ISSN 1066-369X, Russian Mathematics, 2017, Vol. 61, No. 11, pp. 34–39. © Allerton Press, Inc., 2017.
Original Russian Text © M. Mirsaburov, S.T. Chorieva, 2017, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2017, No. 11, pp. 39–45.

A Problem With an Analog of Frankl Condition on the Characteristics for Gellerstedt Equation With Singular Coefficient

M. Mirsaburov* **and S. T. Chorieva***

Termez State University
ul. F. Khodzhaeva 43, Termez, 190111 Republic of Uzbekistan

Received June 26, 2016

Abstract—We investigate the problem with an analog of Frankl condition on boundary characteristics for generalized Tricomi equation. We prove that the formulated problem is correct.

DOI: 10.3103/S1066369X17110056

Keywords: *analog of Frankl condition on characteristics, uniqueness of solution, singular integral equation, index.*

1. Statement of Problem A. Let Ω be finite simply connected domain of the complex plane $z = x + iy$ bounded for $y > 0$ by normal curve $\sigma_0(y = \sigma_0(x)) : x^2 + 4(m+2)^{-2}y^{m+2} = 1$ with ends at points $A(-1,0)$ and $B(1,0)$, and for $y < 0$ by characteristics AC and BC of the equation

$$(\operatorname{sgn} y)|y|^m u_{xx} + u_{yy} + (\beta_0/y)u_y = 0, \tag{1}$$

where $m > 0$, $\beta_0 \in (-m/2, 1)$. We denote by Ω^+ and Ω^- the parts of domain Ω lying in half-planes $y > 0$ and $y < 0$, respectively.

In the present paper we prove uniqueness and existence of solution to a problem with analog of the Frankl condition [1–4] on characteristic AC and segment AB of line of change of type of Eq. (1).

Problem A. *Find function* $u(x,y) \in C\left(\overline{\Omega}\right)$ *satisfying the following requirements*:

1) $u(x,y)$ *belongs to* $C^2(\Omega^+)$ *and satisfies Eq.* (1) *in this domain*;

2) $u(x,y)$ *is a generalized solution of class* R_1 ([5], P. 129) (*the cited below expression* (8) *is called generalized solution of class* R_1 *in domain* Ω^- *if* $\tau(x), \nu(x) \in C(-1,1)$) *in* Ω^-;

3) *on the interval of conjugation there is valid the condition*

$$\lim_{y\to-0}(-y)^{\beta_0}\frac{\partial u}{\partial y} = \lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y}, \quad x \in I, \tag{2}$$

and these limits for $x \to \pm 1$ *can have singularities of order lesser* $1 - 2\beta$, *where* $\beta = (m + 2\beta_0)/2(m+2)$, $I = (-1,1)$ *is interval of axis* $y = 0$;

4) *there are fulfilled conditions*

$$u(x, \sigma_0(x)) = \varphi(x), \ \ x \in \overline{I}, \tag{3}$$

$$a(x)D^{1-\beta}_{-1,x}u[\theta(x)] + b(x)D^{1-\beta}_{x,1}u[\theta(-x)] = \psi(x), \ \ x \in I, \tag{4}$$

$$u(-x,0) - u(x,0) = f(x), \ \ x \in \overline{I}, \tag{5}$$

where $D^{1-\beta}_{-1,x}$ *and* $D^{1-\beta}_{x,1}$ *are operators of fractional differentiation,* $\theta(x_0) = (x_0-1)/2 - i[(m+2)(1+x_0)/4]^{2/(m+2)}$ *is affix of intersection of characteristic* AC *with characteristic originating from point* $(x_0, 0)$, $x_0 \in I$, $\varphi(x)$, $a(x)$, $b(x)$, $\psi(x)$, $f(x)$ *are given functions, and*

$$(1-x)^\beta a(x) - (1+x)^\beta b(x) \neq 0, \ \ x \in I. \tag{6}$$

*E-mail: mirsaburov@mail.ru.

Note that in the boundary-value problem with delay [6] shift (4) is defined on the both characteristics AC and BC. In problem A condition (4) is given only on characteristic AC, i.e., the characteristic BC is free of boundary-value condition. Conditions (4) and (5) are analogs of the Frankl condition on the characteristic AC and on the line of degeneration AB, respectively.

