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TO TAKE MASTERS DEGREE OF THE EDUCATION DIRECTION “5130100 – MATHEMATICS”

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MASTER’S DISSERTATION WORK

“FOURIER TRANSFORMS OF THE DELTA FUNCTION ON THE SURFACE”

The work is examined and given permission to defense the thesis
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MAGISTRLIK DISSERTATSIYASINING ANNOTATSIYASI

ANNOTATSIYA

Ushbu magistrlik ishida delta funksiyaning asosiy xossalari keltirilgan. Dissertatsiyadagi asosiy urg’u esa Sirt ustidagi delta funksiyaning Furye almashtirishlariga qaratilgan. Umumiy holda taqsimot bo’ladigan va kompleks tekislikda analitik davom ettirish mumkin bo’lgan Furye almashtirishini qaraymiz. Shuningdek, odatiy Furye almashtirishi umumiy ma’noda integrallanuvchi bo’lmaydi. Sirt ustidagi delta funksiyasining Furye almashtirishining jamlash ko'rsatkichi uchun pastki chegarani olamiz. Bu chegara, shuningdek, unga mos keluvchi tebranuvchi integralning jamlash ko’rsatkichi uchun pastki chegarani aniqlaydi.

ANNOTATION OF MASTER’S DISSERTATION WORK

ANNOTATION

In this dissertation work it is considered basic property of the delta function. The main task of the thesis is the behavior of the Fourier transform of the delta functions supported on surfaces. We consider the Fourier transform which is a distribution in the general case and also it has analytic continuation to the complex space. But, in general the Fourier transform is not integrable in general. We obtain a lower bound for the summation exponent of the Fourier transform of the delta function on the surface. The bound also gives lower bound for the summation exponent for the corresponding oscillatory integral.

Contents

INTRODUCTION

Substantiation and actuality of problems in the final qualifying work.

Delta function are used in mathematical physics, harmonic analysis, geophysics, applied mathematics and so on. Therefore, studying properties of Delta function is one of the actual problem of analysis.

Purpose and tasks of the research.

The purpose of the research is description of Fourier transforms of the Delta function on the surface. The main tasks of the research is description of basic properties of the delta function on the surfaces and Fourier transform of that function. Also investigation of behavior of the Fourier transform of the delta function when frequency takes large.

Scientific results of the thesis.

In the thesis there are given definition of delta function, derivatives and integration of the delta function, Delta function on the surface, Fourier transforms of the Delta function on the surface. Asymptotic behavior of the Fourier transform of the characteristic function of a convex set is given. Also character of fourier transform of the delta function when frequency takes large.

Description of methodology used in research.

In the thesis are used methods of mathematical and functional analysis, including topological vector spaces, linear spaces with countable many norms. Also, we use methods of differential geometry in order to investigate behavior of the Fourier transform of delta function on the surfaces when frequency gets large. The methods of harmonic analysis and asymptotic analysis are used.

Theoretical and practical importance of research results.

The results of the research have fundamental character. But, the methods and results can be used in order to solve the problems of mathematical physics, including spectral properties of discrete Shcrodinger operator. It can be used in analytic number theory. In particular, counting number of points with integer coordinates in homothetic domains, which is important on numerical computations of multiple integrals.

Description of work structure.

The thesis contains Introduction, 3 chapters, 10 paragraphs, Conclusion and References including 10 references.

In the First Chapter there are given general concepts of the Delta functions. It contains elementary concepts of step function and the Delta function, local properties and derivatives and integration of the Delta function.

The Second Chapter consist of 3 paragraphs. There are statement of stationary phase method, information about contribution from stationary boundary points and degenerate stationary points, asymptotic of the Fourier transform of the characteristic function of a convex set and similar problems.

The Fourier transform of the Delta function on the surface is one of the most important concept in the generalized functions theory. And this concept settle down in the Third Chapter. This Chapter have Delta function on the surface, Fourier transforms of the Delta function on the surface, definition and examples, distributions which are defined by surface measures.

Chapter I. The Dirac Delta function

I.1. Step functions. Derivative of the unit step function

Step functions. We shall define the **Heaviside unit step function,** u, as that function which is equal to 1 for every positive value of t and equal to 0 for every negative value of t. This function could equally well have been defined in terms of a specific formula; for example

$$u(t) = \frac{1}{2}\left[1 + \frac{t}{|t|}\right].$$

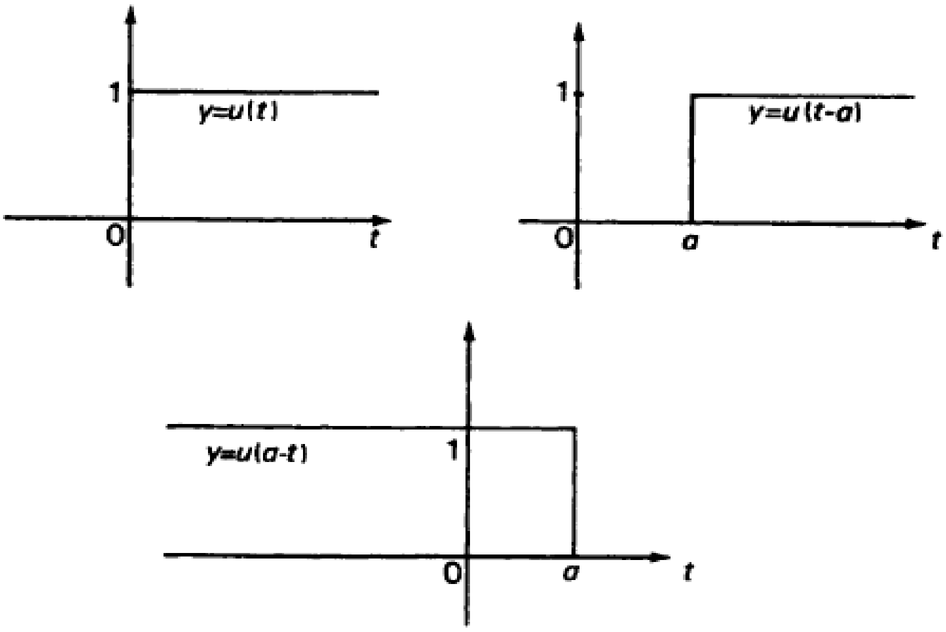


Figure 1.1 shows the graph of $y = u(t)$, together with those of functions $u(t - a)$ and $u(a - t)$ obtained by translation and reflection.

The symbol u will always refer to the unit step function, defined as above, with $u(0)$ left unspecified. For any given real number c we shall understand by u_c the function obtained when we complete the definition by assigning the value c at the origin. Thus we have,

$$u_c(t) = u(t) \quad \text{for all} \quad t \neq 0 \quad \text{and} \quad u_c(0) = c.$$

The Heaviside unit step function is of particular importance in the context of control theory, electrical network theory and signal processing. However its immediate general significance can be gauged from the following considerations: suppose that

ϕ is a function which is continuous everywhere except for the point $t = a$, at which it *has* a simple discontinuity(Fig. 1.2.). Then ϕ can always be represented as a linear combination of functions ϕ_1 and ϕ_2 which are continuous everywhere but which have been truncated at $t = a$:

$$\phi(t) = \phi_1(t)u(a - t) + \phi_2(t)u(t - a).$$

Then $\phi(t) = \phi_1(t)$ for all $t < a$ and $\phi(t) = \phi_2(t)$ for all $t > a$.

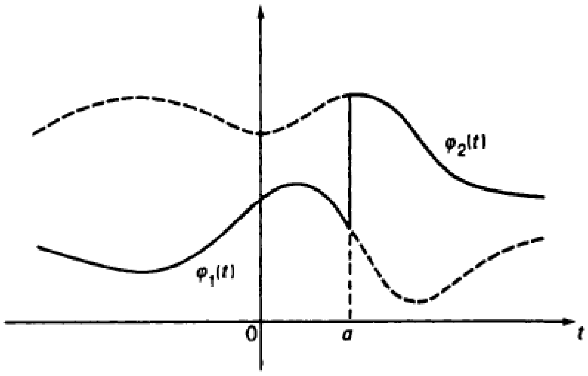


Figure 1.2.

In particular, note the linear combination of unit step functions which produces the so-called signum function:

$$sgn(t) = u(t) - u(-t).$$

Sectionally continuous functions.

Any function which is continuous in an interval except for a finite number of simple discontinuities is said to be **sectional continuous or piecewise continuous** on that interval. **In** particular suppose that a finite interval $[a; b]$ is sub-divided by points $t_0, t_1, \dots, t_n$ where

$$a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = b.$$

If s is a function which is constant on each of the (open) sub-intervals $(t_{k-1}; t_k)$, for example, if

$$s(t) = \alpha_k, \quad for\ t_{k-1} < t < t_k \quad (k = 1,2, \dots, n).$$

then s is certainly piecewise continuous on $[a; b]$. Such a function is often called a **step-function** or a **simple function** on $[a; b]$. The particular function $u(t)$ is called the Heaviside unit step in honour of the English electrical engineer and applied mathematician **Oliver Heaviside,** and is often denoted by the alternative symbol, $H(t)$. Other notations such as $U(t), \sigma(t)$ and $\eta(t)$ are also to be found in the literature.

Derivative of the unit step function
Naive definition of delta function.

Suppose that we try to extend the definition of differentiation in such a way that it applies to functions with jump discontinuities. In particular we would need to define a *'derivative'* for the unit step function, *u*. For all $t \neq 0$ this is of course well-defined in the classical sense as:

$$u'(t) = 0 \quad for \;\; all \quad t \neq 0.$$

corresponding to the obvious fact that the graph of $y = u(t)$ has zero slope for all non-zero values of *t*. At $t = 0$, however, there is a jump discontinuity, and the definition of derivative accordingly fails. A glance at the graph suggests that it would not be unreasonable to describe the slope as 'infinite' at this point. Moreover, if we take any specific representation, u_c, of the unit step, then the ratio $\frac{u_c(h)-u_c(0)}{h}$ becomes arbitrarily large as h approaches 0, or using the familiar convention,

$$\lim_{h\to 0} \frac{u_c(h) - u_c(0)}{h} = +\infty.$$

Thus, from a descriptive point of view at least, the derivative of u would appear to be a function which (for the moment) we shall denote by $\delta(t)$ and which has the following pointwise specification:

$$\delta(t) \equiv u'(t) = 0, \;\; for \; all \;\; t \neq 0.$$

and, once again adopting the conventional use of the ∞ sign,

$$\delta(0) = +\infty.$$

However, as will shortly become clear, we shall need eventually to consider more carefully the significance of the symbol $\delta(t)$. Does it really denote a function, in the proper sense of the word, and in what sense does it represent the derivative of the function $u(t)$?

Now let f be any function continuous on a neighborhood of the origin, say for $-a < t < +a$. Then we should have

$$\int_{-a}^{+a} f(t)u'(t)dt = \int_{-a}^{+a} f(t) \lim_{h\to 0} \left[\frac{u(t+h) - u(t)}{h}\right] dt\,.$$

Assuming that it is permissible to interchange the operations of integration and of taking the limit as h tends to 0, this gives

$$\int_{-a}^{+a} f(t)\delta(t)dt = \lim_{h\to 0} \frac{1}{h} \int_{-h}^{0} f(t)dt = \lim_{h\to 0} f(\xi),$$

where ξ is some point lying between $-h$ and 0. Since f is assumed continuous in the neighborhood of $t = 0$, it follows that

$$\int_{-a}^{+a} f(t)\delta(t)dt = f(0). \qquad (1.1)$$

which, since $u(t) = 0 \ for \ t < 0,$ reduces to

$$f(a) - \int_{0}^{a} f'(t)dt = f(a) - [f(a) - f(0)] = f(0).$$

Sampling property of the Delta function.

The result (1.1) for arbitrary continuous functions f is usually referred to as the **sampling property** of δ. If we follow the usual conventional language, still to be found in much of the engineering and physics literature then we would refer to 6 as the **delta function,** or more specifically as the **Dirac delta function.** This sampling property is clearly independent of the actual value of the number a, and depends only on the behavior of the integrand at (or near) the point $t = 0$. Accordingly it is most usually stated in the following form:

The sampling property of the Dirac delta function: if f is any function which is continuous on a neighborhood of the origin, then

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0). \qquad (1.2)$$

In particular, let $f(t) = u(a - t)$, where a is any fixed number other than 0. Then $f(t) = 0$ for all $t > a$ and $f(t) = 1$ for all $t < a$; moreover f is certainly continuous on a neighborhood of the origin. Hence we should have

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\infty}^{+\infty} u(a - t)\delta(t)dt = u(a), \qquad for\ any\ a \neq 0.$$

This shows that if we were to assume the existence of a function $\delta(t)$ at the outset, with the sampling property (1.2), then it would at least be consistent with the role of $\delta(t)$as a derivative, in some sense, of the unit step function: indeed, the relation $u'(t) = \delta(t)$, from which we first started, could have been deduced as a consequence of that sampling property. Nevertheless, it must still be remembered that the derivation given above of the sampling property is a purely formal one, and that there is as yet no evidence that any function *8(t)* which exhibits such a property actually does exist.

I.2. The Delta function as a limit. Developments of delta function theory

Failure of naive definition of $\boldsymbol{\delta(t)}$.

It is not difficult to show that in point of fact no function defined on the real line $\mathbb{R}$ can enjoy the properties attributed above to δ. The requirement that $\delta(t) = u'(t)$ necessarily implies that $\delta(t) = 0$ for all $t \neq 0$. Hence, for any continuous function $f(t)$, and any positive numbers $\varepsilon_1, \varepsilon_2$, however small, we must have

$$\int_{-\infty}^{-\varepsilon_1} f(t)\delta(t)dt = \int_{\varepsilon_2}^{\infty} f(t)\delta(t)dt = 0.$$

so that

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{\varepsilon_1 \to -0} \int_{-\infty}^{-\varepsilon_1} f(t)\delta(t)dt + \lim_{\varepsilon_2 \to -0} \int_{\varepsilon_2}^{+\infty} f(t)\delta(t)dt = 0.$$

This does not agree with the sampling property (1.2) except in the trivial case of a function f for which $f(0) = 0$.

Furthermore, for any $t > 0$ we actually get

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \int_{-\infty}^{t} u'(\tau)d\tau = 0 \neq u(t).$$

and this is certainly not consistent with the initial assumption that $\delta(t)$ is to behave as the derivative of *u(t).*

The derivation of the alleged sampling property given above involved the interchange of the order of the operations of integration and of taking the limit, and the subsequent failure of $\delta(t)$ to exhibit this property shows that the interchange cannot be justified in this situation. Similarly we cannot justify the application of the usual integration by parts formula in the alternative derivation of the sampling property.

Simple sequential definition of $\boldsymbol{\delta(t)}$.

