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SAMARQAND DAVLAT UNIVERSITETI

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TUTASH MUHIT MEXANIKASIDAN MASALA VA MASHQLAR TO‘PLAMI

1-qism. Vektor va tenzor maydon tushunchalari

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ta’lim yo‘nalishlari bakalavr talabalari uchun

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Ushbu uslubiy qo‘llanma «Tutash muhit mexanikasi asoslari» fani bo‘yicha «5140300 – Mexanika» ta’lim yo’nalishi talabalari uchun mo‘ljallangan bo‘lib, unda shu fanning namunaviy o‘quv dasturidan kelib chiqib, vektor va tenzor maydon elementlarining qisqacha nazariy asoslari, namunaviy misollar yechimlari, mustaqil ish topshiriqlari, sinov savollari, mustaqil o‘zlashtirishga oid adabiyotlar, ulardan foydalanishga oid uslubiy tavsiyalar va boshqa tarqatma materiallar keltirilgan. Bular bakalavr talabalarga shu fanni yanada chuqurroq o‘zlashtirishga yaqindan yordam beradi. Ushbu uslubiy qo‘llanmadan turdosh ixtisoslik va mutaxassisliklar talabalari, yosh ilmiy xodimlar va tadqiqotchilar ham foydalanishlari mumkin.

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MUNDARIJA

KIRISH

Quyida tutash muhit mexanikasining matematik asosini tashkil qiluvchi asosiy tushunchalar bilan tanishamiz.

Tutash muhit mexanikasining tushunchalari tavsiflanishi qo‘llanilayotgan va tanlangan koordinatalar sistemasidan bog‘iq bo‘lmagan fizik miqdorlarga asoslangan. Ko‘pgina hollarda bu fizik miqdorlarni tanlangan koordinatalar sistemasiga tegishli holda o‘rganish qulay. Matematik jihatdan bunday miqdorlar *tenzorlar* (xususan, skalyar va vektor) deb ataladi.

Tenzor koordinatalar sistemasidan bog‘liq bo‘lmagan holda mavjud. Har bir sistemada tenzorni uning *komponentalari* deb ataluvchi niqdorlar to‘plami bilan berish mumkin. Agar tenzorning barcha komponentalari bitta koordinatalar sistemasida berilgan bo‘lsa, uni boshqa koordinatalar sistemasida ham *koordinatalarni almashtirish qoidasi* orqali ifodalab olish mumkin. Tenzorlar to‘g‘risida ta’riflar quyida keltirilgan.

Qulaylik, soddalik va ixchamlilik uchun tutash muhit mexanikasining ko‘pgina tenglamalari tenzorlar orqali ifodalangan. Tenzor almashtirishlarning chiziqliligi va birjinsliligi sababli bir sistemada ishonchli bo‘lgan tenzor tenglamalari ikkinchisida ham ishonchli bo‘ladi. Tenzor munosabatlarning almashtirishlarga nisbatan bunday *invariantliligi* tenzor hisobning tutash muhit mexanikasini o‘rganishda muhim ahamiyatga egaligini anglatadi.

Vektorlar va tenzorlar tushunchalari tutash muhitlar mexanikasida keng qo‘llaniladi. Ular ko‘pgina miqdorlar va ifodalarni ixcham ko‘rinishda yozish imkonini beradi. Shuning uchun ushbu o‘quv qo‘llanmada dastlab tutash muhit mexanikasining matematik asoslari bo‘limini ikki qismga ajratgan holda o‘rganish ma’qul ko‘rindi: vektor va tenzor maydonlar tushunchalari.

1. Skalyar va vektor maydonlar. Bazis vektorlar. Vektor maydon elementlari ustida amallar. Vektorning komponentalarini almashtirish

Skalyar, vektor, maydon va tenzor tushunchalari. Tutash muhit mexanikasining tushunchalari tavsiflanishi qo‘llanilayotgan va tanlangan koordinatalar sistemasidan bog‘iq bo‘lmagan fizik miqdorlarga asoslangan. Ko‘pgina hollarda bu fizik miqdorlarni tanlangan koordinatalar sistemasiga tegishli holda o‘rganish qulay. Matematik jihatdan bunday miqdorlar *tenzorlar* (xususan, skalyar va vektor) deb ataladi.

Har bir nuqtasida to‘la aniqlangan biror miqdorlar maydoni *fazo* deb ataladi. Qiymati haqiqiy sonlar bilan ifodalanuvchi miqdorlar *skalyarlar* deb ataladi, masalan, massa, zaryad, zichlik, temperatura, energiya, ish, hajm, bosim va boshqalar. Bunday qiymatlar maydoni *skalyar maydon* (masalan, zichliklar maydoni, temperaturalar maydoni va hokazo) deb ataladi. Qiymati va fazodagi yo‘nalishi bilan xarakterlanuvchi miqdorlar *vektorlar* deb ataladi, masalan, tezlik, tezlanish, kuch, elektr maydon kuchlanganligi va boshqalar. Boshqacha qilib aytganda, skalyarni tavsiflash uchun bitta sonning o‘zi yetarli, vektor miqdor tavsifi uchun esa uchta son zarur.

Miqdorni biror koordinatalar sistemasida to‘la aniqlovchi sonlar (yoki funksiyalar) shu miqdorning *komponentalari* (*proyeksiyalari*) deb ataladi.

Skalyar miqdor bir komponentali bo‘lib, uning qiymati koordinatalar sistemasini tanlashdan bog‘liq emas.

Murakkabroq obyektlarni tavsiflash uchun ko‘proq sondagi komponentalar talab qilinadi. Shularga ko‘ra, fazoning har bir nuqtasida miqdori va yo‘nalishi bilan xarakterlanadigan maydon *vektor maydon* (masalan, tezliklar maydoni, kuchlar maydoni va hokazo) deb ataladi.

Har qanday tutash muhitning holatini xarakterlovchi kattaliklar, xususan fizik va geometrik kattaliklar, koordinatalar sistemasining tanlanishiga bog‘liq emas, ya’ni ular *invariant kattaliklar* yoki *ob‘yektlardir*. Ammo bu kattaliklarni tenzor hisobi elementlaridan foydalanib o‘rganish, ya’ni ularni biror koordinatalar sistemasiga bog‘lab o‘rganish ancha qulay.

Bunda yuqoridagi invariant ob‘yektning o‘zi koordinatalar sistemasiga bog‘liq bo‘lmagan holda, koordinat sistemasiga bevosita bog‘liq bo‘lgan bir qancha kattaliklarning majmuasi bilan aniqlanadiki, kattaliklar *invariant ob‘ektning tuzuvchilari* yoki *komponentalari* deb ataladi. Ana shunday ko‘p komponentali invariant ob‘yektlarni *tenzorlar* (xususan, skalyar va vektor) deb ataydilar.

Tenzor koordinatalar sistemasidan bog‘liq bo‘lmagan holda mavjud. Har bir sistemada tenzorni uning *komponentalari* deb ataluvchi niqdorlar to‘plami bilan berish mumkin. Agar tenzorning barcha komponentalari bitta koordinatalar sistemasida berilgan bo‘lsa, uni boshqa koordinatalar sistemasida ham *koordinat almashtirish qonuni* orqali ifodalab olish mumkin. Tenzorlar to‘g‘risida ta’riflar quyiroqda batafsil keltirilgan.

Qulaylik, soddalik va ixchamlilik uchun tutash muhit mexanikasining ko‘pgina tenglamalari quyida tenzorlar orqali ifodalangan. Tenzor almashtirishlarning chiziqliligi va birjinsliligi sababli bir sistemada ishonchli bo‘lgan tenzor tenglamalari ikkinchisida ham ishonchli bo‘ladi. Tenzor munosabatlarning almashtirishlarga nisbatan bunday *invariantliligi* tenzor hisobning tutash muhit mexanikasini o‘rganishda muhim ahamiyatga egaligini anglatadi.

Vektorlar va tenzorlar tushunchalari tutash muhitlar mexanikasida keng qo‘llaniladi. Ular ko‘pgina miqdorlar va ifodalarni ixcham, sodda va oson tushunish mumkin bo‘lgan ko‘rinishda yozish imkonini beradi.

Vektorlar va Dekart koordinatalari sistemasida bazis vektorlar. Vektorlar uchun $\vec{a}$, $\vec{b}$, … (xususan, $\vec{\text{Э}}$ birlik vektor); ularning uzunliklari (modullari), ya’ni skalyarlar uchun a, b , … kabi belgilashlarni qabul qilamiz.

Umumiy holda, $\vec{u}$ vektorni *bazis vektorlar* deb ataluvchi, o‘zaro komplonar bo‘lmagan uchta $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlarning chiziqli kombinatsiyasi ko‘rinishida ifodalash mumkin: $\vec{u} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$, bu yerda λ, μ, ν - skalyarlar. Bu bazis vektorlar chiziqli bog‘lanmagan, ya’ni ushbu $\lambda\vec{a} + \mu\vec{b} + \nu\vec{c} = 0$ tenglama faqat $\lambda = 0$, $\mu = 0$, $\nu = 0$ bo‘lgandagina o‘rinli. Bunday bazis vektorlar to‘plami shu sistemaning bazislarini hosil qiladi. Bu vektorlarning chiziqli kombinatsiyasi ko‘paytmalar yig‘indisining barcha traditsion algebraik xossalariga ega.

Koordinata chiziqlari o‘zaro perpendikulyar bo‘lgan koordinatalar sistemasi *ortogonal koordinatalar sistemasi* deyiladi. Umuman *vektorning ortogonal sistemasi* deb shunday $\{\vec{x}_\alpha\}$ vektorlar to‘plamiga aytiladiki, bu to‘plamning ixtiyoriy vektorlari orasidagi skalyar ko‘paytma nolga teng bo‘lishi kerak, ya’ni

$$(\vec{x}_\alpha \cdot \vec{x}_\beta) = 0, \quad (\alpha \neq \beta).$$

Agar bu to‘plamga kiruvchi har bir vektorning normasi birga teng bo‘lsa, bunday sistema *ortonormal sistema* deyiladi.

Agar vektorlarning ortogonal sistemasida unga kiruvchi vektorning hammasiga ortogonal bo‘lgan vektor topilmasa, bunday sistema *to‘liq ortogonal sistema* deyiladi.

Uch o‘lchovli Evklid fazosidagi Dekart koordinatalari sistemasi o‘qlarini x_1, x_2, x_3 lar bilan, ularning ortonormal bazisini esa $\vec{э}_i$ (i =1, 2, 3) lar bilan belgilaymiz. U holda ta’rifga ko‘ra

$$э_i \cdot э_j = \delta_{ij} = \begin{cases} 1, \text{ agar } i = j \text{ bo'lsa;} \\ 0, \text{ agar } i \neq j \text{ bo'lsa,} \end{cases} \tag{1.1}$$

bu yerda δ_{ij} - Kroneker simvoli.

Agar x_1, x_2, x_3 – to‘g‘ri burchakli dekart koordinatalari sistemasining (xususan $x_1 = x$, $x_2 = y$, $x_3 = z$) o‘qlari bo‘lsa (1.1-rasm), bunda $(x_1, x_2, x_3) \in R$, u holda ixtiyoriy $\vec{a}$ vektorni bu o‘qlarning $э_1$, $э_2$, $э_3$ *ortalari* (*birlik yoki basis vektorlari*) bo‘yicha yoyib (xususan, ortogonal to‘g‘ri burchakli dekart koordinatalari sistemasida ushbu $э_1 = \vec{i}$, $э_2 = \vec{j}$, $э_3 = \vec{k}$ - bazis vektorlar *birlik vektorlarning o‘ng triedr*ini hosil qiladi), uni quyidagicha yozish mumkin

$$\vec{a} = a_1 э_1 + a_2 э_2 + a_3 э_3 = \sum_{i=1}^{3} a_i э_i = (a_1, a_2, a_3), \tag{1.2}$$

bu yerda $э_1 = (1, 0, 0)$, $э_2 = (0, 1, 0)$, $э_3 = (0, 0, 1)$; α_i (i=1,2,3) – berilgan $\vec{a}$ vektorning tuzuvchilari (komponentalari). Ushbu yozuvning qulayligi uchun oxirgi tenglikning o‘ng tomonidagi yig‘indi belgisi ($\sum$) ni tashlab yuborib, tenglik quyidagicha yoziladi

$$\vec{a} = a_i \vec{э}_i, \quad (i = 1, 2, 3). \tag{1.3}$$

Ushbu ifodadagi yig‘indi hisoblanuvchi va ikki marta takrorlanuvchi i-indeksi *«gung» indeks* deb ataladi («gapirmasa ham yig‘ndi olish kerakligini anglatadi»). Fazo uch o‘lchovli bo‘lganda gung indeks 1, 2, 3 qiymatlarning har uchalasini ham qabul qiladi. Gung indeksni har qanday i, j, k, ... harflari bilan belgilash mumkin, masalan,

$$\vec{\alpha} = \alpha_i \vec{э}_i = \alpha_j \vec{э}_j = \alpha_k \vec{э}_k = \alpha_1 \vec{э}_1 + \alpha_2 \vec{э}_2 + \alpha_3 \vec{э}_3.$$

Indeksning gung yoki gung emasligini aniqlash uchun uning takrorlanishiga etiborni qaratish kerak. Agar indeks takrorlanmasa u *«erkin» indeks* deb yuritiladi. Misol uchun ushbu

$$\beta_{ij} \vec{э}_j = \vec{b}_i$$

yozuvida i-erkin, j-esa gung indeksdir.

Ortonormal bazisi $\vec{э}_i$ (i =1, 2, 3) bo‘lgan eski x_1, x_2, x_3 Dekart koordinatalari sistemasi o‘qlariga nisbatan biror burchakka burilgan va ortonormal bazisi $\vec{э}_i^{'}$ (i =1, 2, 3) bo‘lgan yangi $x_1^{'}$, $x_2^{'}$, $x_3^{'}$ Dekart koordinatalari sistemasini qaraymiz. Yangi $x_i^{'}$

va eski x_j o‘qlari orasidagi burchak kosinusini α_{ij} deb belgilaymiz. U holda α_{ij} yangi $\vec{э}_i^{\,'}$ va eski $\vec{э}_j$ bazis vektorlarning skalyar ko‘paytmasiga teng bo‘ladi, ya’ni

$$\vec{э}_i^{\,'} \cdot \vec{э}_j = \alpha_{ij} \tag{1.4}$$

Kiritilgan α_{ij} kattalik $\vec{э}_i^{\,'}$ birlik vektorining $\vec{э}_j$ birlik vektori yo‘nalishiga tushirilgan proyeksiyasiga teng, va aksincha, α_{ji} kattalik $\vec{э}_j$ birlik vektorining $\vec{э}_i^{\,'}$ birlik vektori yo‘nalishiga tushirilgan proyeksiyasiga teng, hamda $\alpha_{ij} = \alpha_{ji}$ bo‘lganligi uchun $\vec{э}_i^{\,'}$ birlik vektorining $\vec{э}_j$ bazisdagi yoyilmasi

$$\vec{э}_i^{\,'} = \alpha_{i1}\vec{э}_1 + \alpha_{i2}\vec{э}_2 + \alpha_{i3}\vec{э}_3 = \alpha_{ij}\vec{э}_j \tag{1.5}$$

kabi va $\vec{э}_j$ birlik vektorining yangi $\vec{э}_i^{\,'}$ bazisdagi yoyilmasi esa

$$\vec{э}_j = \alpha_{j1}\vec{э}_1^{\,'} + \alpha_{j2}\vec{э}_2^{\,'} + \alpha_{j3}\vec{э}_3^{\,'} = \alpha_{ij}\vec{э}_i^{\,'} \tag{1.6}$$

kabi bo‘ladi. Ushbu ifodada j indeks erkin, i indeks esa gungdir, (1.6) ifodaning teskarisi (1.5) ifodada i erkin, j gung indeksdir.

Vektorning moduli (*uzunligi*) deb shu vektor komponentalari kvadratlari yig‘indisidan chiqarilgan kvadrat ildizning qiymatiga aytiladi, ya’ni

$$a = |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Xususiy holda, $\vec{a}$ vektorga nisbatan uzunligi bir xil, ammo yonalishi qarama-qarshi bo‘lgan vektor *manfiy vektor* deb ataladi va $-\vec{a}$ kabi yoziladi; birlik vektorning uzunligi birga teng; *nol vektor* esa uzunligi nol va yo‘nalisi noaniq bo‘lgan vektor; $\vec{r} = \vec{r}(x_1, x_2, x_3)$ - *radius vektor*ni quyidagicha yozamiz:

$$\vec{r} = x_1\vec{э}_1 + x_2\vec{э}_2 + x_3\vec{э}_3\,.$$

Vektor maydon elementlari ustida amallar. *Vektor miqdorlarni qo‘shish* (yoki *ayirish*) parallelogramm qoidasiga muvofiq amalga oshiriladi, komponentalari mos ravishda qo‘shiladi (yoki ayriladi):

$$\vec{a} \pm \vec{b} = \left(a_1 \pm b_1,\ a_2 \pm b_2,\ a_3 \pm b_3\right).$$

Vektorlarni qo‘shish (yoki ayirish) kommutativlik va assosiativlik xossasiga ega.

Ikki vektorning skalyar ko‘paytmasi. Ikkita $\vec{a}$ и $\vec{b}$ *vektorlarning skalyar ko‘paytmasi* deb ular modullarining shu vektorlar orasidagi burchak kosinusiga ko‘paytmasiga aytiladi, ya’ni

$$\vec{a} \cdot \vec{b} = ab\cos(\vec{a}\,\hat{}\,\vec{b})\,,$$

bu yerda $\vec{a}\hat{}\vec{b}=\theta$ - vektorlar orasidagi kichik burchak. Skalyar ko‘paytmaning xossalari: 1) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$; 2) $\vec{a}\cdot\vec{a}\geq 0$; 3) $\vec{a}\cdot(\alpha\vec{b}_1+\beta\vec{b}_2)=\alpha\vec{a}\cdot\vec{b}_1+\beta\vec{a}\cdot\vec{b}_2$.

Shunday qilib, rangi birga teng ikkita tenzorning skalyar ko‘paytmasi bu tenzorlarni rangi nolga teng bitta tenzorga, ya’ni skalyarga almashtiradi.

Xususan,

$$a_1=\vec{a}\cdot\vec{э}_1=a\cos\alpha;\quad a_2=\vec{a}\cdot\vec{э}_2=a\cos\beta;\quad a_3=\vec{a}\cdot\vec{э}_3=a\cos\gamma,$$

bu yerda α, β, γ - berilgan $\vec{a}$ vektorning x_1, x_2, x_3 koordinata o‘qlari bilan tashkil etgan kichik burchaklari; a_1, a_2, a_3 lar esa $\vec{a}$ vektorning x_1, x_2, x_3 koordinata o‘qlaridagi mos proyeksiyalari.

Ikki vektorning skalyar ko‘paytmasi shu vektorlarning koordinata o‘qlaridagi proyeksiyalari ko‘paytmasi orqali quyidagicha yoziladi:

$$\vec{a}\cdot\vec{b}=\vec{e}_1(a_1\vec{э}_1+a_2\vec{э}_2+a_3\vec{э}_3)\cdot(b_1\vec{э}_1+b_2\vec{э}_2+b_3\vec{э}_3)=a_1b_1+a_2b_2+a_2b_2.$$

Adabiyotlarda $\vec{a}\vec{b}\equiv\vec{a}\cdot\vec{b}\equiv(\vec{a},\vec{b})$ belgilashlar qabul qilingan.

Boshqacha aytganda, *ikki vektorning skalyar ko‘paytmasi* deb ushbu

$$\vec{a}\cdot\vec{b}=ab\cos\theta,$$

skalyarga aytiladi. $\vec{a}$ vektorni $\vec{э}$ birlik vektorga skalyar ko‘pytirish $\vec{a}$ vektorning $\vec{э}$ vektor yo‘nalishidagi proeksiyasini beradi.

Xususan, *vektorni skalyarga ko‘paytirish* – bu uning har bir komponentasini shu skalyarga ko‘paytirish demakdir:

$$k\cdot\vec{a}=ka_1\vec{э}_1+ka_2\vec{э}_2+ka_3\vec{э}_3=(ka_1,ka_2,ka_3).$$

Boshqacha aytganda, vektorni skalyarga ko‘paytirish yo‘nalishi o‘zgarmagan (agar skalyar musbat bo‘lsa) yoki qarama-qarshisiga o‘zgargan (agar skalyar manfiy bo‘lsa), ammo uzunligi boshqa yangi vektorni beradi; vektorni nolga ko‘paytirish nol vektorni beradi; vektorni birga ko‘paytirish esa uni o‘zgartirmaydi; vektorni uning moduliga teskari miqdorga ko‘paytirish uning yo‘nalishiga mos birlik vektorni beradi. Vektorni skalyarga ko‘paytiriash assosiativlik va distributivlik xossalariga ega.

Ixtiyoriy $\vec{a}$ vektorning birlik vektori quyidagiga teng

$$\vec{э}_a=(\cos\alpha)\vec{э}_1+(\cos\beta)\vec{э}_2+(\cos\gamma)\vec{э}_3.$$

Bu yerdan $\vec{a}$ vektor ixtiyoriy bo‘lganligi uchun ixtiyoriy birlik vektor uchun uning yo‘naltiruvchi kosinuslari uning dekart komponentalari bo‘ladi.

Ikki vektorning vektor ko‘paytmasi deb shu vektorlar tekisligiga perpendikulyar ushbu

$$\vec{c}=\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}=(ab\sin\theta)\vec{э}$$

vektorga aytiladi (1.2-rasm), bunda $\vec{e}$ birlik vektor $\vec{a}$ и $\vec{b}$ vektorlar tekisligiga perpendikulyar yo‘nalgan bo‘lib, birinchi ko‘paytuvchi ikkinchi ko‘paytuvchiga qarab soat strelkasi yo‘nalichiga qarama-qarshi kichik burchakka burilgan.

Adabiyotlarda $\vec{a}\times\vec{b}\equiv[\vec{a}\vec{b}]\equiv[\vec{a}\times\vec{b}]$ belilashlar qabul qilingan.

Vektor ko‘paytmaning moduli ($\vec{c}$ vektorning moduli) $\vec{a}$ va $\vec{b}$ vektorlardan qurilgan parallelogramning yuzasiga teng, ya’ni.

$$|\vec{c}| = |\vec{a}\times\vec{b}| = ab\sin\theta\,.$$

Vektor ko‘paytma, skalyar ko‘paytmadan farqli, rangi o‘zgarmaydigan tenzorni beradi.

Ikki vektorning vektor ko‘paytmasi quyidagi xossalarga ega:

1) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (yoki $[\vec{a}\vec{b}]=-[\vec{b}\vec{a}]$);

2) $\vec{a}\times(\alpha\vec{b}_1+\beta\vec{b}_2)=\alpha\vec{a}\times\vec{b}_1+\beta\vec{a}\times\vec{b}_2$ (yoki $[\vec{a}(\alpha\vec{b}_1+\beta\vec{b}_2)]=\alpha[\vec{a}\vec{b}_1]+\beta[\vec{a}\vec{b}_2]$);

3) Vektor ko‘paytmaning qoidasini e’tiborga olib, koordinatalar sistemasining birinchi choragida $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ vektorlar uchun: $[\vec{e}_1\vec{e}_2]=\vec{e}_3$, $[\vec{e}_2\vec{e}_3]=\vec{e}_1$, $[\vec{e}_3\vec{e}_1]=\vec{e}_2$

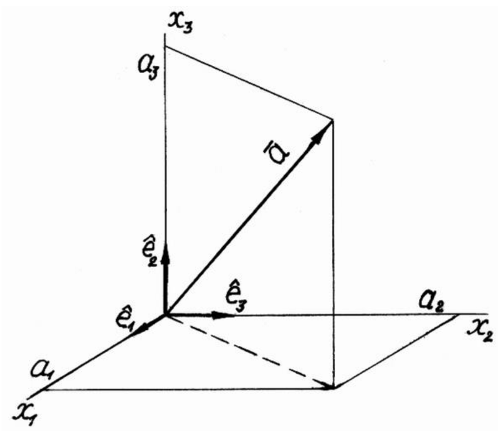
$$\vec{e}_1\times\vec{e}_2=\vec{e}_3,\quad \vec{e}_2\times\vec{e}_3=\vec{e}_1,\quad \vec{e}_3\times\vec{e}_1=\vec{e}_2,$$

$$\vec{e}_1\times\vec{e}_1=0,\quad \vec{e}_2\times\vec{e}_2=0,\quad \vec{e}_3\times\vec{e}_3=0,$$

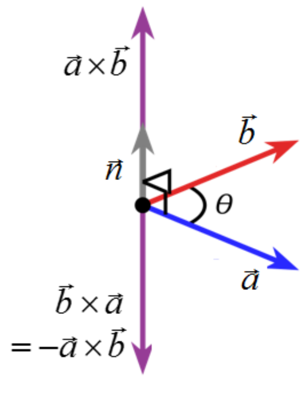
yoki

$$\vec{e}_1\cdot\vec{e}_2=0,\quad \vec{e}_2\cdot\vec{e}_3=0,\quad \vec{e}_3\cdot\vec{e}_1=0,$$

$$\vec{e}_1\cdot\vec{e}_1=1,\quad \vec{e}_2\cdot\vec{e}_2=1,\quad \vec{e}_3\cdot\vec{e}_3=1.$$



1.1-rasm. Uch o‘lchovli to‘g‘ri burchakli dekart koordinatalari sistemasi, uning o‘qlari va bu o‘qlarning mos ortalari hamda vektor va uning komponentalari.



1.2-rasm. Ikki vektorning vektor ko‘paytmasi.

Ushbu qoidani qo‘llab, vektor ko‘paytmaning proyeksiyalarini ko‘paytuvchilarning proyeksiyalari orqali ifodalaymiz:

$$\vec{a}\times\vec{b}=(a_1\vec{e}_1+a_2\vec{e}_2+a_3\vec{e}_3)\times(b_1\vec{e}_1+b_2\vec{e}_2+b_3\vec{e}_3)=$$
$$=a_1b_2\vec{e}_3-a_1b_3\vec{e}_2-a_2b_1\vec{e}_3+a_2b_3\vec{e}_1+a_3b_1\vec{e}_2-a_3b_2\vec{e}_1=$$
$$=(a_2b_3-a_3b_2)\vec{e}_1+(a_3b_1-a_1b_3)\vec{e}_2+(a_1b_2-a_2b_1)\vec{e}_3.$$

Bu yerda siklik o‘rin almastirishda vektor ko‘paytmaning ishorasi qarama-qarshisiga o‘zgartirilgan.

Qulaylik uchun *vektor ko‘paytmani determinant ko‘rinishida yozish* mumkin

$$\vec{c}=\vec{a}\times\vec{b}=[\vec{a}\vec{b}\,]=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

Vektorlarning aralash ko‘paytmasi. Skalyar ko‘paytma ta’rifiga ko‘ra $(\vec{a}\times\vec{b})\cdot\vec{c}$ - *vektor-skalyar ko‘paytmani* qiyidagicha yozish mumkin:

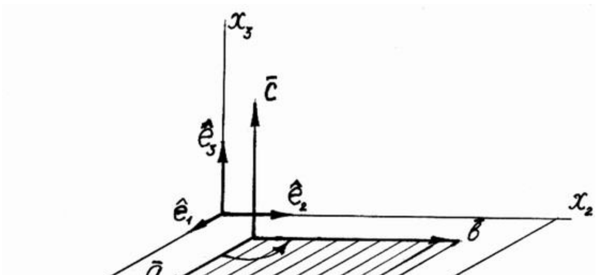
$$(\vec{a}\times\vec{b})\cdot\bar{c}=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot(c_1\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\pm V,$$

bu yerda V – uchta $\vec{a},\,\vec{b},\,\vec{c}$ vektor-lardan tuzilgan parallelepiped hajmi (agar uchlangan vektorlar o‘ng turgan bo‘lsa, ishora “+”, aksincha, ishora “–” olinadi). Bu yerdan vektor-skalyar ko‘paytmaning qiymati ko‘paytuvchi-larning o‘rnini siklik almashtirishdan bog‘liq emasligi va aksincha almashti rishda esa ishora almashinishi haqidagi muhim natijalar keib chiqadi:

$$(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}=(\vec{c}\times\vec{a})\cdot\vec{b}=$$
$$-(\vec{a}\times\vec{c})\cdot\vec{b}=-(\vec{c}\times\vec{b})\cdot\vec{a}=-(\vec{b}\times\vec{a})\cdot\vec{c}\,.$$

Xususan, agar vektor-skalyar ko‘paytmadagi ikki vektor bir xil (yoki o‘zaro parallel) bo‘lsa, u holda bu ko‘paytma nolga teng bo‘ladi:

$$(\vec{a}\times\vec{b})\cdot\vec{a}=(\vec{a}\times\vec{b})\cdot\vec{b}=$$
$$=(\vec{a}\times\vec{a})\cdot\vec{b}=(\vec{b}\times\vec{b})\cdot\vec{a}=0.$$



1.3-rasm. Ikki vektorlar vektor ko‘paytmasining geometrik talqini.

Agar uchta vektor bir tekislikda yotsa (ya’ni ular komplanar bo‘lsa), u holda $\vec{a},\,\vec{b},\,\vec{c}$ vektorlarning komplanarlik shartini quyidagicha yozish mumkin:

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=0,$$

ya'ni shu uchta vektordan tuzilgan parallelepipedning hajmi nolga teng.

Ikkilangan vektor ko'paytma $\vec{a}\times(\vec{b}\times\vec{c})$ = $[\vec{a}[\vec{b}\vec{c}]]$ - bu $\vec{b}$ va $\vec{c}$ vektorlar tekisligida yotuvchi va $\vec{a}$ vektorga perpendikulyar bo'lgan vektorga aytiladi.

Bu ko'paytma quyidagi formula bo'yicha ochib chiqiladi

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})=\begin{vmatrix}\vec{b} & \vec{c}\\ (\vec{a}\cdot\vec{b}) & (\vec{a}\cdot\vec{c})\end{vmatrix}.$$

Buning natijasi sifatida undan quyidagini olish mumkin

$$(\vec{a}\times\vec{b})\times\vec{c}=\vec{b}(\vec{a}\cdot\vec{c})-\vec{a}(\vec{b}\cdot\vec{c})=\begin{vmatrix}\vec{b} & \vec{a}\\ (\vec{b}\cdot\vec{c}) & (\vec{a}\cdot\vec{c})\end{vmatrix}.$$

Vektorlarning skalyat-vektor ko'paytmasi uchun esa quyidagi munosabatlar o'rinli:

$$\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{b}\times\vec{c}=\lambda,$$

bu yerda λ - skalyar. Shuning uchun $\vec{a}\cdot(\vec{b}\times\vec{c})$ = $(\vec{a},[\vec{b}\vec{c}])$,

$$\left(\vec{a},\left[\vec{b}\vec{c}\right]\right)=\left(\vec{b},\left[\vec{c}\vec{a}\right]\right)=\left(\vec{c},\left[\vec{a}\vec{b}\right]\right)=-\left(\vec{a},\left[\vec{c}\vec{b}\right]\right)=-\left(\vec{b},\left[\vec{a}\vec{c}\right]\right)=-\left(\vec{c},\left[\vec{b}\vec{a}\right]\right).$$

$\vec{H}$ (a,b,c) vektorga perpendikulyar va $\vec{r}_0(x_0,y_0,z_0)$ nuqtadan o'tuvchi tekislik tenglamasi vektor shaklida quyidagicha yoziladi:

$$\left((\vec{r}-\vec{r}_0),\vec{H}\right)=0$$

yoki komponentalarda yozsak,

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

$\vec{H}$ (a,b,c) vektorga parallel va $\vec{r}_0(x_0,y_0,z_0)$ nuqtadan o'tuvchi tekislik tenglamasi vektor shaklida quyidagicha yoziladi:

$$\vec{r}=\vec{r}_0+\alpha\vec{H},$$

bu yerda α - ixtiyoriy haqiqiy son. Bunda α ning miqdori barcha koordinatalar sistemasi uchun bir xil bo'lganligidan bunday to'g'ri chiziq tenglamasining komponentalarga nisbatan yozilishi quyidagicha:

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z}{c}$$

Vektorning komponentalarini almashtirish. Eski bazisdan yangi bazisga o'tish formulasi (1.4) dagi α_{ij} koeffisiyentlar o'tish matritsasini tashkil qiladilar. O'tish matritsasi ortogonal matritsadir, chunki

$$1)\ \left[\alpha_{ij}\right]^T\cdot\left[\alpha_{ij}\right]=E;\quad 2)\ \left[\alpha_{ij}\right]\cdot\left[\alpha_{ij}\right]^T=E;\quad 3)\ \left[\alpha_{ij}\right]^T=\left[\alpha_{ij}\right]^{-1},$$

bu yerda $\left[\alpha_{ij}\right]^T$ - transponirlangan, $[\alpha_{ij}]^{-1}$ – teskari va E - matritsaning elementlari

$$\alpha_{ik}\cdot\alpha_{jk}=\delta_{ij} \text{ va } \alpha_{ki}\cdot\alpha_{kj}=\delta_{ij}$$

shartlarni qanoatlantiradi, hamda uning determinanti ±1 ga teng, ya’ni

$$\left|\alpha_{ij}\right| = \pm 1,$$

bu yerdagi musbat ishora olinadi, agar $\vec{э}_i^{\,'}$ va $\vec{э}_j$ bazislar bir xil o‘ng yoki chap bazislar bo‘lsa, aks holda manfiy ishora olinadi.

Endi ixtiyoriy $\vec{a}$ vektorining komponentalarini almashtirish formulalarini topamiz. Buning uchun $\vec{э}_i^{\,'}$ va $\vec{э}_j^{\,'}$ bazislardagi yoyilmalaridan foydalanamiz:

$$\vec{a} = a_j\,\vec{э}_j \quad \text{va} \quad \vec{a} = a_i^{'}\cdot\vec{э}_i^{\,'}, \quad \text{demak} \quad a_j\,\vec{э}_j = a_i^{'}\cdot\vec{э}_i^{\,'}.$$

Oxirgi tenglikda $\vec{э}_j$ o‘rniga uning (1.5) ifodasini qo‘yamiz

$$\alpha_j\cdot\vec{э}_j = \alpha_j\cdot\alpha_{ij}\vec{э}_i^{\,'} =_i^{'} \alpha\vec{э}_i^{\,'}$$

bundan

$$a_i^{'} = \alpha_{ij}\cdot a_j, \quad (\text{chunki} \quad \alpha_{ij} = \alpha_{ji}) \tag{1.7}$$

ifodaga ega bo‘lamiz.

Aksincha, agar $\vec{э}_i^{\,'}$ ning o‘rniga uning (1.5) ifodasini qo‘ysak, u holda

$$a_j = \alpha_{ij}a_i^{'} \tag{1.8}$$

ifodaga ega bo‘lamiz.

Olingan (1.7) formula eski koordinat sistemasidan yangisiga o‘tishda vektor komponentalarini almashtirish qonunidir. Aksincha, (1.8) formula yangi koordinata o‘qlaridan eskilariga o‘tishda vektor komponentalarini almashtirish qonunidir.

Ortonormal bazislar

$$\left|\hat{e}_1\right| = \left|\hat{e}_2\right| = \left|\hat{e}_3\right| = 1,$$

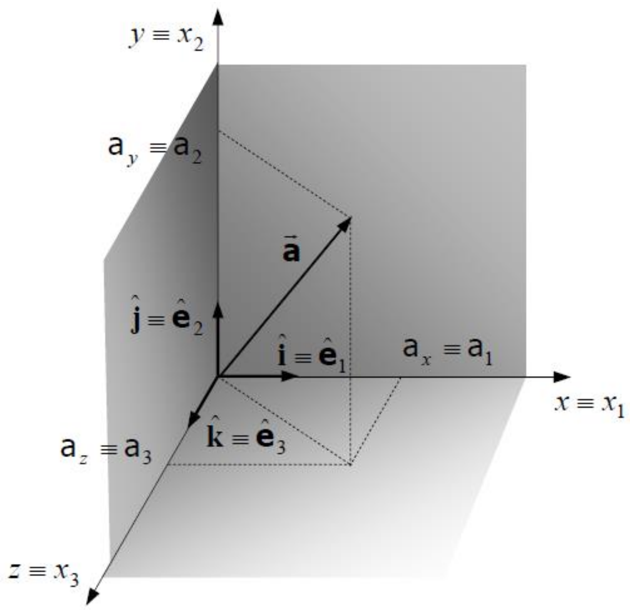
$$\hat{e}_1\cdot\hat{e}_2 = \hat{e}_2\cdot\hat{e}_3 = \hat{e}_3\cdot\hat{e}_1 = 0.$$

Vektor ifodasi

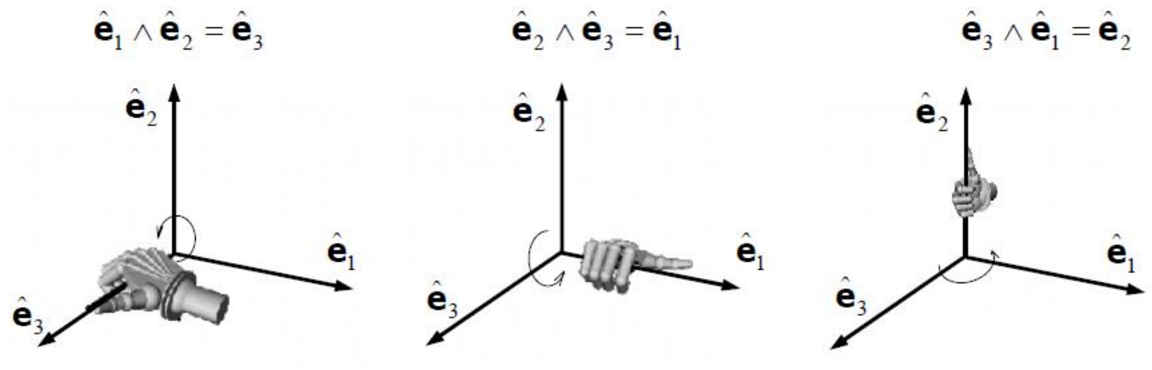
$$\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3.$$

Vektor komponentalari

$$(\vec{a})_i = \vec{a}_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (\vec{a})\cdot\hat{e}_i\,.$$



***a*)**



***b*)**

Kartezian koordinatalari sistemasi:

a) ortonormal bazislar va vektor; *b*) o‘ng qo‘l qoidasi.

Koordinata o‘qlarini burishdan bog‘liq bo‘lmagan ob’ekt sifatida vektorning ta’rifini uning komponentalarini almashtirish qonunidan kelib chiqqan holda berish mumkin: *Dekart koordinatalari sisitemasida koordinata o‘qlarini burganda (1.7) qonun bo‘yicha almashtiriluvchi* a_j *sonlar uchligi bilan tavsiflanuvchi invariant ob’ekt vektor yoki birinchi rang tenzor deyiladi, sonlar uchligi esa uning komponentalari deyiladi.*

Kartezian basis vektorlari Kroneker belgisi orasidagi bog‘lanish ifodasi

$$\hat{e}_i \cdot \hat{e}_j = \hat{e}_j \cdot \hat{e}_i = \delta_{ij}.$$

Kartezian basis vektorlari Kroneker belgisi orasidagi bog‘lanish ifodasining yoyilmasi

$$\hat{e}_i \cdot \hat{e}_j = \begin{bmatrix} \hat{e}_1 \cdot \hat{e}_1 & \hat{e}_1 \cdot \hat{e}_2 & \hat{e}_1 \cdot \hat{e}_3 \\ \hat{e}_2 \cdot \hat{e}_1 & \hat{e}_2 \cdot \hat{e}_2 & \hat{e}_2 \cdot \hat{e}_3 \\ \hat{e}_3 \cdot \hat{e}_1 & \hat{e}_3 \cdot \hat{e}_2 & \hat{e}_3 \cdot \hat{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{ij} = \delta_{ji}.$$

To‘g‘ri burchakli dekart koordinatalari sistemasida vektorlar ustida amallar.

A. Bazis vektorlarning skalyar ko‘paytmasi.

Uch o‘lchovli ***L*₃** fazoda bazis o‘zaro orthogonal $\vec{e}_1,\ \vec{e}_2, \vec{e}_3$ birlik vektorlarni qaramiz.Bu bazis ortonormal (no‘gri burchakli) bazis deyiladi va $\vec{e}_1,\ \vec{e}_2, \vec{e}_3$ - vektorlar birlik vektorlarni tashkil qiladi.Ixtiyoriy $\vec{x}$ ortonormal bazislar orqali yoyilmasi

$$\vec{x} = x_i \vec{e}_i$$

ko‘rinishida bo‘ladi. x_i son $\boldsymbol{x}$ vektorni to‘g‘ri burchakli koordinatalar sistemasida $\vec{e}_1,\ \vec{e}_2, \vec{e}_3$ bazis o‘ng bazis deyiladi, agar $\vec{e}_3$ vektorni soat strelkasiga qarama qarshi yonalishda 90^0 ga burganda $\vec{e}_1$ vektor bilan ustma ust tushsa, agar soat strelkasi yonalishi bo‘yicha burganda $\vec{e}_1$ vektorni oxiri $\vec{e}_2$ vektorni oxiri bilan ustma ust tushsa bazis chap bazis deyiladi.

Analitik geometriya kursidan vektorlarni skalyar ko‘paytmasi

$$\vec{x}\vec{y} = |\vec{x}||\vec{y}|\cos\varphi$$

formula bilan ifodalanishi ma’lum. Bu yerda $|\vec{x}|$ va $|\vec{y}|$- $\vec{x}$ va $\vec{y}$ vektorlarning uzunliklari, φ ular orasidagi burchak.

Ckalyar ko‘paytma quyidagi xossalarga ega:

1) $\vec{x}\vec{y} = \vec{y}\vec{x}$;
2) $(\lambda\vec{x})\vec{y} = \lambda(\vec{y}\vec{x})$;
3) $(\vec{x} + \vec{y})\vec{z} = \vec{x}\vec{z} + \vec{y}z$;
4) $\vec{x}\ \vec{x} > 0$, $\vec{x} \neq 0$.

Bazis vektorlarning skalyar ko‘payitmasi ortonormal bazislarda quyidagicha bo‘ladi:

	e1	***e2***	***e3***
e1	1	0	0
e2	0	1	0
e3	0	0	1

bu tenglik δ_{ij} kattalik bilan ifodalanganda $\delta_{ij} = \{ {}^1_0$ $i=j$ bo‘ganda 1 ga teng aks holda 0 ga teng bo‘ladi.

Bazis vektorlarni skalyar ko‘paytmasi

$$\vec{e}_i\vec{e}_j = \delta_{ij}$$

ko‘rinishida bo‘ladi. δ_{ij} bu simvol Kroneker simvoli deyiladi.

$\vec{x} = x_i\vec{e}_i$ va $\vec{y} = y_j\vec{e}_j$ fazoning ikkita vektori berilgan bo‘lsa,

$$\vec{x}\vec{y} = (x_i\vec{e}_i)(y_j\vec{e}_j) \quad \text{yoki} \quad \vec{x}\vec{y} = x_i y_j(\vec{e}_i\vec{e}_j),$$

bu skalyar ko‘paytmada yig‘indi berilgan i va j lar 1dan 3 gacha o‘zgaradi. 9 ta qo‘shiluvchidan iborat yig‘indini 3tasi noldan farqli $\vec{e}_i\vec{e}_j = 0$, $i \neq j$, $\vec{e}_i\vec{e}_i = 1$ bundan

$$\vec{x}\vec{y} = x_1y_1 + x_2y_2 + x_3y_3$$

yoki buni qisqacha $\vec{x}\vec{y} = x_k y_k$ ko‘rinishida yozish mumkin.

Ixtiyoriy vektorni bazis vektorlar bilan skalyar ko‘paytmasi $\vec{x} = x_i\vec{e}_i$ ni $\vec{e}_k$ bazis bilan ko‘paytmasi

$$\vec{x}\vec{e}_k = x_i(\vec{e}_i\vec{e}_k) = x_i\delta_{ik}$$

uchta qo‘shiluvchidan iborat yigindini bildiradi, bundan ikkitasi $i \neq k$ uchun nolga teng $\delta_{ik} = 0$. noldan farqli bittasi $i=k$ da $\delta_{kk} = 1$, $x_i\delta_{ik} = x_k$, demak

$$\vec{x}\vec{e}_k = x_k .$$

Bu $\vec{x}$ vektorni to‘g‘ri burchakli koordinatalarda o‘qqa mos bo‘lgan orthogonal proeksiyasini ifodalaydi.

Skalyar ko‘paytma orqali geometriyadan ma’lum kattaliklarni yangi ko‘rinishda quyidagicha ifodalash mumkin.

1) $\vec{x} = x_i\vec{e}_i$ vektorni uzunligini hisoblash $|x| = \sqrt{x^2} = \sqrt{\delta_{ik}x_i y_j}$ yoki $|\boldsymbol{x}| = \sqrt{\boldsymbol{x}_i\boldsymbol{x}_i}$.

2) Vektorlar orasidagi burchak kosinusi $\vec{x} = x_i\vec{e}_i$ va $\vec{y} = y_j\vec{e}_j$ $\cos\varphi = \dfrac{x_i y_i}{\sqrt{x_i x_i}\sqrt{y_i y_i}}$

$\vec{x}$ va $\vec{y}$ vektorlarning ortogonallik sharti $x_i y_i = 0$

3) Agar birlik $\vec{a}$ vektor a_k koordinataga ega bo‘lsa uning burchak kosinusi koopdinatasiga teng, yani

$$a_k = \vec{a}\vec{e}_k = cos\,\alpha_k .$$

$a^2=1$ ekanligidan

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1.$$

4) $\vec{x} = x_i\vec{e}_i$ vektorni $\vec{a} = a_i\vec{e}_i$ vektordagi proeksiyasi quyidagicha bo‘ladi.

$$Pr_{\boldsymbol{a}}\boldsymbol{x} = \frac{a_i x_i}{\sqrt{a_i a_i}} .$$

B. Vektorlarnimn vektorli ko‘paytmasi.

1. $\vec{x}$ va $\vec{y}$ vektorlarning vektor ko‘paytmasi quyidagi shartlarni qanoatlantiruvchi uchinchi $\vec{z}$ vektor orqali ifodalanadi.