2. Uniqueness of solution to Problem A. Solution to the modified Cauchy problem with data

$$u(x,0) = \tau(x),\ \ x \in \overline{I};\ \ \lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y} = \nu(x),\ \ x \in I, \tag{7}$$

for Eq. (1) in domain Ω^- is determined by Darboux formula ([7], P. 34)

$$u(x,y) = \gamma_1 \int_{-1}^{1} \tau\left[x + \frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1+t)^{\beta-1}(1-t)^{\beta-1}dt$$
$$+ \gamma_2(-y)^{1-\beta_0}\int_{-1}^{1} \nu\left[x + \frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1+t)^{-\beta}(1-t)^{-\beta}dt, \tag{8}$$

where

$$\gamma_1 = \frac{2^{1-2\beta}\Gamma(2\beta)}{\Gamma^2(\beta)},\ \ \gamma_2 = -\frac{2^{2\beta-1}\Gamma(2-2\beta)}{(1-\beta_0)\Gamma^2(1-\beta)}.$$

We conclude from the boundary-value condition (4) by virtue of (8) and (5) that ($\tau(-x) = \tau(x) + f(x)$), and

$$(1-x)^{\beta}a(x)\nu(x) + (1+x)^{\beta}b(x)\nu(-x)$$
$$= \gamma\Big((1-x)^{\beta}a(x)D^{1-2\beta}_{-1,x}\tau(x) + (1+x)^{\beta}b(x)D^{1-2\beta}_{x,1}\tau(x)\Big) + \Psi(x), \tag{9}$$

where

$$\gamma = \frac{2\Gamma(1-\beta)\Gamma(2\beta)}{\Gamma(\beta)\Gamma(1-2\beta)}\left(\frac{m+2}{4}\right)^{2\beta},\ \Psi(x) = \frac{(1-x^2)^{\beta}\psi(x)}{\gamma_2\Gamma(1-\beta)((m+2)/2)^{1-2\beta}} - \gamma(1+x)^{\beta}b(x)D^{1-2\beta}_{x,1}f(x).$$

Theorem 1 (analog of A. V. Bitsadze's extremum principle, [8], P. 301). *Under conditions $\psi(x)\equiv 0$, $f(x)\equiv 0$, $\varphi(x)\equiv 0$ and*

$$a(x) > 0,\ \ b(x) > 0,\ \ \ x \in \overline{I}, \tag{10}$$

a solution $u(x,y)$ to problem A *attains its most positive meaning (MPM) and least negative meaning (LNM) in closed domain $\overline{\Omega}^+$ at points of curve σ_0.*

Proof. Let function $u(x,y)$ attain its MPM at point $(x_0,y_0)\in\Omega$. Then by virtue of the Hopf principle ([7], P. 25) $(x_0,y_0)\notin\Omega^+$.

Let solution $u(x,y)$ attains its MPM in an inner point $P(x_0,0)$ of interval $I = AB$. Then by virtue of corresponding homogeneous condition (5) the solution $u(x,y)$ also attains its MPM at point $(-x_0,0)$. Hence, by the Zaremba–Giraud principle ([7], P. 74) we have $\nu(x_0) < 0$, $\nu(-x_0) < 0$ at these points. Whence, by means of (10) we obtain

$$(1-x_0)^{\beta}a(x_0)\nu(x_0) + (1+x_0)^{\beta}b(x_0)\nu(-x_0) < 0,\ \ x_0 \in I. \tag{11}$$

On the other hand, it is well-known that at a point of MPM of function $\tau(x)$ the operators of fractional differentiation satisfy the following inequalities ([5], P. 19):

$$D^{1-2\beta}_{-1,x}\tau(x)|_{x=x_0} > 0,\ \ D^{1-2\beta}_{x,1}\tau(x)|_{x=x_0} > 0.$$

We obtain from these inequalities by means of corresponding homogeneous relation (9) with $\Psi(x)\equiv 0$

$$(1-x_0)^{\beta}a(x_0)\nu(x_0) + (1+x_0)^{\beta}b(x_0)\nu(-x_0)$$

$$= \gamma \Big((1-x)^{\beta} a(x) D^{1-2\beta}_{-1,x} \tau(x) + (1+x)^{\beta} b(x) D^{1-2\beta}_{x,1} \tau(x) \Big) |_{x=x_0} > 0, \ \ x \in I. \quad (12)$$

Clearly, (12) contradicts (11).

