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A PROBLEM OF IDENTIFICATION OF A SPECIAL 2D MEMORY KERNEL IN AN INTEGRO–DIFFERENTIAL HYPERBOLIC EQUATION

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Abstract We consider an inverse problem for a partial integro–differential equation of the second order related to recovering a kernel (memory) in the integral term of this equation. It is supposed that the unknown kernel is a trigonometric polynomial with respect to the spatial variables with coefficients continuous with respect to the time variable. The direct problem for a hyperbolic integro–differential equation is the initial-boundary value problem for the half-space $x > 0$ with the zero initial Cauchy data and a special Neumann data at $x = 0$. Local existence theorem and stability estimates for the solution to the inverse problem are obtained.

Key words: kernel, Neumann data, Fourier series, Heaviside step-function, Bessel function, Dirac function, integro–differential equation, Kronecker symbol.

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1 Introduction

Inverse problems for hyperbolic PDEs arise naturally in geophysics, oil prospecting, in the design of optical devices, in many others areas where the interior of an object is to be imaged using the response of to acoustic waves (satisfying hyperbolic PDEs). Problems of identification of memory kernels in hyperbolic equations have been intensively studied starting at the end of the last century (see [1]-[6]). In many cases equations of the electrodynamics and elasticity with integral convolution terms are reduced to one second-order hyperbolic integro–differential equation. Equations of such type describe in the electrodynamics processes with a dispersion while in the elasticity they describe an influence of the viscosity of a material.

In [2], [3] the local existence and the uniqueness of some multidimensional inverse problems for the second-order hyperbolic integro–differential equations in the class of functions having certain smoothness in the time variable and analyticity with respect to the spatial variables were studied. Problems of determining the spatial part of the multidimensional kernel were investigated in the works [4], [5], [7].

Below, using the method of the work [9], we solve the inverse problem of recovering a kernel in the integral term of hyperbolic integro–differential equation. The presented results give a convenient approach for numerical solving the inverse problem.

We consider the integro–differential equation for $(x,\ y,\ t) \in \mathbb{R}^3_+$

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = \int\limits_0^t k(y,\tau) u(x,y,t-\tau) d\tau \tag{1.1}$$

with the initial and boundary conditions

$$u\,|_{t<0} \equiv 0,\ \frac{\partial u}{\partial x}\,|_{x=0} = \delta'(t). \tag{1.2}$$

Here $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\mathbb{R}^3_+ = \{(x,y,t) \in \mathbb{R}^3 \mid x > 0\}$, $\delta'(t)$ is the derivative of the Dirac function.

Boundary condition in (1.2) is generalized function and, that is why the solution to the problem (1.1), (1.2) is understood as a linear functional over the space of infinitely differentiable functions with compact support (test functions), i.e., $u(x,y,t) \in D'\left(\mathbb{R}^3_+\right)$. Therefore, equalities in equations (1.1), (1.2) are understood as the equality of values of the left and right sides for test functions. We note that the kernel in equation (1.1) does not depend on the spatial variable which is perpendicular to the axis where the boundary condition in (1.2) is given. Moreover, we assume that this kernel can be represented in the form of a finite Fourier series

$$k(y,t) = \sum_{s=-N}^{N} k_s(t) e^{isy} \tag{1.3}$$

with a fixed integer $N \geq 0$. Denote by $\Omega\,(N,T,K)$ the set of functions $k(y,t)$ for which the coefficients k_s, $|s| \leq N$, are continuous functions on the interval $[0,\ T]$ and satisfy the conditions

$$|k_s(t)| \leq K,\ t \in [0,\ T],\ -N \leq s \leq N. \tag{1.4}$$

For $k(y,t) \in \Omega\,(N,T,K)$ the solution of problem (1.1), (1.2) is a $2\pi-$periodic function of y and can be represented by the Fourier series

$$u(x,y,t) = \sum_{s=-\infty}^{\infty} u_s(x,t) e^{isy}, \tag{1.5}$$

where as it follows from relations (1.1) - (1.3) the coefficients satisfy the following equations

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right) u_m(x,t) = \int\limits_0^t \sum_{s=-N}^{N} k_s(\tau) u_{m-s}(x,t-\tau) d\tau,$$

$$(x,t) \in \mathbb{R}^2_+ = \left\{(x,t) \in \mathbb{R}^2 \mid x > 0\right\};$$

$$u_m\,|_{t<0} \equiv 0,\ \frac{\partial u_m}{\partial x}\,|_{x=0} = \delta_{0m}\delta'(t),\ m = 0, \pm 1, \pm 2, \ldots. \tag{1.6}$$

In the latter equation δ_{0m} is the Kronecker symbol.

For given functions $k_s(t)$, $s = 0, \pm1, \pm2\pm, ..., \pm N$ we call **the direct problem** the problem of finding the functions $u_m(x,t)$, $m = 0, \pm1, \pm2, ...$ that satisfy (in a generalized meaning) the relations (1.6).

Now let us state the inverse problem that will be studied in the present paper.

The inverse problem. Find the coefficients $k_s(t)$, $s = 0, \pm1, \pm2\pm, ..., \pm N$ by using the conditions

$$u_m(0,t) = f_m(t),\ t \in \mathbb{R}_+,\ \mathbb{R}_+ = \{t \in \mathbb{R} \mid t > 0\},\ m = 0, \pm1, \pm2, ..., \pm N. \tag{1.7}$$

Definition. *Functions* $k_s(t) \in C[0,\infty)$, $s = 0, \pm1, \pm2, ..., \pm N$ *are called a solution to inverse problem (1.6), (1,7), if the the corresponding solution to direct problem (1.7)* $u_m(x,t) \in D'\left(\mathbb{R}^2_+\right)$ *(from the class of generalized functions,i.e., distributions) satisfies equalities (1.7) for* $f_m(t) \in D'(\mathbb{R}_+)$, $m = 0, \pm1, \pm2, ..., \pm N$.

Remark. It turns out that the solution to direct problem (1.6) at $m = 0$ consists of a singular and a regular generalized function. With that at $m = \pm1, \pm2, \dots$ the solution to this problem are only regular generalized functions. The regular parts of the solutions are smooth in some domain (see Lemma 2.1).