1) $\vec{z}$ vektorning kattaligi $|z| = |x||y|\sin\varphi$, ya’ni $\vec{x}$ va $\vec{y}$ vektorlardan qurilgan parolelogramning yuziga teng bo‘ladi. Bu erda $|x|$ va $|y|$- $\vec{x}$ va $\vec{y}$ vektorlarning uzunliklari, φ ular orasidagi burchak.

2) $\vec{z}$ vektor $\vec{x}$ va $\vec{y}$ vektorlarning har biriga orthogonal bo‘ladi.

3) $\vec{z}$ vektor $\vec{x}$ va $\vec{y}$ vektorlar bilan o‘ng uchlik vektor tashkil qiladi. $\vec{x}$ va $\vec{y}$ vektorlarning ko‘paytmasi $\vec{x}\times\vec{y}$ kabi belgilanadi, va quyidagi xossalarga bo‘ysunadi.

a) $\vec{x}\times\vec{y} = \vec{y}\times\vec{x}$;

b) $(\lambda\vec{x})\times\vec{y} = \lambda(\vec{y}\times\vec{x})$;

c) $(\vec{x}+\vec{y})\times\vec{z} = \vec{x}\times\vec{z} + \vec{y}\times z$.

***L*₃** fazoda ortonormal bazis vektorlarning vektor ko‘paytmasini jadval ko‘rinishida yozamiz. Bu vektorli ko‘paytmaning o‘ng va chap bazislarda yozilishi:

Chap bazislarda O‘ng bazislarda

	e_1	e_2	e_3
e_1	0	$-e_3$	e_2
e_2	e_3	0	$-e_1$
e_3	$-e_2$	e_1	0

	e_1	e_2	e_3
e_1	0	e_3	$-e_2$
e_2	$-e_3$	0	e_1
e_3	e_2	$-e_1$	0

Jadvaldagi vektorlar chap qismidagi vektorlar birinchi, balandagilari ikkinchi ko‘paytuvchi sanaladi.

Bazis vektorlarning vektorli ko‘paytmasi ixxtiyoriy ortonormal bazislar uchun, agar o‘ng bazis bo‘lsa +1 ga, chap bazis bo‘lsa -1 ga teng ε kattalik bilan ifodalanadi.

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = \varepsilon \;; \qquad \varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -\varepsilon \,,$$

i, j, k indekslarning ikkitasi o‘zaro teng bo‘lganda nolga teng bo‘ladi. Bu kattalik ham bazislarning tanlanishiga bog‘liq va antisimmetrik *Kroneker simvoli* deyiladi.

Ushbu ε_{ijk} yordamida bazis vektorlarning vektorli ko‘paytmasi hamma vaqt ixtiyoriy bazislarda quyidagi ko‘rinishda yoziladi:

$$\vec{e}_i \times \vec{e}_j = \varepsilon_{ijk}\vec{e}_k \,,$$

bu yerda tenglikning o‘ng qismida *k* indeks bo‘yicha yig‘indi olinadi.

$$\vec{e}_1 \times \vec{e}_2 = \varepsilon_{12k}\vec{e}_k = \varepsilon_{121}\vec{e}_1 + \varepsilon_{122}\vec{e}_2 + \varepsilon_{123}\vec{e}_3 \,,$$

bu yerda birinchi ikkita had nolga teng. Bunda ushbu $\varepsilon_{123} = \varepsilon$ tenglik o‘rinli bo‘ladi. Demak,

$$\vec{e}_1 \times \vec{e}_2 = \varepsilon\, \vec{e}_3 \,,$$

o‘ng sistema uchun

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3 \,,$$

chap sistema uchun esa

$$\vec{e}_1 \times \vec{e}_2 = -\vec{e}_3 \,,$$

bu bog‘liqlik jadvalda ko‘rsatilgan.

$\vec{x} = x_i\vec{e}_i$ va $\vec{y} = y_j\vec{e}_j$ ixtiyoriy ikki vektorning vektor ko‘paytmasi quyidagicha ifodalanadi.

$$\vec{x} \times \vec{y} = (x_i\,\vec{e}_i)\mathrm{x}(y_j\vec{e}_j)$$

vektor ko‘paytmani 2) va 3) xossalaridan

$$\vec{x} \times \vec{y} = x_i y_j\,(\vec{e}_i, \vec{e}_j) = \varepsilon_{ijk}\,\mathrm{x_i}\,y_j\,\vec{e}_k$$

tenglikning o‘ng qismida hamma *i, j, k* indekslar bo‘yicha yig‘indi berilgan, undan nolga teng bo‘lgan qo‘shiluvchilarni tashlab quyidagi ko‘rinishda olish mumkin. Shunday qilib,

$$\vec{x} \times \vec{y} = \varepsilon\,\{(x_2y_3 - x_3y_2)\,\vec{e}_1 + (x_3y_1 - x_1y_3)\,\vec{e}_2 + +(x_1y_2 - x_1y_1)\,\vec{e}_3\,\}$$

yoki qisqachi

$$\vec{x}\times\vec{y}=\varepsilon\begin{vmatrix}\bar{e}_1 & \bar{e}_2 & \bar{e}_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}.$$

Agar $\vec{x}\times\vec{y}$ vektor ko‘paytmani $\bar{z}$ vektor orqali belgilasak va koordinatalarini z_k orqali ifodalasak, u holda

$$z_k=\varepsilon_{kij}x_i y_i$$

$(\varepsilon_{ijk}=\varepsilon_{kij})$ yoki to‘liq ko‘rinishda

$$z_1=\varepsilon(x_2y_3-x_3y_2)\ \ z_2=\varepsilon(x_3y_1-y_3x_1)\ \ z_3=\varepsilon(x_1y_2-x_3y_1),$$

bunda $\varepsilon=1$ bo‘lgani uchun bu formula analitik geometriya kursidagi bu koordinata vektorlar ko‘paytmasi formulasiga to‘g‘ri kelishi aniq ko‘rinadi.

C. Vektorlarning aralash ko‘paytmasi.

Vektorlarning aralash ko‘paytmasi quyidagi formula bilan ifodalangan. $(\vec{x},\vec{y},\vec{z})=(\vec{x}\times\vec{y})\vec{z}$ va $\vec{x},\vec{y},\vec{z}$ vektorlardan parallellipedning hammasi teng musbat ishora bilan olinadigan, agar $\vec{x},\vec{y},\vec{z}$ vektorlar uchligi o‘ng uchlik bo‘lsa; aks holda esa manfiy ishora olinadi.

Aralsh ko‘paytma quyidagi xossalarga bo‘ysunadi:

1) $(\vec{x},\vec{y},\vec{z})=-(\vec{y},\vec{x},\vec{z})$; 2) $(\vec{x},\vec{y},\vec{z})=(\vec{y},\vec{z},\vec{x},)=(\vec{z},\vec{x},\vec{y})$;

3) $(\lambda\vec{x},\vec{y},\vec{z})=\lambda(\vec{x},\vec{y},\vec{z})$; 4) $(\vec{x}+\vec{y},\vec{z},\vec{u})=(\vec{x},\vec{z},\vec{u})+(\vec{y},\vec{z},\vec{u})$.

Bazis vektorlarni aralash ko‘paytmasi quyidagicha bo‘ladi:

$$(\vec{e}_i,\vec{e}_j,\vec{e}_k)=\varepsilon_{ijk}\ \text{ yoki }\ (\vec{e}_i,\vec{e}_j,\vec{e}_k)=(\vec{e}_i\times\vec{e}_j)\vec{e}_k=\varepsilon_{ijk}(\vec{e}_j,\vec{e}_k).$$

Bu tenglikning o‘ng qismida ℓ indeks bo‘yicha yig‘indi berilgan $\vec{e}_i,\vec{e}_k$ skalyar ko‘paytma $\ell=k$ da noldan farqli. Bundan yig‘indini noldan farqli va u $\varepsilon_{ijk}(\vec{e}_k\vec{e}_k)$ kabi yoziladi, bu erda $\vec{e}_k\vec{e}_k=1$.

Endi $\vec{x}=x_i\vec{e}_i$, $\vec{y}=y_j\vec{e}_j$ va $\vec{z}=z_k\vec{e}_k$ ixtiyoriy uchta vektopning aralash ko‘paytmasini qaraymiz va uni quyidagicha yozamiz:

$$(\vec{x},\vec{y},\vec{z})=(x_i\vec{e}_i,y_j\vec{e}_j,z_k\vec{e}_k)\ \text{ yoki }\ (\vec{x},\vec{y},\vec{z})=x_iy_jz_k(\vec{e}_i\,\vec{e}_j\,\vec{e}_k)=\varepsilon_{ijk}x_iy_jz_k,$$

bu yerda i,j,k indekslar bir biriga bog‘liqmas holda 1,2,3 qiymatlarni qabul qiladi va yig‘indini tashkil etadi. Bu tenglikning o‘ng qismi 3^3=27 ta qo‘shiluvchidan iborat yig‘indini bildiradi. Ularning faqatgina 6 tasi noldan farqli, qolgan qo‘shiluvchilarda takrorlanuvchi indekslar mavjud. Bu tenglikni to‘laligicha quyidagicha yozamiz:

$$(\vec{x},\vec{y},\vec{z})=(x_1y_2z_3+x_2y_3z_1+x_3y_1z_2-x_2y_1z_3-x_3y_2z_1-x_1y_3z_2).$$

Bu ifodani oddiyroq quyidagicha yozish mumkin.

$$(\vec{x},\vec{y},\vec{z}) = \varepsilon \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Uchta $\vec{x},\vec{y},\vec{z}$ vektorlarning vektorli ko‘paytmasi

$$\vec{x}\times(\vec{y}\times\vec{z}) = \vec{y}(\vec{x}\vec{z}) - \vec{z}(\vec{x}\vec{y}). \tag{1.9}$$

Agar $\vec{y}$ va $\vec{z}$ vektorlar kollinuar bo‘lsa (1.9) tenglikni chap va o‘ng qismlari nolga tengligini ko‘rish mumkin. $\vec{y}$ va $\vec{z}$ vektorlar kollinuar bo‘lmasin, $\vec{u}$ vektor $\vec{y}\times\vec{z}$ vektorga orthogonal va π tekislikda yotsin va $\vec{u} = \vec{x}\times(\vec{y}\times\vec{z})$. U holda $\vec{u}$ vektor $\vec{y}$ va $\vec{z}$ vektorlar orqali quyidagicha ifodalanadi:

$$\vec{u} = \lambda\vec{y} + \mu\vec{z}\,, \tag{1.10}$$

bunda $\vec{z}^*$ vektor π tekislikda yotsa va $\vec{z}$ vektorni soat strelkasi bo‘yicha 90^0 ga burishdan hosil qilinsa va agar $\vec{y}\times\vec{z}$ vektor oxirlari bilan bir nuqtaga keltirilsa, $\vec{z}^*$, $\vec{z}$ va $\vec{y}\times\vec{z}$ orthogonal vektorlar uchligi hosil qilinadi. U holda

$$\vec{u}\vec{z}^* = \lambda(\vec{y}\vec{z}^*) \quad \text{yoki} \quad \vec{u}\vec{z}^* = [\vec{x}\times(\vec{y}\times\vec{z})]\vec{z}^* = [(\vec{y}\times\vec{z})\times\vec{z}^*]\vec{x} \tag{1.11}$$

$\vec{v} = (\vec{y}\times\vec{z})\times\vec{z}^*$ deb $\vec{v}$ vektor yo‘nalishini $\vec{z}$ vektorga mos holda olsak, $\vec{y}\times\vec{z}$ va $\vec{z}^*$ vektorlar orthogonal, ya’ni $|\vec{v}| = |\vec{y}\times\vec{z}||\vec{z}^*|$

$$|\vec{v}| = |\vec{y}||\vec{z}|^2 \sin(\vec{y},\vec{z}^*) = |\vec{y}||\vec{z}|^2 \cos(\vec{y},\vec{z}^*) = |\vec{z}|(\vec{y}\vec{z}^*)\,.$$

Bundan $\vec{v} = (\vec{y}\vec{z}^*)\vec{z}$. U holda $\vec{u}\vec{z}^* = (\vec{y}\vec{z}^*)(\vec{x}\vec{z})$. (1.11) tenglikda $\lambda = \vec{x}\vec{z}$ ekanligini olamiz.

Agar (1.10) tenglikni λ ga skalyar ko‘paytirsak, $\lambda(\vec{x}\vec{y}) + \mu(\vec{x}\vec{z}) = 0$, bu yerda $\mu = -\vec{x}\vec{y}$, bu esa (1) formulani beradi.

Namunaviy masalalar va ularning yechimlari

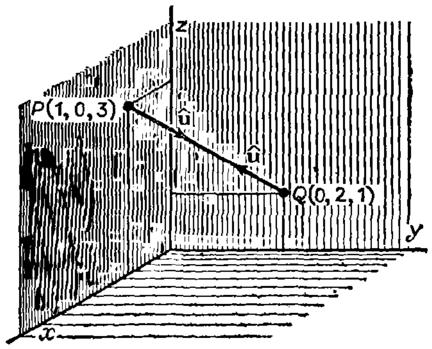
1.1. Rasmda tasvirlangan ortogonal dekart koordinatalar sistemasida quyidagilarni aniqlang: a) $\vec{v} = 2\vec{i} + 3\vec{j} - 6\vec{k}$ vektorga parallel birlik vektorni; b) $P(1,0,3)$ va $Q(0,2,1)$ nuqtalardan o‘tuvchi birlik vektorni;

Yechish: a) $|\vec{v}| = v = \sqrt{(2)^2 + (3)^2 + (-6)^2} = 7$, $\quad \dfrac{\vec{v}}{v} = \dfrac{2}{7}\vec{i} + \dfrac{3}{7}\vec{j} - \dfrac{6}{7}\vec{k}$.

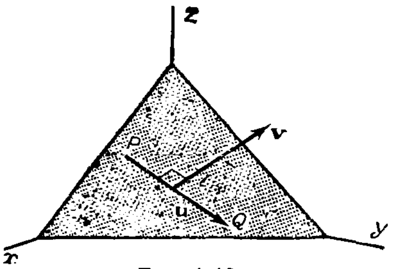
b) $P(1,0,3)$ va $Q(0,2,1)$ nuqtalardan o‘tuvchi vektor

$$\vec{u} = (0-1)\vec{i} + (2-0)\vec{j} - (1-3)\vec{k} = -\vec{i} + 2\vec{j} - 2\vec{k}\,;$$

$|\vec{u}| = \sqrt{(-1)^2 + (2)^2 + (-2)^2} = 3$, Bundan P dan Q ga yo‘nalgan birlik vektor $\frac{\vec{u}}{u} = -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$ yoki Q dan P ga yo‘nalgan birlik vektor esa $\frac{\vec{u}}{u} = \frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$ ko‘rinishda bo‘ladi.



1.2. Rasmda tasvirlangan $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ vektor $ax + by + cz = \lambda$ tenglama bilan berilgan tekislikka perpendikulyar ekanligini isbotlang.



Isbot. $P(x_1, y_1, z_1)$ va $Q(x_2, y_2, z_2)$ lar shu tekislikning ixtiyoriy ukkita nuqtasi bo‘lsin. U holda $ax_1 + by_1 + cz_1 = \lambda$ va $ax_2 + by_2 + cz_2 = \lambda$, o‘rinli bo‘ladi. Bu nuqtalardan o‘tuvchi vektor $\vec{u} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$ ko‘rinishda bo‘ladi. $\vec{v}$ vektorning $\vec{u}$ vektor yo‘nalishidagi proeksiyasi quyidagicha bo‘ladi:

$$\frac{\vec{u} \cdot \vec{v}}{u} = \frac{1}{u}[(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}] \cdot [a\vec{i} + b\vec{j} + c\vec{k}] =$$

$$= \frac{1}{u}(ax_2 + by_2 + cz_2 - ax_1 - by_1 - cz_1) = \frac{\lambda - \lambda}{u} = 0.$$

1.3. Boshi biror $P(x, y, z)$ nuqtada bo‘lgan $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ vektor va ixtiyoriy $\vec{d} = a\vec{i} + b\vec{j} + c\vec{k}$ o‘zgarmas vektor berilgan bo‘lsin. $(\vec{r} - \vec{d}) \cdot \vec{r} = 0$ tenglama sfera tenglamasi ekanligini ko‘rsating.

Yechish. Skalyar ko‘paytmani quyidagicha yozamiz:

$$(\vec{r} - \vec{d}) \cdot \vec{r} = [(x-a)\vec{i} + (y-b)\vec{j} + (z-c)\vec{k}] \cdot [x\vec{i} + y\vec{j} + z\vec{k}] = x^2 + y^2 + z^2 - ax - by - cz) = 0.$$

Ushbu $\frac{d^2}{4} = \frac{(a^2 + b^2 + c^2)}{4}$ belgilashni kiritsak bu tenglamaning ko‘rinishi quyidagicha bo‘ladi: $(x - \frac{a}{2})^2 + (y - \frac{b}{2})^2 + (z - \frac{c}{2})^2 = (\frac{d}{2})^2$, bu esa radiusi $\frac{d}{2}$ ga teng bo‘lgan sfera tenlamasidir.

1.4. Agar $\vec{a}, \vec{b}, \vec{c}$ vektorlar chiziqli bog‘langan bo‘lsa, $\vec{a} \cdot \vec{b} \times \vec{c} = 0$ tenglik o‘rinli bo‘lishini isbotlang. Quyidagi uchta vektorni chiziqli bog‘langanligini (bo‘lanmaganligini) tekshiring.

$$\vec{u} = 3\vec{i} + \vec{j} - 2\vec{k}, \quad \vec{v} = 4\vec{i} - \vec{j} - \vec{k}, \quad \vec{w} = \vec{i} - 2\vec{j} + \vec{k}$$

Isbot. $\vec{a},\vec{b},\vec{c}$ vektorlar chiziqli bog‘langan bo‘lsa, u holda $\lambda,\ \mu,\ \nu$ noldan farqli o‘zgarmaslar uchun $\lambda\vec{a}+\mu\vec{b}+\nu\vec{c}=0$ tenglik o‘rinli bo‘ladi. Bu vektorli ko‘rinishdagi tenglama koordinatalari orqali quyidagicha yoziladi:

$$\lambda a_x+\mu b_x+\nu c_x=0,\quad \lambda a_y+\mu b_y+\nu c_y=0,\quad \lambda a_z+\mu b_z+\nu c_z=0.$$

Bu $\lambda,\ \mu,\ \nu$ ga nisbatan sistema noldan farqli yechimga ega bo‘lishi uchun ular oldidagi koeffiesientlardan tuzilgan determinant nolga teng bo‘lishi kerak, ya’ni,

$$\begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}=0.$$

Bu esa $\vec{a}\cdot\vec{b}\times\vec{c}=0$ tenglikka ekvivalentdir.

$\vec{u},\vec{v},\vec{w}$ vektorlar uchligi uchun esa

$$\begin{vmatrix} 3 & 1 & -2 \\ 4 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix}=0.$$

Bundan ko‘rinadiki, $\vec{u},\vec{v},\vec{w}$ vektorlar chiziqli boglangan. Haqiqatan $\vec{v}=\vec{u}+\vec{w}$.

1.5. Uch o‘lchovli fazoda δ_{ij} - Kroneker belgisini o‘z ichiga olgan quyidagi ifodalarni hisoblang: $a)\,\delta_{ii},\ b)\ \delta_{ij}\delta_{ij},\ c)\,\delta_{ij}\delta_{ik}\delta_{jk},\ d)\,\delta_{ij}\delta_{jk}\delta_{kl}...\delta_{km}.$.

Yechish.

$$a)\,\delta_{ii}=\delta_{11}+\delta_{22}+\delta_{33}=3;$$

$$b)\ \delta_{ij}\delta_{ij}=\delta_{1j}\delta_{1j}+\delta_{2j}\delta_{2j}+\delta_{3j}\delta_{3j}=\delta_{11}\delta_{11}+\delta_{12}\delta_{12}+\delta_{13}\delta_{13}+\delta_{21}\delta_{21}+$$
$$+\delta_{22}\delta_{22}+\delta_{23}\delta_{23}+\delta_{31}\delta_{31}+\delta_{32}\delta_{32}+\delta_{33}\delta_{33}=\delta_{11}+\delta_{22}+\delta_{33}=3;$$

$$c)\,\delta_{ij}\delta_{ik}\delta_{jk}=\delta_{ij}(\delta_{ik}\delta_{jk})=\delta_{ij}\delta_{ij}=3;$$

$$d)\ \delta_{ij}\delta_{jk}\delta_{kl}...\delta_{nm}=\delta_{im}.$$

1.6. Levi-Chivita tenzori xossalaridan foydalanib, quyidagi tengliklarning to‘g‘riligini ko‘rsating: $a)\,\varepsilon_{ijk}\varepsilon_{kij}=6,\ b)\ \varepsilon_{ijk}a_j a_k=0.$

Yechish.

a) Dastlab *i* bo‘yicha yig‘amiz:

$$\varepsilon_{ijk}\varepsilon_{kij}=\varepsilon_{1jk}\varepsilon_{k1j}+\varepsilon_{2jk}\varepsilon_{k2j}+\varepsilon_{3jk}\varepsilon_{k3j}.$$

Endi *j* indeks bo‘yicha yig‘amiz va faqat noldan farqli hadlarni yozamiz:

$$\varepsilon_{ijk}\varepsilon_{kij}=\varepsilon_{12k}\varepsilon_{k12}+\varepsilon_{13k}\varepsilon_{k13}+\varepsilon_{21k}\varepsilon_{k12}+\varepsilon_{23k}\varepsilon_{k23}+\varepsilon_{31k}\varepsilon_{k31}+\varepsilon_{32k}\varepsilon_{k32}.$$

Nihoyat xuddi shunday *k* indeks bo‘yicha yig‘amiz va faqat noldan farqli hadlarni yozamiz:

$$\varepsilon_{ijk}\varepsilon_{kij} = \varepsilon_{123}\varepsilon_{312} + \varepsilon_{132}\varepsilon_{213} + \varepsilon_{213}\varepsilon_{312} + \varepsilon_{231}\varepsilon_{123} + \varepsilon_{312}\varepsilon_{231} + \varepsilon_{321}\varepsilon_{132} =$$
$$= 1\cdot 1 + (-1)\cdot(-1) + (-1)\cdot(-1) + 1\cdot 1 + 1\cdot 1 + (-1)\cdot(-1) = 6$$

b) Dastlab *j* indeks bo‘yicha, keyin *k* bo‘yicha yig‘amiz:

$$\varepsilon_{ijk}a_j a_k = \varepsilon_{i1k}a_1 a_k + \varepsilon_{i2k}a_2 a_k + \varepsilon_{i3k}a_3 a_k =$$
$$= \varepsilon_{i12}a_1a_2 + \varepsilon_{i13}a_1a_3 + \varepsilon_{i21}a_2a_1 + \varepsilon_{i23}a_2a_3 + \varepsilon_{i31}a_3a_1 + \varepsilon_{i32}a_3a_2.$$

Bu ifodadagi hadlar uchun quyidagilar o‘rinli:

$i=1$ bo‘lganda, $\varepsilon_{1jk}a_j a_k = a_2a_3 - a_3a_2 = 0,$

$i=2$ bo‘lganda, $\varepsilon_{2jk}a_j a_k = a_1a_3 - a_3a_1 = 0,$

$i=3$ bo‘lganda, $\varepsilon_{3jk}a_j a_k = a_1a_2 - a_1a_2 = 0.$

1.7. Ushbu $\varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$ tenglikning quyidagi hollar uchun o‘rinli bo‘lishini ko‘rsating.

a) $i=1,\ j=q=2,\ p=3$, b) $i=q=1,\ j=p=2.$

Yechish.

a) $i=1,\ j=2,\ p=3,\ q=2$, bo‘lganligi uchun, *k* indeks bo‘yicha yig‘amiz:

$$\varepsilon_{ijk}\varepsilon_{kpq} = \varepsilon_{12k}\varepsilon_{k32} = \varepsilon_{121}\varepsilon_{132} + \varepsilon_{122}\varepsilon_{232} + \varepsilon_{123}\varepsilon_{332} = 0.$$
$$\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} = \delta_{13}\delta_{22} - \delta_{12}\delta_{23} = 0$$

b) $i=1,\ j=2,\ p=2,\ q=1$, bo‘lgan holda $\varepsilon_{ijk}\varepsilon_{kpq} = \varepsilon_{123}\varepsilon_{321} = -1$ va

$$\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} = \delta_{12}\delta_{21} - \delta_{11}\delta_{22} = -1.$$

1.8. Quyida keltirilgan hollarda $D_{ij}x_ix_j$ ifodani yoyib chiqing va imkon bo‘lgan hollarda soddalashtiring: (a) $D_{ij} = D_{ji}$, (b) $D_{ij} = -D_{ji}$.

Yechish: Avval $D_{ij}x_ix_j$ ifodani yoyib chiqamiz:

$$D_{ij}x_ix_j = D_{1j}x_1x_j + D_{2j}x_2x_j + D_{3j}x_3x_j = D_{11}x_1x_1 + D_{12}x_1x_2 + D_{13}x_1x_3 +$$
$$+ D_{21}x_2x_1 + D_{22}x_2x_2 + D_{23}x_2x_3 + D_{31}x_3x_1 + D_{32}x_3x_2 + D_{33}x_3x_3,$$

(a) $D_{ij}x_ix_j = D_{11}(x_1)^2 + D_{22}(x_2)^2 + D_{33}(x_3)^2 + 2D_{12}x_1x_2 + 2D_{23}x_2x_3 + 2D_{13}x_1x_3,$

(b) $D_{ij}x_ix_j = 0,$ chunki $D_{11} = -D_{11},\quad D_{12} = -D_{21}$, va hokazo.

1.9. Quyidagi ifodalarning to‘liq yozilishini ko‘rsating: $a^i\delta_i^j;\ \delta_i^i;\ a^\alpha b_{\alpha j}$.

Yechish. Berilgan ifodalarda α indeks erkli, ya’ni bu indeks bo‘yicha yig‘indi olinmaydi. Shuning uchun natija quyidagicha:

1) $$a^i\delta_i^j = a^1\delta_1^j + a^2\delta_2^j + a^3\delta_3^j = \begin{Bmatrix} a^1, & j=1 \\ a^2, & j=2 \\ a^3, & j=3 \end{Bmatrix} = a^j;$$

2) $\delta_1^1 + \delta_2^2 + \delta_3^3 = 3$; 3) $a^1 b_{1j} + a^2 b_{2j} + a^3 b_{3j}$.

1.10. Kuchlar muvozanatining quyidagi statik tenglamalarini gung indeks qoidasiga asoslanib yoyib yozing: $\dfrac{\partial\sigma_{ij}}{\partial x_j} = 0.$

Yechish. Bu yerda j – gung indeks bo‘lib, u 1,2,3 qiymatlarni qabul qiladi va fiksirlangan i indeksda j bo‘yicha yig‘indi olinadi. i indeksning ketma-ket 1,2,3 qiymatlarida muvozanat tenglamalari quyidagicha yoyib yoziladi:

$$i=1: \quad \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} = 0,$$
$$i=2: \quad \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} = 0,$$
$$i=3: \quad \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} = 0.$$

Agar bu holat Dekart koordinatalari sistemasida qaralsa, u holda bu muvozanat tenglamalari sistemasi quyidagicha yoziladi:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0,$$
$$\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = 0,$$
$$\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = 0.$$

1.11. Quyidagi Koshi tenglamalarini gung indeks qoidasiga ko‘ra soddalashtirib yozing:

$$S_x = \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z,$$
$$S_y = \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z,$$
$$S_z = \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z.$$

Yechish. Bu yerda x, y, z indekslarni mos ravishda 1,2,3 indekslar bilan almashtiramiz. Tengliklardan ko‘rinadiki, ularni qisqa qilib quyidagicha yozish mumkin: $S_i = \sigma_{ij}n_j$, bunda j – gung indeks.

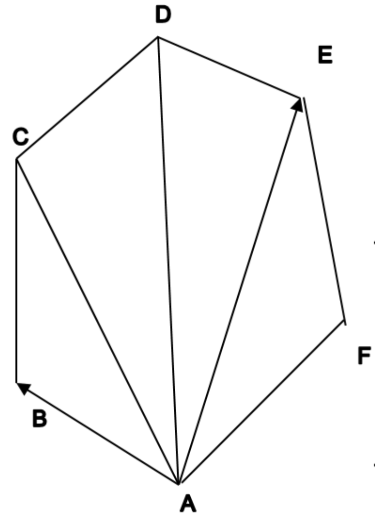
1.12. Quyidagi ifodani gung indeks qoidasiga ko‘ra yoyib yozing: $\frac{\partial x_i}{\partial x_j}$

Yechish. Bu ifoda gung indeks qoidasiga ko‘ra quyidagicha yoziladi:

$$\frac{\partial x_i}{\partial x_j}=\begin{pmatrix}\frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \frac{\partial x_1}{\partial x_3}\\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_3}{\partial x_3}\\ \frac{\partial x_3}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_3}{\partial x_3}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\delta_{ij}.$$

1.13. $ABCDEF$ muntazam oltiburchakda $\overline{AB}=\overline{e_1}$, $\overline{AE}=\overline{e_2}$ bazis vektorlar bo‘lsin. $Б=\{\bar{e}_1,\bar{e}_2\}$ bazisda $\overline{AC}, \overline{AD}, \overline{AF}$ va $\overline{EF}$ vektorlar ifodalansin.

Yechish. $\overline{ED}=\overline{AB}=\overline{e_1}$ ni e’tiborga olinsa $\overline{AD}=\overline{AE}+\overline{ED}=\overline{e_1}+\overline{e_2}$;



$$\overline{BC}=\frac{1}{2}\overline{AD}=\frac{1}{2}(\bar{e}_1+\bar{e}_2);$$

$$\overline{AC}=\overline{AB}+\overline{BC}=\bar{e}_1+\frac{1}{2}(\bar{e}_1+\bar{e}_2)=\frac{3}{2}\bar{e}_1+\frac{1}{2}\bar{e}_2\ ;$$

$$\overline{CD}=\overline{AD}-\overline{AC}=(\bar{e}_1+\bar{e}_2)-\frac{3}{2}\bar{e}_1-\frac{1}{2}\bar{e}_2=\frac{1}{2}(\bar{e}_2-\bar{e}_1)\ ;$$

$$\overline{AF}=\overline{CD}=\frac{1}{2}(\bar{e}_2-\bar{e}_1);$$

$$\overline{EF}=\overline{AF}-\overline{AE}=\frac{1}{2}(\bar{e}_2-\bar{e}_1)-\bar{e}_2=\frac{1}{2}(\bar{e}_1+\bar{e}_2).$$

Javob:

$$\overline{AC}=\frac{3}{2}\bar{e}_1+\frac{1}{2}\bar{e}_2,\quad \overline{AD}=\bar{e}_1+\bar{e}_2,\quad \overline{AF}=\frac{1}{2}(\bar{e}_2-\bar{e}_1),\quad \overline{EF}=-\frac{1}{2}(\bar{e}_1+\bar{e}_2)$$

1.14. $\bar{a}\{2,3,-1\}$, $\bar{b}\{0,1,4\}$, $\bar{c}\{1,0,-3\}$ vektorlar bo‘yicha $\bar{P}=\frac{1}{3}(\bar{a}-2\bar{b}+\bar{c})$ vektor koordinatalari aniqlansin.

Yechish. $\bar{P}=\frac{1}{3}(2\bar{e}_1+3\bar{e}_2-\bar{e}_3-2\bar{e}_2-8\bar{e}_3+\bar{e}_1-3\bar{e}_3)=\frac{1}{3}(3\bar{e}_1+\bar{e}_2-12\bar{e}_3)$.

Javob: $\bar{P}\left\{1,\frac{1}{3},-4\right\}$.

1.15. Uchburchakli $ABCA_1B_1C_1$ prizmada $\overline{AB}=\overline{e_1}$, $\overline{AC}=\bar{e}_2$ va $\overline{AA_1}=\overline{e_3}$ vektorlar bazis tashkil etsin. Agar K_1 nuqta $A_1B_1C_1$ uchburchakning og‘irlik markazi bo‘lsa, $\overline{AK_1}$ vektor koordinatalarini aniqlang.

Yechish. $K-\Delta ABC-$ning og‘irlik markazi bo‘lsin.

$$\overline{AK}=\frac{1}{3}(\overline{AB}+\overline{AC})=\frac{1}{3}(\bar{e}_1+\bar{e}_2)\ \overline{AK_1}=\overline{AK}+\overline{KK_1}=\overline{AK}+\overline{AA_1}=\frac{1}{3}(\bar{e}_1+\bar{e}_2)+\bar{e}_3.$$

Javob: $\overline{AK_1}\left\{\frac{1}{3},\frac{1}{3},1\right\}$.

1.16. $\overline{P}\{2,2,9\}$ kuch $A(4,2,-3)$ nuqtaga qo‘yilgan bo‘lsin. Kuchning $B(2,4,0)$ nuqtaga nisbatan momentining miqdori va yo‘naltiruvchi kosinuslarini aniqlang.

Yechish. $\overline{M}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & -2 & -3\\ 2 & 2 & 9\end{vmatrix}=-12\bar{i}-24\bar{j}+8\bar{k}$; $\overline{BA}=\{2,-,2-3\}$;

$\left|\overline{M}\right|=\sqrt{144+64+4\cdot 144}=28$; $\cos\alpha=-\frac{12}{28}=-\frac{3}{7}$, $\cos\beta=-\frac{6}{7}$, $\cos\beta=\frac{2}{7}$

Mashqlar

1.17. Berilgan $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak kosinusini shu vektorlarning yo‘naltiruvchi kosinuslari orqali ifodalang (yo‘naltiruvchi kosinuslar – bu vektor va koordinata o‘qlari orasidagi burchaklar kosinusi).

1.18. Uchlari $A(0,1,1)$, $B(1,2,3)$, $C(3,1,0)$ va $D(2,1,3)$ nuqtalarda berilgan *ABCD* tetraedr uchun quyidagilarni toping: 1) $\overrightarrow{AB}$ vektorning koordinatalarini; 2) *AB* tomon uzunligini; 3) $\overrightarrow{AB}$ va $\overrightarrow{AC}$ vektorlar orasidagi burchakni; 4) *ABC* yoq yuzasini; 5) *ABC* yoqqa normal vektorni; 6) *ABC* va *ABD* yoqlar orasidagi burchakni; 7) *ABCD* tetraedr hajmini; 8) *ABC* tekislikka parallel va *D* nuqtadan o‘tuvchi tekislik tenglamasini; 9) *AB* to‘g‘ri chiziqqa parallel va *C* nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini; 10) *D* nuqtadan *ABC* yoqqacha bo‘lgan masofani; 11) *C* nuqtadan *AB* to‘g‘ri chiziqqacha bo‘lgan masofani; 12) *D* nuqtaning *ABC* tekislikdagi proyeksiyasi bo‘lgan *O* nuqtaning koordinatalarini; 13) *C* nuqtaning *AB* to‘g‘ri chiziqdagi proyeksiyasi bo‘lgan *P* nuqtaning koordinatalarini.

1.19. x va y koordinatalari quyidagi shartlar bilan berilgan nuqtaning berilgan vaqt momentidagi tezligi va tezlanishi proyeksiyalarini hamda tezligi va tezlanishi orasidagi burchakni toping: 1) $x = \sin(t^2)$, $y = \cos(t^2)$, $t = 1$; 2) $x = \sin(t)\text{-}\cos(2t)$, $y = \cos(t^2)$, $t = 1$; 3) $x = \sin^2(t)$, $y = \cos(t)$, $t = 2$.

1.20. A_1, A_2 va A_3 nuqtalarda joylashgan va massalari m_1, m_2 va m_3 bo‘lgan uchta zarrachalar sistemasining og‘rlik markazi koordinatalarini toping: 1) $m_1 = 2$, $m_2 = 4$, $m_3 = 3$, $A_1\,(0,0,2)$, $A_2\,(1,1,0)$, $A_3\,(0,1,1)$; 2) $m_1 = 1$, $m_2 = 2$, $m_3 = 2$, $A_1\,(1,0,1)$, $A_2\,(0,2,3)$, $A_3\,(1,1,1)$; 3) $m_1 = 3$, $m_2 = 4$, $m_3 = 1$, $A_1\,(2,0,0)$, $A_2\,(1,2,1)$, $A_3\,(-1,1,1)$; 4) $m_1 = 3$, $m_2 = 1$, $m_3 = 2$, $A_1\,(-1,0,1)$, $A_2\,(2,3,0)$, $A_3\,(1,0,2)$.

1.21. Quyidagi ifodalarni soddalashtiring:

1) $(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})$; 2) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{a})$; 3) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{a})$; 4) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{a})$.

1.22. Quyidagi ayniyatning to‘g‘riligini isbotlang:

$$(\vec{a}\times\vec{b})\cdot((\vec{a}\times\vec{c})\times(\vec{b}\times\vec{c}))=(\vec{a}\cdot(\vec{b}\times\vec{c}))^2.$$

1.23 *A* nuqtadan o‘tuvchi va $\vec{H}$ vektorga perpendikulyar bo‘lgan tekislikning tenglamasini hamda *A* nuqtadan o‘tuvchi va $\vec{H}$ vektorga parallel bo‘lgan to‘g‘ri chiziqning tenglamasini yozing: 1) *A*(1,2,-3,), $\vec{H}$ (5,7,-6); 2) *A*(-2,0,1,), $\vec{H}$ (-1,2,4); 3) *A*(1,2,-1,), $\vec{H}$ (0,1,-1).

1.24. Ushbu $-\vec{a}+\vec{b}+\vec{c},\quad \vec{a}-\vec{b}+\vec{c},\quad \vec{a}+\vec{b}-\vec{c}$ vektorlardan tuzilgan parallelepipedning hajmini toping.

1.25. Berilgan $\vec{a}$ va $\vec{b}$ vektorlar uchun $\lambda=(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{a})+(\vec{a}\cdot\vec{b})^2=(ab)^2$ tenglik o‘rinli ekanligini ko‘rsating.

1.26. Berilgan $\dot{\vec{u}}=\vec{\omega}\times\vec{u}$ va $\dot{\vec{v}}=\vec{\omega}\times\vec{v}$ vektorlar uchun ushbu $\frac{d}{dt}(\vec{u}\times\vec{v})=\vec{\omega}\times(\vec{u}\times\vec{v})$ tenglik o‘rinli ekanligini ko‘rsating.

1.27. $\mathbf{u}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ va $\mathbf{v}=\hat{\mathbf{i}}-\hat{\mathbf{j}}$ vektorlarni o‘zaro perpendikulyar ekanligini ko‘rsating hamda bu vektorlar bilan o‘ng uchlikni tashkil qiluvchi ***w*** vektorni toping.

1.28. Yuqoridagi 1.27-topshiriqdagi ***u, v*** va ***w*** birlik ortagonal vektorlar uchun almashtirish matrisasini tuzing.

1.29. Quyidagi matritsalar berilgan:

$$\left(a_{ij}\right)=\begin{pmatrix}2&0&3\\5&1&2\\4&5&7\end{pmatrix},\ \left(x^i\right)=(2,1,4),\ \left(y^i\right)=(3,7,-1)$$

Quyidagilarni hisoblang:

а) $a_{ij}\,x^j$, б) $a_{ij}\,x^i$, в) $a_{ij}\,x^i\,y^j$, г) $a_{ij}\,y^i\,x^j$, д) $a_{ij}\,\delta_i^j$,

е) $\left(a_{ij}-\frac{2}{5}\delta_j^l\,a_{ll}\right)x^i\,y^j$.

1.30. Ushbu $g_{ij}=Э_iЭ_j$ - metrik matritsa ifodasi nechta har xil munosabatlarni o‘z ichiga oladi?

1.31. Quyidagi basis vektorlar uchun qo‘shma basis vektorini toping:

$$Э_1=3\vec{i}+8\vec{k},\ Э_2=\vec{i}+2\vec{k},\ Э_3=2\vec{i}+5\vec{j}+8\vec{k}.$$

1.32. Berilgan g_{ij} - metrik matritsa va $э_i$ – basis vektorning mos qo‘shmalari g^{ji} va $э^j$ uchun quyidagilarni isbotlang: 1) $g^{ij} = э^i э^j$; 2) $э^i э_j = \delta^i_j$; 3) $э_i = g_{ij} э^j$.

1.33. Ushbu $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ vektor $ax + by + cz = \lambda$ tenglama bilan berilgan tekislikka perpendikulyar ekanligini isbotlang.

1.34. Indeksli belgilashlardan foydalanib, ushbu $\nabla \cdot (\vec{a} \times \vec{x}) = 0$ tenglik o‘rinli ekanligini isbotlang, bunda $\vec{x}$ - radius-vektor, $\vec{a}$ - biror o‘zgarmas vektor.

1.35. Faraz qilaylik, ixtiyoriy $\vec{v}$ vektor va $\vec{э}$ birlik vektor berilgan bo‘lsin. Berilgan $\vec{v}$ vektorni $\vec{э}$ vektorga parallel va perpendikulyar bo‘lgan ikkita komponentalarga ajratish mumkinligini, ya’ni $\vec{v} = (\vec{v} \cdot \vec{э})\vec{э} + \vec{э} \times (\vec{v} \times \vec{э})$ ekanligini ko‘rsating.

1.36. Quyidagi tenglikni to‘g‘riligini isbotlang:

$$\varepsilon_{pqs}\varepsilon_{mnr} = \begin{vmatrix} \delta_{mp} & \delta_{mq} & \delta_{ms} \\ \delta_{np} & \delta_{nq} & \delta_{ns} \\ \delta_{rp} & \delta_{rq} & \delta_{rs} \end{vmatrix}.$$

1.37. Quyidagi tengliklarni o‘rinli ekanligini ko‘rsating:

$$\text{а)}\ \varepsilon_{pqs}\varepsilon_{snr} = \delta_{pn}\delta_{qr} - \delta_{pr}\delta_{qn}, \quad \text{б)}\ \varepsilon_{pqs}\varepsilon_{sqr} = -2\delta_{pr}.$$

Sinov savollari

1. Skalyar deb nimaga aytiladi? Misollar keltiring.
2. Vektor deb nimaga aytiladi? Misollar keltiring.
3. Vektorning komponentalari deganda nimani tushunasiz?
4. Skalyar va vektor maydonlar nima? Misollar keltiring.
5. Tenzor deb nimaga aytiladi?
6. Tenzorning rangi deganda nimani tushunasiz?
7. Tenzorning rangiga qarab uning komponentalari soni qanday aniqlanadi?
8. Vektorning ortalari nima? Ularni koordinat fazoda geometrik tasvirlang.
9. Radius-vektor nima va u qanday yoziladi? Uni koordinat fazoda geometrik tasvirlang.
10. Vektorning moduli va komponentalarini ayting. Ularni koordinat fazoda geometrik tasvirlang.
11. Vektorlarning skalyar ko‘paytmasi deb nimaga aytiladi?
12. Skalyar va vektorning tenzor bila nima aloqasi bor?
13. Vektor miqdorlarni qo‘shishni tushuntiring va geometrik tasvirlang.
14. Ikki vektorning skalyar ko‘paytmasini tushuntiring va geometrik tasvirlang.

15. Vektorni skalyarga ko‘paytirishni tushuntiring va geometrik tasvirlang.
16. Ikki vektorning vektor ko‘paytmasi deb nimaga aytiladi. Uni koordinat fazoda geometrik tasvirlang.
17. Vektor-skalyar ko‘paytmani tushuntiring va geometrik tasvirlang.
18. Gung indeklarni va ularni yoyib yozishni tushuntiring.
19. Uch o‘lchovli fazoda to‘g‘ri burchakli bazis vektori deb nimaga aytiladi?
20. Vektorlarning skalyar ko‘paytmasi qanday aniqlanadi?
21. Vektorlarning skalyar ko‘paytmasi xossalarini ayting.
22. Vektorlarni vektorli ko‘paytmasi qanday aniqlanadi?
23. Vektorlarni vektorli ko‘paytmasi xossalarini ayting.
24. Vektorlarni aralash ko‘paytmasi qanday aniqlanadi?
25. Vektorlarni aralash ko‘paytmasi xossalarini ayting.