If $P(x_0, 0) = O(0,0)$, then in just the same way we obtain contradictory of inequalities (11) and (12), where $x_0 = 0$. Consequently, $P(x_0, 0) \notin AB$. The desired solution attains its MPM in domain $\overline{\Omega}^+$ on the curve $\overline{\sigma}_0$.

Analogously, the solution attains its LNM in the closed domain $\overline{\Omega}^+$ on the curve σ_0, too. □

Corollary. Problem A has no more than one solution.

Proof. If boundary data (3)–(5) are homogeneous, then Theorem 1 implies that $u(x, y) \equiv 0$ in domain $\overline{\Omega}^+$. Hence, we have by virtue of continuity of solution to Problem A in mixed domain $\overline{\Omega}$ and conjugation condition (2)

$$u(x, 0) = 0, \ \ x \in \overline{I}; \ \ \lim_{y \to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} = 0, \ \ x \in I. \quad (13)$$

Then we recover the solution to Problem A in domain Ω^- as solution of the modified Cauchy problem with homogeneous data (13). We conclude by formula (8) that $u(x, y) \equiv 0$ in domain $\overline{\Omega}^-$. Consequently, $u(x, y) \equiv 0$ in the whole mixed domain $\overline{\Omega}$.

3. Existence of solution to Problem A.

Theorem 2. *If* $a(x), b(x), \psi(x), f(x) \in C(\overline{I}) \cap C^{1,\alpha}(I)$, $c(x), \varphi(x) \in C^{0,\alpha}(\overline{I})$, $\varphi(x) = (1-x^2)\widetilde{\varphi}(x)$, $\widetilde{\varphi}(x) \in C^{0,\alpha}(\overline{I})$, $f(-1) = f(0) = f(1) = 0$, *and conditions* (6), (10) *are fulfilled, then Problem* A *has a solution.*

In domain Ω^+ solutions to the modified Holmgren problem with boundary data

$$\lim_{y \to +0} y^{\beta_0} \frac{\partial u}{\partial y} = \nu(x), \ \ x \in I; \ \ u(x, y)|_{\sigma_0} = \varphi(x), \ \ x \in \overline{I},$$

and the Dirichlet problem with boundary data

$$u(x, 0) = \tau(x), \ \ x \in \overline{I}; \ \ u(x, y)|_{\sigma_0} = \varphi(x), \ \ x \in \overline{I},$$

are determined by formulas

$$u(x, y) = -k_1 \int_{-1}^{1} \nu(t) \bigg\{ \Big[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2} \Big]^{-\beta} - \Big[(1-xt)^2 + \frac{4t^2}{(m+2)^2} y^{m+2} \Big]^{-\beta} \bigg\} dt - k_1 \beta (m+2)(1-R^2) \times \int_0^l \varphi(\xi(s)) \eta^{\beta_0 - 1}(s) (r_1^2)^{-\beta-1} F(\beta, \beta+1, 2\beta; 1-\sigma) d\xi(s), \quad (14)$$

$$u(x, y) = k_2 (1-\beta_0) y^{1-\beta_0} \int_{-1}^{1} \tau(t) \bigg\{ \Big[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2} \Big]^{\beta-1} - \Big[(1-xt)^2 + \frac{4t^2}{(m+2)^2} y^{m+2} \Big]^{\beta-1} \bigg\} dt - k_2 (1-\beta)(m+2)(1-R^2) y^{1-\beta_0} \times \int_0^l \varphi(\xi(s)) (r_1^2)^{\beta-2} F(1-\beta, 2-\beta, 2-2\beta; 1-\sigma) d\xi(s), \quad (15)$$

correspondingly, where s is length of arc of the curve σ_0 counted from point B, l is the length of whole curve σ_0,

