O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI SAMARQAND DAVLAT UNIVERSITETI

Qo'lyozma huquqida

UDK: 517.3

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Karrali Laplas integrallari va ularning ba'zi tadbiqlari.

5A130101- «Matematika (yo'nalishlar bo'yicha)»

mutaxasisligi bo'yicha magistr akademik darajasini olish uchun yozilgan

DISSERTATSIYA

Ish ko'rib chiqildi va himoyaga ruxsat berildi.

“Matematik analiz” kafedrasi mudiri

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____ _________ 2020 yil

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SAMARQAND-2020

O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGISAMARQAND DAVLAT UNIVERSITETI

Fakultet: Mexanika-matematika Magistratura talabasi: Jo'raqulov A

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O'quv yili: 2019-2020 Mutaxassisligi: Matematika (y. b.)

MAGISTRLIK DISSERTATSIYASINING ANNOTATSIYASI

Annotatsiya

Mazkur magistrlik dissertatsiyasida parametrga bog'liq karrali xosmas integrallar qaraladi. Dissertatsiyaning 2-bobida parametrga bog'liq karrali integrallarni hisoblashning Laplas usuli o`rganilgan.. Dastlab, ikki karrali Laplas integralining asimptotikasi o`rganilib olingan mulohazalar *n* karrali Laplas integrali uchun umumlashtiriladi. Dissertatsiyaning 3-bobi parametrga bog'liq karrali Laplas integralining tadbiqlariga bag'ishlanadi. Unda dastlab plastinkada issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimining vaqtning kichik qiymatlarida asimptotikasi olinadi. So'ngra olingan bu mulohazalar *n* o'lchamli fazoda issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimi uchun umumlashtiriladi.

Resume

In the dissertation, improper integrals depending on the parameters are studied. In the second chapter, we study the Laplace method for multiple improper integrals with parameters that are generalized for Laplace integrals. The third chapter discusses some applications of Laplace integrals. In this chapter, we study the asymptotics of the solution of the Cauchy problem for the heat equation in small parameter values. The resulting asymptotic relations are generalized for the multiple case.

Ilmiy rahbar:

“Matematik analiz” kafedrasi dotsenti D.D.To'raqulov _____

“Matematik analiz” kafedrasi magistranti A.Jo'raqulov _____

Mundarija.

Kirish

Parametrga bog'liq karrali integrallarni asimptotik hisoblashning usullaridan biri P.Laplas taklif qilgan usuldir.

Xo'sh bu usul qanchalik muhim? Nega bu usul ko'p tadqiq etiladi? Bunday integrallarni hisoblash masalasi haqida nima deyish mumkin? Qanday qilib bunday integrallarning asimptotikasini o'rganish mumkin?

Bu usulnig qanday tatbiqlari bor va hokazo bunday savollarni juda ko'plab keltirish mumkin. Biz mazkur dissertatsiyada shu savollarga qisman javob berishga harakat qildik.

Parametrga bog'liq integrallarni asimptotik hisoblashda Laplas integrali deb ataluvchi integral o'rganiladi. Dissertatsiya ishida quyidagicha aniqlangan

$$I(\lambda) := \int_{\mathbb{R}^n} e^{-\lambda g(\mathrm{x})}\psi(\mathrm{x})\mathrm{dx} \qquad (1)$$

integralni qaraymiz.

Bu yerda $\psi(\mathrm{x})$ amplituda deb ataluvchi funksiya va $g(\mathrm{x})$ faza funksiyasi deyiladi, λ esa katta qiymatlar qabul qiluvchi musbat haqiqiy parametr.

Biz odatda $\psi(\mathrm{x}) \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ deb faraz qilamiz, ya'ni $\psi(\mathrm{x})$ cheksiz marta differensiallanuvchi bo'lib, shunday $r(\psi) > 0$ chekli son topiladiki, barcha $|\mathrm{x}| > r$ shartni qanoatlantiruvchi nuqtalar uchun ψ funksiya nolga aylanadi. Bunday funksiyalar to'plami finit, cheksiz silliq funksiyalar fazosi deyiladi. Faza funksiyasi esa cheksiz marta differensiallanuvchi haqiqiy qiymatlarni qabul qiluvchi funksiyadir.

Shuni ta'kidlash kerakki, (1) integral odatda elemantar funksiyalar orqali kvatraturalarda hisoblanmaydi. Bu kabi integrallarni hisoblash usullari bilan uning qiymatini taqriban topishdagi qiyinchiliklar λ parametrning cheksizga intilishi bilan bog'liq. Ma'lumki, kvadratura va kubatura formulalaridagi xatoliklar integral ostidagi funksiyaning hosilalari bilan baholanadi. (1) integralda esa integrallanuvchi funksiya λ parametrning cheksizga intilishi bilan uning hosilalari cheksiz katta qiymatlarni qabul qiladi. Shunga qaramasdan (1) integral $\lambda \to +\infty$ da yetarlicha kichik bo'lishi mumkin. Bunday integrallarning cheksizdagi xarakteri ba'zi hollarda aniq topilishi mumkin. Laplas ta'biri bilan aytganda qaysi intregralning hisoblanishi qiyin bo'lsa uning asimptotikasini topish shunchalik oson bo'ladi. Bu tasdiq albatta majoziy ma'noda aytilgan.

Biz (1) intagralni $\lambda \to +\infty$ dagi xarakteri, asimptotik yoyilmasi va baholari hamda (1) integralni tadbiqlari bilan shug'ullanamiz.

Avvalo, parametrga bog'liq karrali integrallarni qarab chiqamiz. So'ngra, dastlab sodda hol, ikki karrali Laplas integralini qaraymiz. Olingan mulohazalarni umumiy holda n karrali holga davom etiramiz va natijalarning fizik masalaga tadbiqini qaraymiz.

Umumiy masalaning qo'yilishi. Ma'lumki, integralni hisoblash qiyin bo'lganda integralni taqribiy hisoblash qo'llaniladi. Shu bilan integral talab qilingan aniqlik bilan hisoblanadi. Albatta, bu taqribiy hisoblashda xatolik integral ostidagi funksiyaning hosilalari bilan baholanadi. Agar integral ostidagi funksiyaning hosilalari biror sababga ko'ra chegaralanmagan bo'lsa, u holda bu taqribiy hisoblash o'z ahamiyatini yo'qotadi. Xo'sh bu holda integralni hisoblash uchun qanday yo'l tutish kerak bo'ladi, degan savol tug'ilishi tabiiy. Bu savolga ma'lum shartlarda javobni beradigan usulni P.Laplas tomonidan taklif qilingan g'oya beradi. Aslida bu usulga kelishga sababchi bo'lgan qandaydir aytaylik, fizik yoki mexanik va shunga o'xshash jarayonlar bo'lishi mumkin. Bu esa masala qo'yilishining ikkinchi sababchisidir, ya'ni ishning tadbiqi. Yuqoridagi sabablar ma'lum shartlarda qaraladi va javoblar olinadi.

Konkret masalaning qo'yilishi. Mazkur magistrlik dissertatsiyasida karrali Laplas integrallining asimptotik yoyilmasi qaraladi. Uning tadbiqi sifatida issiqlik tarqalish tenglamasi uchu qo'yilgan Koshi masalasi yechimi qaraladi, amaliyot uchun yetarlicha bo'lgan natija olinadi.

Mavzuning dolzarbligi. Yuqorida ko'rdikki, integral ostidagi funksiyaning hosilalari chegaralanmaganda integralni hisoblash uchun qo'llaniladigan taqribiy usullar ahamiyatini yo'qotishi Laplas usulining shunchalik muhimligini bildiradi. Matematik fizikadagi issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimini boshlang'ich shartni ko'rsatish va shu kabi masalalar mavzuning dolzarbligini ko'rsatadi.

Ishning maqsad va vazifalari. Parametrga bog'liq integrallarni asimptotik hisoblashning Laplas usulini o'rganish. Laplas integrali asimptotik yoyilmasi, xususan yoyilmaning bosh hadini olish. Bu asimptotik usulni matematik fizikadagi issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimini umumiy holda n o'lchamli fazoda asimptotikasini olish.

Ilmiy tadqiqot metodlari. Dissertatsiya ishida matematik analiz va analizning asimptotik usullaridan foydalaniladi.

Ishning ilmiy ahamiyati. Laplas integrali asimptotik yoyilmasidan matematik fizikadagi issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimini vaqtning kichik qiymatlarida asimptotikasi.

Ishning amaliy ahamiyati. Dissertatsiya nazariy va amaliy xarakterga ega bo'lib, uning natijalari parametrga bog'liq integrallarni asimptotik hisoblash va issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimini asimptotik hisoblash.