2. Egri chiziqli koordinatalarni almashtirish. Skalyar va vektor maydonlarning differensial xarakteristikalari

2.1. Egri chiziqli koordinatalarni almashtirish

Yuqorida biz Dekart koordinatalar sistemasida koordinatalarni almashtirish bilan tanishdik. Endi ixtiyoriy ikkita $\zeta^1, \zeta^2, \zeta^3$ va η^1, η^2, η^3 koordinat sistemalari (egri chiziqli) koordinatalarini almashtirish masalasini qaraymiz. Faraz qilaylik, bu ikki sistema orasida uzluksiz, o‘zaro bir qiymatli moslik

$$\zeta^i = \zeta^i\,(\eta^1, \eta^2, \eta^3), \qquad (i = 1, 2, 3) \tag{2.1}$$

mavjud bo‘lsin. Bu funksiya (moslik)ni η^1, η^2, η^3 lar boyicha differensiallaymiz

$$d\zeta^i = \frac{\partial\zeta^i}{\partial\eta^1}d\eta^1 + \frac{\partial\zeta^i}{\partial\eta^2}d\eta^2 + \frac{\partial\zeta^i}{\partial\eta^3}d\eta^3. \tag{2.2}$$

Yuqorida indekslarga doir keltirilgan mulohazalarga asosan (2.2) ni

$$d\zeta^i = \frac{\partial\zeta^i}{\partial\eta^j}d\eta^j, \quad (i, j = 1,2,3) \tag{2.3}$$

ko‘rinishda yozish mumkin, bu yerda j -gung indeks. Agar

$$a^i_j = \frac{\partial\zeta^i}{\partial\eta^j}, \quad (i, j = 1, 2, 3) \tag{2.4}$$

deb belgilab olsak, (2.2) ifodaga ko‘ra a^i_j miqdorlar uchinchi tartibli matrisani tashkil etishlarini ko‘ramiz:

$$A = \left\| a^i_j \right\| = \begin{Vmatrix} a^1_1 & a^1_3 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{Vmatrix} = \begin{Vmatrix} \dfrac{\partial\zeta^1}{\partial\eta^1} & \dfrac{\partial\zeta^1}{\partial\eta^2} & \dfrac{\partial\zeta^1}{\partial\eta^3} \\ \dfrac{\partial\zeta^2}{\partial\eta^1} & \dfrac{\partial\zeta^2}{\partial\eta^2} & \dfrac{\partial\zeta^2}{\partial\eta^3} \\ \dfrac{\partial\zeta^3}{\partial\eta^1} & \dfrac{\partial\zeta^3}{\partial\eta^2} & \dfrac{\partial\zeta^3}{\partial\eta^3} \end{Vmatrix}.$$

O‘zaro bir qiymatlilik shartidan bu matrisaning noldan farqli ekanligi kelib chiqadi, ya’ni

$$\Delta = \left| a^i_j \right| \neq 0 \qquad (i, j = 1, 2, 3).$$

U holda (2.2) sistemani $d\eta^i$ larga nisbatan yechsak, quyidagi munosabatni olamiz

$$d\eta^i = \frac{\partial\eta^i}{\partial\zeta^j}d\zeta^j, \quad (i, j = 1,2,3). \tag{2.5}$$

Quyidagicha belgilash kiritamiz

$$b_j^i = \frac{\partial \eta^i}{\partial \zeta^j}. \tag{2.6}$$

U holda (2.5) ifodaning koeffisiyentlaridan tuzilgan $B=\left\| b_j^i \right\|$ matrisaga ega bo‘lamiz. A va B matrisalar *to‘g‘ri va teskari almashtirishlarning o‘tish matritsalari* deyiladi. Bu matritsalar o‘zaro teskari matritsalardir. Haqiqatan, (2.4) va (2.6) ifodalarga asosan

$$A \cdot B = \left\| a_j^i \right\| \left\| b_k^j \right\| = \left\| a_j^i \cdot b_j^i \right\| = \left\| \frac{\partial \zeta^i}{\partial \zeta^k} \right\| = \begin{cases} 1, & \text{agar } i = k \text{ bo'lsa} \\ 0, & \text{agar } i \neq k \text{ bo'lsa.} \end{cases}$$

Chunki ζ^1, ζ^2, ζ^3 lar o‘zaro bog‘lanmagan koordinatalardir. Demak,

$$A \cdot B = \left\| \delta_k^i \right\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} = E, \tag{2.7}$$

bu yerda

$$\delta_k^i = \begin{cases} 1, & \text{agar } i = k \text{ bo'lsa;} \\ 0, & \text{agar } i \neq k \text{ bo'lsa,} \end{cases}$$

ya’ni yuqorida ko‘rilgan Kroneker simvoli. Oxirgi (2.7) tenglik A va B matrisalarning o‘zaro teskari matrisalar ekanligini ko‘rsatadi. U holda B matrisaning determinanti quyidagi formula bilan topiladi:

$$\left| b_j^i \right| = \frac{1}{\Delta} = \frac{1}{\left| a_j^i \right|}.$$

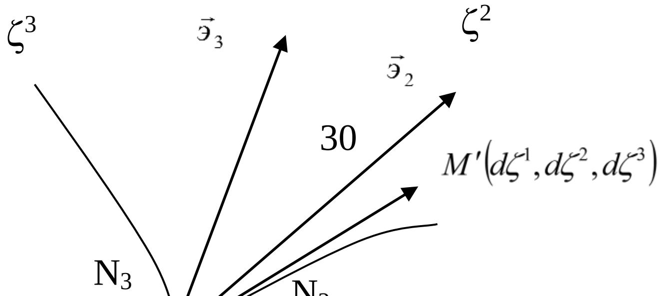
Endi egri chiziqli koordinatalar sistemasida bazis vektorlarni kiritish zaruriyati paydo bo‘ladi. Buning uchun boshi biror M nuqtada bo‘lgan ζ^1, ζ^2, ζ^3 koordinatalar sistemasini olamiz va uning $M(0,0,0)$ hamda $M'(d\zeta^1, d\zeta^2, d\zeta^3)$ nuqtalarini bog‘lovchi $d\vec{r} = \vec{MM}$ ob’yektni (2.1-chizmada strelka bilan ko‘rsatilgan) qaraymiz.

Shundan keyin M nuqtadan koordinata chiziqlarini o‘tkazamiz va ularda faqat $d\zeta^1$, $d\zeta^2$, $d\zeta^3$ koordinata orttirmalaridan biri bilangina aniqlanadigan N_1, N_2 va N_3 nuqtalarni belgilaymiz.

Ushbu

$$\frac{\partial \bar{r}}{\partial \zeta^i} = \vec{э}_i, (i = 1,2,3) \tag{2.15}$$

geometrik ob’yektlarni *bazis vektorlari* deb ataymiz.



2.1 – chizma.

Ko‘rinib turibdiki, bunday bazis vektorlari koordinata chiziqlarining urinmalari bo‘ylab yo‘naladilar (2.1-chizma). Umuman olganda d$\vec{r}$ ixtiyoriy yo‘nalgandir, lekin har qanday holda ham (2.15) ga ko‘ra d$\vec{r}$ ni

$$d\vec{r} = d\zeta^1\vec{\text{э}}_1 + d\zeta^2\vec{\text{э}}_2 + d\zeta^3\vec{\text{э}}_3 \qquad (2.16)$$

yoki

$$d\vec{r} = d\zeta^i\,\vec{\text{э}}_i$$

ko‘rinishda yozish mumkinkin (ushbuni Dekart koordinatalari sisitemasidagi ifodasi (2.2) bilan solishtiring). Bu yerdagi $d\zeta^i$ lar $d\vec{r}$ ning *komponentalari* deyiladi.

Bazis vektorilari $\vec{\text{э}}_i$ larning ζ^1, ζ^2, ζ^3 koordinata sistemasidagi koordinatalari mos ravishda (1,0, 0), (0, 1, 0), (0, 0, 1) lardan iborat. Shu bazis vektorlarning ζ^1, ζ^2, ζ^3 koordinata sistemasidan farqli, boshqa koordinatalar sistemasidagi koordinatalari albatta boshqacha bo‘ladi. Biror η^1, η^2, η^3 koordinata sistemasidagi bazis vektorlarini $\vec{\text{э}}_i{}'$ lar bilan belgilaymiz. U holda qaralayotgan ob’yekt uchun

$$d\vec{r} = d\eta^i \cdot \vec{\text{э}}_i{}' \qquad (2.17)$$

formulaga ega bo‘lamiz. Ta’rifga ko‘ra bu yerda

$$\vec{\text{э}}_i{}' = \frac{\partial\vec{r}}{\partial\eta^i}\,. \qquad (2.18)$$

Oxirgi formulani quyidagicha o‘zgartiramiz:

$$\vec{\text{э}}_i{}' = \frac{\partial\vec{r}}{\partial\eta^i} = \frac{\partial\vec{r}}{\partial\zeta^j}\cdot\frac{\partial\zeta^j}{\partial\eta^i} = \vec{\text{э}}_i \cdot a_i^j$$

yoki

$$\vec{\text{э}}_i{}' = a_i^j \cdot \vec{\text{э}}_j\,. \qquad (2.19)$$

Endi $d\eta^i$ komponentalar uchun (2.12) va (2.13) ifodalarga ko‘ra

$$d\eta^i = b_j^i d\zeta^j\,. \qquad (2.20)$$

Oxirgi (2.19) va (2.20) formulalardan foydalanib $d\vec{r}$ ning koordinatalar sistemasini almashtirishga nisbatan invariantligini isbotlash qiyin emas. Haqiqatan (2.17) dan

$$d\vec{r} = d\eta^i \vec{э}_i' = b_j^i d\zeta^i \cdot a_i^s \vec{э}_s' = a_i^s b_j^i d\zeta^j \vec{э}_s' = d\zeta^j \vec{э}_j \, ,$$

chunki (2.11) va (2.13) larga ko‘ra

$$a_i^s \cdot b_j^i = \delta_j^s = \begin{cases} 1, & j = s, \\ 0, & j \neq s. \end{cases}$$

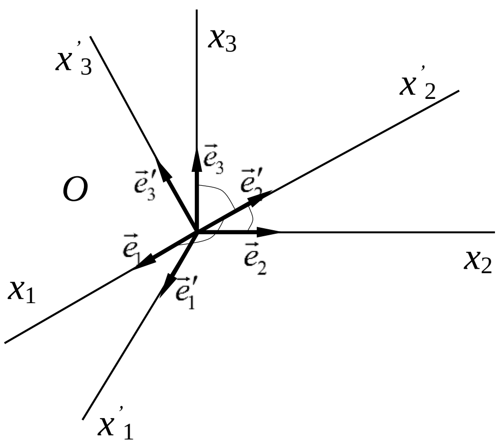
Koordinatalar sistemalari almashtirilganda xuddi $\vec{э}_i'$ bazislariga o‘xshash (2.19) formula bilan almashtiriluvchi kattaliklar *kovariant kattaliklar*, (2.20) formulalar bilan almashtiriluvchi kattaliklar esa *kontravariant kattaliklar* deyiladi.

Koordinatalar sistemasini markazi atrofida burish. Dekart koordinatalari sistemasini markazi atrofida burish natijasida yangi koordinatalar sistemasi hosil bo‘ladi (2.2-chizma). Bunday sistemani $\vec{e}_1$, $\vec{e}_2$ va $\vec{e}_3$ ortonormallashgan bazislarda berish mumkin. Burish natijasida hosil bo‘lgan koordinatalar sistemasining $\vec{e}_i'$ bazis vektorlarini eski $\vec{e}_j$ bazis vektorilari orqali quyidagicha ifodalash mumkin:

$$\vec{e}_i' = \alpha_{ij}\vec{e}_j, \quad \alpha_{ij} = \cos(\vec{e}_i', \vec{e}_j) \, ,$$

bu yerda α_{ij} - yangi bazis vektorilarining eski bazis bilan tashkil qilgan burchaklari kosinuslari. Xuddi shunday, eski bazislarni ham yangi bazislar orqali ifodalanadi:

$$\vec{e}_j = \alpha_{ij}'\vec{e}_i', \quad \alpha_{ij}' = \cos(\vec{e}_j, \vec{e}_i'),$$



2.2-chizma

bu yerda α_{ij}' - eski bazis vektorilarining yangi basis vektorlar bilan tashkil qilgan burchaklari kosinuslari.

Bunday almashtirish *orthogonal almashtirish* deyiladi va matritsa shaklida quyidagicha yoziladi:

$$A = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}.$$

Ortogonal almashtirishlarning xossalar:

1) $\alpha_{ik}\alpha_{jk} = \delta_{ij}$; Ushbu xossaga ega A matritsa *orthogonal matritsa* deb ataladi.

2) $\alpha_{ki}\alpha_{kj} = \delta_{ij}$;

3) A^T – transponirlangan matritsa orthogonal bo‘lib, $A^{-1} = A^T$, $|A| = |A^{-1}| = |A^T| = \pm 1$.

Har xil koordinatalar sistemasida bazis komponentalari orasidagi bog‘lanish quyidagicha:

$$\vec{a}' = \vec{a}, \Rightarrow \left. \begin{array}{l} \vec{a} = a_i\vec{e}_i, \\ \vec{a}' = a_i'\vec{e}_i'. \end{array} \right| \Rightarrow \begin{array}{l} a_i' = \alpha_{ij}a_j, \\ a_i = \alpha_{ji}a_j'. \end{array}$$

Dekart, silindrik va sferik koordinatalari sistemasi va ularning biridan ikkinchisiga o‘tish.

Dekart koordinatalari sistemasi (*x, y, z*) da *birlik vektorlar* (ortalar) $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ kabi belgilanadi (*x, y, z* $\in$R). U holda $\vec{u}$ vektorni quyidagicha ifodalash mumkin:

$$\vec{u} = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z \ ,$$

bunda $u_x = u_x(x, y, z)$, $u_y = u_y(x, y, z)$, $u_z = u_z(x, y, z)$ - berilgan $\vec{u}$ vektorning mos koordinat o‘qlaridagi proeksiyalari (komponentalar).

Xususiy holda, $\vec{r} = \vec{r}(x, y, z)$ - radius vektorni quyidagicha yozamiz:

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z .$$

Birinchi tartibli operatsiyalar (maydonning differensial xarakteristikalari).

Maydon nazariyasida uchta birinchi tartibli operatsiyalar qaraladi. Bu operatsiyalar ma’lum bir matematik amallarni bajarib,

- skalyar miqdorni vektor miqdorga;
- vektor miqdorni skalyar miqdorga;
- vektor miqdorni boshqa vektor miqdorga

aylantirish imkonini beradi.

Bu operatsiyalar mos ravishda *gradiyent*, *divergensiya* va *rotor (uyurma)* deb ataladi.

Ularning har birini yana bir bor qarab chiqamiz.

Biror $\varphi = \varphi(x, y, z)$ skalyar funksiyaning *gradiyenti* vektor bo‘lib, u quyidagicha ifodalanadi:

$$\mathrm{grad}\,\varphi = \frac{\partial \varphi}{\partial x}\vec{e}_x + \frac{\partial \varphi}{\partial y}\vec{e}_y + \frac{\partial \varphi}{\partial z}\vec{e}_z \ .$$

Gradiyent – bu vektor-differensial operator (yoki Gamil’ton operatori) hamdir, ya’ni $\varphi = \varphi(x, y, z)$ skalyar funksiya uchun

$$\nabla\varphi = \mathrm{grad}\,\varphi \quad \text{yoki} \quad \nabla\varphi = = \frac{\partial \varphi}{\partial x}\vec{e}_x + \frac{\partial \varphi}{\partial y}\vec{e}_y + \frac{\partial \varphi}{\partial z}\vec{e}_z \ ,$$

bunda ∇ - nabla deb o‘qiladi.

Gradiyentning mexanik manosi shundan iboratki, maydonning berilgan nuqtasida funksiya maksimal tezlik bilan o‘zgaradi.

$\vec{u}$ vektorning *divergensiyasi* deb ushbu

$$\mathrm{div}\,\vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \vec{u}$$

ifodaga aytiladi. Bundan kelib chiqadiki, ixtiyoriy vektor maydon biror skalyar maydonni, ya’ni aynan o‘zining divergensiya maydonini (uzoqlashishini) beradi.

Agar $\mathrm{div}\,\vec{u} = 0$ bo‘lsa, u holda bunday maydon *solenoidal maydon* deb ataladi.

Maydon uyurmasi (rotori) – bu quyidagi amallarning bajarilishidan hosil bo‘ladigan

$$\mathrm{rot}\vec{u}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ u_x & u_y & u_z\end{vmatrix}=\nabla\times\vec{u}=\left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right)\vec{e}_x+\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)\vec{e}_y+\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)\vec{e}_z.$$

Agar $\mathrm{rot}\vec{u}=0$ bo‘lsa, u holda bu maydon *uyurmasiz maydon* deyiladi.

Uchta operatsiyaning har biri gidrodinamik talqinga ega va bular keyingi bandlarda keltiriladi.

Silindrik koordinatalar sistemasi. Bizga ma’lumki, (r,θ,z) - silindrik va (x,y,z) -dekart koordinatalar sistemalari orasidagi bog‘lanish ifodasi quyidagicha:

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z,$$

bunda $r\in]0,\infty[$, $\theta\in[0,2\pi[$, $z\in\mathrm{R}$. Xuddi yuqoridagidek, (1.1) ga ko‘ra $\vec{r}(r,\theta,z)$ - radius vektorni quyidagicha yozamiz:

$$\vec{r}=r\cos\theta\vec{e}_x+r\sin\theta\vec{e}_y+z\vec{e}_z\ .$$

Silindrik koordinatalar sistemasida birlik vektorlar $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_z$ kabi belgilanadi, bunda ularning dekart koordinatali sistemasidagi $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ birlik vektorlar bilan bog‘lanish ifodalari quyidagicha bo‘ladi:

$$\vec{e}_r=\cos\theta\vec{e}_x+\sin\theta\vec{e}_y\ ,\ \ \vec{e}_\theta=-\sin\theta\vec{e}_x+\cos\theta\vec{e}_y\ ,\ \ \vec{e}_z=\vec{e}_z$$

yoki aksincha

$$\vec{e}_x=\cos\theta\vec{e}_r-\sin\theta\vec{e}_\theta\ ,\ \ \vec{e}_y=\sin\theta\vec{e}_r+\cos\theta\vec{e}_\theta\ ,\ \ \vec{e}_z=\vec{e}_z.$$

Shunga ko‘ra $\varphi(r,\theta,z)$ skalyar funksiyaning *gradiyenti*

$$\mathrm{grad}\,\varphi=\frac{\partial\varphi}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\vec{e}_\theta+\frac{\partial\varphi}{\partial z}\vec{e}_z$$

kabi, $\vec{u}=u_r\vec{e}_r+u_\theta\vec{e}_\theta+u_z\vec{e}_z$ vektorning *divergensiyasi*

$$\mathrm{div}\vec{u}=\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}$$

kabi, $\vec{u}=u_r\vec{e}_r+u_\theta\vec{e}_\theta+u_z\vec{e}_z$ vektor maydon *uyurmasi* (*rotori*) quyidagicha

$$\mathrm{rot}\vec{u}=\frac{1}{r}\begin{vmatrix}\vec{e}_r & r\vec{e}_\theta & \vec{e}_z\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z}\\ u_r & ru_\theta & u_z\end{vmatrix}$$

Sferik koordinatalar sistemasi. Bizga ma’lumki, (r,θ,ϕ) -sferik va (x, y, z) - dekart koordinatalar sistemalari orasidagi bog‘lanish ifodasi quyidagicha:

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta,$$

bunda $r \in]0,\infty[$, $\theta \in]0,\pi[$, $\phi \in [0,2\pi[$.

Bu yerda ham, (1.1) ga ko‘ra $\vec{r}(r,\theta,\phi)$ - radius vektorni quyidagicha yozamiz:

$$\vec{r} = r\sin\theta\cos\phi\,\vec{e}_x + r\sin\theta\sin\phi\,\vec{e}_y + r\cos\theta\,\vec{e}_z \ .$$

Sferik koordinatalar sistemasida birlik vektorlar $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$ kabi belgilanadi, bunda ularning dekart koordinatali sistemasidagi $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ birlik vektorlar bilan bog‘lanish ifodalari quyidagicha bo‘ladi:

$$\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z \ ,\\
\vec{e}_\theta &= \cos\theta\cos\phi\,\vec{e}_x + \cos\theta\sin\phi\,\vec{e}_y - \sin\theta\,\vec{e}_z \ ,\\
\vec{e}_\phi &= -\sin\phi\,\vec{e}_x + \cos\phi\,\vec{e}_y
\end{aligned}$$

yoki aksincha

$$\begin{aligned}
\vec{e}_x &= \sin\theta\cos\phi\,\vec{e}_r + \cos\theta\cos\phi\,\vec{e}_\theta - \sin\phi\,\vec{e}_\phi \ ,\\
\vec{e}_y &= \sin\theta\sin\phi\,\vec{e}_r + \cos\theta\sin\phi\,\vec{e}_\theta + \cos\phi\,\vec{e}_\phi \ ,\\
\vec{e}_z &= \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta \ .
\end{aligned}$$

Shunga ko‘ra $\varphi(r,\theta,\phi)$ skalyar funksiyaning *gradiyenti*

$$\operatorname{grad}\varphi = \frac{\partial\varphi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\vec{e}_\phi$$

kabi, $\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_\phi\vec{e}_\phi$ vektorning *divergensiyasi*

$$\operatorname{div}\vec{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

kabi, $\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_\phi\vec{e}_\phi$ vektor maydon *uyurmasi* (*rotori*) quyidagicha

$$\operatorname{rot}\vec{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ u_r & ru_\theta & r\sin\theta\, u_\phi \end{vmatrix}.$$

Ikkinchi tartibli operatsiyalar. $\varphi(x,y,z)$ skalyar funksiya va $\vec{u}$ vektor berilgan, deb faraz qilaylik. Skalyarni vektorga, vektorni skalyarga va vektorni vektorga aylantiruvchi mos $\operatorname{grad}\varphi$, $\operatorname{div}\vec{u}$, $\operatorname{rot}\vec{u}$ operatsiyalar beshta ikkinchi tartibli operatsiyalarni hosil qiladi:

- skalyar miqdorni vektor miqdorga aylantirish: $\operatorname{grad}(\operatorname{div}\vec{u})$;
- vektor miqdorni skalyar miqdorga aylantirish: $\operatorname{div}(\operatorname{grad}\varphi)$; $\operatorname{div}(\operatorname{rot}\vec{u})$;
- bir vektor miqdorni boshqa bir vektor miqdorga aylantirish

$$\operatorname{rot}(\operatorname{grad}\varphi); \quad \operatorname{rot}(\operatorname{rot}\vec{u}).$$

Maydon nazariyasi ko‘rsatadiki, bu ifodalardan ikkitasi aynan nolga teng:

$$\mathrm{div}(\mathrm{rot}\,\vec{u}) \equiv 0 \quad \text{va} \quad \mathrm{rot}(\mathrm{grad}\,\varphi) \equiv 0 .$$

Ushbu $\mathrm{div}(\mathrm{grad}\,\varphi)$ operatsiya skalyar maydon uchun *Laplas operatori* deb yuritiladi, u $\Delta\varphi$ kabi belgilanadi va quyidagicha ifodalanadi:

$$\Delta\varphi = \mathrm{div}(\mathrm{grad}\,\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} .$$

$\varphi = \varphi(x, y, z)$ skalyar funksiya uchun Laplas operatorining Gamil'ton operatori bilan bog‘lanish ifodasi quyidagicha:

$$\Delta\varphi = \nabla\varphi \cdot \nabla\varphi = \nabla^2\varphi .$$

Laplas operatorining silindik koordinatalari sistemasidagi ifodasi quyidagicha:

$$\Delta\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\varphi}{\partial\theta^2} + r\frac{\partial^2\varphi}{\partial z^2}\right],$$

sferik koordinatalari sistemasidagi ifodasi esa quyidagicha:

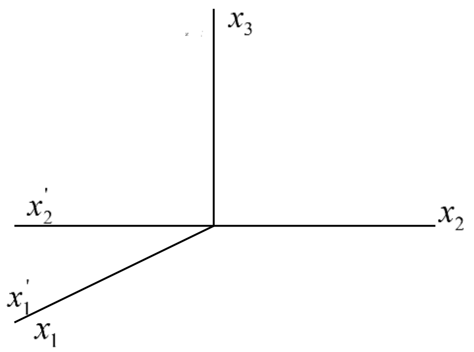
$$\Delta\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} .$$

Namunaviy masalalar va ularning yechimlari

2.1.1. Quyidagi chizmada keltirilgan o‘qlarning yo‘nalishi va joylashishiga qarab almashtirish ortoganalligini ko‘rsating.

Yechish. Chizmadan ko‘rinadiki, o‘tish matrisasi quyidagi korinishga ega

$$\alpha_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



Ortogonallik shartlari $\alpha_{ij}\alpha_{ik} = \delta_{jk}$ yoki $\alpha_{ji}\alpha_{ki} = \delta_{jk}$ aynan bajariladi va matritsaviy shakldа quyidagi ko‘rinishga ega

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.1.2. $Ox_1'x_2'x_3'$ va $Ox_1x_2x_3$ koordinatalar sistemasini orasidagi bog‘lanish munosabat bog‘lovchi koordinat almashtirishlar quyidagi jadval shaklida berilgan:

	X_1	X_2	X_3
X_1'	$-\dfrac{3}{5\sqrt{2}}$	$\dfrac{1}{\sqrt{2}}$	$-\dfrac{4}{5\sqrt{2}}$
X_2'	4/5	0	-3/5
X_3'	$\dfrac{3}{5\sqrt{2}}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{4}{5\sqrt{2}}$

1. Ortogonallik sharti bajarilishini ko‘rsating.
2. Berilgan $A(1,2,4)$ nuqtaning shtrixli koordinatalar sistemasidagi koordinatalarini aniqlang.
3. $a(a_1, a_2, a_3)$ vektorni shtrixli koordinatalar sistemasida ifodalang.
4. $Ax+By+Cz+D=0$ tekislik tenglamasini shtrixli koordinatalar sistemasida ifodalang.

Yechish. *a*) Ortogonallikni ixtiyoriy satr(ustun)ning komponentalarini boshqa ixtiyoriy satr(ustun)ning mos komponentalariga ko‘paytmalari yig‘indisi nolga tengligidan topamiz

$$-\frac{3}{5\sqrt{2}}\cdot\frac{1}{\sqrt{2}}+\frac{4}{5}\cdot 0+\frac{3}{5\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=0;$$

$$-\frac{3}{5\sqrt{2}}\cdot\left(-\frac{4}{5\sqrt{2}}\right)+\frac{4}{5}\cdot\left(-\frac{3}{5}\right)+\frac{3}{5\sqrt{2}}\cdot\frac{4}{5\sqrt{2}}=0;$$

$$-\frac{3}{5\sqrt{2}}\cdot\left(-\frac{4}{5}\right)+\frac{4}{5}\cdot 0+\frac{3}{5\sqrt{2}}\cdot\left(-\frac{3}{5}\right)=0;$$

$$-\frac{3}{5\sqrt{2}}\cdot\frac{3}{5\sqrt{2}}+\left(\frac{1}{\sqrt{2}}\right)^2-\frac{4}{5\sqrt{2}}\cdot\frac{4}{5\sqrt{2}}=0.$$

Demak ortogonallik sharti bajarilar ekan.

b) Shtrixli koordinatalar sistemasida *A* nuqtaning koordinatalarini topamiz

$$x_1'=-\frac{3}{2\sqrt{5}}\cdot 1+\frac{1}{\sqrt{2}}\cdot 2-\frac{4}{5\sqrt{2}}\cdot 4=-\frac{23}{5\sqrt{2}};$$

$$x_2'=-\frac{4}{5}\cdot 1+0\cdot 2-\frac{3}{5}\cdot 4=-\frac{16}{5};$$

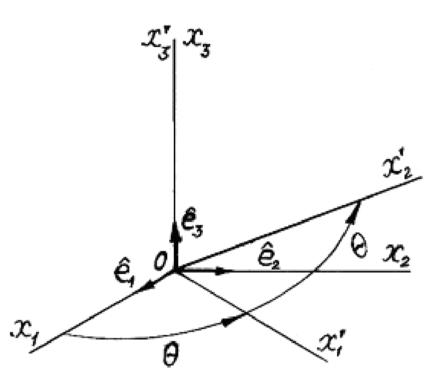
$$x_3'=\frac{3}{2\sqrt{5}}\cdot 1+\frac{1}{\sqrt{2}}\cdot 2+\frac{4}{5\sqrt{2}}\cdot 4=\frac{19+\sqrt{5}}{5\sqrt{2}}.$$

Demak, shtrixli koordinatalar sistemasida $A\left(-\dfrac{23}{5\sqrt{2}},-\dfrac{16}{5},\dfrac{19+\sqrt{5}}{5\sqrt{2}}\right)$.

2.1.3. Ushbu $Ox_1^{'}x_2^{'}x_3$ dekart koordinatalari sistemasining o‘qlari $Ox_1\,x_2x_3$ dekart koordinatalari sistemasini x_3 o‘qi atrofida θ burchakka burish natijasida hosil qilingan. Ko‘rsatilgan o‘qlarda a_{ij} almashtirish koeffisiyentlarini aniqlang va $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$ vektorning komponentalarini shtrixli koordinatalar sistemasida toping.

Yechish. Ta’rifga ko‘ra $a_{ij} = \cos(x_i^{'}, x_j)$. O‘qlar orasidagi mos burchaklar 1.4-rasmda tasvirlangan, bu esa yo‘naltiruvchi kosinuslarning jadvalini quyidagi tartibda tuzish imkonini beradi.

Tenzor almashtirishning ko‘rinishi quyidagicha yoziladi:



1.4-rasm.

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Vektorlarni almashtirish qoidasiga ko‘ra

$$v_1^{'} = a_{1j}v_j = a_{11}v_1 + a_{12}v_2 + a_{13}v_3 = v_1\cos\theta + v_2\sin\theta,$$

$$v_2^{'} = a_{2j}v_j = a_{21}v_1 + a_{22}v_2 + a_{23}v_3 = -v_1\sin\theta + v_2\cos\theta,$$

$$v_3^{'} = a_{3j}v_j = a_{31}v_1 + a_{32}v_2 + a_{33}v_3 = v_3.$$

2.1.4. Faraz qilaylik, shtrixli va strixsiz koordinatalar sistemalari o‘qlari yo‘nalishlari orasidagi burchaklar quyidagi jadval shaklida berilgan bo‘lsin

	x_1	x_2	x_3
$x_1^{'}$	135^0	60^0	120^0
$x_2^{'}$	90^0	45^0	45^0
$x_3^{'}$	45^0	60^0	120^0

Ushbu a_{ij} almashtirishning koeffisiyentlarini aniqlang va ular uchun ortogonallik shartining bajarilishini ko‘rsating.

Yechish. Ushbu a_{ij} koeffisiyentlar yo‘naltiruvchi kosinuslar bo‘lib, ularni burchaklar uchun berilgan jadvalga ko‘ra hisoblash mumkin. Shunday qilib,

$$a_{ij} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Ushbu $a_{ij}a_{ik} = \delta_{jk}$ ortogonallik sharti quyidagi shartlarning bajarilishini talab qiladi:

1) j=k=1 da $a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31} = 1$ tenglik bajarilishi zarur; bu tenglikning chap tomoni, ko‘rinib turibdiki, jadvalning birinchi ustuni elementlarining kvadratlari yig‘indisini ifodalaydi;

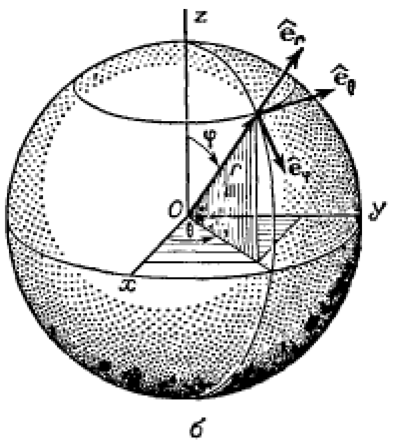
2) j=2 va j=3 da $a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0;$ tenglik bajarilishi zarur; bu tenglikning chap tomoni, ko‘rinib turibdiki, jadvalning ikkinchi va uchinchi ustunlari mos elementlarining ko‘paytmasi yig‘indisini ifodalaydi;

3) jadvalning ixtiyoriy ikkita ustunlari mos elementlari ko‘paytmasining yig‘indisi nolga teng bo‘lishi zarur; ixtiyoriy ustun elementlari kvadratlarining yig‘indisi birga teng bo‘lishi zarur.

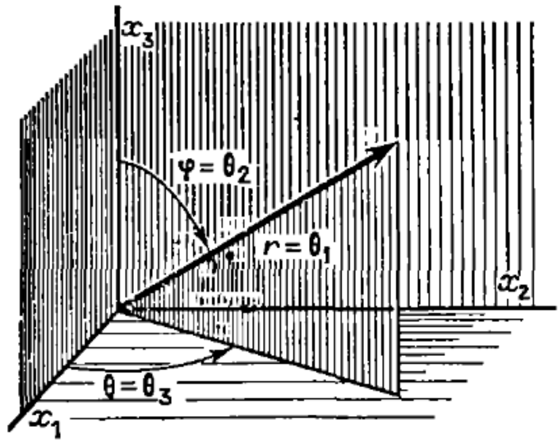
Agar ortogonallik sharti ushbu $a_{ij}a_{ik} = \delta_{jk}$ ko‘rinishda yozilgan bo‘lsa, u holda ustunlar o‘rniga satrlar ko‘paytiriladi.

Yuqorida keltirilgan yechim bu shartlarning barchasini bajaradi.

2.1.5. Chizmada tasvirlangan sferik koordinatalar sistemasi uchun metrik tenzor komponentalarini toping.



2.3-chizma



2.4-chizma

Yechish. Metrik tenzor komponentalarini $g_{pq}=\frac{\partial x_i}{\partial\theta_p}\cdot\frac{\partial x_i}{\partial\theta_q}$. Koordinatalarni nomerlashni quyidagi chizmada tasvirlangan ko‘rinishda ($r=\theta_1,\ \varphi=\theta_2,\ \theta=\theta_3$) kiritamiz. U holda

$$x_1=\theta_1\sin\theta_2\cos\theta_3,\ x_2=\theta_1\sin\theta_2\sin\theta_3,\ x_3=\theta_1\cos\theta_2$$

$$\frac{\partial x_1}{\partial\theta_1}=\sin\theta_2\cos\theta_3,\ \frac{\partial x_1}{\partial\theta_2}=\theta_1\cos\theta_2\cos\theta_3,\ \frac{\partial x_1}{\partial\theta_3}=-\theta_1\sin\theta_2\sin\theta_3,$$

$$\frac{\partial x_2}{\partial\theta_1}=\sin\theta_2\sin\theta_3,\ \frac{\partial x_2}{\partial\theta_2}=\theta_1\cos\theta_2\sin\theta_3,\ \frac{\partial x_2}{\partial\theta_3}=\theta_1\sin\theta_2\cos\theta_3,$$

$$\frac{\partial x_3}{\partial\theta_1}=\cos\theta_2,\qquad \frac{\partial x_3}{\partial\theta_2}=-\theta_1\sin\theta_2,\qquad \frac{\partial x_3}{\partial\theta_3}=0$$

o‘rinli bo‘ladi. Bu ifodalardan foydalanib metrik tenzor komponentalarini quyidagicha topamiz.

$$g_{11}=\frac{\partial x_i}{\partial\theta_1}\cdot\frac{\partial x_i}{\partial\theta_1}=\sin^2\theta_2\cos^2\theta_3+\sin^2\theta_2\sin^2\theta_3+\cos^2\theta_2=1,$$

$$g_{22}=\frac{\partial x_i}{\partial\theta_2}\cdot\frac{\partial x_i}{\partial\theta_2}=\theta_1^2\cos^2\theta_2\cos^2\theta_3+\theta_1^2\cos^2\theta_2\sin^2\theta_3+\theta_1^2\sin^2\theta_2=\theta_1^2,$$

$$g_{33}=\frac{\partial x_i}{\partial\theta_3}\cdot\frac{\partial x_i}{\partial\theta_3}=\theta_1^2\sin^2\theta_2\sin^2\theta_3+\theta_1^2\sin^2\theta_2\cos^2\theta_3=\theta_1^2\sin^2\theta_2$$

$p\neq q$ bo‘lgan hol uchun g_{pq} tenzorning komponentalari nolga teng bo‘ladi. Masalan:

$$g_{12}=\frac{\partial x_i}{\partial\theta_1}\cdot\frac{\partial x_i}{\partial\theta_2}=(\sin\theta_2\cos\theta_3)\cdot(\theta_1\cos\theta_2\cos\theta_3)+$$

$$=(\sin\theta_2\sin\theta_3)\cdot(\theta_1\cos\theta_2\sin\theta_3)-(\cos\theta_2)\cdot(\theta_1\sin\theta_2)=0.$$

Sferik koordinatalarda ds chiziqli element usunligini aniqlash uchun quyidagi tenglik o‘rinli bo‘laqdi.

$$(ds)^2=(d\theta_1)^2+\theta_1^2(d\theta_2)^2+(\theta_1\ \sin\theta_2)^2(d\theta_3)^2,$$

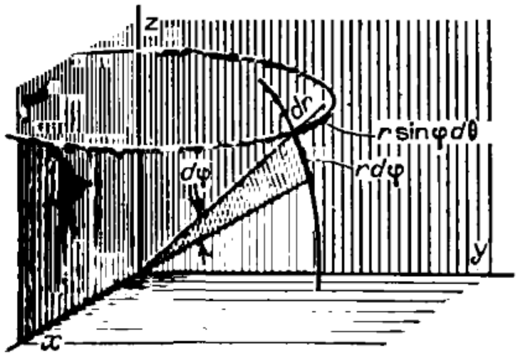
yoki

$$(ds)^2=(dr)^2+r^2(d\varphi)^2+(r\sin\varphi)^2(d\theta)^2.$$

Silindrik koordinatalar sistemasida masalani yechishni mustaqil yechish uchun qoldiramiz.

2.1.6. Egri chiziqli koordinatalarning $d\theta_i$ orttirmalariga mos chiziqli ds element uzunligi $ds=\sqrt{g_{ii}}d\theta_i$ (yig‘indi olinmasin) ga teng ekanligini ko‘rsating.

Yechish. Bizga $(ds)^2 = g_{pq} d\theta_p d\theta_q$ ko‘rinishida aniqlanishi ma’lum. Bunga ko‘ra $(d\theta_1, 0, 0)$ chiziqli element uchun $(ds)^2 = g_{11}(d\theta_1)^2$ va $ds = \sqrt{g_{11}} d\theta_1$. Xuddi shunday $(0, d\theta_2, 0)$ uchun $ds = \sqrt{g_{22}} d\theta_2$ va $(0, 0, d\theta_3)$ uchun $ds = \sqrt{g_{33}} d\theta_3$ o‘rinli bo‘ladi. Bu ifodalarni sferik koordinatalar sistemasida (2.5-chizma) quyidagicha aniqlaymiz.



2.5-chizma

1) $(d\theta_1, 0, 0)$ uchun $ds = d\theta_1 = dr;$

2) $(0, d\theta_2, 0)$ uchun $ds = \theta_1 d\theta_2 = rd\varphi;$

3) $(0, 0, d\theta_3)$ uchun $ds = \theta_1 \sin\theta_2 d\theta_3 = r\sin\varphi d\theta.$

2.1.7. Egri chiziqli koordinatalarda $(d\theta_1, 0, 0)$ va $(0, d\theta_2, 0)$ chiziqli elementlar orasidagi burchak β_{12} berilgan. $\cos\beta_{12} = \dfrac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$ tenglik o‘rinli ekanligini ko‘rsating.

Yechish. Bizga $(d\theta_1, 0, 0)$ chiziqli element uzunligi $ds_1 = \sqrt{g_{11}} d\theta_1$ va $(0, d\theta_2, 0)$ chiziqli element uzunligi $ds_2 = \sqrt{g_{22}} d\theta_2$ ekanligi avvalgi masaladan ma’lum. Bundan $(d\theta_1, d\theta_2, 0)$ chiziqli element uzunligi kvadratini $(ds)^2 = (ds_1)^2 + (ds_2)^2 + 2\cos\beta_{12} ds_1 ds_2.$ kabi hisoblash mumkin. Boshqa tomondan, $(ds)^2 = dx_i dx_i = g_{11} d\theta_1^2 + g_{22} d\theta_2^2 + 2 g_{12} d\theta_1 d\theta_2.$ Bu ifodalardan $\cos\beta_{12} = g_{12} \dfrac{d\theta_1}{ds_1} \dfrac{d\theta_2}{ds_{21}}$. 2.5-masala natijasidan foydalanib ushbu $\cos\beta_{12} = \dfrac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$ ifodani hosil qilamiz.

2.1.8. K koordinatalar sistemasida $\vec{b} = \{0, \sqrt{2}, 2\}$ vektor berilgan. K koordinatalar sistemasini x_3 o‘q atrofida 45^0 burchakka burishdan hosil qilingan K' koordinatalar sistemasida koordinatalari $\vec{c}' = \{1-\sqrt{2}, 1+\sqrt{2}, 1\}$ bo‘lgan vektor koordinatalarini aniqlang. Bu vektorlarning skalyar ko‘paytmasini toping.

Yechish. K koordinatalar sistemasida $\vec{c}$ vektorning c_k koordinatalarini aniqlash uchun yuqoridagi masaladagi a_{ij} almashtirish matritsasining teskarisidan foydalanamiz:

$$c_k = a_{ki}^T c_i \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} & & \\ & a_{ki}^T & \\ & & \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \\ c_3' \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1-\sqrt{2} \\ 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Topilgan $\vec{c}$ vektor bilan berilgan $\vec{b}$ vektorning skalyar kopaytmasini topamiz: $(\vec{b},\vec{c}) = 0\cdot(-2) + \sqrt{2}\cdot\sqrt{2} + 2\cdot 1 = 4.$

2.1.9. Jadvalda ikkita ortogonal dekart koordinatalar sistemasi o‘qlari orasidagi burchaklarning yo‘naltiruvchi kosinuslari qiymatlarining bir qismi berilgan. $Ox_1'x_2'x_3'$ o‘ng sistema bo‘lganda, jadvalning quyi qatorini aniqlang.

	x_1	x_2	x_3
x_1'	$3/5$	$-4/5$	0
x_2'	0	0	1
x_3'			

Yechish. Jadvalning birinchi qatoridan $\hat{e}_1' = {}^3\!/_5\,\hat{e}_1 - {}^4\!/_5\,\hat{e}_2$, $\hat{e}_2' = \hat{e}_3$ ekanligini aniqlash mumkin. O‘ng sistema uchun $\hat{e}_3' = \hat{e}_1' \times \hat{e}_2'$ tenglik o‘rinli bo‘lishi kerak yoki $\hat{e}_3' = \left({}^3\!/_5\,\hat{e}_1 - {}^4\!/_5\,\hat{e}_2\right) \times \hat{e}_3 = -{}^3\!/_5\,\hat{e}_2 - {}^4\!/_5\,\hat{e}_1$. U holda jadvalning uchinchi qatori quyidagi ko‘rinishni oladi:

x_3'	$-4/5$	$-3/5$	0

Mashqlar

2.1.10. $0x_1x_2x_3$ dekart koordinatalar sistemasida $\vec{a}$ vektor componentalari bilan berilgan. Koordinata o‘qlarini quyidagi variantlarda ko‘rsatilgan koordinata o‘qi atrofida berilgan burchakka burishdan hosil qilingan koordinatalar sistemasida bu vektorning komponentalarini aniqlang.

a) $\vec{a} = \{1, 1, \sqrt{3}\}$, $0x_1$ o‘q atrofida 30^0 burchakka;

b) $\vec{a} = \{0, 3, \sqrt{3}\}$, $0x_1$ o‘q atrofida 120^0 burchakka;

c) $\vec{a} = \{2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}\}$, $0x_2$ o‘q atrofida 15^0 burchakka;

d) $\vec{a} = \{0, 4, -4\sqrt{2}\}$, $0x_2$ o‘q atrofida 135^0 burchakka;

e) $\vec{a} = \{0, 1, \sqrt{3}\}$, $0x_3$ o‘q atrofida 45^0 burchakka;

f) $\vec{a} = \{1, -\sqrt{3}, 0\}$, $0x_3$ o‘q atrofida 150^0 burchakka;

2.1.11. Koordinatalar sistemasini quyida variantlarda berilgan o‘q atrofida berilgan burchakka burishdan hosil qilingan koordinatalar sistemada $\vec{a}'$ vektorning komponentalari ko‘rsatilgan. Berilgan (burishdan oldingi) koordinatalar sistemasida bu vektorning komponentalarini aniqlang.

a) $\vec{a}'=\{2,\ 0,\ -2\}$, $0x_1$ o‘q atrofida 45^0 burchakka;

b) $\vec{a}'=\{\sqrt{2},\ -1,\ 0\}$, $0x_1$ o‘q atrofida 150^0 burchakka;

c) $\vec{a}'=\{0,\ 1,\ 2\}$, $0x_2$ o‘q atrofida 60^0 burchakka;

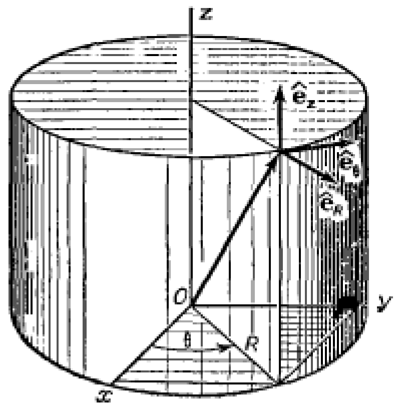
d) $\vec{a}'=\{6,\ -\sqrt{3},\ -2\sqrt{3}\}$, $0x_2$ o‘q atrofida 150^0 burchakka;

e) $\vec{a}'=\{\sqrt{3}/2,\ -1/2,\ 1\}$, $0x_3$ o‘q atrofida 75^0 burchakka;

f) $\vec{a}'=\{-1-\sqrt{2}, -1+\sqrt{2},\ 3\}$, $0x_3$ o‘q atrofida 135^0 burchakka.

2.1.12. K koordinatalar sistemasini x_3 o‘q atrofida 30^0 burchakka burishdan hosil qilingan K' koordinatalar sistemada $\vec{a}' = \{1,\ 1,\ \sqrt{3}\}$ vektor K koordinatalar sistemasini x_3 o‘q atrofida 45^0 burchakka burishdan hosil qilingan K'' koordinatalar sistemada $\vec{b}'' = \{\sqrt{2},\ \sqrt{2},\ 3\}$ vektor komponentalarini K koordinatalar sistemasida aniqlang. Bu vektorlarning skalyar ko‘paytmasini toping.

2.1.13. Chizmada tasvirlangan silindrik koordinatalar sistemasi uchun metrik tenzor komponentalarini toping.



a

2.1.14. Agar $(a_{ij}) = \begin{pmatrix} 2 & 0 & 3 \\ 5 & 1 & 2 \\ 4 & 5 & 7 \end{pmatrix}$,

$(x_i) = (2,1,4)$, $(y_i) = (3,7,-1)$ bo‘lsa, quyidagilarni toping. a) $a_{ij}x_j$, $a_{ij}x_i$,

$a_{ij}, y_j,\ \ a_{ij}, y_i, a_{ij}, x_i y_j, a_{ij}, \delta_{ij},\ a_{ij} - \frac{2}{5}\delta_{ij}a_{ll},\ \left(a_{ij} - \frac{2}{5}\delta_{ij}a_{ll}\right)x_i,\ \left(a_{ij} - \frac{2}{5}\delta_{ij}a_{ll}\right)x_i y_j$

2.1.15. Eski va yangi koordinatalar sistemasi orasidagi burchaklar quyidagicha berilgan almashtirish koeffitsiyentlarini aniqlang va ortogonalligini tekshiring.

	x_1	x_2	x_3
x'_1	135º	60º	120º
x'_3	90º	45º	45º
x'_3	45º	60º	120º

Sinov savollari

1. To‘la differensial gung indeklarda qanday yoziladi?
2. To‘g‘ri va teskari almashtirishlar matritsasini yozing.

4. Kroneker simvolini gung indekslarda yozing.
5. O‘zaro teskari matritsalar determinantlari orasidagi bog‘lanish formulasi.
6. Bazis vektorlarni gung indeklarda yozing.
7. Radius vektor differensialini gung indeklarda yozing.
8. Kovariant va kontravariant kattaliklar deb nimaga aytiladi?
9. Orthogonal almashtirish deb nimaga aytiladi?
10. Orthogonal matritsa deb nimaga aytiladi va u qanday xossalarga ega?
11. Koordinatalar sistemasini markazi atrofida burishni qanday izohlaysiz?
12. Dekart, silindrik va sferik koordinatalari sistemasida divergensiya, gradiyent va rotor operatorlari qanday yoziladi?
13. Solenoidal (uyurmasiz) va uyurmali maydonlarga tushuncha bering.

