{A(x-2)+B(x-3)}{(x-2)(x-3)}$$

ko'rinishda yozib, ushbu

$$2x-1=A(x-2)+B(x-3) \quad \text{yoki} \quad 2x-1=(A+B)x-(2A+3B)$$

tenglikka kelamiz. Ikki ko'phadning tengligidan foydalanib, A va B larga nisbatan ushbu

$$\begin{cases} A+B=2 \\ 2A+3B=1 \end{cases} \tag{1.12}$$

sistemaga kelamiz. (1.12) dan $A=5,\ B=-3$ bo'ladi. Shunday qilib, berilgan to'g'ri kasr sodda kasrlar orqali quyidagicha ifodalanadi:

$$\frac{2x-1}{x^2-5x+6}=\frac{5}{x-3}+\frac{-3}{x-2}\ . \ ▶$$

1.13-misol. $\int\frac{dx}{x^4-1}$ ni hisoblansin.

◀Integral ostidagi $\frac{1}{x^4-1}$ kasrni sodda kasrlarga ajratamiz:

$$\frac{1}{x^4-1}=\frac{1}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

Bu tenglikni quyidagicha yozib olmiz:

$$\frac{1}{x^4-1}=\frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)}{(x-1)(x+1)(x^2+1)}$$

U holda

$$1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

ya'ni

$$1=(A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+(A-B-D)$$

bo'ladi. Natijada A,B,C,D larni topish uchun

$$\begin{aligned}A+B+C&=0,\\A-B+D&=0,\\A+B-C&=0,\\A-B-D&=1.\end{aligned}$$

sistemaga kelamiz. Bu sistemani echib,

$$A=\frac{1}{4},\ B=-\frac{1}{4},\ C=0,\ D=-\frac{1}{2}$$

bo'lishini topamiz. Demak,

$$\frac{1}{x^4-1}=\frac{1}{4}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x+1}-\frac{1}{2}\cdot\frac{1}{x^2+1}$$

bo'lib,

$$\int\frac{dx}{x^4-1}=\frac{1}{4}\cdot\int\frac{dx}{x-1}-\frac{1}{4}\cdot\int\frac{dx}{x+1}-\frac{1}{2}\cdot\int\frac{dx}{x^2+1}=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}arctgx+C$$

bo'ladi. ►

1.14 – m i s o l . Ushbu

$$I=\int\frac{2x^5+6x^3+1}{x^4+3x^2}dx$$

integral hisoblansin.

◄Integral ostidagi kasrning suratini maxrajiga bo`lib, butun qismini ajratamiz:

$$\frac{2x^5+6x^3+1}{x^4+3x^2}=2x+\frac{1}{x^4+3x^2}$$

so`ng bu tenglikning o`ng tomonidagi to`g`ri kasrni sodda kasrlarga yoyamiz:

$$\frac{1}{x^4+3x^2}=\frac{1}{x^2\left(x^2+3\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+3} \tag{1.13}$$

Bundan

$$I=Ax\left(x^2+3\right)+B\left(x^2+3\right)+\left(Cx+D\right)x^2=\left(A+C\right)x^3+\left(B+D\right)x^2+3Ax+3B$$

bo`lishi kelib chiqadi. Keyingi tenglikda esa

$$A=0,\ \ \mathrm{B}=\frac{1}{3},\ \ \mathrm{C}=0,\ \ \mathrm{D}=-\frac{1}{3}$$

bo`lishini topamiz. (1.13) tenglikka ko`ra

$$\frac{1}{x^2\left(x^2+3\right)}=\frac{1}{3x^2}-\frac{1}{3\left(x^2+3\right)}$$

bo`ladi. endi berilgan integralni hisoblaymiz:

$$\int \frac{2x^5+6x^3+1}{x^4+3x^2}dx = \int\left(2x+\frac{1}{x^4+3x^2}\right)dx = \int\left[2x+\frac{1}{3x^2}-\frac{1}{3\left(x^2+3\right)}\right]dx =$$

$$= x^2-\frac{1}{3x}-\frac{1}{3\sqrt{3}}arctg\frac{x}{\sqrt{3}}+C. ▶$$

1.15 – m i s o l . Ushbu

$$\mathrm{I} = \int \frac{3x+1}{x\left(1+x^2\right)^2}dx$$

integral hisoblansin.

◀Integral ostidagi kasrni sodda kasrlarga yoyamiz:

$$\frac{3x+1}{x\left(1+x^2\right)^2} = \frac{A}{x}+\frac{Bx+C}{1+x^2}+\frac{Dx+F}{\left(1+x^2\right)^2}$$

Umumiy maxrajga keltirib topamiz:

$$3x+1 = A\left(1+x^2\right)^2+\left(Bx+C\right)x\left(1+x^2\right)+\left(Dx+F\right)x =$$

$$= \left(A+B\right)x^4+Cx^3+\left(2A+B+D\right)x^2+\left(C+F\right)x+A.$$

A,B,C,D,F larni topish uchun quyidagi

$$\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+F=0 \\ A=1 \end{cases}$$

sistema hosil bo`ladi. Bu sistemani yechsak,

$$A=1,\ \ \mathrm{B}=\text{-}1,\ \ \mathrm{C}=0,\ \ \mathrm{D}=\text{-}1,\ \ \mathrm{F}=3$$

bo`lishi kelib chiqadi. Demak, integral ostidagi funksiya

$$\frac{3x+1}{x\left(1+x^2\right)^2} = \frac{1}{x}-\frac{x}{1+x^2}+\frac{-x+3}{\left(1+x^2\right)^2}$$

bo`ladi. Uning integralini hisoblaymiz:

$$I = \int\frac{3x+1}{x\left(1+x^2\right)^2}dx = \int\frac{dx}{x}-\int\frac{xdx}{1+x^2}-\int\frac{xdx}{\left(1+x^2\right)^2}+3\int\frac{dx}{\left(1+x^2\right)^2} =$$

$$= \ln|x|-\frac{1}{2}\ln\left(1+x^2\right)+\frac{1}{2\left(1+x^2\right)}+3\int\frac{dx}{\left(1+x^2\right)^2}.$$

Keyingi tenglikning o`ng tomonidagi integralni rekkurent formulaga ko`ra topamiz:

$$I_2 = \int\frac{dx}{\left(x^2+1\right)^2} = \frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}I_1 = \frac{x}{2\left(x^2+1\right)}+\frac{1}{2}\int\frac{dx}{x^2+1} =$$

$$= \frac{x}{2\left(x^2+1\right)}+\frac{1}{2}arctgx+C$$

Demak,

$$I=\int\frac{3x+1}{x(1+x^2)}dx=\ln|x|-\frac{1}{2}\ln(1+x^2)+\frac{3x+1}{2(x^2+1)}+\frac{3}{2}arctgx+C\,. ▶$$

1.16-misol . Ushbu

$$\int\sqrt{\frac{1+x}{1-x}}\frac{dx}{1-x}$$

integral hisoblansin.

◀ Bu integralni hisoblash uchun

$$t=\sqrt{\frac{1+x}{1-x}}$$

deb olamiz. U holda

$$x=\frac{t^2-1}{t^2+1}\,,\quad dx=\frac{4tdt}{(t^2+1)^2}$$

bo'lib ,

$$\int\sqrt{\frac{1+x}{1-x}}\frac{dx}{1-x}=2\int\frac{t^2dt}{t^2+1}$$

bo'ladi . Natijada berilgan integral uchun topamiz :

$$\int\sqrt{\frac{1+x}{1-x}}\frac{dx}{1-x}=2\int\frac{t^2dt}{t^2+1}=2t-2arctgt+C=2\sqrt{\frac{1+x}{1-x}}-2arctg\sqrt{\frac{1+x}{1-x}}+C\,. ▶$$

1.17-misol. Ushbu

$$\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$$

integral hisoblansin .

◀ Bu integralda $t=\sqrt[6]{x}$ almashtirish bajaramiz . Natijada

$$x=t^6\,,\quad dx=6t^5\,dt$$

bo'lib, berilgan integral uchun topamiz:

$$\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}=6\int\frac{t^3dt}{t+1}=6\int\left((t^2-t+1)-\frac{1}{t+1}\right)dt=6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right)+C=$$

$$=6\left(\frac{\sqrt{x}}{3}-\frac{\sqrt[3]{x}}{2}+\sqrt[6]{x}-\ln\left|\sqrt[6]{x}+1\right|\right)+C=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln\left|\sqrt[6]{x}+1\right|+C \;▶$$

1.18-misol. $\int\frac{1}{x+\sqrt{x^2+x+1}}dx$ integral hisoblansin.

◀. Bu integral uchun $(a=1)$ bajaramiz:

$$t=x+\sqrt{x^2+x+1}\,.$$

U holda

$$x=\frac{t^2-1}{1+2t},\sqrt{x^2+x+1}=\frac{t^2+t+1}{1+2t},\ dx=2\frac{t^2+t+1}{(1+2t)^2}dt$$

bo'lib, berilgan integral uchun

$$\int \frac{1}{x+\sqrt{x^2+x+1}}dx = 2\int \frac{t^2+t+1}{t(1+2t)^2}dt$$

bo'ladi. Endi

$$2\frac{t^2+t+1}{t(1+2t)^2} = \frac{2}{t} - \frac{3}{1+2t} - \frac{3}{(1+2t)^2}$$

bo'lishini e'tiborga olib, topamiz:

$$\int \frac{1}{x+\sqrt{x^2+x+1}}dx = 2\ln|t| - \frac{3}{2}\ln|1+2t| + \frac{3}{2}\cdot\frac{1}{1+2t} + C =$$

$$= 2\ln\left|x+\sqrt{x^2+x+1}\right| - \frac{3}{2}\ln\left|1+2x+2\sqrt{x^2+x+1}\right| + \frac{3}{2}\cdot\frac{1}{1+2x+2\sqrt{x^2+x+1}} + C.$$

►

1.19–misol. Ushbu

$$\int \frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2}dx$$

integral hisoblansin.

◄ Bu integralda $p=-2$ (butun son) ekanligini aniqlaymiz. Yuqoridagilarga ko'ra $x=t^6$ $(t=\sqrt[6]{x}\,)$ almashtirish bajarib topamiz:

$$\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx = 6\int \frac{t^8}{(1+t^2)^2}dt\,.$$

Bu tenglikning o'ng tomonidagi integral ostidagi funksiyani

$$\frac{t^8}{(1+t^2)^2} = t^4 - 2t^2 + 3 - 4\frac{1}{t^2+1} + \frac{1}{(t^2+1)^2}$$

ko'rinishda yozish mumkin ekanini e'tiborga olsak, u holda

$$\int \frac{t^8}{(1+t^2)^2}\,dt = \frac{t^5}{5} - 2\frac{t^3}{3} + 3t - 4arctgt + \int \frac{dt}{(t^2+1)^2}$$

bo'ladi. Oxirgi integral bizga ma'lum bolgan (17) rekurrent munosabat yordamida osongina hisoblanadi.

$$\int \frac{dt}{(t^2+1)^2} = \frac{1}{2}\cdot\frac{1}{t^2+1} + \frac{1}{2}arctgt + C.$$

Natijada quyidagiga ega bo'lamiz:

$$\int \frac{t^8}{(1+t^2)^2}dt = \frac{1}{5}t^5 - \frac{2}{3}t^3 + 3t - \frac{7}{2}arctgt + \frac{1}{2}\cdot\frac{1}{t^2+1} + C.$$

Demak , $t=\sqrt[6]{x}$ ekanini e'tiborga olib, uzil kesil yozamiz.

$$\int\frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx=\frac{6}{5}\sqrt[6]{x^5}-4\sqrt{x}+18\sqrt[6]{x}-21arctg\sqrt[6]{x}+3\cdot\frac{\sqrt[6]{x}}{\sqrt[3]{x}+1}+C \quad ►$$

1.20- misol. $\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$ integralni hisoblang.

◄ Bu integralni $\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}=\int x(1+x^{\frac{2}{3}})^{-\frac{1}{2}}dx$ ko'rinishda yozib, $m=1,\ n=\frac{2}{3},\ p=-\frac{1}{2}$ bo'lishini topamiz. Bu holda $\frac{m+1}{n}=3$ bo'lib,

$$t=(1+x^{\frac{2}{3}})^{\frac{1}{2}}$$

almashtirishni bajaramiz. Unda

$$1+x^{\frac{2}{3}}=t^2,\quad x=(t^2-1)^{\frac{3}{2}}\quad \text{va}\quad dx=\frac{3}{2}(t^2-1)^{\frac{1}{2}}\cdot 2tdt$$

bo'lib, berilgan integral uchun ushbu

$$\int x(1+x^{\frac{2}{3}})^{-\frac{1}{2}}dx=3\int(t^2-1)^2t^2dt=3\frac{t^7}{7}-6\frac{t^6}{5}+t^3+C,\ \ t=\sqrt{1+x^{\frac{2}{3}}}$$

ifoda topiladi. ►

1.21 – m i s o l . Ushbu

$$I=\int\frac{dx}{(1-x)\sqrt{1-x^2}}$$

integral hisoblansin.

◄Ravshanki,

$$(1-x)\sqrt{1-x^2}=(1-x)\sqrt{(1-x)(1+x)}=(1-x)(1+x)\cdot\sqrt{\frac{1-x}{1+x}}\ .$$

Demak, integral ostidagi funksiya x va $\sqrt{\frac{1-x}{1+x}}$ larning ratsional funksiyasi bo`ladi.

Bu integralda $\frac{1-x}{1+x}=t^2$ $\left(t=\sqrt{\frac{1-x}{1+x}}\right)$ almashtirish bajaramiz. Unda

$$x=\frac{1-t^2}{1+t^2},\ \ \mathrm{dx}=\frac{-4\mathrm{tdt}}{\left(1+\mathrm{t}^2\right)^2},\ \ 1-\mathrm{x}=\frac{2\mathrm{t}^2}{1+t^2},\ \ 1+\mathrm{x}=\frac{2}{1+\mathrm{t}^2}$$

bo`lib,

$$I=-\int\frac{4tdt}{\left(1+t^2\right)^2\frac{2t^2}{1+t^2}\cdot\frac{2}{1+t^2}\cdot t}=-\int\frac{dt}{t^2}=\frac{1}{t}+C$$

bo`ladi. Demak,

$$I=\sqrt{\frac{1+x}{1-x}}+C\ .►$$

1.21– m i s o l . Ushbu

$$I=\int\frac{dx}{x\sqrt{4x^2+4x+3}}$$

integral hisoblansin.

◀Bu integralda $a=4>0$ bo`lgani uchun $\sqrt{4x^2+4x+3}=t-2x$ almashtirish bajaramiz. Unda

$$4x^2+4x+3=t^2-4tx+4x^2,\quad 4x+3=t^2-4xt,\quad x=\frac{t^3-3}{4(1+t)},\quad dx=\frac{t^2+2t+3}{4(1+t)^2}dt,$$

$$\sqrt{4x^2+4x+3}=t-\frac{2(t^2-3)}{4(1+t)}=\frac{t^2+2t+3}{2(1+t)}$$

bo`lib,

$$I=\int\frac{\frac{t^2+2t+3}{4(1+t)^2}dt}{\frac{t^3-3}{4(1+t)}\cdot\frac{t^2+2t+3}{2(1+t)}}=2\int\frac{dt}{t^2-3}$$

bo`ladi. Ravshanki,

$$2\int\frac{dt}{t^2-3}=\frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C$$

Demak,

$$I=\frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C=\frac{1}{\sqrt{3}}\ln\left|\frac{2x+\sqrt{4x^2+4x+3}-\sqrt{3}}{2x+\sqrt{4x^2+4x+3}+\sqrt{3}}\right|+C.\ \blacktriangleright$$

1.22 – misol . $\int\frac{dx}{3+\sin x}$ integral hisoblansin .

◀ Bu integralda $t=tg\frac{x}{2}$ almashtirish bajarib topamiz :

$$\int\frac{dx}{3+\sin x}=\int\frac{1}{3+\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2}=2\int\frac{dt}{3t^2+2t+3},$$

$$2\int\frac{dt}{3t^2+2t+1}=\frac{2}{3}\int\frac{d(t+\frac{1}{3})}{(t+\frac{1}{3})^2+\frac{8}{9}}=\frac{2}{3}\int\frac{d(t+\frac{1}{3})}{(t+\frac{1}{3})^2+(\frac{2\sqrt{2}}{3})^2}=\frac{\sqrt{2}}{2}arctg3\frac{t+\frac{1}{3}}{2\sqrt{2}}+C.$$

Demak ,

$$\int\frac{dx}{3+\sin x}=\frac{\sqrt{2}}{2}arctg\frac{3tg\frac{x}{2}+1}{2\sqrt{2}}+C.\ \blacktriangleright$$

Shuni ta`kidlash lozimki, $\int R(\sin x,\cos x)dx$ integralda $t=tg\frac{x}{2}$ almashtirish universal almashtirish bo’lib, u integralni har doim ratsional funksiya integraliga keltirsada ko’pincha bu almashtirish murakab hisoblashlarga olib keladi .

Ayrim hollarda trigonometrik funksiyalarni integrallashda $t=tgx$, $t=\sin x$, $t=\cos x$ almashtirishlar qulay bo’ladi .