Tadqiqot ob'ekti va predmeti. Mazkur magistrlik dissertatsiyasining ob'ekti karrali xosmas integrallar. Tadqiqotning predmeti karrali Laplas integralning katta parametr bo'yicha asimptotikasini o'rganish.

Dissertatsiyaning tarkibi. Dissertatsiya kirish, uchta bob, xulosa, foydalanilgan adabiyotlar ro'yxatidan iborat.

Birinchi bobda parametrga bog'liq integral tushunchasi va uning asosiy xossalari bayon etiladi. Shuningdek, parametrga bog'liq xosmas integrallar qaralib uzoqlashuvchi karrali xosmas integrallarni Abel jamlash usuli keltirilgan.

Ikkinchi bobda karrali Laplas integralining asimptotikasi qaralib, ikki karrali Laplas integrali uchun asimptotik yoyilma olinadi.

Ikkinchi bobning 1-qismida quyidagi

$$\iint_{\Omega} e^{-\lambda g(t_1,t_2)}\, dt_1 dt_2 \qquad (2.1.1)$$

integral qaralgan.

Ma'lum shartlarda (2.1.1) integralni quyidagi

$$\iint_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)}\, f(x_1,x_2) dx_1 dx_2 \qquad (2.1.2)$$

ko'rinishga olib kelinishi bayon qilingan. So`ngra

$$\iint_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)}\, |f(x_1,x_2)| dx_1 dx_2 < +\infty \qquad (2.1.3)$$

shart asosida parametr cheksizlikka intilganda (2.1.2) integral o'zining asosiy qiymatini koordinatalar boshining yetarlicha kichik atrofida olishi haqidagi quyidagi tasdiq keltirilgan.

2.1.1-tasdiq: *Faraz qilaylik, f funksiya (2.1.3) shartni qanoatlantirsin. U holda istalgan* $\delta > 0$ *uchun (2.1.2) integral*

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 =$$

$$= \iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 + O(1)e^{-\lambda\delta^2}, \qquad \lambda \geq 1 \quad (2.1.6)$$

bahoni qanoatlantiradi.

Yuqoridagi tasdiqdan yanada umumiyroq bo'lga quyidagi tasdiq keltirilgan.

2.1.2-tasdiq: *Har qanday* $\delta > 0$ *va ixtiyoriy k*≥ 0*,* $l\geq 0$ *uchun*

$$\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} x_1^{2k}x_2^{2l}dx_1dx_2 = \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\lambda^{k+l+1}} + O\left(e^{-\lambda\delta^2}\right) \quad (2.1.8)$$

baho o'rinli.

2.1.1-2.1.2-tasdiqlar natijasi sifatida quyidagi teorema keltirilgan.

2.1.1-teorema: *Faraz qilaylik, f funksiya (2.1.3) shartni qanoatlantirib, koordinatalar boshining biror atrofida 2m marta uzluksiz xususiy hosilalarga ega bo'lsin. U holda λ→ +∞ da*

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 = \lambda^{-1}\left[\sum_{i=0}^{m-1} a_i\lambda^{-i} + \frac{O(1)}{\lambda^m}\right] \quad (2.1.12)$$

asimptotik tenglik o'rinli.

Bunda a_i *koeffitsiyent quyidagicha aniqlanadi:*

$$a_i = \frac{\sum_{j=0}^{i} \frac{\partial^{(2i)} f(0,0)}{\partial x_1^{2(i-j)}\partial x_2^{2j}} C_{2i}^{2j}\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!}$$

2.1.1-natija: *Koordinatalar boshining biror atrofida ikki marta uzluksiz xususiy hosilalarga ega bo'lgan (2.1.3) shartni qanoatlantiruvchi f funksiya uchun*

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 = \frac{\pi}{\lambda}\left[f(0,0) + \frac{O(1)}{\lambda}\right], \lambda \to +\infty \ (2.1.19)$$

asimptotik tenglik o'rinli.

(2.1.2) integral uchun olingan natijalar asosida (2.1.1) integral uchun quyidagilar olingan.

2.1.3-natija: *Agar biror yagona* $(t_1^0, t_2^0) \in \Omega$ *da* $\nabla g(t_1^0, t_2^0) = 0$ *va*

$$det g''(t_1^0, t_2^0) = \begin{vmatrix} \frac{\partial^2 g}{\partial t_1^2} & \frac{\partial^2 g}{\partial t_1 \partial t_2} \\ \frac{\partial^2 g}{\partial t_2 \partial t_1} & \frac{\partial^2 g}{\partial t_2^2} \end{vmatrix}_{(t_1^0, t_2^0)} > 0$$

bo'lib bu nuqta minimum nuqtasi bo'lsa, u holda quyidagi

$$\iint\limits_{\Omega} e^{-\lambda g(t_1, t_2)}\, dt_1 dt_2 =$$

$$= e^{-\lambda g(t_1^0, t_2^0)} \frac{\pi}{\lambda}\left[\frac{2}{\sqrt{\det(g''(t_1^0, t_2^0))}} + \frac{O(1)}{\lambda}\right], \lambda \to +\infty \quad (2.1.35)$$

asimptotik tenglik o'rinli.

2.1.2-teorema: *Faraz qilaylik,* $g(t_1, t_2)$ *funksiya* Ω *sohada yotuvchi yagona* (t_1^0, t_2^0) *nuqtada xosmas kritik nuqtaga ega bo'lsin. Agar shu* (t_1^0, t_2^0) *nuqta* $g(t_1, t_2)$ *funksiya uchun lokal minimum nuqta bo'lsa, u holda ,* $\lambda \to +\infty$ *da quyidagi asimptotik yoyilma o'rinli bo'ladi:*

$$\iint\limits_{\Omega} e^{-\lambda g(t_1, t_2)}\, dt_1 dt_2 \approx e^{-\lambda g(t_1^0, t_2^0)} \lambda^{-1} \sum_{i=0}^{\infty} a_i \lambda^{-i} \qquad 2.1.38$$

bunda $a_i = \dfrac{\sum_{j=0}^{i} \frac{\partial^{(2i)} f(0,0)}{\partial x_1^{2(i-j)} \partial x_2^{2j}} C_{2i}^{2j} \Gamma\left(i-j+\frac{1}{2}\right) \Gamma\left(j+\frac{1}{2}\right)}{(2i)!}$ *kabi aniqlanadi.*

Ikkinchi bobning 2-paragrafida quyidagi

$$I(\lambda) = \iint\limits_{\Omega} e^{-\lambda g(t_1, t_2)}\, \psi(t_1, t_2) dt_1 dt_2 \qquad (2.2.1)$$

integral qaralib uning asimptotikasi uchun quyidagicha tasdiq o`rinli ekanligi isbotlanadi.

2.2.1-teorema: *Faraz qilaylik* Ω *sohada yotuvchi yagona* (t_1^0, t_2^0) *nuqta faza funksiyasi uchun xosmas kritik nuqta (minimum nuqtasi) bo'lib,* $\psi(t_1, t_2)$ *funksiyani tashuvchisi* (t_1^0, t_2^0) *nuqtaning atrofida bo'lsin. U holda* $\lambda \to +\infty$ *da quyidagi asimptotik yoyilma o'rinli bo'ladi:*

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,\psi(t_1,t_2)dt_1dt_2 \approx e^{-\lambda g(t_1^0,t_2^0)}\lambda^{-1}\sum_{i=0}^{\infty} a_i\lambda^{-i} \qquad (2.2.2)$$

Eslatma: Yuqoridagi asimptotik yoyilma quyidagi ma'noda tushuniladi: istalgan nomanfiy butun r va N sonlari uchun quyidagi munosabat o'rinli:

$$\left(\frac{d}{d\lambda}\right)^r\left[I(\lambda) - e^{-\lambda g(t_1^0,t_2^0)}\lambda^{-1}\sum_{i=0}^{N} a_i\lambda^{-i}\right] =$$

$$= O\left(\lambda^{-r-(N+1)}\right)\,\lambda \to +\infty$$

Yuqoridagi teoremaning natijasi sifatida yoyilmaning bosh hadi keltirilgan.

2.2.1-natija: *(2.2.2) yoyilma uchun quyidagi formula o'rinli:*

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,\psi(t_1,t_2)dt_1dt_2 =$$

$$= e^{-\lambda g(t_1^0,t_2^0)}\frac{2\pi}{\lambda}\left[\frac{\psi(t_1^0,t_2^0)}{\sqrt{\det(g''(t_1^0,t_2^0))}} + \frac{O(1)}{\lambda}\right], \lambda \to +\infty$$

Ikkinchi bobni 3-paragrafida quyidagi

$$I(\lambda) = \int\limits_{\mathbb{R}^n} e^{-\lambda g(\mathrm{t})}\psi(\mathrm{t})\mathrm{dt} \qquad (2.3.8)$$

integral qaralgan.