2.2. Skalyar maydon gradiyenti. Yo‘nalish bo‘yicha hosila

Maydon nazariyasida uchta birinchi tartibli operatsiyalar qaraladi. Bu operatsiyalar (maydonning differensial xarakteristikalari) ma’lum bir matematik amallarni bajarib: skalyar miqdorni vektor miqdorga; vektor miqdorni skalyar miqdorga; vektor miqdorni boshqa vektor miqdorga aylantirish imkonini beradi. Bu operatsiyalar mos ravishda *gradiyent, divergensiya* va *rotor (uyurma)* deb ataladi. Ularning har birini qarab chiqamiz.

Agar fazoning har bir $\vec{r}$ nuqtasida $\varphi(\vec{r})$ skalyar berilgan bo‘lsa – bu *skalyar maydon* deyiladi. Agar fazoning har bir $\vec{r}$ nuqtasida $\vec{a}(\vec{r})$ vektor berilgan bo‘lsa – bu *vektor maydon* deyiladi. $d\vec{r}$ vektorga ko‘chgan $d\varphi$ – *skalyar maydon orttirmasi* quyidagiga teng:

$$d\varphi = \varphi(\vec{r} + d\vec{r}) - \varphi(\vec{r}) = \frac{\partial\varphi}{\partial x_1}dx_1 + \frac{\partial\varphi}{\partial x_2}dx_2 + \frac{\partial\varphi}{\partial x_3}dx_3 = \mathrm{grad}\varphi \cdot d\vec{r}\ .$$

Gradiyent – bu $\dfrac{\partial\varphi}{\partial x_1}, \dfrac{\partial\varphi}{\partial x_2}, \dfrac{\partial\varphi}{\partial x_3}$ komponentalar bilan berilgan ushbu vektor:

$$\mathrm{grad}\varphi \equiv \nabla\varphi \equiv \frac{\partial\varphi}{\partial\vec{r}} = \frac{\partial\varphi}{\partial x_1}\vec{e}_1 + \frac{\partial\varphi}{\partial x_2}\vec{e}_2 + \frac{\partial\varphi}{\partial x_3}\vec{e}_3\,,$$

yani *gradiyent* – bu vektor-differensial operator (yoki Gamil’ton operatori) hamdir, bunda ∇ - *nabla* deb o‘qiladi. Bunga ko‘ra skalyar maydon orttirmasini quyidagicha yozamiz:

$$d\varphi = \mathrm{grad}\varphi \cdot d\vec{r} = \left|\mathrm{grad}\varphi\ \right| \cdot \left|d\vec{r}\right| \cdot \cos\theta\,,$$

bu yerda θ- gradiyent va $d\vec{r}$ vektorlar orasidagi burchak. Bu yerdan kelib chiqadiki, *gradiyentning yo'nalishi* – bu skalyar maydonning berilgan nuqtadagi tezkor o'sish yo'nalishi, *gradiyentning moduli* – bu maydonning shu yo'nalishdagi o'sish tezligi.

Ushbu $\varphi(x,y,z)=const$ - *sath sirti* (xususan, tekislikda $\varphi(x,y)=const$ - *sath chiziqlari*) deb ataladi. Ba'zi qollarda sath chiziqlari nuqtalarda, sath sirtlari esa nuqtalarda va chiziqlarda ifodalanishi mumkin. Ushbu $\varphi(x,y,z)$ skalyar maydonning birlik vektori $\vec{l}=(\cos\alpha,\cos\beta,\cos\gamma)$ bo'lsin. *Yo'nalish bo'yicha hosila* berilgan maydonning shu yo'nalishdagi o'zgarishini ifodalaydi va ushbu

$$\frac{\partial\varphi(x,y,z)}{\partial l}=\frac{\partial\varphi}{\partial x}\cos\alpha+\frac{\partial\varphi}{\partial y}\cos\beta+\frac{\partial\varphi}{\partial z}\cos\gamma$$

formula bilan hisoblanadi. Bu yerdan ko'rinadiki, yo'nalish bo'yicha hosila bu $\vec{l}=(\cos\alpha,\cos\beta,\cos\gamma)$ va $\left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\right)$ vektorlarning skalyar ko'paytmasi ekan, bunda $\left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\right)$ vektor $\varphi(x,y,z)$ funksiyaning gradiyenti. Shularga ko'ra

$$\frac{\partial\varphi}{\partial l}=|grad\varphi|\cdot|\vec{l}|\cdot\cos\phi,$$

bunda ϕ- $grad\varphi$ *va* $\vec{l}$ vektorlar orasidagi burchak. Gradiyent berilgan funksiyaning tezkor o'sishi yo'nalishini ko'rsataydi.

Gradiyentning mexanik manosi shundan iboratki, maydonning berilgan nuqtasida funksiya maksimal tezlik bilan o'zgaradi.

Skalyar maydonning ekstremal nuqtalari – bu ko'chish bo'yicha chiziqli hadlarigacha aniqlik bilan ko'chishlari maydoni o'zgarmaydigan nuqtalar. Bunday tuqtalarda $\mathrm{grad}\varphi=0$ tenglik o'rinli.

Kuch maydoni $\vec{F}(\vec{r})$ – bu vektor maydon bo'lib, fazoning har bir nuqtasida uning qiymati shu nuqtadagi zarrachaga ta'sir etayotgan kuchga teng.

Potensial kuch maydoni – bu ixtiyoriy yopiq kontur bo'ylab zarracha ko'chishida bajarilgan ish nolga teng bo'ladigan kuch maydoni. Bu holda $U(\vec{r})$ - *skalyar maydonning potensial energiyasi* tushunchasini kiritish mumkinki, u kuch maydoni bilan $\vec{F}(\vec{r})=-\mathrm{grad}U(\vec{r})$ kabi bog'langan.

Issiqlik oqimi zichligi $\vec{q}(\vec{r})$ – bu vaqt birligi ichida issiqlik oqimiga perpendikulyar orientatsiyalangan birlik yuza orqali o'qib o'tuvchi issiqlik energiyasi miqdori. Bu vektor miqdor bo'lib, temperatura gradiyenti bilan quyidagicha bog‘langan: $\vec{q}(\vec{r}) = -\kappa \cdot \operatorname{grad} T(\vec{r})$, bu yerda $T(\vec{r})$ - temperaturaning skalyar maydoni, κ - issiqlik o‘tkazuvchanlik koeffisiyenti.

Namunaviy masalalar va ularning yechimlari

2.2.1-masala. Ushbu $u = xy^2z^3$ funksiyaning $M(3, 2, 1)$ nuqtadagi $\overline{MN}$ yonalish bo‘yicha hosilasini toping, bunda $N(5, 4, 2)$.

Yechish. Dastlab $\overline{MN}$ vektorni topamiz:

$$\overline{MN} = \bar{l} = (5-3)\bar{i} + (4-2)\bar{j} + (2-1)\bar{k} = 2\bar{i} + 2\bar{j} + \bar{k}.$$

Yo‘naltiruvchi kosinuslar:

$$\cos\alpha = \frac{MN_x}{|MN|} = \frac{2}{\sqrt{2^2+2^2+1^2}} = \frac{2}{3},$$

Xuddi shunday, $\cos\beta = 2/3$; $\cos\gamma = 1/3$. Xususiy hosilalar:

$$\frac{\partial u}{\partial x} = y^2z^3, \qquad \frac{\partial u}{\partial y} = 2xyz^3, \qquad \frac{\partial u}{\partial z} = 3xy^2z^2 .$$

Xususiy hosilalarning M nuqtadagi qiymatlari:

$$\left(\frac{\partial u}{\partial x}\right)_M = 4, \quad \left(\frac{\partial u}{\partial y}\right)_M = 12, \quad \left(\frac{\partial u}{\partial z}\right)_M = 36.$$

Bu yerdan: $$\frac{\partial u}{\partial l} = 4\cdot\frac{2}{3} + 12\cdot\frac{2}{3} + 36\cdot\frac{1}{3} = \frac{68}{3}.$$

2.2.2-masala. Ushbu $u = x^3 + y^3 + z^3 - 3xyz$ skalyar maydon gradientining $M_0(2,1,-1)$ nuqtadagi miqdori va yo‘nalishini toping.

Yechish. Berilgan funksiyaning xususiy hosilalarini va ularning M_0 nuqtadagi qiymatlarini topamiz:

$$\frac{\partial u}{\partial x} = 3x^2 - 3yz, \quad \frac{\partial u}{\partial y} = 3y^2 - 3xz, \quad \frac{\partial u}{\partial z} = 3z^2 - 3xy,$$

$$\left.\frac{\partial u}{\partial x}\right|_{M_0} = 15, \quad \left.\frac{\partial u}{\partial y}\right|_{M_0} = 9, \quad \left.\frac{\partial u}{\partial z}\right|_{M_0} = -3.$$

Yo‘nalish bo‘yicha hosila formulasiga ko‘ra $\operatorname{grad} u(M_0) = 15\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}.$

Endi bu graientning berilgan M_0 nuqtadagi miqorini topamiz:

$$|\operatorname{grad} u(M_0)| = \sqrt{15^2 + 9^2 + (-3)^2} = 3\sqrt{35}.$$

2.2.3-masala. Ushbu $u = x^2y - 3xyz + xz^2y^2$ maydonning M_0 nuqtadagi hosilasini koordinata o‘qlari bilan α, $\beta = \pi/4$, $\gamma = \pi/3$ o‘tkir burchaklar tashkil etuvchi *i***e** yo‘nalish bo‘yicha toping. Maydon xarakterining ushbu yo‘nalishda o‘zgarishini toping.

Yechish. Berilgan u funksiyaning M_0 nuqtadagi xususiy hosilalari:

$$\left(\frac{\partial u}{\partial x}\right)_0 = (2xy - 3yz + y^2z^2)\Big|_{M_0} = 14, \quad \left(\frac{\partial u}{\partial y}\right)_0 = (x^2 - 3xz + 2xyz^2)\Big|_{M_0} = 8,$$

$$\left(\frac{\partial u}{\partial z}\right)_0 = (-3xy + 2xy^2z)\Big|_{M_0} = -14.$$

Masalaning shartiga ko‘ra $\cos\beta = \cos\pi/4 = \sqrt{2}/2, \cos\gamma = \cos\pi/3 = 1/2.$ Ma’lumki, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, α - o‘tkir burchak, u holda $\cos\alpha = \sqrt{1 - 1/2 - 1/4} = 1/2.$ Yo‘nalish bo‘yicha hosila formulasiga ko‘ra

$$\frac{\partial u}{\partial e} = 14 \cdot \frac{1}{2} + 8\frac{\sqrt{2}}{2} + (-14)\frac{1}{2} = 4\sqrt{2}.$$

Chunki $\frac{\partial u}{\partial e} > 0,$ shuning uchun $u(M)$ skalyar maydon shu yo‘nalishda o‘sadi.

2.2.4-masala. Ushbu $f(r) = r^3$ maydon o‘sishining $A(1,2,2)$ nuqtadagi eng katta tezligini toping.

Yechish. Maydonning gradiyentini topamiz:

$$\operatorname{grad} f(r) = \operatorname{grad} r^3 = 3r^2 \operatorname{grad} r = 3r^2\frac{\vec r}{r} = 3r\vec r.$$

Berilgan maydon o‘sishining A(1,2,2) nuqtadagi eng katta tezligi quyidagiga teng:

$$\left|\operatorname{grad} f(r)\right|_A = \left|3r\vec r\right|_A = 3r^2\Big|_A = 3\left(x^2 + y^2 + z^2\right)\Big|_A = 27.$$

2.2.5-masala. Ushbu $u = f(x, y, z) = xy^2 + z^2$ skalyar maydonning $M_0(2, 1, -1)$ nuqtadagi gradiyentini va bu maydonning shu nuqtadagi gradiyent yo‘nalishidagi hosilasini toping.

Yechish. Ma’lumki,

$$\frac{\partial f(M_0)}{\partial x} = y^2\Big|_{M_0} = 1, \quad \frac{\partial f(M_0)}{\partial y} = 2xy\Big|_{M_0} = 4, \quad \frac{\partial f(M_0)}{\partial z} = 2z\Big|_{M_0} = -2.$$

Natijada, grad$f(M_0) = 1\cdot\bar i + 4\cdot\bar j - 2\cdot\bar k$.

Agar $\bar l$ yo‘nalish gradiyentning yo‘nalishi bilan mos tushsa, u holda

$$\frac{\partial f(M_0)}{\partial l} = \left|\operatorname{grad} f\right| = \sqrt{1+16+4} = \sqrt{21}.$$

Maple matematik paketdagi yechim quyidagicha:

```
> restart: with(plots):with(VectorCalculus):
> u:=(x,y,z)->x·y^2+z^2: u(x,y,z); # funksiyaning berilishi
> v:=<cos(a),cos(b),cos(c)>; # yo‘nalishning berilishi
> d:=simplify(DirectionalDiff( u(x,y,z), v, [x,y,z] )); # yo‘nalish bo‘yicha hosila
> d1 := evalf(subs({x = 2, y = 1, z = -1, cos(a) = 1, cos(b) = 4, cos(c) = -2}, d));
```

#berilgan skalyar maydonning M_0 (2,1,-1) nuqtada u = {1,4,-2} vektor yo‘nalishidagi hosilasi.

2.2.6-masala. Ushbu

$$u = f(x, y, z) = sin(xy/5)exp(-x^2-y^2) \text{ va } u = f(x, y) = x\, exp(-x^2-y^2/4)$$

skalyar maydonlarning sath chiziqlarini Maple matematik paketi yordamida chizing.

Yechish.

```
> restart: with(plots):
> u:=(x,y)->sin(x*y/5)*exp(-x^2-y^2); #skalyar maydonning berilishi
> plot3d(u(x,y),x=-2..2,y=-2..2, axes=frame); #maydon tenglamasiga mos sirt
> r:=0.04: n:=20: #skalyar maydon (-0.04;0,04) intervalda qiymat qabul qiladi
> display(seq(implicitplot(u(x,y)=r-2*(i-1)*r/n,
   x=-2..2,y=-2..2,style=line,axes=normal, color=COLOR(RGB,(r-(i-
1)*r/n)/r,0,0)),
   i=1..n),insequence=false);
> contourplot(u(x,y),x=-3..3,y=-3..3,contours=40); # sath chiziqlari
> v:=(x,y)->x*exp(-x^2-y^2/4); #skalyar maydonning berilishi
> plot3d(v(x,y),x=-3..3,y=-5..5, axes=frame); #maydon tenglamasiga mos sirt
> r:=0.45: n:=40: #skalyar maydon (-0.45;0,45) intervalda qiymat qabul qiladi
> display(seq(implicitplot(v(x,y)=r-2*(i-1)*r/n,
  x=-3..3,y=-4..4,style=line,axes=normal, color=COLOR(RGB,(r-(i-1)*r/n)/r,0,0)),
  i=1..n),insequence=false);
> contourplot(v(x,y),x=-3..3,y=-4..4,contours=40); # sath chiziqlari
```

2.2.7-masala. Ushbu $\varphi(\vec{r}) = (x^2 - 5x + 7y^2 - y + 6z^2 + 3)$ berilgan funksiya uchun (x_0, y_0, z_0) nuqtada gradiyentni, ekstremum nuqtalarni, eng tezkor o‘sish yo‘nalishini toping hamda funksiya o‘zgarmas bo‘ladigan sirtning shu nuqtaga o‘tkazilgan urinma tekislik tenglamasini tuzing.

Yechish. Gradiyent vektori komponentalarini va $\varphi(\vec{r})$ fuksiyaning ekstremum nuqtalarini topish uchun quyidagilarni hisoblaymiz:

$$\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}(x^2-5x+7y^2-y+6z^2+3)=2x-5;$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial}{\partial y}(x^2-5x+7y^2-y+6z^2+3)=14y-1;$$

$$\frac{\partial\varphi}{\partial z}=\frac{\partial}{\partial z}(x^2-5x+7y^2-y+6z^2+3)=12z.$$

Ekstremum nuqtalarni topishga imkon beruvchi tenglamalar sistemasi:

$$2x-5=0;$$
$$14y-1=0;$$
$$12z=0.$$

Bu sistemaning yechimi $\varphi(\vec{r})$ funksiyaning ekstremum nuqtalari koordinatalarini beradi: $x=2{,}5;\ y=\frac{1}{14};\ z=0.$

Gradiyentning vektor maydoni uchun ifoda quyidagicha:

$$grad\,\varphi(\vec{r})=\frac{\partial\varphi}{\partial x}\cdot\vec{e}_x+\frac{\partial\varphi}{\partial y}\cdot\vec{e}_y+\frac{\partial\varphi}{\partial z}\cdot\vec{e}_z=(2x-5)\cdot\vec{e}_x+(14y-1)\cdot\vec{e}_y+12z\cdot\vec{e}_z\,,$$

$\vec{r}_0(x0,y_0,z_0)$ nuqtada esa:

$$grad\,\varphi(\vec{r}_0)=(2x_0-5)\cdot\vec{e}_x+(14y_0-1)\cdot\vec{e}_y+12z_0\cdot\vec{e}_z\,.$$

$\varphi(\vec{r})$ funksiyaning $\vec{r}_0(x_0,y_0,z_0)$ nuqtadagi qiymati o‘zgarmas bo‘ladigan tekislikka urinma qilib o‘tkazilgan tekislik tenglamasini topish uchun gradiyent vektorni hisoblash ifodasidan uning komponentalarini topishdan foydalanib quyidagi natijaga kelamiz:

$$(2x_0-5)\left(x-x_0\right)+(14y_0-1)\left(y-y_0\right)+12z\left(z-z_0\right)=0\,.$$

2.2.8-masala. Quyidagi gradiyent vektorning komponentalarini toping:

$a)$ $grad\,(r)$, bu yerda $r\equiv\left|\vec{r}\right|=\left(x^2+y^2+z^2\right)^{1/2}$.

Ko‘rsatma. Ushbu masalada $\varphi(\vec{r})=r$.

$$\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}[\left(x^2+y^2+z^2\right)^{1/2}]=\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}};$$

Natijada: $\dfrac{\partial\varphi}{\partial x}=\dfrac{x}{\sqrt{x^2+y^2+z^2}};$

Xuddi shunday: $\dfrac{\partial\varphi}{\partial y}=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$ va $\dfrac{\partial\varphi}{\partial z}=\dfrac{z}{\sqrt{x^2+y^2+z^2}}.$

Bularni gradiyent formulasiga qo‘yamiz:

$$grad\,\varphi(\vec{r}) = \frac{\partial\varphi}{\partial x}\cdot\vec{e}_x + \frac{\partial\varphi}{\partial y}\cdot\vec{e}_y + \frac{\partial\varphi}{\partial z}\cdot\vec{e}_z = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\cdot\vec{e}_x + y\cdot\vec{e}_y + z\cdot\vec{e}_z) = \frac{\vec{r}}{r};$$

Javob: $grad\,\varphi(\vec{r}) = \dfrac{\vec{r}}{r}$.

$b)$ $grad\,(\rho)$, где $\rho \equiv |\vec{\rho}| = \left(x^2+y^2\right)^{1/2}$;

Ko‘rsatma. Bu masalaning yechimi xuddi yuqoridagi kabi, ammo bu yerda faqat $\vec{\rho} = x\cdot\vec{e}_x + y\cdot\vec{e}_y$. Ma’lumki, $\varphi(\vec{r}) = \rho$.

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}[\left(x^2+y^2\right)^{1/2}] = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}};$$

Natijada: $\dfrac{\partial\varphi}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$; Xuddi shunday: $\dfrac{\partial\varphi}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$; $\dfrac{\partial\varphi}{\partial z} = 0$.

Bularni gradiyent formulasiga qo‘yamiz:

$$grad\,\varphi(\vec{r}) = \frac{\partial\varphi}{\partial x}\cdot\vec{e}_x + \frac{\partial\varphi}{\partial y}\cdot\vec{e}_y + \frac{\partial\varphi}{\partial z}\cdot\vec{e}_z = \frac{1}{\sqrt{x^2+y^2}}(x\cdot\vec{e}_x + y\cdot\vec{e}_y) = \frac{\vec{\rho}}{\rho};$$

Javob: $grad\,\varphi(\vec{r}) = \dfrac{\vec{\rho}}{\rho}$.

$c)$ $grad\,(f(r))$;

Ko‘rsatma. Ushbu masalada $\varphi(\vec{r}) = f(r)$.

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}f(r) = \frac{df(r)}{dr}\cdot\frac{\partial r}{\partial x} = \frac{df(r)}{dr}\cdot\frac{\partial}{\partial x}[\left(x^2+y^2+z^2\right)^{1/2}] = \frac{df(r)}{dr}\cdot\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}};$$

Natijada: $\dfrac{\partial f(r)}{\partial x} = \dfrac{df(r)}{dr}\dfrac{x}{\sqrt{x^2+y^2+z^2}}$;

Xuddi shunday: $\dfrac{\partial\varphi}{\partial y} = \dfrac{df(r)}{dr}\cdot\dfrac{y}{\sqrt{x^2+y^2+z^2}}$; $\dfrac{\partial\varphi}{\partial z} = \dfrac{df(r)}{dr}\cdot\dfrac{z}{\sqrt{x^2+y^2+z^2}}$.

Bularni gradiyent formulasiga qo‘yamiz:

$$grad\,\varphi(\vec{r}) = \frac{\partial\varphi}{\partial x}\cdot\vec{e}_x + \frac{\partial\varphi}{\partial y}\cdot\vec{e}_y + \frac{\partial\varphi}{\partial z}\cdot\vec{e}_z =$$

$$= \frac{df(r)}{dr}\frac{1}{\sqrt{x^2+y^2+z^2}}(x\cdot\vec{e}_x + y\cdot\vec{e}_y + z\cdot\vec{e}_z) = \frac{df(r)}{dr}\frac{\vec{r}}{r};$$

Javob: $grad\,\varphi(\vec{r}) = \dfrac{df(r)}{dr}\cdot\dfrac{\vec{r}}{r}$.

2.2.9. Hisoblang: $grad\,(f(r)g(r))$;

Ko'rsatma. Ushbu masalada $\varphi(\vec{r}) = f(r)\cdot g(r)$.

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}(f(r)\cdot g(r)) = \frac{d(f(r)\cdot g(r))}{dr}\cdot\frac{\partial r}{\partial x} =$$

$$= (\frac{df(r)}{dr}\cdot g(r) + f(r)\cdot\frac{dg(r)}{dr})\cdot\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}};$$

Natijada: $$\frac{\partial\varphi(r)}{\partial x} = (\frac{df(r)}{dr}\cdot g(r) + f(r)\cdot\frac{dg(r)}{dr})\cdot\frac{x}{\sqrt{x^2+y^2+z^2}};$$

Xuddi shunday: $$\frac{\partial\varphi}{\partial y} = (\frac{df(r)}{dr}\cdot g(r) + f(r)\cdot\frac{dg(r)}{dr})\cdot\frac{y}{\sqrt{x^2+y^2+z^2}};$$

va $$\frac{\partial\varphi}{\partial z} = (\frac{df(r)}{dr}\cdot g(r) + f(r)\cdot\frac{dg(r)}{dr})\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Bularni gradiyent formulasiga qo'yamiz:

$$grad\,\varphi(\vec{r}) = (\frac{df(r)}{dr}\cdot g(r) + f(r)\cdot\frac{dg(r)}{dr})\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}(x\cdot\vec{e}_x + y\cdot\vec{e}_y + z\cdot\vec{e}_z) =$$

$$= (\frac{df(r)}{dr}\cdot g(r) + f(r)\cdot\frac{dg(r)}{dr})\cdot\frac{\vec{r}}{r};$$

Javob: $grad\,\varphi(\vec{r}) = (\frac{df(r)}{dr}\cdot g(r) + f(r)\cdot\frac{dg(r)}{dr})\cdot\frac{\vec{r}}{r}$.

2.2.10. Hisoblang: $grad(\vec{\alpha}\vec{r})$, bu yerda $\vec{\alpha}$ - o'zgarmas vektor.

Ko'rsatma. Ushbu masalada $\varphi(\vec{r}) = (\vec{\alpha},\vec{r}) = \alpha_x x + \alpha_y y + \alpha_z z$.

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}(\alpha_x x + \alpha_y y + \alpha_z z) = \alpha_x;$$

Natijada: $\frac{\partial\varphi}{\partial x} = \alpha_x$; Xuddi shunday: $\frac{\partial\varphi}{\partial y} = \alpha_y$; $\frac{\partial\varphi}{\partial z} = \alpha_z$.

Bularni gradiyent formulasiga qo'yamiz:

$$grad\,\varphi(\vec{r}) = \frac{\partial\varphi}{\partial x}\cdot\vec{e}_x + \frac{\partial\varphi}{\partial y}\cdot\vec{e}_y + \frac{\partial\varphi}{\partial z}\cdot\vec{e}_z = (\alpha_x\vec{e}_x + \alpha_y\vec{e}_y + \alpha_z\vec{e}_z) = \vec{\alpha};$$

Javob: $grad\,\varphi(\vec{r}) = \vec{\alpha}$.

2.2.11. Hisoblang: $grad(\vec{a},[\vec{\omega}\times\vec{r}])$, bu yerda $\vec{a}$ *u* $\vec{\omega}$ - o'zgarmas vektorlar.

Ko'rsatma. Ushbu masalada

$$\varphi(\vec{r}) = (\vec{a},[\vec{\omega}\times\vec{r}]) = \alpha_x(\omega_y z - \omega_z y) + \alpha_y(\omega_z x - \omega_x z) + \alpha_z(\omega_x y - \omega_y x).$$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}(\alpha_x(\omega_y z - \omega_z y) + \alpha_y(\omega_z x - \omega_x z) + \alpha_z(\omega_x y - \omega_y x)) = \alpha_y\omega_z - \alpha_z\omega_y = [\vec{\alpha}\times\vec{\omega}]_x,$$

bu yerda $[\vec{\alpha}\times\vec{\omega}]_x$ - vektor ko‘paytmaning x o‘qdagi komponentasi.

Izoh: $\varphi(\vec{r})$ funksiyaning ifodasini quyidagicha yozish hamda vektor ko‘paytmaning vektorlarni siklik o‘rin almashtirish xossalaridan foydalanish orqali yuqorida natijani tezkor olish mumkin:

$$\varphi(\vec{r}) = (\vec{a},[\vec{\omega}\times\vec{r}\,]) = (\vec{r},[\vec{a}\times\vec{\omega}]) = x\cdot[\vec{\alpha}\times\vec{\omega}]_x + y\cdot[\vec{\alpha}\times\vec{\omega}]_y + z\cdot[\vec{\alpha}\times\vec{\omega}]_z$$

Natijada: $\dfrac{\partial\varphi}{\partial x} = [\vec{\alpha}\times\vec{\omega}]_x$; $\dfrac{\partial\varphi}{\partial y} = [\vec{\alpha}\times\vec{\omega}]_y$; $\dfrac{\partial\varphi}{\partial z} = [\vec{\alpha}\times\vec{\omega}]_z$.

Bularni gradiyent formulasiga qo‘yamiz: $grad\,\varphi(\vec{r}) = [\vec{\alpha}\times\vec{\omega}]$.

Javob: $grad\,\varphi(\vec{r}) = [\vec{\alpha}\times\vec{\omega}]$.

2.2.12. Ushbu $\exp(-\alpha r^2)$, r=$|\vec{r}\,|$ funksiyaning tezkor o‘sishi yo‘nalishida A(3, 2, 1) nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

Ko‘rsatma. $\vec{r}_0(x_0, y_0, z_0)$ nuqtadan o‘tuvchi va (a,b,c) vektorga parallel to‘g‘ri chiziq tenglamasini yo‘nalish bo‘yicha hosila formulasidan foydalanib topamiz. $\exp(-\alpha r^2)$, r=$|\vec{r}\,|$ funksiyaning $\vec{r}_0(x_0, y_0, z_0)$ nuqtadagi tezkor o‘sishi yo‘nalishi bu funksiyaning shu nuqtadagi gradiyent vektori orqali topiladi. Buning uchun dastlab berilgan funksiya gradiyentining vektor maydonini topib, bu qiymatlardan foydalanamiz.

Ushbu masalada $\varphi(\vec{r}) = \exp(-\alpha r^2)$.

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}[\exp(-\alpha r^2)] = \frac{d[\exp(-\alpha r^2)]}{dr}\cdot\frac{\partial r}{\partial x} = -2\alpha\cdot r\cdot\exp(-\alpha r^2)\frac{x}{r}.$$

Natijada: $\dfrac{\partial\varphi}{\partial x} = -2\alpha\cdot\exp(-\alpha r^2)\cdot x;$

Xuddi shunday: $\dfrac{\partial\varphi}{\partial x} = -2\alpha\cdot\exp(-\alpha r^2)\cdot y;$ $\dfrac{\partial\varphi}{\partial x} = -2\alpha\cdot\exp(-\alpha r^2)\cdot z.$

Bularni gradiyent formulasiga qo‘yamiz:

$$\begin{aligned} grad\,\varphi(\vec{r}) &= \frac{\partial\varphi}{\partial x}\cdot\vec{e}_x + \frac{\partial\varphi}{\partial y}\cdot\vec{e}_y + \frac{\partial\varphi}{\partial z}\cdot\vec{e}_z = \\ &= -2\alpha\cdot\exp(-\alpha r^2)\cdot(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z) = -2\alpha\cdot\exp(-\alpha r^2)\cdot\vec{r} \end{aligned}.$$

Gradiyent vektorning A(3, 2, 1) nuqtagi qiymatini topish uchun $\vec{r}_0(3,2,1)$ radius-vektorni kiritib, uninng uzunligini hisoblaymiz: $r_0 = \sqrt{14}$. Topilganlarni gradiyent-ning ifodasiga qo‘yamiz:

$$grad\,\varphi(\vec{r}_0) = -2\alpha\exp(-14\alpha)\cdot(3\vec{e}_x + 2\vec{e}_y + \vec{e}_z)$$

$$(-2\alpha\exp(-14\alpha))^{-1}\frac{x-3}{3}=(-2\alpha\exp(-14\alpha))^{-1}\frac{y-2}{2}=(-2\alpha\exp(-14\alpha))^{-1}\frac{z-1}{1}$$

Bu tenglamani noldan farqli ifodaga bo‘lib, izlanayotgan to‘g‘ri chiziq tenglamasini topamiz.

Javob: $\frac{x-3}{3}=\frac{y-2}{2}=z-1.$

Mashqlar

2.2.13. Quyida berilgan funksiyalar uchun (x_0, y_0, z_0) nuqtada gradiyentni, ekstremum nuqtalarni, eng tezkor o‘sish yo‘nalishini toping hamda funksiya o‘zgarmas bo‘ladigan sirtning shu nuqtaga o‘tkazilgan urinma tekislik tenglamasini tuzing: 1) (x^2-y^2+x-y+z); 2) (x^3-3y^2z+y^2+3z^2); 3) (x^2-5x+7y^2-y+6z^2+3); 4) (5x^3-8y^2z+4y^2+4z^2+6); 5) (6x^4+8xyz^2+7x^2z+y+6).

2.2.14. Gradiyent vektorining komponentalarini toping: 1) grad(r), bu yerda $r=|\vec r|=\left(x^2+y^2+z^2\right)^{1/2}$; 2) grad(ρ), где $\rho=|\vec\rho|=\left(x^2+y^2\right)^{1/2}$; 3) grad ($\frac{1}{r}$); 4) grad (ln($\rho$)); 5) grad ($\frac{1}{|\vec r-\vec R|}$), bu yerda $\vec R$- o‘zgarmas vektor, $\vec r=(x,y,z)$; 6) grad (ln$|\vec\rho-\vec\rho_0|$), bu yerda $\vec\rho_0$- o‘zgarmas vektor, $\vec\rho=(x,y,0)$; 7) grad (f(r)); 8) grad(f(ρ)); 9) grad (f($\vec k\vec r$)), bu yerda $\vec k$ - o‘zgarmas vektor; 10) grad(f(ρ, z)); 11) grad (f($\vec r$)g($\vec r$)); 12) grad ($\vec\alpha\vec r$), bu yerda $\vec\alpha$ - o‘zgarmas vektor; 13) grad$(\vec a,[\vec\omega,\vec r])$, bu yerda $\vec a$ va $\vec\omega$ - o‘zgarmas vektorlar; 14) grad (exp(-αr)); 15) grad (exp(-αr^2)); 16) grad (exp(-$\alpha\rho$)); 17) grad (exp(-αρ^2))*;* 18) grad (sin($\vec k\vec r$)); 19) grad (sin($\vec k\vec\rho$)). 20) $\mathrm{grad}\frac{e^{-cr}}{r}$, $\mathrm{grad}\sin(\vec k\vec r)$; 21) $\mathrm{grad}([\vec a\vec r],[\vec b\vec r])$; 22) $\mathrm{grad}\left(\frac{\vec d\vec r}{r^3}\right)$, $\vec d=const$; 23) $\mathrm{grad}\left((\vec a\vec r)(\vec b\vec r)\right)$.

2.2.15. exp (-r^2), r=|$\vec r$ | funksiyaning eng tezkor o‘sish yo‘nalishidagi A(3, 2, 1) nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini yozing.

2.2.16. (x^2+y^2-3z) funksiyaning A(-1, 2, -1) nuqtadagi o‘zgarmas sirtiga o‘tkazilgan urinma tekislik tenglamasini yozing.

2.2.17. A(-1, 1, 1) nuqtada (x^2+2y^2-z^2) va r=|$\vec r$ | funksiyalarning tezkor o‘sish yo‘nalishlari orasidagi burchakni toping.

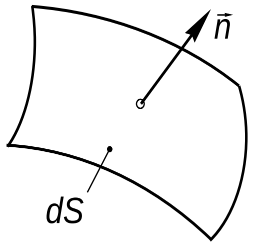
2.2.18. Agar potensial energiya (2x+y^2-z) ga teng bo‘lsa, A(-1, -2, -1) nuqtadagi zarrachaga ta’sir etuvchi kuchni toping.

2.2.19. $A(1, 1, 2)$ nuqtadan o‘tuvchi, $|\vec{r}-\vec{a}|$ funksiyaning qiymati o‘zgarmas bo‘ladigan nuqtasiga urinma yo‘nalgan tekislik tenglamasini yozing, bu yerda $\vec{r}$ - radius-vektor, $\vec{a}$ - koordinatalari (1, 2, 0) bo‘lgan o‘zgarmas vektor.

2.2.20. Ushbu $u = x^2 + y^2 + z^2 - xyz$ skalyar maydon gradientining $M_0(-2,1,1)$ nuqtadagi miqdori va yo‘nalishini toping.

2.3. Vektor maydon divergensiyasi. Ostrogradskiy-Gauss teoremasi

Yuzacha vektori $\Delta\vec{S} \equiv \Delta\vec{f} \equiv \vec{n}\Delta S$ - bu yuzaga perpendikulyar yo‘nalgan va moduli uning yuzasiga teng bo‘lgan vektor. Agar yuza yopiq sirt bo‘lsa, bu vektor $\vec{n}$ normal vektor bo‘ylab yo‘nalgan bo‘lib, u o‘ng vint qoidasi bo‘yicha yuzani aylanib o‘tuvchi yo‘nalish bilan bog‘liq vektor.



$\vec{r}$ nuqtada $\Delta\vec{S}$ yuzacha orqali o‘tuvchi $\vec{a}(\vec{r})$ *vektor maydon-ning* $\Delta\Phi$ *oqimi* (1.1-rasm): $\Delta\Phi = \vec{a}(\vec{r})\cdot\Delta\vec{S} = \vec{a}(\vec{r})\cdot\vec{n}\Delta S$.

1.1-rasm. Sirt elementi va normal birlik vektor.

S sirt orqali o‘tuvchi $\vec{a}(\vec{r})$ *vektor maydonning* Φ *oqimi* – bu S sirtning barcha $\Delta\vec{S}_i$ yuzachalari orqali o‘tuvchi maydonning oqimlari yig‘indisiga teng. $\Delta S_i \to 0$ da bu yig‘indi ushbu $\Phi = \sum_i \vec{a}(\vec{r}_i)\Delta\vec{S}_i = \int_S \vec{a}(\vec{r})d\vec{S} = \int_S \vec{a}(\vec{r})\cdot\vec{n}dS$ sirt bo‘yicha integralga qaylanadi, bu yerda $\vec{r}_i$ - $\Delta\vec{S}_i$ yuzachaning o‘rta nuqtasi.

Yopiq S sirt orqali o‘tuvchi $\vec{a}(\vec{r})$ *vektor maydonning* Φ_S *oqimi* – bu S sirt bilan chegaralangan chekli V yopiq hajmning differensial kichik ΔV_m hajmlari sirtlari orqali o‘tuvchi $\Delta\Phi_m$ oqimlar yig‘indisi bo‘lib, uni quyidagicha yozish mumkin: $\Phi_S = \sum_m \Delta\Phi_m$. Ushbu yig‘indi integral yig‘indiga aylanishi (va uning uchun $m\to\infty$ da limit mavjud bo‘lishi) uchun ΔV_m hajmga mos $\Delta\Phi_m$ oqimlar proporsional bo‘lishi zarur.

$\vec{r}_m$ nuqtada $\vec{a}(\vec{r})$ *vektor maydonning divergensiyasi* – bu skalyar bo‘lib, u quyidagiga teng: $\mathrm{div}\vec{a}(\vec{r}_m) = \dfrac{\Delta\Phi_m}{\Delta V_m}$, bu yerda $\vec{r}_m$ - ΔV_m hajmning o‘rta nuqtasi. Bu yerdan kelib chiqadiki, m bo‘yicha yig‘indining $\Delta V_m \to 0$ dagi limiti V hajm bo‘yicha olingan integralga aylanadi, ya’ni

$$\Phi_S = \sum_m \Delta\Phi_m = \sum_m \operatorname{div}\vec{a}(\vec{r}_m)\Delta V_m \Rightarrow \int_V \operatorname{div}\vec{a}(\vec{r})\,dV .$$

Bu oqimni V hajmni chegaralovchi S sirt bo‘yicha $\Phi_S = \int_S \vec{a}(\vec{r})d\vec{S} = \int_S \vec{a}(\vec{r})\cdot\vec{n}dS$ integral ko‘rinishida ifodalasak, quyidagi Gauss teoremasiga (Ostrogradskiy-Gauss teoremasiga) kelamiz:

$$\int_S \vec{a}(\vec{r})d\vec{S} = \int_S \vec{a}(\vec{r})\cdot\vec{n}dS = \int_V \operatorname{div}\vec{a}(\vec{r})dV ,$$

bunda $dV=dx_1dx_2dx_3$. Bu munosabat *Ostrogradskiy-Gauss formulasi* deb atalib, u yopiq sirt bo‘ylab olingan sirt integralini shu sirt bilan chegaralangan soha bo‘yicha olingan uch karrali integral bilan bog‘laydi. Bu formula ko‘rsatadiki, yopiq sirt orqali o‘tayotgan vektor maydon oqimi shu sirt bilan chegaralangan hajm bo‘ylab maydon divergensiyasidan olingan uch karrali integralga teng.

Tutash muhit mexanikasida, xususan, suyuqlik va gazlar mexanikasida skalyar maydon uchun Ostrogradskiy-Gauss formulasidan kelib chiqadigan ushbu $\int_S \varphi\vec{n}dS = \int_V \operatorname{grad}\varphi dV$, formula keng qo‘llaniladi, bu yerda φ - biror skalyar funksiya. Bulardan tashqari yana ikkita formulani kiritamiz: $\int_S \varphi\vec{k}\cdot\vec{n}dS = \int_V \operatorname{grad}\varphi\cdot\vec{k}dV$, bunda $\vec{k}$ - birlik vektor ; $-\int_S \vec{u}\wedge\vec{n}dS = \int_V \operatorname{rot}\vec{u}dV$.

Vektor maydon divergensiyasi va uning komponentalari xususiy hosilalari orasidagi bog‘lanish quyidagicha: $\operatorname{div}\vec{a}(\vec{r}) \equiv \nabla\vec{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \equiv \sum_{\alpha=1}^{3}\frac{\partial a_\alpha}{\partial x_\alpha}$.

Bundan kelib chiqadiki, ixtiyoriy vektor maydon divergensiyasi biror skalyar maydonni, ya’ni aynan o‘zining uzoqlashishini beradi.

Agar $\operatorname{div}\vec{a}(\vec{r}) \equiv 0$ bo‘lsa, u holda bunday maydon *solenoidal* deb ataladi.

Uzviylik tenglamasi – bu suyuqlik hajmining har bir nuqtasida ρ zichlik o‘zgarishi tezligining shu nuqtadagi suyuqlik ρ zichligi va $\vec{v}(\vec{r})$ tezligi ko‘pasytmasining divergensiyasi orasidagi bog‘lanishni ifodalovchi xususiy hosilali differensial tenglama: $\frac{\partial\rho}{\partial t} = -\operatorname{div}(\rho\vec{v})$. Bu uzviylik tenglamasi Gauss teoremasidan foydalanib, massaning saqlanish qonunidan chiqariladi.

Elektr zaryadi saqlanishi qonunining differensial shakli – bu har bir nuqtada ρ elektr zaryadi zichligi o‘zgarishi tezligining shu nuqtadagi $\vec{j}(\vec{r})$ elektr toki zichligi divergensiyasi orasidagi bog‘lanishni ifodalovchi xususiy hosilali differensial tenglama: $\dfrac{\partial\rho}{\partial t} = -\mathrm{div}\,\vec{j}$. Bu uzviylik tenglamasi Gauss teoremasidan foydalanib, zaryadning saqlanish qonunidan chiqariladi.

Issiqlik o‘tkazuvchanlik tenglamasi – bu issiqlik uzatilishi jarayonida issiqlik energiyasi saqlanishi qonunidan va Gauss teoremasidan foydalanib chiqarilgan temperatura o‘zgarishining differensial tenglamasi. Bir jinsli muhitda bu tenglama $\dfrac{\partial T(\vec{r})}{\partial t} = a\,\mathrm{div}\,\mathrm{grad}\,T(\vec{r})$ ko‘rinishda yoziladi, bu yerda a – issiqlik o‘tkazuvchanlik koeffisiyenti.

Namunaviy masalalar va ularning yechimlari

2.3.1. Dekart koordinatalari sistemasida berilgan ushbu $a_1 = 20\sin(x/\pi)$ yagona tashkil etuvchiga ega vektor maydon divergensiyasini toping.

Yechish. Divergensiya formulasiga ko‘ra

$$\mathrm{div}\,\vec{a} \equiv \nabla\vec{a} = \frac{\partial a_1}{\partial x} = \frac{\partial}{\partial x}\left(20\sin\frac{x}{\pi}\right) = \frac{20}{\pi}\cos\frac{x}{\pi}.$$

2.3.2. Dekart koordinatalari sistemasida berilgan ushbu $\vec{a} = \cos(ay)\vec{e}_x + \sin(ax)\vec{e}_y + tg(az)\vec{e}_z$ va $\vec{b} = 6x\vec{e}_x + 5z\vec{e}_y + 10y\vec{e}_z$ vektor maydonlarning divergensiyasini toping.

Yechish. Divergensiya formulasiga ko‘ra

$$\mathrm{div}\,\vec{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} =$$

$$= \frac{\partial}{\partial x}\big(\cos(ay)\big) + \frac{\partial}{\partial y}\big(\sin(ax)\big) + \frac{\partial}{\partial z}\big(tg(az)\big) = \frac{a}{\cos^2(az)};$$

$$\mathrm{div}\,\vec{b} = \frac{\partial b_1}{\partial x} + \frac{\partial b_2}{\partial y} + \frac{\partial b_3}{\partial z} = \frac{\partial}{\partial x}(6x) + \frac{\partial}{\partial y}(5z) + \frac{\partial}{\partial z}(10y) = 6.$$

2.3.3. Ushbu $\vec{a} = x\vec{e}_x + y^2\vec{e}_y + z^3\vec{e}_z$ vektor maydonning M(-2,4,5) nuqtadagi divergensiyasini toping.

Yechish. $\mathrm{div}\,\vec{a} = \dfrac{\partial a_1}{\partial x} + \dfrac{\partial a_2}{\partial y} + \dfrac{\partial a_3}{\partial z} = \left(1 + 2y + 3z^2\right)_M = 1 + 8 + 75 = 84.$

2.3.4. Ushbu $\vec{a} = f(r)\vec{r}$, $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$, $r = |\vec{r}|$ sferik vektor maydonning divergensiyasini toping. $\vec{a}$ maydonning solenoidallik holatidan $f(r)$ funksiyaning ko‘rinishini toping.

Yechish. Dastlab berilgan maydonni koordinatalari orqali yozib olamiz: $\vec{a} = f(r)\vec{r} = \{f(r)x, f(r)y, f(r)z\}$. Divergensiyaning ta’rifiga ko‘ra

$$\operatorname{div}\vec{a} = \frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y) + \frac{\partial}{\partial z}(f(r)z) =$$

$$= f'(r)\frac{x^2}{r} + f(r) + f'(r)\frac{y^2}{r} + f(r) + f'(r)\frac{z^2}{r} + f(r) = f'(r)r + 3f(r).$$

Solenoidallik sharti $\operatorname{div}\vec{a} = 0$ ga ko‘ra $f'(r)r + 3f(r) = 0$. O‘zgaruvchilarni ajratish orqali quyidagi natijaga kelamiz: $\frac{df}{f} = -3\frac{dr}{r}$. Bu integrallasak, $\ln|f(r)| = -3\ln|r| + \ln C$, bu yerdan esa $f(r) = \frac{C}{r^3}$, bu yerda C – ixtiyoriy o‘zgarmas. Shunday qilib, agar $f(r) = \frac{C}{r^3}$ bo‘lsagina, u holda $\vec{a} = f(r)\vec{r}$ sferik vektor maydonning divergensiyasi nolga teng. Bu maydon koordinata boshiga ega bo‘lmagan ixtiyoriy sohada solenoidal bo‘ladi.

2.3.5. Ostrogradskiy-Gauss formulasidan foydalanib, ushbu V jismning to‘la S sirti orqali tashqi normal yo‘nalishida $\vec{a}$ vektor maydonning oqimini toping: $\frac{H}{R^2}\left(x^2 + y^2\right) \le z \le H$ - jism hajmi tenglamasi; $\vec{a} = x^3\vec{e}_x + y^3\vec{e}_y + R^2 z\vec{e}_z$ - vektor.