One way of dealing with this situation is to abandon the assumption that $\delta(t)$ represents an actual function, and to construct instead a suitable sequence of proper functions which will approximate the desired behavior of $\delta(t)$ as closely as we wish. Then we could treat all expressions involving so-called delta functions in terms of a symbolic shorthand for certain limiting processes. There are many ways in which we could do this, but among the simplest and most straightforward examples are the sequences $(s_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ of functions illustrated in Fig. 1.3 and defined formally below:

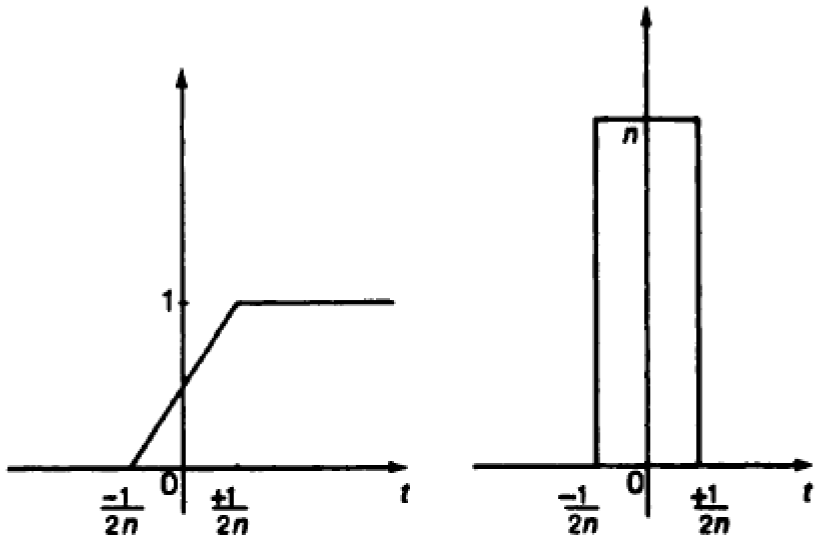


Figure 1.3.

$$s_n(t) = \begin{cases} 1 & for\ t > \dfrac{1}{2n} \\ nt + \dfrac{1}{2} & for -\dfrac{1}{2n} < t < \dfrac{1}{2n} \\ 0 & for\ t < -\dfrac{1}{2n} \end{cases}$$

$$d_n(t) = \begin{cases} 0 & for\ t > \dfrac{1}{2n} \\ n & for -\dfrac{1}{2n} < t < \dfrac{1}{2n} \\ 0 & for\ t < -\dfrac{1}{2n} \end{cases}$$

Then,

(1) $d_n(t) = \frac{d}{dt} s_n(t)$, for all t except $t = \pm\frac{1}{2n}$,

(2) $s_n(t) = \int_{-\infty}^{t} d_n(\tau)d\tau$, for all t,

(3) if f is any function continuous on some neighborhood of the origin then,

$$\int_{-\infty}^{+\infty} f(t)d_n(t)dt = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(t)dt = f(\xi_n).$$

where ξ_n is some point such that $-\frac{1}{2n} < \xi_n < \frac{1}{2n}$.

If we allow n to tend to infinity then the sequences $\{s_n\}$ and $\{d_n\}$ tend in the ordinary pointwise sense to limit functions u and d_∞:

$$\lim_{n\to\infty} s_n(t) = u(t) = \begin{cases} 1 & for\ t > 0 \\ 0 & for\ t < 0 \end{cases}$$

(In actual fact, $\lim_{n\to\infty} s_n(t) = u_{\frac{1}{2}}(t)$ since $s_n(0) = \frac{1}{2}$ for all n.)

$$\lim_{n\to\infty} d_n(t) = d_\infty(t) = \begin{cases} +\infty & for\ t = 0 \\ 0 & for\ t \neq 0 \end{cases}$$

and it follows that, for any function f continuous on a neighborhood of the origin we actually get

$$\int_{-\infty}^{+\infty} f(t) d_\infty(t) dt = 0.$$

However, on the other hand we have,

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(t) d_n(t) dt = \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{+\frac{1}{2n}} n f(t) dt = \lim_{n\to\infty} f(\xi_n)\,,\ \ where -\frac{1}{2n} < \xi_n < +\frac{1}{2n},$$

so that, by continuity,

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(t) d_n(t) dt = f(0).$$

Thus, for n = 1,2,3, ..., we can construct functions d_n whose operational behavior approximates more and more closely those properties required of the so-called delta function. The symbolic expression $\delta(t)$, when it appears within an integral sign, must be understood as a convenient way of indicating that the integral in question is itself symbolic and really denotes a limit of a sequence of bona fide integrals. Further, $\delta(t)$ must be clearly distinguished from the ordinary pointwise limit function $d_\infty(t)$. More precisely, when we use expressions like $\int_{-\infty}^{+\infty} f(t0\ \delta(t) dt$, they are to be understood simply as a convenient way of denoting the limit

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(t) d_n(t) dt = f(0).$$

and should not be interpreted at their face value, namely as

$$\int_{-\infty}^{+\infty} f(t) \left[\lim_{n\to\infty} d_n(t)\right] dt \equiv \int_{-\infty}^{+\infty} f(t) d_\infty(t) dt = 0.$$

I.3. Delta function as a functional.

Function and functional. Recall that, as stated we reserve the term "function" to denote a real, single-valued function of a single real variable. That is to say, we understand a function to be a mapping of (real) numbers into numbers, and therefore to be completely characterized as such. It has been made clear in I.2. that the so-called delta function is not a function in this strict sense at all, and is not determined by the values which may be attributed to $\delta(t)$ as t ranges from $-\infty$ to $+\infty$. The situation can be made somewhat clearer if we consider first the sense in which we are really to understand the Heaviside unit step:

For each given value of the parameter c the symbol u_c denotes a proper function which has a well-defined point wise specification over the entire range $-\infty < t < +\infty$. Now let f be an arbitrary bounded continuous function which vanishes identically outside some finite interval. Then we can write

$$\int_{-\infty}^{+\infty} f(t)\, u_c(t)\, dt = \int_{0}^{+\infty} f(t) dt. \tag{1.3}$$

Equation (1.3) expresses a certain operational property associated with u_c in the sense that it defines a mapping or transformation: to each continuous function f which vanishes outside some finite interval there corresponds a well-defined number $\int_0^{+\infty} f(t)dt$. This number is independent of the particular value $c = u_c(0)$ and hence of the particular function $u_c(t)$. Recalling our convention that *u* is to stand for the unit step function left undefined at the origin, we could equally well write (1.3) in the form,

$$\int_{-\infty}^{+\infty} f(t)\, u(t)\, dt = \int_{0}^{+\infty} f(t) dt. \tag{1.4}$$

The symbol *u* appearing on the left-hand side of (1.4) may now be given an alternative interpretation. Instead of standing for some one particular function, it could be regarded as the representative of an entire family of equivalent functions, u_c, anyone of which would suffice to characterize the specific operation on f which we have in mind. From this point of view we have no real need to specify the precise point wise behavior of the unit step. It is enough that the symbol *u* appearing in (1.4) is known to define a certain mapping of a specific class of functions into numbers, and we could indicate this by adopting an appropriate notation:

To each continuous function f which vanishes outside some finite interval there corresponds a certain number which we usually write as $u(f)$, or sometimes as $u(f)$, and which is given by the mapping

$$f \to u(f) \equiv < u, f >= \int_{0}^{+\infty} f(t)dt.$$

In the same way the one important feature of the so-called delta function, $\delta(t)$, is that it provides a convenient means of representing an operation, defined at least for all functions continuous on a neighborhood of the origin, which maps or transforms each such function f into the value $f(0)$ which it assumes at the origin. Following the convention suggested in the above equation we could adopt a corresponding notation, $\delta(f)$ or $< \delta, f >$, and write this operation in the form

$$f \to \delta(f) \equiv < \delta, f >= f(0).$$

Symbols such as u and ~ when regarded as specifying operations on certain classes of functions (rather than as supposedly standing for actual functions *u(*t) and $\delta(t)$ in their own right) are properly referred to as **functionals.**

However, it is usually more agreeable, and often more convenient, to retain the familiar notation of the integral calculus and to write the mappings defined by *u* and δ in the form

$$f \to \int_{0}^{+\infty} f(t)dt = \int_{-\infty}^{+\infty} f(t)u(t)dt. \qquad (1.5a)$$

and

$$f \to f(0) = \int_{-\infty}^{+\infty} f(t)\delta(t)dt. \qquad (1.5b)$$

The significance of the integral $\int_{-\infty}^{+\infty} f(t)u(t)dt$ in (1.5a) then appears primarily as its capacity to express the number $u(f)$ in a familiar canonical form, although a genuine integration process is indeed involved. In (1.5b) however, both the integral sign and the expression $\delta(t)$, which seems to imply existence of an actual function, must be understood in a purely symbolic sense. This use of conventional notation, appropriately borrowed from elementary calculus, is perhaps the main reason why u, δ, and, as we shall see later, certain other functionals, are more often described as **generalized functions** - possibly a somewhat misleading term but one which is now very generally accepted.

I.4. Derivatives and Integration of the Delta function.

Definition of $\boldsymbol{\delta'(t)}$.

To obtain a meaningful 'derivative' for the delta function we have to define a generalized function with an appropriate sampling property. For any number $a \neq 0$ the generalized function $\frac{1}{a}[\delta(t+a) - \delta(t)]$ certainly has a well-defined sampling property, and we need to examine what happens to this in the limit as a tends to zero. Suppose, then, that f is any function which is continuously differentiable in some neighborhood of the origin. If a is sufficiently small in absolute magnitude, we have:

$$\int_{-\infty}^{+\infty} f(t)\frac{\delta(t+a) - \delta(t)}{a}dt = \frac{1}{a}\left[\int_{-\infty}^{+\infty} f(t)\delta(t+a)dt - \int_{-\infty}^{+\infty} f(t)\delta(t)dt\right] = \frac{1}{a}[f(-a) - f(0)].$$

Hence

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f(t)\frac{\delta(t+a) - \delta(t)}{a}dt = -f'(0).$$

This suggests that $\delta'(t)$ should be characterized by the following property:

If $\boldsymbol{f}$ is any function which has a continuous derivative $\boldsymbol{f'}$, at least in some neighborhood of the origin, then

$$\int_{-\infty}^{+\infty} f(\tau)\delta'(\tau)d\tau = -f'(0). \qquad (1.6)$$

This might have been inferred from a formal integration by parts:

$$\int_{-\infty}^{+\infty} f(t)\delta'(t)dt = [\mathrm{f(t)\delta(t)}]\,|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(t)f'(t)dt = -f'(0).$$

Strictly this does involve a tacit appeal to the point wise behavior of the delta function. However, all that we are really using is the fact that the operational property of 6 is confined absolutely to the point $t = 0$. At all other points it has no effect, and this is what is really meant by saying that it 'evaluates to zero' at such points. A straightforward generalization of the argument leading to (1.6) gives the following rule for the sampling property associated with the nth derivative of the delta function:

For each given positive integer *n* the generalized function $D^n\delta \equiv \delta^n$, (the n^{th} derivative of the delta function), is defined by the characteristic property

$$\int_{-\infty}^{+\infty} f(\tau)\delta^n(\tau)d\tau = (-1)^n f^n(0).$$

where f is any function with continuous derivatives at least up to the n^{th} order in some neighborhood of the origin.

Properties of $\delta'(t)$.

For the most part the properties of the derivatives $\delta^{(n)}$ are fairly obvious generalizations of the corresponding properties of the delta function itself, and can be established by using similar arguments. We list them briefly here, enlarging on those of particular difficulty and/or importance.

(i) **Translation.** For any continuously differentiable function f and any scalar a we have

$$\int_{-\infty}^{+\infty} f(\tau)\delta'(\tau)d\tau \equiv \int_{-\infty}^{+\infty} f(\tau)\delta'(\tau-a)d\tau = \int_{-\infty}^{+\infty} f(t+a)\delta'(t)dt =$$
$$= [-f'(t+a)]_{t=0} = -f'(a).$$

More generally,

$$\int_{-\infty}^{+\infty} f(\tau)\delta_a^{(n)}(\tau)d\tau \equiv \int_{-\infty}^{+\infty} f(\tau)\delta_a^{n}(\tau-a)d\tau = (-1)^n f^n(a).$$

for any function f which has continuous derivatives at least up to the n^{th} order in some neighborhood of the point $t = a$.

(ii) Addition. Let n and m be positive integers. If f is any function which has continuous derivatives at least up to the n^{th} order on a neighborhood of the point $t = a$, and at least up to the m^{th} order in some neighborhood of the point t = b, then

$$\int_{-\infty}^{+\infty} f(\tau)\left[\delta_a^{(n)}(\tau) + \delta_b^{(m)}(\tau)\right]d\tau = (-1)^n f^{(n)}(a) + (-1)^m f^{(m)}(b).$$

(iii) Multiplication by a scalar. The product $k\delta^{(n)}$, where k is any fixed number, has a well defined sampling operation given by the following equivalence:

$$\int_{-\infty}^{+\infty} f(\tau)\left[k\delta^{(n)}(\tau)\right]d\tau = \int_{-\infty}^{+\infty} [kf(\tau)]\delta^{(n)}(\tau)d\tau.$$

This has the value $(-1)^n k f^{(n)}(0)$. In particular if $k = 0$ then we may write $k\delta^{(n)}(t) = 0$.

(iv) Multiplication by a function. If ϕ is a fixed function which is continuously differentiable in a neighborhood of the origin, then a meaning can be given to the product $\phi(t)\delta'(t)$. If we assume that associativity holds for all the products involved then we ought to have

$$\int_{-\infty}^{+\infty} f(\tau)[\phi(\tau)\delta'(\tau)]d\tau = \int_{-\infty}^{+\infty} [f(\tau)\phi(\tau)]\delta'(\tau)d\tau =$$

$$= -\left[\frac{d}{d\tau}[f(\tau)\phi(\tau)]\right]_{\tau=0} = -f'(0)\phi(0) - f(0)\phi'(0).$$

This gives the formal equivalence

$$\phi(t)\delta'(t) \equiv \phi(0)\delta'(t) - \phi'(0)\delta(t).$$

In the same way, if ϕ is a fixed function which is twice continuously differentiable in a neighborhood of the origin, then we should get

$$\int_{-\infty}^{+\infty} f(\tau)\big[\phi(\tau)\delta^{(2)}(\tau)\big]d\tau = \int_{-\infty}^{+\infty} [f(\tau)\phi(\tau)]\delta^{(2)}(\tau)d\tau = \left[\frac{d^2}{d\tau^2}[f(\tau)\phi(\tau)]\right]_{\tau=0} =$$

$$= f^{(2)}(0)\phi(0) + 2f'(0)\phi'(0) + f(0)\phi^{(2)}(0).$$

so that

$$\phi(t)\delta^{(2)}(t) \equiv \phi(0)\delta^{(2)}(t) - 2\phi'(0)\delta'(t) + \phi^{(2)}(0)\delta(t).$$

In general, if ϕ has continuous derivatives at least up to order n, then $\phi(t)\delta^{(n)}(t)$ reduces to

$$\phi(0)\delta^{(n)}(t) - n\phi'(0)\delta^{(n-1)}(t) + \frac{n(n-1)}{2!}\phi^{(2)}(0)\delta^{(n-2)}(t) - \cdots + (-1)^n\phi^{(n)}(0)\delta(t)$$

Exercise.