Dastlab $\lambda \to +\infty$ da quyidagi integral qaraladi.

$$F(\lambda) = \int\limits_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})\mathrm{dx}, \qquad (\mathrm{x} = (x_1, x_2, \dots, x_n)) \qquad (2.3.1)$$

$$\int\limits_{\mathbb{R}^n} e^{-|\mathrm{x}|^2}|f(\mathrm{x})|\mathrm{dx} < +\infty \qquad (2.3.2)$$

(2.3.1) integral (2.3.2) shartda parametr cheksizlikka intilganda o'zining asosiy qiymatini koordinatalar boshining yetarlicha kichik atrofida olishini ko'rsatuvchi tasdiq keltirilgan.

2.3.1-tasdiq: *Faraz qilaylik, f funksiya (2.3.2) shartni qanoatlantirsin. U holda istalgan* $\delta > 0$ *uchun (2.3.1) integral*

$$\int\limits_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})\mathrm{dx} = \int\limits_{B[0,\delta]} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})\mathrm{dx} + O(1)e^{-\lambda\delta^2}, \qquad \lambda \geq 1 \quad (2.3.3)$$

bahoni qanoatlantiradi.

Yanada umumiyroq bo'lgan quyidagi tasdiq keltirilgan.

2.3.2-tasdiq: *Har qanday* $\delta > 0$ *va ixtiyoriy* $k_1 \geq 0, k_2 \geq 0, \dots, k_n \geq 0$ *uchun*

$$\int\limits_{B[0,\delta]} e^{-\lambda|\mathrm{x}|^2} x_1^{2k_1} x_2^{2k_2} \dots x_n^{2k_n} \mathrm{dx} =$$

$$= \frac{\Gamma\left(k_1 + \frac{1}{2}\right)\Gamma\left(k_1 + \frac{1}{2}\right)\dots\Gamma\left(k_1 + \frac{1}{2}\right)}{\lambda^{k_1+k_2+\dots+k_n+\frac{n}{2}}} + O\left(e^{-\lambda\delta^2}\right) \quad (2.3.4)$$

baho o'rinli.

So'ngra quyidagi sodda tasdiq keltirilgan.

2.3.3-tasdiq: *Hech bo'lmaganda biri toq bo'lgan* $k_1 > 0, k_2 > 0, \dots, k_n > 0$ *sonlar uchun quyidagi tenglik o'rinli.*

$$\int\limits_{B[0,\delta]} e^{-\lambda|\mathrm{x}|^2} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \mathrm{dx} = 0 \qquad (2.3.5)$$

Yuqoridagi mulohazalarning natijasi sifatida quyidagi teorema keltirilgan.

2.3.1-teorema: *Faraz qilaylik, f funksiya (2.3.2) shartni qanoatlantirib, koordinatalar boshining biror atrofida 2m marta uzluksiz xususiy hosilalarga ega bo'lsin. U holda* $\lambda \to +\infty$ *da*

$$\int\limits_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})\mathrm{dx} = \lambda^{-\frac{n}{2}}\left[\sum_{i=0}^{m-1} a_i\lambda^{-i} + \frac{O(1)}{\lambda^m}\right] \qquad (2.3.6)$$

asimptotik tenglik o'rinli.

Bu yerda a_i *koeffisiyent quyidagicha aniqlanadi:*

$$a_i = \frac{\sum_{j_1=0}^{i}\sum_{j_2=0}^{j_1}\dots\sum_{j_{n-1}=0}^{j_{n-2}} \frac{\partial^{2i} f(0)}{\partial x_1^{2(i-j_1)}\partial x_2^{2(j_1-j_2)}\dots\partial x_{n-1}^{2(j_{n-2}-j_{n-1})}\partial x_n^{2j_{n-1}}} c(i, j_1, j_2, \dots, j_{n-1})}{(2i)!}$$

bunda $c(i, j_1, j_2, \dots, j_{n-1})$ *quyidagicha aniqlanadi:*

$$c(i, j_1, j_2, \dots, j_{n-1}) = C_{2i}^{2j_1} C_{2j_1}^{2j_2} \dots C_{2j_{n-2}}^{2j_{n-1}} \Gamma\left(i - j_1 + \frac{1}{2}\right) \Gamma\left(j_1 - j_2 + \frac{1}{2}\right) \dots \Gamma(j_{n-1} + \frac{1}{2})$$

2.3.1-natija: *Koordinatalar boshining biror atrofida ikki marta uzluksiz xususiy hosilalarga ega bo'lgan (2) shartni qanoatlantiruvchi f funksiya uchun*

$$\int\limits_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})\mathrm{dx} = \left(\frac{\pi}{\lambda}\right)^{\frac{n}{2}} \left[f(0) + \frac{O(1)}{\lambda}\right], \lambda \to +\infty \quad (2.3.7)$$

asimptotik tenglik o'rinli.

Xosmas kritik nuqtaga ega bo'lgan funksiyani shu nuqta atrofida normal shaklga keltirish haqidagi Mors lemmasi keltirilgan.

2.3.2- teorema (*Mors lemmasi*). *Faraz qilaylik, $\mathrm{x} = \mathrm{x}^0$ xosmas kritik nuqta bo'lsin. U holda shu nuqtaning shunday U, $y = 0$ nuqtaning biror V atrofi hamda $\varphi : V \mapsto U$ diffeomorf akslantirish topiladiki, biror $0 \le m \le n$ butun son uchun*

$$\Phi(\varphi(\mathrm{y})) = \sum_{j=1}^{m} y_j^2 - \sum_{j=m+1}^{n} y_j^2,$$

tenglik o'rinli bo'ladi. Bu yerda m soni

$$\left[\frac{\partial^2 \Phi}{\partial x_k \partial x_j}\right](\mathrm{x}^0)$$

matrisaning musbat xos qiymatlari soni bilan ustma-ust tushadi.

Yuqoridagi mulohazalarga asoslangan holda quyidagi

$$I(\lambda) = \int\limits_{\mathbb{R}^n} e^{-\lambda g(\mathrm{t})} \psi(\mathrm{t}) \mathrm{dt} \qquad (2.3.8)$$

integral uchun parametr cheksizlikka intilganda asimptotik yoyilma olingan.

2.3.3-teorema: *Faraz qilaylik, $g(\mathrm{t})$ funksiya t^0 nuqtada xosmas kritik nuqtaga ega bo'lib bu nuqta minimum nuqtasi bo'lsin. Agar $\psi(\mathrm{t})$ funksiyaning tashuvchisi t^0 nuqtaning yetarli kichik atrofida joylashgan bo'lsa, u holda $\lambda \to +\infty$ da quyidagi asimptotik formula o'rinli bo'ladi:*

$$I(\lambda) \sim e^{-\lambda g(\mathrm{t}^0)} \lambda^{-\frac{n}{2}} \sum_{i=0}^{\infty} a_i \lambda^{-i}, \qquad (2.3.9)$$

bu yerda a_i koeffitsiyentlar quyidagicha aniqlanadi:

$$a_i = \frac{\sum_{j_1=0}^{i}\sum_{j_2=0}^{j_1}\dots\sum_{j_{n-1}=0}^{j_{n-2}}\frac{\partial^{2i}f(0)}{\partial y_1^{2(i-j_1)}\partial y_2^{2(j_1-j_2)}\dots\partial y_{n-1}^{2(j_{n-2}-j_{n-1})}\partial y_n^{2j_{n-1}}}c(i,j_1,j_2,..,j_{n-1})}{(2i)!}$$

bunda o'z navbatida $c(i,j_1,j_2,..,j_{n-1})$ *koeffisiyentlar va f(y) funksiya quyidagicha aniqlanadi:*

$$c(i,j_1,j_2,..,j_{n-1}) = C_{2i}^{2j_1}C_{2j_1}^{2j_2}\dots C_{2j_{n-2}}^{2j_{n-1}}\Gamma\left(i-j_1+\frac{1}{2}\right)\Gamma\left(j_1-j_2+\frac{1}{2}\right)\dots\Gamma(j_{n-1}+\frac{1}{2})$$

$$f(\mathrm{y}) = 2^{\frac{n}{2}}\frac{\psi(\mathrm{t}^0+\varphi(\mathrm{y}))}{\sqrt{det\Phi''_{\mathrm{xx}}(\varphi(\mathrm{y}))}}$$

bu yerda $\Phi(\mathrm{x}) = g(\mathrm{x}+\mathrm{t}^0) - g(\mathrm{t}^0)$, φ *esa 2.3.2-teoramadagi diffeomorf akslantirish.*

Eslatma. a_0 koeffisiyent uchun quyidagi formula o'rinli:

$$a_0 = \frac{(2\pi)^{\frac{n}{2}}\psi(\mathrm{t}^0)e^{-\lambda g(\mathrm{t}^0)}}{\sqrt{|det g''_{\mathrm{tt}}(\mathrm{t}^0)|}}$$

2.3.3-natija: *Faraz qilaylik,* $g(\mathrm{t})$ *funksiya* t^0 *nuqtada xosmas kritik nuqtaga ega bo'lib bu nuqta minimum nuqtasi bo'lsin. Agar* $\psi(\mathrm{t})$ *funksiyaning tashuvchisi* t^0 *nuqtaning yetarli kichik atrofida joylashgan bo'lsa, u holda* $\lambda \to +\infty$ *da quyidagi asimptotik tenglik o'rinli bo'ladi:*

$$\int_{\mathbb{R}^n} e^{-\lambda g(\mathrm{t})}\psi(\mathrm{t})\mathrm{dt} = e^{-\lambda g(\mathrm{t}^0)}\left(\frac{2\pi}{\lambda}\right)^{-\frac{n}{2}}\left[\frac{\psi(\mathrm{t}^0)}{\sqrt{|det g''_{\mathrm{tt}}(\mathrm{t}^0)|}} + \frac{O(1)}{\lambda}\right]$$

Uchinchi bobda karrali Laplas integrali asimptotik yoyilmasining tadbiqlari qaraladi.