1.23-misol. $\int \frac{dx}{\cos^4 x}$ integral hisoblansin .

◀ Agar bu integralda $t = tg\frac{x}{2}$ universal almashtirish bajarilsa , u holda

$$\int \frac{dx}{\cos^4 x} = 2\int \frac{(1+t^2)^3}{(1-t^2)^4}dt$$

bo'ladi . Biroq qaralayotgan integralda $t = tgx$ almashtirish bajarilsa , u holda

$$\int \frac{dx}{\cos^4 x} = \int (1+tg^3 x)d(tgx) = \int (1+t^2)dt$$

bo'lib , undan

$$\int \frac{dx}{\cos^4 x} = t + \frac{t^3}{3} + C = tgx + \frac{tg^3 x}{3} + C$$

bo'lishini topamiz. ▶

2. ANIQ INTEGRAL VA INTEGRALLANUVCHI FUNKSIYALAR SINFI.

2.1.Aniq integral ta'rifi.

Biror $[a, b] \complement R$ segment berilgan bo'lsin .

Uning ushbu

$$a_0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

munosabatda bo'lgan chekli sondagi ixtiyoriy $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ nuqtalari sistemasini olaylik . Agar $A_1 = [x_{i-1}, x_i]$, $i = 1,2,3 \ldots n$ deb belgilasak , u holda ravshanki ,

1) $A_1 \cup A_2 \cup \ldots \cup A_n = [a, b]$;

2) $A_k \cap A_j = \emptyset, (k, j = 1,2, \ldots n)$

Bizga malum bo'lgan to'plamni bo'laklash tushunchasi ta`rifiga binoan $\{ A_1, A_2, \ldots, A_n\}$ sistema $[a, b]$ da bo'laklash bajargan bo'ladi , va aksincha, agar bizga $[a\ , b]$ segmentning biror chekli $\{ B_1, B_2, \ldots, B_n\}$ bo'laklashi berilgan bo'lsa, u ushbu

$$a_0 = y_0 < y_1 < y_2 < \ldots < y_m = b$$

munosabatda bo'lgan chekli sondagi $y_0, y_1, y_2, \ldots, y_{m-1}, y_m$ nuqtalar sistemasini aniqlaydi . Binobarin , biz to'plamni bo'laklash ta`rifiga ekvivalent bo'lgan quyidagi ta`rifni kirita olamiz .

2.1-ta`rif . $[a, b]$ segmentning ushbu

$$a_0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

munosabatda bo'lgan ixtiyoriy chekli sondagi $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ nuqtalari sistemasi $[a, b]$ segmentda bo'laklash bajaradi deyiladi .

Uni

$$P = (x_0, x_1, x_2, \ldots, x_{n-1}, x_n)$$

kabi belgilanadi .

Har bir x_k $(k = 0,1,2, \ldots, n)$ nuqta bo'laklashning bo'luvchi nuqtasi,[x_k, x_{k+1}] $(k = 0,1,2, \ldots, n)$ segment esa P bo'laklashning oralig'i deyiladi.

P bo'laklash oraliqlari uzunligi $\Delta x_k = x_{k+1} - x_k$ $(k = 0,1,2, \dots, n)$ larning eng kattasi , ya`ni ushbu

$$\lambda_p = ma\,x\{\Delta x_k\} = \max\{\Delta x_0, \Delta x_1, \Delta x_2, \dots, \Delta x_{n-1}\}$$

miqdor P bo'laklashning diametri deb ataladi. [a,b] segment berilgan holda bu segmentni turli usullar bilan istalgan sondagi bo'laklashlarni tuzish mumkin ekan. Bu bo'laklashlardan iborat to'plamni F bilan belgilaymiz : F={P}.

2.2. Integral yig'indi.

$[a,b]$ segmentda $f(x)$ funksiya aniqlangan bo'lsin . Shu segmentni

$$P = \{x_0, x_1, x_2 \dots, x_k, \dots, x_n\} \in F$$

bo'laklashi va bu bo'laklashning har bir $[x_k, x_{k+1}]$ $(k = 0,1, \dots, n-1)$ oralig'ida ixtiyoriy ξ_k $(\xi_k \in [x_k, x_{k+1}])$ nuqta olamiz . Berilgan funksiyaning ξ_k nuqta-dagi qiymati $f(\xi_k)$ ni $\Delta x_k = x_{k+1} - x_k$ ga ko'paytirib, quyidagi yig'indini tuzamiz:

$$\sigma = f(\xi_0)\Delta x_0 + f(\xi_1)\Delta x_1 + \dots + f(\xi_k)\Delta x_k + \dots + f(\xi_{n-1})\Delta x_{n-1} = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k .$$

2.2-ta`rif . Ushbu

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k . \tag{2.1}$$

yig'indi $f(x)$ funksiyaning integral yig'indisi (Riman yig'indisi) deb ataladi .

Masalan, 1) $f(x) = x$ funksiyaning $[a\,,\,b]$ segmentdagi integral yig'indisi (Riman yig'indisi)

$$\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k = \sum_{k=0}^{n-1} \xi_k \Delta x_k$$

bo'ladi , bunda

$$x_k \le \xi_k \le x_{k-1} \qquad (k = 0,1, \dots, n-1) .$$

2.3-ta'rif. (Aniq integral ta'rifi) $f(x)$ funksiya $[a,b]$ segmentda aniqlangan bo'lsin. $[a,b]$ segmentning shunday

$$P_1, P_2, \dots, P_m, \dots \tag{2.2}$$

$(P_m \in F,\ m = 1,2, \dots)$ bo'laklashlarni qaraymizki, ularni mos diametrlaridan tashkil topgan

$$\lambda_{p_1}, \lambda_{p_2}, \lambda_{p_3}, \ldots, \lambda_{p_m}, \ldots$$

Ketma-ketlik nolga intilsin: $\lambda_{p_m} \to 0$.

Bunday $P_m(m = 1, 2, 3, \ldots)$ bo'laklashlarga nisbatan $f(x)$ funksiyaning integral yig'indilarini tuzamiz. Natijada $[a,b]$ segmentni (2.2) bo'laklashlarga mos $f(x)$ funksiyaning integral yig'indilari qiymatlaridan iborat quyidagi:

$$\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$$

Ketma-ketlik hosil bo'ladi. Ravshanki bu ketma-ketlikning har bir hadi ξ_k nuqtalarga bog'liqdir.

2.4-ta'rif. Agar $[a,b]$ segmentni har qanday (2.2) bo'laklashlar ketma-ketligi $\{P_m\}$ olinganda ham unga mos integral yig'indi qiymatlaridan iborat $\{\sigma_m\}$ ketma-ketlik ξ_k nuqtalarning tanlab olinishiga bog'liq bo'lmagan ravishda hamma vaqt bitta J songa intilsa, bu J son σ yig'indining $\lambda_p \to 0$ dagi limiti deb ataladi. U

$$\lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k$$

kabi belgilanadi.

Yig'indi limitini quyidagicha ham ta'riflash mumkin.

2.5–ta'rif. Agar $\forall \varepsilon > 0$ son olingnda ham shunday $\delta > 0$ son topilsaki, $[a,b]$ segmentni diametri $\lambda_p < \delta,$ bo'lgan har qanday P bo'laklash uchun tuzilgan σ yig'indi ixtiyoriy ξ_k nuqtalarda

$$|\sigma - J| < \varepsilon$$

tengsizliklarni qanoatlantirsa, J son σ yig'indining $\lambda_p \to 0$ dagi limiti deb ataladi.

2.6-ta'rif. Agar $\lambda_p \to 0$ da $f(x)$ funksiyaning integral yig'indisi (2.1) chekli limitga ega bo'lsa, u holda $f(x)$ funksiya $[a, b]$ segmentda integrallanuvchi deyiladi, σ yig'indining chekli limiti J esa $f(x)$ funksiyaning [a,b] segmentdagi aniq integrali deb ataladi. Funksiyaning aniq integrali

$$\int_a^b f(x)dx$$

kabi belgilanadi.

Demak,

$$\int_a^b f(x)dx = \lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k \ .$$

Bunda a son integralning quyi chegarasi, $b$son esa integralning yuqori chegarasi,$[a, b]$segment integrallash oralig'i deb ataladi.

Agar $\lambda_p \to 0$ da yig'indining limiti mavjud bo'lmasa yoki uning limiti cheksiz bo'lsa, u holda funksiya $[a, b]$ segmentda integrallanmaydi deyiladi.

2.1–misol. $f(x) = C = const$ funksiyaning $[a,b]$ segmentdagi integrali hisoblansin.

◀ $[a,b]$ segmentni ixtiyoriy

$$P = \{x_0, x_1, \ldots, x_n\} \qquad (a = x_0 < x_1 < \ldots < x_n = b)$$

bo'laklashni olib, $f(x) = C$ funksiyaning integral yig'indisini topamiz:

$$\sigma = \sum_{k=0}^{n-1} C \cdot \Delta x_k = C\sum_{k=0}^{n-1} \Delta x_k = C[(x_1 - x_0) + (x_2 - x_1) + ... + (x_n - x_{n-1})] = C(x_n - x_0) = C(b-a)$$

Ravshanki,

$$\lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} C(b-a) = C(b-a).$$

Demak,

$$\int_a^b Cdx = C(b-a) \ . \ ▶$$

Xususan, $f(x) = 1$ bo'lganda quyidagiga egamiz:

$$\int_a^b 1 \cdot dx = \int_a^b dx = b - a \ .$$

2.2-misol**.** Ushbu $f(x) = x$ funksiyaning $[a,b]$ segmentdagi integrali hisoblansin.

◀ Ma'lumki, $[a,b]$ segmentda $f(x) = x$ funksiyaning integral yig'indisi

$$\sigma = \sum_{k=0}^{n-1} \xi_k \Delta x_k$$

bo'lib, bunda $\Delta x_k = x_{k+1} - x_k$ va

$$x_k \le \xi_k \le x_{k+1} .$$

Bu tengsizlikni $\Delta x_k > 0$ ga ko'paytirib topamiz:

$$x_k \cdot \Delta x_k \le \xi_k \cdot \Delta x_k \le x_{k+1} \cdot \Delta x_k \qquad (k = 0, 1, \ldots, n-1) .$$

Keyingi tengsizliklardan esa

$$\sum_{k=0}^{n-1} x_k \Delta x_k \le \sum_{k=0}^{n-1} \xi_k \Delta x_k \le \sum_{k=0}^{n-1} x_{k+1} \Delta x_k$$

tengsizliklar kelib chiqadi.

Demak,

$$\sum_{k=0}^{n-1} x_k \Delta x_k \le \sigma \le \sum_{k=0}^{n-1} x_{k+1} \Delta x_k .$$

Endi $\sum_{k=0}^{n-1} x_k \Delta x_k$ va $\sum_{k=0}^{n-1} x_{k+1} \Delta x_k$ yig'indilarni quyidagicha o'zgartirib yozib olamiz:

$$\sum_{k=0}^{n-1} x_k \Delta x_k = \sum_{k=0}^{n-1} x_k (x_{k+1} - x_k) = \frac{1}{2}\sum_{k=0}^{n-1} (x_{k+1}^2 - x_k^2) - \frac{1}{2}\sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 =$$

$$= \frac{1}{2}(x_n^2 - x_0^2) - \frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2 = \frac{b^2 - a^2}{2} - \frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2 .$$

Agar $x_{k+1} = x_k + \Delta x_k$ ekanini e'tiborga olsak, u holda

$$\sum_{k=0}^{n-1} x_{k+1} \Delta x_k = \sum_{k=0}^{n-1} (x_k + \Delta x_k) \Delta x_k = \sum_{k=0}^{n-1} x_k \Delta x_k + \sum_{k=0}^{n-1} \Delta x_k^2 = \frac{b^2 - a^2}{2} + \frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2 .$$

Demak,

$$\frac{b^2 - a^2}{2} - \frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2 \le \sigma \le \frac{b^2 - a^2}{2} + \frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2 .$$

Bu munosabatdan

$$\left| \sigma - \frac{b^2 - a^2}{2} \right| \le \frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2$$

tengsizlik kelib chiqadi. So'ngra $\frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2$ uchun

$$\frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k^2 \le \lambda_p \frac{1}{2}\sum_{k=0}^{n-1} \Delta x_k \le \frac{b-a}{2} \lambda_p$$

(bunda $\lambda_p = \max_k \{\Delta x_k\}$) bo'lishidan $\lambda_p \to 0$ da

$$\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 \to 0$$

bo'lishini topamiz.

Demak,

$$\lim_{\lambda_P\to 0}\sigma = \frac{b^2-a^2}{2}.$$

Bu esa ta'rifga ko'ra

$$\int_a^b x dx = \frac{b^2-a^2}{2}$$

ekanini bildiradi. ►

2.3—misol. $[a,b]$ segmentda Dirixle funksiyasi uchun aniq integral mavjud emasligi ko'rsatilsin.

◄ Dirixle funksiyasi $D(x)$ uchun integral yig'idini xususan quyidagicha bo'lishini ko'rgan edik:

$$\sigma = \begin{cases} b-a, & \text{agar barcha } \xi_k \text{ ratsional son bo'lsa,} \\ 0, & \text{agar barcha } \xi_k \text{ irratsional son bo'lsa.} \end{cases}$$

Ravshanki, $\lambda_P \to 0$ da σ yig'indi limitga ega emas. Demak, Dirixle funksiyasi $[a,b]$ segmentda integrallanmaydi. ►

Odatda, yuqorida keltirilgan aniq integral Riman integrali, integral yig'indini Riman yig'indisi deyiladi.

2.1-eslatma. Agar $f(x)$ funksiya $[a,b]$ segmentda chegaralanmagan bo'lsa, u shu segmentda integrallanmaydi.

2.3. Darbu yig'indilari.

$f(x)$ funksiya $[a,b]$ oraliqda aniqlangan bo'lib, shu oraliqda chegaralangan bo'lsin:

$$m \le f(x) \le M, \qquad x\in[a,b].$$

$[a,b]$ oraliqda biror

$$P = \{x_0, x_2, \dots, x_n\} \in F$$

$(a = x_0 < x_1 < \dots < x_n = b)$ bo'laklashni olaylik. Bu funksiyaning aniq chegaralari

$$m_k = \inf\{f(x)\}, \quad x \in [x_k, x_{k+1}],$$
$$M_k = \sup\{f(x)\}, \quad x \in [x_k, x_{k+1}]$$

mavjud .

Ravshanki, ixtiyoriy $\xi_k \in [x_k, x_{k+1}]$ uchun

$$m_k \le f(\xi_k) \le M_k$$

bo'ladi. Endi m_k va M_k sonlarni $[x_k, x_{k+1}]$ oraliqning uzunligi $\Delta x_k = x_{k+1} - x_k$ $(k = 0, \dots, n-1)$ ga ko'paytirib quyidagi

$$\sum_{k=0}^{n-1} m_k \Delta x_k = m_0 \Delta x_0 + m_1 \Delta x_1 + \dots + m_k \Delta x_k + \dots + m_{n-1} \Delta x_{n-1} ,$$
$$\sum_{k=0}^{n-1} M_k \Delta x_k = M_0 \Delta x_0 + M_1 \Delta x_1 + \dots + M_k \Delta x_k + \dots + M_{n-1} \Delta x_{n-1}$$

yig'inilarni tuzamiz.

2.7-ta'rif. Ushbu

$$s = \sum_{k=0}^{n-1} m_k \Delta x_k , \qquad S = \sum_{k=0}^{n-1} M_k \Delta x_k$$

yig'indilar mos ravishda Darbuning quyi hamda yuqori yig'indilari deb ataladi.

Darbu yig'indilari, funksiyaga hamda P bo'laklashga bog'liq:

$$s = s(P),\ S = S(P)$$

va har doim

$$s(P) \le S(P)$$

bo'ladi.

2. 4.Aniq integralning mavjudligi.

Aytaylik, $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan bo'lsin .

2.1-teorema . $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lishi uchun $\forall \varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topilib, $[a,b]$ oraliqni diametri $\lambda_P < \delta$ bo'lgan har qanday P bo'laklashi uchun Darbu yig'ndilari

$$S(P) - s(P) < \varepsilon$$

tengsizlikni qanoatlantirishi zarur va yetarli.

Agar avvalgidek $f(x)$ funksiyaning $[x_k, x_{k+1}]$ $(k=0,1,\dots,n-1)$ oraliqdagi tebranishini ω_k orqali belgilasak, u holda

$$S(P)-s(P)=\sum_{k=0}^{n-1}(M_k - m_k)\Delta x_k = \sum_{k=0}^{n-1}\omega_k \Delta x_k$$

bo'lib, yuqorida keltirilgan teorema quyidagicha ifodalanadi.

2.2-teorema. $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lishi uchun $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ son topilib, $[a,b]$ oraliqni diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'laklashda

$$\sum_{k=0}^{n-1}\omega_k \Delta x_k < \varepsilon \tag{2.3}$$

tengsizlikning bajarilisi zarur va yetarli.

Ravshanki, (2.3) munosabatni quyidagi

$$\lim_{\lambda_P\to 0}\sum_{k=0}^{n-1}\omega_k \Delta x_k = 0$$

ko'rinishda ham yozish mumkin.