The expressions $\sin(t)\,\delta'(t)$ and $\cos(t)\,\delta'(t)$ may be replaced by simpler equivalent expressions as follows:

$$\sin(t)\,\delta'(t) = (\sin(0))\delta'(t) - (\cos(0))\delta(t) = -\delta(t),$$

$$\cos(t)\,\delta'(t) = (\cos(0))\delta'(t) - (-\sin(0))\delta(t) = \delta'(t).$$

This can be confirmed by examining typical integrals containing the products concerned and then reducing them to simpler equivalent forms. Thus, if *f(t)* is an arbitrary continuously differentiable function, we have

$$\int_{-\infty}^{+\infty} f(\tau)[\sin(\tau)\,\delta'(\tau)]d\tau = \int_{-\infty}^{+\infty} [f(\tau)\sin(\tau)]\delta'(\tau)d\tau = \left[-\frac{d}{d\tau}(f(\tau)\sin(\tau))\right]_{\tau=0} =$$

$$= -f(0) = -\int_{-\infty}^{+\infty} f(\tau)\delta(\tau)d\tau.$$

which shows that $\sin(t)\,\delta'(t)$ is operationally equivalent to $-\delta(t)$. In the same way for $\cos(t)\,\delta'(t)$ we have

$$\int_{-\infty}^{+\infty} f(\tau)[\cos(\tau)\,\delta'(\tau)]d\tau = \int_{-\infty}^{+\infty} [f(\tau)\cos(\tau)]\delta'(\tau)d\tau = \left[-\frac{d}{d\tau}(f(\tau)\cos(\tau))\right]_{\tau=0} =$$

$$= -f'(0) = -\int_{-\infty}^{+\infty} f(\tau)\delta'(\tau)d\tau.$$

Integration of the Delta function.

The significance of the delta function (and that of each of its derivatives) is intimately bound up with a certain conventional use of the notation for integration. This convention is consistent with the classical role of integration as a process which is, in some sense, inverse to that of differentiation.

If $t \neq 0$ the function $u(t-\tau)$ is a continuous function of τ in a neighborhood of $\tau \equiv 0$. Accordingly the formal product $u(t-\tau)\delta(\tau)$ is meaningful and we may write

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \int_{-\infty}^{+\infty} u(t-\tau)\delta(\tau)d\tau = u(t).$$

In particular this gives

$$\int_{-\infty}^{+\infty} \delta(\tau)d\tau = 1.$$

More generally, recall that we define $\delta^{(n)}$ as the derivative of $\delta^{(n-1)}$:

$$\delta^{(n)}(t) = \frac{d}{dt}\delta^{(n-1)}(t),$$

and is consistent with the definitions of $\delta^{(n)}$ and $\delta^{(n-1)}$ to write

$$\int_{-\infty}^{t} \delta^{(n)}(\tau)d\tau = \int_{-\infty}^{+\infty} u(t-\tau)\delta^{(n)}(\tau)d\tau = \delta^{(n-1)}(t),\ \ n \geq 1$$

Finally, since the function $f(t) \equiv 1$ is, trivially, infinitely differentiable we have the result

$$\int_{-\infty}^{+\infty} \delta^{(n)}(\tau)d\tau = 0,\ \ for\, n \geq 1.$$

Chapter II. Stationary phase method. Contribution from stationary boundary points.

II.1. Contribution of the stationary boundary points

Boundary stationary points of the II type.
Consider the integral

$$F(\lambda) = \int_{\Omega} f(x) e^{i\lambda S(x)} dx \qquad (2.1)$$

where Ω is a domain with smooth boundary in $\mathbb{R}^n$ and analogous $f(x), S(x) \in C^\infty(\Omega) \cap C([\Omega])$, $S(x)$ is real value function.

We introduce the concept of the contribution from the boundary $\partial\Omega$ of the Ω to the integral (2.1). For the sake of simplicity, let's assume the phase S have a finite number of stationary points $x^1, \ldots .x^m \in \Omega$. We construct C^∞ partition of unity in $\mathbb{R}^n_x$:

$$1 \equiv \sum_{j=1}^{m} \varphi_j(x) + \varphi_{\partial\Omega}(x) + \sum_{j=1}^{N} \psi_j(x) \qquad (2.2)$$

here $\varphi_j, \psi_j \in C_0^\infty(\Omega)$ and $\varphi_j \equiv 1$ in the neighborhood of the point x^j, $\varphi \equiv 0$ in the neighborhood of the point x^0 for $k \neq j$.
The function $\varphi_{\partial\Omega} \neq 1$ in some ε-neighborhood is the set $\partial\,\Omega$ and outside the domain Ω . Then

$$F(\lambda) = \sum_{j=1}^{n} F\left(\lambda, x^j\right) + F(\lambda, \partial\Omega) + \Phi(\lambda).$$

where $\Phi(\lambda) = O(\lambda^{-\infty})$ as $\lambda \to +\infty$ and

$$F(\lambda) = \int_{\Omega} f(x)\varphi_j(x) e^{i\lambda S(x)} dx,$$

$$F(\lambda, \partial\Omega) = \int_{\Omega} f(x)\varphi_{\partial\Omega}(x) e^{i\lambda S(x)} dx \qquad (2.3)$$

It's obvious that

$$\Phi(\lambda) = \sum_{j=1}^{N} \int_{\Omega} f\psi_j e^{i\lambda S} dx \,.$$

By construction, supp ψ_j does not contain stationary points of the function S and intersects $\partial\,\Omega$. By virtue of Lemma, each of the integrals which has no critical points of phase function e.g. Φ has the order $O(\lambda^{-\infty})$ as $\lambda \to +\infty$.

The integral F (λ, ∂ Ω) will be called the contribution from the boundary ∂ Ω to the integral F (λ). Formula (2.2) means that the asymptotic behavior of F (λ) is equal to the sum of the contributions from the stationary points of the phase S lying in the domain Ω and from the boundary of the domain ∂ Ω.

The choice of the function $\varphi_{\partial\Omega}(x)$ determines the contribution does not matter:

Integrals of the form (2.3) with different functions $\varphi_{\partial\Omega}$ differ by an order of magnitude $O(\lambda^{-\infty})$.

If ∂ Ω the phase S has no stationary points, then the integral F (λ, ∂ω) reduces to the integral over ∂ Ω (up to $O(\lambda^{-\infty})$).

Lemma 2.1. Let $\Omega -$ be a bounded domain in $\mathbb{R}^n$ with boundary of class C^∞, functions $f(x), S(x) \in C^\infty([\Omega])$, the phase S is real-valued and has no stationary points on ∂ Ω . Then, for any integer $N \geq 0$, the expansion

$$F(\lambda, \partial\Omega) = \sum_{j=0}^{N} (i\lambda)^{-j} \int_{\partial\Omega} e^{i\lambda S(x)} \omega_j(x) + R_N(\lambda) \tag{2.4}$$

here $\omega_j(x) -$ the differential forms of degree n-1 of class C^∞ on ∂ Ω ,

$$R_N(\lambda) \leq C_N \lambda^{-N-1} \tag{2.5}$$

as $\lambda \geq 1$.

As usual, the decomposition (2.4) can be differentiated by λ any number of times.

The forms $\omega_j(x)$ have the form

$$\omega_j(x) = |\nabla S(x)|^{-2} \sum_{k=1}^{n} \frac{\partial S(x)}{\partial x_k} (L^t)^{j-1} f(x) dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n \tag{2.6}$$

(cap means that the corresponding factor is absent), where the L-operator.

In particular, $\omega_1(x) = f(x)\omega_S(x)$, where ω_S is the Leray-Gelfand differential form corresponding to the function S. Let us write the first term in the expansion:

$$F(\lambda, \partial\Omega) = \frac{1}{i\lambda} \int_{\partial\Omega} \frac{f(x)}{|\nabla S(x)|^2} e^{i\lambda S(x)} \sum_{k=1}^{n} \frac{\partial S(x)}{\partial x_k} \times dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n + O(\lambda^{-2}) \tag{2.7}$$

Considering, that

$$\sum_{k=1}^{n} S'_{x_k}(x)dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n = \frac{\partial S(x)}{\partial n_x} d\sigma,$$

where dσ is an element of the surface area ∂ω, $\frac{\partial}{\partial n_x}$ is the derivative in the direction of the external normal to ∂ Ω at x, the leading term of the asymptotic can be written in the form

$$F(\lambda, \partial\Omega) = \frac{1}{i\lambda} \int_{\partial\Omega} e^{i\lambda S(x)} \frac{f(x)}{|\nabla S(x)|^2} \frac{dS(x)}{dn_x} d\sigma + O(\lambda^{-2}). \qquad (2.7)'$$

Proof.

For brevity, we denote $\varphi_{\partial\Omega}$ by φ. The integral (2.3) is taken of some ε-neighborhood Ω_ε of the boundary ∂Ω. We have $\partial\Omega_\varepsilon = \partial\Omega \cup \Gamma$, where Γ intersects ∂ Ω . By construction, φ ≡ 1 on ∂Ω, φ ≡ 0 on Γ.

Applying formula (2.4) to the integral (2.3) and integrating by parts, we obtain

$$F(\lambda, \partial\Omega) = \frac{1}{i\lambda} \int_{\partial\Omega} f(x)\varphi(x) L\, e^{i\lambda S(x)} dx = \frac{1}{i\lambda} \int_{\partial\Omega} e^{i\lambda S(x)} f(x) |\nabla S(x)|^{-2} \times$$

$$\times \sum_{k=1}^{n} S'_{x_k}(x)dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n - \frac{1}{i\lambda} F_1(\lambda, \partial\Omega).$$

The integral F_1 is obtained from F by replacing $f\varphi \to L^t(f\varphi)$, where the L-operator. Since the integrand in the integral F1 is bounded, then $\lambda^{-1}F_1 = O(\lambda^{-1})$.

Integrating the integral F_1 by parts, we obtain $F_1 = (i\lambda)^{-1}\widetilde{F_1} - (i\lambda)^{-1}F_2$, where $\widetilde{F_1}$ is the integral over ∂ Ω , F_2 is the integral over $\partial\Omega$, which is obtained from F by replacing $f\varphi \to L^2(f\varphi)$.

Since $\widetilde{F_1} = O(1), F_2 = O(1)$, we have proved (2.7). Continue Integration by parts we obtain (2.4), (2.5).

Let us prove (2.6). For $j = 1$, this formula is proved. Let $j > 1$, Then the corresponding integral is

$$\int_{\partial\Omega} e^{i\lambda S(x)} \sum_{k=1}^{n} a_k(x)(L^t)^{k-1}\big(f(x)\varphi(x)\big)\, dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n,$$

where $a_k(x) = S'_{x_k}(x)|\nabla S(x)|^{-2}$.

Since φ (x) ≡ 1 in some neighborhood of the boundary $\partial\Omega$, all its derivatives are equal to zero on $\partial\Omega$. Therefore, for $x \in \partial\Omega$, we have $(L^t)^{k-1}\big(f(x)\varphi(x)\big) = (L^t)^{k-1}\big(f(x)\big)$ and (2.6) is proved.

Let us analyze the results of Lemma 2.1. We showed that the contribution from the boundary F (λ,$\partial\Omega$) is asymptotically equal to the sum of the integrals over the boundary. But each of these integrals is an integral over the manifold $\partial\Omega$ of the rapidly oscillating function. To obtain the final asymptotic formulas, it is necessary to calculate the asymptotic of these integrals, which we will do.

Consider the function $S(x)$ on the manifold $\partial\Omega$. By hypothesis, ∇S (x) ≠ 0; however, this function as a function on $\partial\Omega$ has stationary points on $\partial\Omega$ (for example, it reaches the largest and smallest values on $\partial\Omega$). The stationary points of the function S (x) on $\partial\Omega$, as a function on the manifold $\partial\Omega$ (i.e., S (x) is considered only for $x \in \partial\Omega$), we will call *stationary points of the II type or boundary stationary points*.

Let the manifold $\partial\Omega$ in a neighborhood of the point x^0 be defined parametrically, i.e.,

$$x_1 = \psi_1(u_1, \dots, u_{n-1}), \dots, x_n = \psi_n(u_1, \dots, u_{n-1}),$$
$$(u_1, \dots, u_{n-1}) \in U,$$

where $U -$ is neighborhood of the point $(0, \dots, 0)$. In the vector notation, we have

$$x = \psi(u), \;\; x^0 = \psi(0).$$

The point x^0 is a stationary point of the II type of function S, if

$$\nabla_u \tilde{S}(0) = 0, \;\; \tilde{S}(u) = (S \circ \psi)(u).$$

A stationary point of the II type x^0 is called non-degenerate if

$$\det \left\| \frac{\partial^2 (S \circ \psi)(u)}{\partial u_i \partial u_j} \right\|_{u=0} \neq 0$$

It is easy to verify that both of these definitions are invariant with respect to the choice of local coordinates on $\partial\Omega$. The stationary boundary point of the I type, obviously, is the point x at which $\nabla S(x^0) = 0$.

Let $\partial\Omega$ in a neighborhood of x^0 be given by the equation

$$g(x) = 0, \ \nabla g(x^0) \neq 0, \tag{2.8}$$

where g has in class of function C^∞. Then x^0 will be stationary point of the II type, if and only if $a \neq 0$ exists and

$$\nabla S(x^0) = a\nabla g(x^0). \tag{2.9}$$

Geometrically, this means that a manifold of equation $S(x) = S(x^0)$ is tangent to $\partial\Omega$ at the point $x^0(Figure\ 2.1)$:

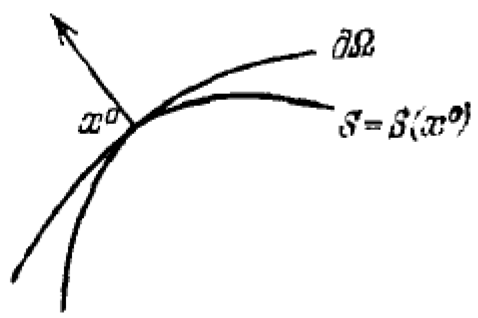


Figure 2.1.