Uchinchi bobni 1- paragrafida ikki karrali Laplas integrali asimptotik yoyilmasining tadbiqi sifatida plastinkada issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasining asimptotik yechimi qaralgan.

Plastinkada issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi quyidagicha aniqlanadi:

$$\begin{cases} \dfrac{\partial u(x_1,x_2,t)}{\partial t} = a^2\left(\dfrac{\partial^2 u(x_1,x_2,t)}{\partial x_1^2} + \dfrac{\partial^2 u(x_1,x_2,t)}{\partial x_2^2}\right) & (3.1.1) \\ u(x_1,x_2,t)|_{t=+0} = \psi(x_1,x_2) & (3.1.2) \end{cases}$$

3.1.1-teorema: *Faraz qilaylik $\psi(x_1,x_2)$ funksiya $C_0^\infty(\mathbb{R}^2)$ sinfga tegishli bo'lsin. U holda (3.1.2) shartni qanoatlantiruvchi (3.1.1) tenglamani yechimi uchun $t \to 0$ da quyidagi yoyilma o'rinli:*

$$u(x_1,x_2,t) \approx \sum_{i=0}^{\infty} b_i t^i \qquad (3.1.4)$$

bu yerda b_i koeffisiyent quyidagicha aniqlanadi:

$$b_i = \frac{(4a^2)^i}{(2i)!\,\pi}\sum_{j=0}^{i} \frac{\partial^{2i}\psi(x_1,x_2)}{\partial x_1^{2(i-j)}\partial x_2^{2j}} C_{2i}^{2j}\,\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)$$

Natija sifatida (3.1.2) shartni qanoatlantiruvchi (3.1.1) tenglamani beshta hadgacha olingan asimptotik yechimi keltirilgan.

3.1.1-natija: *Faraz qilaylik $\psi(x_1,x_2)$ funksiya $C_0^\infty(\mathbb{R}^2)$ sinfga tegishli bo'lsin. U holda (2) shartni qanoatlantiruvchi (1) tenglamani yechimi uchun $t \to 0$ da*

$$u(x_1,x_2,t) = \psi(x_1,x_2) + a^2\left(\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}\right)t + \frac{a^4}{2}\left(\frac{\partial^4\psi}{\partial x_1^4} + 2\frac{\partial^4\psi}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\psi}{\partial x_2^4}\right)t^2$$
$$+ \frac{a^6}{6}\left(\frac{\partial^6\psi}{\partial x_1^6} + 3\frac{\partial^6\psi}{\partial x_1^4\partial x_2^2} + 3\frac{\partial^6\psi}{\partial x_1^2\partial x_2^4} + \frac{\partial^6\psi}{\partial x_2^6}\right)t^3 +$$
$$+ \frac{a^8}{168}\left(\frac{\partial^8\psi}{\partial x_1^8} + 28\frac{\partial^8\psi}{\partial x_1^6\partial x_2^2} + 48\frac{\partial^8\psi}{\partial x_1^4\partial x_2^4} + 28\frac{\partial^8\psi}{\partial x_1^2\partial x_2^6} + \frac{\partial^8\psi}{\partial x_2^8}\right)t^4 + O(1)t^5$$

asimptotik tenglik o'rinli.

Uchinchi bobning 3- paragrafida *n* karrali Laplas integrali asimptotik yoyilmasining tadbiqi sifatida $\mathbb{R}^n$fazosida issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasining asimptotik yechimi qaralgan.

$\mathbb{R}^n$fazosida issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi quyidagicha aniqlanadi:

$$\begin{cases} \dfrac{\partial u(\mathrm{x},t)}{\partial t} = a^2\left(\dfrac{\partial^2 u(\mathrm{x},t)}{\partial x_1^2} + \dfrac{\partial^2 u(\mathrm{x},t)}{\partial x_2^2} + \cdots + \dfrac{\partial^2 u(\mathrm{x},t)}{\partial x_n^2}\right) & (3.2.1) \\ \qquad u(\mathrm{x},t)|_{t=+0} = \psi(\mathrm{x}) & (3.2.2) \end{cases}$$

(3.2.2) shartni qanoatlantiruvchi (3.2.1) tenglamani yechimi ushbu

$$u(\mathrm{x},t) = \frac{1}{(2a\sqrt{\pi t})^n} \iint\limits_{\mathbb{R}^n} \psi(\xi) e^{-\frac{r^2}{4a^2 t}} d\xi; \ (3.2.3)$$

$$(r = |\mathrm{x} - \xi|)$$

formula bilan aniqlanadi.

3.2.1-teorema: *Faraz qilaylik $\psi(\mathrm{x})$ funksiya $C_0^\infty(\mathbb{R}^n)$ sinfga tegishli bo'lsin. U holda (3.2.2) shartni qanoatlantiruvchi (3.2.1) tenglamaning yechimi uchun $t \to 0$ da quyidagi yoyilma o'rinli:*

$$u(\mathrm{x},t) \approx \sum_{i=0}^{\infty} b_i t^i \qquad (3.2.4)$$

bu yerda b_i koeffisiyent quyidagicha aniqlanadi:

$$b_i = \frac{(4a^2)^i}{(2i)!\,\pi^{\frac{n}{2}}} \sum_{j=0}^{i} \sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} \dots \sum_{j_{n-1}=0}^{j_{n-2}} \frac{\partial^{2i}\psi(\mathrm{x})}{\partial x_1^{2(i-j_1)} \partial x_2^{2(j_1-j_2)} \dots \partial x_{n-1}^{2(j_{n-2}-j_{n-1})} \partial x_n^{2j_{n-1}}} c(i, j_1, j_2, \dots, j_{n-1})$$

bunda $c(i, j_1, j_2, \dots, j_{n-1})$ quyidagicha aniqlanadi:

$$c(i, j_1, j_2, \dots, j_{n-1}) = C_{2i}^{2j_1} C_{2j_1}^{2j_2} \dots C_{2j_{n-2}}^{2j_{n-1}} \Gamma\left(i - j_1 + \frac{1}{2}\right) \Gamma\left(j_1 - j_2 + \frac{1}{2}\right) \dots \Gamma(j_{n-1} + \frac{1}{2})$$

Teoremaning natijasi sifatida $\mathbb{R}^3$ fazosida (3.2.2) shartni qanoatlantiruvchi (3.2.1) tenglamani to'rtta hadgacha olingan asimptotik yechimi keltirilgan.

3.2.1-natija: *Faraz qilaylik $\psi(\mathrm{x})$ funksiya $C_0^\infty(\mathbb{R}^3)$ sinfga tegishli bo'lsin. U holda (3.2.2) shartni qanoatlantiruvchi (3.2.1) tenglamani yechimi uchun $t \to 0$ da*

$$u(\mathrm{x},t) = \psi(\mathrm{x}) + a_1 t + a_2 t^2 + a_3 t^3 + O(1) t^4$$

asimptotik tenglik o'rinli bo'lib, bunda a_1, a_2 va a_3 koeffisiyentlar quyidagicha aniqlanadi:

$$a_1 = a^2 \left(\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2} \right)$$

$$a_2 = \frac{a^4}{2} \left(\frac{\partial^4\psi}{\partial x_1^4} + \frac{\partial^4\psi}{\partial x_2^4} + \frac{\partial^4\psi}{\partial x_3^4} + 2\frac{\partial^4\psi}{\partial x_1^2 \partial x_2^2} + 2\frac{\partial^4\psi}{\partial x_1^2 \partial x_3^2} + 2\frac{\partial^4\psi}{\partial x_2^2 \partial x_3^2} \right)$$

$$a_3 = \frac{a^6}{6} (\frac{\partial^6\psi}{\partial x_1^6} + \frac{\partial^6\psi}{\partial x_2^6} + \frac{\partial^6\psi}{\partial x_3^6} + 3(\frac{\partial^6\psi}{\partial x_1^4 \partial x_2^2} + \frac{\partial^6\psi}{\partial x_1^4 \partial x_3^2} + \frac{\partial^6\psi}{\partial x_2^4 \partial x_3^2} +$$

$$+\frac{\partial^6\psi}{\partial x_2^4\partial x_1^2}+\frac{\partial^6\psi}{\partial x_3^4\partial x_1^2}+\frac{\partial^6\psi}{\partial x_3^4\partial x_2^2})+15\frac{\partial^6\psi}{\partial x_1^2\partial x_2^2\partial x_3^2})$$

I BOB. Parametrga bog'liq karrali integrallar.