2.5. Integrallanuvchi funksiyalar sinfi.

Aniq integralning mavjudligi haqidagi teoremadan foydalanib, bazi funksiyalarning sinfi integrallanuvchi bo'lishini ko'ramiz. $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan bo'lsin .

2.3-teorema. Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u shu oraliqda integrallanuvchi bo'ladi.

2.4-teorema. Agar $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan va monoton bo'lsa, funksiya shu oraliqda integrallanuvchi bo'ladi.

2.5-teorema. Agar $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan va bu oraliqning chekli sondagi nuqtalarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa , funksiya shu oraliqda integrallanuvchi bo'ladi.

Aniq integralning ta'rifi bo'yicha , ya'ni uni, integral yig'indining limiti sifatida qarab , ba'zi bir soda funksiyalarning integrallarini hisoblash mumkin. Lekin, ko'pincha har qanday uzluksiz funksiyaning aniq integralini, integral yig'indining limiti sifatida qarab, hisoblash ancha noqulayliklarga va qiyinchiliklarga olib keladi.

2.2-eslatma. $f(x)$ funksiya $[a, b]$ oraliqda integrallanuvchi bo'lsin. Biz

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

hamda

$$\int_a^a f(x)dx = 0$$

tengliklar o'rinli deb kelishib olamiz.

2.6. Aniq integralning xossalari.

Endi $f(x)$ funksiya aniq integralining xossalarini o'rganamiz.

1-xossa. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, u istalgan $[\alpha,\beta]\subset[a,b]$ oraliqda ham integrallanuvchi bo'ladi.

2-xossa . Agar $f(x)$ funksiya $[a,c]$ hamda $[c,b]$ oraliqlarda integrallanuvchi bo'lsa, u holda funksiya $[a,b]$ oraliqda ham integrallanuvchi va ushbu

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

formula o'rinli bo'ladi.

3-xossa. Agar $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lsa, u holda $cf(x)$ $(c = const)$ ham shu oraliqda integrallanuvchi bo'ladi va ushbu

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

formula o'rinli bo'ladi.

4-xossa. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi va $f(x) \geq d > 0$ bo'lsa, u holda $\frac{1}{f(x)}$ funksiya ham shu oraliqda integrallanuvchi bo'ladi.

5-xossa. Agar $f(x)$ va $g(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda $f(x)$ $\pm$ $g(x)$ funksiya ham shu oraliqda integrallanuvchi bo'ladi va ushbu

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

formula o'rinli bo'ladi.

6-xossa. $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda $f(x)g(x)$ funksiya ham shu oraliqda integrallanuvchi bo'ladi.

4) – va 5) – xossalardan quyidagi natija kelib chiqadi.

1-natija. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi va $g(x) \geq d > 0$ bo'lsa, u holda $\dfrac{f(x)}{g(x)}$ funksiya ham $[a,b]$ oraliqda integrallanuvchi bo'ladi.

7-xossa Agar $f(x)$ $[a,b]$ oraliqda integrallanuvchi bo'lib, $\forall x \in [a,b]$ lar uchun $f(x) \geq 0$ bo'lsa, u holda

$$\int_a^b f(x)dx \geq 0 \qquad\qquad (a < b)$$

bo'ladi.

2-natija. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lib, $\forall x \in [a,b]$ lar uchun $f(x) \leq g(x)$ tengsizlik o'rinli bo'lsa, u holda ushbu

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

tengsizlik ham o'rinli bo'ladi.

3-natija (Koshi–Bunyakovskiy tengsizligi). Agar $f(x)$ va $g(x)$ funksi-yalar $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda (yuqoridagi xossalarga ko'ra) ushbu $f(x) - \alpha\, g(x)$ (α – ixtiyoriy o'zgarmas) funksiya ham $[a,b]$ oraliqda integrallanuvchi va

$$\int_a^b [f(x)-\alpha\, g(x)]^2 dx \geq 0$$

tengsizlik o'rinli bo'ladi.

Demak, ixtiyoriy o'zgarmas α son uchun

$$\alpha^2 \int_a^b g^2(x)dx - 2\alpha \int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \geq 0$$

tensizlik o'rinli. Bu tengsizlikning chap tomonidagi ifoda α ga nisbatan kvadrat uchhad bo'lib, u α ning barcha haqiqiy qiymatlarida manfiy emas. Demak, bu kvadrat uchhadning diskriminanti musbat emas, ya'ni

$$\left[\int_a^b f(x)g(x)dx\right]^2 - \int_a^b f^2(x)dx \int_a^b g^2(x)dx \leq 0.$$

Natijada quyidagi

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f^2(x)dx}\sqrt{\int_a^b g^2(x)dx}$$

tengsizlikka kelamiz. Bu tengsizlik Koshi–Bunyakovskiy tengsizligi deb ataladi.

8-xossa. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda $|f(x)|$ funksiya ham shu oraliqda integrallanuvchi bo'ladi va

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

tengsizlik o'rinli bo'ladi.

2.8. O'rta qiymat haqidagi teoremalar.

$f(x)$ funksiya $[a,b]$ oraliqda aniqlangan va chegaralangan bo'lsin. U holda $[a,b]$ oraliqda

$$m = \inf\{f(x)\}, \quad M = \sup\{f(x)\}$$

mavjud va $\forall x \in [a,b]$ uchun

$$m \leq f(x) \leq M \qquad (2.4)$$

tengsizliklar o'rinli bo'ladi.

2.6-teorema. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda shunday o'zgarmas μ $(m \le \mu \le M)$ son mavjudki, ushbu

$$\int_a^b f(x)dx = \mu(b-a)$$

tenglik o'rinli bo'ladi.

◀ (2.4) tengsizliklardan foydalanib topamiz:

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$$

Bundan

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)\,.$$

Bu tengsizliklarni $b-a>0$ songa bo'lamiz:

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M\,.$$

Agar

$$\mu = \frac{\int_a^b f(x)dx}{b-a}$$

deb olsak, u holda izlangan tenglik kelib chiqadi. ▶

4-natija. Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u holda bu oraliqda shunday c $(c \in [a,b])$ nuqta topiladiki,

$$\int_a^b f(x)dx = f(c)(b-a)$$

tenglik o'rinli bo'ladi.

7-teorema. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integral-lanuvchi bo'lib, $g(x)$ funksiya shu oraliqda o'z ishorasini o'zgartirmasa, u holda shunday o'zgarmas μ $(m \le \mu \le M)$ son mavjudki,

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx \tag{2.5}$$

tenglik o'rinli bo'ladi.

◀ Aniq integrallning 5-xossasiga asosan $f(x)g(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'ladi. Endi $g(x)$ funksiya $[a,b]$ oraliqda manfiy bo'lmasin, ya'ni $\forall x\in[a,b]$ lar uchun $g(x)\geq 0$ bo'lsin deylik. U holda $m\leq f(x)\leq M$ tengsizliklarni $g(x)$ ga ko'paytirib, so'ngra hosil bo'lgan ushbu

$$mg(x)\leq f(x)g(x)\leq Mg(x)$$

tengsizliklarni $[a,b]$ oraliqda integrallab topamiz:

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx. \tag{2.6}$$

Ikki holni qaraylik:

a) $\int_a^b g(x)dx = 0$ bo'lsin. U holda

$$\int_a^b f(x)g(x)dx = 0$$

bo'lib, bunda μ deb $m\leq\mu\leq M$ tengsizliklarni qanoatlantiruvchi ixtiyoriy sonni olish mumkin.

b) $\int_a^b g(x)dx > 0$ bo'lsin. Bu holda (2.6) tengsizliklardan

$$m\leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}\leq M$$

bo'lishi kelib chiqadi.

$$\mu = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$$

deb olsak, unda

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$$

bo'ladi.

$[a,b]$ oraliqda $g(x)\le 0$ bo'lganda (2.5) formula huddi shunga o'xshash isbotlanadi

5-natija: Agar $[a,b]$ oraliqda $f(x)$ funksiya uzluksiz, $g(x)$ funksiya integrallanuvchi bo'lsa, hamda shu oraliqda $g(x)$ funksiya o'z ishorasini o'zgartirmasa, u holda shunday c $(c\in[a,b])$ nuqta topiladiki,

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

tenglik o'rinli bo'ladi.

2.9. Aniq integrallarni hisoblash

1.Nyuton-Leybnis formulasi.

Aytaylik, $f(x)$ funk-siya $[a,b]$ segmentda berilgan va shu segmentda uzluksiz bo‘lsin. U holda $f(x)$ boshlang‘ich funksiya

$$F(x) = \int_a^x f(t)dt$$

ga ega bo‘ladi.

Ravshanki, $\Phi(x)$ funksiya $f(x)$ ning ixtiyoriy boshlang‘ich funksiyasi bo‘lsa, u holda

$$\Phi(x) = F(x) + C \qquad (C = const)$$

bo‘ladi.

Bu tenglikda, avval $x = a$ deb $\Phi(a) = C$, so‘ngra $x = b$ deb

$$\Phi(b) = \int_a^b f(x)dx + C$$

bo‘lishini topamiz.

Demak,

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a). \qquad (2.7)$$

(2.10) formula Nyuton-Leybnis formulasi deyiladi.

Odatda, $\Phi(b) - \Phi(a)$ ayirma $\Phi(x)\Big|_a^b$ kabi yoziladi. Demak,

$$\int_a^b f(x)\,dx = \Phi(x)\Big|_a^b = \Phi(b) - \Phi(a).$$

Masalan,

$$\int_a^b \frac{1}{x}\,dx = \ln x\Big|_a^b = \ln b - \ln a = \ln\frac{b}{a}. \qquad (a > 0, b > 0)$$

2. O‘zgaruvchilarini almashtirish formulasi**.**

Faraz qilaylik, $f(x) \in C[a,b]$ bo‘lsin. Ravshanki, bu holda

$$\int_a^b f(x)\,dx$$

integral mavjud bo‘ladi.

Ayni paytda, bu funksiya $[a,b]$ da boshlang‘ich $\Phi(x)$ funksiyaga ega bo‘lib,

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a)$$

bo‘ladi.

Aytaylik, aniq integralda x o‘zgaruvchi ushbu

$$x = \phi(t)$$

formula bilan almashtirilgan bo‘lib, bunda $\phi(t)$ funksiya quyidagi shartlarni bajarsin:

1) $\phi(t) \in C[\alpha,\beta]$ bo‘lib, $\phi(t)$ funksiyaning barcha qiymatlari $[a,b]$ ga tegishli;

2) $\phi(\alpha) = a, \quad \phi(\beta) = b$;

3) $\phi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\phi'(t)$ hosilaga ega bo‘lsin.

U holda

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(t))\cdot\phi'(t)\,dt \qquad (2.8)$$

bo‘ladi.

◄ Ravshanki, $\Phi(\varphi(t))$ murakkab funksiya $[\alpha,\beta]$ segmentda uzluksiz bo‘lib,

$$(\Phi(\varphi(t)))' = \Phi'(\varphi(t))\cdot\varphi'(t)$$

bo‘ladi.

Agar $\Phi'(x) = f(x)$ ekanini e’tiborga olsak, unda

$$(\Phi(\varphi(t)))' = f(\varphi(t))\cdot\varphi'(t)$$

bo‘lishini topamiz. Bu esa $\Phi(\varphi(t))$ funksiya $[\alpha,\beta]$ da $f(\varphi(t))\cdot\varphi'(t)$ funksiyaning boshlang‘ich funksiyasi ekanini bildiradi. Nyuton-Leybnis formulasiga ko‘ra

$$\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt = \Phi(\varphi(\beta)) - \Phi(\varphi(\alpha)) = \Phi(b) - \Phi(a) \qquad (2.9)$$

bo‘ladi.

(2.8) va (2.9) munosabatlardan

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (2.10)$$

bo‘lishi kelib chiqadi. ►

(2.10) formula aniq integralda o‘zgaruvchini almashtirish formulasi deyiladi.

2.5.-misol. Ushbu

$$\int_0^1 \sqrt{1-x^2}\,dx$$

integral hisoblansin.

Yechilishi. Berilgan integralda $x = \sin t$ almashtirishni bajaramiz.

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t dt = \int_0^{\frac{\pi}{2}} \cos^2 t dt =$$

$$= \int_0^{\frac{\pi}{2}} (\frac{1}{2}+\frac{1}{2}\cos 2t)dt = (\frac{1}{2}t+\frac{1}{4}\sin 2t)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

3. Bo‘laklab integrallash formulasi.

Aytaylik , $u\,(x)$ va $v\,(x)$ funksiyalarning har biri $[a,b]$ segmentda uzluksiz $u'(x)$ va $v'\,(x)$ hosilalarga ega bo’lsin. U holda

$$\int_a^b u\,(x)\,d\,v\,(x) = (u\,(x)\cdot v\,(x))\Big|_a^b - \int_a^b v(x)\,d\,u\,(x) \qquad (2.11)$$

bo‘ladi.

(2.11) formula aniq integrallarda bo‘laklab integrallash formulasi deyiladi.

2.6-misol. Ushbu

$$\int_1^2 x\ln x\,d\,x$$

integral hisoblansin.

Yechilishi. Bu intervalda $u\left(x\right) = \ln x,\ d\,v\left(x\right) = x$ deb $d\,u\left(x\right) = \frac{1}{x}\,d\,x,\ \ v\left(x\right) = \frac{x^2}{2}$ bo‘lishini topamiz. Unda (2.11) formulaga ko‘ra:

$$\int_1^2 x\ln x d\,x = \left(\frac{x^2}{2}\ln x\right)\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\,d\,x = 2\ln 2 - \frac{1}{2}\int_1^2 x d\,x = 2\ln 2 - \frac{3}{4}$$

bo‘ladi.

2.7-misol. Ushbu

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x dx \qquad (n = 0,1,2,...)$$

integral hisoblansin.

◄ Ravshanki,

$$J_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}\qquad,\qquad J_1=\int_0^{\frac{\pi}{2}}\sin x dx=(-\cos x)\Big|_0^{\frac{\pi}{2}}=1.$$

$n\geq 2$ bo‘lganda berilgan integralni

$$J_n=\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,d(-\cos x)$$

ko‘rinishida yozib, unga bo‘laklab integrallash formulasini qo‘llaymiz. Natijada

$$J_n=(-\sin^{n-1}x\cdot\cos x)\Big|_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 xdx=$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)dx=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}xdx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n xdx=$$

$$=(n-1)J_{n-2}-(n-1)J_n$$

bo‘lib, undan ushbu

$$J_n=\frac{n-1}{n}J_{n-2}$$

rekurrent formula kelib chiqadi.

Bu formula yordamida berilgan integralni $n=1,2,3,........$ bo‘lganda ketma-ket hisoblash mumkin.

Aytaylik, $n=2m$- juft son bo‘lsin. Unda

$$J_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}......\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot J_0=\frac{(2m-1)!!}{(2m)!!}\cdot\frac{\pi}{2}$$

bo‘ladi.

Aytaylik, $n=2m+1$- toq son bo‘lsin. Unda

$$J_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}......\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot J_1=\frac{(2m)!!}{(2m+1)!!}$$

bo‘ladi. ($m!!$ simvol m dan katta bo‘lmagan va u bilan bir xil juftlikka ega bo‘lgan natural sonlarning ko‘paytmasini bildiradi.) ►

2.8 – m i s o l . Ushbu

$$\int_0^{10\pi} \sqrt{1-\cos 2x}\,dx$$

hisoblansin.

◀Ma'lumki,

$$\sqrt{1-\cos 2x} = \sqrt{2\sin^2 x} = \sqrt{2}\cdot|\sin x|$$

Unda

$$\int_0^{10\pi} \sqrt{1-\cos 2x}\,dx = \sqrt{2}\int_0^{10\pi} |\sin x|\,dx$$

bo'ladi. Aniq integralning xossalaridan foydalanib topamiz:

$$\int_0^{10\pi} |\sin x|dx = \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx + \int_{2\pi}^{3\pi} \sin dx - \int_{3\pi}^{4\pi} \sin x dx + \cdots + \int_{8\pi}^{9\pi} \sin x dx - \int_{9\pi}^{10\pi} \sin x dx =$$
$$= \underbrace{2+2+\cdots+2}_{10ma} = 20.$$

Demak,

$$\int_0^{10\pi} \sqrt{1-\cos 2x}\,dx = 20\cdot\sqrt{2}\,. ▶$$

2.9– m i s o l . Ushbu

$$\int_0^{a} x^2\sqrt{a^2-x^2}\,dx$$

integral hisoblansin.

◀Bu integralni o'zgaruvchini almashtirish usulidan foydalanib hisoblaymiz:

$$\int_0^{a} x^2\sqrt{a^2-x^2}\,dx = \left[x = a\sin t\,,\ dx = a\cos t dt\,,\ t\in\left[0,\frac{\pi}{2}\right]\right] = a^4\int_0^{\frac{\pi}{2}} \sin^2 t\cdot\cos^2 t dt =$$

$$= \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t dt = \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4t}{2} dt = \frac{a^4}{8}\left(t - \frac{\sin 4t}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{a^4}{16}\pi ▶$$

2.10 – m i s o l . Ushbu

$$I = \int_1^{e} (x\ln x)^2\,dx$$

integral hisoblansin.