At least one of the components of the vector $\nabla g(x^0)$ is nonzero;

Let $\frac{\partial g(x^0)}{\partial x_n} \neq 0$ for definiteness. Then from equation (2.8) we can express x_n in terms of the remaining variables:

$$x_n = \psi(x'), \qquad x' \in U, \qquad x' = (x_1, \dots, x_{n-1}),$$

where U is neighborhood of the point $x^{0'}$, so that we can take $x_1, \dots x_{n-1}$ as local coordinates on $\partial\Omega$. For $x \in \partial\Omega$ we have

$$S(x) = S\big(x', \psi(x')\big) \equiv \tilde{S}(x').$$

For simplify, $x^0 = 0, S(0) = g(0) = 0.$ Then

$$\tilde{S}(x') = \sum_{j=1}^{n-1} \widetilde{S_j} x_j + \frac{1}{2}\sum_{i,j=1}^{n-1} \widetilde{S_{ij}} x_i x_j + \cdots$$

The coefficients of this decomposition are of the form

$$\tilde{S}_j = S_j - S_n \frac{g_j}{g_n} \tag{2.10}$$

$$\tilde{S}_{ij} = S_{ij} + g_n^{-1} S_n\big(-g_{ij} + 2g_i g_{jn} g_n^{-1} - g_{nn} g_i g_j g_n^{-2}\big) - 2S_{in} g_j g_n^{-1} + S_{nn} g_i g_j g_n^{-2}.$$

here $S_j = S'_{x_j}(x^0), \ S_{ij} = S''_{x_i x_j}(x^0)$ and similarly defined g_j, g_{ij}.

From the condition $d\tilde{S} = 0$, we obtain $\frac{s_j}{s_n} = \frac{g_j}{g_n}$, i.e. (2.9). Non-degeneracy of a stationary point means that $\det \tilde{S}''_{x'x'} \neq 0$ is at this point.

Theorem 2.1. Let the conditions of Lemma 2.1 be satisfied and let $\partial\Omega$ have exactly one and, moreover, non degenerate stationary point of the second type x^0 of the function $S(x^0)$. Then, as λ → + ∞, the asymptotic expansion

$$F(\lambda, \partial\Omega) \sim \lambda^{-\frac{n+1}{2}} e^{i\lambda S(x^0)} \sum_{j=0}^{\infty} a_j \lambda^{-j}. \qquad (2.11)$$

This decomposition can be differentiated by λ any number of times. We write out the leading term of the asymptotic. Moreover, we assume that, $\partial\Omega$ is given by the equation $g(x) \equiv 0$ for x close to x^0, and that $g'_{x_n}(x^0) \neq 0$. Then

$$F(\lambda, \delta\Omega) = i(2\pi)^{\frac{n-1}{2}} \lambda^{-\frac{n+1}{2}} e^{\left(i\lambda S(x^0) + \frac{i\pi}{4} sgn \tilde{S}''_{x'x'}(x^0)\right)} \left|\det \tilde{S}''_{x'x'}(x^0)\right|^{-\frac{1}{2}} \times$$

$$\times \left(\frac{\partial S(x^0)}{\partial x_n}\right)^{-1} [f(x^0 + O(\lambda^{-1})]. \qquad (2.11)'$$

Here $\tilde{S}''_{x'x'}(x^0)$ – matrix with elements $\tilde{S}_{ij}$ $\left(see\ (2.10)\right)$.

Proof. We construct a C^{∞} – partition of unity into ∂: $1 = \varphi_0(x) + \varphi_1(x)$, $x \in \partial\Omega$. Here $\varphi_0 = 1$ for point x close to x^0, and $supp\ \varphi_0$ concentrated in a small neighborhood of x^0. Then each of the integrals on the right-hand side of equality (2.4) is divided into two terms. Integrals containing φ_1 are of order $O(\lambda^{-\infty})$. In the remaining Integrals, it remains to go to the local coordinates on $\partial\Omega$ and use the above Theorem 2.1.

Obviously, if $\partial\Omega$ has a finite number of nondegenerate stationary points of the second type, then the asymptotic behavior of the contribution from the boundary $F(\lambda, \partial\Omega)$ is equal to the sum of contributions of the form (2.11) from these points.

The contribution from the boundary to the Integral F(λ) may be of a greater order than $O\left(\lambda^{-\frac{n}{2}}\right)$. Consider

Example. Let the conditions of Lemma be satisfied and $S(x) \equiv S_0 \equiv const$ on $\partial\Omega$. Then, as λ → + ∞, by $(2,4), (2.7)'$

$$F(\lambda, \partial\Omega) \sim e^{i\lambda S_0} \frac{1}{i\lambda} \left[\int_{\partial\Omega} \frac{\partial S(x)}{\partial n} |\nabla S(x)|^{-2} d\sigma + \sum_{k=1}^{\infty} a_k (i\lambda)^{-k} \right],$$

$d\sigma -$ element of the surface $\partial\Omega$, so that the leading term of the asymptotic is of order λ^{-1}, regardless of the dimension $\partial\Omega$. Such a situation takes place, for example, in the well-known experiment of Arago during diffraction by a circular disk of the field of a point light source lying on a straight line perpendicular to the disk and passing through its center.

Remark 2.1. Lemma 2.1 and Theorem 2.1. without any changes are transferred to integrals containing additional parameters

$$F(\lambda, \alpha) = \int\limits_{\Omega(\alpha)} e^{i\lambda S(x^0)} f(x, \alpha) dx,$$

If all conditions are satisfied uniformly in α.

Contribution from the boundary stationary point of I type.

Let an $\Omega -$ bounded domain in $\mathbb{R}^n$ be a boundary $\partial\Omega$ of class C^∞, the functions f (x), $S(x) \in C^\infty([\Omega])$, and the function S (x) be real-valued.

Let $x^0 \in \partial\Omega, \nabla S(x) \in C^\infty([\Omega])$. We call x^0 a non-degenerate boundary stationary point if the matrix $B(x^0) = ||S''_{\xi\xi}(x^0)||$ is non-degenerate where $\xi = (\xi_1, \dots, \xi_{n-1}) -$ coordinates are in the orthonormal basis located in the tangent plane of $T\, \partial\Omega_{x^0}$ to $\partial\Omega$ at the point x^0.

Theorem 2.2. Let $x^0 \in \partial\Omega -$ be a non-degenerate stationary boundary point of the function S (x) and $f(x) \equiv 0$ outside some sufficiently small neighborhood of the point x^0. Then, as λ → + ∞, the asymptotic expansion

$$F(\lambda) \sim \lambda^{-\frac{n}{2}} e^{i\lambda S(x^0)} \sum_{k=0}^{\infty} a_k \lambda^{-\frac{k}{2}}. \tag{2.12}$$

This decomposition can be differentiated by λ any number of times. the leading term of the asymptotic is equal to the right one, multiplied by $\frac{1}{2}$, i.e., half the contribution from the internal stationary point is simply equal.

Proof. We assume that $supp\, f$ does not contain non-degenerate stationary points (of the first and second type) other than x^0. Let $x^0 = 0, S(x^0) = 0$ for simplicity. In the neighborhood of the point x^0, we introduce the local coordinates $u = (u_1, \dots, u_n), x = \psi(u),$ so that $\partial\Omega$ has the form $u_n = 0$ and that the point u = 0 corresponds to the point x = 0. Then

$$F(\lambda) = \int_V \varphi(u) e^{i\lambda \tilde{S}(u)} du,$$

where indicated

$$\tilde{S}(u) = (S\varphi)(u), \qquad \varphi(u) = (f\psi)(u) \det \psi'_u(u),$$

and the V-semi-neighborhood of the point u = 0 let $u_n > 0$ for $u \in V$ for definiteness.

We apply the stationary phase method to the integral F (λ) with respect to the variables $u_1, \dots, u_{n-1}$, thereby reducing the integral to one-dimensional. Without loss of generality, we can assume that V is a cube $V = I \times \tilde{V}$, where I-interval $0 < u_n < \delta, \tilde{V} - cube\ -\delta < u_j < \delta, 1 \leq j \leq n-1\ and\ \delta > 0$ how small is needed. This statement follows from the principle of localization. Then

$$F(\lambda) = \int_0^\infty F_1(\lambda, u_n) du_n,$$

$$F_1(\lambda, u_n) = \int_{\tilde{V}} \psi(u) e^{i\lambda \tilde{S}(u)}\, du'.$$

where $u' = (u_1, \dots, u_{n-1})$. The stationary points of the function $\tilde{S}$ as functions of u', are determined from the equation $\tilde{S}'_{u'}(u) = 0$. We have for small u

$$\tilde{S}(u) = \frac{b_{nn}}{2} u_n^2 + u_n < b, u' > + \frac{1}{2} < Bu', u' > + \cdots,$$

where b is an n-vector B – symmetric matrix of order n.

Therefore, the equation $\tilde{S}'_{u'}(u) = 0$ has the form

$$u_n b + Bu' + \cdots = 0.$$

Since, by condition, detB ≠ 0, then

$$u'(u_n) = -u_n B^{-1} b + \cdots,$$

and this stationary point is non-degenerate for small δ since

$$\tilde{S}(u', 0) = \frac{1}{2} < Bu', u' > + \cdots$$

Applying Theorem 2.2 to the integral F_1, we obtain the asymptotic expansion

$$F_1(\lambda, u_n) \sim \lambda^{-\frac{n-1}{2}} e^{i\lambda\tilde{S}(u'(u_n),u_n)} \sum_{j=0}^{\infty} a_j(u_n),$$

where $a_j(u_n) \in C^\infty([0,\delta])$. From this function a_j vanish at $u_n = \delta$ along with all derivatives. Further,

$$\tilde{S}(u'(u_n), u_n) = \frac{1}{2}(b_{nn} - < b, B^{-1}b >)u_n^2 + \cdots$$

The coefficient for u_n^2equals to $\det \tilde{S}''_{uu}(0)(\det B)^{-1}$ and therefore is nonzero. Applying above theorem, we obtain the expansion (2.12).

II.2. Asymptotic of the Fourier transform of the characteristic function of a convex set and similar problems

We consider the asymptotic for $|\xi| \to \infty$ of the integral

$$F(\xi) = \int_\Omega f(x) e^{-i<x,\xi>} dx,$$

where $\Omega -$ is a bounded region in $\mathbb{R}^n$ with boundary $\partial\Omega \in C^\infty$, $\xi \in \mathbb{R}^n$, function $f(x) \in C^\infty([\Omega])$. If $f(x) \equiv 1$, then $F(\xi)$ is the Fourier transform of the characteristic function of the set Ω (this function is 1 for $x \in \Omega$ and equal to 0 outside Ω) for simplicity, let the origin lie inside Ω.

The phase function $S =< x, \xi >$ does not have stationary points at $\xi \neq 0$, since $S_x = \xi$, but it has stationary boundary points of the second type on $\partial\Omega$. Namely, it is precisely these points $x(\xi)$ at which the hypersurface $< x, \xi >= const$ touches $\partial\Omega$.

Lemma 2.2. A stationary point of the second type $x(\xi) \in \partial\Omega$ is non-degenerate if and only if the Gaussian curvature of $\partial\Omega$ at this point is nonzero.

Proof. Without loss of generality, we can assume that $\xi = (0, \dots, 0, \xi_n), \xi_n \neq 0$. Let $x^0(\xi)$ be one of the stationary boundary points, then the normal n_{x^0} to $\partial\Omega$ at this point is parallel or anti-parallel to the vector ξ. In a neighborhood of the point $x^0(\xi)$, we choose the local Cartesian coordinates y so that the axis Oy is directed along the external normal to $\partial\Omega$ and so that the points $(y', 0), y' = (y_1, \dots, y_{n-1})$ are puddled in the tangent plane $T_{x^0}\,\partial\Omega$ at $\partial\Omega$. Here $y = 0 \leftrightarrow x = x^0(\xi)$. Then the equation $\partial\Omega$ for small y takes the form

$$y_n = \frac{1}{2} < By', y' > + \cdots, \tag{2.13}$$

where is a B-symmetric square matrix of order n-1, and

$$S(x,\xi) =< x^0(\xi), \xi > +y_n\xi_n. \tag{2.14}$$

It follows from these formulas and (2.10) that the non-degeneracy of the point $x^0(\xi)$ is equivalent to the non-degeneracy of the matrix B. But from (2.13) it follows that det B is equal to the Gaussian curvature of the hypersurface $\partial\Omega$ at the point $x^0(\xi)$. *The lemma is proved.*

Let $K -$ cone

$$K = \left\{\xi \in \mathbb{R}^n : 0 < |\xi| < \infty, \frac{\xi}{|\xi|} \in U\right\},$$

$U -$ area on a unit sphere $|\xi| = 1$.

Theorem 2.3. Let K be a simply connected cone and, for any $\xi \in [K], \xi \neq 0$, let the Gaussian curvature of the boundary $\partial\Omega$ be nonzero at all stationary points of the second type of function $S =< x, \xi >$. Then:

1^0. The function S for all $\xi \in [K], \xi \neq 0$ has the same number $m = m(K)$ of stationary points of the second type $x^{(1)}(\xi), \dots, x^{(m)}(\xi)$ and all of them are non-degenerate.

2^0. The asymptotic behavior of $F(\xi)$ for $\xi \in [K]$ is equal to the sum of the contributions from these points

$$F(\xi) \sim \sum_{j=1}^{m} F\left(\xi, x^{(j)}(\xi)\right).$$

We write the formula for the contribution from the point $x(\xi)$:

$$F\big(\xi, x(\xi)\big) \sim i\varepsilon e^{i<x(\xi),\xi>}(2\pi)^{\frac{n-1}{2}}|\xi|^{-\frac{n+1}{2}}|\chi_1 \dots \chi_{n-1}|^{-\frac{1}{2}}e^{\frac{i\pi}{4}\sum_{j=1}^{n-1} sgn(\varepsilon\chi_j)} \times$$

$$\times\left[(f \circ x)(\xi) + \sum_{k=1}^{\infty} a_k(\omega)|\xi|^{-k}\right], \ \ \omega = \frac{\xi}{|\xi|}. \tag{2.15}$$

Here $\chi_1, \chi_2, \dots, \chi_{n-1} -$ principal curvatures of $\partial\Omega$ at point $x(\xi)$,

$$-\varepsilon = sgn < \xi, n_{x(\xi)} >, \tag{2.16}$$

where $n_{x(\xi)}$ is the outward normal of $\partial\Omega$ at a point $x(\xi)$.