1.1-§ Parametrga bog'liq karrali xosmas integrallar.

Dastlab parametrga bog'liq karrali integrallarni ta'rifi va asosiy xossalarini keltiramiz.

Bizga $G \subset \mathbb{R}^m$ va $\Omega \subset \mathbb{R}^n$ to'plamlar hamda $f: G \times \Omega \to \mathbb{R}$ funksiya berilgan bo'lsin. Bu degani, $f(x, y)$ ikki ko'p o'lchovli o'zgaruvchilarning, ya'ni *m* o'lchovli *x* ning va *n* o'lchovli *y* ning funksiyasi ekanligini anglatadi.

Faraz qilaylik, $\Omega \subset \mathbb{R}^n$ to'plam Jordan ma'nosida o'lchovli bo'lsin. Bundan tashqari, $f(x, y)$ har bir tayinlangan $x \in G$ da *y* ning funksiyasi sifatida Ω to'plamda integrallanuvchi bo'lsin. Bu holda

$$\Phi(x) = \int\limits_{\Omega} f(x, y)\, dy \qquad (1.1.1)$$

integral $x \in G$ parametrga bog'liq karrali integral deb ataladi.

Quyida keltirilgan tasdiqlarda biz G va Ω to'lamlarni sodda deb, ya'ni ochiq bog'lamli to'plam deb hisoblaymiz. Jordan bo'yicha o'lchovli ekanligidan ularning chegaralanganligi kelib chiqadi. Biz $\bar{G}$ simvol bilan G to'plamning yopig'ini belgilaymiz.

1.1.1-tasdiq. *Istalgan $f \in C(\bar{G} \times \bar{\Omega})$ funksiya uchun (1.1.1) integral $\bar{G}$ yopiq to'plamda uzluksiz funksiyadir.*

Isbot: Shartga ko`ra $f \in C(\bar{G} \times \bar{\Omega})$ ekan demak, ixtiyoriy $\varepsilon > 0$ soni uchun shunday $\delta(\varepsilon) > 0$ topiladiki, $\rho\big((x', y'), (x'', y'')\big) < \delta$ tengsizlikni qanoatlantiruvchi barcha $(x', y') \in \mathbb{R}^{n+m}$ va $(x'', y'') \in \mathbb{R}^{n+m}$ nuqtalar uchun

$$|f(x', y') - f(x'', y'')| < \varepsilon$$

tengsizlik bajariladi. Xususan, $(x', y') = (x + h, y)$, $(x'', y'') = (x, y)$ nuqtalar uchun $|h| < \delta$ bo'lganda $|f(x + h, y) - f(x, y)| < \varepsilon$ tengsizlik o'rinli bo'ladi.

Quyidagi ayirmani baholaymiz:

$$|\Phi(x + h) - \Phi(x)| \leq \int\limits_{\Omega} |f(x + h, y) - f(x, y)|\, dy < \varepsilon |\Omega|$$

Demak, ixtiyoriy $\varepsilon > 0$ soni uchun shunday $\delta(\varepsilon) > 0$ topiladiki,

$\rho\big((x+h,y),(x,y)\big)<\delta$ tengsizlikni qanoatlantiruvchi barcha $(x+h,y)\in\mathbb{R}^{n+m}$ va $(x,y)\in\mathbb{R}^{n+m}$ nuqtalar uchun

$$|\Phi(x+h)-\Phi(x)|<\varepsilon|\Omega|$$

tengsizlik bajarilar ekan. Bu esa $\Phi(x)$ funksiyaning $\bar{G}$ da uzluksizligini anglatadi.

1.1.2-tasdiq. *Agar G kvadratlanuvchi soha bo'lsa, istalgan $f\in C(\bar{G}\times\bar{\Omega})$ funksiya uchun (1.1.1) tenglik bilan aniqlangan Φ funksiya G to'plamda integrallanuvchi bo'lib,*

$$\int\limits_G \Phi(x)\,dx=\int\limits_\Omega dy\int\limits_G f(x,y)\,dx \qquad (1.1.2)$$

tenglik bajariladi.

Isbot: Tasdiqning shartidan kelib chiqib quyidagilarni yozamiz:

$$\int\limits_G f(x,y)\,dx=f(x',y)|G|, \qquad \int\limits_\Omega f(x,y)\,dy=f(x,y')|\Omega|$$

Yuqoridagi tengliklarning bunday yozilishini sababchisi o'rta qiymat haqidagi teoremani natijasidir. Bu natijani yana bir bor qo'llaymiz:

$$\int\limits_\Omega dy\int\limits_G f(x,y)\,dx=\int\limits_\Omega f(x',y)|G|\,dy=f(x',y')|G\times\Omega|,$$

$$\int\limits_G dx\int\limits_\Omega f(x,y)\,dy=\int\limits_G f(x,y')|\Omega|\,dx=f(x',y')|G\times\Omega|,$$

yuqoridagi ikki tenglikdan (1.1.2) ning o'rinli ekanligi kelib chiqadi.

1.1.3-tasdiq. *Agar $f\in C(\bar{G}\times\bar{\Omega})$ va har bir $1\le j\le m$ uchun $\frac{\partial f}{\partial x_j}\in C(\bar{G}\times\bar{\Omega})$ bo'lsa, u holda (1.1.1) integral G sohada x_j o'zgaruvchi bo'yicha uzluksiz differensiallanuvchi bo'lib,*

$$\frac{\partial\Phi(x)}{\partial x_j}=\int\limits_\Omega \frac{\partial f(x,y)}{\partial x_j}dy \qquad (1.1.3)$$

tenglik bajariladi.

Isbot: Buning uchun $\tilde{x} = (x_1, x_2, \dots, x_{m-1})$ deb belgilash kiritamiz. U holda, $x = (\tilde{x}, x_m) \in G$.Endi $\tilde{x}$ ni tayinlab, shunday a son tanlaymizki, bunda $(\tilde{x}, a)$ va $= (\tilde{x}, x_m)$ nuqtalarni tutashtiruvchi kesma butunligicha G sohada yotsin.

Agar

$$\psi(x_m) = \int_{\Omega} \frac{\partial f(\tilde{x}, x_m, y)}{\partial x_m} dy \qquad (1.1.4)$$

bir o'zgaruvchili funksiyani kiritsak, u holda bu funksiya integrali uchun, 1.1.2-tasdiqni qo'llab va Nyuton-Leybnits formulasidan foydalanib,

$$\int_a^{x_m} \psi(t)\, dt = \int_{\Omega} dy \int_a^{x_m} \frac{\partial f(\tilde{x}, t, y)}{\partial t} dt =$$

$$= \int_{\Omega} [f(\tilde{x}, x_m, y) - f(\tilde{x}, a, y)] dy = \Phi(\tilde{x}, x_m) - \Phi(\tilde{x}, a)$$

tenglikni olamiz.

Bu tenglikni ikki tarafini x_m bo'yicha differensiallasak,

$$\psi(x_m) = \frac{\partial \Phi(x)}{\partial x_j}$$

munosabatga ega bo'lamiz.

Hosil bo'lgan tenglikni (1.1.4) bilan taqqoslab, talab qilingan (1.1.3) formulani j=m uchun olamiz. Boshqa j lar uchun ham isbot xuddi shu kabi bo'ladi.

(1.1.1) ko'rinishdagi karrali xosmas integrallar xuddi bir karrali parametrga bog'iq integrallar kabi o'rganiladi. Bunda quyidagi ikki holni qarash kerak: 1) integral ostidagi funksiya maxsusliklar to'plami parametrga bog'liq emas; 2) bu to'plam parametrga bog'liq.

Birinchi holda (1.1.1) xosmas integral mahkamlangan maxsuslikka ega bo'lgan integral deb ataladi. Ushbu bandda aynan shunday xosmas integrallarni o'rganamiz.