◀Bu integralni bo'laklab integrallash formulasidan foydalanib hisoblaymiz:

$$I=\int_1^e (x\ln x)^2 dx=\left[u=\ln^2 x\,,\ du=2\ln x\cdot\frac{1}{x}dx\,,\ dv=x^2dx\,,\ v=\frac{x^3}{3}\right]=$$
$$=\frac{x^3}{3}\ln^2 x\Big|_1^e-\frac{2}{3}\int_1^e x^2\ln x dx=\frac{e^3}{3}-\frac{2}{3}\int_1^e x^2\ln x dx$$

Keyingi integral ham bo‘laklab integrallash usuli yordamida hisoblanadi:

$$\int_1^e x^2\ln x dx=\left[u=\ln x\,,\ du=\frac{1}{x}dx\,,\ dv=x^2dx\,,\ v=\frac{x^3}{3}\right]=$$
$$=\frac{x^3}{3}\ln x\Big|_1^e-\frac{1}{3}\int_1^e x^2dx=\frac{e^3}{3}-\frac{x^3}{9}\Bigg|_1^e=\frac{e^3}{3}-\frac{e^3}{9}+\frac{1}{9}=\frac{2}{9}e^3+\frac{1}{9}$$

Demak,

$$I=\frac{e^3}{3}-\frac{2}{3}\left(\frac{2}{9}e^3+\frac{1}{9}\right)=\frac{e^3}{3}-\frac{4}{27}e^3-\frac{2}{27}=\frac{1}{27}\left(5e^3-2\right).\blacktriangleright$$

2.12- misol. Ushbu

$$\int_2^3\frac{dx}{x^2-2x-8}$$

Integralni hisoblansin.

► Bu integralni ko’paytuvchilarga ajratish yo’li bilan hisoblaymiz.

$$\int_2^3\frac{dx}{x^2-2x-8}=\int_2^3\frac{dx}{(x-4)(x+2)}=\frac{1}{6}\int_2^3(\frac{1}{x-4}-\frac{1}{x+2})dx=\frac{1}{6}\int_2^3\frac{dx}{x-4}-\frac{1}{6}\int_2^3\frac{dx}{x+2}$$

$$=\frac{1}{6}\ln|x-3|\Bigg|_2^3-\frac{1}{6}\ln|x+2|\Bigg|_2^3=\frac{\ln|-1|}{6}-\frac{\ln|-2|}{6}-\frac{\ln|5|}{6}+\frac{\ln|4|}{6}=\frac{1}{6}(\ln\frac{4}{10});$$

3. ANIQ INTEGRAL YORDAMIDA TEKIS SHAKLNING YUZINI HISOBLASH.

3.1. Tekis shaklning yuzi tushunchasi.

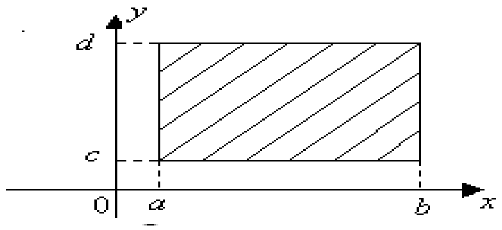
Ma’lumki, (x, y) juftlik, $(x \in R, y \in R)$tekislikda nuqtani ifodalaydi. Koordinatalari ushbu

$$a \leq x \leq b, \quad c \leq y \leq d \qquad (a \in R, b \in R, c \in R, d \in R)$$

tengsizliklarni qanoatlantiruvchi tekislik nuqtalaridan hosil bo‘lgan D_0 to‘plam :

$$D_0 = \{(x, y); x \in [a, b], y \in [a, b]\}$$

1. to‘g‘ri to‘rtburchak deyiladi (3.1-chizma)



3.1-chizma

Bu to‘g‘ri to‘rtburchakning tomonlari (chegaralari) mos ravishda koordinatalar o‘qiga parallel bo‘ladi.

D_0 to‘g‘ri to‘rtburchakning yuzi deb uning chegarasining, ya’ni

$$x = a, \; y = c \qquad (c \leq y \leq d)$$

$$x = b, \; y = d \qquad (a \leq x \leq b)$$

to‘g‘ri chiziq kesmalarining D_0 ga tegishli bo‘lishi yoki tegishli bo‘lmasligidan qat’i nazar) ushbu

$$\mu(D_0) = (b - a)(d - c)$$

miqdorga aytiladi.

Aytaylik, tekislik nuqtalaridan iborat biror Q to‘plam berilgan bo‘lsin.

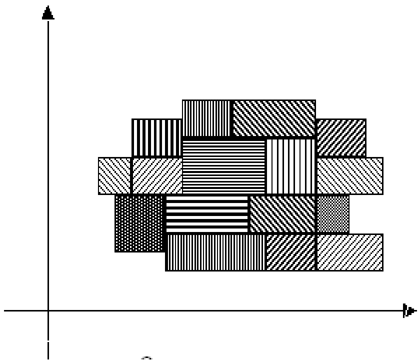
Agar shunday D_0 to‘g‘ri to‘rtburchak topilsaki,

$$Q \subset D_0$$

bo‘lsa, Q chegaralangan to‘plam deyiladi.

Har qanday chegaralangan tekislik nuqtalaridan iborat to‘plam tekis shakl deyiladi.

Agar tekis shakl chekli sondagi kesishmaydigan to‘g‘ri to‘rtburchaklarning birlashmasi sifatida ifodalansa, uni to‘g‘ri ko‘pburchak deymiz.(3.2-chizma)



3.2-chizma

Bunday to‘g‘ri ko‘pburchakning yuzi deb, uni tashkil etgan to‘g‘ri to‘rtburchaklar yuzalari yig‘indisiga aytiladi.

To‘g‘ri ko‘pburchak yuzi quyidagi xossalarga ega:

1) To‘g‘ri ko‘pburchak yuzi har doim manfiy bo‘lmaydi: $\mu(D) \geq 0$

2) Kesishmaydigan ikki D_1 va D_2 to‘g‘ri ko‘pburchaklardan tashkil topgan to‘g‘ri ko‘pburchak yuzi D_1 va D_2 larning yuzalari yig‘indisiga teng:

$$\mu(D_1 \cup D_2 = \mu(D_1) + \mu(D_2);$$

3) Agar D_1 va D_2 to‘g‘ri ko‘pburchaklar uchun

$$D_1 \complement D_2$$

bo‘lsa, u holda

$$\mu(D_1) \leq \mu(D_2)$$

bo‘ladi.

Tekislikda biror chegaralangan Q shakl berilgan bo‘lsin. Bu shaklning ichiga A to‘g‘ri ko‘pburchak $(A\complement Q)$ so‘ngra Q shaklni o‘z ichiga olgan B to‘g‘ri ko‘pburchak $(Q\complement B)$ lar chizamiz. Ularning yuzlari mos ravishda $\mu(A)$ va $\mu(B)$ bo‘lsin.

Ravshanki, bunday to‘g‘ri ko‘pburchaklar ko‘p bo‘lib, ularning yuzalaridan iborat$\{\ \mu(A)\}$ va $\{\mu(A)\}$ to‘plamlar hosil bo‘ladi.

Ayni paytda, bu sonli to‘plamlar chegaralangan bo‘ladi. Binobarin, ularning aniq chegaralari

$$\text{Sup}\{\ \mu(A)\},\ \ \inf\{\ \mu(A)\}$$

lar mavjud.

3.1-ta'rif. Agar

$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo'lsa, Q shakl yuzaga ega deyiladi. Ularning umumiy qiymati Q shaklning yuzi deyiladi va $\mu(Q)$ kabi belgilanadi:

$$\mu(Q) = \sup\{\mu(A)\} = \inf\{\mu(B)\}$$

3.1-teorema. Tekis shakl Q yuzaga ega bo'lish uchun $\forall\varepsilon > 0$ son olinganda ham shunday $A\ (Q \subset A)$ va $B\ (Q \subset B)$ to'g'ri ko'pburchaklar topilib, ular uchun

$$\mu(A) - \mu(B) < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

◄ **Zarurligi.** Aytaylik, Q shakl yuzaga ega bo'lsin. Unda ta'rifga binoan

$$\sup\{\mu(A)\} = \inf\{\mu(B)\} = \mu(Q)$$

bo'ladi.

Modomiki,

$$\sup\{\mu(A)\} = \mu(Q),$$
$$\inf\{\mu(B)\} = \mu(Q)$$

ekan, unda $\forall\varepsilon > 0$ olinganda ham shunday to'g'ri ko'pburchak $A(A \subset Q)$ hamda shunday to'g'ri ko'pburchak $B(Q \subset B)$ topiladiki,

$$\mu(Q) - \mu(A) < \frac{\varepsilon}{2},$$

$$\mu(B) - \mu(Q) < \frac{\varepsilon}{2}$$

bo'ladi. Bu tengsizliklardan

$$\mu(B) - \mu(A) < \varepsilon$$

bo'lishi kelib chiqadi.

Yetarliligi. Aytaylik, $A\ (A \subset Q)$ va $B\ (Q \subset B)$ to'g'ri ko'pburchaklar uchun $\mu(B) - \mu(A) < \varepsilon$ tengsizligi bajarilsin.

Ravshanki,

$$\mu(A) \le \sup\{\mu(A)\},$$
$$\mu(B) \ge \inf\{\mu(B)\}.$$

Bu munosabatlardan

$$\inf\{\mu(B)\} - \sup\{\mu(A)\} \le \mu(B) - \mu(A) < \varepsilon$$

bo‘lishini topamiz.

ε -ixtiyoriy musbat son bo‘lganligidan

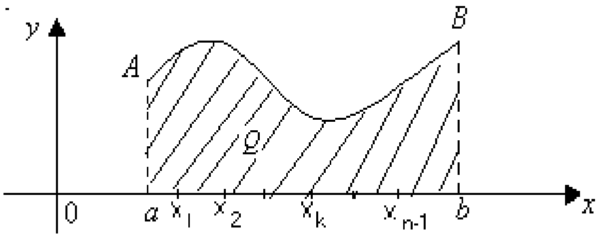
$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo‘lishi kelib chiqadi. Demak, Q shakl yuzaga ega. ►

3.2. Egri chiziqli trapetsiyaning yuzini hisoblash.

Faraz qilaylik, $f(x) \in C[a,b]$ bo‘lib, $\forall x \in [a,b]$ da $f(x) \ge 0$ bo‘lsin.

Yuqoridan $f(x)$ funksiya grafigi, yon tomonlardan $x = a, x = b$ vertikal chiziqlar hamda pastdan absissa o‘qi bilan chegaralangan Q shaklni qaraylik. (3.3-chizma)



3.3-chizma

Odatda, bu shakl egri chiziqli trapetsiya deyiladi. $[a,b]$ segmentni ixtiyoriy

$$P = \{x_0, x_1, x_2, ..., x_n\} \quad (a = x_0 < x_1 < x_2 < ... < x_n = b)$$

bo‘laklashni olamiz. Bu bo‘laklashning har bir $[x_k, x_{k+1}]$ oralig‘ida

$$\inf\{f(x)\} = m_k, \quad \sup\{f(x)\} = M_k \quad (k = 0,1,2,...,n-1)$$

mavjud bo‘ladi.

Endi asosi $\Delta x_k = x_{k+1} - x_k$, balandligi m_k bo‘lgan $(k = 0,1,2,...,n-1)$ to‘g‘ri to‘rtburchaklarning birlashmasidan tashkil topgan to‘g‘ri ko‘pburchakni A deylik.

Shuningdek, asosi $\Delta x_k = x_{k+1} - x_k$, balandligi M_k bo‘lgan $(k = 0,1,2,...,n-1)$ to‘g‘ri to‘rtburchaklarning birlashmasidan tashkil topgan to‘g‘ri ko‘pburchakni B deylik. Ravshanki,

$$A \subset Q \ , \quad Q \subset B$$

bo‘lib, ularning yuzalari

$$\mu(A)=\sum_{k=0}^{n-1} m_k\cdot\Delta x_k,\ \ \mu(B)=\sum_{k=0}^{n-1} M_k\cdot\Delta x_k$$

bo‘ladi.

Bu yig‘indilarni $f(x)$ funksiyaning $[a,b]$ segmentining P bo‘laklashiga nisbatan Darbuning quyi hamda yuqori yig‘indilari ekanini payqash qiyin emas:

$$\mu(A)=s(f;P)\quad,\qquad \mu(B)=S(f;P).$$

$f(x)\in C[a,b]$ bo‘lgani uchun $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo‘ladi. Unda integrallanuvchilik mezoniga ko‘ra, $\forall\varepsilon>0$ olinganda ham $[a,b]$ segmentning shunday P bo‘laklashi topiladiki,

$$S(f;P)-s(f;P)<\varepsilon$$

bo‘ladi. Birobarin, ushbu

$$\mu(B)-\mu(A)<\varepsilon$$

tengsizlik bajariladi. Bu esa, 3.1-teoremaga muvofiq, qaralayotgan egri chiziqli trapetsiyaning yuzaga ega bo‘lishini bildiradi. Unda ta’rifga ko‘ra

$$\sup\{\mu(A)\}=\inf\{\mu(B)\}$$

bo‘ladi.

Ayni paytda,

$$\sup\{\mu(A)\}=\underline{\int_a^b} f(x)dx,$$

$$\inf\{\mu(B)\}=\overline{\int_a^b} f(x)dx$$

bo‘lganligi sababli Q egri chiziqli trapetsiyaning yuzi

$$\mu(Q)=\int_a^b f(x)dx \tag{3.1}$$

ga teng bo‘ladi.

3.1-misol**.** Tekislikda ushbu

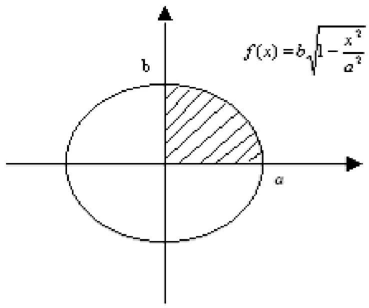
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ellips bilan chegaralangan Q shaklning yuzi topilsin.

◀Ellips bilan chegaralangan Q shaklning yuzi OX va OY koordinata o‘qlari hamda

$$f(x)=b\cdot\sqrt{1-\frac{x^2}{a^2}} \qquad , \qquad 0\le x\le a$$

chiziqlar bilan chegaralangan egri chiziqli trapetsiya yuzining 4 tasiga teng bo‘ladi. (3.4-chizma).



3.4-chizma

Unda (3.1) formuladan foydalanib topamiz:

$$\mu(Q)=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=$$

$$=\left|\begin{matrix} x=a\sin t\ , & \\ & 0\le t\le\frac{\pi}{2} \\ dx=a\cos t dt\ , & \end{matrix}\right|=$$

$$=\frac{4b}{a}\cdot a^2\int_0^{\frac{\pi}{2}}\cos^2 tdt=4ab\cdot\frac{\pi}{4}=ab\pi. ▶$$

Aytaylik, $f_1(x)\in C[a,b]$, $f_2(x)\in C[a,b]$ bo‘lib, $\forall x\in[a,b]$ da

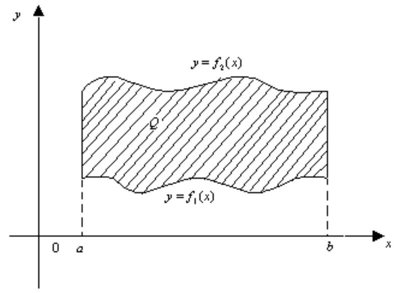
$$0\le f_1(x)\le f_2(x)$$

bo‘lsin.

Tekislikdagi Q shakl quyidagi

$$y=f_1(x) \quad , \quad y=f_2(x) \qquad , \qquad x=a \quad , \quad x=b$$

chiziqlar bilan chegaralangan shaklni ifodalasin (3.5-chizma)



3.5-chizma

Bu shaklning yuzi

$$\mu(Q)=\int_a^b f_2(x)dx-\int_a^b f_1(x)dx=\int_a^b[f_2(x)-f_1(x)]dx \qquad (3.2)$$

bo‘ladi.

3.2-misol. Tekislikda ushbu

$$y=4-x^2 \quad , \qquad y=x^2-2x$$

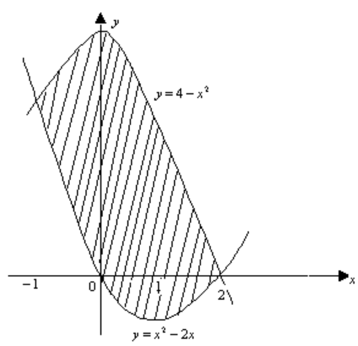
chiziqlar (parabolalar) bilan chegaralangan Q shaklning yuzi topilsin.

◀ Parabolalarning tenglamalari

$$y=4-x^2 \quad ,$$
$$y=x^2-2x$$

ni birgalikda yechib, ularning kesishish nuqtalarini topamiz:

$$4-x^2=x^2-2x \ ,$$
$$x_1=-1\ ,\ x_2=2\ ;\ y_1=3\ ,\ y_2=0:\ \ A(-1;3)\ ,\ B(2;0).$$



3.6-chizma

Bu shaklning yuzini (3.2) formuladan foydalanib hisoblaymiz:

$$\mu(Q)=\int_{-1}^{2}[(4-x^2)-(x^2-2x)]dx=\int_{-1}^{2}(4+2x-2x^2)dx=(4x+x^2-\frac{2}{3}x^3)\Big|_{-1}^{2}=9. \ ▶$$

Eslatma. Agar $f(x) \in C[a,b]$ funksiya $[a,b]$ da ishora saqlamasa, (3.1) integral egri chiziqli trapetsiyalar yuzalarining yig‘indisidan iborat bo‘ladi. Bunda OX o‘qining yuqoridagi yuza musbat ishora bilan, OX o‘qining pastdagi yuza manfiy ishora bilan olinadi.