2 Direct problem

We begin studying the inverse problem with consideration of some properties of solution to direct problems (1.6). Concerning the solution to direct problems (1.6) the following lemma is valid.

Lemma 2.1. *Let* T *be an arbitrary positive number,* $k_s(y,t) \in \Omega(N,T,K)$ *and* $D(T) = ((x,t)|\ 0 \le x \le T - t)$. *Then the solution to problem (1.6) exists and can be represented in* $D(T)$ *in the form*

$$u_m(x,t) = -\delta_{0m}\delta(t-x) + v_m(x,t)\theta(t-x), m = 0, \pm1, \pm2, \dots, \tag{2.1}$$

where $\theta(t)$ *is the Heaviside step-function:* $\theta(t) = 1$ *for* $t \ge 0$*;* $\theta(t) = 0$ *for* $t < 0$ *and* $v_m(x,t)$ *are continuously differentiable functions in the domain* $D'(T) = ((x,t)|\ 0 \le x \le t \le T - x)$. *Moreover, this solution is unique and there exist positive constants* $C_1 = C_1(N,T,K) \ge 1$ *and* $C_2 = C_2(N,T,K)$, *continuously depending on* K, T *such that*

$$|v_m(x,t)| \le C_1 \frac{KT^2}{2},\ -N \le m \le N,$$
$$|v_m(x,t)| \le C_1 \frac{\left(KT^2\right)^{n+1}(2N+1)^n t^n}{2^{n+1} \cdot n!}, \tag{2.2}$$

if

$$(x,t) \in D'(T),\ Nn < |m| \le (n+1)N,\ n = 1, 2, \dots,$$

and

$$\left|\frac{\partial v_m(x,t)}{\partial t}\right| \le \frac{KT(1+m^2T^2)}{2}[1 + (2N+1)C_2],\ -N \le m \le N,$$
$$\left|\frac{\partial v_m(x,t)}{\partial t}\right| \le (1+m^2T^2)\frac{(KT)^{n+1}(2N+1)^n t^n}{2^{n+1} \cdot n!} \max\{C_1, (2N+1)C_2\}, \tag{2.3}$$

if

$$(x,t) \in D'(T),\ Nn < |m| \le (n+1)N,\ n = 1,2,\dots.$$

Proof. It follows from the hyperbolic equation theory that the solution to the problem (1.1),(1.2) vanishes for all (x, y, t) satisfying the condition $x > t > 0$ because the initial data are zero and the boundary source is located on the axis $x = 0, t = 0$. Hence all $u_m(x,t) = 0$ for $x > t > 0$. To separate the singular part of the solution to direct problem, we represent it in the form

$$u_m(x,t) = \alpha_m \delta(t-x) + v_m(x,t)\theta(t-x), m = 0, \pm1, \pm2, \dots,$$

where α_m are unknown constants, $v_m(x,t)$ are unknown regular functions. Substituting this equality into equations (1.6) and using the method of separation of singularities [8, p. 611-629], we find $\alpha_m = -\delta_{0m}$.

In view of representation (2.1), we rewrite equations (1.6) with respect to $v_m(x,t)$. For this, we use the property of the Dirac function, from which it follows equality

$$\sum_{s=-N}^{N} \delta_{0(m-s)} \int_0^t k_s(\tau)\delta(t-x-\tau)d\tau = \theta(N-|m|)k_m(t-x),\ t > x > 0.$$

Taking this fact into consideration, we conclude that equations (1.6) with respect to $v_m(x,t)$ can be rewritten in the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right) v_m(x,t)$$

$$= -\theta(N-|m|)k_m(t-x) + \sum_{s=-N}^{N} \int_0^{t-x} k_s(\tau)v_{m-s}(x,t-\tau)d\tau,$$

$$t > x > 0,\ m = 0, \pm1, \pm2, \dots,$$

$$v_m \mid_{t \le x} \equiv 0,\ \frac{\partial v_m}{\partial x} \mid_{x=0} = 0,\ m = 0, \pm1, \pm2, \dots. \tag{2.4}$$

For the sake of convenience, we continue all functions $v_m(x,t)$, for $x < 0$ as even functions: $v_m(-x,t) = v_m(x,t)$. Then problem (2.4) is equivalent to the following integral equation

$$v_m(x,t) = -\frac{\theta(N-|m|)}{2} \iint_{\diamondsuit(x,t)} J_0\left(m\sqrt{(t-\tau)^2 - (x-\xi)^2}\right) k_m(\tau - |\xi|)d\tau d\xi$$

$$+\frac{1}{2} \iint_{\diamondsuit(x,t)} J_0\left(m\sqrt{(t-\tau)^2 - (x-\xi)^2}\right) \int_0^{\tau-|\xi|} \sum_{s=-N}^{N} k_s(\alpha)v_{m-s}(\xi, \tau-\alpha)d\alpha d\tau d\xi,$$

$$(x,t) \in D'(T),\ m = 0, \pm1, \pm2, \dots, \tag{2.5}$$

where $J_0(\zeta)$ is the Bessel function and

$$\diamondsuit(x,t)=\left\{(\xi,\tau)\mid |\xi|\le\tau\le t-|x-\xi|,\ \frac{x-t}{2}\le\xi\le\frac{x+t}{2}\right\}.$$

Recall that the Bessel function $J_\nu(\zeta)$ for a fixed integer $\nu\ge 0$ is defined by the formula

$$J_\nu(\zeta)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+\nu)!}\left(\frac{\zeta}{2}\right)^{2j+\nu}.$$

From this formula follows the estimate

$$\left|\frac{J_\nu(\zeta)}{\zeta^\nu}\right|\le\frac{1}{2^\nu\nu!},\ \ |\zeta|\le 2. \tag{2.6}$$