Yechish. Divergensiya formulasiga ko‘ra $\operatorname{div}\vec{a} = 3(x^2 + y^2) + R^2$. Shuning uchun $\int\limits_{(S)} (\vec{a}, \vec{n}_0)dS = \int\limits_{(V)} \left[3(x^2 + y^2) + R^2\right]dV$. Bu uch karrali integralni hisoblash uchun silindrik koordinatalari sistemasiga o‘tiladi. Jism sirtining tenglamasi $z = \frac{Hr^2}{R^2}$, shularga ko‘ra

$$\int\limits_{(V)} \left[3(x^2 + y^2) + R^2\right]dV = \int\limits_0^{2\pi} d\varphi \int\limits_0^R (3r^2 + R^2)rdr \int\limits_{Hr^2/R^2}^{H} dz =$$

$$= 2\pi\int\limits_0^R (3r^2 + R^2)(H - Hr^2/R^2)rdr = \frac{2\pi H}{R^2}\int\limits_0^R (R^4 + 2r^2R^2 - 3r^4)rdr = \pi HR^4.$$

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2.3.6. Quyidagilarni toping: 1) $div(x^3 - 2xy^2; 2x^4 z + y^2; 2z^3 x^2 y\exp(x))$; 2) $div(\vec{r})$; 3) $div(xy^2\sin(zx); \cos(xy^2 z)\ln(z); \ln(yz)\exp(x - 2y))$; 4) $div(\vec{\rho})$;

5) $div(x\exp(x^2)+\cos y\cdot\sin x;\sin(zy);\cos(zy)\exp(zx))$; 6) $div(grad(1/r))$;

7) $div(x^4+5xy^2z-\exp(x);\ln(y)z^2-7y;\sin(25xyz^2))$; 8) $div(grad(\ln(\rho)))$;

9) $div(f(r)\vec{r})$; 10) $div(f(\rho)\vec{\rho})$; 11) $div[\vec{\omega},\vec{r}]$, bu yerda $\vec{\omega}$ - o'zgarmas vektor; 12) $div[\vec{\alpha},[\vec{\omega},\vec{r}]]$, bu yerda $\vec{\alpha}$ va $\vec{\omega}$ - o'zgarmas vektorlar; 13) $div(\vec{\alpha},(\vec{\omega},\vec{r}))$, bu yerda $\vec{\alpha}$ va $\vec{\omega}$ - o'zgarmas vektorlar; 14) $div(grad(f(\vec{r})g(\vec{r})))$; 15) $div(f(\vec{r})\vec{A}(\vec{r}))$; 16) $div(f(r)[\vec{\alpha},\vec{r}])$, bu yerda $\vec{\alpha}$ - o'zgarmas vektor; 17) $div(f(\rho)[\vec{\alpha},\vec{\rho}])$, bu yerda $\vec{\alpha}$ - o'zgarmas vektor; 18) $\operatorname{div}[\vec{a}[\vec{b}\vec{r}]]$; 19) $\operatorname{div} z\vec{r}$; 20) $\operatorname{div}(r[\vec{a}\vec{r}])$; 21) $\operatorname{div}\dfrac{[\vec{a}\vec{r}]}{r}$;

22) $\operatorname{div}\dfrac{\vec{r}-\vec{R}_0}{|\vec{r}-\vec{R}_0|}$, $\vec{R}_0,\vec{d}=const, \vec{r}(x,y,z), r=|\vec{r}|$; 23) $\operatorname{div}\left(\dfrac{(\vec{d}\vec{r})\vec{r}}{r^5}\right)$.

2.3.7. S sirt orqali o'tuvchi $\vec{A}(x+1;2z-y+2;3x-z+1)$ maydon oqimini toping, bunda S sirt quyidagicha berilgan: 1) S - birlik kvadrat bo'lib, xOy tekislikda joylashgan (kvadratning tomonlari x va y o'qlarga parallel); $\vec{n}(0,0,1)$ - musbat normal; 2) S - xOy tekislikda joylashgan, markazi koordinata boshida bo'lgan, radiusi R ga teng aylana; $\vec{n}(0,0,1)$ - musbat normal. 3) Markazi koordinata boshida bo'lgan r radiusli sfera sirti orqali o'tuvchi $\vec{a}(\vec{r})(x-z,y+2x-z,x+y)$ maydon oqimini toping.

2.3.8. Qirralari *x*, *y*, *z*, koordinat o'qlariga parallel bo'lgan birlik kub va $\vec{A}$ maydon uchun Ostrogratskiy-Gauss teoremasini tekshiring, bunda $\vec{A}$: 1) (*x*-*y*; *z*+*y*-*x*; 2*z*); 2) (2*x*-*z*; *y*+*z*-*x*; 2*x*+*y*-*z*).

2.3.9. Qirralari *x*, *y*, *z*, koordinat o'qlariga parallel bo'lgan birlik kub va $\vec{A}$ maydon uchun Ostrogratskiy-Gauss teoremasini tekshiring, bunda $\vec{A}$:

1) $(x^2-2yz+1;2xy+y^2+2;2yz+1)$; 2) $(x^2+y^2-z;xyz+5;xy^2+1)$;

3) $(x-y-z^2-1;2zy+y^2-4;2y^2z^2+x+2)$; 4) $(2xy-3;2z^2+1;x-zy^2+1)$.

2.3.10. Markazi koordinata boshida bo'lgan R radiusli sfera va $\vec{A}$ maydon uchun Ostrogratskiy-Gauss teoremasini tekshiring, bunda $\vec{A}$: 1) $\vec{A}=\vec{r}$; 2) $\vec{A}=r\vec{r}$; 3) $\vec{A}=r^2\vec{r}$; 4) $\vec{A}=r^3\vec{r}$; 5) $\vec{A}=\vec{r}/r$.

2.3.11. Radiusi R va balandili h bo'lgan silindr (silindrning o'qi Oz o'q bilan mos va uning quyi asosi xOy tekislikda yotadi) va $\vec{A}$ maydon uchun Ostrogratskiy-

Gauss teoremasini tekshiring, bunda $\vec{A}$: 1) $\vec{A}=\vec{\rho}$; 2) $\vec{A}=\rho\vec{\rho}$; 3) $\vec{A}=\rho^2\vec{\rho}$; 4) $\vec{A}=\rho^3\vec{\rho}$; 5) $\vec{A}=\vec{\rho}/\rho$.

2.3.12. Radiusi *R* va balandili *h* bo‘lgan silindr (silindrning o‘qi *Oz* o‘q bilan mos va uning quyi asosi *xOy* tekislikda yotadi) va $\vec{A}$ maydon uchun Ostrogratskiy-Gauss teoremasini tekshiring, bunda $\vec{A}$: 1) $\vec{A}=\vec{r}$; 2) $\vec{A}=\rho\vec{r}$; 3) $\vec{A}=\rho^2\vec{r}$; 4) $\vec{A}=\rho^3\vec{r}$; 5) $\vec{A}=\vec{r}/\rho$.

2.3.13. Markazi koordinata boshida bo‘lgan *R* radiusli sfera va $\vec{A}$ maydon uchun Ostrogratskiy-Gauss teoremasini tekshiring, bunda $\vec{A}$: 1) $\vec{A}=\vec{r}/r^3$; 2) $\vec{A}=f(r)\vec{r}$.

2.3.14. Radiusi *R* va balandili *h* bo‘lgan silindr (silindrning o‘qi *Oz* o‘q bilan mos va uning quyi asosi *xOy* tekislikda yotadi) va $\vec{A}$ maydon uchun Ostrogratskiy-Gauss teoremasini tekshiring, bunda $\vec{A}$: 1) $\vec{A}=\vec{\rho}/\rho^2$; 2) $\vec{A}=f(\rho)\vec{\rho}$.

2.3.15. Radiusi *R* va balandili *h* bo‘lgan silindr (silindrning o‘qi *Oz* o‘q bilan mos va uning quyi asosi *xOy* tekislikda yotadi) va $\vec{A}$ maydon uchun Ostrogratskiy-Gauss teoremasini tekshiring, bunda $\vec{A}$: 1) $\vec{A}=\vec{r}/\rho^2$; 2) $\vec{A}=f(\rho)\vec{r}$.

2.4. Vektor maydor rotori va Stoks teoremasi

Yopiq *L* kontur bo‘ylab $\vec{a}(\vec{r})$ *vektor maydon A_L sirkulyatsiyasi* – bu skalyar miqdor bo‘lib, *L* yopiq konturni bo‘luvchi $\Delta\vec{r}_n$ uchastkalar vektorlarining shu uchastkalar o‘rta nuqtasi tezligi $\vec{a}(\vec{r})$ vektoriga skalyar ko‘paytmalari yig‘indisiga teng $A_L=\sum\limits_n \vec{a}(\vec{r}_n)\Delta\vec{r}_n$. Bu yerda yig‘indidan $\Delta\vec{r}_n \to 0$ da limitga o‘tsak, sirkulyatsiyaning *L* kontur bo‘ylab integral ifodasiga kelamiz: $A_L=\int\limits_L \vec{a}(\vec{r})d\vec{r}$. Konturni aylanib o‘tish yo‘nalishi sirkulyatsiyaning ishorasini o‘zgartiradi. *L* kontur bo‘ylab sirkulyatsiyani *L* konturli *S* yuzani L_k konturli $\Delta\vec{S}_k$ kichik yuzachalarga bo‘lishdan olingan sirkulyatsiyalar yig‘indisi ko‘rinishida ifodalash mumkin, bunda *L* va L_k konkurlar bo‘ylab yo‘nalish o‘ng vint qoidasiga ko‘ra olinadi: $A_L=\sum\limits_k \Delta A_{L_k}$. Ushbu yig‘indini integral ifodaga almashtirish mumkin bo‘lishi uchun yig‘indi olinayotgan miqdorlar ΔS_k ga proporsional bo‘lisi zarur. Agar $\vec{a}(\vec{r})$ *maydonning rotori* deb ataluvchi $\operatorname{rot}\vec{a}(\vec{r})$ vektor maydon $\Delta A_{L_k}=\operatorname{rot}\vec{a}(\vec{r}_k)\cdot\Delta\vec{S}_k$ mavjud bo‘lsa buning imkoni bo‘ladi. Bu yerda yig‘indidan $\Delta S_k \to 0$ da limitga o‘tsak, sirkulyatsiyaning *S* sirt bo‘ylab integral ifodasiga kelamiz: $A_L = \int_S \operatorname{rot} \vec{a}(\vec{r})\, d\vec{S}$.

Shunday qilib, sirkulyatsiyani kontur bo‘ylab integral ko‘rinishida hamda shu konturga tortilgan sirt orqali o‘tuvchi vektor maydon rotori oqimi ko‘rinishida ifodalash mumkin ekan. Ana shu natija *Stoks teoremasi* deb ataladi:

$$A_L = \int_L \vec{a}(\vec{r})d\vec{r} = \int_S \operatorname{rot} \vec{a}(\vec{r})d\vec{S}.$$

Sirt uchun normalning musbat yo‘nalishi kontur bo‘ylab o‘ng vint qoidasiga asoslangan yo‘nalish bilan bog‘liq.

Vektor maydon rotori maydon komponentalaridan olingan xususiy hosilalar orqali ifodalanadi:

$$\operatorname{rot} \vec{a}(\vec{r}) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix}.$$

Tenglikning o‘ng tarafidagi determinantni birinch satr bo‘yicha formal tarzda yoyib yozamiz. Natijada rotorning birlik ortalardagi chiziqli kombinatsiyali yoyilmasi ifodasi hosil bo‘ladi:

$$\operatorname{rot} \vec{a}(\vec{r}) = = \vec{e}_1\left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) + \vec{e}_2\left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) + \vec{e}_3\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)$$

Agar $\operatorname{rot} \vec{a}(\vec{r}) = 0$ bo‘lsa, u holda bunday maydon *uyurmasiz* deyiladi.

Grin formulasi (*x*,*y*) tekislikda berilgan ikki o'lchovli maydon $\vec{a}(x, y, z)(P(x, y), Q(x, y), 0)$ va (*x*,*y*) tekislikda berilgan *L* yopiq kontur holi uchun tuzilgan Stoks teoremasidan kelib chiqadi. U holda $A_L = \int_L \vec{a}(\vec{r})d\vec{r} = \int_L Pdx + Qdy$, $\operatorname{rot} \vec{a}(\vec{r})(0,0,(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}))$ va $d\vec{S}(0,0,dxdy)$. Natijada Stoks teoremasidan quyidagi Grin formulasi kelib chiqadi: $A_L = \int_L Pdx + Qdy = \int_S (\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y})dxdy$.

Namunaviy masalalar va ularning yechimlari

2.4.1. Ushbu $\vec{a} = z^2\vec{e}_x + x^2\vec{e}_y + y^2\vec{e}_z$ vektor maydonning *M*(1,2,3) nuqtadagi rotorini toping.

Yechish. Rotorning ta’rifiga ko‘ra

$$\operatorname{rot}\vec{a}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ z^2 & x^2 & y^2\end{vmatrix}=\Big[(2y-0)\vec{e}_x+(2z-0)\vec{e}_y+(2x-0)\vec{e}_z\Big]_M=4\vec{e}_x+6\vec{e}_y+2\vec{e}_z.$$

2.4.2. Ushbu $\vec{a}=f(r)\vec{r},\ \vec{r}=x\vec{e}_x+y\vec{e}_y+z\vec{e}_z,\ r=|\vec{r}|$ sferik vektor maydonning rotorini toping.

Yechish. Dastlab berilgan maydonni koordinatalari orqali yozib olamiz: $\vec{a}=f(r)\vec{r}=\{f(r)x,\,f(r)y,\,f(r)z\}$. Rotorning ta’rifiga ko‘ra

$$\operatorname{rot}\vec{a}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f(r)x & f(r)y & f(r)z\end{vmatrix}=\left[\left(\frac{\partial}{\partial y}f(r)z-\frac{\partial}{\partial z}f(r)y\right)\vec{e}_x+\left(\frac{\partial}{\partial z}f(r)x-\frac{\partial}{\partial x}f(r)z\right)\vec{e}_y+\right.$$

$$\left.+\left(\frac{\partial}{\partial x}f(r)y-\frac{\partial}{\partial y}f(r)x\right)\vec{e}_z\right]=f'(r)\left[\left(\frac{yz}{r}-\frac{zy}{r}\right)\vec{e}_x+\left(\frac{zx}{r}-\frac{xz}{r}\right)\vec{e}_y+\left(\frac{xy}{r}-\frac{yx}{r}\right)\vec{e}_z\right]=0.$$

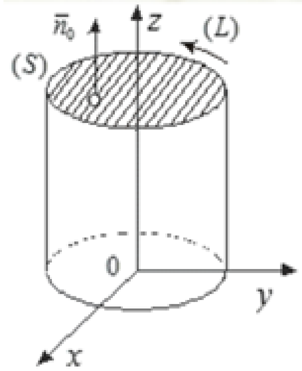
Bu yerdan ixtiyoriy sferik vektor maydonning rotori nolga teng, ya’ni sferik vektor maydon uyurmasiz ekan.

2.4.3. Ushbu (L): $x^2+y^2=4,\ z=3$ kontur bo‘ylab $\vec{a}=y\vec{e}_x+x^2\vec{e}_y-z\vec{e}_z$ vektor maydon sirkulyatsiyasini *a*) to‘g‘ridan-to‘g‘ri va *b*) Stoks teoremasidan foydalanib hisoblang.

Yechish. *a*) (L) – kontur radiusi R = 2 bo‘lgan aylana bo‘lib, z = 3 tekislikda yotadi. Rasmda ko‘rsatilgandan yo‘nalishni tanlaylik. Berilgan chiziqning parametric tenglamasi (L): $x = 2\cos t$, $y = 2\sin t$, $z = 3$, $0 \le t \le 2\pi$, bunga ko‘ra dx = -2sintd$t$, dy = 2costd$t$, dz = 0. Berilgan vektorning sirkulyatsiyasi quyidagicha:

$$\Gamma=\int_0^{2\pi}\Big[2\sin t\cdot(-2\sin t)+4\cos^2 t\cdot(2\cos t)-3\cdot 0\Big]dt=-4\pi.$$

b) Sirkulyatsiyani Stiks teoremasi bo‘yicha hisoblash uchun (L) konturga tortilgan biror (S) sirtni tanlaylik. Tabiiyki, bu konturi (L) aylanadan iborat (S) sirt doira. Konturning tanlangan yo‘nalishiga ko‘ra doiraga o‘tkazilgan $\vec{n}_0$ normalni quyidagicha olish zarur: $\vec{k}:\vec{n}_0=\vec{k}$. Hisoblashlarni bajaramiz:



$$\operatorname{rot}\vec{a}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y & x^2 & -z\end{vmatrix}=(2x-1)\vec{e}_z.$$

Stoks teoremasi bo‘yicha:

$$\Gamma=\int\limits_{(S)}(\operatorname{rot}\vec{a},\vec{n}_0)dS=\int\limits_{(S)}(2x-1)dS=\int\limits_0^{2\pi}d\varphi\int\limits_0^2(2\rho\cos\varphi-1)\rho d\rho=-2\pi\left.\frac{\rho^2}{2}\right|_0^2=-4\pi.$$

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2.4.4. Quyidagilarni toping: 1) $rot(xz\exp(y);\exp(x^2)z\cos(y);\ln(z)\exp(xy))$; 2) $rot(xy^2\sin(z);2xyz;\ln(xyz))$; 3) $rot(xy^2\sin(zx);\exp(xyz)\ln(xyz);\cos(xy^2)z^3)$; 4) $rot(yz\exp(x^2);\cos(xy^2z)\exp(xyz);xy^2\sin(z)\ln(y))$; 5) $rot(\vec{r})$; 6) $rot(\vec{\rho})$; 7) $rot(f(r)\vec{r})$; 8) $rot(f(\rho)\vec{\rho})$; 9) $rot[\vec{\omega},\vec{r}]$, bu yerda $\vec{\omega}$ - o‘zgarmas vektor; 10) $rot[\vec{\alpha},[\vec{\omega},\vec{r}]]$, bu yerda $\vec{\alpha}$ va $\vec{\omega}$ - o‘zgarmas vektorlar; 11) $rot(\vec{\alpha},(\vec{\omega},\vec{r}))$, bu yerda $\vec{\alpha}$ va $\vec{\omega}$ - o‘zgarmas vektorlar; 12) $rot(grad(f(\vec{r})))$; 13) $div(rot(\vec{A}(\vec{r})))$; 14) $rot(f(\vec{r})\vec{A}(\vec{r}))$; 15) $\operatorname{rot} y\vec{r}$; 15) $\operatorname{rot}[\vec{a}[\vec{b}\vec{r}]]$; 16) $\operatorname{rot}(\vec{d}\sin(\vec{k}\vec{r}))$; 17) $\operatorname{rot}(r[\vec{a}\vec{r}])$; 18) $\operatorname{rot}\dfrac{[\vec{a}\vec{r}]}{r}$; 19) $\operatorname{rot}\dfrac{\vec{d}}{r}$; 20) $\operatorname{rot}\dfrac{[\vec{d}\vec{r}]}{r^3}$; 21) $\operatorname{rot}\left(\dfrac{(\vec{d}\vec{r})\vec{r}}{r^5}\right)$.

2.4.5. Hisoblang: $rot[\vec{a},\vec{b}]$, $div[\vec{a},\vec{b}]$, $rot(c\vec{a})$, $div(c\vec{a})$, $grad(\vec{a},\vec{b})$, bu yerda $\vec{a}$, $\vec{b}$ va c quyidagilarga teng: 1) $\vec{a}(y;z;x)$, $\vec{b}(2x;y-z;0)$, $c=1/r$; 2) $\vec{a}(2x;z;y)$, $\vec{b}(b_1;b_2;b_3)$, $c=r^2$; 3) $\vec{a}(y;-z;x)$, $\vec{b}(y;x;x)$, $c=\ln(r)$; 4) $\vec{a}(x^2;y^2;z^2)$, $\vec{b}(z;2x;y)$, $c=r^{1/2}$; 5) $\vec{a}(a_2;a_2;a_3)$, $\vec{b}(y;-y;zx)$, $c=r$.

2.4.6. $\vec{\omega}(\omega_1;\omega_2;\omega_3)=const$ uchun quyidagi ifodalarni hisoblang: 1) $rot[\rho\vec{\omega},\vec{r}]$; 2) $rot[\rho\vec{\omega},\vec{\rho}]$; 3) $rot[r\vec{\omega},\vec{\rho}]$; 4) $rot[r\vec{\omega},\vec{r}]$; 5) $rot[\vec{\omega},\vec{\rho}]$; 6) $rot[\vec{\omega},\vec{r}]$; 7) $rot[\vec{\rho},\vec{r}]$.

2.4.7. Markazi koordinata boshida bo‘lgan va (y,z) tekislikda yotgan birlik radiusli aylana bo‘yicha $\vec{a}(\vec{r})(x-z,y+2x-z,x+y)$ maydonning sirkulyatsiyasini toping.

2.4.8. xOy, xOz, yOz tekisliklarda yotgan birlik kvadratlar va $\vec{A}$ maydonning quyidagi ifodalari uchun Stoks teoremasini tekshiring:

1) $(x+y+z;\ x-z;\ z+y)$; 2) $(x-y;\ 2z-x;\ x+y+2z)$.

2.4.9. Birlik kub qirralari va $\vec{A}$ maydonning quyidagi ifodalari uchun Stoks teoremasini tekshiring:

1) $(xy^2+1;\ 2zy+1;\ xyz+2)$; 2) $(y+z^2x-3;\ z^2+x+y+4;\ zx-3)$;

3) $(xyz+x+1;\ 2xy-z-2;\ z+1)$; 4) $(2z-1;\ 2y-3x+z-1;\ xyz+1)$.

2.4.10. xOy tekislikda yotuvchi, markazi O nuqtada bo‘lgan R radiusi aylana va $\vec{A}=[\vec{b}\vec{r}]$ maydon uchun Stoks teoremasini tekshiring, bunda $\vec{b}$ - o‘zgarmas vektor.

2.4.11. Ushbu (L) kontur bo‘ylab $\vec{a}$ vektor maydon sirkulyatsiyasini *a*) to‘g‘ridan-to‘g‘ri va *b*) Stoks teoremasidan foydalanib hisoblang:

a) (L): $x^2+y^2=9,\ x+y+z=1$; $\vec{a}=xy\vec{e}_x+yz\vec{e}_y+zx\vec{e}_z$,

b) (L): $x^2+y^2=4,\ z=0$; $\vec{a}=z\vec{e}_x+x\vec{e}_y+y\vec{e}_z$,

c) (L): $x^2+y^2+z^2=4,\ x^2+y^2=z^2,\ z\ge 0$; $\vec{a}=y\vec{e}_x-x\vec{e}_y+z\vec{e}_z$,

d) (L): $x+y+2z=2,\ x\ge 0,\ y\ge 0,\ z\ge 0$; $\vec{a}=2xz\vec{e}_x-y\vec{e}_y+z\vec{e}_z$.

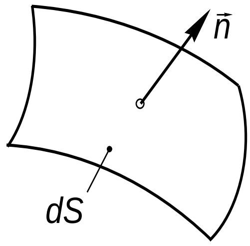
2.5. Maydon nazariyasining integral munosabatlari. Vektor analizning nabla operatori usulisiz yechiladigan aralash masalalari. Nabla operatori usulidan foydalanib, masalalar yechish

Maydon nazariyasining integral munosabatlari. Quyida skalyar va vektor maydon nazariyasining keyinchalik ishlatiladigan integral munosabatlari bilan qisqacha tanishamiz.

Faraz qilaylik, V hajm S sirt bilan qoplangan, dS – shu S sirt elementi, $\vec{n}$ - sirt elementiga tashqi normal yo‘nalishidagi birlik vektor bo‘lsin (2.5.1-rasm).

1) Vektor maydon oqimi. *Vektor maydon (masalan, $\vec{u}$) oqimi* deb quyidagi sirt integraliga aytiladi:

$$\iint\limits_S \vec{u}\cdot\vec{n}\,dS\,.$$



2.5.1-rasm. Sirt elementi va normal birlik vektor.

Agar rotor ($\mathrm{rot}\,\vec{u}$)ning vektor maydoni qaralsa, u holda bu maydonning oqimi quyidagicha ifodalanadi:

$$\iint_S \left(\mathrm{rot}\,\vec{u}\right)\cdot\vec{n}dS\,.$$

Agar S yopiq sirt bo‘lsa, u holda (1.10) ifoda nolga teng bo‘ladi.

2) Vektor maydon sirkulyatsiyasi Faraz qilaylik, biror $\vec{u}$ miqdorning vektor maydoni qaralayotgan bo‘lsin. *L* kontur bo‘ylab $\vec{u}$ *vektor sirkulya-tsiyasi* deb quyidagi egri chiziqli integralga aytiladi:

$$\Gamma = \int_L \vec{u}\cdot d\vec{l}\,.$$

Ba’zida bu integral vektor maydonning *L* kontur bo‘ylab «ishi» deb talqin qilinadi. Agar yopiq yo‘l (kontur) bo‘ylab vektor sirkulyatsiyasi nolga teng bo‘lsa, u holda bu maydon *potensial maydon* deb ataladi.

3) Stoks formulasi. Ushbu

$$\oint_L \vec{u}\cdot d\vec{l} = \iint_S \left(\mathrm{rot}\,\vec{u}\right)\cdot\vec{n}dS$$

Stoks formulasi yopiq fazoviy egri chiziq bo‘yicha olingan egri chiziqli integralni shu egri chiziqqa tortilgan sirt bo‘yicha olin-gan sirt integraliga almashtirish imkonini beradi, boshqacha aytganda, kontur bo‘ylab vektor sirkulyatsiyasi shu kontur bi-lan chegaralangan sirt orqali o‘tayotgan uyurma oqimiga teng.

Agar *S* sirt yopiq bo‘lsa, u holda shu sirt orqali o‘tayotgan uyurma oqimi nolga teng bo‘ladi, ya’ni

$$\iint_S \left(\mathrm{rot}\,\vec{u}\right)\cdot\vec{n}dS = 0\,.$$

4) Gauss-Ostrogradskiy formulasi. Ushbu

$$\iint_S \vec{u}\cdot\vec{n}dS = \iiint_V \mathrm{div}\,\vec{u}dV\,,$$

bunda *dV=dxdydz*, munosabat *Gauss-Ostrogradskiy formulasi* deb atalib, u yopiq sirt bo‘ylab olingan sirt integralini shu sirt bilan chegaralangan soha bo‘yicha olingan uch karrali integral bilan bog‘laydi. Bu formula ko‘rsatadiki, yopiq sirt orqali o‘tayotgan vektor maydon oqimi shu sirt bilan chegaralangan hajm bo‘ylab maydon divergensiyasidan olingan uch karrali integralga teng.

Suyuqlik va gazlar mexanikasida skalyar maydon uchun Gauss-Ostrogradskiy formulasidan kelib chiqadigan ushbu

$$\iint_S \varphi \vec{n} dS = \iiint_V \text{grad}\, \varphi dV ,$$

formula keng qo‘llaniladi, bu yerda φ - biror skalyar funksiya.

Bulardan tashqari yana ikkita formulani kiritamiz:

$$\iint_S \varphi \vec{k} \cdot \vec{n} dS = \iiint_V \text{grad}\, \varphi \cdot \vec{k} dV ,$$

bunda $\vec{k}$ - birlik vektor ;

$$-\iint_S \vec{u} \wedge \vec{n} dS = \iiint_V rot \vec{u} dV .$$

Nabla operatori usulidan foydalanib, masalalar yechish. «Nabla» vektor-differensial operator uchta $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ komponentalarga ega va uni quyidagi ko‘rinishda ifodalash mumkin: $\nabla = \frac{\partial}{\partial x_1}\vec{e}_1 + \frac{\partial}{\partial x_2}\vec{e}_2 + \frac{\partial}{\partial x_3}\vec{e}_3$. Ifodani algebraik akslantirish vektorlar algebrasi qoidasiga ko‘ra amalga oshiriladi, faqat bunda «nabla» differensial operator va ta’rifiga ko‘ra differensiallanuvchi funksiyadan chapda turishi zarurligini unutmaslik kerak. Shuning uchun, $\nabla \cdot \vec{a}$ - bu $\vec{a}(\vec{r})$ maydonning divergensiyasi, $\vec{a} \cdot \nabla$ - bu skalyar differensial operator ekanligini esdan chiqarmang: $\vec{a} \cdot \nabla \equiv \sum_{\alpha=1}^{3} a_\alpha(\vec{r}) \cdot \frac{\partial}{\partial x_\alpha}$. Ma’lumki, $\nabla \varphi = \text{grad}\, \varphi$, $\nabla \cdot \vec{a} = \text{div} \vec{a}$, $\nabla \times \vec{a} = \text{rot} \vec{a}$.

Laplas operatori – bu quyidagi ikkinchi tartibli skalyar differensial operator: $\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. «Nabla» operatorning xossasini e’tiborga olib, ixtiyoriy maydon uchun quyidagi munosabatlarning o‘rinli ekanligiga ishonch hosil qilish mumkin: $\text{divrot} \vec{a} = (\nabla, [\nabla \vec{a}]) = 0$, $\text{rotgrad} \varphi = [\nabla, (\nabla \varphi)] = [\nabla, \nabla] \varphi = 0$, $\text{rotrot} \vec{a} = [\nabla [\nabla \vec{a}]] = \nabla(\nabla, \vec{a}) - (\nabla, \nabla)\vec{a} = \text{graddiv} \vec{a} - \Delta \vec{a}$. Bunda oxirgi had uchta $\Delta a_1, \Delta a_2, \Delta a_3$ komponentali vector ekanligini e’tiborga olish lozim.

Vektor analizda nabla operatori usuli yordamida ifodalarni akslantirish quyidagi sxema bo‘yicha amalga oshiriladi:

1) grad, div, rot va Δ operatsiyalar o‘rniga nabla operatoridan foydalanilgan quyidagi operatsiyalar kiritiladi:

$$\text{grad} f \equiv \nabla f ; \;\; \text{div} \vec{A} \equiv \nabla \vec{A}; \;\; \text{rot} \vec{A} \equiv [\nabla \vec{A}]; \; \Delta \equiv (\nabla, \nabla) .$$

2) Nabla operator differensial operator sifatida qaysi funksiyalarga ta’sir qilishini tahlil qilaylik. Ta’rifga ko‘ra bu funksiyalar nabla operatoridan o‘ngda

joylashishi kerak. Agar bunday funksiyalar ikkita bo‘lib, ular o‘zaro ko‘pytirilsa, unda nabla operatorni har biri “o‘z funksiyasi”ga ta’sir etuvchi ikkita operatorlar yig‘indisi ko‘rinishida ifodalash mumkin. Masalan,

$$\mathrm{rot}[\vec{A}\vec{B}] = [\nabla[\vec{A}\vec{B}]] = [\nabla^{(A)}[\vec{A}\vec{B}]] + [\nabla^{(B)}[\vec{A}\vec{B}]]\,.$$

3) Hosil qilingan ifodalarni akslantirish uchun vektorla algebrasi usullaridan foydalanamiz. Bunda nabla operatorning differensial xarakteriga e’tibor qaratmaymiz va uni odatdagi vektor deb qaraymiz. Qaralayotgan misol uchun:

$$[\nabla^{(A)}[\vec{A}\vec{B}]] = \vec{A}(\nabla^{(A)}\vec{B}) - \vec{B}(\nabla^{(A)}\vec{A})$$

$$[\nabla^{(B)}[\vec{A}\vec{B}]] = \vec{A}(\nabla^{(B)}\vec{B}) - \vec{B}(\nabla^{(B)}\vec{A})$$

4) Vektorlar algebrasining standart qoidalaridan foydalanib, hosil qilinga ifodani shunday akslantiramizki, nabla operator ta'siri bajarilayotgan funksiyadan chap tomonda tursin va u ta’sir qilmayotgan funksiyadan o‘ng tarafda tursin U holda:

$$\vec{A}(\nabla^{(A)}\vec{B}) - \vec{B}(\nabla^{(A)}\vec{A}) = (\vec{B}\nabla^{(A)})\vec{A} - \vec{B}(\nabla^{(A)}\vec{A})$$

$$\vec{A}(\nabla^{(B)}\vec{B}) - \vec{B}(\nabla^{(B)}\vec{A}) = \vec{A}(\nabla^{(B)}\vec{B}) - (\vec{A}\nabla^{(B)})\vec{B}$$

5) Nabla operatorlari differensial operator sifatida ta’sir qilayotgan holatlarni to‘g‘ri amal deb qabul qilib, bunday belgilashlardan nabla operatorlarni olib tashlaymiz va zarur bo‘lgan hollarda da nabla grad, div, rot va Δ belgilashlarni kiritamiz. Qaralayotgan misolimizda quyidagi yakuniy natijaga kelamiz:

$$\mathrm{rot}[\vec{A}\vec{B}] = (\vec{B}\nabla^{(A)})\vec{A} - \vec{B}(\nabla^{(A)}\vec{A}) + \vec{A}(\nabla^{(B)}\vec{B}) - (\vec{A}\nabla^{(B)})\vec{B} =$$

$$= (\vec{B}\nabla)\vec{A} - \vec{B}(\nabla\vec{A}) + \vec{A}(\nabla\vec{B}) - (\vec{A}\nabla)\vec{B} = \vec{A}\,\mathrm{div}\vec{B} - \vec{B}\,\mathrm{div}\vec{A} + (\vec{B}\nabla)\vec{A} - (\vec{A}\nabla)\vec{B}\,,$$

bu yerda

$$(\vec{A}\nabla)\vec{B} = (A_1\frac{\partial}{\partial x_1} + A_2\frac{\partial}{\partial x_2} + A_3\frac{\partial}{\partial x_3})(B_1\vec{e}_1 + B_2\vec{e}_2 + B_3\vec{e}_3) =$$

$$= (A_1\frac{\partial B_1}{\partial x_1} + A_2\frac{\partial B_1}{\partial x_2} + A_3\frac{\partial B_1}{\partial x_3})\vec{e}_1 + (A_1\frac{\partial B_2}{\partial x_1} + A_2\frac{\partial B_2}{\partial x_2} + A_3\frac{\partial B_2}{\partial x_3})\vec{e}_2 +$$

$$+ (A_1\frac{\partial B_3}{\partial x_1} + A_2\frac{\partial B_3}{\partial x_2} + A_3\frac{\partial B_3}{\partial x_3})\vec{e}_3$$

Bu tushunchalarni soddaroq tarzda tayyor formulalar ko‘rinishida keltirib o‘tamiz. Yana bir bor ta’kidlash joizki, **grad** operatori differensiallanuvchi skalyar maydonda aniqlangan va ularni vektor maydonga aylantiradi, xuddi shunday **div** va **rot** operatorlari differensiallanuvchi vektor maydonlarda aniqlangan bo‘lib, mos ravishda **div** operatori vektor maydonni skalyar maydonga, **rot** operatori esa vektor maydonni vektor maydonga aylantiradi. Buni nabla (∇) operatori bilan aytadigan bo‘lsak, $\varphi(x,y,z)$ skalyar maydon **A**=grad φ vektor maydonga, vektor maydon **A** esa φ = div **A** skalyar maydonga yoki **B** = rot **A** vektor maydonga aylanadi.

Endi vektor maydon operatorlari va nabla (∇) operatori xossalari bilan bog‘liq asosiy formulalarni foydalanishga qulay jadval shaklida keltirib o‘taylik:

grad $u = \nabla u$	div $\mathbf{A} = \nabla\cdot\mathbf{A}$	rot $\mathbf{A} = \nabla\times\mathbf{A}$
grad$(c) = 0$, c = const grad$(cu) = c$ grad u $\nabla(cu) = c\,\nabla u$	div $(c\mathbf{A}) = c$ div$\mathbf{A}$ $\nabla(c\mathbf{A}) = c\,\nabla\cdot\mathbf{A}$	rot $(c\mathbf{A}) = c$ rot$\mathbf{A}$ $\nabla\times(c\mathbf{A}) = c\,\nabla\times\mathbf{A}$
grad$(u+v)$=grad u + grad v $\nabla(u+v)= \nabla u + \nabla v$	div$(\mathbf{A}+\mathbf{B})$ = div$\mathbf{A}$ + div$\mathbf{B}$ $\nabla(\mathbf{A}+\mathbf{B}) = \nabla\mathbf{A} + \nabla\mathbf{B}$	rot$(\mathbf{A}+\mathbf{B})$ = rot$\mathbf{A}$ + rot$\mathbf{B}$ $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$
grad$(u\cdot v)=v\cdot$grad $u+u\cdot$grad v $\nabla(u\cdot v)= v\cdot\nabla u + u\cdot\nabla v$	div$(u\mathbf{A})$ = $u\cdot$div$\mathbf{A}+\mathbf{A}\cdot$grad$u$ $\nabla(u\mathbf{A}) = u\,\nabla\mathbf{A} + \mathbf{A}\,\nabla u$	rot$(u\mathbf{A}) = u$ rot$\mathbf{A}$ + grad$u\times\mathbf{A}$ $\nabla\times(u\mathbf{A}) = u\,\nabla\times\mathbf{A} + \nabla u\times\mathbf{A}$
grad$(u/v)=(v\cdot$grad$u$ – –$u\cdot$grad$v)/v^2$ $\nabla(u/v) = (v\cdot\nabla u - u\cdot\nabla v)/v^2$	div$(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot$rot$\mathbf{A}$– $\mathbf{A}\cdot$rot$\mathbf{B}$ $\nabla(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$	rot$(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}$– – $(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{A}div\mathbf{B}$ – $\mathbf{B}$div$\mathbf{A}$
grad $f(u) = f'(u)$ grad u		

Mashqlar

A) Vektor analizning nabla operatori usulisiz yechiladigan aralash masalalari.

2.5.1. Quyidagi f skalyar maydonlar uchun div grad f ni hisoblang:

1) $f = \sin(\vec{k}\vec{r})$, $\vec{k}$ - o‘zgarmas vektor; 2) $f = r^{-1}\sin(\vec{k}\vec{r})$, $\vec{k}$ - o‘zgarmas vektor; 3) $f = \sin(\vec{k}\vec{\rho})$, $\vec{k}$ - o‘zgarmas vektor; 4) $f = (\vec{k}\vec{r})^2$, $\vec{k}$ - o‘zgarmas vektor; 5) $f = (\vec{k}\vec{\rho})^2$, $\vec{k}$ - o‘zgarmas vektor; 6) $f = 1/r$; 7) $f = \ln(\rho)$; 8) $f = \exp(-\alpha r)$; 9) $f = \exp(-\alpha r^2)$; 10) $f = \exp(-\alpha\rho)$; 11) $f = \exp(-\alpha\rho^2)$.

2.5.2. Quyidagi $\vec{a}$ vektor maydonlar uchun rot rot $\vec{a}$ ni hisoblang:

1). $\vec{a}(x^2;\ xy+y^2;\ xz+z^2)$; 2) $\vec{a}(x^2+y^2;\ xz;\ yz)$;

3) $\vec{a}(z^2;\ xy;\ yz+z^2)$; 4) $\vec{a}(2xz;\ x^2+y^2;\ 2z^2)$.

2.5.3. Quyidagi f skalyar maydonlar uchun rot grad f ni hisoblang:

1) $f = \exp(-\alpha r)$; 2) $f = \ln(\rho)$; 3) $f = 1/r$.

2.5.4. Quyidagi $\vec{a}$ vektor maydonlar uchun div rot $\vec{a}$ ni hisoblang:

1) $\vec{a} = [\vec{\omega}, \vec{r}\,]$, $\vec{\omega}$ - o‘zgarmas vektor; 2) $\vec{a}(x^2 + y^2;\ xz;\ yz)$.

2.5.5. Vektorlar algebrasi usullaridan foydalanib, quyida berilgan ifodalarni soddalashtiring va hisoblang:

1) $rot[[\vec{a},\vec{\rho}],[\vec{b},\vec{\rho}]]$; 2) $rot[\vec{a},[\vec{\rho},\vec{b}\,]]$; 3) $div[[\vec{a},\vec{\rho}],[\vec{b},\vec{\rho}]]$; 4) $div[\vec{a},[\vec{\rho},\vec{b}\,]]$;

5) $rot[[\vec{a},\vec{r}\,],[\vec{b},\vec{r}\,]]$; 6) $rot[\vec{a},[\vec{r},\vec{b}\,]]$; 7) $div[[\vec{a},\vec{r}\,],[\vec{b},\vec{r}\,]]$; 8) $div[\vec{a},[\vec{r},\vec{b}\,]]$.

B) Nabla operatori usulidan foydalanib, masalalar yechish.

2.5.6. Nabla (∇) operatori o‘rniga grad, div, rot va Δ operatsiyalardan foydalanib, quyidagi ifodalarni yozing:

1) $(\nabla[\vec{A}\vec{B}])$; 2) $\nabla(\vec{A}\vec{B})$; 3) $[\nabla[\vec{A}\vec{B}]]$; 4) $(\nabla f\vec{A})$; 5) $[\nabla f\vec{A}]$; 6) $\nabla(fg)$;

7) $(\nabla, \nabla(fg))$; 8) $[\nabla[\vec{\omega}\vec{r}\,]]$; 9) $(\nabla[\vec{\omega}\vec{r}\,])$; 10) $\nabla(\nabla\vec{A})$; 11) $(\nabla,\nabla)\vec{A}$;

12) $(\nabla[\nabla\vec{A}])$; 13) $[\nabla[\nabla\vec{A}]]$; 14) $(\nabla,\nabla)f$; 15) $[\nabla \times \nabla f\,]$.

2.5.7. Quyidagi ifodalarni nabla (∇) operatori usulidan foydalanib akslantiring va keyin ularni xususiy hosilalarda yozing:

1) $div[\vec{A}\vec{B}]$; 2) $grad(\vec{A}\vec{B})$; 3) $rot[\vec{A}\vec{B}]$; 4) $div(f\vec{A})$; 5) $rot(f\vec{A})$; 6) $grad(f\vec{b})$;

7) $div\ grad(f\vec{b})$; 8) $rot[\vec{\omega}\vec{r}\,]$, bu yerda $\vec{\omega}$ - o‘zgarmas vektor; 9) $div[\vec{\omega}\vec{r}\,]$, bu yerda $\vec{\omega}$ - o‘zgarmas vektor;

2.5.8. Quyidagi ifodalarni xususiy hosilalarda yozing: 1) $grad\ div\vec{A}$; 2) $(\Delta\vec{A})_x$; $(\Delta\vec{A})_y$; $(\Delta\vec{A})_z$; 3) $div\ rot\vec{A}$; 4) $rot\ rot\vec{A}$; 5) $div\ gradf$; 6) $rot\ gradf$; 7) (Δf).

2.5.9. Agar φ $(\vec{E} = -grad\varphi)$ potensial quyidagicha berilgan bo‘lsa, $\vec{E}$ - elektr maydonning kuchlanganligini toping:

1) $(x^2 + 2y^2 z + \sin(x))\exp(-(x^2 + y^2 + z^2))$;

2) $(x^2 + \sin(zx) + y^2 x\cos(z))\exp(-x^2)$.

2.5.10. Agar $\vec{E}$ $(div\vec{E} = 4\pi\rho)$ - elektr maydonning kuchlanganligi quyidagicha berilgan bo‘lsa, vakuumda elektr zaryadlar zichligi ρ ni toping:

1) $(x^2 + 4\sin(z)\exp(xy);\ \cos(x) + \ln(xyz);\ xy^2 z)$;

2) $(x\exp(-x^2) + y + z;\ \ln(xy)\sin(z);\ x + y + z + 1)$.

Sinov savollari

1. Skalyar o‘zgaruvchining vektor funksiyasi limiti, formal xossalari. Skalyar o‘zgaruvchining vektor funksiyasi uzluksizligi.
2. Skalyar o‘zgaruvchining vektor funksiyasi hosilasi, uning koordinatalardagi ifodasi. Hosilaning xossalari.
3. Skalyar maydon. Skalyar maydonning sirti va sath chizig‘i. Skalyar maydonning differensiallanuvchanligi, gradiyent.
4. Skalyar maydonning yo‘nalish bo‘yicha hosilasi. Gradiyentning yo‘nalish bo‘yicha bilan bog‘liqligi.
5. Gradiyentning algebraik va geometrik xossalari. Dekart koordinatalarida gradiyentni va yo‘nalish bo‘yicha hosilani hisoblash. Gamilton belgilash sistemasi.
6. Vektor maydon. Vektor maydonning differensiallanuvchanligi, differensial operator. Dekart koordinatalari sistemasida differensial operator matritsasi. Vektor maydonning yo‘nalishi bo‘yicha hosilasi.
7. Vektor maydon divergensiyasi, uning xossalari va uni dekart koordinatalarida hisoblash. Gauss-Ostrogradskiy teoremasi. Oqim orqali divergensiya ifodasi. Solenoidal vektor maydonlar va ularning belgilari.
8. Vektor maydon rotori, uning xossalari va uni dekart koordinatalarida hisoblash. Rotorning formal xossalari. Stoks teoremasi.
9. Potensial vector maydonlar. Potensiallikning har xil belgilari.
10. Vektor analizning ba’zi formulalari: vektor maydonlar skalyar ko‘paytmasining gradiyenti.
11. Vektor analizning ba’zi formulalari: vektor maydonlar vektor ko‘paytmasining divergensiyasi va rotori
12. Egri chiziqli koordinatalar sistemasida gradiyent, divergensiya va rotorni hisoblash.

3. Tenzorlar ustida amallar. Tenzorning skalyar invariantlari

Ixtiyoriy bir egri chiziqli koordinatalar sistemasidan ikkinchisiga almashtirish amalga oshirilganda tenzorlar *odatdagi tenzorlar* deb ataladi; agar birjinsli koordinatalar sistemasida almashtirish chegaralangan bo‘lsa bunday tenzorlar *dekart tenzorlari* deb ataladi. Agar tutash muhitlar mexanikasida maxsus va umumlshgan holatlar qaralmasa, odatda, bu fan tushunchalari dekart tenzorlari yordamida o‘rganiladi. Tenzorlarni ular bo‘ysunadigan almashtirish qonunlariga mos rangi yoki tartibiga qarab turlash mumkin. Shu nuqtai nazardan, miqdorlarni har xil rangdagi tenzorlar deb qarash qulay: skalyar – bu rangi nolga teng tenzor ($3^0=1$ komponentali); vektor – bu rangi birga teng tenzor ($3^1=3$ komponentali); $3^2=9$ komponentali miqdor – bu rangi ikkiga teng tenzor va hokazo. Shuning uchun har xil rangli tenzorlar bo‘lishi mumkin: rangi *n* ga teng tenzorning komponentalari soni 3^n ta, bunda *n – tenzorning tartibi*.