Masalan, OX o‘qi hamda $f(x) = \sin x$, $0 \le x \le 2\pi$ funksiya grafigi bilan chegaralangan shaklning yuzi

$$\mu(Q) = \int_0^{\pi} \sin x + (-\int_{\pi}^{2\pi} \sin x dx) = (-\cos x)\Big|_0^{\pi} - (-\cos x)\Big|_{\pi}^{2\pi} = 4$$

bo‘ladi.

3.3. Egri chiziqli sektorning yuzini hisoblash.

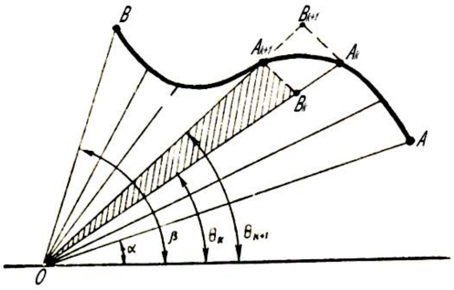
Aytaylik, $A\breve{B}$ egri chiziq qutb koordinatalar sistemasida ushbu

$$\rho = \rho(\theta) \quad , \quad \alpha \le \theta \le \beta \quad (\alpha \in R, \beta \in R)$$

tenglama bilan berilgan bo‘lsin. Bunda

$$\rho(\theta) \in C[\alpha, \beta] \ , \quad \forall \theta \in [\alpha, \beta] \quad \text{да} \qquad \rho(\theta) \ge 0 \ .$$

Tekislikda $A\breve{B}$ egri chiziq hamda OA va OB radius-vektorlar bilan chegaralangan Q shaklni qaraymiz. (3.7 -chizma).



3.7- chizma

$[\alpha, \beta]$ segmentni ixtiyoriy

$$P = \{\theta_0, \theta_1, ..., \theta_n\} \qquad (\alpha = \theta_0 < \theta_1 < ... < \theta_n = \beta)$$

bo‘laklashini olamiz. O nuqtadan har bir qutb burchagi θ_k ga mos OA_k radius-vektor o‘tkazamiz. Natijada OAB -egri chiziqli sektor

$$OA_k A_{k+1} \left(k = 0,1,2,...,n-1 \quad ; \ A_0 = A \ , \ A_n = B\right)$$

egri chiziqli sektorchalarga ajraladi.

Ravshanki, $\rho = \rho(\theta) \in C[\alpha, \beta]$

bo‘lganligi uchun $[\theta_k \ , \theta_{k+1}]$ da $(k = 0,1,2,..., n-1)$

$$m_k = \inf\{\rho(\theta)\} \ \ , \quad M_k = \sup\{\rho(\theta)\}$$

lar mavjud.

Endi har bir $[\theta_k, \theta_{k+1}]$ segment uchun radius-vektorlari mos ravishda m_k hamda M_k bo‘lgan doiraviy sektorlarni hosil qilamiz. Bunday doiraviy sektorlar yuzaga ega bo‘lib, ularning yuzi mos ravishda

$$\frac{1}{2} m_k^2 \cdot \Delta\theta_k \quad , \ \frac{1}{2} M_k^{2} \cdot \Delta\theta_k \qquad (\Delta\theta_k = \theta_{k+1} - \theta_k)$$

bo‘ladi.

Radius-vektorlari $m_k \quad (k = 0,1,2,..., n-1)$ bo‘lgan barcha doiraviy sektorlar birlashmasidan hosil bo‘lgan shaklni Q_1 desak, unda $Q_1 \subset Q$ bo‘lib, uning yuzi

$$\mu(Q_1) = \frac{1}{2} \sum_{k=0}^{n-1} m_k^2 \cdot \Delta\theta_k \tag{3.3}$$

bo‘ladi.

SHuningdek, radius-vektorlari $M_k \quad (k = 0,1,2,..., n-1)$ bo‘lgan barcha doiraviy sektorlar birlashmasidan hosil bo‘lgan shaklni Q_2 desak, unda $Q \subset Q_2$ bo‘lib, uning yuzi

$$\mu(Q_2) = \frac{1}{2} \sum_{k=0}^{n-1} M_k^2 \cdot \Delta\theta_k \tag{3.4}$$

bo‘ladi.

(3.3) va (3.4) yig‘indilar $\frac{1}{2}\rho^2(\theta)$ funksiyaning Darbu yig‘indilari bo‘ladi. Ayni paytda, $\frac{1}{2}\rho^2(\theta)$ funksiya $[\alpha, \beta]$ da uzluksiz bo‘lgani uchun u integrallanuvchidir. Demak, $\forall \varepsilon > 0$ olinganda ham $[\alpha, \beta]$ segmentning shunday P bo‘laklashi topiladiki,

$$S(\frac{1}{2}\rho^2(\theta); P) - s(\frac{1}{2}\rho^2(\theta); P) < \varepsilon$$

bo‘ladi. Binobarin, ushbu

$$\mu(Q_2) - \mu(Q_1) < \varepsilon$$

tengsizlik bajariladi. Bu esa, 3.2-teoremaga muvofiq, qaralayotgan egri chiziqli sektorning yuzaga ega bo‘lishini bildiradi. Unda ta’rifga ko‘ra

$$\sup\{\mu(Q_1)\} = \inf\{\mu(Q_2)\}$$

bo‘ladi. Ayni paytda,

$$\sup\{\mu(Q_1)\} = \underline{\int_{\alpha}^{\beta}} \rho^2(\theta)d\theta, \quad \inf\{\mu(Q_2)\} = \overline{\int_{\alpha}^{\beta}} \rho^2(\theta)d\theta$$

bo‘lgani sababli Q egri chiziqli sektorning yuzi

$$\mu(Q) = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\theta)d\theta$$

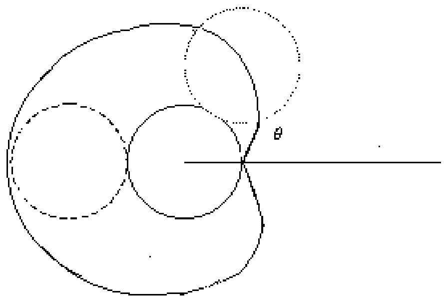
ga teng bo‘ladi.

3.3-misol**.** Ushbu

$$\rho = \rho(\theta) = a(1 - \cos\theta) \quad (a \in R,\ 0 \le \theta \le 2\pi)$$

funksiya grafigi bilan chegaralangan shaklning yuzi topilsin.

◄ Bu funksiya grafigi kardioidani ifodalaydi. Ma’lumki, kardioida radiusi r ga teng bo‘lgan aylananing shu radiusli ikkinchi qo‘zg‘almas aylana bo‘ylab harakati (sirpanmasdan dumalashi) natijasida birinchi aylana ixtiyoriy nuqtasining chizgan chizig‘idir. (3.8-chizma).



3.8-chizma

Kardioida qutb o‘qiga nisbatan simmetrik bo‘lganligi sababli yuqori yarim tekislikdagi shaklning yuzini topib, so‘ngra uni 2 ga ko‘paytirsak, izlanayotgan yuza kelib chiqadi.

θ o‘zgaruvchi $[0, \pi]$ da o‘zgarganda ρ radius vektor kardioidaning yuqori yarim tekislikdagi qismini chizadi. SHuning uchun

$$\mu(Q) = 2\cdot\frac{1}{2}\int_0^{\pi}\rho^2(\theta)d\theta = \int_0^{\pi}a^2(1-\cos\theta)^2d\theta =$$

$$= a^2\int_0^{\pi}\left[\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right]d\theta =$$

$$= a^2\left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi} = \frac{3}{2}\pi a^2$$

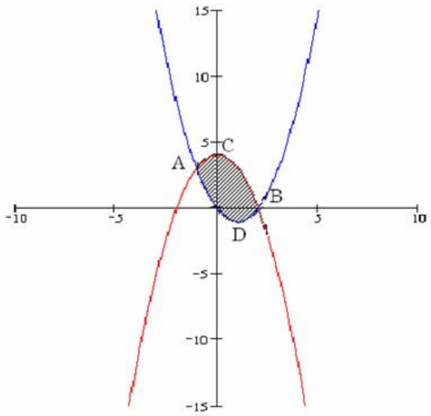
bo‘ladi. ►

3.4 – misol. Ushbu $y = 4\text{-}x^2$ va $y = x^2\text{-}2x$

egri chiziqlar (parabolalar) bilan chegaralangan shakilni yuzi topilsin.

◄ Parabolalar $A(-1;3)$ va $B(2;0)$ nuqtalarda kesishadi. Izlanayotgan shaklning yuzasi

$S = S_{A_1ACB} + S_{OBD} - S_{A_1AO}$ bo’ladi. (3.9-chizma)



3.9-chizma.

Ravshanki,

$$S_{A_1ACB} = \int_{-1}^{2}(4 - x^2)dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-1}^{2} = 9,$$

$$S_{OBD} = \int_{2}^{0}(x^2 - 2x)dx = \left(\frac{x^3}{3} - x^2\right)\Big|_{2}^{0} = \frac{4}{3},$$

$$S_{A_1AO} = \int_{-1}^{0}(x^2 - 2x)dx = \left(\frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} = \frac{4}{3}.$$

Demak, $S = 9 + \frac{4}{3} - \frac{4}{3} = 9$.►

3.5-misol . $\begin{cases} x = acost \\ y = bsint \end{cases}$ $(0 \le t \le 2\pi)$ ellipsni yuzi topilsin.

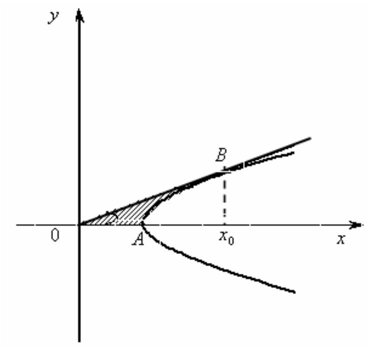
◀ Ellips koordinata o'qlariga nisbatan simmetrik bo'lganligi sababli, uning birinchi chorakdagi qismining yuzasini topish yetarli.

Endi $x = acost = 0$, $x = acost = a$ deb, t o'zgaruvchining $t_1 = \frac{\pi}{2}$ dan $t_2 = 0$ gacha o'zgaruvchini topamiz. Demak,

$$\frac{1}{4}S = \int_{\frac{\pi}{2}}^{0} ab\sin t(-\sin t)dt = ab\int_{0}^{\frac{\pi}{2}} \sin^2 tdt = \frac{\pi ab}{4}$$

bo'lib, $S = \pi ab$ bo'ladi.▶

3.5-misol . Tenglamaning $x^2\text{-}y^2 = a^2$ bo'lgan giperboladan $M(x_0, y_0)$ nuqta



3.10-chizma.

olingan. 3.10-chizmada ko'rsatilgan egri chiziqli OAB uchburchak yuzi topilsin.

◀ $x = rcos\varphi$, $y = rsin\varphi$ deb qutb koordinatalar sistemasida berilgan giperbolaning tenglamasini tuzamiz:

$$x^2\text{-}y^2 = r^2cos^2\varphi\text{-}r^2sin^2\varphi = r^2cos^2 2\varphi .$$

Demak, $r^2 = \dfrac{a^2}{cos2\varphi}$

Unda izlanaytgan shaklning yuzasi

$$S = \frac{1}{2}\int_0^a r^2(\varphi)d\varphi = \frac{a^2}{2}\int_0^a \frac{d\varphi}{\cos 2\varphi} = \frac{a^2}{4}\cdot \ln\frac{(x_0 + y_0)^2}{a^2} = \frac{a^2}{2}\ln\frac{x_0 + y_0}{a} . ▶$$

4. ANIQ INTEGRAL YORDAMIDA YOY UZUNLIGI VA AYLANMA JISM YUZINI HISOBLASH.

4.1. Yoy uzunligi tushunchasi.

Ma'lumki, tekislikdagi ikki $A(x_1, x_2)$ va $B(x_1, x_2)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi l_0 uzunlikka ega va uning uzunligi

$$\mu(l_0) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

ga teng bo'ladi.

Aytaylik, tekislikdagi l chiziq $A_0(x_0, y_0)$,$A_1(x_1, y_1)$,…,$A_n(x_n, y_n)$ nuqtalarni $(n \in N)$ birin-ketin to'g'ri chiziq kesmalari bilan birlashtirishidan hosil bo'lgan bo'lsin. Odatda, bunday chiziq siniq chiziq deyiladi.

Siniq chiziq uzunligi (perimetri) deb, uni tashkil etgan to'g'ri chiziq kesmalari uzunliklarining yig'indisiga aytiladi.

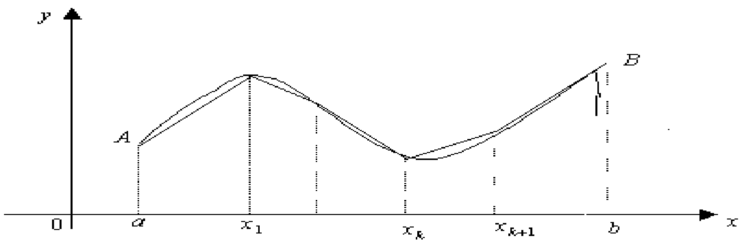
$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$

Faraz qilaylik, tekislikdagi $\breve{AB}$ egri chizig'i (uni $\breve{AB}$ yoyi deb ham ataymiz) ushbu

$$y=f(x) \qquad a \le x \le b$$

tenglama bilan berilgan bo'lsin.

Bunda f(x)∈ $C[a, b]$.



4.1-chizma

$[a, b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \qquad (a = x_0 < x_1 < ... < x_n = b$$

bo'laklashni olib, bo'luvchi x_k $(k = 0,1,2,...,n)$ nuqtalar orqali OY o'qiga parallel to'g'ri chiziqlar o'tkazamiz. Bu to'g'ri chiziqlarning $A\breve{B}$ yoyi bilan kesishgan nuqtalari

$$A_k(x_k, f(x_k)) \qquad (k = 0,1,2,...,n \ ; \quad A_0 = A \ , \ A_n = B)$$

bo‘ladi.

$\breve{AB}$ yoyidagi bu $A_k(x_k, f(x_k))$ nuqtalarni bir-biri bilan to‘g‘ri chiziq kesmalari yordamida birlashtirib, l siniq chiziqni hosil qilamiz. (4.2-chizma)

Odatda, l siniq chiziq $\breve{AB}$ yoyiga chizilgan siniq chiziq deyiladi. U uzunlikka ega bo‘lib, uzunligini (perimetrini) $\mu(l)$ deylik.

Agar P_1 va P_2 lar $[a,b]$ segmentning ikkita bo‘laklashi bo‘lib, $P_1 \subset P_2$ bo‘lsa, u holda bu bo‘laklashlarga mos $\breve{AB}$ yoyiga chizilgan siniq chiziq l_1 , l_2 larning perimetrlari uchun

$$\mu(l_1) \leq \mu(l_2)$$

bo‘ladi.

Demak, P bo‘laklashning bo‘luvchi nuqtalari sonini orttirib borilsa, $\breve{AB}$ yoyiga chizilgan ularga mos siniq chiziqlar perimetrlari ham ortib boradi.

4.1-ta’rif. Agar $\lambda_p \to 0$ da $\breve{AB}$ yoyiga chizilgan siniq chiziq perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + \left[f(x_{k+1}) - f(x_k)\right]^2}$$

chekli limitga ega bo‘lsa, $\breve{AB}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_P \to 0} \mu(l) = \mu(\breve{AB})$$

Limit $\breve{AB}$ yoyining uzunligi deyiladi.

Masalan, agar

$$f(x) = kx + C \qquad (a \leq x \leq b)$$

bo‘lsa, unda $\breve{AB}$ ning uzunligi

$$\mu(\breve{AB}) = \lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + k^2 (x_{r+1} - x_k)^2} =$$

$$= \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{1 + k^2} \cdot (x_{k+1} - x_k) = \sqrt{1 + k^2} \cdot (b - a)$$

bo‘ladi.

Aytaylik, $\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo‘lsin.