$$\sigma = \left.\begin{matrix} \dfrac{r^2}{r_1^2}, & \begin{matrix} r^2 \\ r_1^2 \end{matrix} \end{matrix}\right\} = (x-\xi(s))^2 + \frac{4}{(m+2)^2}\left(y^{\frac{m+2}{2}} \mp (\eta(s))^{\frac{m+2}{2}}\right)^2,$$

$$(\xi(s), \eta(s)) \in \sigma_0,$$

$$k_1 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2\beta}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)}, \quad k_2 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2(1-\beta)}\frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)},$$

$$R^2 = x^2 + \frac{4}{(m+2)^2}y^{m+2}.$$

It is not difficult to deduce from (14) and (15) the following relations between unknown functions $\tau(x)$ and $\nu(x)$ transferred onto I from domain Ω^+ ([7], pp. 113, 152):

$$\tau(x) = -k_1\int_{-1}^{1}\left[|x-t|^{-2\beta} - (1-xt)^{-2\beta}\right]\nu(t)dt + \Phi_1(x), \quad x \in I, \tag{16}$$

$$\nu(x) = -k_2(1-\beta_0)\frac{m+2}{2}\left[\int_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\beta}} + (1-2\beta)\int_{-1}^{1}\frac{\tau(t)dt}{(1-xt)^{2-2\beta}}\right] + \Phi_2(x), \quad x \in I, \tag{17}$$

where

$$\Phi_1(x) = 2\beta((m+2)/2)^{2\beta}k_1(1-x^2)\int_{-1}^{1}\varphi(t)(1-t^2)^{\beta-1/2}(1-2xt+x^2)^{-1-\beta}dt,$$

$$\Phi_2(x) = (1-\beta_0)(1-\beta)(m+2)k_2(1-x^2)\int_{-1}^{1}\varphi(t)(1-2xt+x^2)^{\beta-2}dt.$$

Let us note that relations (16), (17) are valid for the whole segment I.

We change in relation (17) variable x by $-x$, and by means of boundary-value conditions $\tau(-x) = \tau(x) + f(x)$, $\tau'(-x) = -\tau'(x) - f'(x)$ obtain

$$\nu(-x) = \nu(x) + F_0(x), \tag{18}$$

where

$$F_0(x) = -\frac{k_2(1-\beta_0)(m+2)}{2}\left[\int_{-1}^{1}\frac{(x-t)f'(t)dt}{|x-t|^{2-2\beta}} + (1-2\beta)\int_{-1}^{1}\frac{f(t)dt}{(1-xt)^{2-2\beta}}\right] + \Phi_2(-x) - \Phi_2(x).$$

We rewrite relation (9) in terms of (18) in the form

$$\left[(1-x)^\beta a(x) + (1+x)^\beta b(x)\right]\nu(x) = \gamma\big[(1-x)^\beta a(x)D_{-1,x}^{1-2\beta}\tau(x) + (1+x)^\beta b(x)D_{x,1}^{1-2\beta}\tau(x)\big] + F_1(x), \tag{19}$$

where $F_1(x) = \Psi(x) - (1+x)^\beta b(x)F_0(x)$.

Then we apply operators $D_{-1,x}^{1-2\beta}$ and $D_{x,1}^{1-2\beta}$ to equality (16), and obtain

$$D_{-1,x}^{1-2\beta}\tau(x) = -k_1\Gamma(1-2\beta)\Bigg[(1-\cos 2\beta\pi)\nu(x) + \frac{\sin 2\beta\pi}{\pi}\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x} - \frac{1}{1-xt}\right)\nu(t)dt\Bigg] + D_{-1,x}^{1-2\beta}\Phi_1(x),$$

$$D_{x,1}^{1-2\beta}\tau(x) = -k_1\Gamma(1-2\beta)\Bigg[(1-\cos 2\beta\pi)\nu(x) - \frac{\sin 2\beta\pi}{\pi}\int_{-1}^{1}\left(\frac{1-t}{1-x}\right)^{1-2\beta}\left(\frac{1}{t-x}+\frac{1}{1-xt}\right)\nu(t)dt\Bigg] + D_{x,1}^{1-2\beta}\Phi_1(x).$$