Ushbu fan doirasida keyingi tushunchalarimizda rangi ikkidan yuqori bo‘lmagan tenzorlar ustida amallar bajaramiz. Quyida *rangi birga teng tenzorlar* (*vektorlar*) va *rangi ikkiga teng tenzorlar* (*diadiklar*) hamda ular ustida amallar bajarisjga oid masalalar qaraladi (faqat bazi hollardagina, *rangi uchga teng tenzor* (*triadik*), masalan Levi-Chivita tenzori, *rangi to‘rtga teng tenzor* (*tetradik*) qaraladi).

Tenzorlar uchun esa *A*, *B*, … kabi belgilashlarni qabul qilamiz.

Diad, diadik va ular ustida amallar. *Diad* deb ikki vektorning $\vec{a}\vec{b}$ aniqmas ko‘paytmasiga aytiladi, bunda ko‘paytiriluvchi vektorlar orasiga hech qanday belgi qo‘yilmaydi. Usbu diadni $\vec{a}$ va $\vec{b}$ vektorlarning komponentalari orqali ifodalalik:

$$\vec{a}\vec{b} = (a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3)(b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3) = a_1b_1\vec{e}_1\vec{e}_1 + a_1b_2\vec{e}_1\vec{e}_2 + a_1b_3\vec{e}_1\vec{e}_3 +$$
$$+ a_2b_1\vec{e}_2\vec{e}_1 + a_2b_2\vec{e}_2\vec{e}_2 + a_2b_3\vec{e}_2\vec{e}_3 + a_3b_1\vec{e}_3\vec{e}_1 + a_3b_2\vec{e}_3\vec{e}_2 + a_3b_3\vec{e}_3\vec{e}_3.$$

Bu ifoda to‘qqizta haddan iborat bo‘lganligi uchun u $\vec{a}\vec{b}$ *diadning to‘qqizhadli shakli* deb ataladi.

D diadik deb ikkinchi rangli tenzorga ataladi; uni har doim chekli sondagi diadlar yig‘indisi shaklida quyidagicha ifodalash mumkin:

$$D = \vec{a}_1\vec{b}_1 + \vec{a}_2\vec{b}_2 + \ldots + \vec{a}_n\vec{b}_n.$$

Agar *D* ning formuladagi har bir diadda birinchi va ikkinchi ko‘paytuvchilar o‘rni almashtirilsa, u holda hosil qilingan diadik (tenzor) dastlabkisiga *o‘z-o‘ziga qo‘shma* yoki *simmetrik diadik* (*tenzor*) deb ataladi va u quyidagicha yoziladi:

$$D_q = \vec{b}_1\vec{a}_1 + \vec{b}_2\vec{a}_2 + \ldots + \vec{b}_n\vec{a}_n.$$

Xususan, $D = (D_q)_q$ tenglik o‘rinli.

Agar D diadik yi’gindilaridagi gar bir diadni vektorlarning skalyar ko‘paytmasi bilan almashtirsak, u holda D diadikning skalyari deb ataluvchi skalyarga ega bo‘lamiz

$$D_s = \vec{a}_1 \cdot \vec{b}_1 + \vec{a}_2 \cdot \vec{b}_2 + \ldots + \vec{a}_n \cdot \vec{b}_n.$$

Agar D diadik yi’gindilaridagi gar bir diadni vektorlarning vektor ko‘paytmasi bilan almashtirsak, u holda natija D diadikning vektori deb ataladi

$$D_v = \vec{a}_1 \times \vec{b}_1 + \vec{a}_2 \times \vec{b}_2 + \ldots + \vec{a}_n \times \vec{b}_n.$$

D_q, D_s va D_v lar D diadik ifodasining tanlanishidan bog‘liq emas.

Vektorlarning aniqmas ko‘paytmasi distributivlik xossasiga ega, ya’ni

$$\vec{a}(\vec{b} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c},$$

$$(\vec{a} + \vec{b})\vec{c} = \vec{a}\vec{c} + \vec{b}\vec{c},$$

$$(\vec{a} + \vec{b})(\vec{c} + \vec{d}) = \vec{a}\vec{c} + \vec{a}\vec{d} + \vec{b}\vec{c} + \vec{b}\vec{d},$$

Xususan, λ va μ - skalyarlar bo‘lsa, u holda

$$(\lambda + \mu)\vec{a}\vec{b} = \lambda\vec{a}\vec{b} + \mu\vec{a}\vec{b},$$

$$(\lambda\vec{a})\vec{b} = \vec{a}(\lambda\vec{b}) = \lambda\vec{a}\vec{b}.$$

Aniqmas ko‘paytma umumiy holda nokommutativ, ya’ni $\vec{a}\vec{b} \neq \vec{b}\vec{a}$.

Agar $\vec{c}$ - ixtiyoriy vektor bo‘lsa, u holda $\vec{c} \cdot D$ va $D \cdot \vec{c}$ skalyar ko‘paytmalar vektorlar bo‘lib, ular quyidagi formulalardan aniqlanadi:

$$\vec{c} \cdot D = (\vec{c} \cdot \vec{a}_1)\vec{b}_1 + (\vec{c} \cdot \vec{a}_2)\vec{b}_2 + \ldots + (\vec{c} \cdot \vec{a}_n)\vec{b}_n = \vec{u},$$

$$D \cdot \vec{c} = \vec{a}_1(\vec{b}_1 \cdot \vec{c}) + \vec{a}_2(\vec{b}_2 \cdot \vec{c}) + .. + \vec{a}_n(\vec{b}_n \cdot \vec{c}) = \vec{w}.$$

D va B *diadiklar o‘zaro teng* deyiladi, agar ixtiyoriy $\vec{c}$ vektor uchun quyidagi tengliklar bajarilsa:

$$\vec{c} \cdot D = \vec{c} \cdot B \quad \text{yoki} \quad D \cdot \vec{c} = B \cdot \vec{c}\,.$$

Birlik diadik I – bu shunday diadikki, uni quyidagicha ifodalash mumkin:

$$I = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3,$$

bu *birlik diadning to‘qqizhadli shakli* ham deb ataladi, bu yerda $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ vektorlar uch o‘lchovli Yevklid fazosining ortonormallashtirilgan ixtiyoroy bazis vektorlari. Birlik diadik quyidagi xossaga ega, ya’ni barcha vektorlar uchun quyidagi tenglik o‘rinli

$$I \cdot \vec{c} = \vec{c} \cdot I = \vec{c}\,.$$

Ushbu $\vec{c}\times D$ va $D\times\vec{c}$ vektor ko‘paytmalar diadiklar bo‘lib, quyidagi formulalardan aniqlanadi:

$$\vec{c}\times D=(\vec{c}\times\vec{a}_1)\vec{b}_1+(\vec{c}\times\vec{a}_2)\vec{b}_2+...+(\vec{c}\times\vec{a}_n)\vec{b}_n=F,$$

$$D\times\vec{c}=\vec{a}_1(\vec{b}_1\times\vec{c})+\vec{a}_2(\vec{b}_2\times\vec{c})+..+\vec{a}_n(\vec{b}_n\times\vec{c})=G.$$

Ushbu $\vec{a}\vec{b}$ va $\vec{c}\vec{d}$ ikki diadlarning skalyar ko‘paytmasi, ta’rifga ko‘ra, diad bo‘lib, quyidagicha aniqlanadi

$$\vec{a}\vec{b}\cdot\vec{c}\vec{d}=(\vec{b}\cdot\vec{c})\vec{a}\vec{d}.$$

Ushbu formuladan foydalanilsa, ixtiyoriy ikkita *D* va *B* diadiklarning skalyar ko‘paytmasi ham diadik bo‘lib, u quyidagicha aniqlanadi:

$$D\cdot B=(\vec{a}_1\vec{b}_1+\vec{a}_2\vec{b}_2+...+\vec{a}_n\vec{b}_n)\cdot(\vec{c}_1\vec{d}_1+\vec{c}_2\vec{d}_2+...+\vec{c}_n\vec{d}_n)=$$
$$=(\vec{b}_1\cdot\vec{c}_1)\vec{a}_1\vec{d}_1+(\vec{b}_2\cdot\vec{c}_2)\vec{a}_2\vec{d}_2+...+(\vec{b}_n\cdot\vec{c}_n)\vec{a}_n\vec{d}_n=K.$$

D va *B* diadlar *o‘zaro teskari* deyiladi, agar

$$D\cdot B=B\cdot D=I.$$

O‘zaro teskari diadiklar uchun ko‘pincha $D=B^{-1}$ va $B=D^{-1}$ belgilashlar ishlatiladi.

D diadik o‘z-o‘ziga *qo‘sma* yoki *simmetrik* deyiladi, agar $D=D_c$ bo‘lsa va *antisimmetrik* deyiladi, agar $D=-D_c$ bo‘lsa.

Har bir diadikni simmetrik va antisimmetrik diadiklar yig‘indisi ko‘rinishida ifodalash mumkin va bu ifodalanish yagona

$$D=\frac{1}{2}(D+D_c)+\frac{1}{2}(D-D_c)=G+H,$$

bu yerda $G=\frac{1}{2}(D+D_c)$ - simmetrik diadik; $H=\frac{1}{2}(D-D_c)$ - antisimmetrik diadik.

D tenzor (diadik)ni quyidagicha yoyish mumkin:

$$D=I_1I+(D-I_1I)=D^{(s)}+D^{(d)},$$

bu yerda $D^{(s)}=I_1I$ - sferik tenzor (shar qismi); $D^{(d)}=(D-I_1I)$ - deviator qismi; I_1 - berilgan *D* tenzorning birinchi invariant.

Chiziqli vektorli funksiyalar. $\vec{a}$ vektor boshqa $\vec{b}$ vektorning funksiyasi deyiladi, agar $\vec{b}$ vektorning berilishi bilan $\vec{a}$ vektor aniqlangan bo‘lsa. Bu funksional bog‘lanish quyidagi formula bilan ifodalanadi: $\vec{a}=f(\vec{b})$. Bu yerdagi *f* funksiya chiziqli deyiladi, agar ixtiyoriy $\vec{b}$ va $\vec{c}$ vektorlar hamda ixtiyoriy λ skalyar uchun quyidagi munosabatlar o‘rinli bo‘lsa $f(\vec{b}+\vec{c})=f(\vec{b})+f(\vec{c})$; $f(\lambda\vec{b})=\lambda f(\vec{b})$.

Agar $\vec{a}$ vektorni komponentalari bo‘yicha yoyib chiqsak, $\vec{a} = f(b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3)$.

Agar f funksiya chiziqli bo‘lsa, $\vec{a} = b_1 f(\vec{e}_1) + b_2 f(\vec{e}_2) + b_3 f(\vec{e}_3)$. Agar bunda $\vec{u} = f(\vec{e}_1),\ \vec{v} = f(\vec{e}_2),\ \vec{w} = f(\vec{e}_3)$ desak, $\vec{a} = \vec{u}\,(\vec{e}_1 \cdot \vec{b}) + \vec{v}\,(\vec{e}_2 \cdot \vec{b}) + \vec{w}\,(\vec{e}_3 \cdot \vec{b}) = (\vec{u}\,\vec{e}_1 + \vec{v}\,\vec{e}_2 + \vec{w}\,\vec{e}_3) \cdot \vec{b}$. Bu ifoda diadikning vektor bilan skalyar ko‘paytmasini ifodalaydi, ya’ni $\vec{a} = D \cdot \vec{b}$, bu yerda $D = \vec{u}\,\vec{e}_1 + \vec{v}\,\vec{e}_2 + \vec{w}\,\vec{e}_3$. Bu, o‘z navbatida, ixtiyoriy f chiziqli funksiyani diadik va vektorning ko‘paytmasi shaklida ifodalash mumkin ekanligini ko‘rsatadi. Shunday qilib, D diadik chiziqli vektor operator bo‘lib, argument deb ataluvchi $\vec{b}$ vektorga ta’sir etganda uni $\vec{a}$ vektor-funksiyaga almashtiradi.

Indeksli belgilashlar. Ixtiyoriy rangli tenzor komponentalarini va tenzorning o‘zini ham qulaylik uchun ko‘rgazmali va qisqa holda indeksli belgilar bilan ifodalash mumkin.

Birinchi rangli tenzorlar (vektorlar) bitta erkin indeksli asosiy harflar bilan belgilanadi, masalan, $\vec{a}$ vaktorni a_i yoki $\vec{b}$ vektorni b^i kabi. Ikkinchi rangli tenzorlar ikkita erkin indeksli harflar bilan belgilanadi, masalan, D_{ij} yoki D^{ij}. Uchinchi rangli tenzorlar uchta erkin indeksli harflar bilan belgilanadi, masalan, T_{klm} yoki ε_{ijk} va hokazo. Indeks bilan bog‘liq bo‘lmagan belgilar skalyarni ifodalaydi, masalan, λ, μ, ρ yoki nolinchi rangli tenzor.

Agar indeksdan bir marta foydalanilgan bo‘lsa, u 1, 2, …, N qiymatlarni qabul qiladi, bunda N – musbat butun son bo‘lib, *indeksning o‘lchami* deyiladi.

Takrorlanmaydigan indekslar *erkin indekslar* deb ataladi. Bu son shu hadning *tenzor rangi* hisoblanadi.

Demak, indekslar bilan belgilangan tenzorning rangi bu indekslarning erkinlariga qarab aniqlanadi.

Agar indeksdan ikki marta foydalanilsa, u holda bu indeks intervaldagi barcha qiymatlarini va bu to‘plamdagi shunday indeksli barcha hadlarni qabul qiladi hamda ularning yig‘indisi olinadi deb tushunish kerak.

Bu aytilganlarni yana oydinlashtiraylik:

$a_{ij}b_j$, F_{ikk} , $R^p_{.qp}$, $\varepsilon_{ijk}u_jv_k$ – birinchi rang tenzorlar;

D^{ij}, $D_i^{\cdot j}$ yoki $D^j_{.i}$, D_{ij} ; A_{ijip} , $B^{ij}_{..jk}$, $\delta_{ij}u_kv_k$ – ikkinchi rang tenzorlar va hokazo (aralash shaklda nuqta j – ikkinchi indeks ekanligini bildiradi) va hokazo.

a_i – rangi nolga teng tenzorning uchta komponentasi bor: $a_i = (a_1, a_2, a_3)$.

A_{ij} - rangi ikkiga teng tenzorning to‘qqizta hadi bor: $A_{ji} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$.

Umuman olganda N-o‘lchovli fazoda n-rangli tenzorning N^n ta komponentlari mavjud bo‘ladi.

Takroriy indekslar *gung indekslar* deb ataladi, chunki uni erkin indeks sifatida foydalanilmagan boshqa ixtiyoriy indeks bilan almashtirish ular orqali ifodalangan hadning qiymatini almashtirmaydi. To‘g‘ri yozilgan ifodada hech bir indeks ikki martadan ortiq uchramaydi.

Demak, indekslar bilan belgilangan tenzorning rangi bu indekslarning erkinlariga qarab aniqlanadi, masalan, bitta erkin indeksli had F_i – birinchi rangli tenzor; ikkita erkin indeksli had T_{ij} – ikkinchi rangli tenzor; to‘rtta erkin indeksli had S_{ijkm} – to‘rtinchi rangli tenzor va hokazo.

Gung indekslardan foydaanishga misol tariqasida quyidagi yozuvlarni keltiramiz:

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = a_i\vec{e}_i \text{ - vektor;}$$

$$T = \frac{1}{3}\sum_{i=1}^{3}\sigma_{ii} = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) = \frac{1}{3}\sigma_{ii} \text{ - tenzor;}$$

$$\vec{a}\vec{b} = (a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3)(b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3) = (a_i\vec{e}_i)(b_j\vec{e}_j) = a_i b_j \vec{e}_i\vec{e}_j \text{ - diad;}$$

$$D = D_{ij}\vec{e}_i\vec{e}_j \text{ - diadik.}$$

Gung indekslarni boshqa harflar bilan belgilash yig‘indining qiymatini o‘zgartirmaydi:

$$a_i b_i = a_k b_k = a_n b_n.$$

Simmetrik tenzorlar uchun, masalan, T_{ijk} tenzorda i va j indeklari bo‘yicha simmetriklik sharti bajarilgan bo‘lsa, $T_{ijk} = T_{kji}$ tenglik o‘rinli. Antisimmetrik tenzorlar uchun, masalan, T_{ijkl} tenzorda i va l indeklari bo‘yicha antisimmetriklik sharti bajarilgan bo‘lsa, $T_{ijkl} = -T_{ljki}$ tenglik o‘rinli. Barcha indekslari bo‘yicha to‘la antisimmetrik tenzor a_{prs}; p, r, s = 1, 2, 3 ni misol keltiraylik: $a_{prs} = -a_{psr} = a_{spr} = -a_{srp} = a_{rsp} = -a_{rps}$.

Xususan, uch o‘lchovli fazoda δ_{ij} - Kroneker belgisi gung indeksli birlik tenzor bo‘lib, tenzorli belgilashlarda keng qo‘llaniladi:

$$\delta_{ij} = \begin{cases} 1, \text{ agar } i = j; \\ 0, \text{ agar } i \neq j. \end{cases}$$

Bu o‘z navbatida quyidagi tenglikning to‘g‘riligini ko‘rsatadi:

$$\delta_{ij} = \vec{e}_i \cdot \vec{e}_j, \quad i, j = 1, 2, 3.$$

Tengliklar sistemasini ixcham shaklda yozish uchun indeksli belgilardan foydalanish juda qulay, masalan,

$$x_i = D_{ij} z_j, \quad (i = 1, 2, 3).$$

Bu birhadda i - erkin indeks, j - gung indeks. Buning indekslar bo‘yicha yoyib yozilishi quyidagicha:

$$x_1 = D_{11} z_1 + D_{12} z_2 + D_{13} z_3,$$
$$x_2 = D_{21} z_1 + D_{22} z_2 + D_{23} z_3,$$
$$x_3 = D_{31} z_1 + D_{32} z_2 + D_{33} z_3.$$

Tenzor komponentalarini x_k koordinata bo‘yicha differensiallash ushbu $\dfrac{\partial}{\partial x_k}$ differensial operator orqali belgilanadi. Bu amalni ixcham yozish uchun $\vec{\nabla}$ belgidan foydalaniladi va u quyidagicha kengaytirib yoziladi:

$$\vec{\nabla} = \frac{\partial}{\partial x_k} \vec{e}_k = \frac{\partial}{\partial x_1} \vec{e}_1 + \frac{\partial}{\partial x_2} \vec{e}_2 + \frac{\partial}{\partial x_3} \vec{e}_3.$$

Tenzorlar ustida bir qator amalllar bajarish mumkin (yozuvlar gung indekli belgilashlarda):

1) Bir xil rangli ikkita tenzor o‘zaro teng deyiladi, agar ularning mos komponentalari uchun $A_{ij} = B_{ij}$ tenglik o‘rinli bo‘la.

2) Tenzor skalyarga ko‘paytirilsa, uning rangi o‘zgarmaydi: λA_{ij} .

3) Bir xil rangli ikkita tenzor o‘zaro qo‘shilganda (yoki ayrilganda) ularning mos komponentalari qo‘shiladi (yoki ayriladi): $A_{ij} + B_{ij} = C_{ij}$ (yoki $A_{ij} - B_{ij} = C_{ij}$).

Tenzorlarni qo‘shish va skalyarga ko‘paytirish. Tenzorlarni qo‘shish amali faqat bir xil tuzilishga ega hamda ranglari teng bo‘lgan tenzorlar ustidagina o‘rinlidir. Masalan,

$$A = A^{ijk} \vec{э}_i \vec{э}_j \vec{э}_k \ \ va \ \ B = B^{ijk} \vec{э}_i \vec{э}_j \vec{э}_k$$

uchinchi rang tenzorlarni qo‘shsak

$$C = A + B = \left(A^{ijk} + B^{ijk}\right) \vec{э}_i \vec{э}_j \vec{э}_k$$

yana uchinchi rang tenzor hosil qilamiz. Ayirsak ham huddi shunday.

Quyidagi beshta tenzorni qaraylik

$$T_1 = T_j^{\,i} \vec{э}_i \vec{э}^{\,j}, \quad T_2 = T_{pq} \vec{э}^{\,p} \vec{э}^{\,q}, \quad T_3 = T_{mn} \vec{э}^{\,m} \vec{э}^{\,n},$$
$$T_4 = T_l^{\,k} \vec{э}_k \vec{э}^{\,l}, \quad T_5 = T_{l\cdot}^{\cdot k} \vec{э}_k \vec{э}^{\,l}.$$

Bu tenzorlardan faqat T_1+T_4 va T_2+T_3 yig‘indilarnigina tuzish mumkin T_1 va T_2; T_3 va T_4 larning yig‘indilarini tuzib bo‘lmaydi, chunki ushbu tenzorlarning tuzilishlari har xil. Xuddi shu sababga ko‘ra T_5 tenzorini qolgan tenzorlarning hech biri bilan qo‘shib bo‘lmaydi.

Tenzorlarni qo‘shish amalini ketma-ket qo‘llash natijasida tenzorni songa (skalyarga) ko‘paytirish amalini asoslash mumkin. Demak, agar A-tenzor bo‘lsa, ixtiyoriy $k \geq 0$ soni uchun

$$\mathbf{C = k\,A}$$

ob’yekt ham tenzordir.

Tenzorlarni simmetriklash va antisimmetriklash.

Ta’rif 1. $T = T^{ijkl}\vec{э}_i\vec{э}_j\vec{э}_k\vec{э}_l$ tenzor *i* va *j* indekslari bo‘yicha *simmetrik* deyiladi, agar

$$T^{ijkl} = T^{jikl}$$

bo‘lsa.

Ta’rif 2. $T = T^{ijkl}\vec{э}_i\vec{э}_j\vec{э}_k\vec{э}_l$ tenzor *i* va *j* indekslari bo‘yicha *antisimmetrik* deyiladi, agar

$$T^{ijkl} = -T^{jikl}$$

bo‘lsa.

Yuqorida keltirilgan qo‘shish qoidasidan foydalanib, ixtiyoriy $T = T^{ij}\vec{э}_i\vec{э}_j$ ikkinchi rang tenzorga doimo simmetrik

$$T_0 = \frac{1}{2}\left(T^{ij} + T^{ji}\right)\vec{э}_i\vec{э}_j \tag{3.1}$$

va antisimmetrik

$$T_1 = \frac{1}{2}\left(T^{ij} - T^{ji}\right)\vec{э}_i\vec{э}_j \tag{3.2}$$

tenzorlarni mos qo‘yish mumkin. Ushbu (3.1) va (3.2) amallarga mos ravishda tenzorni *simmetriklash* va *antisimmetriklash* deyiladi.

Ko‘rinib turibdiki, agar T simmetrik tenzor bo‘lsa $T_0 = T$ va $T_1 = 0$, va aksincha agar T antisimmetrik tenzor bo‘lsa, $T_0 = 0$ va $T_1 = T$ bo‘ladi. Yuqoridagi (3.1) va (3.2) formulalardan

$$T^{ij} = \frac{1}{2}\left(T^{ij} + T^{ji}\right) + \frac{1}{2}\left(T^{ij} - T^{ji}\right), \tag{3.3}$$

ya’ni

$$T = T_0 + T_1.$$

Demak, (4.3) formuladan ko‘rinadiki tenzorning T^{ij} matrisasini simmetrik va antisimmetrik matrisalar yig‘indisi kabi tasvirlash mumkin.

Eslatma. Ranglari va tuzilishlaridan qat’iy nazar ixtiyoriy tenzorlarni ularning berilish tartibida ko‘paytirish mumkin. Masalan, $A = A^i \vec{э}_i$ vektor va $T = T_{\cdot j}^{k\cdot} \vec{э}_k \vec{э}^j$ tenzorlarni quyidagicha ko‘paytirish mumkin

$$B = A^i T_{\cdot j}^{k\cdot} \vec{э}_i \vec{э}_k \vec{э}^j ; \quad C = T_{\cdot j}^{k\cdot} A^i \vec{э}_k \vec{э}^j \vec{э}_i ,$$

lekin hosil qilingan B va C ob’yektlar uchinchi rang tenzorlari bo‘lmaydilar. Shuning uchun bunday ko‘paytalarga *tenzorlarning noaniq ko‘paytmalari* deyiladi.

Tenzorning indekslarini ko‘tarish va tushirish. Tenzorning rangini pasaytirish. O‘tgan 3-ma’ruzadagi (2.14) formulaga asosan g fundamental metrik tenzor yordamida tenzor komponentalarining yuqori indekslarini tushirish mumkin degan xulosaga kelamiz. Xuddi shunday tenzor komponentalarining pastki indekslarini ko‘tarish ham mumkin. Masalan,

$$T = T_{ijk} \vec{э}^i \vec{э}^j \vec{э}^k = T_{ijk}\, g^{im} \vec{э}_m g^{jn} \vec{э}_n g^{kp} \vec{э}_p = T_{ijk}\, g^{im} g^{jn} g^{kp} \vec{э}_m \vec{э}_n \vec{э}_p =$$

$$= T^{mnp} \vec{э}_m \vec{э}_n \vec{э}_p = T^{mnp} \vec{э}_m \vec{э}_n \vec{э}^q g_{pq} = T^{mnp} g_{pq} \cdot \vec{э}_m \vec{э}_n \vec{э}^q = T_{\cdot\cdot q}^{mn\cdot} \vec{э}_m \vec{э}_n \vec{э}^q$$

Bu yerda T tenzor komponentalarining indekslari ko‘tarildi va yana qisman tushirildi. Fundamental metrik tenzor yordamida bajarilgan yuqoridagi amal *indekslarni ko‘tarish va tushirish* amali deyiladi.

Tenzorlarning ranglarini pasaytirish ham mumkin. Lekin bu amal faqat aralash komponentali tenzorlar uchungina o‘rinlidir. Ushbu amal bilan xususiy misolda tanishamiz. Faraz qilaylik, aralash komponentali uchinchi rang tenzor $T_{i\cdot\cdot}^{\cdot jk} \vec{э}^i \vec{э}_j \vec{э}_k$ berilgan bo‘lsin. Bu tenzorning hamma komponentalari ichidan faqat $i = j$ bo‘lganlarini tanlab olib, ularning yig‘indisini qaraymiz:

$$T_{i\cdot\cdot}^{\cdot ik} = T_{1\cdot\cdot}^{\cdot 1k} + T_{2\cdot\cdot}^{\cdot 2k} + T_{3\cdot\cdot}^{\cdot 3k}$$

Ko‘rinib turibdiki, bu yig‘indi faqat k indeksgagina bog‘liq, chunki i-«gung» indeksdir. $T_{i\cdot\cdot}^{\cdot ik}$ yig‘indi $T_{i\cdot\cdot}^{\cdot ik}$ komponentalar ustida i va j indekslar bo‘yicha tenzorning rangini pasaytirish amali natijasidir. Bunday amalni bajarish natijasida tenzorning rangi birlikka kamayadi. Demak,

$$T_i = T_{i\cdot\cdot}^{\cdot ik} \vec{э}_k$$

tenzorining rangi birga teng. Agar tenzor o‘zining faqat konrtavariant yoki faqat kovariant komponentalari bilan berilgan bo‘lsa uning rangini pasaytirish g fundamemtal metrik tenzor yordamida bajariladi:

$$T = T^{ij} \vec{э}_i \cdot \vec{э}_j = T^{ij} g_{ik} \vec{э}^k \cdot \vec{э}_j = T_{k\cdot}^{\cdot j} \vec{э}^k \cdot \vec{э}_j$$

demak,

$$T^{ij} g_{ik} = T_{k\cdot}^{\cdot j}$$

ya’ni bundan keyin tenzorning rangini pasaytirish uchun uning komponentasini avvalo fundamental metrik tenzor yordamida aralash holga keltirish va shundan keyin rangini pasaytirish mumkin.

Tenzorning skalyar invariantlari. Tutash muhit mexanikasi qoidalarining matematik ifodalari koordinat sistemasining tanlanishiga bog‘liq bo‘lmasligi kerak. Ushbu talab matematik ifodalar faqat invariant ob’ektlar orqali yozilishi kerakligini taqozo qiladi. Bu esa o‘z navbatida vektor va tenzorlarning invariantlarini hosil qilish zaruriyatini keltirib chiqaradi.

Faraz qilaylik

$$\vec{A} = A^i \vec{э}_i = A_j \vec{э}^{\,j} = A^i g_{ij} \vec{э}^{\,j}$$

vektori berilgan bo‘lsin. Bu vektorning uzunligini qaraymiz

$$\left|\vec{A}\right|^2 = \vec{A} \cdot \vec{A} = A^i A^j \vec{э}_i \vec{э}_j = A^i A^j g_{ik} \vec{э}^{\,k} \vec{э}_j = A^i A^j g_{ij} = A^i A_i\,,$$

bu yerdan $A^i A_i$ ifoda invariant ekanligi kelib chiqadi, chunki vektorning kontravariant va kovariant komponentalari o‘zaro teskaridirlar. Demak, vektorning faqat bitta mustaqil invarianti mavjud bo‘lib, u ham bo‘lsa uning uzunligidir. Endi $T^{ij} \vec{э}_i \vec{э}_j = T$ tenzorning rangini g_{ij} lar yordamida pasaytiraylik

$$T^{ij} g_{ij} = T_{\cdot i}^{i\cdot} = T_{\cdot 1}^{1\cdot} + T_{\cdot 2}^{2\cdot} + T_{\cdot 3}^{3\cdot}\,,$$

ko‘rinib turibdiki, bunday amal natijasida songa (skalyarga) ega bo‘ldik. Ma’lumki skalyar miqdorlar koordinat sistemasining tanlanishga bog‘liq emas. Demak, $T_{\cdot i}^{i\cdot}$ miqdor tenzorning invariantidan iboratdir va uni *ikkinchi rang tenzorning birinchi invariant deb* ataymiz. Quyidagi ifodalar

$$T_{\cdot j}^{i\cdot} T_{\cdot i}^{j\cdot}, \quad T_{\cdot j}^{i\cdot} \cdot T_{\cdot p}^{j\cdot} T_{\cdot i}^{p\cdot}$$

invariantlarni tashkil qilishini tekshirish qiyin emas. Shunday qilib, ikkinchi rang tenzorning uchta invarianti mavjud

$$T_{\cdot i}^{i\cdot}; \quad T_{\cdot j}^{i\cdot} T_{\cdot i}^{j\cdot}; \quad T_{\cdot j}^{i\cdot} T_{\cdot p}^{j\cdot} T_{\cdot i}^{p\cdot} \tag{3.4}$$

Tenzorning bosh o‘qlari va komponentalari tushunchalarini tenzor sirti tushunchasi bilan bog‘liq holda kiritamiz. Buning uchun ζ^1, ζ^2, ζ^3 koordinatalar sistemasining boshiga juda yaqin tyurgan nuqtani olamiz va $\overrightarrow{OM} = d\vec{r} = d\zeta^i \vec{э}_i$

vektori hamda $T = T_{ij}\vec{э}^i\vec{э}^j$ tenzorning komponetalaridan tuzilgan $T_{ij}d\zeta^i d\zeta^j$ ifodani qaraymiz. Ushbu ifoda invariant bo‘lganligidan

$$T_{ij}d\zeta^i d\zeta^j = T_{ij}^{'}d\eta^i d\eta^j = C, \tag{3.5}$$

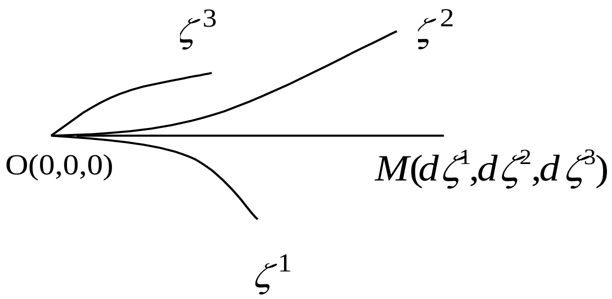
bu yerda C- biror skalyar miqdor. Belgilangan O nuqtaning cheksiz kichik atrofida C ning ma’lum qiymati va T_{ij} larning O nuqtada olingan qiymatlarida (3.5) ikkinchi tartibli sirt tenglamasidir. Bu sirt *tenzor sirti* deyiladi.

Demak, har qanday ikkinchi rang tenzorga har bir nuqtada (3.5) ikkinchi taztibli sirtni mos qo‘yish mumkin. Ma’lumki (3.5) tenglamani koordinat sistemasini almashtirish yo‘li bilan quyidagi kanonik

$$T_{11}(dx^{'})^2 + T_{22}(dx^2)^2 + T_{33}(dx^3)^2 = C$$

ko‘rinishga keltirish mumkin. Bunda x^1, x^2, x^3 koordinatalar sistemasi ortogonal bo‘ladi. Demak, fazoning har bir nuqtasi uchun shunday koordinatalar o‘qlarini kiri tish mumkinki, bunda *ikkinchi rang simmetrik tenzorning faqat uchta T_{11}, $T_{22,}$ T_{33} kooponentalari noldan farqli* bo‘ladi. Bunday *koordinat o‘qlari tenzorning bosh o‘qlari deyiladi.*

O‘qlari bosh o‘qlar bo‘ylab yo‘nalgan koordinatalar sistemasi esa tenzorning bosh koordinat sistemasi deyiladi. *Bosh koordinatalar sistemasidagi tenzorning har qanday noldan farqli har xil komponentalari bosh komponentalar deyiladi.*



4.1-chizma

Namunaviy masalalar va ularning yechimlari

3.1. Biror $D = a\vec{e}_1\vec{e}_1 + b\vec{e}_2\vec{e}_2 + c\vec{e}_3\vec{e}_3$ diad va $\vec{r} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$ radius-vektor berilgan. Ushbu $\vec{r}\cdot D\cdot\vec{r} = 1$ tenglama $ax_1^2 + bx_2^2 + cx_3^2 = 1$ - ellipsni ifodalashini ko‘rsating.

Yechish. Ko‘rsatilgan amallarni bajaramiz: avval D tenzorni o‘ngdagi $\vec{r}$ vektorga ko‘paytiramiz, keyin esa hosil bo‘lgan vektorlarni skalyar ko‘paytiramiz:

$$\vec{r}\cdot D\cdot\vec{r} = (x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3)\cdot(a\vec{e}_1\vec{e}_1 + b\vec{e}_2\vec{e}_2 + c\vec{e}_3\vec{e}_3)\cdot(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3) =$$
$$= (x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3)\cdot(ax_1\vec{e}_1 + bx_2\vec{e}_2 + cx_3\vec{e}_3) = ax_1^2 + bx_2^2 + cx_3^2.$$

Agar hosil qilingan ifodaning qiymati birga teng chiqishi zarurligini e'tiborga olsak, ya'ni $ax_1^2 + bx_2^2 + cx_3^3 = 1,$ u holda bu ellipsoidning tenglamasini ifodalaydi.

3.2. Agar ushbu

$$D = 3\vec{e}_1\vec{e}_1 + 2\vec{e}_2\vec{e}_2 - \vec{e}_3\vec{e}_3 + 5\vec{e}_3\vec{e}_3 \quad \text{va} \quad F = 4\vec{e}_1\vec{e}_3 + 6\vec{e}_2\vec{e}_2 - 3\vec{e}_3\vec{e}_2 + \vec{e}_3\vec{e}_3$$

munosabatlar o'rinli bo'lsa, $G = D \cdot F$ va $H = F \cdot D$ diadlarni aniqlang.

Yechish. Diadlarni ko'paytirishning ushbu $\vec{a}\vec{b} \cdot \vec{c}\vec{d} = (\vec{b} \cdot \vec{c})\vec{a}\vec{d}$ qoidasidan foydalanamiz. U holda

$$G = D \cdot F = (3\vec{e}_1\vec{e}_1 + 2\vec{e}_2\vec{e}_2 - \vec{e}_3\vec{e}_3 + 5\vec{e}_3\vec{e}_3) \cdot (4\vec{e}_1\vec{e}_3 + 6\vec{e}_2\vec{e}_2 - 3\vec{e}_3\vec{e}_2 + \vec{e}_3\vec{e}_3) =$$
$$= 12\,\vec{e}_1\vec{e}_3 + 12\vec{e}_2\vec{e}_2 + 3\vec{e}_2\vec{e}_2 - 15\vec{e}_3\vec{e}_2 - \vec{e}_2\vec{e}_3 + 5\vec{e}_3\vec{e}_3 =$$
$$= 12\,\vec{e}_1\vec{e}_3 + 15\vec{e}_2\vec{e}_2 - 15\vec{e}_3\vec{e}_2 - \vec{e}_2\vec{e}_3 + 5\vec{e}_3\vec{e}_3.$$

Xuddi shu tarzda

$$H = F \cdot D = (4\vec{e}_1\vec{e}_3 + 6\vec{e}_2\vec{e}_2 - 3\vec{e}_3\vec{e}_2 + \vec{e}_3\vec{e}_3) \cdot (3\vec{e}_1\vec{e}_1 + 2\vec{e}_2\vec{e}_2 - \vec{e}_3\vec{e}_3 + 5\vec{e}_3\vec{e}_3) =$$
$$= 20\,\vec{e}_1\vec{e}_3 + 12\vec{e}_2\vec{e}_2 - 6\vec{e}_3\vec{e}_2 - 6\vec{e}_3\vec{e}_2 + 3\vec{e}_3\vec{e}_3 + 5\vec{e}_3\vec{e}_3 =$$
$$= 20\,\vec{e}_1\vec{e}_3 + 12\vec{e}_2\vec{e}_2 - 6\vec{e}_2\vec{e}_3 - 6\vec{e}_3\vec{e}_2 + 8\vec{e}_3\vec{e}_3.$$

Natijalardan ko'rinib turibdiki, olingan diadlar har xil.

3.3. Ikkinchi rang tenzor D ning to'qqizta hadli yozuvidan kelib chiqib, uni quyidagicha ifodalash mumkin ekanligini ko'rsating:

$$D = (D \cdot \vec{e}_1)\vec{e}_1 + (D \cdot \vec{e}_2)\vec{e}_2 + (D \cdot \vec{e}_{33})\vec{e}_3.$$

Xuddi shunday, ushbu

$$\vec{e}_1 \cdot D \cdot \vec{e}_1 = D_{11}, \quad \vec{e}_1 \cdot D \cdot \vec{e}_2 = D_{12} \text{ va hokazo}$$

munosabatlarning ham o'rinli ekanligini ko'rsating.

Yechish. D diadni to'qqizhadli shaklda yozamiz va uning hadlarini guruhlashtiramiz:

$$D = D_{11}\vec{e}_1\vec{e}_1 + D_{12}\vec{e}_1\vec{e}_2 + D_{13}\vec{e}_1\vec{e}_3 + D_{21}\vec{e}_2\vec{e}_1 + D_{22}\vec{e}_2\vec{e}_2 + D_{23}\vec{e}_2\vec{e}_3 + D_{31}\vec{e}_3\vec{e}_1 +$$
$$+ D_{32}\vec{e}_3\vec{e}_2 + D_{33}\vec{e}_3\vec{e}_3 = (D_{11}\vec{e}_1 + D_{21}\vec{e}_2 + D_{31}\vec{e}_3)\vec{e}_1 + (D_{12}\vec{e}_1 + D_{22}\vec{e}_2 + D_{32}\vec{e}_3)\vec{e}_2 +$$
$$+ (D_{13}\vec{e}_1 + D_{23}\vec{e}_2 + D_{33}\vec{e}_3)\vec{e}_3 = \vec{d}_1\vec{e}_1 + \vec{d}_2\vec{e}_2 + \vec{d}_3\vec{e}_3 =$$
$$= (D \cdot \vec{e}_1)\vec{e}_1 + (D \cdot \vec{e}_2)\vec{e}_2 + (D \cdot \vec{e}_3)\vec{e}_3,$$

bu yerda

$$\vec{d}_1 = D_{11}\vec{e}_1 + D_{21}\vec{e}_2 + D_{31}\vec{e}_3 = D \cdot \vec{e}_1,$$
$$\vec{d}_2 = D_{12}\vec{e}_1 + D_{22}\vec{e}_2 + D_{32}\vec{e}_3 = D \cdot \vec{e}_2,$$
$$\vec{d}_3 = D_{13}\vec{e}_1 + D_{23}\vec{e}_2 + D_{33}\vec{e}_3 = D \cdot \vec{e}_3.$$

Bu yerdan kelib chiqadiki,

$$\vec{e}_1 \cdot \vec{d}_1 = \vec{e}_1 \cdot (D \cdot \vec{e}_1) = \vec{e}_1 \cdot (D_{11}\vec{e}_1 + D_{21}\vec{e}_2 + D_{31}\vec{e}_3) = D_{11},$$
$$\vec{e}_2 \cdot \vec{d}_1 = \vec{e}_2 \cdot (D \cdot \vec{e}_1) = \vec{e}_2 \cdot (D_{11}\vec{e}_1 + D_{21}\vec{e}_2 + D_{31}\vec{e}_3) = D_{21},$$
$$\vec{e}_2 \cdot \vec{d}_2 = \vec{e}_2 \cdot (D_{12}\vec{e}_1 + D_{22}\vec{e}_2 + D_{32}\vec{e}_3) = D_{22}.$$

3.4. Ushbu
$$D = 3\vec{e}_1\vec{e}_1 + 4\vec{e}_1\vec{e}_3 + 6\vec{e}_2\vec{e}_1 + 7\vec{e}_2\vec{e}_2 + 10\vec{e}_3\vec{e}_1 + 2\vec{e}_3\vec{e}_2$$
tenzorni simmetrik va antisimmetrik qismlarga ajrating.

Yechish. Tenzorning quyidagi yoyilma formulasidan foydalanamiz:
$$D = \frac{1}{2}(D + D_c) + \frac{1}{2}(D - D_c) = E + F,$$
bu yerda $E = E_c -$ berilgan D tenzorning simmtrik qismi; $F = -F_c$ - berilgan D tenzorning antisimmtrik qismi.