Bunda:

1) $\varphi(t) \in C[\alpha, \beta]$, $\quad \psi(t) \in C[\alpha, \beta]$;

2) $\forall t_1, t_2 \in [\alpha, \beta]$, $t_1 \ne t_2$ uchun (4.1)

$$A_1(x_1, y_1) = A_1(\varphi(t_1), \psi(t_1)) ,$$
$$A_2(x_2, y_2) = A_2(\varphi(t_2), \psi(t_2))$$

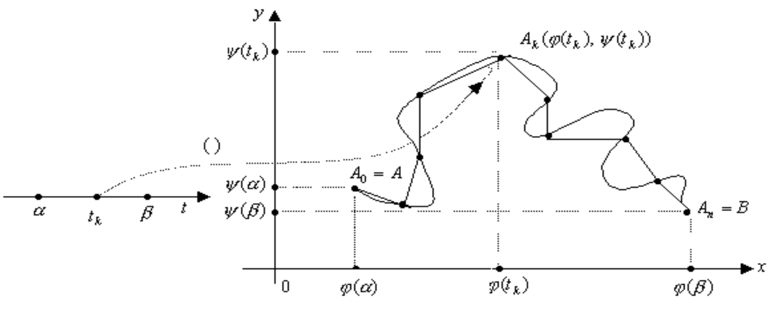
nuqtalar turlicha ;

3) $t = \alpha$ ga A nuqta, $t = \beta$ ga B nuqta mos kelsin.

$[\alpha, \beta]$ segmentning ixtiyoriy

$$P = \{t_0, t_1, ..., t_n\} \qquad (\alpha = t_0 < t_1 < ... < t_n = \beta$$

bo‘laklashni olib, bu bo‘laklashning bo‘luvchi t_k $(k = 0,1,2,...,n)$ nuqtalariga mos kelgan $\breve{AB}$ yoydagi $A_k = A_k(x_k, y_k)$ $(x_k = \varphi(t_k)$, $y_k = \psi(t_k)$; $k = 0,...,n)$ nuqtalarni bir-biri bilan to‘g‘ri chiziq kesmalari yordamida birlashtirib, $\breve{AB}$ yoyga chizilgan siniq chiziq l ni hosil qilamiz.



4.2-chizma

Bu siniq chiziq perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{[\varphi(t_{k+1}) - \varphi(t_k)]^2 + [\psi(t_{k+1}) - \psi(t_k)]^2}$$

bo‘ladi.

4.2-ta’rif. Agar $\lambda_p \to 0$ da $\breve{AB}$ yoyiga chizilgan siniq chiziq perimetri $\mu(l)$ chekli limitga ega bo‘lsa, $\breve{AB}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p \to 0} \mu(l) = \mu(\breve{AB})$$

limit $\breve{AB}$ yoyining uzunligi deyiladi.

Yuqorida keltirilgan ta'riflardan yoy uzunligining (agar u mavjud bo'lsa) musbat bo'lishi kelib chiqadi.

Endi yoy uzunligining ikkita xossasini isbotsiz keltiramiz:

1) Agar $\breve{AB}$ yoyi uzunlikka ega bo'lib, u $\breve{AB}$ yoydagi nuqtalar yordamida n ta $A_k\breve{A}_{k+1}$ yoylarga $(k = 0,1,2,...,n$; $A_0 = A, B = A_{n+1})$ ajralgan bo'lsa, u holda har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega va

$$\mu(\breve{AB}) = \sum_{k=0}^{n} \mu(A_k\breve{A}_{k+1})$$

bo'ladi.

2) Agar $\breve{AB}$ yoyi n ta $A_k\breve{A}_{k+1}$ yoylarga ajralgan bo'lib, har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega bo'lsa, u holda $\breve{AB}$ yoyi ham uzunlikka ega bo'ladi.

4.2. y=f(x) tenglama bilan berilgan egri chiziq uzunligini hisoblash.

Faraz qilaylik, $\breve{AB}$ egri chiziq ushbu

$$y = f(x) \quad , \quad a \le x \le b$$

tenglama bilan berilgan bo'lsin. Bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x)$ hosilaga ega.

$[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \quad (a = x_0 < x_1 < ... < x_n = b)$$

bo'laklashini olib, unga mos $\breve{AB}$ yoyiga chizilgan l siniq chiziqni hosil qilamiz. Bu siniq chiziqning perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

bo'ladi.

Har bir $[x_k, x_{k+1}]$ segmentda $f(x)$ funksiyaga Lagranj teoremasini qo‘llab topamiz:

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f'(\tau_k) \cdot (x_{k+1} - x_k)]^2} =$$

$$= \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)} \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)}\, \Delta x_k \quad ,$$

bunda $\tau_k \in [x_k, x_{r+1}]$.

Bu tenglikdagi yig‘indining $\sqrt{1+f'^2(x)}$ funksiyaning integral yig‘indisidan farqi shuki, integral yig‘indida $\xi_k \in [x_k, x_{k+1}]$ nuqta ixtiyoriy bo‘lgan holda yuqoridagi yig‘indida esa τ_k nuqta Lagranj teoremasiga muvofiq olingan tayin nuqta bo‘lishidadir. Ammo $\sqrt{1+f'^2(x)}$ funksiya integrallanuvchi bo‘lganligi sababli $\xi_k = \tau_k$ deb olinishi mumkin. Natijada

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k$$

bo‘lib, undan

$$\lim_{\lambda_p \to 0} \mu(l) = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k = \int_a^b \sqrt{1 + f'^2(x)}dx$$

bo‘lishi kelib chiqadi.

Demak, $A\breve{B}$ yoyining uzunligi

$$\mu(A\breve{B}) = \int_a^b \sqrt{1 + f'^2(x)}dx \qquad (4.2)$$

bo‘ladi. Bu formula yordamida yoy uzunligi hisoblanadi.

4.1-misol**.** Ushbu

$$f(x) = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \qquad (a > 0 \quad , \quad -a \le x \le a)$$

tenglama bilan berilgan $A\breve{B}$ egri chizig‘ining uzunligi topilsin.

Bu tenglama bilan aniqlanadigan chiziq zanjir chizig‘i deyiladi.

◀ Ravshanki,

$$f'(x)=\frac{1}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}}),\quad 1+f'^2(x)=\frac{1}{4}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2,$$

$$\sqrt{1+f'^2(x)}=\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$$

bo‘ladi. (4.2) formuladan foydalanib, zanjir chizig‘ining uzunligini topamiz:

$$\mu(A\breve{B})=\int_{-a}^{a}\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})dx=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})\Big|_{-a}^{a}=a(e-\frac{1}{e}).\ \blacktriangleright$$

4.3. Parametrik ko‘rinishda berilgan egri chiziq uzunligini hisoblash.

Faraz qilaylik, $A\breve{B}$ egri chiziq ushbu

$$\begin{cases} x=\varphi(t), \\ y=\psi(t) \end{cases}\qquad (\alpha\le t\le\beta)$$

tenglamalar sistemasi bilan berilgan bo‘lib, (4.1) shartlarning bajarilishi bilan birga $\varphi(t),\psi(t)$ funksiyalari $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hamda $\psi'(t)$ hosilalarga ega bo‘lsin.

$[\alpha,\beta]$ segmentning ixtiyoriy

$$P=\{t_0,t_1,...,t_n\}\qquad (\alpha=t_0<t_1<...<t_n=\beta)$$

bo‘laklashini olib, ularga mos $A\breve{B}$ yoyining $A_k=A_k(x_k,y_k)$ $(x_k=\varphi(t_k)$, $y_k=\varphi(t_k))$ nuqtalarini bir-biri bilan to‘g‘ri chiziq kesmasi yordamida birlashtirishdan hosil bo‘lgan l siniq chiziq perimetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{[\varphi(t_{k+1})-\varphi(t_k)]^2+[\psi(t_{k+1})-\psi(t_k))]^2}$$ ni qaraymiz.

Lagranj teoremasidan foydalanib topamiz:

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\tau_k)\cdot(t_{k+1}-t_k)^2+\psi'^2(\theta_k)\cdot(t_{k+1}-t_k)^2}=$$

$$=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}\cdot\Delta t_k\qquad (\Delta t_k=t_{k+1}-t_k)$$

bunda

$$\tau_k\in[t_k,t_{k+1}].\qquad \theta_k\in[t_k,t_{k+1}].$$

Keyingi tenglikni quyidagicha yozib olamiz:

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\cdot\Delta t_k+$$

$$+\sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\cdot\Delta t_k \qquad (4.3)$$

bunda,

$$\xi_k\in[t_k,t_{k+1}].$$

Modomiki,

$$\sqrt{\varphi'^2(t)+\psi'^2(t)}\in C[\alpha,\beta]$$

ekan unda

$$\sqrt{\varphi'^2(t)+\psi'^2(t)}\in R[\alpha,\beta]$$

bo‘lib,

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\cdot\Delta t_k=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (4.4)$$

bo‘ladi.

Ixtiyoriy a, b, c, d haqiqiy sonlar uchun ushbu

$$\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right|\le|a-c|+|b-d|$$

tengsizlik o‘rinli bo‘ladi.

Bu tengsizlikdan foydalanib topamiz:

$$\left|\sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\Delta t_k\right|\le$$

$$\le\sum_{k=0}^{n-1}|\varphi'(\tau_k)-\varphi'(\xi_k)|\Delta t_k+\sum_{k=0}^{n-1}|\psi'(\theta_k)-\psi'(\xi_k)|\Delta t_k\le$$

$$\le\sum_{k=0}^{n-1}\omega_k(\varphi')\cdot\Delta t+\sum_{k=0}^{n-1}\omega_k(\psi')\cdot\Delta t.$$

$$\varphi'(t)\in R[\alpha,\beta],\qquad \psi'(t)\in R[\alpha,\beta]$$

bo‘lganligi sababli

$$\lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} [\sqrt{\varphi'^2(\tau_k) + \psi'^2(\theta_k)} - \\ - \sqrt{\varphi'^2(\xi_k) + \psi'^2(\xi_k)}]\Delta t_k = 0 \tag{4.5}$$

bo‘ladi.

(4.4) va (4.5) munosabatlarni e’tiborga olib, $\lambda_p \to 0$ da (4.3) tenglikda limitga o‘tsak, u holda $A\breve{B}$ yoyining uzunligi uchun

$$\mu\,(A\breve{B}) = \int_{\alpha}^{\beta} \sqrt{\phi'^2\,(t)\, + \psi'^2\,(t)}\; dt \tag{4.6}$$

bo‘lishi kelib chiqadi. Bu formula yordamida yoy uzunligi hisoblanadi.

4.2-misol. Ushbu

$$\begin{cases} x = a(t\text{-}sint) \\ y = a(1\text{-}cost) \qquad (0 \le t \le 2\pi) \end{cases}$$

Tenglamalar sistemasi bilan berilgan egri chiziq yoyining (sikloidaning) uzunligi topilsin.

◀ Bu egri chiziqning uzunligini topishda quydagicha topamiz.

Ravshanki, $\quad x'(t) = a(1\text{-}cost), \;\; y'(t) = asint,$

$$x'^2(t) + y'^2(t) = a^2(1\text{-}cost)^2 + a^2 sin^2 t = 2a^2(1\text{-}cost)$$

bo’lib,

$$\sqrt{x'^2(t) + y'^2(t)} = a\sqrt{2(1-cost)}$$

bo’ladi. Izlanayotgan egri chiziqning uzunligi

$$l = \int_0^{2a} a\sqrt{2(1-cost)}dt$$

bo’laadi. Bu teglikning o’ng tomonidagi integralni hisoblaymiz:

$$\int_0^{2\pi} a\sqrt{2(1-cost)}dt = a\int_0^{2\pi} \sqrt{4 \cdot sin^2 \frac{t}{2}} = 4a\int_0^{2\pi} sin\frac{t}{2}d\left(\frac{t}{2}\right) = -4a \cdot cos\frac{t}{2} \Bigg|_0^{2\pi} = 8a\,.$$

Demak, $l = 8a$.▶

4.3–misol. Ushbu

$$f(x) = \frac{3}{2}(x^{\frac{1}{3}} - \frac{1}{5}x^{\frac{5}{3}}) \qquad 1 \le x \le 8$$

Egri chiziqli yoyni uzunligi topilsin.

Yechilishi. ► Bu egri chiziqning uzunligini topishda (4.2) formuladan foydalanamiz.

$$f'(x)\frac{3}{2}\left(\frac{1}{3}x^{-\frac{2}{3}}-\frac{5}{3}\frac{1}{5}x^{\frac{2}{3}}\right)=\frac{1}{2}(x^{-\frac{2}{3}}-x^{\frac{2}{3}})=\frac{1}{2}(\frac{1-x^{\frac{4}{3}}}{x^{\frac{2}{3}}}).$$

Ravshanki,

$$\sqrt{1+\frac{1}{4}(\frac{1-x^{\frac{4}{3}}}{x^{\frac{2}{3}}})^2}=\sqrt{\frac{1+2x^{\frac{4}{3}}+x^{\frac{8}{3}}}{4x^{\frac{4}{3}}}}=\sqrt{\frac{(1+x^{\frac{4}{3}})^2}{(2x^{\frac{2}{3}})^2}}=\frac{(1+x^{\frac{4}{3}})}{2x^{\frac{2}{3}}}\,.$$

bo’ladi. Izlanayotgan egri chiziqning uzunligi

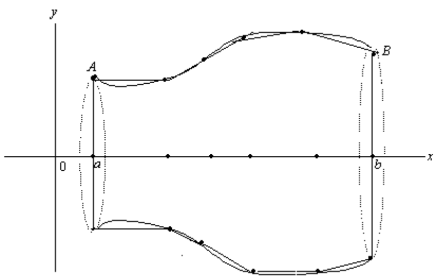
$$I=\int_1^8\frac{1+x^{\frac{4}{3}}}{2x^{\frac{2}{3}}}dx=\int_1^8(\frac{1}{x^{\frac{2}{3}}}+x^{\frac{2}{3}})dx=\frac{1}{2}\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1}\Bigg|_1^8+\frac{1}{2}\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\Bigg|_1^8=\frac{1}{2}\Big(3\,8^{\frac{1}{3}}+$$

$$\frac{3}{5}8^{\frac{5}{3}}-3-\frac{3}{5}\Big)=\frac{1}{2}\frac{18}{5}=\frac{9}{5};$$

4.4. Aylanma sirtning yuzi va uni hisoblash

1. Aylanma sirt va uning yuzi tushunchasi**.** Ma’lumki, to‘g‘ri chiziq kesmasini biror o‘q atrofida aylantirishdan silindrik, konus (kesik konus) sirtlar hosil bo‘ladi. Bu sirtlar yuzaga ega va ular ma’lum formulalar yordamida topiladi.

Aytaylik, $f(x)\in C[a,b]$ bo‘lib, $\forall x\in[a,b]$ da $f(x)\geq 0$ bo‘lsin. Bu funksiya grafigi $\breve{AB}$ yoyini tasvirlasin (4.3-chizma)



(4.3-chizma)

$\breve{AB}$ yoyni Ox o‘qi atrofida aylantirishdan hosil bo‘lgan sirt aylanma sirt deyiladi. Uni Π deylik. $[a,b]$ segmentni ixtiyoriy

$$P=\{x_0,x_1,...,x_n\}\qquad\qquad(a=x_0<x_1<...<x_n=b)$$

bo‘laklashni olaylik. Bu bo‘laklashning har bir

$$x_k \, (k = 0,1,2,...,n)$$

bo‘luvchi nuqtalari orqali Oy o‘qiga parallel to‘g‘ri chiziqlar o‘tkazib, ularning $\breve{AB}$ yoyi bilan kesishish nuqtalarini $A_k = A_k(x_k, f(x_k))$ bilan belgilaylik. $(A_0 = A,\ A_n = B;\quad k = 0,1,2,...,n)$ Bu nuqtalarni o‘zaro to‘g‘ri chiziq kesmalari bilan birlashtirib, $\breve{AB}$ yoyiga L siniq chiziq chizamiz.

$\breve{AB}$ yoyini Ox o‘qi atrofida aylantirish bilan birga L siniq chiziqni ham shu o‘q atrofida aylantiramiz. Natijada kesik konus sirtlarining birlashmasidan tashkil topgan K sirt hosil bo‘ladi. Bu K sirt yuzaga ega va uning yuzi

$$\mu(K) = 2\pi \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2} \cdot \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

ga teng. (Bunda kesik konusning yon sirtining yuzini topish formulasidan foydalanildi).

Ravshanki, K sirt, binobarin uning yuzi $\mu(K)$ $[a,b]$ segmentning bo‘laklashlariga bog‘liq bo‘ladi.

4.3-ta’rif. Agar $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki, $[a,b]$ segmentning diametri $\lambda_p < \delta$ bo‘lgan ixtiyoriy P bo‘laklashi uchun

$$|\mu(K) - S| < \varepsilon \quad (S \in R)$$

tengsizlik bajarilsa, S son $\mu(K)$ ning $\lambda_p \to 0$ dagi limiti deyiladi:

$$\lim_{\lambda_p \to 0} \mu(K) = S\,.$$

4.4-ta’rif. Agar $\lambda_p \to 0$ da $\mu(K)$ yig‘indi chekli S limitga ega bo‘lsa, $\varPi$ aylanma sirt yuzaga ega deyiladi.

Bunda S son $\varPi$ aylanma sirtning yuzi deyiladi: $S = \mu(\varPi)$.

Demak,

$$\mu(\varPi) = \lim_{\lambda_p \to 0} 2\pi \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2} \cdot \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}\,.$$

2.Aylanma sirt yuzini hisoblash**.** Faraz qilaylik, $f(x)\in C[a,b]$ bo‘lib, u $[a,b]$ segmentda uzluksiz $f'(x)$ hosilaga ega bo‘lsin.