Hereafter we shall also use another estimate for J_ν [9]:

$$|J_\nu(\zeta)|\le 1$$

for all $\zeta\in\mathbb{R}$. Consider for equations (2.5) the method of successive approximations. Define

$$v_m(x,t)=\sum_{j=0}^{\infty}v_m^j(x,t), \tag{2.7}$$

where

$$v_m^0(x,t)=-\frac{\theta(N-|m|)}{2}\iint\limits_{\diamondsuit(x,t)}J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)k_m(\tau-|\xi|)d\tau d\xi,$$

$$v_m^j(x,t)$$

$$=\frac{1}{2}\iint\limits_{\diamondsuit(x,t)}J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\int\limits_0^{\tau-|\xi|}\sum_{s=-N}^{N}k_s(\alpha)v_{m-s}^{j-1}(\xi,\tau-\alpha)d\alpha d\tau d\xi, \tag{2.8}$$

$$(x,t)\in D'(T),\ j=1,2,\dots,\ m=0,\pm1,\pm2,\dots.$$

Obviously all functions $v_m^j(x,t)$ are continuous in $D'(T)$. Moreover, estimating these functions, we get the following estimates

$$\left|v_m^0(x,t)\right|\le\frac{\theta(N-|m|)}{2}$$

$$\times\iint\limits_{\diamondsuit(x,t)}\left|J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\right|\left|k_m(\tau-|\xi|)\right|d\tau d\xi\le\theta(N-|m|)\frac{KT^2}{2},$$

$$\left|v_m^1(x,t)\right|\le\frac{KT^2}{2}\int\limits_0^t\sum_{s=-N}^{N}\max_{|\xi|\le\frac{T}{2}}\left|v_{m-s}^0(\xi,\tau)\right|d\tau\le\theta(2N-|m|)\frac{K^2T^4(2N+1)t}{2^2},$$

$$(x,t)\in D'(T).$$

Continuing these estimates, we easily obtain that

$$\left|v_m^j(x,t)\right|\le\theta\left((j+1)N-|m|\right)\frac{KT^2}{2}\left(\frac{KT^2(2N+1)t}{2}\right)^j\frac{1}{j!},$$

$$(x,t)\in D'(T),\ j=0,1,2,\ldots.$$

Since $t\le T$ series (2.7) uniformly converges in $(x,t)\in D'(T)$ for all m. Hence, its sum is a continuous function in $D'(T)$. Moreover, the following estimates hold

$$|v_0(x,t)|\le\sum_{j=0}^{\infty}\left|v_0^j(x,t)\right|\le\frac{KT^2}{2}\sum_{j=0}^{\infty}\left(\frac{KT^2(2N+1))t}{2}\right)^j\frac{1}{j!}\le\frac{KT^2}{2}C_1,$$

$$|v_m(x,t)|\le\sum_{j=n}^{\infty}\left|v_m^j(x,t)\right|\le\frac{KT^2}{2}\sum_{j=n}^{\infty}\left(\frac{KT^2(2N+1)t}{2}\right)^j\frac{1}{j!}$$

$$\le\frac{\left(KT^2\right)^{n+1}(2N+1)^n t^n}{2^{n+1}n!}C_1,\ (x,t)\in D'(T), \tag{2.9}$$

$$Nn<|m|\le(n+1)N,\ n=0,1,2,\ldots,$$

where $C_1=\exp\left(KT^2(2N+1)/2\right).$

Now differentiating equations (2.5) with respect to t and x, we easily check that functions $v_m(x,t)$ are continuously differentiable in $D'(T)$. We check it for the derivatives with respect to t only. The expressions for these derivatives will be useful in the analysis of the inverse problem. From (2.5), we find

$$\frac{\partial v_m(x,t)}{\partial t}=-\frac{\theta(N-|m|)}{2}\int\limits_{\frac{x-t}{2}}^{\frac{x+t}{2}}\Bigg[k_m(t-|\xi|-|x-\xi|)$$

$$+\int\limits_{|\xi|}^{t-|x-\xi|}H_m(t-\tau,x-\xi)k_m(\tau-|\xi|)d\tau\Bigg]d\xi$$

$$+\frac{1}{2}\int\limits_{\frac{x-t}{2}}^{\frac{x+t}{2}}\int\limits_{0}^{t-|\xi|-|x-\xi|}\sum_{s=-N}^{s=N}k_s(\tau)v_{m-s}(\xi,t-|x-\xi|-\tau)d\tau d\xi$$

$$+\frac{1}{2}\int\limits_{\diamondsuit(x,t)}H_m(t-\tau,x-\xi)\int\limits_{0}^{\tau-|\xi|}\sum_{s=-N}^{s=N}k_s(\alpha)v_{m-s}(\xi,\tau-\alpha)d\alpha d\tau d\xi, \tag{2.10}$$

$$(x,t)\in D'(T),\ m=0,\pm1,\pm2,\ldots.$$

where

$$H_m(t,x)=\frac{\partial}{\partial t}J_0\left(m\sqrt{t^2-x^2}\right)=-m^2t\frac{J_1(\zeta)}{\zeta}\Big|_{\zeta=m\sqrt{t^2-x^2}}.$$

It follows from the definition of the Bessel function that $J_1(\zeta)/\zeta$ is a continuous function for all $\zeta \in [0,\infty)$ and

$$\lim_{\zeta\to 0}\frac{J_1(\zeta)}{\zeta}=\frac{1}{2}. \tag{2.11}$$

From relation (2.10) we see that the derivatives $\partial v_m(x,t)/\partial t$ are, indeed, continuous in $D'(T)$ for all m.

Since from (2.6) it follows that $|J_1(\zeta)/\zeta| \le 1/2$ for all $\zeta\in\mathbb{R}$, we have $|H_m(t,x)| \le m^2T/2$. Therefore we can estimate $\partial v_m(x,t)/\partial t$ as follows

$$\left|\frac{\partial v_m(x,t)}{\partial t}\right|$$

$$\le \frac{KT(1+m^2T^2)}{2}\left[\theta(N-|m|)+\int_0^t\sum_{s=-N}^{N}\max_{\xi\in\Sigma(x,t,\tau)}|v_{m-s}(\xi,\tau)|\,d\tau\right],$$

$$(x,t)\in D'(T),\ m=0,\pm1,\pm2,\ldots,$$

where $\Sigma(x,t,\tau)=\{\xi:(\xi,\tau)\in\Diamond(x,t)\}$. Using (2.2), we get the following estimates

$$\left|\frac{\partial v_m(x,t)}{\partial t}\right| \le \frac{KT(1+m^2T^2)}{2}$$

$$\times\left[1+(2N+1)\int_0^t\max\left\{\max_{\xi\in\Sigma(x,t,\tau),|s|\le N}|v_s(\xi,\tau)|,\max_{\xi\in\Sigma(x,t,\tau),N<|s|\le 2N}|v_s(\xi,\tau)|\right\}d\tau\right]$$