Proof. The non-degeneracy of stationary points follows from Lemma 2.2. Let $\omega^0 = \frac{\xi^0}{|\xi^0|} \in U$ then have a finite number of stationary points of the second type of $x^{(j)}(\omega^0), 1 \leq j \leq m.$

Indeed, if there were infinitely many of them, then, due to the compactness of $\partial\Omega$, they would have a limit point, which would also be a stationary point. But non-degenerate stationary points are isolated. By the implicit function theorem for all

$\omega \in S^{n-1}$ sufficiently close to ω^0, there are also exactly m stationary points of $x^j(\omega)$, and $x^j(\omega) \to x^j(\omega^0)$ for $\omega \to \omega^0$. Using the Heine-Borel lemma, we find that the number of stationary points is the same for all $\xi \in [K]$, $\xi \neq 0$. Therefore, the asymptotic behavior of $F(\xi)$ is the sum of contributions from points $x^j(\xi)$.

Let $x(\xi)$ be the bottom of these points. In the notation of Lemma (See (2.13), (2.14)) for the leading term of the contribution $F\big(\xi, x(\xi)\big)$, we obtain from the formula (2.7)′

$$F\big(\xi, x(\xi)\big) \sim i\varepsilon e^{i<x(\xi),\xi>}(2\pi)^{\frac{n-1}{2}}|\xi|^{-\frac{n+1}{2}}|\det B|^{-\frac{1}{2}}e^{\frac{i\pi}{4}sgn(\varepsilon B)}(f \circ x)(\xi),$$

where is indicated in (2.16). This implies (2.15). The theorem is proved.

The manifold dimensions $(n-1)$ in $\mathbb{R}^n$ are called strictly convex if all its principal curvatures at any point are positive. If an Ω −bounded domain in $\mathbb{R}^n$ with a strictly convex boundary, then for any $\xi \neq 0$ the function $S = < x, \xi >$ has exactly 2 stationary points of the second type $x^{\pm}(\xi)$. The normal to $\partial\Omega$ at point $x^{+}(\xi)$ (respectively, $x^{-}(\xi)$) is parallel (antiparallel) to the vector. Denote $k^{\pm}(\xi)$ by the Gaussian curvature $\partial\Omega$ at points $x^{\pm}(\xi)$. By Theorem 2.3 we have

Corollary 2.1. Let $\Omega -$ be a bounded domain with a strictly convex boundary of class C^∞. Then for $|\xi| \to \infty$

$$F(\xi) \sim -i(2\pi)^{\frac{n-1}{2}}|\xi|^{-\frac{n+1}{2}}\left[e^{i<x^{+}(\xi),\xi>}|k^{+}(\xi)|^{-\frac{1}{2}}\right]e^{-\frac{i\pi}{4}(n-1)} \times$$

$$\times\left[(f \circ x^{+})(\xi) + \sum_{j=1}^{\infty} a_j^{+}(\omega)|\xi|^{-j}\right] -$$

$$-e^{i<x^{-}(\xi),\xi>}|k^{-}(\xi)|^{-\frac{1}{2}}e^{\frac{i\pi}{4}(n-1)}\left[(f \circ x^{-})(\xi) + \sum_{j=1}^{\infty} a_j^{-}(\omega)|\xi|^{-j}\right].$$

Coefficients $a_j^{\pm}(\omega) \in C^\infty(S^{n-1})$, where S^{n-1}-sphere $|\xi| = 1$.

Corollary 2.2. Let the region Ω-unbounded conditions of Theorem 2.3 be satisfied and property $\xi \in K$ of Theorem 2.3 holds for all $\xi \in K$. Then if $f(x)$ is a compactly supported function then conclusion 2 of Theorem 2.3. holds true.

4. Asymptotic behavior of the principal values of the integrals.

Let $P(x)$ be a real-valued function of the class $C^\infty(\mathbb{R}^n)$ with real zeros, and $f(x) \in C_0^\infty(\mathbb{R}^n)$. Then the integral $I = \int_{\mathbb{R}^n}\frac{f(x)}{P(x)}dx$, generally speaking, diverges. We give one way to regularize this integral.

Let the set of zeros $\{x: P(x) = 0\}$ of function P contain a bounded component of M_0 and let $\nabla P(x) \neq 0$ be on M_0. Then M_0 is a $C^\infty -$ manifold of dimension n-1.

Moreover, for all sufficiently small ε the set $\{x: P(x) = \varepsilon\}$ contains the component M_ε with the same properties as M_0 and $M_\varepsilon \to M_0$ for $\varepsilon \to 0$. Let $f(x) = C_0^\infty(\mathbb{R}^n)$ be concentrated in a sufficiently small neighborhood of M_0.

The main value of the integral I, by definition, is called the limit

$$v.p. \int_{\mathbb{R}^n} \frac{f(x)}{P(x)} dx = \lim_{\varepsilon \to +0} \int_{|P(x)|>\varepsilon} \frac{f(x)}{P(x)} dx. \qquad (2.17)$$

Let the $\omega_P -$ differential Leray-Gelfand form correspond to $P: dP \wedge \omega_P = dx$. Then

$$v.p. \int_{\mathbb{R}^n} \frac{f(x)}{P(x)} dx = \lim_{\varepsilon \to +0} \int_{|c|>\varepsilon} \Phi_f(c) \frac{dc}{c} = v.p. \int_{-\infty}^{\infty} \Phi_f(c) \frac{dc}{c},$$

$$\Phi_f(c) = \int_{P(x)=c} f(x)\, \omega_P(x),$$

where does the existence of the limit immediately (2.17).
Consider the integral

$$F(\lambda) = v.p. \int \frac{f(x)}{P(x)} e^{i\lambda S(x)} dx \qquad (2.18)$$

We list the conditions on the functions f, S, P.
1^0. A function $P(x) \in C^\infty(\mathbb{R}^n)$ and a real-valued set $\{x \in \mathbb{R}^n: P(x) = 0\}$ of its real zeros contains a compact $C^\infty -$ manifold M_0^{n-1} of dimension $n-1, \nabla P(x) \neq 0$ for $x \in M_0^{n-1}$.
2^0. The function $f(x) \in C_0^\infty(\mathbb{R}^n)$ and is concentrated in a small neighborhood of the set M_0^{n-1}. The function in some region containing $supp\, f$ is real-valued.
3^0. $\nabla S(x) \neq 0$ for $x \in supp\, f$ and manifold M_0^{n-1}, the function $S(x)$ has a finite number and, moreover, nondegenerate stationary points of the second type $x^1, \dots x^m$.

We calculate the asymptotic behavior of the integral $F(\lambda)$ under these conditions.
We consider the integral at the beginning

$$\Phi(0, \lambda) = \int_{P(x)=0} f(x) e^{i\lambda S(x)} \omega_P(x),$$

where is the $\omega_P -$ differential form of Leray-Gelfand. The asymptotic behavior of this integral is the sum of the contributions of $\Phi_j(0, \lambda)$ from the stationary points.

$$\Phi(0, \lambda) \sim \sum_{j=1}^{m} \Phi_j(0, \lambda). \qquad (2.19)$$

Theorem 2.4. Let conditions 1^0-3^0 be satisfied. Then, for $\lambda \to +\infty$, the integral (2.18) has the asymptotic expansion

$$F(\lambda) \sim \pi i \sum_{j=1}^{m} \Phi_j(0,\lambda)\, sgn\, [< \nabla S(x^j), \nabla P(x^j) >].$$

This decomposition can be differentiated by λ any number of times.

The formula for contributions $\Phi_j(0,\lambda)$ will be given below.

Proof. We put

$$\Phi(c,\lambda) = \int_{P(x)=0} e^{i\lambda S(x)} f(x) \omega_P(x),$$

Then

$$F(\lambda) = v.p. \int_{-\infty}^{\infty} \frac{\Phi(c,\lambda)}{c} dc.$$

For small $c > 0$, the set $M_c = \{x: P(x) = c\} \cap suppf$ is a $C^\infty -$ manifold of dimension $n-1$, and the function S has exactly m on M_c and, moreover, non-degenerate stationary points of the second type $x^1(c), \dots, x^m(c)$.

Moreover, $x^j(c) \in C^\infty$ is small for $c, x^j(0) = x^j$. The asymptotic behavior of $\Phi(c,\lambda)$ at $\lambda \to +\infty$ is equal to the sum of contributions $\Phi_1(c,\lambda)$ from points of $x^j(c)$ uniformly in $c \in [-c_0, c_0]$ if $c_0 > 0$ is sufficiently small. Each of these contributions has the form

$$\Phi_j(c,\lambda) \sim \lambda^{-\frac{n-1}{2}} e^{i\lambda (S \circ x^j)(c)} \sum_{k=0}^{\infty} a_{kj}(c) \lambda^{-k},$$

where are the functions $a_{kj} \in C^\infty$ for small $|c|$. Let us show that,

$$sgn \frac{d}{dc}(S \circ x^j)(c)||_{c=0} = \pm 1$$

depending on whether the vectors $\nabla S, \nabla P$ at the point x are parallel or antiparallel. Since x^j is a stationary point of the second type, then $\nabla S = \alpha \nabla P, \alpha \neq 0$, is at this point. Differentiating the triumph of $P(x) = c$, we get with $x = x^j$: $< \nabla, \frac{dx}{dc} >= 1$, whence it follows that, $\frac{dS}{dc} =< \nabla S, \frac{dx}{dc} >= \alpha |\nabla P|^2 \neq 0$. Applying Theorem in the above to the integrals $v.p. \int \frac{\Phi_j(c,\lambda)}{c} dc$, we obtain (2.19).

We write the formula for the main term of contribution $\Phi_j(0,\lambda)$.

Let $\frac{\partial P(x^j)}{\partial x_n} \neq 0, x' = (x_1, \dots, x_{n-1}),$

$$\tilde{S}(x') = S(x), \qquad x \in M_0^{n-1}.$$

Then similarly $(2.11)'$ we have

$$\Phi_j(0,\lambda) = (2\pi)^{\frac{n-1}{2}} \lambda^{-\frac{n-1}{2}} e^{i\lambda S(x^j) + \frac{i\pi}{4} sgn\, \tilde{S}''_{x'x'}(x^j)} \left|\det \tilde{S}''_{x'x'}(x^j)\right|^{-\frac{1}{2}} \times$$

$$\times \left[\frac{\partial P(x^j)}{\partial x_n}\right]^{-1} \left[f(x^j) + O(\lambda^{-1})\right].$$

Corollary 2.3. Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \Omega$ where $\partial\Omega$ is the area in $\mathbb{R}^k$.

$$F(\lambda, \alpha) = v.p. \int_{\mathbb{R}^n} \frac{f(x,\alpha) e^{i\lambda S(x,\alpha)}}{P(x)} dx. \qquad (2.20)$$

Then all the conclusions of Theorem 2.4 remain valid for the integral (2.20) uniformly with respect to $\alpha \in K$, if $f, S \in C^\infty$ with respect to (x, α), conditions 2^0, 3^0 are satisfied uniformly in $\alpha \in \Omega$. Here K is an arbitrary compact lying in Ω.

Asymptotic behavior of the fundamental solutions of certain classes of differential equations with constant coefficients and radiation conditions.

We consider the equations

$$P(D)E(x) = \delta(x). \qquad (2.21)$$

here $x \in \mathbb{R}^n, D = (D_1, \dots, D_n), D_j = -\frac{i\partial}{\partial x_j}$ and $P(\xi) -$ is polynomial from $\xi = (\xi_1, \dots, \xi_n)$ with regular coefficients. The solution $E(x)$ of equation (2.21) is called the fundamental (or elementary) solution of the operator P(D).

We will assume that the conditions:

1. $P(\xi) -$ hypoelliptic polynomial, $P(\xi) = T(\xi)Q(\xi)$, where $T(\xi) -$ polynomial with real coefficients, the polynomial $Q(\xi)$ has no real zeros.
2. The real zeros of the polynomial $T(\xi)$ form $m \geq 1$ of smooth closed strictly convex varieties $K_1, \dots, K_m$ of dimension $n-1$. These varieties do not intersect and $\nabla T(\xi) \neq 0$ under $\xi \in K_j, 1 \leq j \leq m$.

We are interested in the asymptotic behavior of the fundamental solution $E(x)$ for $|x| \to \infty$.

The simplest example is the Helmholtz operator $\Delta + k^2, k > 0$.

We obtain the integral representation for $E(x)$. Using the Fourier transform in (2.21) we get $P(\xi)\tilde{E} = 1$, whence $\tilde{E} = \frac{1}{P(\xi)}$ and applying the inverse Fourier transform, we get $E(x) = (2\pi)^{-n} \int \frac{e^{i<x,\xi>} d\xi}{P(\xi)}$. However, this integral diverges, since P has real zeros, and it needs to be regularized.

Let $n = 1$ and $\xi_1, \dots, \xi_m -$ be real zeros of P all of them simple. Then the function

$$E(x) = \frac{1}{2\pi}\int\limits_{\gamma} \frac{e^{ix\xi}}{P(\xi)} d\xi.$$

it is a fundamental solution. Here, the $\gamma -$ contour in the complex plane ξ, which coincides with the weight axis everywhere except small neighborhoods of points ξ_k. In these neighborhoods γ goes along a semicircle that goes around point ξ_k from below or from above. The formula for E can also be written as follows:

$$E(x) = \frac{1}{2\pi}\ v.p. \int\limits_{-\infty}^{\infty} e^{ix\xi} \frac{d\xi}{P(\xi)} + i\sum_{j=1}^{m} \varepsilon_j \frac{e^{ix\xi_j}}{P'(\xi_j)}. \qquad (2.22)$$

Here $\varepsilon_j = +1(-1)$, if γ goes around point ξ_j from above (from below).

The function E(x) is a solution of equation (2.21) in the following sense. It is a functional over the space K of functions belonging to $C_0^{\infty}(-\infty, \infty)$. Its Fourier transform E(ξ) satisfies the equation P$(\xi)E(\xi) = 1$, i.e., for any function ψ(ξ)∈Z this is the space of functions that are Fourier transforms of functions from K), the identity

$$(P(\xi)E(\xi), \psi(\xi)) = \int_{-\infty}^{\infty} \psi(\xi) d\xi. \qquad (2.23)$$

Any function $\psi \in Z$, obviously, analytically extends to the entire complex plane ξ and kills faster than any degree for real ξ → ∞. a-priory,

$$(\widetilde{E}\ (\xi)\ \psi(\xi)) = \int_{\gamma} \frac{\psi(\xi)}{\mathrm{P}(\xi)} d\xi.$$

Hence,

$$((P(\xi)E(\xi), \psi(\xi)) = \int_{\gamma} \psi(\xi)\delta\xi = \int_{-\infty}^{\infty} \psi(\xi)\ d\xi\ ,$$

Since $\psi -$ is an entire function and (4.23) is proved.

Note that formula (2.22) defines 2^m fundamental solutions (each zero can be circumvented either from below or from above).

For n> 1, a formula similar to (2.22) holds for E(x). We bring her. Let a function h$(\xi) \in C_0^{\infty}(R^n)$ and h = 1 in some neighborhood of all the varieties K_j.

