On uniqueness of Gibbs measure and fixed point of integral operator Hammerstein's type

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Abstract

In this paper we consider function of potential, which the Hamiltonian has unique translation-invariant Gibbs measure for all k.

Keywords: Cayley tree, translation-invariant Gibbs measure, fixed point, Hamiltonian, potential, graph, Hammerstein's integral operator.

In the theory of Gibbs measures there are a lot of papers which devoted to Gibbs measures on a Cayley tree [1]. Translation-invariant Gibbs measures for models with a continuum set of spin values is reduced to find positive fixed points of nonlinear integral operators Hammerstein's type [2]. In [3],[4] proven theorems, that integral operators Hammerstein's type can have one, two and three positive fixed points.

In this paper given function of potential, which the Hamiltonian has unique translation-invariant Gibbs measure for all $k \geq 2$. As well as shown, that integral operator Hammerstein's type has a unique nontrivial positive fixed point.

A Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each vertices. Here V is the set of vertices and L that of edges.

Consider models where the spin takes values in the set $[0,1]$, and is assigned to the vertices of the tree. For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to [0,1]$. Denote $\Omega_A = [0,1]^A$ is the set of all configurations on A.

We consider the Hamiltonian of the model in [4]:

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x),\sigma(y)}, \quad \sigma \in \Omega_V$$

where $J \in R \setminus \{0\}$ and $\beta = \frac{1}{T}$, $T > 0$ is temperature, $\xi : (u,v) \in [0,1]^2 \to \xi_{uv} \in \mathbb{R}$ is a given bounded, measurable function. As usually, $\langle x,y\rangle$ stands for the nearest neighbor vertices on the Cayley three of order four.

We put

$$C_+[0,1] = \{f \in C[0,1] : f(x) \geq 0\}, \quad C_>[0,1] = \{f \in C[0,1] : f(x) > 0\}.$$

We consider the Hamiltonian H on the Cayley tree of order k with the function of potential

$$\xi_{t,u} = \frac{1}{J\beta} \ln\left(a + b(1 - \sin \pi t)(1 - \cos \pi u)\right), \qquad (1)$$

where $a > 0, b > 0$.

For every $k \in \mathbb{N}$ we consider an integral operator H_k acting in the cone $C_+[0,1]$ by the rule

$$(H_k f)(t) = \int_0^1 \exp(J\beta\xi_{tu}) f^k(u) du, \quad k \in \mathbb{N}.$$

The operator H_k is called Hammerstein's integral operator of order homogeneously k.

Denote $N_+^{fix}(H_k)$- a number of nontrivial positive fixed points of the operator H_k i.e.,

$$N_+^{fix}(H_k) = |\{f(t) \in C_>[0,1] : (H_k f)(t) = f(t)\}|.$$

Here and further on |A| denotes the cardinality of A.

For the potential $\xi_{t,u}$ the following equality holds, as in the [2]-[4]:

$$N_>^{fix}(H_k) = N^{tigm}(H),$$

where $N^{tigm}(H)$-number of translation-invariants Gibbs measures for the Hamiltonian H.

Lemma. *Integral operator Hammerstein's type H_k has a unique nontrivial positive fixed point for all $k \geq 2$ and It's following form:*

$$f(t) = \alpha(a + b(1 - \sin \pi t)),$$

where

$$\alpha = \frac{1}{\sqrt[k-1]{\sum_{i=0}^{k} \mathcal{C}_k^i a^{k-i} b^i \alpha_i}}, \quad \alpha_i = \int_0^1 (1 - \sin \pi u)^i du, \quad \mathcal{C}_k^i = \frac{k!}{(k-i)!i!}.$$

Corollary. *The Hamiltonian H with the function of potential $\xi_{t,u}(1)$ on the Cayley tree of order $k \in \mathbb{N}$ has the unique translation-invariant Gibbs measure for all k, i.e. $N^{tigm}(H) = 1$.*

References

[1] Rozikov U.A.: Gibbs mesaures on Cayley tree. *World Scientific.***404** (2013).

[2] Eshkabilov Yu. Kh., Haydarov F. H., Rozikov U. A. Uniqueness of Gibbs Measure for Models With Uncountable Set of Spin Values on a Cayley Tree. *Math. Phys. Anal. Geom.* **16** (2013), 1-17.

[3] Eshkabilov Yu. Kh., Nodirov Sh. D., Haydarov F. H. Positive fixed points of quadratic operators and Gibbs measures. *Positivity,* **20** No.4, (2016), 929-943.

[4] Eshkabilov Yu. Kh., Nodirov Sh. D. Positive Fixed Points of Cubic Operators on $\mathbb{R}^2$ and Gibbs Measures. *Journal of Siberian Federal University. Mathematics Physics.* **12**(6),(2019), 663-673.