$$\le \frac{KT(1+m^2T^2)}{2}\left[1+(2N+1)\max\left\{\frac{KT^2t}{2\cdot 1!},\frac{\left(KT^2\right)^2(2N+1)t^2}{2^2\cdot 2!}C_1\right\}\right],|m|\le N, \tag{2.12}$$

$$\left|\frac{\partial v_m(x,t)}{\partial t}\right| \le \frac{KT(1+m^2T^2)}{2}\int_0^t\sum_{s=-N}^{N}\max_{\xi\in\Sigma(x,t,\tau)}|v_{m-s}(\xi,\tau)|\,d\tau$$

$$\le \frac{KT(1+m^2T^2)}{2}(2N+1)\int_0^t\max_{\xi\in\Sigma(x,t,\tau),N(n-1)<|s|\le(n+2)N}|v_s(\xi,\tau)|\,d\tau$$

$$\le \frac{KT(1+m^2T^2)}{2}(2N+1)\int_0^t\max\Big\{\max_{\xi\in\Sigma(x,t,\tau),N(n-1)<|s|\le nN}|v_s(\xi,\tau)|,$$

$$\max_{\xi\in\Sigma(x,t,\tau),Nn<|s|\le(n+1)N}|v_s(\xi,\tau)|,\max_{\xi\in\Sigma(x,t,\tau),N(n+1)<|s|\le(n+2)N}|v_s(\xi,\tau)|\Big\}d\tau$$

$$\le \frac{KT(1+m^2T^2)}{2}(2N+1)C_1\max\Big\{\frac{(KT^2)^n(2N+1)^{n-1}t^n}{2^n\cdot n!},$$

$$\frac{(KT^2)^{n+1}(2N+1)^nt^{n+1}}{2^{n+1}\cdot(n+1)!},\frac{(KT^2)^{n+2}(2N+1)^{n+1}t^{n+2}}{2^{n+2}\cdot(n+2)!}\Big\},$$

$$(x,t)\in D'(T),\ Nn<|m|\le(n+1)N,\ n=1,2,\ldots. \tag{2.13}$$

Introducing the notation

$$C_2 = C_1 \max\left\{\frac{KT^3}{2}, \frac{\left(KT^3\right)^2(2N+1)}{2^3}\right\},$$

we come to the estimates

$$\left|\frac{\partial v_m(x,t)}{\partial t}\right| \le \frac{KT(1+m^2T^2)}{2}[1+(2N+1)C_2],\ -N\le m\le N,$$

$$\left|\frac{\partial v_m(x,t)}{\partial t}\right| \le (1+m^2T^2)\frac{(KT)^{n+1}(2N+1)^n t^n}{2^{n+1}\cdot n!}\max\left\{C_1,(2N+1)C_2\right\},$$

$$(x,t)\in \bar{D}'(T),\ Nn<|m|\le (n+1)N,\ n=1,2,\dots,$$

which coincide with (2.3). Similarly one can check that the derivative $\frac{\partial v_m(x,t)}{\partial x}$ is continuous functions in $D'(T)$.

By (2.1), the uniqueness of the solution to problem (1.6) reduces the uniqueness of the solution to (2.4). Since problem (2.4) is equivalent to the integral equation (2.5), the uniqueness of the solution to this equation follows from the general theory of integral equations. The lemma is proven. □

Lemma 2.2. Let $k(y,t)$, $\hat{k}(y,t)$ *be two arbitrary functions of the set* $\Omega(N,T,K)$ *and* $u_m(x,t)$, $\hat{u}_m(x,t)$, $m=0,\pm1,\pm2,...$, *be the solutions to problem (1.6) which correspond to* $k(y,t)$, $\hat{k}(y,t)$ *respectively, and* $\tilde{v}_m(x,t)=v_m(x,t)-\hat{v}_m(x,t)$, $\tilde{k}_s(t)=k_s(t)-\hat{k}_s(t)$. *Then there exist constants* $C_3=C_3(N,T,K)\ge 1$ *and* $C_4=C_4(N,T,K)$, *continuously depending on* K, T, *such that*

$$|\tilde{v}_0(x,t)|\le C_3\frac{\tilde{K}T^2}{2},$$

$$|\tilde{v}_m(x,t)|\le C_3\frac{\tilde{K}T^2}{2}\frac{\left(KT^2(2N+1)t\right)^n}{2^n\cdot n!},$$

if

$$(x,t)\in \bar{D}'(T),\ nN<|m|\le (n+1)N,\ n=0,1,2,.... \tag{2.14}$$

and

$$\left|\frac{\partial \tilde{v}_m(x,t)}{\partial t}\right| \le \tilde{K}\Big[\frac{T(1+m^2T^2)}{2}+C_4\frac{KT^2}{2}\frac{t}{1!}\Big],\ |m|\le N,$$

$$\left|\frac{\partial \tilde{v}_m(x,t)}{\partial t}\right| \le C_4\tilde{K}\frac{\left(KT^2(2N+1)\right)^n}{2^2}\frac{t^n}{n!}, \tag{2.15}$$

if

$$(x,t)\in D'(T), Nn<|m|\le (n+1)N,,\ n=1,2,\dots,$$

where $\tilde{K}=\max\limits_{-N\le s\le N}\max\limits_{0\le t\le T}\left|k_s(t)-\hat{k}_s(t)\right|$.