Mahkamlangan maxsuslikka ega (1.1.1) xosmas integralni, parametrga bog'liqmas ravishda tanlangan, qamrab oluvchi sohalar bo'yicha integrallar bilan yaqinlashtirish mumkin. Biz integral ostidagi funksiya maxsusliklari Ω sohasining chegarasida yotadi deb faraz qilamiz. Masalan, agar ψ funksiyaning maxsusligi bo'lmasa,

$$\int\limits_{|y|<1} \frac{\psi(x,y)}{|y|^{n-1}\sqrt{1-|y|^2}}dy \qquad (1.1.5)$$

xosmas integral uchun Ω soha sifatida, tabiiyki,

$\Omega = \{y \in \mathbb{R}^n : 0 < |y| < 1\}$ to'plamni olish kerak.

Bu sohaning chegarasi radiusi 1 va markazi *y=0* nuqtada bo'lgan sfera va *y=0* nuqtaning o'zidan iborat.

Ravshanki,

$$\Omega_k = \left\{y \in \mathbb{R}^n : \frac{1}{k} < |y| < 1 - \frac{1}{k}\right\}$$

to'plamlar Ω ni qamarab oluvchi $\{\Omega_k\}$ sohalar ketma-ketligini tashkil qiladi.

Yana bir misol sifatida

$$\int\limits_{\mathbb{R}^n} \frac{\psi(x,y)}{|y-a|^{n-1}}dy \qquad (1.1.6)$$

integralni olishimiz mumkin, bu yerda *a* orqali $\mathbb{R}^n$ fazosining ixtiyoriy tayinlangan nuqtasi belgilangan. (1.1.6) integral uchun Ω soha sifatida $\mathbb{R}^n$ fazodan *a* nuqtani chiqarib tashlash natijasida hosil bo'lgan to'plamni olish mumkin, ya'ni

$$\Omega = \{y \in \mathbb{R}^n : y \neq a\}.$$

Ravshanki,

$$\Omega_k = \left\{y \in \mathbb{R}^n : \frac{1}{k} < |y| < k\right\}$$

sohalar *x* parametrga bog'liq bo'lmay, Ω ni qamrab oluvchi ketma-ketlikni tashkil qiladi.

Har bir (1.1.5) va (1.1.6) integral mahkamlangan maxsuslikka ega va *x* parametrga bog'liq xosmas integraldir. Shuning uchu integrallash sohasini qamrab oluvchi sohalar ketma-ketligini *x* parametrga bog'liqmas ravishda tanlash mumkin. E'tibor bersak, agar (1.1.6) integralda *a* ni parametr deb hisoblasak, u holda Ω_k sohalarni *a* ga bog'liq qilib tanlashga to'g'ri kelar edi.

Faraz qilayliki, (1.1.1) xosmas integral bo'lib, $x \in G$ parametrning har bir qiymatida yaqinlashsin. Bundan tashqari, Ω_k sohalar ketma-ketligi Ω ni qamrab olsin. Quyidagi

$$\Phi_k(x) = \int_{\Omega_k} f(x,y)\,dy \qquad (1.1.7)$$

Funksiyalar ketma-ketligini kiritamiz.

Agar (1.1.7) ko'rinishdagi ixtiyoriy $\Phi_k(x)$ ketma-ketlik $\Phi(x)$ ga G da tekis yaqinlashsa, u holda.

$$\Phi(x) = \int_{\Omega} f(x,y)dy$$

xosmas integralni $x \in G$ *parametr bo'yicha tekis yaqinlashadi* deymiz.

Xuddi bir o'zgariluvchi holidagidek, karrali xosmas inegrallar uchun ham Veyershtrassning tekis yaqinlashish alomati o'rinli: agar integral ostidagi $f(x,y)$ funksiya yuqoridan *x* ga bog'liqmas *g(y)* funksiya bilan baholansa, u holda *g* dan olingan integral yaqinlashishidan *f* dan olingan integralning tekis yaqinlashishi kelib chiqadi. Maslan, agar ψ funksiya biror o'zgarmas bilan chegaralangan bo'lsa, u holda (1.1.5) integral *x* bo'yicha tekis yaqinlashadi.

Parametrga bog'liq (1.1.1) ko'rinishdagi tekis yaqinlashuvchi karrali xosmas integrallar uchun ham bir o'zgariluvchili holda isbotlangan kabi tasdiqlar o'rinli. Xususan,

1) *agar integral ostidagi $f(x,y)$ funksiya $G \times \Omega$ da uzluksiz bo'lsa, u holda tekis yaqinlashuvchi (1.1.1) integral G da uzluksiz funsiya bo'ladi;*
2) *agar (1.1.1) integral barcha $x \in G$ larda yaqinlashsa hamda integral ostidagi funksiya $G \times \Omega$ da uzluksiz $\frac{\partial f(x,y)}{\partial x_j}$ hosilaga ega bo'lib, bu hosiladan olingan integral tekis yaqinlashsa, u holda integral ostida differensiallash formulasi o'rinli:*

$$\frac{\partial}{\partial x_j}\int_{\Omega} f(x,y)\,dy = \int_{\Omega} \frac{\partial f(x,y)}{\partial x_j}$$

1.1.1-misol. Quyidagi

$$u(x) = \int\limits_{\mathbb{R}^n} \frac{\psi(x,y)}{|y|^{\lambda}} dy, \quad \lambda < n, \qquad (1.1.8)$$

integralni qaraymiz.

Aytaylik, $\psi(x,y)$ funksiya markazi koordinatalar boshida bo'lgan biror shardan tashqarida nolga teng bo'lsin, ya'ni shunday $R > 0$ son topilsinki,

$$\psi(x,y) = 0, \quad |y| > R$$

shart bajarilsin.

U holda integral, aslida, shu shar bo'yicha olinadi, ya'ni

$$u(x) = \int\limits_{|y| \leq R} \frac{\psi(x,y)}{|y|^{\lambda}} dy$$

Ravshanki, $g(y) = |y|^{-\lambda}$ funksiya $\lambda < n$ da shu shar bo'yicha integrallanuvchi. Shunday ekan, biz Veyershtrass alomatini qo'llashimiz mumkin. Bu alomatga ko'ra:

1) agar ψ ikki o'zgaruvchining uzluksiz funksiyasi bo'lsa, u holda (1.1.8) integral uzluksiz funksiya bo'ladi:
2) agar ψ funksiya uzluksiz $\frac{\partial \psi}{\partial x_j}$ hosilaga ega bo'lsa, u holda

$$\frac{\partial u(x)}{\partial x_j} = \int\limits_{\mathbb{R}^n} \frac{\partial \psi(x,y)}{\partial x_j} \frac{dy}{|y|^{\lambda}}, \quad \lambda < n,$$

tenglik bajariladi.

Eslatma. Agar $\psi(x,y)$ funksiya cheksiz differensiallanuvchi bo'lib, (1.1.9) shartni qanoatlantirsa, u holda (1.1.8) integral ham parametrga nisbatan cheksiz differensiallanuvchi bo'lib, istalgan α multiindeks uchun

$$D^{\alpha} u(x) = \int\limits_{\mathbb{R}^n} \frac{D_x^{\alpha} \psi(x,y)}{|y|^{\lambda}} dy, \quad 0 < \lambda < n, \qquad (1.1.10)$$

tenglik bajarialdi.

1.2-§ Diagonalda maxsuslikka ega karrali xosmas integrallar.

Bu banda biz integral ostidagi funksiya parametrga bog'liq to'plamda cheksizlikka aylanishi mumkin bo'lgan xosmas integrallarni qaraymiz. Bunday turdagi xosmas integrallardan eng ko'p ahamiyatga ega bo'lgani quyidagi

$$u(x) = \int_{\mathbb{R}^n} F(y-x)\,\psi(y)dy \qquad (1.2.1)$$

ko'rinishdagi integraldir. Bu integralda $\psi(y)$ funksiya $\mathbb{R}^n$ fazoda aniqlangan va yetarlicha sillia bo'lib, $F(y)$ esa, faqat $y = 0$ nuqtada maxsuslikka ega bo'lishi mumkin bo'lgan, $\mathbb{R}^n$ fazoda uzluksiz funksiyadir.

Bu degani, integral ostidagi funksiyaning maxsusligi $x = y$ <<diagonalda>> joylashgan. Parametr rolini $x \in \mathbb{R}^n$ o'zgaruvchi o'ynaydi.

Agar F funksiyani tayinlab qo'ysak, (1.2.1) o'zi aniqlangan har bir $\psi(x)$ funksiyaga biror $u(x)$ funksiyani mos qo'yadi. Bunday akislantirish *integral operator* va $F(y-x)$ funksiya esa, uning *yadrosi* deb ataladi. Yadroni biz lokal integrallanuvchi deb hisoblaymiz, ya'ni istalgan $R > 0$ uchun

$$\int_{|y|<R} |F(y)|\,dy = C_R < +\infty \qquad (1.2.2)$$

xosmas integral yaqinlashadi deymiz.