Bu funksiya grafigi $\breve{AB}$ yoyini Ox o‘qi atrofida aylantirishdan hosil bo‘lgan Π aylanma sirtning yuzini topamiz.

◄ $[a,b]$ segmentning ixtiyoriy P bo‘laklashini olib, yuqoridagidek

$$\mu(K)=2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

yig‘indini tuzamiz.

Lagranj teoremasiga ko‘ra

$$f(x_{k+1})-f(x_k)=f'(\xi_k)(x_{k+1}-x_k)=f'(\xi_k)\cdot\Delta x_k$$

bo‘ladi, bunda $\xi_k\in[x_k,x_{k+1}]$. Natijada

$$\mu(K)=2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\sqrt{1+f'^2(\xi_k)}\Delta x_k$$

bo‘ladi.

Keyingi tenglikni quyidagicha yozib olamiz:

$$\mu(K)=2\pi\sum_{k=0}^{n-1}f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k+\pi\left\{\sum_{k=0}^{n-1}[(f(x_k)-\right.$$
$$\left.-f(\xi_k))+(f(x_{k+1})-f(\xi_k))]\sqrt{1+f'^2(\xi_k)}\Delta x_k\right\}. \tag{4.7}$$

$f'(x)\in C[a,b]$ bo‘lganligi sababli

$$f(x)\sqrt{1+f'^2(x)}\in R[a,b]$$

bo‘ladi. Demak, $\lambda_p\to 0$ da

$$2\pi\sum_{k=0}^{n-1}f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k\to 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx \tag{4.8}$$

Ravshanki,

$$\sqrt{1+f'^2(x)}\in C[a,b].$$

Demak, bu funksiya $[a,b]$ da o‘zining maksimum qiymatiga ega bo‘ladi. Uni M deylik:

$$M=\max_{a\le x\le b}\sqrt{1+f'^2(x)}.$$

$f(x)$ funksiya $[a,b]$ segmentda tekis uzluksiz. Unda $\forall\varepsilon>0$ olinganda ham, $\frac{\varepsilon}{2M(b-a)}$ ga ko‘ra shunday $\delta>0$ son topiladiki, $\lambda_p<\delta$ bo‘lganda

$$|f(x_k)-f(\xi_k)|<\frac{\varepsilon}{2M(b-a)},\quad |f(x_{k+1})-f(\xi_k)|<\frac{\varepsilon}{2M(b-a)}$$

bo‘ladi. Shularni e’tiborga olib topamiz:

$$\left|\left\{\sum_{k=0}^{n-1}\left[(f(x_k)-f(\xi_k))+(f(x_{k+1})-f(\xi_k))\right]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k\right\}\right|\le$$

$$\le\sum_{k=0}^{n-1}\left[|f(x_k)-f(\xi_k)|+|f(x_{k+1})-f(\xi_k)|\right]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k<$$

$$<M\left[\frac{\varepsilon}{2M(b-a)}+\frac{\varepsilon}{2M(b-a)}\right]\cdot\sum_{k=0}^{n-1}\Delta x_k<\varepsilon\ .$$

Bundan $\lambda_p\to 0$ da

$$\sum_{k=0}^{n-1}\left[(f(x_k)-f(\xi_k))+(f(x_{k+1})-f(\xi_k))\right]\cdot\sqrt{1+f'^2(\xi_k)}\,\Delta x_k\to 0 \qquad (4.9)$$

bo‘lishi kelib chiqadi.

$\lambda_p\to 0$ da (4.1) tenglikda limitga o‘tib, (bunda (4.2) va (4.4) munosabatlarni e’tiborga olib) aylanma sirtning yuzi uchun

$$\mu(\Pi)=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx \qquad (4.10)$$

bo‘lishini topamiz. ►

4.2-misol. Ushbu

$$f(x)=\frac{a}{2}(e^{\frac{x}{a}}+e^{\frac{-x}{a}}),\quad a>0,\ 0\le x\le a$$

zanjir chizig‘ini Ox o‘q atrofida aylantirishdan hosil bo‘lgan aylanma sirtning yuzi topilsin.

◄ Ravshanki,

$$f'(x)=\frac{1}{2}(e^{\frac{x}{a}}-e^{\frac{-x}{a}})$$

(4) formuladan foydalanib, izlanayotgan aylanma sirtning yuzini topamiz:

$$\mu(\Pi) = 2\pi\int_0^a \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})\sqrt{1 + \frac{1}{4}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})^2}\, dx =$$

$$= \frac{\pi a}{2}\int_0^a (e^{\frac{x}{a}} + e^{-\frac{x}{a}})^2 dx = \frac{\pi a}{2}\int_0^a (e^{\frac{2x}{a}} + 2 + e^{-\frac{2x}{a}})dx =$$

$$= \frac{\pi a}{2}\left[\frac{a}{2}e^{\frac{2x}{a}} + 2x - \frac{a}{2}e^{-\frac{2x}{a}}\right]\Bigg|_0^a = \frac{\pi a^2}{4}(e^2 - e^{-2} + 4)$$ ►

Aytaylik, $\breve{AB}$ egri chiziq yuqori yarim tekislikda joylashgan bo‘lib, u ushbu

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \quad (\alpha \le t \le \beta)$$

parametrik tenglamalar sistemasi bilan berilgan bo‘lsin. Bunda $\varphi(t),\ \psi(t)$ funksiyalari $[\alpha, \beta]$ da uzluksiz va uzluksiz $\varphi'(t), \psi'(t)$ hosilalarga ega. Bu egri chiziqni Ox o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma sirtning yuzi

$$\mu(\Pi) = 2\pi\int_\alpha^\beta \psi(t)\sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt \qquad (4.11)$$

bo‘ladi.

4.5-misol. Ushbu

$$x^2 + (y-2)^2 = 1$$

aylanani Ox o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma sirtning (torning) yuzi topilsin.

◄ Aylananing tenglamasini quyidagicha

$$\begin{aligned} x &= \varphi(t) = \cos t \\ y &= \psi(t) = 2 + \sin t \end{aligned} \qquad (0 \le t \le 2\pi)$$

parametrik ko‘rinishda yozamiz.

Izlanayotgan aylanma sirtning yuzi, (4.11) formulaga ko‘ra

$$\mu(\Pi) = 2\pi\int_0^{2\pi} (2 + \sin t)\sqrt{(\cos t)'^2 + (2 + \sin t)'^2}\, dt =$$

$$= 2\pi\int_0^{2\pi} (2 + \sin t)dt = 8\pi^2$$

bo‘ladi. ►

4.6-misol. Ushbu

$$x = a(t - sint),\ \ y = a(t - cost) \quad\ 0 \le t \le 2\pi,$$

chiziqning OX o'q atrofida aylanishi natijasida hosil bo'lgan sirtni yuzini toping.

Yechilishi. Izlanayotgan sirtning yuzi, (4.11) formula orqali hisoblanadi

$$x' = a(t - sint), y' = a(t - cost)$$

ds=$\sqrt{(x')^2 + (y')^2}dt = \sqrt{(a(1 - \mathrm{cost}))^2 + (a\mathrm{sint})^2} = a\sqrt{2}\sqrt{1 - cost}dt =$

$2asin\frac{t}{2}dt,$

$\mu(\Pi) = 2\pi\int_0^{2\pi} a\,(1 - cost)2asin\frac{t}{2}dt = 2\pi\int_0^{2\pi} 4a^2\sin^3\frac{t}{2}dt = 16\pi a^2\int_0^{2\pi}\sin^3 v dv = \frac{64}{3}\pi a^2.$

5. ANIQ INTEGRALNING FIZIKA VA MEXANIKAGA TATBIQLARI.

5.1. Inersiya momenti.

Mexanikada moddiy nuqta harakati muhim tushunchalardan biri hisoblanadi.

Odatda, o‘lchami yetarli darajada kichik va massaga ega bo‘lgan jism moddiy nuqta deb qaraladi.

Aytaylik, tekislikda m massaga ega bo‘lgan A moddiy nuqta berilgan bo‘lib, bu nuqtadan biror l o‘qqacha (yoki O nuqtagacha) bo‘lgan masofa r ga teng bo‘lsin.

Ushbu

$$J = mr^2$$

miqdor A moddiy nuqtaning l o‘qqa (O nuqtaga) nisbatan inersiya momenti deyiladi.

Masalan, $A = A(x, y)$ moddiy nuqtaning koordinata o‘qlariga hamda koordinata boshiga nisbatan inersiya momentlari mos ravishda

$$J_x = my^2 \ , \ J_y = mx^2 \ , \ J_0 = m\sqrt{x^2 + y^2}$$

bo‘ladi.

Tekislikda, har biri mos ravishda

$$m_0 , m_1 , m_2 ,...., m_{n-1}$$

massaga ega bo‘lgan moddiy nuqtalar sistemasi

$$\{A_0 , A_1 , A_2 ,...., A_{n-1}\}$$

ning biror l o‘qqa (O nuqtaga) nisbatan inersiya momenti ushbu

$$J_n = \sum_{k=0}^{n-1} m_k r_k^{\ 2}$$

yig‘indi bilan ta’riflanadi, bunda $r_k - A_k$ nuqtadan l o‘qqacha (O nuqtagacha) bo‘lgan masofa $(k = 0,1,2,...,n-1)$.

Faraz qilaylik, $y = f(x)$ egri chiziq yoyi $\breve{AB}$ bo‘yicha zichligi $\rho = 1$ ga teng massa tarqatilgan bo‘lib, bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz hamda uzluksiz $f'(x)$ hosilaga ega bo‘lsin.

Ravshanki, bu holda massa yoy uzunligiga teng bo‘ladi:

$$m=\int_a^b \sqrt{1+f'^2(x)}\,dx.$$

$[a,b]$ segmentning ixtiyoriy

$$P=\{x_0,x_1,...,x_n\} \quad (a=x_0<x_1<...<x_n=b)$$

bo‘laklashini olamiz. Bu bo‘laklash $A\breve{B}$ yoyni

$$A_k=A_k(x_k,f(x_k)) \quad (k=0,1,2,...,n-1)$$

nuqtalar bilan n ta $A_k\breve{A}_{k+1}$ $(A_0=A,\ A_{n-1}=B)$ bo‘lakka ajratadi. Bunda $A_k\breve{A}_{k+1}$ bo‘lakning massasi

$$m_k=\int_{x_k}^{x_{k+1}} \sqrt{1+f'^2(x)}\,dx \qquad (k=0,1,2,...,n-1)$$

bo‘ladi. O‘rta qiymat haqidagi teoremadan foydalanib topamiz:

$$m_k=\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

bunda,

$$\xi_k\in[x_k,x_{k+1}],\Delta x_k=x_{k+1}-x_k.$$

Ma’lumki,

$$(\xi_k,f(\xi_k)) \quad (k=0,1,2,...,n-1)$$

moddiy nuqtaning koordinata o‘qlariga hamda kordinata boshiga nisbatan inersiya momentlari mos ravishda

$$J'_{x_k}=m_k\cdot f^2(\xi_k)=f^2(\xi_k)\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

$$J'_{y_k}=m_k\cdot\xi_k^2=\xi_k^2\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

$$J'_0=m_k(\xi_k^2+f^2(\xi_k))=(\xi_k^2+f^2(\xi_k))\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k$$

bo‘ladi. Unda ushbu

$$\{(\xi_0,f(\xi_0)),\ (\xi_1,f(\xi_1),...,(\xi_{n-1},f(\xi_{n-1}))\}$$

moddiy nuqtalar sistemasining inersiya momentlari mos ravishda

$$J_x^{(n)} = \sum_{k=0}^{n-1} f^2\left(\xi_k\right) \cdot \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k ,$$

$$J_y^{(n)} = \sum_{k=0}^{n-1} \xi_k^2 \cdot \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k ,$$

$$J_0^{(n)} = \sum_{k=0}^{n-1} (\xi_k^2 + f^2(\xi_k)) \cdot \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k ,$$

tengliklar bilan ifodalanadi.

Agar P bo‘laklashning diametri λ_p nolga intila borsa, unda har bir $A_k \breve{A}_{k+1}$ yoyning uzunligi ham nolga intila borib, yuqoridagi

$$J_x^{(n)}, \; J_y^{(n)}, \quad J_0^{(n)},$$

yig‘indilarning limitini massaga ega bo‘lgan $A\breve{B}$ egri chiziqning mos ravishda koordinata boshi hamda koordinata o‘qlariga nisbatan inersiya momentlarini ifodalaydi deb qarash mumkin.

Ayni paytda,

$$\lim_{\lambda_p \to 0} J_x^{(n)} = \int_a^b f^2\left(x\right)\sqrt{1 + f'^2(x)}\, dx \;\; ,$$

$$\lim_{\lambda_p \to 0} J_y^{(n)} = \int_a^b x^2 \sqrt{1 + f'^2(x)}\, dx \;\; ,$$

$$\lim_{\lambda_p \to 0} J_0^{(n)} = \int_a^b (x^2 + f^2(x))\sqrt{1 + f'^2(x)}\, dx$$

bo‘ladi.

Demak, massaga ega bo‘lgan $A\breve{B}$ egri chiziqning koordinata o‘qlariga hamda koordinata boshiga nisbatan inersiya momentlari aniq integrallar yordamida topiladi:

$$J_x = \int_a^b f^2\left(x\right)\sqrt{1 + f'^2(x)}\, dx \;\; ,$$

$$J_y = \int_a^b x^2 \sqrt{1 + f'^2(x)}\, dx \;\; ,$$

$$J_0 = \int_a^b (x^2 + f^2(x))\sqrt{1 + f'^2(x)}\, dx .$$

5. 1–misol. Ushbu

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{x=0, y=0, a>0, b>o}$$

Chiziqlar bilan chegaralangan tekis figuraning OX va OY oqlargaa nisbatan statik momentlari va og'irlik markazini koordinatalarini toping.

Yechilishi. Bizga malum bo'lgan $S=\int_a^b \left(f(x)-g(x)\right)\rho(x)dx,$

$$M_x=\frac{1}{2}\int_a^b \left(\left(f^2(x)-g^2(x)\right)\rho(x)dx,\quad M_y=\int_a^b \left(x^2\left(f(x)-g(x)\right)\rho(x)\right)dx,\right.$$

$$x_M=\frac{M_y}{S}; \qquad y_M=\frac{M_x}{S}$$

Formulalarga asosan,

$$M_x=\frac{1}{2}\int_a^b \left(\frac{b^2}{a^2}(a-x)^2\right)dx=\frac{ab^2}{6},\frac{b}{a}\int_a^b \left(\int_0^a (ax-x)dx=\frac{ab}{2}\right.$$

$$M_y=\int_a^b \left(x\frac{b}{a}(a-x)\right)dx=\frac{b}{a}\int_a^b \left(\left(ax-x^2\right)\right)dx=\frac{a^2b}{6},$$

$$S=\int_a^b \left(\frac{b}{a}(a-x)dx=\frac{b}{a}\int_a^b (ax-x)dx=\frac{ab}{2},\right.$$

$$x_M=\frac{M_y}{S}=\frac{\frac{a^2b}{6}}{\frac{ab}{2}}=\frac{a}{3}; \qquad y_M=\frac{M_x}{S}=\frac{\frac{a^2b}{6}}{ab}=\frac{b}{3};$$

Demak, $\quad M(x_M, y_M)=M(\frac{a}{3},\frac{b}{3})$.

5.2. O'zgaruvchi kuchning bajargan ishi.

Biror jismni Ox o'qi bo'ylab, shu o'q yo'nalishida bo'lgan $F=F(x)$ kuch ta'siri ostida a nuqtadan b nuqtaga $(a<b)$ o'tkazish uchun bajarilgan ishni topish lozim bo'lsin.

Ravshanki, jismga ta'sir etuvchi kuch o'zgarmas, ya'ni

$$F(x)=C-const$$

bo'lsa, unda jismni a nuqtadan b nuqtaga o'tkazish uchun bajarilgan ish

$$A=C\cdot(b-a)$$

ga teng bo‘ladi.

Aytaylik, jismga ta’sir etuvchi kuch x ga $(x \in [a,b])$ bog‘liq bo‘lib, u $[a,b]$ da uzluksiz bo‘lsin:

$$F = F(x) \in C[a,b].$$

$[a,b]$ segmentning ixtyoriy

$$P = \{x_0\,,\, x_1\,, ...,\, x_n\} \quad (a = x_0 < x_1 < ... < x_n = b)$$

bo‘laklashini olib, bu bo‘laklashning har bir

$$[x_k\,, x_{k+1}] \qquad\qquad (k = 0,1,2,...,n-1)$$

bo‘lakchasida ixtyoriy ξ_k $\quad \xi_k \in [x_k\,, x_{k+1}];$ $(k = 0,1,2...,n-1)$ nuqta olamiz.

Agar har bir $[x_k\,, x_{k+1}]$ da jismga ta’sir etuvchi kuchni o‘zgarmas va u $F(\xi_k)$ ga teng deyilsa, u holda $[x_k\,, x_{k+1}]$ oraliqda bajarilgan ish (kuch ta’sirida jismni x_k nuqtadan x_{k+1} nuqtaga o‘tkazish uchun bajarilgan ish) taxminan

$$F(\xi_k)\cdot(x_{k+1} - x_k)$$

formula bilan, $[a,b]$ oraliqda bajarilgan ish esa , taxminan

$$A \approx \sum_{k=0}^{n-1} F(\xi_k)\cdot(x_{k+1} - x_k) = \sum_{k=0}^{n-1} F(\xi_k)\cdot \Delta x_k \tag{5.1}$$

formula bilan ifodalanadi.