Proof. Using equations (2.5) with (v_m,k_s) and $(\hat{v}_m,\hat{k}_s)$ subtracting one from the other, we find

$$\tilde{v}_m(x,t)=\frac{\theta(N-|m|)}{2}\iint\limits_{\Diamond(x,t)} J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\tilde{k}_m(\tau-|\xi|)d\tau d\xi$$

$$+\frac{1}{2}\iint\limits_{\diamondsuit(x,t)} J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\int\limits_0^{\tau-|\xi|}\sum_{s=-N}^{N}\Big[k_s(\alpha)\tilde{v}_{m-s}(\xi,\tau-\alpha)$$

$$+\tilde{k}_s(\alpha)\hat{v}_{m-s}(\xi,\tau-\alpha)\Big]d\alpha d\tau d\xi,$$

$$(x,t)\in D'(T),\ m=0,\pm1,\pm2,\dots. \tag{2.16}$$

Represent $\tilde{v}_m(x,t)$ in the form

$$\tilde{v}_m(x,t)=\sum_{j=0}^{\infty}\tilde{v}_m^j(x,t), \tag{2.17}$$

where

$$\tilde{v}_m^0=\frac{\theta(N-|m|)}{2}\iint\limits_{\diamondsuit(x,t)} J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\tilde{k}_m(\tau-|\xi|)d\tau d\xi$$

$$+\frac{1}{2}\iint\limits_{\diamondsuit(x,t)} J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\int\limits_0^{\tau-|\xi|}\sum_{s=-N}^{N}\tilde{k}_s(\alpha)\hat{v}_{m-s}(\xi,\tau-\alpha)d\alpha d\xi d\tau, \tag{2.18}$$

$$\tilde{v}_m^j(x,t)$$

$$=\frac{1}{2}\iint\limits_{\diamondsuit(x,t)} J_0\left(m\sqrt{(t-\tau)^2-(x-\xi)^2}\right)\int\limits_0^{\tau-|\xi|}\sum_{s=-N}^{N}k_s(\alpha)\tilde{v}_{m-s}^{j-1}(\xi,\tau-\alpha)d\alpha d\xi d\tau, \tag{2.19}$$

$$(x,t)\in D'(T),\ j=0,1,2...,\ m=0,\pm1,\pm2,\dots.$$

From here we obtain

$$|\tilde{v}_m^0(x,t)|\le\frac{\tilde{K}T^2}{2}\theta(N-|m|)+\frac{\tilde{K}T^2}{2}\int_0^t\sum_{s=-N}^{N}\max_{|\xi|\le T/2}|\hat{v}_{m-s}(\xi,\tau)|\,d\tau,$$

$$|\tilde{v}_m^j(x,t)|\le\frac{KT^2}{2}\int_0^t\sum_{s=-N}^{N}\max_{|\xi|\le T/2}\left|\tilde{v}_{m-s}^{j-1}(\xi,\tau)\right|d\tau, \tag{2.20}$$

$$(x,t)\in D'(T),\ j=1,2,...,\ m=0,\pm1,\pm2,\dots.$$

Using the estimates (2.2) for functions $\hat{v}_m(x,t)$ here, as result we get

$$|\tilde{v}_0^0(x,t)|\le\frac{\tilde{K}T^2}{2}+\frac{\tilde{K}T^2}{2}(2N+1)\int_0^t\max_{|\xi|\le T/2,|s|\le 2N}|\hat{v}_s(\xi,\tau)|$$

$$\le\frac{\tilde{K}T^2}{2}\left(1+(2N+1)C_2\right),$$

$$|\tilde{v}_m^0(x,t)|\le\frac{\tilde{K}T^3}{2}+\frac{\tilde{K}T^2}{2}(2N+1)\int_0^t\max_{|\xi|\le T/2,Nn<|s|\le(n+1)N}|\hat{v}_s(\xi,\tau)|$$

$$\leq \frac{\tilde{K}T^2}{2}\left(T+(2N+1)C_2\right),\ |m|\leq N,$$

$$|\tilde{v}_m^0(x,t)| \leq \frac{\tilde{K}T^2}{2}(2N+1)\int_0^t \max_{|\xi|\leq T/2,|s|\leq T}|\hat{v}_{m-s}(\xi,\tau)|$$

$$\leq \tilde{K}\frac{T^{2(n+1)}K^n(2N+1)^n t^n}{2^{n+1}\cdot n!}\max\left\{C_1,(2N+1)C_2\right\},$$

$$(x,t)\in D'(T),\ nN<|m|\leq(n+1)N,\ n=1,2,....$$

In view of $C_1 \geq 1$, the last estimates we can write in the more compact form as follows

$$|\tilde{v}_m^0(x,t)| \leq (C_1+(2N+1)C_2)\,\frac{\tilde{K}T^2}{2}, -N\leq m\leq N,$$

$$|\tilde{v}_m^0(x,t)| \leq (C_1+(2N+1)C_2)\,\frac{\tilde{K}T^2}{2}\frac{K^nT^{2n}(2N+1)^n t^n}{2^n\cdot n!},$$

$$(x,t)\in D'(T),\ nN<|m|\leq(n+1)N,\ n=1,2,....$$

Further calculations for $j \geq 1$ show that estimates

$$|\tilde{v}_0^j(x,t)| \leq (C_1+(2N+1)C_2)\,\frac{\tilde{K}T^2}{2}\frac{\left(KT^2(2N+1)t\right)^j}{2^j\cdot j!},$$

$$|\tilde{v}_m^j(x,t)|$$

$$\leq (C_1+(2N+1)C_2)\,\frac{\tilde{K}T^2}{2}\frac{\left(KT^2(2N+1)t\right)^{n+j}}{2^{n+j}\cdot(n+j)!}\max\left(1,\frac{KT^2t}{2(n+j+1)}\right),$$

$$(x,t)\in D'(T),\ nN<|m|\leq(n+1)N,\ n=0,1,2,...$$

hold. Hence,

$$|\tilde{v}_0(x,t)| \leq C_1\left(C_1+(2N+1)C_2\right)\frac{\tilde{K}T^2}{2},$$

$$|\tilde{v}_m(x,t)| \leq C_1\left(C_1+(2N+1)C_2\right)\frac{\tilde{K}T^2}{2}\frac{\left(KT^2(2N+1)t\right)^n}{2^n\cdot n!}\max\left(1,\frac{KT^3}{2}\right),$$

$$(x,t)\in D'(T),\ nN<|m|\leq(n+1)N,\ n=0,1,2,....$$

Thus, estimate (2.14) holds with

$$C_3 = C_1\left(C_1+(2N+1)C_2\right)\max\left(1,\frac{KT^3}{2}\right).$$

For proving (2.15), we use the relation (2.10) for (v_m, k_s) and $(\hat{v}_m, \hat{k}_s)$. Subtracting one from the other, we get

$$\frac{\partial\tilde{v}_m(x,t)}{\partial t}=\frac{\theta(N-|m|)}{2}\int\limits_{\frac{x-t}{2}}^{\frac{x+t}{2}}\Big[\tilde{k}_m(t-|\xi|-|x-\xi|)$$