Then

$$E(x) = (2\pi)^{-n}\, v.p. \int\limits_{\mathbb{R}^n} \frac{h(\xi)e^{i<x,\xi>}}{P(\xi)} d\xi + (2\pi)^{-n} \sum_{j=1}^{m} \varepsilon_j \pi i \int\limits_{K_j} e^{i<x,\xi>} \omega_P(\xi) +$$

$$+F^{-1}_{\xi\to x}\left(\frac{1-h(\xi)}{P(\xi)}\right) \equiv E_1(x) + E_2(x) + E_3(x).$$

The numbers ε_j are ± 1, the sign depends on the orientation of K_j. Namely, $\varepsilon_j = +1$, if K_j is oriented so that the direction of the vector $\nabla T(\xi)$ and $\varepsilon_j = -1$ otherwise are chosen as the positive direction of the normal to K_j at each point x. Below we assume that the orientations K_j are fixed, so that the set $(\varepsilon_1, \dots, \varepsilon_m)$ is fixed.

We show that $E_3(x)$ decreases faster than any degree as | x | → ∞ and we calculate the asymptotic behavior of $E_{1,2}(x)$ using Theorem 2.3 and 2.4.

Lemma 2.3. For any $N \geq 0$ and for any multi-index α there exists a constant C_N, a such that

$$|D_x^\alpha E_3(x)| \leq C_{N,\alpha}(1+|x|)^{-N}, \;\; x \in \mathbb{R}^n. \tag{2.24}$$

Proof. For any integer $M \geq 0$ we have

$$E_3(x) = (-i|x|^2)^{-M} F^{-2}\left(\Delta^M \frac{1-h(\xi)}{P(\xi)}\right),$$

where is the Δ-Laplace operator. Since the polynomial P is hypoelliptic, there exist constants c, C> 0 such that for any β

$$\left|\frac{D^\beta P(\xi)}{P(\xi)}\right| \leq C(1+|\xi|)^{-c|\beta|}, \qquad \xi \in \mathbb{R}^n.$$

Therefore, if $M > 0$ is sufficiently large, then $\left|\Delta^M\left(\frac{1-h}{P}\right)\right| \leq C'(1+|\xi|)^{-n-1}$, so that for $|x| \geq 1$.

$$|E_3(x)| \leq C'|x|^{-2M}(2\pi)^{-n} \int\limits_{\mathbb{R}^n} (1+|\xi|)^{-n-1} d\xi \leq C''|x|^{-2M}.$$

Thus, (2.24) was proved for $|\alpha| = 0$; the case $|\alpha| > 0$ is investigated in a similar way.

The phase function $S =< x, \xi >$ has exactly 2 stationary points of the second kind $\xi_j^{\pm}(x)$ on each surface K_j by Lemma 2.2. Let $\xi_j^{\pm}(x)$ (respectively, $\xi_j^{-}(x)$ th of these points In which the positive direction of the normal to K_j coincides with x (respectively, s-x)). We introduce the notation $k_j(x)$ is the Gaussian curvature of the manifold K_j at the point $\xi_j^{+}(x)$, $n_j -$ is the direction of the external normal to K_j at the point $\xi_j^{+}(x), \omega = \frac{x}{|x|}$.

Theorem 2.5. Let conditions 1^0-3^0 be satisfied. Then for $|x| \to \infty$

$$E(x) = (2\pi)^{\frac{1-n}{2}} |x|^{\frac{1-n}{2}} \sum_{j=1}^{m} \left(\sqrt{k_j(x)} \frac{\partial P(\xi_j^{+}(x))}{\partial n_j} \right)^{-1} \times$$
$$\times e^{i< x, \xi_j^{+}(x) > + \frac{i\pi}{4}(n-3)\varepsilon_j} \left(1 + \sum_{l=1}^{\infty} a_{lj}(\omega) |x|^{-l} \right),$$

where the function $a_{lj} \in C^{\infty}$ for $|\omega| = 1$.

This decomposition can be differentiated by x any number of times.

Proof. It suffices to consider the case $m = 1$. Let K be the set of real zeros of the polynomial P. Since a K -compact strictly convex manifold, then, by Lemma 2.2, the function $S =< x, \xi >$ has exactly 2 stationary points of the second type $\xi^{\pm}(x)$ on K and both of them are non-degenerate. Suppose that a point such that the vector x is parallel to the external normal to K at this point. Then, for $|x| \to \infty$, the asymptotic behavior of $E_2(x)$ is equal to the sum of the contributions from the points of $\xi^{\pm}(x)$. Similarly by Theorem 2.3, the asymptotic behavior of $E_1(x)$ is equal to the sum of the same contributions multiplied by $\pi i sgn < \xi, \nabla T >$, so that when summing we get the doubled contribution from the point $\xi^{+}(x)$.

Additions. The leading term of the expansion (2.11) can be written in a more invariant form. Let the conditions of the theorem be satisfied. We introduce the Lagrange function $L(x, \mu) = S(x) + \mu g(x)$. If x^0 is the stationary point of the Lagrange function for μ_0 such that $\nabla S(x^0) + \mu_0 \nabla g(x^0)$. We show that the main term of the asymptotic expression is expressed in terms of the values of ∇g and the matrix

$$Q(x, \mu) = \det \frac{\partial^2 L(x, \mu)}{\partial x \partial \mu} = \left\| \begin{matrix} S''_{xx} + \mu g''_{xx} & (\nabla g)^t \\ \nabla g & 0 \end{matrix} \right\|.$$

at the point (x^0, μ_0). If $x^0 = 0, g(x^0) = 0, \frac{\partial g(x^0)}{\partial x_n} \neq 0$; we pass to the coordinates of y: $y_1 = x_1, \dots, y_{n-1} = x_{n-1}, y_n = g(x)$ and denote by S^*, g^* the functions S, g written in the variables y. Then

$$\tilde{y} = (y_1, \dots, y_{n-1}), \qquad \tilde{Q}(y) = \frac{\partial^2 S^*(y)}{\partial \tilde{y}^2}.$$

Then the identities are valid

$$\det Q = -\left(\frac{\partial g}{\partial x_n}\right)^2 \det \tilde{Q}\,, \qquad sgnQ = sgn\tilde{Q}. \tag{2.25}$$

when $x = x^0, \mu = \mu^0$. It's true that

$$Q = \begin{Vmatrix} T^t & 0 \\ 0 & E \end{Vmatrix} Q^* \begin{Vmatrix} T & 0 \\ 0 & E \end{Vmatrix},$$

Where the matrix Q^* is constructed along S^*, g^* in the same way as the matrix $Q, T = \frac{\partial y(0)}{\partial x}$ and the E-zero and unit $(n \times n)$ matrices.

We have

$$\det Q = \det Q^* (\det T)^2, \qquad sgnQ = sgnQ^*.$$

Since $g^* = y_n, \det Q^* = -\det \tilde{Q}$, then $\det Q = -g_n^2 \det \tilde{Q}$ and the first of the identities (2.25) are proved. Further, the matrix Q^* is reduced by a linear transformation to the form

$$\begin{Vmatrix} \tilde{Q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix},$$

whence follows the second of identities (2.36). therefore, the coefficient a_0 in the expansion (2.13) is equal to

$$a_0 = (2\pi)^{\frac{n-1}{2}} \frac{\nabla g}{\nabla S} |detQ|^{-\frac{1}{2}} e^{\frac{i\pi}{4}(sgnQ+2)}.$$

where $x = x^0,\ \mu = \mu_0$ and the orientation of the border $\partial\Omega$ is such that the vector $-\nabla g(x^0)$ is directed along the external normal to $\partial\Omega$.

Consider another important example of the integral

$$\Phi(\lambda) = \int_0^\infty dy \int_{-\infty}^\infty dx\, e^{-i\lambda xy} \varphi(x, y). \tag{2.26}$$

In this case, the conditions of Theorem 2.2 are not satisfied; namely, all points of the boundary – axis $y = 0$ — are stationary boundary points with the phases I, II type. In addition to that, the point (0,0) is a stationary point of the first type phase.

Proposition 2.1. Let the function $\varphi \in C^\infty$ with $y \geq 0$ be compactly supported. Then for $\lambda \to +\infty$ the asymptotic expansion

$$\Phi(\lambda) \sim \sum_{n=0}^\infty a_n \lambda^{-n-1}, \tag{2.27}$$

whose coefficients are of the form

$$a_n = i^{n+1}\, v.p. \int_{-\infty}^{\infty} x^{-n-1} \left(\frac{\partial}{\partial y}\right)^n \varphi(x,0)dx + \frac{i^n \pi}{n!}\left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^n \varphi(0,0) \qquad (2.28)$$

Proof. We represent the integral (2.26) in the form

$$\Phi(\lambda) = \lambda^{-1} \int_0^{\infty} I dt\,, \quad I = \int_{-\infty}^{\infty} e^{itx} \varphi(x,\varepsilon)dx,$$

where is $\varepsilon = t\lambda^{-1}$. By the Taylor formula, we have

$$\varphi(x,\varepsilon) = \sum_{n=0}^{N} \frac{\varepsilon^n}{n!}\left(\frac{\partial}{\partial y}\right)^n \varphi(x,0) + R_N,$$

$$R_N = \frac{1}{N!}\int_0^{\varepsilon} (\varepsilon-\tau)^N \left(\frac{\partial}{\partial \tau}\right)^{N+1} \varphi(x,\tau)d\tau,$$

so that

$$\Phi(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n-1} + \Phi_N(\lambda),$$

where the coefficients a_n have the form

$$a_n = \frac{1}{n!}\int_0^{\infty} y^n dy \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial y}\right)^n \varphi(x,0) e^{-ixy} dx, \qquad (2.29)$$

and the remainder is

$$\Phi_N(\lambda) = \lambda^{-1} \int_0^{\infty} I_N dt,$$

$$I_N = \int_{-\infty}^{\infty} e^{-itx} R_N dx = \frac{1}{N!}\int_0^{\varepsilon} (\varepsilon-\tau)^N \left(\int_{-\infty}^{\infty} e^{-itx} \psi(x,\tau)dx\right) d\tau.$$

Here $\psi = \left(\frac{\partial}{\partial \tau}\right)^{N+1} \varphi(x,\tau)$. Integrals can be rearranged due to the finiteness of φ. Since φ is a smooth compactly supported function, for any $k > 0$ the estimate

$$\left|\int_{-\infty}^{\infty} e^{-itx} \psi(x,\tau)dx\right| \le C_k (1+|t|)^{-k}$$

evenly according to $\tau \in [0;\infty]$. Therefore,

$$|I_N| \le \frac{C_k}{N!}(1+t)^{-k}\int_0^{\varepsilon}(\varepsilon-\tau)^N d\tau = C_k'(1+|t|)^{-k},$$

$$|\Phi_N| \le C_k'\lambda^{-N-2}\int_0^{\infty} t^{N+1}(1+t)^{-k}dt \le C_k''\lambda^{-N-2},$$

and thus, formula (2.27) is proved. Further we have

$$\int_0^{\infty} e^{-itx}t^n dt = i^{n+1}n!\,x^{-n-1} + i^n\pi\delta^{(n)}(x),$$

where equality is understood in the sense of generalized functions, and (2.29) implies (2.28).

Recall that,

$$v.p.\int_{-\infty}^{\infty} x^{-2k}\psi(x)dx = \int_0^{\infty} x^{-2k}[\psi(x)+\psi(-x)-$$

$$-2\left(\psi(0)+\frac{x^2}{2!}\psi''(0)+\cdots+\frac{x^{2k-2}}{(2k-2)!}\psi^{(2k-2)}(0)\right),$$

$$v.p.\int_{-\infty}^{\infty} x^{-2k-1}\psi(x)dx = \int_0^{\infty} x^{-2k-1}[\psi(x)-\psi(-x)-$$

$$-2\left(x\psi'(0)+\frac{x^3}{3!}\psi'''(0)+\cdots+\frac{x^{2k-1}}{(2k-1)!}\psi^{(2k-1)}(0)\right).$$

The leading term of the asymptotics has the form

$$\Phi(\lambda) = \lambda^{-1}\left[\pi\varphi(0,0) + i\int_{-\infty}^{\infty}\frac{\varphi(x,0)-\varphi(0,0)}{x}dx\right] + O(\lambda^{-2}).$$

II.3. Degenerate stationary points

The existence of asymptotic expansions.

Let phase $S(x) \in C^{\infty}$ in the neighborhood of the stationary point x^0 and let one of the conditions be satisfied:

1. The function $S(x)$ continues analytically to a complex neighborhood of the point x^0 and the point x^0 is an isolated critical point of $S(z), z \in \mathbb{C}^n$.

2. The local ring of the map $x \to \nabla S(x)$ is finite-dimensional.

The local ring of the map $x \to \nabla S(x)$ is the factor space $\frac{F\left[[x-x^0]\right]}{\frac{\partial S}{\partial x_1},\dots,\frac{\partial S}{\partial x_n}}$, where x^0 is the ring of formal power series in the variables $x_1 - x_1^0, \dots, x_n - x_n^0$ and $\left(\frac{\partial S}{\partial x_1}, \dots, \frac{\partial S}{\partial x_n}\right)$ is an ideal spanned by Taylor series at the point x^0 of $\frac{\partial S}{\partial x_1}$ function.

Then there exists a diffeomorphism $x = \varphi(y)$, $\varphi(0) = x^0$ such, $(S \circ \varphi)(y)$ is a polynomial. Thus, in cases 1, 2, the calculation of the contribution from a degenerate stationary point can be reduced to the case when the phase is a polynomial.

Theorem 2.6. Suppose that an $x^0 -$ real stationary point of phase S, function $f(x) \in C_0^{\infty}(\mathbb{R}^n)$ does not contain stationary points other than x^0 and one of conditions 1, 2 is satisfied. Then, for $\lambda \to +\infty$, the asymptotic expansion

$$F(\lambda) = \int\limits_{\mathbb{R}^n} f(x) e^{i\lambda S(x)} dx \sim \sum_{k=0}^{\infty} \left(\sum_{l=0}^{N} a_{kl} \lambda^{-r_k} (\ln \lambda)^{l-1} \right). \tag{2.30}$$

Here r_k is a rational number, $\frac{n}{2} \leq r_0 < r_1 < \dots < r_m \to \infty$ $(m \to \infty)$ and N are some fixed number.

This theorem follows from Theorem in the above and Erdeyi's lemma.

The numbers r_m, N are invariants of the stationary points. It is with a smooth replacement of $x = \varphi(y)$ that they do not change the value of the integral $F(\lambda)$, and the asymptotic expansion of the form (2.30) is unique. The most important invariant is r_0. The calculation of r_0 is based on the Hironaka theorem on the resolution of the singularities, i.e., on reducing the polynomial S to some canonical form by changing variables. However, it is very difficult to effectively carry out such a reduction.