Ularning qiymatlarini hisoblaymiz:
$$E = \frac{1}{2}(D + D_c) =$$
$$= \frac{1}{2}(6\vec{e}_1\vec{e}_1 + 4\vec{e}_1\vec{e}_3 + 4\vec{e}_3\vec{e}_1 + 6\vec{e}_2\vec{e}_1 + 6\vec{e}_1\vec{e}_2 + 14\vec{e}_2\vec{e}_2 + 10\vec{e}_3\vec{e}_1 + 10\vec{e}_1\vec{e}_3 + 2\vec{e}_3\vec{e}_2 + 2\vec{e}_2\vec{e}_3) =$$
$$= 3\vec{e}_1\vec{e}_1 + 3\vec{e}_1\vec{e}_2 + 7\vec{e}_1\vec{e}_3 + 3\vec{e}_2\vec{e}_1 + 7\vec{e}_2\vec{e}_2 + \vec{e}_2\vec{e}_3 + 7\vec{e}_3\vec{e}_1 + \vec{e}_3\vec{e}_2 = E_c,$$
$$F = \frac{1}{2}(D - D_c) =$$
$$= \frac{1}{2}(4\vec{e}_1\vec{e}_3 - 4\vec{e}_3\vec{e}_1 + 6\vec{e}_2\vec{e}_1 - 6\vec{e}_1\vec{e}_2 + 10\vec{e}_3\vec{e}_1 - 10\vec{e}_1\vec{e}_3 + 2\vec{e}_3\vec{e}_2 - 2\vec{e}_2\vec{e}_3) =$$
$$= -3\vec{e}_1\vec{e}_2 - 3\vec{e}_1\vec{e}_3 + 3\vec{e}_2\vec{e}_1 - \vec{e}_2\vec{e}_3 + 3\vec{e}_3\vec{e}_1 + \vec{e}_3\vec{e}_2 = -F_c.$$

Javobning to‘g‘riligini tekshiramiz:
$$E + F =$$
$$= 3\vec{e}_1\vec{e}_1 + 3\vec{e}_1\vec{e}_2 + 7\vec{e}_1\vec{e}_3 + 3\vec{e}_2\vec{e}_1 + 7\vec{e}_2\vec{e}_2 + \vec{e}_2\vec{e}_3 + 7\vec{e}_3\vec{e}_1 + \vec{e}_3\vec{e}_2 - 3\vec{e}_1\vec{e}_2 - 3\vec{e}_1\vec{e}_3 + 3\vec{e}_2\vec{e}_1 -$$
$$- \vec{e}_2\vec{e}_3 + 3\vec{e}_3\vec{e}_1 + \vec{e}_3\vec{e}_2 = 3\vec{e}_1\vec{e}_1 + 4\vec{e}_1\vec{e}_3 + 6\vec{e}_2\vec{e}_1 + 7\vec{e}_2\vec{e}_2 + 10\vec{e}_3\vec{e}_1 + 2\vec{e}_3\vec{e}_2 = D.$$

3.5. Uch o‘lchovli fazoda ushbu $A_{ii},\ B_{ijj},\ R_{ij},\ a_i T_{ij},\ a_i b_j S_{ij}.$ tenzor belgilarini yoyib yozing:

Yechish. Belgilashlarga ko‘ra A_{ii} bitta yig‘indini ifodalaydi
$$A_{ii} = A_{11} + A_{22} + A_{33}.$$
Ushbu B_{ijj} yozuv quyidagi uchta yig‘indini ifodalaydi:
$$1)\ i = 1 \text{ da } B_{111} + B_{122} + B_{133},$$

$$2)\ i=2 \text{ da } B_{211}+B_{222}+B_{233},$$

$$3)\ i=3 \text{ da } B_{311}+B_{322}+B_{333}.$$

Ushbu R_{ij} ning ifodasi quyidagi matritsa ko‘rinishida yozish mumkin bo‘lgan to‘qqizta komponentani ifodalaydi:

$$R_{ij}=\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}.$$

Ushbu a_iT_{ij} ifoda quyidagi uchta yig‘indini ifodalaydi:

$$1)\ j=1 \text{ da } a_1T_{11}+a_2T_{21}+a_3T_{31};$$

$$2)\ j=2 \text{ da } a_2T_{12}+a_2T_{22}+a_3T_{32};$$

$$3)\ j=3 \text{ da } a_1T_{13}+a_2T_{23}+a_3T_{33}.$$

Ushbu $a_ib_jS_{ij}$ ifoda to‘qqizta hadning yig‘indisini ifodalaydi. Bunda i bo‘yicha birinchi qo‘shiluvchi $a_ib_jS_{ij}=a_1b_jS_{1j}+a_2b_jS_{2j}+a_3b_jS_{3j}$. Endi qo‘shiluvchilarning j bo‘yicha barchasi qo‘shib chiqiladi:

$$a_ib_jS_{ij}=a_1b_1S_{11}+a_1b_2S_{12}+a_1b_3S_{13}+a_2b_1S_{21}+a_2b_2S_{22}+a_2b_3S_{23}+$$
$$+a_3b_1S_{31}+a_3b_2S_{32}+a_3b_3S_{33}.$$

3.6. Uch o‘lchovli fazoda δ_{ij} - Kronekker belgisini o‘z ichiga olgan quyidagi ifodalarni hisoblang:

$$\delta_{ii},\ \delta_{ij}\delta_{ij},\ \delta_{ij}\delta_{ik}\delta_{jk},\ \delta_{ij}\delta_{jk}\delta_{kl}\ldots\delta_{km},\ \sigma_{ij}\delta_{ik}\delta_{jk},\ \delta_{ij}A_{ik\cdot}\ldots$$

Yechish.

$$\delta_{ii}=\delta_{11}+\delta_{22}+\delta_{33}=3;$$

$$\delta_{ij}\delta_{ij}=\delta_{1j}\delta_{1j}+\delta_{2j}\delta_{2j}+\delta_{3j}\delta_{3j}=\delta_{11}\delta_{11}+\delta_{12}\delta_{12}+\delta_{13}\delta_{13}+\delta_{21}\delta_{21}+$$
$$+\delta_{22}\delta_{22}+\delta_{23}\delta_{23}+\delta_{31}\delta_{31}+\delta_{32}\delta_{32}+\delta_{33}\delta_{33}=\delta_{11}+\delta_{22}+\delta_{33}=3;$$

$$\delta_{ij}\delta_{ik}\delta_{jk}=\delta_{ij}(\delta_{ik}\delta_{jk})=\delta_{ij}\delta_{ij}=3;$$

$$\delta_{ij}\delta_{jk}\delta_{kl}\ldots\delta_{nm}=\delta_{im};$$

$$\sigma_{ij}\delta_{ik}\delta_{jk}=\sigma_{ij}\delta_{ij}=\sigma_{ii}=\sigma_{11}+\sigma_{22}+\sigma_{33};$$

$$\delta_{ij}A_{ik}=\delta_{1j}A_{1k}+\delta_{2j}A_{2k}+\delta_{3j}A_{3k}.$$

$$j=1 \text{ da } \delta_{ij}A_{ik}=\delta_{11}A_{1k}+\delta_{21}A_{2k}+\delta_{31}A_{3k}\rightarrow\delta_{11}A_{1k};$$

$k=1$ da $\delta_{11}A_{11}\rightarrow A_{11};$

$k=2$ da $\delta_{11}A_{12}\rightarrow A_{12};$

$k=3$ da $\delta_{11}A_{13}\rightarrow A_{13};$

$j=2$ da $\delta_{ij}A_{ik}=\delta_{12}A_{1k}+\delta_{22}A_{2k}+\delta_{32}A_{3k}\rightarrow\delta_{22}A_{2k};$

$k=1$ da $\delta_{22}A_{21}\rightarrow A_{21};$

$k=2$ da $\delta_{22}A_{22}\rightarrow A_{22};$

$k=3$ da $\delta_{33}A_{23}\rightarrow A_{33};$

$j=3$ da $\delta_{ij}A_{ik}=\delta_{13}A_{1k}+\delta_{23}A_{2k}+\delta_{33}A_{3k};$

$k=1$ da $\delta_{33}A_{31}\rightarrow A_{31};$

$k=2$ da $\delta_{33}A_{32}\rightarrow A_{32};$

$k=3$ da $\delta_{33}A_{33}\rightarrow A_{33}.$

Shunday qilib, quyidagi natijaga kelamiz: $\delta_{ij}A_{ik}=A_{jk}.$

3.7. Ushbu ε_{ijk} - Levi-Chiviti tenzorini indekslari bo‘yicha yoyib yozib, u uchun quyidagi tengliklar o‘rinli ekanligini ko‘rsating: $a)\ \varepsilon_{ijk}\varepsilon_{kij}=6,\ b)\ \varepsilon_{ijk}a_ja_k=0.$

Yechish.

a) Dastlab *i* indeks bo‘yicha yig‘ib chiqamiz:

$$\varepsilon_{ijk}\varepsilon_{kij}=\varepsilon_{1jk}\varepsilon_{k1j}+\varepsilon_{2jk}\varepsilon_{k2j}+\varepsilon_{3jk}\varepsilon_{k3j}.$$

Endi *j* indeks bo‘yicha faqat noldan farqli hadlarni yig‘ib chiqamiz:

$$\varepsilon_{ijk}\varepsilon_{kij}=\varepsilon_{12k}\varepsilon_{k12}+\varepsilon_{13k}\varepsilon_{k13}+\varepsilon_{21k}\varepsilon_{k21}+\varepsilon_{23k}\varepsilon_{k23}+\varepsilon_{31k}\varepsilon_{k31}+\varepsilon_{32k}\varepsilon_{k32}.$$

Oxirida, xuddi shunday, *k* indeks bo‘yicha faqat noldan farqli hadlarni yig‘ib chiqamiz:

$$\varepsilon_{ijk}\varepsilon_{kij}=\varepsilon_{123}\varepsilon_{312}+\varepsilon_{132}\varepsilon_{213}+\varepsilon_{213}\varepsilon_{321}+\varepsilon_{231}\varepsilon_{123}+\varepsilon_{312}\varepsilon_{231}+\varepsilon_{321}\varepsilon_{132}=$$
$$=(1)(1)+(-1)(-1)+(-1)(-1)+(1)(1)+(1)(1)+(-1)(-1)=6.$$

b) Dastlab *j*, keyin esa *k* indeks bo‘yicha faqat noldan farqli hadlarni yig‘ib chiqamiz:

$$\varepsilon_{ijk}a_ka_k=\varepsilon_{i1k}a_1a_k+\varepsilon_{i2k}a_2a_k+\varepsilon_{i3k}a_3a_k=$$
$$=\varepsilon_{i12}a_1a_2+\varepsilon_{i13}a_1a_3+\varepsilon_{i21}a_2a_1+\varepsilon_{i23}a_2a_3+\varepsilon_{i31}a_3a_1+\varepsilon_{i32}a_3a_2.$$

Bu ifodadan quyidagilarni hosil qilamiz:

$i=1$ da $\varepsilon_{1jk}a_ja_k=a_2a_3-a_3a_2=0;$

$i=2$ da $\varepsilon_{2jk}a_ja_k=-a_1a_3+a_3a_1=0;$

$i=3$ da $\varepsilon_{3jk}a_ja_k = a_1a_2 - a_2a_1 = 0.$

Shuni ta'kidlaymizki, ushbu $\varepsilon_{ijk}a_ja_k$ ifoda a vektorni o‘z-o‘ziga vektor ko‘paytirishning indeksli shaklini ifodalaydi va natijada u nolga teng bo‘lib chiqishi zarur: $\vec{a}\times\vec{a}=0.$

3.8. Quyida berilgan vektorlarning f_2 komponentasini hisoblang:

$$a)\ f_i = \varepsilon_{ijk}T_{jk},\ b)\ f_i = \frac{\partial c_i}{\partial x_j}b_j - \frac{\partial c_j}{\partial x_i}b_j.$$

Yechish.

$$a)\ i=2 \text{ da } f_2 = \varepsilon_{2jk}T_{jk} = \varepsilon_{213}T_{13} + \varepsilon_{231}T_{31} = -T_{13} + T_{31}.$$

$$b)\ i=2 \text{ da } f_2 = \frac{\partial c_2}{\partial x_1}b_1 + \frac{\partial c_2}{\partial x_2}b_2 + \frac{\partial c_2}{\partial x_3}b_3 - \frac{\partial c_1}{\partial x_2}b_1 - \frac{\partial c_2}{\partial x_2}b_2 - \frac{\partial c_3}{\partial x_2}b_3 =$$

$$= \left(\frac{\partial c_2}{\partial x_1} - \frac{\partial c_1}{\partial x_2}\right)b_1 + \left(\frac{\partial c_2}{\partial x_3} - \frac{\partial c_3}{\partial x_2}\right)b_3.$$

3.9. Agar $a)\ D_{ij} = D_{ji}$, $b)\ D_{ij} = -D_{ji}$. bo‘lsa, ushbu $D_{ij}x_ix_j$ ifodani yoyib yozing va uni imkoni boricha soddalashtiring.

Yechish. Dastlab *i* indeks bo‘yicha, keyin esa *j* indeks bo‘yicha yigib chiqamiz:

$$D_{ij}x_ix_j = D_{1j}x_1x_j + D_{2j}x_2x_j + D_{3j}x_3x_j =$$

$$= D_{11}x_1x_1 + D_{12}x_1x_2 + D_{13}x_1x_3 + D_{21}x_2x_1 + D_{22}x_2x_2 + D_{23}x_2x_3 +$$

$$+ D_{31}x_3x_1 + D_{32}x_3x_2 + D_{33}x_3x_3.$$

Shuning uchun

$$a)\ D_{ij}x_ix_j = D_{11}(x_1)^2 + D_{22}(x_2)^2 + D_{33}(x_3)^2 + 2D_{12}(x_1x_2) +$$

$$+ 2D_{13}(x_1x_3) + 2D_{23}(x_2x_3).$$

$$b)\ D_{ij}x_ix_j = 0,\ \ D_{11} = -D_{11},\ D_{12} = -D_{21}.$$

3.10. Agar B_{ij} – antisimmetrik va A_{ij} – simmetrik tenzorlar bo‘lsa, u holda $A_{ij}B_{ij}=0$ tenglik o‘rinli. Buni isbotlang.

Yechish. Ma’lumki, agar $A_{ij} = A_{ji}$ va $B_{ij} = -B_{ji}$ bo‘lsa, u holda

$$A_{ij}B_{ij} = -A_{ji}B_{ji} \text{ yoki } A_{ij}B_{ij} + A_{ji}B_{ji} = A_{ij}B_{ij} + A_{pq}B_{pq} = 0.$$

Bu yerda barcha indekslar gung indekslar va $A_{pq}B_{pq} = A_{ij}B_{ij}$, shuning uchun $2A_{ij}B_{ij} = 0$ yoki $A_{ij}B_{ij} = 0.$

3.11. Ushbu $\vec{a} = 3\vec{e}_1 + 4\vec{e}_3$, $\vec{b} = 2\vec{e}_2 - 6\vec{e}_3$ vektorlar va $D = 3\vec{e}_1\vec{e}_1 + 2\vec{e}_1\vec{e}_3 - 4\vec{e}_2\vec{e}_2 - 5\vec{e}_3\vec{e}_2$ diad uchun matritsalarni ko‘paytirish orqali ushbu $\vec{a}\cdot D$, $D\cdot\vec{b}$ va $\vec{a}\cdot D\cdot\vec{b}$ ko‘paytmalarni hisoblang.

Yechish. Faraz qilaylik, $\vec{a}\cdot D = \vec{v};$ u holda

$$(v_1, v_2, v_3) = (3,0,4)\begin{pmatrix} 3 & 0 & 2 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{pmatrix} = (9,-20,6).$$

Faraz qilaylik, $D\cdot\vec{b} = \vec{\omega};$ u holda

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -12 \\ -8 \\ -10 \end{pmatrix}.$$

Faraz qilaylik, $\vec{a}\cdot D\cdot\vec{b} = \vec{\vartheta}\cdot\vec{b} = \lambda;$ u holda

$$(\lambda) = (9,-20,6)\begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix} = (-76).$$

3.12. Dekart koordinatalari sistemasidagi quyidagi matritsa ko‘rinishida berilgan T – ikkinchi rang tenzorning bosh yo‘nalishlarini va bosh qiymatlarini toping

$$T_{ij} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Yechish. Bosh qiymatlarni hisoblash uchun avval quyidagi determinantga ega bo‘lamiz:

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}.$$

Bu determinantni nolga tenglashtirib yechib, quyidagi holat tenglamasiga ega bo‘lamiz:

$$(1-\lambda)((3-\lambda)^2 - 1) = 0.$$

Bu kubik tenglamaning ildizlarini aniqlaymiz:

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = (\lambda - 1)(\lambda - 2)(\lambda - 4) = 0,$$

Bular quyidagilarga teng:

$$\lambda_{(1)} = 1, \quad \lambda_{(2)} = 2, \quad \lambda_{(3)} = 4.$$

Faraz qilaylik, $n_i^{(1)} -$ bosh yo‘nalishning $\lambda_{(1)} = 1$ ga mos birlik vektorlari komponentalari bo‘lsin. Natijada quyidagi tenglamalar sistemasiga ega bo‘lamiz:

$$\left.\begin{array}{c} 2n_1^{(1)} - n_2^{(1)} = 0 \\ -n_1^{(1)} + 2n_2^{(1)} = 0 \\ 0 = 0 \end{array}\right\},$$

Bu yerdan $n_1^{(1)} = n_2^{(1)} = 0$ va $n_i n_i = 1$ shartdan esa $n_3^{(1)} = \pm 1$ tenglikka ega bo‘lamiz.

Xuddi shunday, $\lambda_{(2)} = 2$ uchun sistema quyidagicha yoziladi:

$$\left.\begin{array}{c} n_1^{(2)} - n_2^{(2)} = 0 \\ -n_1^{(2)} + n_2^{(2)} = 0 \\ -n_3^{(2)} = 0 \end{array}\right\},$$

bu yerdan $n_3^{(2)} = 0,$ hamda $n_i n_i = 1$ ekanligidan esa $n_1^{(2)} = n_2^{(2)} = \pm \frac{1}{\sqrt{2}}.$

Bu jarayonni davom ettirib, $\lambda_{(3)} = 4$ uchun

$$\left.\begin{array}{c} -n_1^{(3)} - n_2^{(3)} = 0 \\ -n_1^{(2)} + n_2^{(3)} = 0 \\ -3n_3^{(3)} = 0 \end{array}\right\},$$

bu yerdan $n_3^{(3)} = 0,$ hamda $n_i n_i = 1$ ekanligidan esa $n_1^{(3)} = -n_2^{(3)} = \pm \frac{1}{\sqrt{2}}.$

x_i^* - bosh o‘qlarning x_i - dastlabki sistemaga nisbatan oriyentatsiyasi quyidagi jadvalda berilgan yo‘naltiruvchi kosinuslar orqali aniqlanadi:

	x_1	x_2	x_3
x_1^*	0	0	± 1
x_2^*	$\pm \frac{1}{\sqrt{2}}$	$\pm \frac{1}{\sqrt{2}}$	0

χ_3^*	$\mp\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0

Bu yerdan ko‘rinadiki, almashtirish tenzorining matritsasi quyidagicha yoziladi:

$$A=\begin{pmatrix} 0 & 0 & \pm 1 \\ \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} & 0 \\ \mp\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

3.13. Yuqoridagi 1.14 masaladagi T_{ij} tenzor matritsasi ushbu $T_{ij}^*=a_{ip}a_{jp}T_{pq}$ almashtirish orqali diagonal (bosh) ko‘rinishga keltirilishi mumkinligini ko‘rsating.

Yechish.

$$T_{ij}^*=\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 0 & 0 & 1 \\ \sqrt{2} & \sqrt{2} & 0 \\ -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

3.14. To‘g‘ri burchakli dekart koordinatalari sistemasi x, y, z da $\vec{a}=2\vec{i}+3\vec{j}-\vec{k}$ vektor berilgan. Yangi x', y', z' koordinatalar sistemasida bu vektorning komponentalarini aniqlang, bunda o‘qlarning yo‘nalishlari quyidagi jadvalda keltirilgan yo‘naltiruvchi kosinuslar bilan berilgan:

	x	y	z
x'	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0
y'	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0
z'	0	0	1

Yechish. Ushbu $a_{i'} = \alpha_{i'i} a_i$ formulaga asosan

$$a'_x = 2 \cdot \left(\frac{\sqrt{3}}{2} \right) + 3 \cdot \left(\frac{1}{2} \right) + (-1) \cdot 0 = \sqrt{3} + \frac{3}{2};$$

$$a'_y = 2 \cdot \left(-\frac{1}{2} \right) + 3 \cdot \left(\frac{\sqrt{3}}{2} \right) + (-1) \cdot 0 = \frac{3\sqrt{3}}{2} - 1;$$

$$a'_z = 2 \cdot (0) + 3 \cdot (0) + (-1) \cdot 1 = -1.$$

Endi $\vec{a}$ vektorni yangi x', y', z' koordinatalar sistemasida yozamiz:

$$\vec{a}' = \left(\sqrt{3} + \frac{3}{2} \right) \vec{i} + \left(\frac{3\sqrt{3}}{2} - 1 \right) \vec{j} - \vec{k}.$$

3.15. To‘g‘ri burchakli dekart koordinatalari sistemasi x, y, z da σ_{ij} - ikkinchi rang tenzor berilgan:

$$\sigma_{ij} = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

Yangi x', y', z' koordinatalar sistemasida bu tenzorning komponentalarini aniqlang, bunda o‘qlarning yo‘nalishlari quyidagi jadvalda keltirilgan yo‘naltiruvchi kosinuslar bilan berilgan:

	x	y	z
x'	0	0	1
y'	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
z'	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0

Yechish. Ushbu $a_{i'j'} = \alpha_{i'i} \alpha_{j'j} a_{ij}$ formulaga asosan

$$\sigma'_{11} = \alpha_{11}\alpha_{11}\sigma_{11} + \alpha_{12}\alpha_{11}\sigma_{21} + \alpha_{13}\alpha_{11}\sigma_{31} + \alpha_{11}\alpha_{12}\sigma_{12} + \alpha_{12}\alpha_{12}\sigma_{22} + \alpha_{13}\alpha_{12}\sigma_{32} +$$
$$+ \alpha_{11}\alpha_{13}\sigma_{13} + \alpha_{12}\alpha_{13}\sigma_{23} + \alpha_{13}\alpha_{13}\sigma_{33} = 0 \cdot 0 \cdot 7 + 0 \cdot 0 \cdot 0 + 1 \cdot 0 \cdot (-2) + 0 \cdot 0 \cdot 0 +$$
$$+ 0 \cdot 0 \cdot 5 + 1 \cdot 0 \cdot 0 + 0 \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 0 + 1 \cdot 1 \cdot 4 = 4;$$

$$\sigma'_{12}=\alpha_{11}\alpha_{21}\sigma_{11}+\alpha_{12}\alpha_{21}\sigma_{21}+\alpha_{13}\alpha_{21}\sigma_{31}+\alpha_{11}\alpha_{22}\sigma_{12}+\alpha_{12}\alpha_{22}\sigma_{22}+\alpha_{13}\alpha_{22}\sigma_{32}+$$
$$+\alpha_{11}\alpha_{23}\sigma_{13}+\alpha_{12}\alpha_{23}\sigma_{23}+\alpha_{13}\alpha_{23}\sigma_{33}=0\cdot\left(\frac{1}{\sqrt{2}}\right)\cdot 7+0\cdot\left(\frac{1}{\sqrt{2}}\right)\cdot 0+1\cdot\left(\frac{1}{\sqrt{2}}\right)\cdot(-2)+$$
$$+0\cdot\left(-\frac{1}{\sqrt{2}}\right)\cdot 0+0\cdot\left(-\frac{1}{\sqrt{2}}\right)\cdot 5+1\cdot\left(-\frac{1}{\sqrt{2}}\right)\cdot 0+0\cdot 0\cdot(-2)+0\cdot 0\cdot 0+1\cdot 0\cdot 4=-\sqrt{2}$$

va hokazo. Natijada

$$\sigma'_{ij}=\begin{pmatrix}4 & -\sqrt{2} & -\sqrt{2}\\ -\sqrt{2} & 6 & 1\\ -\sqrt{2} & 1 & 6\end{pmatrix}.$$

Mustaqil yechish uchun masalalar

3.16. Quyidagi T simmetrik tenzor uchun uning bosh qiymatlarini va bosh o‘qlarining yo‘nalishlarini aniqlang:

$$T_{ij}=\begin{pmatrix}7 & 3 & 0\\ 3 & 7 & 4\\ 0 & 4 & 7\end{pmatrix}.$$

3.17. Agar $\vec{\nabla}\cdot\vec{v}=0,\ \vec{\nabla}\times\vec{v}=\dot{\vec{w}}$ va $\vec{\nabla}\times\vec{v}=-\vec{v}$ bo‘lsa, $\Delta\vec{v}=\ddot{\vec{v}}$ ekanligini ko‘rsating.

3.18. Ushbu $Ox_1'x_2'x_3'$ va $Ox_1x_2x_3$ koordinatalar sistemasini bog‘lovchi almashtirish quyidagi jadval bilan berilgan:

	x_1	x_2	x_3
x_1'	$+\frac{3}{5\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5\sqrt{2}}$
x_2'	$\frac{4}{5}$	0	$-\frac{3}{5}$
x_3'	$\frac{3}{5\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{4}{5\sqrt{2}}$

a) bular uchun ortogonallik sharti bajarilishini ko‘rsating.

b) radius-vektori $\vec{r}=2\vec{e}_1-\vec{e}_3$ bo‘lgan nuqtaning shtrixli koordinatalar sistemasidagi koordinatalarini toping. Javob: $\frac{2}{5\sqrt{2}},\ \frac{11}{5},\ -\frac{2}{5\sqrt{2}}$.

3.19. Quyidagi tenzor belgilarni yoyib yozing:

$$A=\sigma_{ij}\varepsilon_{ij}, \quad I=\sigma_{ik}\sigma_{ik}, \quad K_i=A_{ij}B_j, \quad K_{ij}=A_{ijk}B_k,$$

$$T=(0{,}5S_{ij}S_{ij})^{0{,}5}, \quad T=(2\varepsilon_{ij}\varepsilon_{ij})^{0{,}5}.$$

3.20. Quyida ikkinchi rangli ikkita simmetrik tenzorlar berilgan:

$$A=\begin{pmatrix} x+1 & a & b \\ a & 2^x & c \\ b & c & 3x-2 \end{pmatrix}, \quad B=\begin{pmatrix} 3 & x & 2x \\ x & 4 & 3x \\ 2x & 3x & 4 \end{pmatrix}.$$

x ning qanday qiymatida A va B tenzorla o‘zaro teng? Bunda a, b va c ning qiymati nimaga teng?

3.21. Ushbu $\dfrac{\partial\sigma_{ij}}{\partial x_j}=0$ kuchlarning statik muvozanat tenglamasini yoyib yozing.

3.22. Uch o‘lchovli fazoda ushbu $A=B_{ij}n_in_j$ tenglamani yoyib yozing.

3.23. To‘g‘ri burchakli dekart koordinatalari sistemasi (x, y, z) da qiya tekislik uchun to‘la kuchlanishning tashkil etuvchilari quyidagicha yoziladi (Koshi tenglamasi):

$$S_x=\sigma_{xx}n_x+\sigma_{xy}n_y+\sigma_{xz}n_z,$$

$$S_y=\sigma_{yx}n_x+\sigma_{yy}n_y+\sigma_{yz}n_z,$$

$$S_z=\sigma_{zx}n_x+\sigma_{zy}n_y+\sigma_{zz}n_z.$$

Bularni tenzor shaklida yozing.

3.24. Quyidagi formulalarni tenzor shaklida yozing:

$$(\sigma_{xx}-\sigma)n_x+\sigma_{xy}n_y+\sigma_{xz}n_z=0,$$

$$\sigma_{yx}n_x+(\sigma_{yy}-\sigma)n_y+\sigma_{yz}n_z=0,$$

$$\sigma_{zx}n_x+\sigma_{zy}n_y+(\sigma_{zz}-\sigma)n_z=0.$$

3.25. Uch o‘lchovli fazoda δ_{ij} - Kronekker belgisini o‘z ichiga olgan quyidagi ifodalarni hisoblang: $\varepsilon_{ij}\delta_{ij}$, $\frac{1}{3}\sigma_{ij}\delta_{ij}$, $A_j\delta_{ij}$, $\delta_{ii}C_{jj}$.

3.26. Quyidagi tenzor miqdorlarning rangini aniqlang: $a_{ij}b_j$, F_{ikk}, A_{ijlp}, $\sigma_{ij}u_kv_k$.

3.27. Agar $D=3\vec{i}\vec{i}+2\vec{j}\vec{j}-\vec{j}\vec{k}+5\vec{k}\vec{k}$ va $F=4\vec{i}\vec{k}+6\vec{j}\vec{j}-3\vec{k}\vec{j}+\vec{k}\vec{k}$ bo‘lsa, $G=D\cdot F$ diadikni aniqlang.

3.28. Rangi ikkiga ten antisimmetrik A tenzor va ixtiyoriy $\vec{b}$ vektor uchun ushbu $2\vec{b}\cdot A = A_V \times \vec{b}$ formula o‘rinli ekanligini ko‘rsating.

3.29. Faraz qilaylik, x_1, x_2, x_3 koordinatalar sistemasida quyidagi ikkinchi rangli tenzor berilgan bo‘lsin:

$$T = \begin{pmatrix} a & 20 & 30 \\ 20 & b & 40 \\ 30 & 40 & c \end{pmatrix}.$$

Ushbu tenzorning komponentalarini berilgan sistemani x_3 o‘qi atrifida $\pi/6$ burchakka burish natijasida hosil bo‘lgan yangi x'_1, x'_2, x'_3 koordinatalar sistemasida aniqlang, bunda o‘qlarning yo‘nalishlari bo‘yicha yo‘naltiruvchi kosinuslari quyidagi jadvalda berilgan:

	x_1	x_2	x_3
x_1'	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0
x_2'	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0
x_3'	0	0	1

Sinov savollari

1. Skalyar deb nimaga aytiladi? Misollar keltiring.
2. Vektor deb nimaga aytiladi? Misollar keltiring.
3. Vektorning komponentalari deganda nimani tushunasiz?
4. Skalyar va vektor maydonlar nima? Misollar keltiring.
5. Tenzor deb nimaga aytiladi? Tenzorning rangi deganda nimani tushunasiz?
6. Tenzorning rangiga qarab uning komponentalari soni qanday aniqlanadi?
7. Vektorning ortalari nima? Ularni koordinat fazoda geometrik tasvirlang.
8. Radius-vektor nima va u qanday yoziladi? Uni koordinat fazoda geometrik tasvirlang.
9. Vektorning moduli va komponentalarini ayting. Ularni koordinat fazoda geometrik tasvirlang.
10. Vektorlarning skalyar ko‘paytmasi deb nimaga aytiladi?
11. Skalyar va vektorning tenzor bila nima aloqasi bor?
12. Vektor miqdorlarni qo‘shishni tushuntiring va geometrik tasvirlang.

13. Ikki vektorning skalyar ko‘paytmasini tushuntiring va geometrik tasvirlang.
14. Vektorni skalyarga ko‘paytirishni tushuntiring va geometrik tasvirlang.
15. Ikki vektorning vektor ko‘paytmasi deb nimaga aytiladi. Uni koordinat fazoda geometrik tasvirlang.
16. Vektor-skalyar ko‘paytmani tushuntiring va geometrik tasvirlang.
17. Diad deb nimaga aytiladi? Diadning xossalarini ayting.
18. Birlik diad va uning xossalarini ayting.
19. Diadning vektor bilan vektor ko‘paytmasini tushuntiring.
20. Diadlarning skalyar ko‘paytmasini tushuntiring.
21. O‘zaro teskari, qo‘shma, simmetrik va antisimmetrik diadlar nima? Ularning xossalarini ayting.
22. Gung indeklarni va ularni yoyib yozishni tushuntiring.

4. Fundamental metrik tenzor. Kovariant va kontravariant bazis vektorlar. Kovariant hosila. Kovariant hosilaning xossalari. Kristoffel simvollarini hisoblash. Riman-Kristofel tenzori

Fundamental metrik tenzor. Kovariant va kontravariant bazis vektorlar. Bundan oldingi ma'ruzada keltirilgan mulohazalar fazoning bitta ixtiyoriy, ammo fiksirlangan nuqtasiga oid edi. Bunday mulohazalarni butun fazoga yoyish uchun nuqtaning ixtiyoriyligidan tashqari qaralayotgan fazo uchun metrika (o'lchov) tushunchasini ham kiritish zarur bo'ladi. Ma'lumki, fazoning metrikasi deganda odatda shu fazoda uzunlikni aniqlash usuli tushuniladi. Biror vektorning uzunligini aniqlash uchun uning o'zini-o'ziga skalyar ko'paytirish yetarli, ya'ni

$$\left|d\,\vec{r}\right|^2 = ds^2 = d\vec{r}\cdot d\vec{r} = d\zeta^i \vec{э}_i d\zeta^j \cdot \vec{э}_j = d\zeta^i d\zeta^j \vec{э}_i \vec{э}_j \,.$$

Bu yerdagi $\vec{э}_i$ bazis vektorlarining skalyar ko'paytmasini g_{ij} lar orqali belgilaymiz, ya'ni

$$g_{ij} = \vec{э}_i\, \vec{э}_j \,.$$

U holda $d\vec{r}$ vektorning uzunligi uchun

$$\left|d\vec{r}\right|^2 = d\zeta^i d\zeta^j g_{ij} \tag{4.1}$$

formulaga ega bo'lamiz. Kiritilgan yangi g_{ij} kattaliklar yordamida ixtiyoriy $\vec{A}$ vektorning uzunligini quyidagicha yozish mumkin:

$$\left|\vec{A}\right|^2 = A^i A^j g_{ij} \,.$$

Ushbu ifoda istalgan vektorning uzunligini uning komponentalari va bazis vektorlarining skalyar ko'paytmasi orqali ifodalashga imkon beradi.

Vektorning $\left|d\vec{r}\right|$ uzunligi koordinat sistemasini tanlashga nisbatan invariantdir. Ushbu dalilni o'tgan ma'ruzada ham ta'kidlagan edik va bunday invariantlik ifodasi (2.21) dan iborat edi. Ana shu ifodaga asosan vektorning uzunligi

$$\left|d\vec{r}\right|^2 = g_{ij} d\zeta^i d\zeta^j = g'_{pq} d\eta^p d\eta^q = э'_p э'_q \cdot d\eta^p d\eta^q =$$
$$= a_p^i a_q^j э_i\, э_j\, d\eta^p d\eta^q = g_{ij} a_p^i a_q^j d\eta^p d\eta^q$$

ko'rinishni oladi. Bundan

$$g'_{pq} = g_{ij} a_p^i a_q^j \tag{4.2}$$

ya'ni kiritilgan g_{pq} - kattaliklar kovariant yo'l bilan almashtiriladilar. Ko'rinib turibdiki g_{ij} kattaliklar uchinchi tartibli matrisani tashkil qiladilar. Bu matrisaning determinanti noldan farqli bo'lishini talab qilamiz, ya'ni

$$\Delta = Det\|g_{ij}\| \neq 0$$

bo‘lsin. U holda $\|g_{ij}\|$ ga teskari $\|g^{ij}\|$ matrisa mavjud bo‘ladi va algebra kursidan malumki uning elementlari

$$g^{ij} = k^{ij} / \Delta \tag{4.3}$$

formuladan topiladi, bu yerda k^{ij} -$\|g_{ij}\|$ matrisaning to‘ldiruvchi minorlari, g_{ij} kattaliklar (4.2) formulaga asosan kovariant yo‘l bilan almashtiriladilar. U holda g^{ij} kattaliklar xuddi ikkinchi rang tenzorning T^{ij} komponentalari singari (2.24) formulaga ko‘ra kontravariant yo‘l bilan almashtiriladilar, ya’ni

$$g^{'ij} = b_p^i b_q^j g^{pq} \tag{4.4}$$

Hosil qilingan g^{ij} kattaliklari va $\vec{э}_i$ bazis vektorlari yordamida

$$g = g^{ij}\ \vec{э}_i\ \vec{э}_j$$

ikkinchi rang tenzorga ega bo‘lamiz, hamda biror ζ^i koordinatalari sistemasida

$$\vec{э}^i = g^{ij} \vec{э}_j \tag{4.5}$$

obyektlarni kiritamiz. Bu yerda masalan ixtiyoriy $g^{1i}\ \vec{э}_i$ vektor quyidagiga teng

$$g^{1j}\vec{э}_j = g^{11}\vec{э}_1 + g^{12}\vec{э}_2 + g^{13}\vec{э}_3 = \vec{э}^1$$

ya’ni g^{1j} larga ko‘paytirilgan uchta $\vec{э}_1, \vec{э}_2\ va\ \vec{э}_3$ bazis vektorlarining yig‘indisidan iborat.

Shunga o‘xshash boshqa η^1, η^2, η^3 koordinatalari sistemasida ham

$$\vec{э}^{'p} = g^{'pq}\vec{э}_k^{'}$$

kabi ifodani qabul qilish mumkin. Oxirgi (4.4), (4.5) va (2.19) formulalarga asosan $\vec{э}_i$ bazis vektorlarini almashtirish formulalarini keltirib chiqaramiz

$$\vec{э}^{'p} = g^{'pq}\vec{э}_q^{'} = b_i^p b_j^q g^{ij} \cdot a_q^x \cdot \vec{э}_k = b_i^p \left(b_j^q \cdot a_q^k\right) g^{ij} \cdot \vec{э}_k = b_i^p \cdot g^{ij}\ \vec{э}_j = b_i^p \vec{э}^i$$

chunki

$$b_j^q \cdot a_q^k = \delta_j^k,$$

demak

$$\vec{э}^{'p} = b_i^p \vec{э}^i\ . \tag{4.6}$$

Ko‘rinib turibdiki, $\vec{э}^i$ bazis vektorlari kontravariant yo‘l bilan almashtiriladilar va shuning uchun ham *kontravariant bazis vektorlari* deb ataladilar. Mos ravishda $\vec{э}_i$ bazis vektorlari *kovariant bazis vektorlari* deb yuritiladi.

Yuqorida aytilganlardan ma’lumki $\|g^{ij}\|$ matrisa $\|g_{ij}\|$ matrisaga teskari matrisadir. Shuni hisobga olgan holda (4.5) ifodani $\vec{э}_i$ larga nisbatan yechib, ushbu

$$\vec{э}_j = g_{ij}\vec{э}^i \tag{4.7}$$

ifodaga ega bo‘lamiz. Xuddi shunga o‘xshash, ixtiyoriy boshqa η^1, η^2, η^3 koordinatalari sistmasida ham ushbu

$$\vec{э}'_j = g'_{ij}\vec{э}'^i$$

formula o‘rinli bo‘ladi va bu yerdagi g'_{ij} kattaliklar (4.2) formula yordamida almashtiriladilar.

Ko‘rinib turibdiki, $g_{ij}\vec{э}^i\ \vec{э}^j$ ifoda koordinata sistemasiga bog‘liq bo‘lmagan invariant obyekt bo‘ladi, chunki bu yerdagi $\vec{э}^i\ \vec{э}^j$ ko‘paytma $\vec{э}^i$ kontravariant bazis vektorlarining diad ko‘paytmalari va shuning uchun ham (4.6) formulaga asosan

$$\vec{э}'^i\vec{э}'^j = b^i_k b^j_l \cdot \vec{э}^k\vec{э}^l \tag{4.8}$$

kontravariant yo‘l bilan almashtiriladilar. Bundan tashqari

$$g = g^{ij}\vec{э}_i\vec{э}_j = g^{ij}g_{ip}\vec{э}^p g_{jq}\vec{э}^q = g_{ip}g_{jq}g^{ij}\vec{э}^p\vec{э}^q = g_{pq}\vec{э}^p\vec{э}^q$$

bo‘ladi, ya’ni qaralayotgan obyekt ikkinchi rang tenzorni tashkil etadi.

$$g = g_{pq}\vec{э}^p\vec{э}^q$$

Shunday qilib,

$$g = g_{ij}\vec{э}^i\vec{э}^j = g^{pq}\vec{э}_p\vec{э}_q \tag{4.9}$$

Hosil qilingan g- tenzor fundamental metrik tenzor deb ataladi, g_{ij} kattaliklar-fundamental metrik tenzorning $\vec{э}^i$ kontravariant bazisdagi kovariant komponentalari, g^{ij} kattaliklar esa fundamental metrik tenzorning $\vec{э}_i$ kovariant bazisdagi kontravariant komponentalari deyiladi.

Joyi kelganda biz o‘quvchini Evklid fazosining chuqurroq ta’rifi bilan tanishtirib o‘tishni lozim deb hisoblaymiz. Yuqoridagi mulohazalarni fazoning fiksirlangan nuqtasi uchun olib bordik. *Bu holda (3.1) kvadratik shaklning koeffisiyentlari o‘zgarmas bo‘ladi.* Algebra kursidan ma’lumki, *har qanday o‘zgarmas koeffisiyentli kvadratik shaklni kanonik ko‘rinishga keltirish mumkin, ya’ni fazoning har bir tanlangan nuqtasi uchun* shunday ζ^1, ζ^2, ζ^3 koordinatalarni topish mumkinki, bunda (3.1) kvadratik shakl

$$ds^2 = (d\zeta^1)^2 + (d\zeta^2)^2 + (d\zeta^3)^2 \tag{4.10}$$

ko‘rinishga, fundamental metrik tenzorning matrisasi esa quyidagi ko‘rinishga keltiriladi

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

Umuman olganda, bunday ishni fazoning har bir nuqtasi uchun bajarib bo‘lmaydi, ya’ni (4.10) ko‘rinishga keltiradigan $\zeta^1, \zeta^2, \zeta^3$ lar topilmasligi mumkin. Lekin, agar *biror fazoning hamma nuqtalari uchun shunday koordinat sistemasi mavjud bo‘lsa bu fazo Evklid fazosi, aks holda Evklidmas fazo* deyiladi.

Demak, Evklid fazosi uchun fundamental metrik tenzorning matrisasi elementlari 1 lardan iborat bo‘lgan diagonal matrisadir. Bundan tashqari $\|g_{ij}\|$ *ва* $\|g^{kp}\|$ matrisalar o‘zaro teskari matrisalardir.

Aytilganlarni hisobga olgan holda $\vec{э}^i \vec{э}_p$ aralash diad ko‘paytmalarning ba’zi xususiyatlari bilan tanishamiz. Yuqorida keltirilgan (4.5) formulaga asosan

$$\vec{э}^j \vec{э}_p = g^{ij} \vec{э}_i \vec{э}_p = g^{ij} g_{ip} = \delta_p^j = \begin{cases} 1, & p = j; \\ 0, & p \neq j. \end{cases} \qquad (4.11)$$

bu yerdan

$$\vec{э}^1 \cdot \vec{э}_1 = 1,\ \vec{э}^1 \cdot \vec{э}_2 = 0,\ \vec{э}^1 \cdot \vec{э}_3 = 0,\ \vec{э}^2 \cdot \vec{э}_1 = 0,\ \vec{э}^2 \cdot \vec{э}_2 = 1, \quad va \quad h.k.$$

Bu tengliklar $\vec{э}^1$ vektorining $\vec{э}_2$ *va* $\vec{э}_3$ vektorlari tekisligiga, $\vec{э}_2$ vektorning $\vec{э}^1$ *va* $\vec{э}^3$ vektorlari tekisligiga va h.k. ortogonalligini ko‘rsatadi. Ana shu faktlar asosida quyidagi munosabatlarni isbot qilish qiyin emas:

a) kontravariant bazis vektorlari uchun

$$\vec{э}^1 = \frac{\vec{э}_2 \times \vec{э}_3}{\vec{э}_1(\vec{э}_2 \times \vec{э}_3)}; \quad \vec{э}^2 = \frac{\vec{э}_3 \times \vec{э}_1}{\vec{э}_1(\vec{э}_2 \times \vec{э}_3)}; \quad \vec{э}^3 = \frac{\vec{э}_1 \times \vec{э}_2}{\vec{э}_1(\vec{э}_2 \times \vec{э}_3)};$$

b) kovariant bazis vektorlari uchun

$$\vec{э}_1 = \frac{\vec{э}^2 \times \vec{э}^3}{\vec{э}^1(\vec{э}^2 \times \vec{э}^3)}; \quad \vec{э}_2 = \frac{\vec{э}^3 \times \vec{э}^1}{\vec{э}^1(\vec{э}^2 \times \vec{э}^3)}; \quad \vec{э}_3 = \frac{\vec{э}^1 \times \vec{э}^2}{\vec{э}^1(\vec{э}^2 \times \vec{э}^3)};$$

bu yerda “$\times$” belgisi bilan oddiy vektor ko‘paytma belgilangan.

Fundamental metrik tenzordagi kabi bundan oldingi bo‘limlarda kiritilgan vektor va tenzorning ta’riflarida ishlatilgan A^i *va* T^{ij} *komponentalar vektorning va tenzorning kovariant bazisdagi kontravariant komponentalari deyiladi.*

Endi vektor va tenzorning kovariant komponentalarini kiritish masalasi bilan tanishamiz. Buning uchun (4.7) formuladan foydalanamiz. Ushbu formulaga asosan ixtiyoriy vektorni

$$\vec{A} = A^i \vec{э}_i = A^i g_{ij} \vec{э}^j = A_j \vec{э}^j$$

kabi yozish mumkin. Bu yerda

$$A_j = A^i g_{ij} \tag{4.12}$$

kattaliklar $\vec{A}$ *vektorining* $\vec{э}^j$ *kontravariant bazisdagi kovariant komponentalari deyiladi*. Demak, bu kattaliklar kovariant yo‘l bilan almashtiriladilar.

Umumiy holda $A^i \neq A_i$. Oxirgi (4.12) formuladan ko‘rinadiki metrik tenzorning g_{ij} komponentalari yordamida vektorning A^i kontravariant komponentalarining *indeksini pastga tushirish* mumkin. Xuddi shunday g^{ij} komponentalar yordamida A_j larning *indeksini yuqoriga ko‘tarish* mumkin ekan. Vektor uchun qilingan mulohazalarni tenzor uchun ham qo‘llash mumkin, ya’ni

$$T = T^{ijkl} э_i э_j э_k э_l = T^{ijkl} g_{ip} э^p g_{jq} э^q g_{km} э^m g_{\ln} э^n = T_{pqmn} g_{ip} э^p э^q э^m э^n,$$

bu yerda

$$T_{pqmn} = T^{ijkl} g_{ip} g_{jq} g_{km} g_{\ln} \tag{4.13}$$

kattaliklar to‘rtinchi rang T tenzorning $\vec{э}^i$ kontravariant bazisdagi kovariant komponentalari deyiladi. Qaralayotgan T tenzor uchun (4.7) formulani qisman qo‘llab

$$T = T^{ijkl} э_i э_j э_k э_l = T^{ijkl} э_i э_j g_{kp} э^p \cdot g_{lq} э^q = T^{ij}_{pq} э_i э_j э^p э^q$$

ifodaga ega bo‘lamiz. Bu yerda

$$T^{ij}_{pq} = T^{ij..}_{..pq} = T^{ijkl} g_{kp} g_{lq} \tag{4.14}$$

kattaliklar tenzorining aralash komponetalari deyiladi. Yuqoridagi (4.11) formulalardan fundamental metrik tenzorning aralash komponentalari g_i^j lar ixtiyoriy koordinat sistemasida birlik matrisani tashkil qilishini ko‘rish qiyin emas

$$\left\| g_i^j \right\| = \begin{Vmatrix} g_1^1 & g_2^1 & g_3^1 \\ g_1^2 & g_2^2 & g_3^2 \\ g_1^2 & g_2^3 & g_3^3 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} = \left\| \delta_i^j \right\|. \tag{4.15}$$

Kovariant hosila. Oldingi ma’ruzalardan ma’lumki ixtiyoriy egri chiziqli ζ^1, ζ^2, ζ^3 koordinatalar sistemasida $\vec{э}_i$ bazis vektorlari o‘zgaruvchi vektorlar bo‘lib, ular ζ^i koordinatalarning funksiyalari boladilar. Shuning uchun $\vec{A} = A^i \vec{э}_i$ vektorning hosilasini hisoblashda bu faktni hisobga olshga to‘g‘ri keladi.

Demak,

$$\frac{\partial \vec{A}}{\partial \zeta^i} = \frac{\partial A^i}{\partial \zeta^j} \vec{э}_i + A^i \frac{\partial \vec{э}_i}{\partial \zeta^j} \tag{4.16}$$

va bu yerdagi $\dfrac{\partial \vec{э}_i}{\partial \zeta^j}$ kattalik yana vektordan iborat bo‘ladi va uni $\vec{э}_k$ bazis vektorlari bo‘yicha yoyish mumkin. Bu yoyilmaning komponentalarini Γ_{ij}^k deb belgilaymiz, ya’ni

$$\frac{\partial \vec{э}_i}{\partial \zeta^j} = \Gamma_{ij}^k \cdot \vec{э}_k \qquad (4.71)$$

ifodaga ega bo‘lamiz. Bu yerdagi Γ_{ij}^k lar koordinatalarning funksiyalari bo‘ladilar va *Kristoffel simvollari* deb ataladilar. Oxirgi (4.17) ifodani (4.16) ga qo‘yib

$$\frac{\partial \vec{A}}{\partial \zeta^j} = \frac{\partial A^i}{\partial \zeta^j}\vec{э}_i + A^i \Gamma_{ij}^k \vec{э}_k \qquad (4.18)$$

ga ega bo‘lamiz. Bu ifodaning o‘ng tomonidagi ikkinchi qo‘shiluvchi *i* va *k* lar bo‘yicha yigindidan iborat ya’ni *i* va *k* lar gung indekslar. Shuning uchun bu yerdagi *i* va *k* larning o‘rinlarini almashtirib

$$\frac{\partial \vec{A}}{\partial \zeta^j} = \left(\frac{\partial A^i}{\partial \zeta^j} + A^k \Gamma_{kj}^i\right)\vec{э}_i \qquad (4.19)$$

ifodaga ega bo‘lamiz. Bu ifodada $\vec{э}_i$ bazis vektorlari oldilaridagi *koeffisieyentlar vektorning kontrovariant komponentalaridan olingan kovariant hosilalar deyiladi* va $\nabla_j A^i$ kabi belgilanadi.

Demak,

$$\nabla_j A^i = \frac{\partial A^i}{\partial \zeta^j} + A^k T_{kj}^i . \qquad (4.20)$$

Quyida $\nabla_j A^i$ hosilalarning xossalari bilan tanishamiz.

Kovariant hosilaning xossalari.