$$+\int\limits_{|\xi|}^{t-|x-\xi|}H_m(t-\tau,x-\xi)\tilde{k}_m(\tau-|\xi|)d\tau\Big]d\xi$$

$$+\frac{1}{2}\int\limits_{\frac{x-t}{2}}^{\frac{x+t}{2}}\int\limits_{0}^{t-|\xi|-|x-\xi|}\sum_{s=-N}^{N}\Big[k_m(\tau)\tilde{v}_{m-s}(\xi,t-|x-\xi|-\tau)$$

$$+\tilde{k}_m(\tau)\hat{v}_{m-s}(\xi,t-|x-\xi|-\tau)\Big]d\tau d\xi$$

$$+\frac{1}{2}\int\limits_{\diamondsuit(x,t)}H_m(t-\tau,x-\xi)\int\limits_{0}^{\tau-|\xi|}\sum_{s=-N}^{N}\Big[k_m(\alpha)\tilde{v}_{m-s}(\xi,\tau-\alpha)$$

$$+\tilde{k}_m(\alpha)\hat{v}_{m-s}(\xi,\tau-\alpha)\Big]d\alpha d\tau d\xi,$$

$$(x,t)\in D'(T),\ m=0,\pm1,\pm2,\ldots.$$

Estimating each term in this equality, we have

$$\left|\frac{\partial\tilde{v}_m(x,t)}{\partial t}\right|\leq\frac{\tilde{K}T}{2}\left(1+m^2T^2\right)\theta\left(N-|m|\right)$$

$$+\frac{2T+m^2T^3}{4}\sum_{s=-N}^{N}\int\limits_{0}^{t}\max_{\xi\in\Sigma(x,t,\tau)}\Big[K\left|\tilde{v}_{m-s}(\xi,\tau)\right|+\tilde{K}\left|\hat{v}_{m-s}(\xi,\tau)\right|\Big]d\tau.$$

Using estimates (2.2) and (2.14), we find

$$\left|\frac{\partial\tilde{v}_m(x,t)}{\partial t}\right|\leq\tilde{K}\Bigg[\frac{T(1+m^2T^2)}{2}$$

$$+\frac{2T+m^2T^3}{4}\frac{KT^2}{2}\left(C_1+C_3\right)\left(1+\frac{KT^3(2N+1)}{2}\right)\frac{t}{1!}\Bigg],\ |m|\leq N,$$

$$\left|\frac{\partial\tilde{v}_m(x,t)}{\partial t}\right|\leq\tilde{K}\frac{2T+m^2T^3}{4}\left(C_1+C_3\right)\frac{\left(KT^2(2N+1)\right)^n}{2^2}\frac{t^n}{n!}$$

$$\times\max\left[1+\frac{(KT^3)(2N+1)}{2},\ \frac{(KT^3)^3(2N+1)^2}{2^3}\right],$$

$$(x,t)\in D'(T),Nn<|m|\leq(n+1)N,,\ n=1,2,\ldots.$$

Hence, estimates (2.15) hold with

$$C_4=\frac{2T+m^2T^3}{4}\left(C_1+C_3\right)\max\left[1+\frac{(KT^3)(2N+1)}{2},\ \frac{(KT^3)^3(2N+1)^2}{2^3}\right].$$

The lemma is proven. □

3 The existence and uniqueness theorem

Taking into account presentation (2.1) we conclude that for solvability of the inverse problem the functions $f_m(t)$, $|m| \le N$ must be as follows

$$f_m(t) = -\delta_{0m}\delta(t) + \theta(t)g_m(t),\ t \in \mathbb{R},\ |m| \le N, \tag{3.1}$$

where $g_m(t)$ is a given smooth functions. Then, it is obvious, that the additional conditions for $v_m(x,t)$, as it follow from (1.7) and (2.1), have the forms

$$v_m(0,t) = g_m(t),\ t > 0,\ |m| \le N. \tag{3.2}$$

Using these, from (2.10) after some transformations, one gets

$$\begin{aligned} &g_m'(t) \\ &= \frac{\theta(N-|m|)}{2}\int_0^t k_m(\tau)d\tau + \frac{\theta(N-|m|)}{2} + \int_0^{\frac{t}{2}}\int_\xi^{t-\xi} H_m(t-\tau,\xi)k_m(\tau-\xi)d\tau d\xi \\ &+ \int_0^{\frac{t}{2}}\int_0^{t-2\xi}\sum_{s=-N}^{N} k_s(\tau)v_{m-s}(\xi,t-\xi-\tau)d\tau d\xi \\ &+ \int_0^{\frac{t}{2}}\int_\xi^{t-\xi} H_m(t-\tau,\xi)\int_0^{\tau-\xi}\sum_{s=-N}^{N} k_s(\alpha)v_{m-s}(\xi,\tau-\alpha)d\alpha d\tau d\xi. \end{aligned}$$

By differentiating this equality, we obtain the integral equation for $k_m(t)$:

$$\begin{aligned} k_m(t) &= 2g_m''(t) + \frac{m^2}{2}\int_0^t (t-\tau)k_m(\tau)d\tau - 2\int_0^{\frac{t}{2}}\int_\xi^{t-\xi} H_m'(t-\tau,\xi)k_m(\tau-\xi)d\tau d\xi \\ &-2\int_0^{\frac{t}{2}}\int_0^{t-2\xi}\sum_{s=-N}^{N} k_s(\tau)\left[-\frac{m^2\xi}{2}v_{m-s}(\xi,t-\xi-\tau) + \frac{\partial}{\partial t}v_{m-s}(\xi,t-\xi-\tau)\right]d\tau d\xi \\ &-2\int_0^{\frac{t}{2}}\int_\xi^{t-\xi} H_m'(t-\tau,\xi)\int_0^{\tau-\xi}\sum_{s=-N}^{N} k_s(\tau)v_{m-s}(\xi,\tau-\alpha)d\alpha d\tau d\xi,\ |m| \le N. \end{aligned} \tag{3.3}$$

For getting the last equation we have used (2.11) and the relation $v_m(x,|x|+0) = 0$ which follows from (2.5) for all m. In (2.12)

$$H_m'(t,x) = \frac{\partial}{\partial t}H_m(t,x) = -m^2\left[\frac{J_1(\zeta)}{\zeta} - m^2t^2\frac{J_2(\zeta)}{\zeta^2}\right]_{\zeta=\sqrt{t^2-x^2}}. \tag{3.4}$$

Since the functions $J_1(\zeta)/\zeta$ and $J_2(\zeta)/\zeta^2$ are continuous for all $\zeta \in [0,\infty)$, the function $H_m'(t,x)$ is continuous for $(x,t) \in D'(T)$.