(1.2.1) integral operator ta'sir qiluvchi ψ funksiyalar sifatida biror $R > 0$ (ψ ga bog'liq bo'lgan) o'zgarmas bilan

$$\psi(y) = 0, \quad |y| > R, \qquad (1.2.3)$$

sharti qanoatlantiruvchi funksiyalarni olamiz.

Bu shartlarda (1.2.1) da integrallash, aslida, $\mathbb{R}^n$ fazosi bo'yicha emas, balki radiusi R va markazi koordinatalar boshida bo'lgan shar bo'yicha olinayotgani kelib chiqadi. Shunday ekan, (1.2.1) ikkinchi tur xosmas integral bo'lib, integral ostidagi funksiya maxsusligi $x = y$ nuqtada joylashgan. Misol sifatida, matematik fizikada muhim rol o'ynovchi,

$$u(x) = \int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{n-2}}dy$$

integralni olishimiz mumkin.

(1.2.1) ko'rinishdagi xosmas integrallar quyidagi ajoyib xossaga ega: agar integral operator ta'sir qiluvchi $\psi(x)$ funksiya cheksiz differensiallanuvchi bo'lsa, u holda, yadro silliq bo'lmasada, $u(x)$ funksiya ham cheksiz differensiallanuvchi bo'ladi.

Haqiqatdan, o'zgaruvchini almashtirib, (1.2.1) integralni

$$u(x) = \int_{\mathbb{R}^n} F(y)\,\psi(y+x)dy$$

deb yozish mumkin.

Agar $\psi(x)$ funksiya cheksiz differensiallanuvchi bo'lsa, u holda istalgan α multiindeks uchun, formal ravishda quyidagi

$$D^\alpha u(x) = \int_{\mathbb{R}^n} F(y)\,D^\alpha \psi(y+x)dy \qquad (1.2.4)$$

tenglik o'rinli.

Agar F funksiya lokal integrallanuvchi bo'lib, ψ funksiya (1.2.3) shartni qanoatlantirsa, u holda Veyershtrass alomatiga ko'ra, (1.2.4) integrallar tekis yaqinlashadi. Bundan (1.2.4) formulalarning to'g'ri ekanligi va (1.2.1) integralning cheksiz differensiallanuchi ekanligi kelib chiqadi.

teskari almashtirish bajarsak,

$$D^\alpha u(x) = \int_{\mathbb{R}^n} F(y-x)\,D^\alpha \psi(y)dy \qquad (1.2.5)$$

tenglikka ega bo'lamiz.

Ba'zi muhim hollarda (1.2.1) integral operatori yadrosini <<silliqlash>>, ya'ni uni diagonalining yetarlicha kichik atrofidan tashqarida u bilan ustma-ust tushuvchi silliq funksiya bilan almashtirish maqsadga muvofiq bo'ladi.

Buning uchun $\mathbb{R}$ sonlar o'qida cheksiz differensiallanuvchi bo'lgan shunday $\eta(t)$ funksiyani olamizki, u uchun $0 \le \eta(t) \le 1$ shart va

$$\eta(t) = \begin{cases} 0, \text{agar } \ t \le \dfrac{1}{2} \text{ bo'lsa}, \\ 1, \text{agar } \ t \ge 1 \text{ bo'lsa}, \end{cases} \qquad (1.2.6)$$

tenglik bajarilsin.

Bu funksiya yordamida

$$F_k(x) = F(x) \cdot \eta(k|x|), \quad x \in \mathbb{R}^n \backslash 0 \quad (1.2.7)$$

funksiyalar ketma-ketligini kiritamiz.

$F_k(x)$ funksiyalar $|x| < 1/k$ shardan tashqarida $F(x)$ funksiya bilan ustma-ust tushib, $|x| \leq 1/(2k)$ da 0 ga teng. Ravshanki, $k \to +\infty$da har bir $x \neq 0$ nuqtalarda $F_k(x) \to F(x)$. Lekin, F funksiya nolning atrofida chegaralanmagan bo'lishi mumkinligi sababli, bu ketma ketlik $F(x)$ ga tekis yaqinlasha olmaydi. Shunday bo'lsada, (1.2.7) ketma-ketlik F ga o'rtacha yaqinlashadi. Haqiqatdan,

$$\int_{\mathbb{R}^n} |F_k(x) - F(x)|\, dx = \int_{|x|<\frac{1}{k}} |F(x)| \cdot [1 - \eta(k|x|)] dx \leq$$

$$\leq \int_{|x|<\frac{1}{k}} |F(x)|\, dx \to 0, \quad k \to +\infty$$

Oxirgi integralning nolga intilishi F yadroning lokal integrallanuvchi ekanligidan kelib chiqadi.

Demak, (1.2.3) shartni qanoatlantiruvchi istalgan uzluksiz $\psi(x)$ funksiya uchun

$$u_k(x) = \int_{\mathbb{R}^n} F_k(x - y)\, \psi(y) dy = \int_{\mathbb{R}^n} F_k(y)\, \psi(x + y) dy \quad (1.2.8)$$

ketma-ketlik (1.2.1) integralga tekis yaqinlashadi. Aslida bundanda umumiyroq tasdiq o'rinli.

1.2.1-tasdiq. *Faraz qilaylik, α ixtiyoriy multiindeks bo'lsin. Agar (1.2.3) shartni qanoatlantiruvchi ψ funksiya uzluksiz $D^\alpha \psi(x)$ hosilaga ega bo'lsa, u holda (1.2.8) ketma ketlik uchun $x \in \mathbb{R}^n$ bo'yicha tekis ravishda*

$$\lim_{k \to +\infty} D^\alpha u_k(x) = D^\alpha u(x) \qquad (1.2.9)$$

tenglik o'rinli.

Isbot. (1.2.8) ning o'ng tarafidagi ikkinchi integralni differensiallab,

$$D^{\alpha}u_k(x) = \int_{\mathbb{R}^n} F_k(y)\, D^{\alpha}\psi(x+y)dy$$

tenglikni olamiz.

$D^{\alpha}\psi(x)$ funksiya biror M_{α} o'zgarmas bilan chegaralangan bo'lganligi sababli, bu tenglik va (1.2.4) dan talab qilingan munosabatni olamiz:

$$|D^{\alpha}u_k(x) - D^{\alpha}u(x)| \le \int_{\mathbb{R}^n} |F_k(y) - F(x)| \cdot |D^{\alpha}\psi(x+y)|dy \le$$

$$\le M_{\alpha} \int_{\mathbb{R}^n} |F_k(y) - F(x)|\, dy \to 0, \quad k \to +\infty$$

Bu tasdiqqa asosan, biz (1.2.1) xosmas integralni va uning hosilalarini silliq funksiyalardan olingan integrallar bilan yaqinlashtirishimiz mumkin.

Quyidagi

$$u(x) = \int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{\lambda}} dy, \quad 0 < \lambda < n, \quad (1.2.10)$$

ko'rinishdagi integrallar potensial xilidagi integrallar deyiladi.

Agar $n = 3$ va $\lambda = 1$ bo’lsa, yuqoridagi integral gravitatsiya nazaryasida *Nyuton* potensiali deb, elektr nazaryasida esa, *Kulon* potensiali deb ataladi. Bunda integral ostidagi ψ funksiayaga zichlik deyiladi.

Potensial xilidagi integrallarni $F(y) = |y|^{\lambda}$ yadro bilan (1.2.1) ko'rinishda yozish mumkin, bunda, albatta, yadro lokal integrallanuvchi bo'ladi, ya’ni (1.2.2) shartni qanoatlantiradi. Shunday ekan, yuqoridagi mulohazalarga ko'ra, navbatdagi tasdiqni olamiz.

1.2.2-tasdiq. *Faraz qilaylik, α ixtiyoriy mutiindeks bo'lsin. Agar ψ funksiya uzluksiz $D^{\alpha}\psi(x)$ hosilaga ega bo’lib, (1.2.3) shartni qanoatlantirsa, u holda potensial xilidagi (1.2.10) integral ham α -tartibli hosialga ega bo'lib,*

$$D^{\alpha}u(x) = \int_{\mathbb{R}^n} \frac{D^{\alpha}\psi(y)}{|x-y|^{\lambda}} dy, \quad 0 < \lambda < n, \quad (1.2.11)$$

tenglik bajariladi.

Isbot. (1.2.5) fomuladan kelib chiqadi.

E'tibor beradigan bo`lsak, umumiy (1.2.1) holdan farqli o'laroq, potensial xilidagi integrallar yadrosi diagonaldan tashqarida cheksiz differensiallanuvchidir. Shu sababli, (1.2.10) integral silliqligini ψ funksiyaga nisbatan kamroq shartlarda isbotlash mumkin.