Some examples.

Theorem 2.7. Let $S(x)$ be a positive definite homogeneous polynomial of degree $2m$, $f(x) \in C_0^\infty(\mathbb{R}^n)$. Then for $\lambda \to +\infty$ the asymptotic expansion

$$F(\lambda) \sim \lambda^{-\frac{n}{2m}} e^{\frac{i\pi n}{4m}} \sum_{k=0}^{\infty} a_k \lambda^{-\frac{k}{2m}},$$

$$a_k = e^{\frac{i\pi k}{4m}} \int_{S=1} \sum_{|\beta|=1} x^\beta \partial^\beta f(0) \omega_S.$$

Here ω_S is the Leray-Gelfand differential form.

Proof. Follows from Proposition in the above and Erdeyi's lemma.

Lemma 2.4. Let an $x^0 -$ critical point of the phase S and $S''_{xx}(x^0) = r$. Then, using the diffeomorphism $x = \varphi(y)\big(x^0 = \varphi(0)\big)$, we can in a small neighborhood of the point x^0 bring the phase S to the form

$$(S \circ \varphi)(y) = const + \sum_{j=1}^{r} \pm y_j^2 + S_1(y_{r+1}, \dots y_n). \qquad (2.31)$$

Moreover, all partial derivatives of the first and second order of the phase difference S_1 are equal to zero for $y = 0$.

Proof. Let $r \geq 1$ and $x^0 = 0, S(x^0) = 0$. Without loss of generality, we can assume that the second differential of the function S at the point x = 0 is the sum of squares. This can be achieved using a non-degenerate linear transformation. Put $x' = (x_1, \dots, x_r),\ x'' = (x_{r+1}, \dots, x_n)$, then

$$S(x) = S(0', x'') + S_2(x', x'').$$

Function $S(0', x'')$ has zero order ≥ 3 at point $x'' = 0$.
Further,

$$S_2(x', x'') = \sum_{j=1}^{r} \pm x_j^2 + S_3(x', x'').$$

We consider S_2 as a function of the variables x' and the parameters x'' for small $|x''|$. Since, by construction, the function $S_2(x', 0'')$ has zero order ≥ 3 at the point $x' = 0$, then by the inverse function theorem, the equation $\frac{\partial S_2}{\partial x'} = 0$ for small $|x''|$ has, and moreover, the solution is $x'^0 = \psi(x'')$. While $|\psi(x'')| = O(|x''|)$. By Lemma in the above, using the smooth (in y' and in the parameters x'') change of the variable $x' = x'(y', x''), y' = (y_1, \dots, y_r)$ we can bring the function S_2 to the form $S_2 = \sum_{j=1}^{r} \pm y_j^2$ and this proves (2.31).

Corollary 2.2. Let $rank\, S''_{xx}(x^0) = n - 1$. Then, using a smooth change of variables $x = \varphi(y)$, the phase in a neighborhood of x^0 be reduced to

$$(S \circ \varphi)(y) = S(x^0) + \frac{1}{2} \sum_{j=1}^{n-1} \mu_j y_j^2 \pm a y_n^{k+2}, \qquad a > 0. \qquad (2.32)$$

Here μ_j is a nonzero eigenvalue of the matrix $S''_{xx}(x^0)$, and $k \geq 1$ is an integer. At the same time $\det \varphi'(0) = 1$.

Proof. Thus, the problem of reducing a singularity is completely solved in the case $rank\, S''_{xx}(x^0) = n - 1$. The asymptotic behavior of the integral $F(\lambda)$ with a phase function of the form (2.32) is easily calculated and the leading term of the asymptotic has the form

$$F(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}} \frac{2}{n}\, \Gamma\left(\frac{1}{n}\right) (\pm a\lambda)^{-\frac{1}{n}} e^{\frac{i\pi}{4} \sum_{j=1}^{n} sgn \mu_j + \frac{i\pi}{2n}} \times$$
$$\times |\mu_1 \dots \mu_{n-1}|^{-\frac{1}{2}} e^{i\lambda S(x^0)} f(x^0).$$

The difference $n - r, r = rank \det S''_{xx}(x^0)$ is called the corank of the stationary point x^0 (singularities). Peculiarities of coranks 0,1 have already been studied; let us consider the features of corank 2.

Theorem 2.8. Homogeneous real polynomial $(\neq 0)$ of the third degree in two variables can be brought to one of the following four forms using a non-degenerate linear change of variables:

$$x^3 + y^3, x^2 y - y^3, x^2 y, x^3.$$

Proof. We have $P_3(x, y) = ax^3 + bx^2 y + cxy^2 + dy^3$; for definiteness $a \neq 0$. Let the cubic equation $P_3(\lambda, 1) = 0$ have at least one real root λ_1, and the form P_3 be divided by the linear factor $x - \lambda_1 y$. Assuming $x - \lambda_1 y = y', y = x'$, we get $P_3 = y' P_2(x', y')$, where P_2 is a real quadratic form. Bringing it to the sum of squares, we get $P_3 = y'\left[\pm(Ax' + By')^2 + Cy'^2\right]$. Form P_3 is reduced to type y''^3, if $A = C = 0$, to type $y'' x''^2$, if $C = 0, A \neq 0$, and to type $y''\left(x''^2 \pm y''^2\right)$, if $AC \neq 0$. Let $P_3 = y(x^2 + y^2)$. Make a replacement $y = u + v, x = \sqrt{3}(u - v)$, We bring P_3 to type $4(u^3 + v^3)$.

Theorem 2.9. If $S(x)$ have the form

$$(a)\ S = x^3 + y^3 + \cdots; \quad (b)\ S = yx^2 - y^3 + \cdots,$$

where the points denote the terms of degree ≥ 4. Then, in some neighborhood of the point (0, 0), using a smooth change of variables S, we can reduce to

$$(a)\ S = x'^3 + y'^3; \qquad (b) S = y'x'^2 - y'^3$$

Remark 2.2. If the function S is holomorphic in a complex neighborhood of the point (0, 0), then, using a holomorphic change of variables, the function S is reduced to $x'^3 + y'^3$.

In case (a), the integral $F(\lambda)$ can be reduced to

$$F(\lambda) = \int e^{i\lambda(x^3+y^3)}\varphi(x,y)dxdy \sim c\lambda^{-\frac{2}{3}}\left[f(0,0) + \sum_{k=1}^{\infty}\lambda^{-\frac{k}{3}}c_k\right].$$

For the proof, it is sufficient to consistently apply the one-dimensional stationary phase method with respect to the variables x, y.

Note that the constant c is an invariant that is expressed in terms of third-order derivatives of phase S at a stationary point.

Example. We calculate the asymptotic for $\lambda \to +\infty$ of the integral

$$F(\lambda) = \int_{\mathbb{R}^2}\int f(x,y)e^{i\lambda(yx-y^3)}dxdy,$$

where $f \in C_0^\infty(\mathbb{R}^2)$.

The integral $\int_{S=1}\omega$, where the $\omega -$ differential Leray-Gelfand form corresponding to phase $S = yx^2 - y^3$ converges.

$$\Phi_c(\lambda) = \int_{S=c} f\omega \sim c^{\frac{1}{3}}f(0,0)\int_{S=1}\omega \quad (c \to 0).$$

Consequently, by the Erdeyi lemma, the leading term of the asymptotics has the form

$$F(\lambda) = [f(0,0) + o(1)]\lambda^{-\frac{1}{3}}\frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt{3}}\int_{S=1}\omega.$$

We calculate $\int_{S=1}\omega$. Curve $S = 1$ has the form $x = \pm\sqrt{\frac{1+y^3}{y}}$ and consists of three branches. One of them lies in the half-plane $y < -1$, is symmetric about the $y-$ axis and has straight lines $y = \pm x$ with asymptotes. Two other branches lie in the half-plane $y > 0$, are symmetric about the $y -$ axis, and one of them has rays $y = 0, x > 0; y = x > 0$. We have

$$\int\limits_{S=1} \omega = \int\limits_{S=1} \frac{dy}{S'_x} = \int\limits_{-\infty}^{-1} \frac{dy}{\sqrt{y(1+y^3)}} + \int\limits_{0}^{\infty} \frac{dy}{\sqrt{y(1+y^3)}} = \frac{1}{3}\left[B\left(\frac{1}{2},\frac{1}{3}\right) + B\left(\frac{5}{6},\frac{1}{3}\right)\right].$$

Let $P_m(x_1, \dots, x_n)$ be a homogeneous polynomial of degree $m \geq 2$ in variables $n \geq 2$. The $GL(n, \mathbb{R})$-group of non-degenerate matrices of order $n \times n$. The $GL(n, \mathbb{R})$ group is a manifold of real dimension n^2 (= the number of elements of the $n \times n$ matrix). The polynomial P_m has C_{m+n-1}^{n-1} coefficients.

Inequality $n^2 \geq C_{m+n-1}^{n-1}$ holds only for m = 2, $n \geq 2$ or for m = 3, n = 2. Otherwise, the linear group contains fewer parameters than the set of polynomials $\{P_m(x)\}$, so using a linear change of variables $\omega x = Ty$, it is impossible to reduce any polynomial of degree m to one of a finite (or even discrete) number of canonical types for $m = 3, n \geq 3$ and for $m > 3, n \geq 2$. The family of canonical forms in these cases depends on continuous parameters. This already holds for homogeneous polynomials $P_4(x, y)$ of the fourth degree in two variables.

$$S_1(x,y) = xy(x^2 - y^2), \qquad S_2(x,y) = x(x+ty)(x^2 - y^2) + \cdots,$$

where the members of degrees ≥ 5 and $t \neq 0$ are denoted with a malleus enough. We show, that there is no smooth change of variables $(x, y) = \varphi(x', y')$ taking S_1 to S_2 (in a small neighborhood of the point (0,0)). We put $S_2^*(x', y') = S_2(x, y)$ then the expansion of S_2^* by the Taylor formula begins with terms of degree 4, the coefficients for which are determined only by the linear part of the transformation φ at zero. The linear transformation invariant is the double ratio of four straight lines. Namely, let the lines m_1, m_2, m_3, m_4 leave one point O, the line intersect the lines m_j in M_j. The double ratio is called the number

$$d = \frac{\overline{M_1M_4}}{\overline{M_2M_4}} : \frac{\overline{M_1M_3}}{\overline{M_2M_4}},$$

which does not depend on the choice of the line m. Let the $m_j -$ lines $x + y = 0, x + ty = 0, x = 0, x = y$; then $d = \frac{2t}{t+1}$ for the function $S_2, d = 2$ for S_1.

Thus, when classifying degenerate critical points, moduli arise (families of nonequivalent relatively smooth replacement of germs that depend on continuous parameters), and the problem of classifying degenerate critical points is currently not completely solved.

Chapter III. Fourier transform of the delta function on the surface

III.1. Delta function on the sphere

Let us be given a surface $S := \{x: F(x) = 0, dF(x) \neq 0\}$. If $\frac{\partial F(x)}{\partial x_1} \neq 0, F(x) = 0$, then we define the surface as a graph of the function $x_1 = (x_2, \dots, x_n)$.

Now we get a *delta* sequence:

$$f_v(x) = 2v\mathfrak{X}_{\left[-\frac{1}{v};\frac{1}{v}\right]}(x).$$

As determined of distribution

$$[f_v](\varphi) = 2v \int_{-\frac{1}{v}}^{\frac{1}{v}} \varphi(x)dx \to \varphi(0) = \delta, \qquad v \to \infty.$$

So for the surface function we get

$$[F_v](\varphi) = 2v \int_{|F(x)| \leq \frac{1}{v}} \varphi(x)\, dx \quad \to \int_S \frac{\varphi(x)dS(x)}{|\nabla F(x)|}.$$

In other words, we have created a delta function on the surface

$$\delta_S(\varphi) = \int_S \frac{\varphi(x)dS(x)}{|\nabla F(x)|}.$$

Assume that for $\forall \psi \in C_0^\infty(\mathbb{R}^n)$ satisfying following equation:

$$\psi\delta_S(\varphi) = \int_S \psi(x)\varphi(x)dS(x).$$

So we have following Fourier transforms of the function $\psi\delta_S(\varphi)$

$$F[\psi\delta_S(\varphi)](\xi) = \int_S e^{-2\pi i(\xi,x)}\psi(x)dS(x).$$

Now let's look at determining a delta function on the surface

Let's assume we have $S := \{x: F(x) = 0, \nabla F(x) \neq 0\}$ function, then

$$(\psi(x)\delta_S, \varphi) = \lim_{\sigma\to+0}\int_{|F(x)|\le\sigma}\psi(x)\varphi(x)dx = \int_S \psi(x)\varphi(x)dS(x).$$

here $\nabla F(x) = (\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$.

We prove the above equation.

Assume that we have following integral,

$$\int_{|F(x)|\le\sigma}\psi(x)\varphi(x)dx, \qquad \forall\psi \in C_0^\infty(U)$$

here U – sufficiently small neighborhood of the origin.

Let's assume $\frac{\partial F}{\partial x_1} \ne 0$ for $\forall x \in supp(\psi)$, then according to the theorem of implicit functions, $x_1 = F(x_2, \dots, x_n)$ function exists of sufficiently small neighborhood of the origin.

We done following transform $y_1 = F(x), y_2 = x_2, \dots, y_n = x_n$. We know that there is inverse function for y_1. Then the Jacobian matrix of the transform

$$J\left(\frac{y}{x}\right) = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \qquad detJ\left(\frac{y}{x}\right) \ne 0.$$

According to the transform and Fubini's theorem we write (1) integral in the following

$$\int_{|y_1|<\sigma}\int_{\mathbb{R}^{n-1}}\psi(x_1(y), y_2, \dots, y_n)\varphi(x_1(y), y_2, \dots, y_n)\frac{dy_1 \dots dy_n}{|\frac{\partial F}{\partial x_1}(x_1(y), y_2, \dots, y_n)|}$$

Multiply (2) integral by $\frac{1}{2\sigma}$ and move to the limit when $\sigma \to 0$,

$$\lim_{\sigma\to+0}\frac{1}{2\sigma}\int_{|y_1|<\sigma}\int_{\mathbb{R}^{n-1}}\psi(x_1(y), y_2, \dots, y_n)\varphi(x_1(y), y_2, \dots, y_n)\frac{dy_1 \dots dy_n}{\left|\frac{\partial F}{\partial x_1}(x_1(y), y_2, \dots, y_n)\right|}$$

$$= \int_{\mathbb{R}^{n-1}}(\psi\varphi)(x_1(0, y_2, \dots, y_n), y_2, \dots, y_n)\frac{dy_2 \dots dy_n}{\left|\frac{\partial F}{\partial x_1}(x_1(0, y_2, \dots, y_n), y_2, \dots, y_n)\right|}.$$

According to the definition of surface integral from the last integral in the above equation, we have $cosx_1 = \frac{|\frac{\partial F}{\partial x_1}|}{|\nabla F(x)|}$.