Note that integral equations (2.5) and (2.10) determine $v_m(x,t)$, $\frac{\partial v_m(x,t)}{\partial t}$ respectively, as a function of $k_s(t)$, $s = -N,...,N$. Hence, introducing the operator $U =$

$(U_{-N},...,U_N)$ by the right sides of (3.3), we can rewrite relations (3.3) as the operator equations

$$k_m(t) = U_m\left(k_{-N}(t),...,k_N(t)\right), \tag{3.5}$$
$$m = 0, \pm 1, \pm 2, ..., \pm N,\ t \in [0,\ T].$$

Let

$$k_m^0(t) = 2g''_m(t),\ m = 0, \pm 1, \pm 2, ..., \pm N. \tag{3.6}$$

Denote by $\Omega_0\left(N, T, K_0\right)$ the set of functions $k_s(t)$, $-N \le s \le N$, satisfying the conditions

$$\left\|k_m(t) - k_m^0(t)\right\|_{C[0,T]} \le K_0,\ K_0 = \max_{-N\le s\le N}\left\|k_m^0(t)\right\|_{C[0,T]}, \tag{3.7}$$
$$m = 0, \pm 1, \pm 2, ..., \pm N.$$

If functions $k_s(t) \in \Omega_0\left(N, T, K_0\right)$ for $-N \le s \le N$ then, obviously, $k_s(t) \in \Omega\left(N, T, 2K_0\right)$. For operator equations (3.5) the following theorem holds.

Theorem 3.1. *Let presentation (3.1) be valid. Moreover, let data (3.2) satisfy the following conditions:*

$$g_m(t) \in C^2[0,\ T],\ g_m(0) = g'_m(0) = 0,\ m = 0, \pm 1, \pm 2, ..., \pm N.$$

and

$$G = \max_{-N\le s\le N}\|g''_m(t)\|_{C[0,T]}. \tag{3.8}$$

Then there exists a number $T_0 \in (0,\ T)$ such that operator equations (3.5) have a unique solution, belonging to the set $\Omega_0\left(N, T_0, 2G\right)$.

Proof. It is obvious that $\|k_0\| = 2G$. We prove that operator $U = (U_{-N},...,U_N)$ maps the set $\Omega_0\left(N, T, 2G\right)$ into itself and is a contraction operator if T is sufficiently small. The theorem of existence and uniqueness then follows immediately from the Contraction Mapping Principle.

Estimate first the function $H'_m(t-\tau,\xi)$ defined by (3.4)for

$$(\xi,\tau) \in \triangleright(t) = \left\{(\xi,\tau) : 0 \le \xi \le \frac{t}{2},\ \xi \le \tau \le t - \xi\right\}.$$

Since from (2.6) follows that

$$\left|\frac{J_1(\zeta)}{\zeta}\right| \le \frac{1}{2},\ \left|\frac{J_2(\zeta)}{\zeta^2}\right| \le \frac{1}{4}$$

for all $\zeta \in \mathbb{R}$, then the following estimate holds

$$|H'_m(t-\tau,\xi)| \le \frac{m^2(2 + m^2T^2)}{4},\ (\xi,\tau) \in \triangleright(t),\ t \in [0,\ T]. \tag{3.9}$$

Demonstrate that for sufficiently small T the operator $U = (U_{-N},...,U_N)$ is a contraction mapping of the set $\Omega_0\left(N, T, 2G\right)$ into itself. Indeed, for $k_m(t) \in \Omega_0\left(N, T, 2G\right)$, $-N \le m \le N$ the inclusion $k_m(t) \in \Omega\left(N, T, 4G\right)$, $-N \le m \le N$ is valid. Then, in view of (3.9) from relations (3.3) we obtain

$$\left|k_m(t)-k_m^0(t)\right| \le m^2GT^2\left(3+m^2T^2\right)+8G(2N+1)$$
$$\times\iint\limits_{\triangleright(t)}\left[\frac{m^2\xi}{2}\max_{-N\le s\le N}\left|v_{m-s}(\xi,\tau-\xi)\right|+\max_{-N\le s\le N}\left|\frac{\partial}{\partial t}v_{m-s}(\xi,\tau-\xi)\right|\right]$$
$$+8G(2N+1)\iint\limits_{\triangleright(t)}\int_0^{\tau-\xi}\max_{-N\le s\le N}\left|v_{m-s}(\xi,\tau-\alpha)\right|d\alpha d\tau d\xi$$
$$\le T^2N^2G\left(3+N^2T^2\right)+2T^2G(2N+1)$$
$$\times\left[\frac{N^2T}{4}\max_{-2N\le j\le 2N}\max_{(\xi,\tau)\in\bar{D}'(T)}\left|v_j(\xi,\tau)\right|+\max_{-2N\le j\le 2N}\max_{(\xi,\tau)\in\bar{D}'(T)}\left|\frac{\partial}{\partial t}v_j(\xi,\tau)\right|\right]$$
$$+\frac{3T^3G(2N+1)}{4}\max_{-2N\le j\le 2N}\max_{(\xi,\tau)\in\bar{D}'(T)}\left|v_j(\xi,\tau)\right|,$$
$$t\in[0,T],\ -N\le m\le N.$$

For functions $v_m(x,t)$ and $\frac{\partial}{\partial t}v_m(x,t)$ the estimates (2.2) and (2.3) valid with $K=4G$. Using them, we find

$$\left|k_m(t)-k_m^0(t)\right|\le GN^2T^2\left(3+N^2T^2\right)$$
$$+2T^2G(2N+1)\Big[\frac{1}{2}GN^2T^3C_1\max\left(1,\ 2GT^3(2N+1)\right)$$
$$+2GT(1+N^2T^2)\max\left(\max(1,C_1)+(2N+1)C_2\right)\max\left(1,2GT^2(2N+1)\right)\Big]$$
$$+\frac{3}{2}G(2N+1)T^2C_1\max\left(1,\ 2GT^3(2N+1)\right)\equiv 2GT^2C_5(N,T,G),$$
$$t\in[0,T],\ -N\le m\le N,$$

where C_5 is positive constant, continuously depending on T. Choosing $T_1=T_1(N,G)$ as a positive root of the equation

$$T^2C_5(N,T,G)=\sigma,$$

for some $0<\sigma<1$, we obtain that

$$\left|k_m(t)-k_m^0(t)\right|\le 2G,\ t\in[0,T_1],\ -N\le m\le N,$$

i.e. the operator $U=(U_{-N},...,U_N)$ maps the set $\Omega_0\left(N,T_1,2G\right)$ into itself.