1. Dekart koordinatalari sistemasida *($\xi^i = x^i$)* kovariant hosila vektor komponentalaridan koordinata bo‘yicha olingan oddiy hosila bilan bir xil bo‘ladi. Haqiqatan, $\dfrac{\partial \vec{э}_i}{\partial \xi^j} = \dfrac{\partial \vec{э}_i}{\partial x^j}$ $\quad (\vec{э}_1 = \vec{i}, \quad \vec{э}_2 = \vec{j}, \quad \vec{э}_3 = \vec{k})$

bo‘lganligidan $\Gamma_{kj}^i = 0$, u holda (4.10) dan $\nabla_j A^i = \dfrac{\partial A^i}{\partial \xi^j}$ bo‘ladi.

2. Kovariant $\nabla_j A^i$ hosilalar ikkinchi rang tenzorning komponentalari bo‘ladilar. Haqiqatan, faraz qilaylik, η^i – yangi, ξ^j – eski koordinatalar sistemalari bo‘lsin, u holda

$$\frac{\partial\vec{A}}{\partial\eta^k}=\frac{\partial\vec{A}}{\partial\xi^j}\cdot\frac{\partial\xi^j}{\partial\eta^k}$$

va bu yerdan $\dfrac{\partial\vec{A}}{\partial\eta^k}$ lar vektorning kovariant komponentalari kabi almashtirilishi kelib chiqadi. Shuning uchun $T=\dfrac{\partial\vec{A}}{\partial\xi^j}\vec{э}^j$ ob’ekt invariant ob’ektdir. U holda (4.19) va (4.20) formulalarga asosan,

$$T=\nabla_j A^i\vec{э}_i\vec{э}_j$$

ya’ni T – 2-rang tenzor va bu tenzorning komponentalari $\nabla_j A^i$ lardan iborat.

Ixtiyoriy φ skalyar miqdordan olingan kovariant hosila uning oddiy hosilasi bilan bir xil bo‘ladi:

$$\nabla_i\varphi=\frac{\partial\varphi}{\partial\xi^i}.$$

Endi tenzorning kontravariant komponentalaridan olingan kovariant hosilalar bilan tanishamiz. Ushbu $P=P^{ij}\vec{э}_i\vec{э}_j$ tenzor berilgan bo‘lsin:

$$\frac{\partial P}{\partial\xi^k}=\frac{\partial P^{ij}}{\partial\xi^k}\vec{э}_i\vec{э}_j+P^{ij}\frac{\partial\vec{э}_i}{\partial\xi^k}\vec{э}_j+P^{ij}\vec{э}_i\frac{\partial\vec{э}_j}{\partial\xi^k}=$$

$$=\frac{\partial P^{ij}}{\partial\xi^k}\vec{э}_i\vec{э}_j+P^{ij}\Gamma^l_{ik}\vec{э}_l\vec{э}_j+P^{ij}\vec{э}_i\Gamma^l_{jk}\vec{э}_l. \tag{4.21}$$

Yig‘indining ikkinchi hadida *i* va *l* larning, 3 – hadida esa *j* va *l* larning o‘rinlarini almashtiramiz (shunday qilishga haqqimiz bor, chunki bu indekslar bo‘yicha yig‘indilar hisoblanadi)

$$\frac{\partial P}{\partial\xi^k}=\left(\frac{\partial P^{ij}}{\partial\xi^k}+P^{lj}\Gamma^i_{\ell k}+P^{il}\Gamma^j_{\ell k}\right)\vec{э}_i\vec{э}_j=\nabla_k P^{ij}\vec{э}_i\vec{э}_j,$$

bu yerda

$$\nabla_k P^{ij}=\frac{\partial P^{ij}}{\partial\xi^k}+P^{lj}\Gamma^i_{lk}+P^{il}\Gamma^j_{lk}$$

ifoda ikkinchi rang tenzor P^{ij} kontravariant komponentalarining *kovariant hosilasi* deyiladi. Olingan $\nabla_k P^{ij}$ hosilalar ikkinchi rang tenzorining komponentalari bo‘ladilar, ya’ni

$$G_1=\frac{\partial P}{\partial\xi^k}\vec{э}^k=\nabla_k P^{ij}\vec{э}_i\vec{э}_j\vec{э}^k,$$

yoki

$$G_2=\nabla_k P^{ij}\vec{э}_i\vec{э}^k\vec{э}_j,\quad G_3=\nabla_k P^{ij}\vec{э}^k\vec{э}_i\vec{э}_j$$

Kovariant hosila uchun quyidagi formulalar o‘rinli

$$a)\nabla_k(P^{ij}+q^{mn})=\nabla_k P^{ij}+\nabla_k q^{mn},$$

$$b)\nabla_k(P^{ij}\cdot q^{mn})=(\nabla_k P^{ij})q^{mn}+P^{ij}(\nabla_k q^{mn}).$$

Ma’lumki, $\vec{э}_j\vec{э}^k=\delta_j^k$, bu ifodani koordinata bo‘yicha diffirensiallasak,

$$\frac{\partial\vec{э}_j}{\partial\xi^i}\vec{э}^k+\vec{э}_j\frac{\partial\vec{э}_k}{\partial\xi^i}=\vec{э}^k(\Gamma_{ji}^l\vec{э}_l)+\frac{\partial\vec{э}^k}{\partial\xi^i}\vec{э}_j=0.$$

Oxirgi yig‘indi faqat $k = l$ bo‘lgan holdagina 0 dan farqli. U holda

$$\frac{\partial\vec{э}^k}{\partial\xi^k}\vec{э}_j=-\Gamma_{ji}^k$$

ga ega bo‘lamiz, bu yerdan

$$\frac{\partial\vec{э}^k}{\partial\xi^i}=-\Gamma_{ji}^k\vec{э}_j.$$

Endi (5.1) tenglikdan foydalanib, $\vec{A}=A_i\vec{э}^i$ vektorning A_i kovariant komponentalaridan olingan kovariant hosilalarini topamiz

$$\frac{\partial\vec{A}}{\partial\xi^i}=\frac{\partial A_j}{\partial\xi^i}\vec{э}^j+A_j\frac{\partial э^j}{\partial\xi^i}=\frac{\partial A_j}{\partial\xi^i}\vec{э}^j-A_j\Gamma_{ki}^j\vec{э}^k.$$

Oxirgi yig‘indida j va k larning o‘rinlarini almashtirib,

$$\frac{\partial\vec{A}}{\partial\xi^i}=\left(\frac{\partial A_j}{\partial\xi^i}-\Gamma_{ji}^kA_k\right)\vec{э}^j \tag{4.22}$$

ifodaga ega, bu erda

$$\nabla_iA_j=\frac{\partial A_j}{\partial\zeta^i}-\Gamma_{ji}^kA_k$$

formula vektorning kovariant komponentalaridan olingan kovariant hosilalarini beradi. Xuddi shunga o‘xshash istalgan tenzorning kovariant komponentalaridan olingan kovariant hosilalar tushunchasini kiritish mumkin. Oddiy hisoblashlar yordamida

$$\nabla_ig^{jk}=\nabla_ig_{jk}=0$$

ekanligini isbotlash qiyin emas. Bu esa g_{jk} larning belgisidan tashqariga chiqarish mumkinligini, ya’ni

$$\nabla_i(g^{jk}A_k)=g^{jk}\nabla_iA_k \qquad va \qquad \nabla_i(g_{jk}A^k)=g_{jk}\nabla_iA^k$$

ekanligini ko‘rsatadi.

Kristoffel simvollarini hisoblash. Quyida Kristoffel simvollarini hisoblash masalasi bilan shug‘ullanamiz. Oldindan aytish kerakki, simvollar har qanday tenzorning komponentalari bo‘la olmaydilar. Ma’lumki, Evklid fazosida $\vec{э}_i = \dfrac{\partial \vec{r}}{\partial \eta^i}$ bazis vektorlari kabi aniqlangan edi, bundan

$$\frac{\partial \vec{э}_i}{\partial \eta^j} = \frac{\partial \vec{r}}{\partial \eta^i \partial \eta^j} = \frac{\partial \vec{r}}{\partial \eta^j} = \frac{\partial \vec{э}_j}{\partial \eta^j}. \qquad (4.23)$$

Agar (4.18) ifodani hisobga olsak, (4.23) ni quyidagi ko‘rinishda yozish mumkin:

$$\Gamma_{ij}^k \vec{э}_k = \Gamma_{ij}^k \vec{э}_k$$

Bu tenglik Kristoffel simvollarining pastki indekslar bo‘yicha simmetrikligini, ya’ni $\Gamma_{ij}^k = \Gamma_{ij}^k$ ekanligini ko‘rsatadi. Kristoffel simvollarini fundamental metrik tenzor komponentalari orqali ifodalash mumkin. Buning uchun quyidagi munosabatlar qaraladi:

$$\frac{\partial g_{js}}{\partial \eta^k} = \frac{\partial(\vec{э}_j \vec{э}_s)}{\partial \eta^k} = \frac{\partial \vec{э}_j}{\partial \eta^k}\vec{э}_s + \frac{\partial \vec{э}_s}{\partial \eta^k}\vec{э}_j, \quad \frac{\partial g_{ks}}{\partial \eta^j} = \frac{\partial \vec{э}_k}{\partial \eta^j}\vec{э}_s + \frac{\partial \vec{э}_s}{\partial \eta^j}\vec{э}_k,$$

bu tengliklarni

$$\frac{\partial g_{js}}{\partial \eta^k} - \frac{\partial \vec{э}_s}{\partial \eta^k}\vec{э}_j = \frac{\partial \vec{э}_j}{\partial \eta^k}\vec{э}_s = \Gamma_{jk}^l \vec{э}_l \vec{э}_s = \Gamma_{jk}^l g_{ls}; \quad \frac{\partial g_{ks}}{\partial \eta^j} - \frac{\partial \vec{э}_s}{\partial \eta^j}\vec{э}_k = \frac{\partial \vec{э}_k}{\partial \eta^j}\vec{э}_s = \Gamma_{kj}^l \vec{э}_l \vec{э}_s = \Gamma_{kj}^l g_{ls}$$

ko‘rinishda yozish mumkin. Endi bu ifodalarni hadma-had qo‘shib, Kristoffel simvollarining pastki indekslar bo‘yicha simmetrikligini hisobga olsak,

$$\frac{\partial g_{js}}{\partial \eta^k} - \frac{\partial g_{ks}}{\partial \eta^j} - \left(\frac{\partial \vec{э}_s}{\partial \eta^k}\vec{э}_j + \frac{\partial \vec{э}_s}{\partial \eta^j}\vec{э}_k \right) = 2\Gamma_{jk}^l g_{ls} \qquad (4.24)$$

ifodaga ega bo‘lamiz va (4.23) tenglikdan foydalanib, qavslar ichidagi ifodani quyidagicha o‘zgartiramiz:

$$\frac{\partial \vec{э}_s}{\partial \eta^k}\vec{э}_j + \frac{\partial \vec{э}_s}{\partial \eta^j}\vec{э}_k = \frac{\partial \vec{э}_k}{\partial \eta^s}\vec{э}_j + \frac{\partial \vec{э}_j}{\partial \eta^s}\vec{э}_k = \frac{\partial g_{jk}}{\partial \eta^s}.$$

Bu tengliklarni hisobga olgan holda (4.24) ning ikkala tomonini $\frac{1}{2}g^{si}$ ga ko‘paytirsak,

$$\frac{1}{2}g^{si}\left(\frac{\partial g_{js}}{\partial \eta^k} + \frac{\partial g_{ks}}{\partial \eta^j} - \frac{\partial g_{jk}}{\partial \eta^s} \right) = \Gamma_{jk}^l g_{ls} g^{si} = \Gamma_{jk}^l \delta_l^i$$

ifodaga ega bo‘lamiz, bu yerdan izlanayotgan quyidagi munosabat kelib chiqadi:

$$\Gamma_{jk}^i = \frac{1}{2}g^{si}\left(\frac{\partial g_{js}}{\partial \eta^k} + \frac{\partial g_{ks}}{\partial \eta^j} - \frac{\partial g_{jk}}{\partial \eta^s} \right). \qquad (4.25)$$

Quyidagi koordinatalar sistemasini almashtirganda Kristoffel simvollarini almashtirish formulasini isbotsiz keltiramiz:

$$\Gamma'^{\gamma}_{i\,j} = \left(\Gamma^{\omega}_{\alpha\,\beta} \frac{\partial \xi^{\alpha}}{\partial \eta^{i}} \cdot \frac{\partial \xi^{\beta}}{\partial \eta^{j}} + \frac{\partial^2 \xi^{\omega}}{\partial \eta^{j} \partial \eta^{j}} \right) \frac{\partial \eta^{\gamma}}{\partial \xi^{\omega}},$$

bu yerda $\Gamma'^{k}_{i\,j} - \eta^{i}$ koordinatalar sistemasidagi Γ^{k}_{ij} lar esa $\xi^{\,i}$ koordinatalar sistemasidagi Kristoffel simvollaridir.

Riman – Kristoffel tenzori va uning xossalari. Evklid fazosi uchun yaroqli bo‘lgan Dekart koordinatalari sistemasini kiritish mumkinki, bunday sistemada fundamental metrik tenzorning komponentalari o‘zgarmas bo‘ladi, ya’ni $g_{ik} = const$. U holda bu fazoning har bir nuqtasi uchun $\Gamma'^{\gamma}_{ij} = 0$. Agar fazo Riman fazosi bo‘lsa, o‘tish matrisasining determenanti 0 dan farqli, ya’ni

$$Det \left\| \frac{\partial \eta^{k}}{\partial \xi^{\omega}} \right\| \neq 0$$

bo‘lganligi uchun (4.25) formulaga asosan,

$$\Gamma^{\omega}_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial \eta^{i}} \cdot \frac{\partial \xi^{\beta}}{\partial \eta^{j}} + \frac{\partial^2 \xi^{\omega}}{\partial \eta^{i} \partial \eta^{j}} = 0 \qquad (4.26)$$

tenglik bajarilishi kerak bo‘ladi. Demak, (4.26) tenglamalar tenglik bajariladigan koordinatalar sistemasini aniqlovchi tenglamalar sifatida qaralishi mumkin. Agar fazoning Evklid fazosi bo‘lishligini talab qilinsa, (4.26) tenglamalar sistemasi bilan $\xi^{\,\alpha}$ koordinatalar sistemasini $\eta^{\,i}$ Dekart koordinatalar sistemasi bilan almashtirish diffirensial tenglamalar sistemasi bo‘ladilar. Umumiy holda bu sistemani integrallab bo‘lmaydi. Shunday bo‘lsada fazoning Evklid fazosi bo‘lishligi sharti (4.26) tenglamalar sistemasining integrallanish sharti bilan bir xil bo‘ladi. Bu shartlarni esa topish mumkin. Buning uchun (4.26) tenglamalarni η^{k} koordinatalar boyicha differensiallaymiz

$$\frac{\partial \Gamma^{\omega}_{\alpha\beta}}{\partial \eta^{k}} \cdot \frac{\partial \xi^{\alpha}}{\partial \eta^{i}} \cdot \frac{\partial \xi^{\beta}}{\partial \eta^{j}} + \Gamma^{\omega}_{\alpha\beta} \frac{\partial^2 \xi^{\alpha}}{\partial \eta^{i} \partial \eta^{k}} \cdot \frac{\partial \xi^{\beta}}{\partial \eta^{j}} + \Gamma^{\omega}_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial \eta^{i}} \cdot \frac{\partial^2 \xi^{\beta}}{\partial \eta^{j} \partial \eta^{k}} + \frac{\partial^3 \xi^{\omega}}{\partial \eta^{i} \partial \eta^{j} \partial \eta^{k}} = 0$$

va (4.26) ga asosan ikkinchi tartibli hosilalarni quyidagicha almashtiramiz

$$\frac{\partial^2 \xi^{\alpha}}{\partial \eta^{i} \partial \eta^{k}} = -\Gamma^{\alpha}_{\lambda\mu} \cdot \frac{\partial \xi^{\lambda}}{\partial \eta^{i}} \cdot \frac{\partial \xi^{\mu}}{\partial \eta^{k}}; \qquad \frac{\partial^2 \xi^{\beta}}{\partial \eta^{i} \partial \eta^{k}} = -\Gamma^{\beta}_{\lambda\mu} \cdot \frac{\partial \xi^{\lambda}}{\partial \eta^{i}} \cdot \frac{\partial \xi^{\mu}}{\partial \eta^{k}}; \qquad (4.27)$$

va

$$\frac{\partial \Gamma_{\alpha\beta}{}^{\omega}}{\partial \eta^{k}} = \frac{\partial \Gamma_{\alpha\beta}{}^{\omega}}{\partial \eta^{\mu}} \cdot \frac{\partial \xi^{\mu}}{\partial \eta^{k}}$$

ekanligidan foydalanamiz. U holda (4.27) ifodalarning birinchisida α va λ larning, 2-sida esa μ va λ larning orinlarini almashtirib,

$$\Gamma_{\alpha\beta}{}^{\omega} \frac{\partial^2 \xi^{\alpha}}{\partial \eta^{i} \partial \eta^{k}} = -\Gamma_{\alpha\beta}{}^{\omega} \cdot \Gamma_{\alpha\mu}{}^{\lambda} \frac{\partial \xi^{\alpha}}{\partial \eta^{i}} \cdot \frac{\partial \xi^{\beta}}{\partial \eta^{j}} \cdot \frac{\partial \xi^{\mu}}{\partial \eta^{k}},$$

$$\Gamma_{\alpha\beta}^{\ \ \omega}\frac{\partial^2\xi^\beta}{\partial\eta^j\partial\eta^k}=-\Gamma_{\alpha\lambda}^{\ \ \omega}\cdot\Gamma_{\beta\mu}^{\ \ \lambda}\frac{\partial\xi^\alpha}{\partial\eta^i}\cdot\frac{\partial\xi^\beta}{\partial\eta^j}\cdot\frac{\partial\xi^\mu}{\partial\eta^k},$$

ifodalarga ega bolamiz. Bu ifodalarni orniga qo‘ysak,

$$\left(\frac{\partial\Gamma_{\alpha\beta}^{\ \ \omega}}{\partial\xi^\mu}-\Gamma_{\lambda\beta}^{\ \ \omega}\cdot\Gamma_{\alpha\mu}^{\ \ \lambda}-\Gamma_{\alpha\lambda}^{\ \ \omega}\cdot\Gamma_{\beta\mu}^{\ \ \lambda}\right)\cdot\frac{\partial\xi^\alpha}{\partial\eta^i}\cdot\frac{\partial\xi^\beta}{\partial\eta^j}\cdot\frac{\partial\xi^\mu}{\partial\eta^k}+\frac{\partial^3\xi^\omega}{\partial\eta^i\partial\eta^j\partial\eta^k}=0, \qquad (4.28)$$

bu yerda μ va β hamma k va j indekslarning orinlarini almashtirib va $\Gamma_{ij}^{\ \ k}=\Gamma_{ji}^{\ \ k}$ ekanligidan foydalanib (4.28) ga oxshash munosabatni olamiz

$$\left(\frac{\partial\Gamma_{\alpha\mu}^{\ \ \omega}}{\partial\xi^\beta}-\Gamma_{\lambda\mu}^{\ \ \omega}\Gamma_{\alpha\beta}^{\ \ \lambda}-\Gamma_{\alpha\lambda}^{\ \ \omega}\Gamma_{\beta\mu}^{\ \ \lambda}\right)\frac{\partial\xi^\alpha}{\partial\eta^i}\cdot\frac{\partial\xi^\beta}{\partial\eta^j}\cdot\frac{\partial\xi^\mu}{\partial\eta^k}+\frac{\partial^3\xi^\omega}{\partial\eta^i\partial\eta^j\partial\eta^k}=0. \qquad (4.29)$$

Keyingi (4.28) va (4.29) tengliklarning birini ikkinchisidan ayirib, hamda o‘tish determenatli 0 dan farqi bo‘lishi kerakligidan foydalansak, fazoning Evklid fazosi bo‘lishi shartini keltirib chiqaramiz

$$R_{\beta\mu\alpha.}^{\ \ \ \ \ \omega}=\frac{\partial\Gamma_{\alpha\beta}^{\ \ \omega}}{\partial\xi^\mu}-\frac{\partial\Gamma_{\alpha\mu}^{\ \ \omega}}{\partial\xi^\beta}+\Gamma_{\lambda\mu}^{\ \ \omega}\cdot\Gamma_{\alpha\beta}^{\ \ \lambda}-\Gamma_{\lambda\beta}^{\ \ \omega}\cdot\Gamma_{\alpha\mu}^{\ \ \lambda}=0. \qquad (4.30)$$

Agar fazo Evklid fazosi bo‘lsa, bu shartlar istalgan koordinatalar sistemasida bajarilishi kerak.

Hosil qilingan $R_{\beta\mu\alpha}^{\ \ \ \ \omega}$ lar - 4-rang Riman-Kristoffel tenzorining komponentalaridan iborat. Riman-Kristoffel tenzorining sof kovariant komponentalarining ko‘rinishi

$$R_{ij\mu\nu}=g_{\alpha\nu}R_{ij\mu.}^{\ \ \ \ \ \alpha}=\frac{\partial\Gamma_{\nu\mu j}}{\partial\xi^j}-\frac{\partial\Gamma_{\nu\mu j}}{\partial\xi^i}+g^{\alpha\omega}\left(\Gamma_{\omega\mu j}\cdot\Gamma_{\alpha\nu i}-\Gamma_{\omega\mu i}\cdot\Gamma_{\alpha\nu j}\right) \qquad (4.31)$$

dan iborat, bu yerda

$$\Gamma_{\nu\alpha j}=\frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial\xi_j}+\frac{\partial g_{j\nu}}{\partial\xi^\alpha}-\frac{\partial g_{\alpha j}}{\partial\xi^\nu}\right).$$

Ko‘pincha (4.31)

$$R_{ij\mu\nu}=\frac{1}{2}\left[\frac{\partial^2 g_{\nu i}}{\partial x^j\partial x^\mu}+\frac{\partial^2 g_{\mu j}}{\partial x^i\partial x^\nu}-\frac{\partial^2 g_{\mu i}}{\partial x^j\partial x^\nu}-\frac{\partial^2 g_{\nu i}}{\partial x^i\partial x^\mu}\right].$$

ko‘rinishda ishlatiladi. Bu formuladan quyidagi xossalar kelib chiqadi:

$$R_{ij\mu\nu}=-R_{ji\mu\nu};\quad R_{ij\mu\nu}=-R_{ij\nu\mu};\quad R_{ij\mu\nu}=-R_{\mu\nu ij;}$$

$$R_{ij\mu\nu}+R_{\mu ij\nu}+R_{ij\mu\upsilon}=0;\quad R_{ii\mu\nu}=0;\quad R_{ij\nu\nu}=0$$

3-o‘lchovli fazoda Riman-Kristoffel tenzori faqat oltita noldan farqli komponentalarda ega, ular

$$R_{1212},\quad R_{1313},\quad R_{2323},\quad R_{1213},\quad R_{2123},\quad R_{3132}$$

lardan iborat.

Bog‘liqlik koeffitsientlari (birinchi tur Kristoffel simvollari) $\Gamma^{k}_{\alpha i}$ bazis vektorlarni differensiyallahda kiritiladi:

$$\frac{\partial \vec{э}_{\alpha}}{\partial x^{i}} = \Gamma^{k}_{\alpha i}\vec{э}_{\kappa}, \qquad \frac{\partial \vec{э}^{\alpha}}{\partial x^{i}} = -\Gamma^{\alpha}_{ki}\vec{э}^{k}$$

va quyidagi formulalar bilan hisoblash mumkin:

$$\Gamma^{l}_{ik} = \frac{1}{2}g^{ls}\left(\frac{\partial g_{is}}{\partial x^{k}} + \frac{\partial g_{ks}}{\partial x^{i}} - \frac{\partial g_{ki}}{\partial x^{s}}\right). \qquad (4.32)$$

Vektorni koordinata bo‘yicha differensiyallash:

-kontravariant komponentalarning kovariant hosilasi:

$$\frac{\partial \vec{w}}{\partial x^{i}} \equiv (\nabla_{i} w^{\alpha})\vec{э}_{\alpha} = \left(\frac{\partial w^{\alpha}}{\partial x^{i}} + w^{\beta}\Gamma^{\alpha}_{\beta i}\right)\vec{э}_{\alpha},$$

- kovariant komponentalarning kovariant hosilasi:

$$\frac{\partial \vec{w}}{\partial x^{i}} \equiv (\nabla_{i} w^{\alpha})\vec{э}^{\alpha} = \left(\frac{\partial w_{\alpha}}{\partial x^{i}} - w_{\beta}\Gamma^{\beta}_{\alpha i}\right)\vec{э}^{\alpha}$$

Tenzorni koordinata bo‘yicha differensiyallash:

$$\frac{\partial T}{\partial x^{i}} \equiv (\nabla_{i} T^{\alpha\beta})\vec{э}_{\alpha}\vec{э}_{\beta} = \left(\frac{\partial T^{\alpha\beta}}{\partial x^{i}} + T^{k\beta}\Gamma^{\alpha}_{ki} + T^{\alpha k}\Gamma^{\beta}_{ki}\right)\vec{э}_{\alpha}\vec{э}_{\beta}.$$

Gamiltonning vektorli operatori "nabla": $\vec{\nabla} = \nabla_{i}\vec{э}^{i}$.

Skalyar funksiya gradiyenti:

$$grad\psi = \vec{\nabla}\psi(x^{i}) = \nabla_{i}\vec{э}^{i}\psi = \nabla_{i}\psi\vec{э}^{i} = \frac{\partial \psi}{\partial x^{i}}\vec{э}^{i}.$$

funktsiyaning $\vec{n}$ yo‘nalish bo‘yicha hisilasi:

$$\frac{\partial \psi}{\partial \vec{n}} = \vec{\nabla}\psi\frac{\vec{n}}{|\vec{n}|}.$$

Namunaviy masalalar va ularning yechimlari

4.1. Quyidagi munosabat o‘rinli ekanligini ko‘rsating

$$e_{pqs}e_{mnr} = \begin{vmatrix} \delta_{mp} & \delta_{mq} & \delta_{ms} \\ \delta_{np} & \delta_{nq} & \delta_{ns} \\ \delta_{rp} & \delta_{rq} & \delta_{rs} \end{vmatrix}. \qquad (*)$$

Yechish: Faraz qilaylik A_{ij} ning determinanti quyidagicha aniqlangan bo‘lsin

$$\det A = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}.$$

Determinantning qator va ustunlarini o‘zgarishi ishorani o‘zgartiradi, shuning uchun

$$\begin{vmatrix} A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} A_{12} & A_{11} & A_{13} \\ A_{22} & A_{21} & A_{23} \\ A_{32} & A_{31} & A_{33} \end{vmatrix} = -\det A$$

va qatorlar almashtirishning itiyoriy soni uchun

$$\begin{vmatrix} A_{m1} & A_{m2} & A_{m3} \\ A_{n1} & A_{n2} & A_{n3} \\ A_{r1} & A_{r2} & A_{r3} \end{vmatrix} = e_{mnr} \det A$$

yoki ustunlar almashuvi uchun

$$\begin{vmatrix} A_{1p} & A_{1q} & A_{1s} \\ A_{2p} & A_{2q} & A_{2s} \\ A_{3p} & A_{3q} & A_{3s} \end{vmatrix} = e_{pqs} \det A$$

munosabatlar o‘rinli bo‘ladi. Demak, ixtiyoriy sondagi qator va ustun almashishlari quyidagi munasabatga olib keladi

$$\begin{vmatrix} A_{mp} & A_{mq} & A_{ms} \\ A_{np} & A_{nq} & A_{ns} \\ A_{rp} & A_{rq} & A_{rs} \end{vmatrix} = e_{mnr} e_{pqs} \det A .$$

Agarda $A_{ij} = \delta_{ij}$ bo‘lsa $\det A = 1$ bo‘ladi va (*) munosabat o‘rinli bo‘ladi.

4.2. Quyidagi vektor ifodalarning f_2 komponentasini hisoblang:

(a) $f_i = e_{ijk} T_{jk}$, (b) $f_i = c_{i,j} b_j - c_{j,i} b_j$, (c) $f_i = B_{ij} f_j^*$

Yechish. (a) $f_i = e_{ijk} T_{jk}$, $f_2 = e_{2jk} T_{jk} = e_{213} T_{13} + e_{231} T_{31} = -T_{13} + T_{31}$,

(b) $f_i = c_{i,j} b_j - c_{j,i} b_j$, $f_2 = c_{2,1} b_1 + c_{2,2} b_2 + c_{2,3} b_3 - c_{1,2} b_1 - c_{2,2} b_2 - c_{3,2} b_3 = (c_{2,1} - c_{1,2}) b_1 + (c_{2,3} - c_{3,2}) b_3$,

(c) $f_i = B_{ij} f_j^*$, $f_2 = B_{21} f_1^* + B_{22} f_2^* + B_{23} f_3^*$.

4.3. Sferik va silindrsimon koordinatalar sistemasida bog‘liqlik koeffitsientlarini hisoblang.

Yechish. (4.32) munosabatlardan foydalanib, sferik koordinatalar sistemasi komponentalarning noldan farqlilarini topamiz

$$\Gamma_{22}^1 = -x^1 \cos^2 x^3, \quad \Gamma_{33}^1 = -x^1, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x^1},$$

$$\Gamma_{23}^2 = \Gamma_{32}^2 = -tg x^3, \quad \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{x^1}, \quad \Gamma_{22}^3 = \frac{1}{2}\sin(2x^3)$$

silindrik koordinatalar sistemasida quyidagi komponentalar noldan farqli bo‘ladi:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \quad \Gamma_{22}^1 = -r .$$

4.4. Birinchi rang tenzor (vektor) $\vec{a} = a^i \vec{э}_i$ ning hosilasini aniqlang. Uchta koordinataning har biri x^1, x^2, x^3 uchun agar u silindrik koordinatalar sistemasi $(x^1 = r,\ x^2 = \varphi,\ x^3 = z)$ da quyidagicha ko‘rsatilgan bo‘lsa: $a^1 = 2r^2 + 3z,\ a^2 = 0,\ a^3 = 5r + 2z^2.$

Yechish. Birinchi rang tenzorning koordinata boyicha hosilasi ham birinchi rang tenzor hisoblanadi, uning komponentalari $b^i = \dfrac{\partial \vec{a}}{\partial x^j} = (\nabla_i a^i)\vec{э}_i = b^i \vec{э}_i$ boladi.

Dastlabki tenzor komponentalarining kovariant hosilalari quyidagicha aniqlanadi:

$$b^i = \nabla_i a^i = \frac{\partial a^i}{\partial x^j} + a^k \Gamma^i_{kj}.$$

Hosil qilingan vektorning komponentalari quyidagicha yozilishi mumkin:

$$b^1 = \frac{\partial a^1}{\partial x^j} + a^1 \Gamma^1_{1j} + a^2 \Gamma^1_{2j} + a^3 \Gamma^1_{3j} = \frac{\partial a^1}{\partial x^j},$$

$$b^2 = \frac{\partial a^2}{\partial x^j} + a^1 \Gamma^2_{1j} + a^2 \Gamma^2_{2j} + a^3 \Gamma^2_{3j} = a^1 \Gamma^2_{1j},$$

$$b^3 = \frac{\partial a^3}{\partial x^j} + a^1 \Gamma^3_{1j} + a^2 \Gamma^3_{2j} + a^3 \Gamma^3_{3j} = \frac{\partial a^3}{\partial x^j}.$$

Xususan, $j = 1$ uchun quyidagini olamiz:

$$b^1 = \frac{\partial a^1}{\partial x^j} = \frac{\partial (2r^2 + 3z)}{\partial r} = 4r,\ b^2 = a^1 \Gamma^2_{11} = 0,\ b^3 = \frac{\partial a^3}{\partial x^j} = \frac{\partial (5r + 2z^2)}{\partial r} = 5.$$

Hosil qilingan birinchi rang tenzor quyidagi ko‘rinishda bo‘ladi: $\vec{b} = 4r\vec{э}_1 + 5\vec{э}_3$.

Xuddi shunday, $j = 2$ uchun $\vec{b} = (2r + 3\dfrac{z}{r})\vec{э}_2$. $j = 3$ uchun $\vec{b} = 3\vec{э}_1 + 4z\vec{э}_3$ ni hosil qilamiz.

4.5. Tutash muhitning istalgan nuqtasidagi kuchlanganlik holati quyidagi tenzor tomonidan berilgan:

$$\Sigma = \begin{pmatrix} 3xy & 5y^2 & 0 \\ 5y^2 & 0 & 2z \\ 0 & 2z & 0 \end{pmatrix}.$$

$y^2 + z^2 = 4$ silindrik sirtga urinma tekislikdagi $P(2;1;\sqrt{3})$ nuqtada kuchlanish vektorini aniqlang.

Yechish. P nuqtasidagi kuchlanish komponentlari quyidagi qiymatlarni oladi:

$$\Sigma = \begin{pmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{pmatrix}.$$

P nuqtadagi birlik normal vektor quyiudagi vektor bilan aniqlanadi:

$$grad\varphi = \nabla(y^2 + z^2 - 4) = 2y\vec{j} + 2z\vec{k}.$$

Shuning uchun, P nuqtada $\vec{\nabla}\varphi = 2\vec{j} + 2\sqrt{3}\vec{k}$ bo‘ladi.

U holda P nuqtadagi birlik normal vektor $\vec{n} = \frac{1}{2}\vec{j} + \frac{\sqrt{3}}{2}\vec{k}$. P nuqtada n ga perpendikulyar yuzadagi bo‘lgan kuchlani vektori quyidagicha aniqlanadi:

$$\begin{pmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3 \\ \sqrt{3} \end{pmatrix}.$$

4.6. $\lambda = A_{ij}x_i x_j$ (bu erda A_{ij} =*const*) funksiya uchun $\partial\lambda/\partial x_i = (A_{ij} + A_{ji})x_j$ va $\partial^2\lambda/\partial x_i\partial x_j = A_{ij} + A_{ji}$ ekanligini ko‘rsating va bu hosilalarni $A_{ij} = A_{ji}$ hol uchun soddalashtiring.

Yechish: Quyidagi hosilani qarab chiqamiz

$$\frac{\partial\lambda}{\partial x_k} = A_{ij}\frac{\partial x_i}{\partial x_k}x_j + A_{ij}x_i\frac{\partial x_j}{\partial x_k}.$$

Ushbu tenglik $\frac{\partial x_i}{\partial x_k} \equiv \delta_{ik}$ o‘rinli bo‘lgani uchun

$$\frac{\partial\lambda}{\partial x_k} = A_{kj}x_j + A_{ik}x_i = (A_{kj} + A_{jk})x_j.$$

Differensiallashni davom ettirib

$$\frac{\partial^2\lambda}{\partial x_p\partial x_k} = (A_{kj} + A_{jk})\frac{\partial x_j}{\partial x_p} = A_{kp} + A_{pk}$$

ni hosil qilamiz. Agarda $A_{ij} = A_{ji}$ bo‘lsa, u holda $\frac{\partial\lambda}{\partial x_k} = 2A_{kj}x_j$ va $\frac{\partial^2\lambda}{\partial x_p\partial x_k} = 2A_{pk}$ bo‘ladi.

4.7. Quyida keltirilgan tenzorlarning kovariant hosilalarini hisoblang:

(a) $\overline{A} = A_i\overline{\mathfrak{Э}}^i$; (b) $T = T^{ijk}\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k$; (c) $H = H_{ij}^k\overline{\mathfrak{Э}}^i\overline{\mathfrak{Э}}^j\overline{\mathfrak{Э}}_k$

Yechish:

(a) $$\frac{\partial\overline{A}}{\partial\zeta^i} = \frac{\partial A_j}{\partial\zeta^i}\overline{\mathfrak{Э}}^j + A_j\frac{\partial\overline{\mathfrak{Э}}^j}{\partial\zeta^i} = \frac{\partial A_j}{\partial\zeta^i}\overline{\mathfrak{Э}}^j - A_j\Gamma_{ki}^j\mathfrak{Э}^k,$$

$$\frac{\partial T}{\partial\zeta^n} = \frac{\partial T^{ijk}}{\partial\zeta^n}\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k + T^{ijk}\frac{\partial\overline{\mathfrak{Э}}_i}{\partial\zeta^n}\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k + T^{ijk}\overline{\mathfrak{Э}}_i\frac{\partial\overline{\mathfrak{Э}}_j}{\partial\zeta^n}\overline{\mathfrak{Э}}_k + T^{ijk}\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\frac{\partial\overline{\mathfrak{Э}}_k}{\partial\zeta^n} = \frac{\partial T^{ijk}}{\partial\zeta^n}\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k +$$

(b) $$+T^{ljk}\Gamma_{in}^l\overline{\mathfrak{Э}}_l\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k + T^{ljk}\overline{\mathfrak{Э}}_i\Gamma_{jn}^p\overline{\mathfrak{Э}}_p\overline{\mathfrak{Э}}_k + T^{ljk}\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\Gamma_{kn}^q\overline{\mathfrak{Э}}_q = \frac{\partial T^{ijk}}{\partial\zeta^n}\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k + T^{ljk}\Gamma_{\ln}^i\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k +$$

$$+T^{ipk}\Gamma_{pn}^j\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k + T^{ijq}\Gamma_{qn}^k\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k = \left(\frac{\partial T^{ijk}}{\partial\zeta^n} + T^{ljk}\Gamma_{\ln}^i + T^{ipk}\Gamma_{pn}^j + T^{ijq}\Gamma_{qn}^k\right)\overline{\mathfrak{Э}}_i\overline{\mathfrak{Э}}_j\overline{\mathfrak{Э}}_k,$$

$$\frac{\partial H}{\partial \zeta^n} = \frac{\partial H^k_{ij}}{\partial \zeta^n}\bar{э}^i\bar{э}^j\bar{э}_k + H^k_{ij}\frac{\partial \bar{э}^i}{\partial \zeta^n}\bar{э}^j\bar{э}_k + H^k_{ij}\bar{э}^i\frac{\partial \bar{э}^j}{\partial \zeta^n}\bar{э}_k + H^k_{ij}\bar{э}^i\bar{э}^j\frac{\partial \bar{э}_k}{\partial \zeta^n} = \frac{\partial H^k_{ij}}{\partial \zeta^n}\bar{э}^i\bar{э}^j\bar{э}_k -$$

$$\text{(c)}\quad - H^k_{ij}\Gamma^i_{mn}\bar{э}^m\bar{э}^j\bar{э}_k - H^k_{ij}\bar{э}^i\Gamma^j_{ln}\bar{э}^l\bar{э}_k + H^k_{ij}\bar{э}^i\bar{э}^j\Gamma^\nu_{kn}\bar{э}_\nu =$$

$$= \left(\frac{\partial H^k_{ij}}{\partial \zeta^n} - H^k_{mj}\Gamma^m_{in} - H^k_{il}\Gamma^l_{jn} + H^\nu_{ij}\Gamma^k_{\nu n}\right)\bar{э}^i\bar{э}^j\bar{э}_k$$

Mashqlar

4.8. $\vec{a} = a^i\,\vec{g}_i = a_i\,\vec{g}^i$ vektorning (a_i) -kovariant va (a^i) -kontravariant komponentalari berilgan. $\vec{a}$ vektorning (a^i) -kontravariant komponentalarini $\vec{a}$ vektorning $\vec{g}^i$ asosiy bazislari yo‘nalishidagi $a^i\left|\vec{g}_i\right|$ -parallel tashkil etuvchilari yoki $\vec{a}$ vektorning o‘zaro bog‘langan bazislar o‘qidagi $\dfrac{a^i}{\left|\vec{g}^i\right|}$ -ortogonal proyeksiyalaridan topish mumkinligini ko‘rsating.

4.9. $\vec{a}$ vektorning (a_i) -kovariant komponentalarini $\vec{a}$ vektorning o‘zaro bog‘langan bazislar yo‘nalishidagi $a_i\left|\vec{g}^i\right|$ -parallel tashkil etuvchilari yoki $\vec{a}$ vektorning $\vec{g}_i$ asosiy bazislari yo‘nalishidagi $\dfrac{a^i}{\left|\vec{g}_i\right|}$ -ortogonal proyeksiyalaridan topish mumkinligini ko‘rsating.

4.10. $g^{ik} = G^{ik}/g$ ekanligini ko‘rsating, bu yerda $G^{ik} -$ g_{ik} -matritsaning algebraik to‘ldiruvchisi.

4.11. Ikkinchi rang tenzor kovariant komponentalarini kovariant differensiyallash uchun formula hosil qiling.

4.12. Uchinchi rang $\hat{T} = T^{\alpha\beta}_{\gamma}\vec{э}_\alpha\,\vec{э}_\beta\,\vec{э}^\gamma$ tenяorni kovariant differensiyallash formulasini aniqlang.

4.13. Bazis vektorlarning ta'rifiga asoslanib, quyidagi tenglikni isbotlang: $\Gamma^\alpha_{ij} = \Gamma^\alpha_{ji}$.

4.14. Silindrik koordinatalar sistemasida komponentalari quyidagicha berilgan $\vec{W}$ vektorning kovariant komponentalarining kovariant hosilalarini toping:

$$W_1 = 3x^1 \sin x^2,\; W_2 = (x^1)^2 \cos x^3,\; W_3 = (x^3)^2 \ln x^1.$$

4.15. Sferik koordinatalar sistemasidada quyidagi vektor komponentalarining kovariant hosilalarini hisoblang: $w^1 = 2x^1 \cos x^3,\; w^2 = \ln x^1,\; w^3 = tgx^2$.

4.16. Quyidagi tengliklarni isbotlang:

a) $\nabla_i(w^\alpha + v^\alpha) = \nabla_i w^\alpha + \nabla_i v^\alpha$, b) $\nabla_i(w^\alpha v^\beta) = (\nabla_i w^\alpha)v^\beta + w^\alpha \nabla_i v^\beta$.

5.17. Silindrik koordinatalar sistemasida tenzorning kovariant komponentalarining kovariant hosilalarini toping:

$$(T_{ij}) = \begin{pmatrix} x^1 \ln x^3 & 0 & x^1 tgx^2 \\ 0 & x^3 ctgx^2 & 0 \\ x^1 tgx^2 & 0 & (x^1)^2 \sin x^2 \end{pmatrix}.$$

4.18. $\lambda = x^2 + 2xy - z^2$ funktsiyaning berilgan $\vec{n} = (\frac{2}{7}; -\frac{3}{7}; -\frac{6}{7})$ birlik vektor yo‘nalishi boyicha hosilasini toping.

4.19. Skalyar temperaturaharorati maydoni quyidagicha berilgan: $T = T(x, y, z) = xy - 5z$. Fazodagi $(x = 2, y = 3, z = 0)$ nuqtada yo‘nalish bo‘yicha uning hosilasining maksimal va minimal qiymatlarini aniqlang.

4.20. Koordinatalari $x = y = z = 0$ bo‘lgan nuqtada bosim qiymati $p = 1$ bosim gradiyenti $\text{grad}p = 1\vec{i} + 3\vec{j} + 4\vec{k}$ ga teng. Ushbu nuqtaning kichik bir atrofida koordinatalari $x = 0{,}01$; , $y = 0{,}02$; , $z = -0{,}01$ nuqtada bo‘lgan bosim qiymatini aniqlang.

4.21. Tutash muhitning istalgan nuqtasidagi kuchlanganlik holati quyidagi tenzor tomonidan berilgan:

$$\Sigma = \begin{pmatrix} 3x^2 & 4xy & -5z \\ 4xy & 0 & 0 \\ -5z & 0 & 0 \end{pmatrix}.$$

$y^2 + 2x^2 = 8$ silindrik sirtga urinma tekislikdagi $P(\sqrt{2}; 2; 1)$ nuqtada kuchlanish vektorini aniqlang.

Sinov savollari

1. Nima uchun vektorning uzunligini uning o‘zini - o‘ziga skalyar ko‘paytmasi sifatida aniqlash qulay?
2. g_{ij} va g^{kp} belgilashlar nimalarni ifodalaydilar?
3. Fundamental metrik tenzor deb nimaga aytiladi?
4. Fundamental metrik tenzorning kontravariant va kovariant komponentalarini yozib bering.
5. Fundamental metrik tenzor nimaga kerak?
6. Algebra kursida kvadratik shakl va uning kanonik shakli deb nimaga aytiladi?

7. Vektorning va tenzorning kontravariant bazisdagi kontravariant komponentalari deb nimaga aytiladi?
8. Vektorning va tenzorning kontravariant bazisdagi kovariant komponentalari tushunchalarini bilasizmi?
9. Tenzorning aralash komponentalari qanday hosil qilinadi?
10. Indeks qanday tushiriladi va qanday ko‘tariladi?
11. Qanday tenzorlarni qo‘shish mumkin?
12. Qanday tenzorlarni ayirish mumkin?
13. Tenzorning songa ko‘paytmasi nimani beradi?
14. Ikkinchi rang tenzorni simmetriklash va antisimmetriklash usulini yozib bering.
15. Tenzorni simmetrik va antisimmetrik tenzorlar yig‘indisi ko‘rinishida yosing.
16. Qanday tenzorlarni ko‘paytirish mumkin?
17. Tenzorning indeksini ko‘tarish yoki tushirish uchun nima qilmoq kerak?
18. Tenzorning rangi qanday pasaytiriladi?
19. Tenzorning nechta skalyar invarianti mavjud? Ularni yozing.
20. Tenzorning bosh o‘qlari deb qanday o‘qlarga aytiladi?
21. Tenzorning bosh komponentalari deb nimaga aytiladi?
22. Vektorning kovariant hosilasi nima? Ifodasini yozing.
23. Kovariant hosila deb nimaga aytiladi?
24. Kovariant hosilaning xossalaini ayting.
25. Skalyar miqdordan olingan kovariant hosila nimaga teng?
26. Kovariant hosila uchun o‘rinli bo‘lgan formularni yozing.
27. Kristoffel simvollarini yozing.
28. Kristoffel simvollarini almashtirish formulalarini keltiring.
29. Kristoffel simvollarini qanday hisoblash mumkin.
30. Riman-Kristoffel tenzorini yozing.
31. Riman-Kristoffel tenzorining xossalarini ayting.
32. Riman-Kristoffel tenzorini nechanchi rangga ega.
33. Uch o‘lchovli fazoda Riman-Kristoffel tenzorining nechta komponentasi noldan farqli?

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1-qism. Vektor va tenzor maydon tushunchalari

(uslubiy qo‘llanma)

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