Accordingly, we form the following surface integral,

$$\int_{\mathbb{R}^{n-1}} (\psi\varphi)(x_1(0, y_2, \dots, y_n), y_2, \dots, y_n) \frac{dy_2 \dots dy_n}{\left|\frac{\partial F}{\partial x_1}(x_1(0, y_2, \dots, y_n), y_2, \dots, y_n)\right|} =$$

$$= \int_{\mathbb{R}^{n-1}} (\psi\varphi)(x) \frac{dS}{|\nabla F(x)|}.$$

The function $F(x)$ is infinitely differentiable.

III.2. Fourier transform of the delta function on the surface

Now let's look at the following integral

$$\int_{a<t<b} \psi(x)\varphi(x)\, dx = \Phi(t),$$

Consider finding the derivative of $\Phi(t)$.

Finite difference of the function $\Phi(t)$ at the point $t = t_0$. In this case, according to the definition of the derivative, the following equation is valid for the derivative $\Phi'(t_0)$

$$\Phi'(t_0) = \lim_{\Delta t \to 0} \frac{\Phi \quad (t_0 + \Delta t) - \Phi(t_0)}{2\Delta t} = \int_{F(t)=t_0} \psi(x)\varphi(x) \frac{dS}{|\nabla F(x)|}. \qquad (3.1)$$

Thus for arbitrary $a < t < b$

$$\Phi'(t) = \int_{F(t)=t} \psi(x)\varphi(x) \frac{dS}{|\nabla F(x)|}.$$

Hence $\Phi'(t) -$ existence and continuous.

Exercise. Assume that $\frac{1}{2} \leq \mathrm{t} \leq 1$, then we find the derivative of the function depends on the parameter t

Solution.

According to the Equation (3.1), the following equation holds for the derivative of $\mathrm{F}(t)$

$$\mathrm{F}'(t) = \iiint_{x^2+y^2+z^2=t} \varphi(x,y,z) \frac{dS}{4(x^2+y^2+z^2)^{\frac{1}{2}}} =$$

$$= \frac{1}{2\sqrt{t}} \iiint_{x^2+y^2+z^2=t} \varphi(x,y,z)\, ds. \qquad (3.2)$$

If $\varphi(x,y,z) = 1$ then $\mathrm{F}(t) = \frac{4\pi}{3} t^{\frac{3}{2}}$ or for the Equation (3.2) $\mathrm{F}'(t) = \frac{4\pi t}{2\sqrt{t}} = 2\pi\sqrt{t}$.

Let us now consider the Fourier transforms of the surface integral:

$$F[\psi\delta_S](\xi) = \int_S e^{-2\pi i(\xi,x)} \psi(x) \frac{dS}{|\nabla F(x)|}, \qquad (3.3)$$

here $S \coloneqq \{x: F(x) = 0\}$.

If $\xi = \lambda \cdot \eta,\ |\eta| = 1,\ \lambda \in \mathbb{R}_+$ then (3.3) integral looks like this

$$F[\psi\delta_S](\xi) = \int_S e^{-2\pi i\lambda(\eta,x)} \psi(x) \frac{dS}{|\nabla F(x)|}.$$

The characteristic of the Fourier transforms in infinity is a vibrating integral, that is:

$$\lambda \to \infty.$$

Exercise. We calculate the Fourier transforms of the unit sphere,

S is unit sphere. Then we define the following equation for its Fourier transforms:

$$F(\xi) = \int_S e^{-2\pi i(\xi,x)}\, dS(x). \qquad (3.4)$$

Solution.

We get the $\xi -$ orthogonal transforms, $A -$ orthogonal matrix. In this case, the integral (3.4) looks like this:

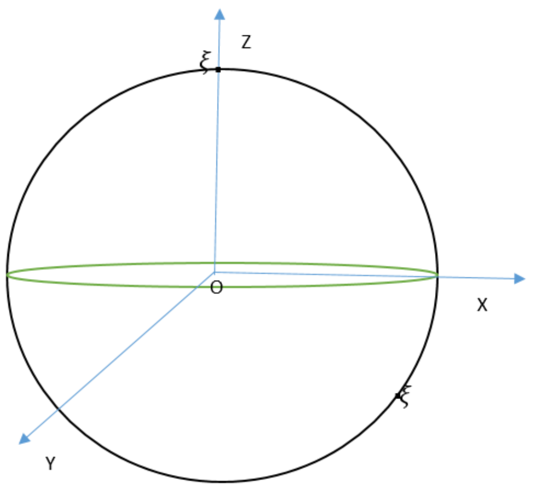
$$\Phi_t(\xi) = \int_{x^2+y^2+z^2\leq t} e^{-2\pi i(\xi,X)}\, dxdydz,$$

We get substitution $X = Ay, y = (x', y', z')$ in the above integral, then $(A\xi, Ay) = (\xi, y)$, here $X = (x.y.z)$. Accordingly, the above integral looks like this:

$$\Phi_t(\xi) = \int_{x^2+y^2+z^2\leq t} e^{-2\pi i(\xi,\cdot)}\, dxdydz.$$

Using this rotation, we move the point $(\xi_1, \xi_2, \ \xi_3)$ to the point $(0, 0, \ |\xi|)$,

This rotation is shown in the figure below:



Accordingly, the integral $\Phi_t(\xi)$ looks like this:

$$\Phi_t(0,0,|\xi|) = \Phi_t(\xi_1,\xi_2,\xi_3) =$$

$$= \int_{x^2+y^2+z^2=t} e^{-2\pi i|\xi|z}\, dz = \int_{-\sqrt{t}}^{\sqrt{t}} e^{-2\pi i|\xi|z} \cdot \pi(t - z^2)\, dz =$$

$$= \pi \int_{-\sqrt{t}}^{\sqrt{t}} \cos(2\pi|\xi|z)\,(t - z^2)\, dz.$$

Using twicly integration by parts of latest integral,

$$\pi \int_{-\sqrt{t}}^{\sqrt{t}} \cos(2\pi|\xi|z)\,(t - z^2)\, dz = -\frac{\sqrt{t}\cos\left(2\pi|\xi|\sqrt{t}\right)}{\pi|\xi|^2} + \frac{\sin\left(2\pi|\xi|\sqrt{t}\right)}{2\pi^2|\xi|^3}.$$

If considering limit by using Teylor development of $\cos\left(2\pi|\xi|\sqrt{t}\right)$ and $\sin\left(2\pi|\xi|\sqrt{t}\right)$ at he point $|\xi| \to$, then we get $\frac{4}{3}\pi\sqrt{t^3}$.

Thus $\Phi_t(0,0,|\xi|) \to \frac{4}{3}\pi\sqrt{t^3},\ |\xi| \to 0.$

Then we come to the formula for the volume of a sphere, if $x^2 + y^2 + z^2 = t$ then we get $t = \mathbb{R}^2$, Of these we get $V = \frac{4}{3}\pi\mathbb{R}^3$.

Thus we have the following equation for the Fourier transforms of the Delta function on the unit sphere

$$F[\delta] = \frac{\partial\Phi_{t=0}(0,0,|\xi|)}{\partial t} = \left(-\frac{\sqrt{t}\cos\left(2\pi|\xi|\sqrt{t}\right)}{\pi|\xi|^2} + \frac{\sin\left(2\pi|\xi|\sqrt{t}\right)}{2\pi^2|\xi|^3}\right)'_{t=1} =$$

$$= \frac{2\sin(2\pi|\xi|)}{|\xi|}.$$

Now let's look at the Fourier transforms of the following dimension

$$\hat{\mu}(\xi) = \int_S e^{-2\pi i(\xi,x)}\,\psi(x)dS(x)$$

here $dS(x)$ is a dimension of surface.

If we make this $= \lambda\eta$ substitution, in this case the integral (3.8) looks like this

$$\hat{\mu}(\lambda\eta) = \int_S e^{-2\pi i(\lambda\eta,x)}\,\psi(x)dS(x)$$

If we assume that $|\eta| = 1$ is a unit vector, then at $\lambda \to +\infty$ it will be a oscillating integral.

III.3. About distributions, defined by surface measures

First, we give the definition of a C^1- smooth hyper surface $S \subset \mathbb{R}^{n+1}$. For the convenience of the entries, we introduce some notation:

Let $X = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ be some point and $k(1 \le k \le n+1)$ a fixed positive integer.

Put $X_k := (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}) \in \mathbb{R}^n$. Let $X_k^0 \in \mathbb{R}^n$ be some point and $\delta > 0$ a fixed positive number. We define δ a neighborhood of this point as follows:

$$U_\delta(X_k^0) := (x_1^0 - \delta, x_1^0 + \delta) \times \dots \times (x_{k-1}^0 - \delta, x_{k-1}^0 + \delta) \times$$
$$\times (x_{k+1}^0 - \delta, x_{k+1}^0 + \delta) \times \dots \times (x_{n+1}^0 - \delta, x_{n+1}^0 + \delta) \subset \mathbb{R}^n$$

And also for positive numbers $\delta > 0, \varepsilon > 0$ and a fixed positive integer $k(1 < k \leq n+1)$ we define the neighborhood $U_{\delta\varepsilon}(X^{k_0})$ of the point $X^0 = (x_1^0, \dots, x_{n+1}^0)$ by the following relation:

$$U_{\delta\varepsilon}(X^{k_0}) := (x_1^0 - \delta, x_1^0 + \delta) \times \dots \times (x_{k-1}^0 - \delta, x_{k-1}^0 + \delta) \times (x_k^0 - \varepsilon, x_k^0 + \varepsilon) \times$$
$$\times (x_{k+1}^0 - \delta, x_{k+1}^0 + \delta) \times \dots \times (x_{n+1}^0 - \delta, x_{n+1}^0 + \delta) \subset \mathbb{R}^{n+1}$$

$S \subset \mathbb{R}^{n+1}$ is called a $C^1 -$ smooth hyper surface, if for each point $X^0 = (x_1^0, \dots, x_{n+1}^0) \in S$ there exists a natural number $1 \leq k \leq n+1$, neighborhoods $U_\delta(X_k^0)$ and $U_{\delta\varepsilon}(X^{k0})$ of points X_k^0 and X^0 respectively, as well as the function $\phi \in C^1\left(U_\delta(X_k^0)\right)$ such that the following relation holds:

$$S \cap U_{\delta\varepsilon}(X^{k0}) = \{X \in U_{\delta\varepsilon}(X^{k0}) : x_k = \phi(X^k)\}.$$

For the cases $k = 0, n+1$, the neighborhoods $U_\delta(X_k^0)$, as well as $U_{\delta\varepsilon}(X^{k0})$, are defined similarly. In other words, a $C^1 -$ smooth hyper surface is locally a graph of a $C^1 -$ smooth function.

Of course, for such hyper surfaces, the surface measure dS is naturally determined, as well as a measure with a compact support of the following form: $d\mu := \varphi(X)dS$, where φ is some continuous function, with compact support, defined on the surface S.

The measure $d\mu := \varphi(X)dS$ naturally determines the distribution defined in the Schwartz space by the following equality:

$$d\mu(\psi) := \int_S \psi(X)\varphi(X)dS \tag{3.5}$$

where $\psi \in \mathbb{S}$ is the main (test) function from the Schwartz space S.

Since the distribution defined by formula (3.5) has a compact support, then, according to the Peli-Wiener-Schwartz theorem, its Fourier transform is an ordinary, locally integrable function. Moreover, this function is equivalent to the function denoted by $\widehat{d\mu}$, which analytically continues in $\mathbb{C}^n$.

In addition, this function is explicitly written by the following surface integral:

$$\widehat{d\mu}(\xi) := \int\limits_S e^{iX\cdot\xi}\varphi(X)dS, \tag{3.6}$$

where $X \cdot \xi$ is the scalar product of the vectors X and ξ.

It follows from formula (3.6) using the trivial estimate of the integral that for each point $\xi \in \mathbb{C}^n$ the inequality holds:

$$\left|\widehat{d\mu}(\xi)\right| \leq \max_{x\in S}|\varphi(X)|e^{C|\Im(\xi)|}, \tag{3.7}$$

where C is some positive number depending on the support φ and $\Im(\xi)$ is the imaginary part of the vector ξ. Since φ has a compact support, the maximum of this function is reached on S.

By $A_p(\mathbb{R}^{n+1})$ we denote the class of distributions $u \in \mathbb{S}'(\mathbb{R}^{n+1})$ such that the Fourier transform $\hat{u} \in \mathbb{S}'(\mathbb{R}^{n+1})$ is regular distribution determined by a function from the space $L^p(\mathbb{R}^{n+1})$. Roughly speaking, $\hat{u} \in L^p(\mathbb{R}^{n+1})$.

Here $L^p(\mathbb{R}^{n+1})$ is the space of integrable functions of degree $p(1 \leq p < \infty)$. Of course, if $p = \infty$, then we are dealing with a space of essentially bounded functions.

From estimate (3.7) it follows that $d\mu \in A_\infty(\mathbb{R}^{n+1})(p \geq 1)$ immediately follows the validity of the inclusion: $d\mu \in A_q(\mathbb{R}^{n+1})$ for $p \leq q$. Therefore, the following number is naturally determined:

$$p_\mu := \inf\{1 \leq p : d\mu \in A_p(\mathbb{R}^{n+1}) \text{ for any nonzero } \varphi \in C_0(S)\}.$$

The main result of the thesis is the following lower bound theorem for the number p_μ:

Theorem 3.1. Let $S \subset \mathbb{R}^{n+1}$ be an arbitrary C^1 −smooth hyper surface, containing the origin and $d\mu$ is the measure defined by formula (3.9), φ is a continuous, non-negative function concentrated in a sufficiently small neighborhood of zero such that $\varphi(0) > 0$. Then $\widehat{d\mu} \notin L^p(\mathbb{R}^{n+1})$ for any $1 \leq p \leq \frac{2(n+1)}{n}$. In other words, the following lower bound holds true: $p_\mu \geq \frac{2(n+1)}{n}$.

CONCLUSION

Studying properties of Delta function is one of the actual problem of analysis. The purpose of the research is description of Fourier transforms of the Delta function on the surface. The main tasks of the research is description of basic properties of the delta function on the surfaces and Fourier transform of that function.

In the thesis there are given definition of delta function, derivatives and integration of the delta function, Delta function on the surface, Fourier transforms of the Delta function on the surface. In the thesis are used methods of mathematical and functional analysis, including topological vector spaces, linear spaces with countable many norms. The results of the research have fundamental character. But, the methods and results can be used in order to solve the problems of mathematical physics, including spectral properties of discrete Shcrodinger operator.

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