Now let $k_m(t),\ \hat{k}_m(t),\ m=-N,...,N$ be two solutions of operator equations (3.5) belonging to the set $\Omega_0\left(N,T,2G\right)$. Denote corresponding solutions of integral equations (2.5), (2.10) by v_m, $\hat{v}_m$ respectively, and its derivatives with respect to t by $\partial v_m/\partial t$ and $\partial\hat{v}_m/\partial t$. Denoting

$$\tilde{v}_m=v_m-\hat{v}_m,\ \frac{\partial\tilde{v}}{\partial t}=\frac{\partial v}{\partial t}-\frac{\partial\hat{v}}{\partial t},\ \tilde{k}_s(t)=k_s(t)-\hat{k}_s(t),\ \tilde{K}=\max_{-N\le s\le N}\max_{0\le t\le T}\left|\tilde{k}_s(t)\right|$$

from (3.3), we find

$$\tilde{k}_m(t) = \frac{m^2}{2}\int_0^t (t-\tau)\tilde{k}_m(\tau)d\tau - 2\int_0^{\frac{t}{2}}\int_{\xi}^{t-\xi} H'_m(t-\tau,\xi)\tilde{k}_m(\tau-\xi)d\tau d\xi$$

$$-2\int_0^{\frac{t}{2}}\int_0^{t-2\xi}\sum_{s=-N}^{N}\left\{\tilde{k}_s(\tau)\left[-\frac{m^2\xi}{2}v_{m-s}(\xi,t-\xi-\tau)+\frac{\partial}{\partial t}v_{m-s}(\xi,t-\xi-\tau)\right]\right.$$

$$\left.+\hat{k}_s(\tau)\left[-\frac{m^2\xi}{2}\tilde{v}_{m-s}(\xi,t-\xi-\tau)+\frac{\partial}{\partial t}\tilde{v}_{m-s}(\xi,t-\xi-\tau)\right]\right\}d\tau d\xi$$

$$-2\int_0^{\frac{t}{2}}\int_{\xi}^{t-\xi} H'_m(t-\tau,\xi)\int_0^{\tau-\xi}\sum_{s=-N}^{N}\left\{\tilde{k}_s(\tau)v_{m-s}(\xi,\tau-\alpha)+\hat{k}_s(\tau)\tilde{v}_{m-s}(\xi,t-\tau)\right\}d\alpha d\tau d\xi,$$

$$t\in[0,T],\ \ |m|\le N.$$

Estimating the terms of this equality, we get

$$\left|\tilde{k}_m(t)\right| \le \frac{T^2m^2\tilde{K}}{8}\left(3+m^2T^2\right)$$

$$+\frac{T^2}{2}(2N+1)\max_{-2N\le j\le 2N}\max_{(\xi,\tau)\in\bar{D}'(T)}\left\{\tilde{K}\left[\frac{m^2T}{4}\left|v_j(\xi,\tau)\right|+\left|\frac{\partial v_j(\xi,\tau)}{\partial t}\right|\right]\right.$$

$$\left.+4G\left[\frac{m^2T}{4}\left|\tilde{v}_j(\xi,\tau)\right|+\left|\frac{\partial \tilde{v}_j(\xi,\tau)}{\partial t}\right|\right]\right\}+\frac{T^3m^2(2+m^2T^2)}{8}(2N+1)$$

$$\times\max_{-2N\le j\le 2N}\max_{(\xi,\tau)\in\bar{D}'(T)}\left[\tilde{K}\left|v_j(\xi,\tau)\right|+4G\left|\tilde{v}_j(\xi,\tau)\right|\right],\ t\in[0,T],\ m=-N,...,N.$$

Using estimates (2.1), (2.2) and (2.14), (2.15), we find

$$\left|\tilde{k}_m(t)\right| \le T^2N^2\tilde{K}\frac{3+T^2N^2}{8}+\frac{T^2}{2}(2N+1)\tilde{K}\left[\frac{T^2N^2}{2}(C_1+C_3)G\right.$$

$$\times\max\left(1,2G(2N+1)T^3\right)+2T\left(1+T^2N^2\right)G\left(C_1+(2N+1)C_2\right)$$

$$\left.\times\max\left(1,2G(2N+1)T^3\right)+4TG\max\left(\frac{1+T^2(N^2+4C_4G)}{2},T^2C_4G(2N+1)\right)\right]$$

$$+\frac{T^5N^2}{4}(2+T^2N^2)(2N+1)\left(C_1+C_3\right)\max\left(1,2G(2N+1)T^3\right)\equiv\tilde{K}T^2C_6\left(N,T,G\right),$$

where C_6 is positive constant, continuously depending on T. Set a fixed $\rho\in(0,1)$ and define $T_2=T_2\left(N,G,\rho\right)$ as the positive root of the equation

$$T^2C_6\left(N,T,G\right)=\rho.$$

Then the estimates

$$\left|\tilde{k}_m(t)\right|\le\rho\tilde{K},\ \ m=-N,...,N$$

implies that the operator $U = (U_{-N}, ..., U_N)$ is contractive on the set $\Omega_0\left(N, T_2, 2G\right)$. Taking $T_0 = \min\left(T_1, T_2\right)$ we get that this operator maps the set $\Omega_0\left(N, T_2, 2G\right)$ into itself and is a contraction operator on this set. By the Banach principle operator equation (3.5) has a unique solution belonging to the set $\Omega_0\left(N, T_2, 2G\right)$. $\square$

Remark 3.1. As in the paper [10], the solution of the inverse problem found in the local sense can be uniquely extended to the segment $[0, T]$.

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