We substitute formulas for $D_{-1,x}^{1-2\beta}\tau(x)$ and $D_{x,1}^{1-2\beta}\tau(x)$ into the right side of (19), and obtain

$$\left[(1-x)^\beta a(x) + (1+x)^\beta b(x)\right]\nu(x) + \lambda(1-x)^\beta a(x)\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\nu(t)dt - \lambda(1+x)^\beta b(x)\int_{-1}^{1}\left(\frac{1-t}{1-x}\right)^{1-2\beta}\left(\frac{1}{t-x}+\frac{1}{1-xt}\right)\nu(t)dt = F_2(x), \quad (20)$$

where $\lambda = \cos\beta\pi/\pi(1+\sin\beta\pi)$,

$$F_2(x) = \frac{\gamma}{1+\sin\beta\pi}\left[F_1(x) + \gamma(1-x)^\beta a(x) D_{-1,x}^{1-2\beta}\Phi_1(x) + \gamma(1+x)^\beta b(x) D_{x,1}^{1-2\beta}\Phi_1(x)\right] \in C^{0,\alpha}(\overline{I}).$$

Then we apply identities

$$\left(\frac{1\pm t}{1\pm x}\right)\left(\frac{1}{t-x}\mp\frac{1}{1-xt}\right) = \frac{1}{t-x}-\frac{t}{1-xt},$$

and transform Eq. (20) to the form

$$\left[(1-x)^\beta a(x) + (1+x)^\beta b(x)\right]\nu(x) + \lambda\int_{-1}^{1}\left(\frac{1}{t-x}-\frac{t}{1-xt}\right)K(x,t)\nu(t)dt = F_2(x), \quad x\in I, \quad (21)$$

where

$$K(x,t) = \left(\frac{1+x}{1+t}\right)^{1-2\beta}(1-x)^\beta a(x) - \left(\frac{1-x}{1-t}\right)^{1-2\beta}(1+x)^\beta b(x).$$

We change variables

$$s = \frac{2t}{1+t^2}, \quad t = \frac{s}{1+\sqrt{1-s^2}}; \quad y = \frac{2x}{1+x^2}, \quad x = \frac{y}{1+\sqrt{1-y^2}},$$

and by means of identity

$$\frac{1}{t-x}-\frac{t}{1-xt} = \frac{2(1-t^2)}{(1+x^2)(1+t^2)\left(\frac{2t}{1+t^2}-\frac{2x}{1+x^2}\right)},$$

rewrite Eq. (21) in the form

$$A(y)\rho(y) + \frac{B(y)}{\pi}\int_{-1}^{1}\frac{\rho(s)ds}{s-y} = \lambda\int_{-1}^{1}\frac{K(x,x)-K(x,s)}{s-y}\rho(s)ds + (1+x^2)F_2(x), \quad y\in I, \qquad (22)$$

$$\rho(y) = (1+x^2)\nu(y), \quad A(y) = (1-x)^\beta a(x) + (1+x)^\beta b(x),$$

$$B(y) = \lambda\pi K(x,x) = \lambda\pi((1-x)^\beta a(x) - (1+x)^\beta b(x)),$$

where $x = y/(1+\sqrt{1-y^2})$, $t = s/(1+\sqrt{1-s^2})$.

Since by virtue of (6) we have $A^2(y)+B^2(y)\neq 0$, it follows that (22) is singular integral equation of normal type ([5], P. 43; [9]). We seek its solution $\rho(y)$ in class of Hölder functions $H(-1,1)$ bounded at the points $y=-1$ and $y=1$.

Consider the function

$$G(y)=\frac{A(y)-iB(y)}{A(y)+iB(y)}=\frac{[(1-x)^\beta a(x)+(1+x)^\beta b(x)]-i\lambda\pi[(1-x)^\beta a(x)-(1+x)^\beta b(x)]}{[(1-x)^\beta a(x)+(1+x)^\beta b(x)]+i\lambda\pi[(1-x)^\beta a(x)-(1+x)^\beta b(x)]}.$$

We evaluate by N. I. Muskhelishvili formula ([5], P. 43)

$$\alpha_k+i\beta_k=\pm\frac{\ln G(c_k)}{2\pi i},\quad k=0,1.$$

Here the sign "$-$" corresponds to meaning $c_0=-1$, and the sign "$+$" to meaning $c_1=1$.