P bo‘laklashning diametri λ_p nolga intila borganda yuqoridagi yig‘indining qiymati izlanayotgan ish miqdorini tobora aniqroq ifodalaydi. Bu hol $\lambda_p \to 0$ da (5.1) yig‘indi-ning chekli limitini bajarilgan ish deyilishi mumkinligini ko‘rsatadi.

Demak,

$$A = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} F(\xi_k)\cdot \Delta x_k\,.$$

Modomiki, $F(x) \in C[a,b]$ ekan,

$$\lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} F(\xi_k)\cdot \Delta x_k = \int_a^b F(x)dx$$

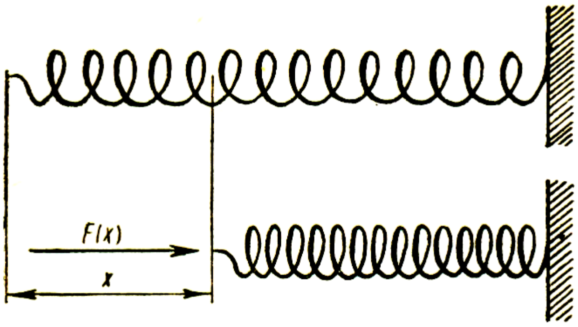
bo‘ladi.

SHunday qilib, o‘zgaruvchi $F(x)$ kuchning $[a,b]$ dagi bajargan ishi

$$A = \int_a^b F(x)dx \qquad (5.2)$$

formula bilan ifodalanadi.

5.2-misol**.** Vintsimon prujinaning bir uchi mustahkamlangan, ikkinchi uchiga esa $F = F(x)$ kuch ta'sir etib, prujina qisilgan (5.1-chizma)



5.1-chizma

Agar prujinaning qisilishi unga ta'sir etayotgan $F(x)$ kuchga proporsional bo‘lsa, prujinani a birlikka qisish uchun $F(x)$ kuchning bajargan ishi topilsin.

◄ Agar $F(x)$ kuch ta'sirida prujinaning qisilish miqdorini x orqali belgilasak, u holda

$$F(x) = kx$$

bo‘ladi, bunda k-proporsionallik koeffitsienti (qisilish koeffitsienti). (5.2) formulaga ko‘ra bajarilgan ish

$$A = \int_0^a kxdx = \frac{ka^2}{2}$$

bo‘ladi. ►

5.3-misol. Jismning harakat tezligi

V=$(t^2 + 4t)$cm/cek

tenglama orqali berilgan . Jismning harakat boshlagandan so'ng 8 sekund vaqt davomida o'tgan yo'lini aniqlang.

Yechilishi. $S = \int_{t_1}^{t_2} f(t)dt$ formulaga asosan

$$S=\int_0^8 ((t^2+4t)dt = (\frac{1}{3}t^3+2t^2)\Big|_0^8 = \frac{1}{3}8^3+2\ 8^2 = \frac{896}{3} = 298{,}7 \text{ cm.}$$

5.4-misol**.** 1 kΓ miqdordagi kuch purjinani 3 cm ga cho’zganda, u qanday ish bajarishini aniqlang.

Yechilishi. Guk qonuniga asosan, kuch purjinaning cho’zilishi yoki qisilishiga to’g’ri proporsanal, ya’ni

F=kx

bunda x-purjinaning cho’zilish yoki qisilishi miqdori, k- proporsionallik kooeffitsiyenti. Qaralyotgan masalada k ning qiymatini topish uchun, berilganlarni, Guk qonunini ifodalovchi, F=kx tenglamaga keltirib qo’yamiz: 1=k o,o3 bu yerdan $k=\frac{1}{0{,}03}$ ekanligi kelib chiqadi.

Demak, purjinani cho’zuvchi kuch

$$F=\frac{1}{0{,}03}x$$

ko’rinishda ifodalanadi.

Kuch tinch holatda bo’lgan purjinaga ta’sir qilgani uchun, $A=\int_a^b f(x)dx$ formuladagi integralning quyi chegarasi a=0 bo’ladi, yuqori chegarasi esa, b=0,03 bo’ladi.

Demak, izlanayotgan ish:

$$A=\int_0^{0{,}03}\frac{1}{0{,}03}xdx = \frac{1}{0{,}03}\frac{x^2}{2}\Big|_0^{0{,}03} = \frac{1}{0{,}03}\frac{(0{,}03)^2}{2} = 0{,}015 \text{ k}\Gamma\text{M.}$$

6. MAPLE DASTURI ASOSIDA MISOLLAR YECHISH.

6.1-misol. $\int x\sqrt{a-x}dx$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

int(x*sqrt(a-x), x); $-\frac{2(2a+3x)(a-x)^{\frac{3}{2}}}{15}$

6.2-misol. $\int(1-4x)sinxdx$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

int((1-4*x)*sin(x), x); -cos(x) - 4 sin(x) + 4 cos(x) x

6.3-misol. $\int \frac{3x^3-16x^2-x+51}{3x^2-7x-20}dx$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

int((3*x^3-16*x^2-x+51)/(3*x^2-7*x-20), x);

$$\frac{1}{2}\mathrm{x}^2-3\mathrm{x}-\ln(\mathrm{x}-4)+\frac{1}{3}\ln(3\mathrm{x}+5).$$

6.4-misol. $\int sinxsin2xsin3xdx$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

int(sin(x)*sin(2*x)*sin(3*x), x);

$$-\frac{1}{16}\mathrm{cox}(4\mathrm{x})-\frac{1}{8}\cos(2\mathrm{x})+\frac{1}{24}\cos(6\mathrm{x}).$$

6.5-misol. $\int_1^e \frac{\cos(\ln x)}{x}dx$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

int(cos(ln(x))/x, x = 1 .. exp(1)); sin(1)

6.6–misol. $\int_3^6 \frac{\sqrt{x^2-9}}{x^4}dx$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

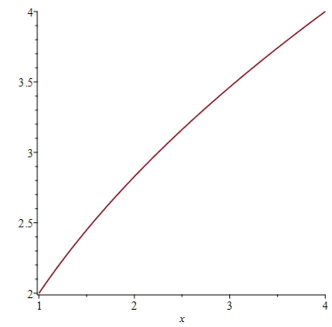
int(sqrt(x^2-9)/x^4, x = 3 .. 6); $\frac{1}{72}\sqrt{3}.$

6.7-misol. $\int_1^4 2\sqrt{x}dx$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

int(2*sqrt(x), x = 1 .. 4); $\frac{28}{3}$

plot(2*sqrt(x), x = 1 .. 4)

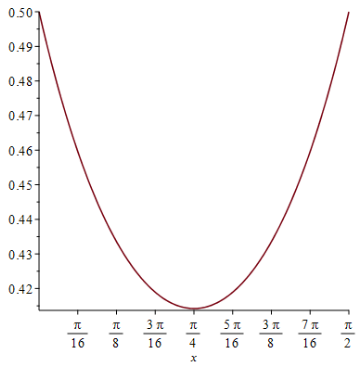


6.7-misol. $\int_{1}^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ integralni hisoblang.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

int(1/(1+sin(x)+cos(x)), x = 0 .. (1/2)*Pi); ln(2)

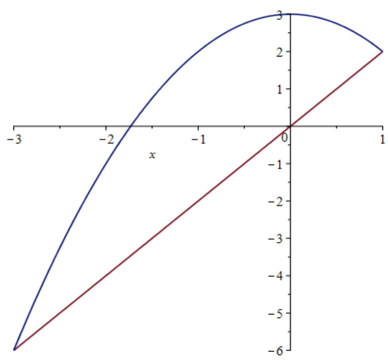
plot(1/(1+sin(x)+cos(x)), x = 0 .. (1/2)*Pi)



6.9-misol. Ushbu y=2x to’g’ri chiziq va $y = 3 - x^2$ parabola bilan chegaralangan sohani yuzini toping.

Yechish. Misolni Maple tizimidan foydalanib yechamiz.

plot([2*x, -x^2+3], x = -3 .. 1); $\frac{32}{3}$



7. MAVZUGA DOIR TESTLAR.

1. Integralni hisoblang; $\int(sinx + 3cosx)dx$.

A) $cosx + 3sinx + C$ B)$-cosx + 3sinx + C$

C) $cosx + sinx + C$ D) $cosx - 3sinx + C$

2. Integralni hisoblang; $\int 2e^{2x}dx$.

A) $e^{2x} + C$ B) $2e^{2x} + C$ C) $e^{x} + C$ D) $e^{-x} + C$

3. Ushbu x=a(t-sint), y=a(1-cost) $0 \le t \le 2\pi$, chiziqning OX o'q atrofida aylanishi natijasida hosil bo'lgan sirtni yuzini toping.

A) $\frac{64}{6}\pi a^2$ B) $\frac{64}{5}\pi a^2$ C) $\frac{64}{3}\pi a^2$ D) $\frac{64}{3}\pi a^2$

4. Integralni hisoblang; $\int \frac{x^2}{x^3+9}dx$.

A) $\frac{1}{3}\ln|x^3 + 9| + 1 + C$ B) $\frac{1}{3}\ln|x^3 + 9| + 12 + C$

C) $\frac{1}{3}\ln|x^3 + 9| + C$ D) $\frac{1}{3}\ln|x^3 + 9| + 2 + C$

5. $y = x$, $y = 3x$ va $x = 2$ to'g'ri chiziqlar bilan chegaralangan shaklning yuzasini hisoblang;

A) 2 B) 4 C) 5 D) 8

6. Integralni hisoblang; $\int_{-1}^{0} 3(2x+1)^2 dx$

A) 2 B) 3 C) 4 D) 1

7. Nyuton-Leybnis formulasi toping.

A) $\int_a^b f(x)dx = \Phi(b) - \Phi(a)$ **B)** $\int_a^b f(x)dx = \Phi(b) + C$

C) $\int_a^b f(x)dx = \Phi(a) + C$ D) $\int_a^b f(x)dx = C$

8. $y = 2^x$, y=2, x=0 chiziqlar bilan chegaralangan sohani yuzini toping.

A) 11+$\frac{\pi^2}{8}$ B) 2+$\frac{\pi^2}{8}$ C) 1+$\frac{\pi^2}{8}$ D) 4+$\frac{\pi^2}{8}$

9. Interalni hisoblang; $\int_{-1}^{0} e^{-x}dx$

A) 1+e B) -1+e C) e+c D) –e

10. Interalni hisoblang; $\int_{0}^{\pi} \sin 2x dx$

A) 0 B) -1 C) 1 D) 2

11. Quydagi jumla qasi integrallanuvchi funksiyalar sinfiga kiradi?.

Agar $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan va bu oraliqning chekli sondagi nuqtalarida uzulishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, funksiya shu oraliqda integrallanuvchi bo'ladi.

A) Monaton funksiyalar B) Uzluksiz funksiyalar

C) Uziladigan funksiyalar D) Hamma javob to'g'ri

12. $\int_{0}^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ ni hisoblang.

A) ln5 B) ln2 C) ln7 D) 9

13. Ellips $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ bilan chegaralangan shaklning yuzini hisoblang;

A) 4ab B) 5ab C) πa D) πab

14. $x^2+y^2+z^2=1$ sfera bilan chegaralangan jismning hajmini hisoblang.

A) $\frac{2}{3}\pi$ B) $\frac{4}{3}\pi$ C) $\frac{2}{5}\pi$ D) $\frac{7}{3}\pi$

15. $\int_{-1}^{0} \sqrt{y+1}dx$ ni hisoblang.

A) $\frac{5}{3}$ B) $\frac{2}{3}$ C) 2 D) 0

16. Jismning harakat tezligi V=$(2t^2+t)$ cm/cek tenglama orqali berilgan . Jismning harakat boshlagandan so'ng 6 sekund vaqt davomida o'tgan yo'lini aniqlang.

A) 162 cm B) 158 cm C) 147 cm D) 169 cm

17. Interalni hisoblang; $\int_{\frac{\pi}{2}}^{\pi} 7\cos x dx.$

A) 7 B) 4 C) 6 D) -7

18. x^2+y^2=4, y$\geq$ 0, berilgan chiziqlar og'irlik markazining x_M va y_M koordinatalarini toping.

A) $x_M = 0,\ y_M = \frac{4}{\pi}$ B) $x_M = 1,\ y_M = \frac{4}{\pi}$

C) $x_M = 0,\ y_M = \frac{14}{\pi}$ D) $x_M = 4,\ y_M = \frac{4}{\pi}$

19. Interalni hisoblang; $\int_1^2 \frac{dx}{x+1}$

A) $ln\frac{3}{2}$ B) $ln\frac{6}{2}$ C) $ln\frac{3}{5}$ D) $ln5$

20. x=2$\sqrt{3}$cost, y=sin2t OX o'qi atrofida aylantirishdan hosil bo'lgan sirtni yuzasini hisoblang.

A) $\frac{15\pi}{8}(1 + ln5)$ B) $\frac{15\pi}{8}(4 + ln5)$

C) $\frac{15}{8}(4 + ln5)$ D) $\frac{15\pi}{14}(4 + ln5)$

Xulosa

Matematik analiz fani matematikaning fundamental bo‘limlaridan biri bo‘lib, u matematikaning poydevori hisoblanadi. Matematik analiz kursi davomida ko‘pgina tushuncha va tasdiqlar, shuningdek, ularning tatbiqlari keltiriladi.

Matematik analiz fanining asosiy vazifasi shu fanning tushuncha, tasdiqlar va boshqa matematik ma’lumotlar majmuasi bilan tanishtiribgina qolmasdan, balki talabalarda mantiqiy fikrlash, matematik usullarni amaliy masalalarni yechishga qo‘llash ko‘nikmalarini shakllantirishdan iborat. Ushbu bitiruv malakaviy ishimda boshlang’ich funksiyalar, aniqmas va aniq integrallar, integrallanuvchi funksiyalar sinfi, aniq integral yordamida tekis shaklni yuzini hisoblash, aniq integral yordamida yoy uzunligi va aylanma jism yuzini hisolash nazaryalarni ochib berdim hamda, hozirgi vaqtda amalyotda yaxshi rivojlangan Maple matematik dasturi matematik masalalarni kompyuter imkonyatlaridan foydalanib yechish samarali natija berishi ham ko’rsatib o’tdim. Jumladan, tatbiqlarini ham ko’p uchratamiz. Bugungi kunda zamonaviy matematikaning turli sohalarida, keng tadbiqqa ega bo‘lgan aniq integralning fizika va mexanika nazariyasi bilan shug‘ullanishmoqda.

Men kelajakdagi izlanishlarimda aniq integrallarni muhandislik sohasida, fizika va mexanikadagi tatbiqlarida yanada ko’proq shug‘ullanmoqchiman.

FOYDALANILGAN ADABIYOTLAR RO‘YXATI.

1. Sh. Mirziyoyev "Erkin va farovon, demokratik O'zbekiston davlatini birgalikda barpo etamiz". T. O'zbekiston, 2016.

2. A, G'oziyev, I, Israilov, M. Yaxshiboyev. " Matematik analizdan misol va masalalar " 2- qism. TOSHKENT-2012.

3. Tao T. Analysis 1, 2. Hindustan Book Agency, India, 2014.

4.Aksoy A. G., Khamsi M. A. A problem book in real analysis.Springerz2010.

5. Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A. Matematik analizdan ma'ruzalar, I, II q. T. "Voris-nashriyot", 2010.

6. Shoimqulov B. A., Tuychiyev T. T., Djumaboyev D. X. Matematik analizdan mustaqil ishlar. T. "O'zbekiston faylasuflari milliy jamiyati", 2008.

7. Фихтенгольц Г. М. Курс дифференциального и интегрального исчисления, 1, 2, 3 т. М. «ФИЗМАТЛИТ», 2001.

8. Садуллаев А., Мансуров Х. Т., Худойберганов Г., Ворисов А. К., Гуломов Р. Математик анализ курсидан мисол ва масалалар тўплами, 1, 2, 3 q. Т. "Ўқитувчи", 1995, 1995, 2000.

9. Шокирова Х. Р. Каррали ва эгри чизиқли интеграллар. Т. "Ўзбекистон", 1990.

10. Демидович Б. П. Сборник задач по математическому анализу. М. «Наука», 1997.

11. Canuto C., Tabacco A. Mathematical Analysis I, II. Springer-Verlag, Italia, Milan, 2008.

12. Ильин В. А., Садовничий В. А., Сендов Б. Х. Математический анализ, 1, 2 т. М. «Проспект», 2007.

13. Zorich V.A. Mathematical Analysis I,II. Springer, 2004, Germany.

14. Азларов Т. А., Мансуров Х. Т. Математик анализ, 1, 2 q. Т. "Ўқитувчи", 1994, 1995.

15. Кудряцев Л. Д. и др. Сборник задач по математическому анализу, 1, 2, 3 т. М. «Наука», 2003.

16. Maruzalar matni.

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1. www.elibrary.com
2. www.nigma.ru
3. www.ziyoNet.uz
4. www.arxiv.uz