It is easy to see that

$$\begin{aligned}\alpha_0+i\beta_0&=-\frac{\ln G(c_0)}{2\pi i}=-\frac{\ln G(-1)}{2\pi i}=-\frac{1}{2\pi i}\ln\frac{1-\lambda\pi i}{1+\lambda\pi i}\\&=-\frac{1}{2\pi i}\left[\ln\left|\frac{1-\lambda\pi i}{1+\lambda\pi i}\right|+i\left(\arg\frac{1-\lambda\pi i}{1+\lambda\pi i}+2k\pi\right)\right]\\&=-\frac{1}{2\pi i}\arg\frac{1+\sin\beta\pi-i\cos\beta\pi}{1+\sin\beta\pi+i\cos\beta\pi}-k=\frac{1}{\pi}\arctan\frac{\cos\beta\pi}{1+\sin\beta\pi}-k\\&=\frac{1}{\pi}\arctan\left(\frac{\pi}{4}-\frac{\beta\pi}{2}\right)-k=\alpha-k,\end{aligned}$$

where $\alpha=(1-2\beta)/4$. Thus, $\alpha_0=\alpha-k$, $\beta_0=0$, where number k is integer. Then we select an integer number λ_0 such that $0<\alpha_0+\lambda_0<1$, i.e., $\lambda_0=k$. As above, one can easily verify that

$$\alpha_1+i\beta_1=\frac{\ln G(c_1)}{2\pi i}=\frac{\ln G(1)}{2\pi i}=\frac{1}{2\pi i}\ln\frac{1+\lambda\pi i}{1-\lambda\pi i}=\alpha+k.$$

Hence, $\alpha_1=\alpha+k$, $\beta_1=0$.

Then we select an integer number λ_1 such that $0<\alpha_1+\lambda_1<1$, i.e., $\lambda_1=-k$. Consequently, Eq. (22) has the index $\chi=-(\lambda_0+\lambda_1)=0$.

Thus, in the class $h(-1,1)$ ([5], P. 44) Eq. (22) has null index. Consequently, this equation is uniquely reducible to Fredholm integral equation of second kind by means of Carleman–Vekua regularization, and unique solubility of the latter equation follows from the uniqueness of solution to Problem A.

REFERENCES

1. Frankl', F. I. "Unterschallstromungen um Profile mit Uberschallzonen, die mit geradem Verdichtungsstoß enden", (Russian) Prikl. Mat. Mekh. **20**, 196–202 (1956) [in Russian].
2. Devingtal', Yu. V. "Über die Existenz und Eindeutigkeit der Lösung einer Aufgabe von F. I. Frankl'" Izv. Vyssh. Uchebn. Zaved., Mat. No. 2, 39–51 (1958) [in Russian].
3. Lin' Czyun'-bin, "Certain Frankl' Problems", Vestnik Leningrad State University. Mathematics, Mechanics and Astronomy **3**, No. 13, 28–39 (1961).
4. Kapustin, N. Yu., Sabitov, K. B. "Solution of a Problem in the Theory of a Frankl's Problem for Equations of Mixed Type", Differ. Equations **27**, No. 1, 47–54 (1991).
5. Smirnov, M. M. *Equations of Mixed Type* (Vyssh. Shkola, Moscow, 1985) [in Russian].
6. Nakhushev, A. M. "Über einige Randwertaufgaben für hyperbolische Gleichungen und Gleichungen vom gemischten Typ", Differ. Uravn. **5**, No. 1, 44–59 (1969).
7. Salakhitdinov, M. S., Mirsaburov, M. *Nonlocal Problems for Equations of Mixed Type With Singular Coefficients* (Universitet, Yangiyo'l Poligraf Servis, Tashkent, 2005) [in Russian].
8. Bitsadze, A. V. *Certain Classes of Equations With Partial Derivatives* (Nauka, Moscow, 1981) [in Russian].
9. Soldatov, A. P. "Noether Theory of Operators. General One-Dimensional Singular Integral Operators", Differ. Equations **14**, 498–508 (1978).

Translated by B. A. Kats