Haqiqatdan, bu holda biz

$$\frac{\partial}{\partial x_j}\frac{1}{|x-y|^{\lambda}} = -\lambda\frac{x_j - y_j}{|x-y|^{\lambda+2}}$$

tenglikdan foydalanishimiz mumkin.

Ravshanki, differensiallangan

$$F_j(x-y) = -\lambda\frac{x_j - y_j}{|x-y|^{\lambda+2}}$$

yadro

$$|F_j(x-y)| \leq \frac{\lambda}{|x-y|^{\lambda+1}}$$

bahoni qanoatlantiradi.

Agar $\lambda + 1$ bo'lsa, $F_j(y)$ funksiya lokal integrallanuvchi bo'ladi. Shuning uchun

$$\frac{\partial u(x)}{\partial x_j} = \int\limits_{\mathbb{R}^n} F_j(x-y)\,\psi(y)dy$$

tenglikdan uzluksiz ψ zichlikka ega potensiallarning barcha qismiy hosilalari uzluksiz ekani kelib chiqadi. Bundan chiqdi, navbatdagi tasdiq o'rinli.

1.2.3-tasdiq. *Agar $\lambda < n-1$ bo'lsa, u holda (1.2.3) shartni qanoatlantiruvchi ixtiyoriy uzluksiz ψ funksiya uchun potensial xilidagi (1.2.10) integral $\mathbb{R}^n$ fazoda uzluksiz differensiallanuvchi funksiya bo'lib,*

$$\nabla u(x) = -\lambda\int\limits_{\mathbb{R}^n}\frac{x-y}{|x-y|^{\lambda+2}}\psi(y)dy$$

tenglik bajariladi.

Eslatma. $\lambda = n - 1$ da bu tasdiq o'rinli emas, ya'ni, shunday uzluksiz ψ zichlik topiladiki, bunda (1.2.10) funksiya uzluksiz bo'lsada, uzluksiz differensiallanuvchi bo'lmaydi.

Yuqorida qayd etganimizdek,

$$u(x) = \int\limits_{\mathbb{R}^3} \frac{\psi(y)}{|x-y|^\lambda} dy = \int\limits_{\mathbb{R}^3} \frac{\psi(x+y)}{|y|^\lambda} dy, x \in \mathbb{R}^3 \quad (1.2.12)$$

integral Nyuton potensiali deb ataladi.

Bu potensialga Laplas operatorining ta'siri natijasini topamiz, ya'ni

$$\Delta u(x) = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

ifodani hisoblaymiz.

1.2.1-teorama. *(1.2.3) shartni qanoatlantiruvchi istalgan ikki marta uzluksiz differensiallanuvchi $\psi(x)$ zichlik uchun (1.2.12) Nyuton potensiali*

$$\Delta u(x) = -4\pi\psi(x) \qquad (1.2.13)$$

tenglamani qanoatlantiradi.

Isbot. F yadroga yaqinlashuvchi

$$F_k(y) = F(x) \cdot \eta(k|x|) = \frac{\eta(k|x|)}{(k|x|)}$$

funksiyalar ketma-ketligini qaraymiz.

Bu yerda η (1.2.6) tenglik bilan aniqlangan silliq funksiyadir.

Endi silliqlangan

$$u_k(x) = \int\limits_{\mathbb{R}^3} F_k(x-y)\,\psi(y) dy = \int\limits_{\mathbb{R}^3} F_k(y)\,\psi(x+y) dy \quad (1.2.14)$$

ketma-ketlikni qaraylik.

1.2.1-tasdiqqa ko'ra, teoremamizning shartlari ostida

$$\lim_{k\to+\infty} \Delta u_k(x) = \Delta u(x)$$

munosabat o'rinli.

Demak, teoremani isbotlash uchun

$$\lim_{k\to+\infty} \Delta u_k(x) = -4\pi\psi(x) \qquad (1.2.15)$$

tenglikni ko'rsatish yetarli.

(1.2.14) dagi birinchi integralni parametr bo'yicha differensiallasak,

$$\Delta u_k(x) = \int\limits_{\mathbb{R}^3} \Delta F_k(x-y)\,\psi(y)dy = \int\limits_{\mathbb{R}^3} \Delta F_k(y)\,\psi(x+y)dy$$

tenglikka ega bo'lamiz.

Bu tenglikni

$$\Delta u_k(x) = \psi(x)\int\limits_{\mathbb{R}^3} \Delta F_k(y)\,dy +$$

$$+\int\limits_{\mathbb{R}^3} \Delta F_k(y)\,[\psi(x+y)-\psi(x)]dy \qquad (1.2.16)$$

ko'rinishda yozamiz.

Ravshanki, bunda

$$\Delta F_k(y) = k^2\frac{\eta''(k|y|)}{|y|}$$

Bu funksiya $|y| \le 1/k$ shardan tashqarida nolga teng. Shuning uchun, $= ky$ almashtirish bajarib va sferik koordinatalarga o'tib,

$$\int\limits_{\mathbb{R}^3} \Delta F_k(y)\,dy = \int\limits_{|y|\le\frac{1}{k}} k^2\frac{\eta''(k|y|)}{|y|}dy = \int\limits_{|x|\le 1} \frac{\eta''(|x|)}{|x|}dx =$$

$$4\pi\int\limits_0^1 \eta''(r)rdr = 4\pi\eta'(r)r\Big|_{r=0}^{r=1} - 4\pi\int\limits_0^1 \eta'(r)\,dr = -4\pi\eta(1) = -4\pi$$

tenglikni olamiz (bo'laklab integrallashda biz $\eta(0) = \eta'(1) = 0$ munosabatni hisobga oldik).

Demak, (1.2.16) tenglik

$$\Delta u_k(x) = -4\pi\psi + \int\limits_{\mathbb{R}^3} \Delta F_k(y)\,[\psi(x+y) - \psi(x)]dy$$

ko'rinishga keladi.

Isbotni yakunlash uchun bu tenglikni o'ng tarafidagi integralni nolga intilishini ko'rsatish yetarli. Buning uchun

$$\varepsilon_k(x) = \sup_{|y|\le 1/k} [\psi(x+y) - \psi(x)]$$

deylik.

ψ funksiyaning uzluksizligiga ko'ra, $k \to +\infty$ da $\varepsilon_k \to 0$. Bundan chiqdi,

$$\left| \int\limits_{\mathbb{R}^3} \Delta F_k(y)\,[\psi(x+y) - \psi(x)] \right| \le \int\limits_{\mathbb{R}^3} |\Delta F_k(y)\,||\psi(x+y) - \psi(x)|dy \le$$

$$\le \varepsilon_k \int\limits_{|y|<\frac{1}{k}} k^2 \frac{M}{|y|} dy = 4\pi\varepsilon_k M k^2 \int\limits_0^{\frac{1}{k}} r dr = 2\pi M \cdot \varepsilon_k \to 0$$

(bu yerda $M = \max|\eta''(t)|$).

Shunday qilib, (1.2.15) tenglikning isboti va shu bilan birga teoremaning ham isboti yakunlanadi.

1.3-§ Uzoqlashuvchi karrali xosmas integrallarni jamlash.

Biz bu paragrafda parametrga bog'liq integrallardan uzoqlashuvchi karrali xosmas integrallarni jamalshda ham foydalanamiz.

Masalan,

$$\int_{\mathbb{R}^n} f(x)\,dx \qquad (1.3.1)$$

xosmas integralni qaraylik.

Abel jamlash usulini ko'p o'lchovli holda kiritish uchun, $\alpha > 0$ parametrga bog'liq quyidagi

$$A(\alpha) = \int_{\mathbb{R}^n} e^{-\alpha|x|} f(x)dx \qquad (1.3.2)$$

integralni qaraymiz. Agar

$$A\int_{\mathbb{R}^n} f(x)\,dx = \lim_{\alpha\to 0}\int_{\mathbb{R}^n} e^{-\alpha|x|} f(x)dx$$

tenglikning o'ng tarafidagi limit mavjud bo'lsa, u holda (1.3.1) xosmas integral Abel usuli bilan jamalanadi deyiladi. Eslatib o'tamiz, agar (1.3.1) integralning bosh qiymati:

$$V.p.\int_{\mathbb{R}^n} f(x)\,dx = \lim_{R\to+\infty}\int_{|x|\le R} f(x)\,dx$$

mavjud bo'lsa Koshi ma'nosida yaqinlashadi deyiladi. Navbatdagi teorema Abel jamlash usuli bilan xosmas integrallarni Koshi bo'yicha integrallashni bog'laydi.

1.3.1-teorema. *Faraz qilaylik, $f(x)$ funksiya $\mathbb{R}^n$ fazosida uzluksiz bo'lib, istalgan $\alpha > 0$ uchun (1.3.2) integral yaqinlashsin. Agar (1.3.1) xosmas integral Koshi bo'yicha yaqinlashsa, u holda*

$$\lim_{\alpha\to 0+0}\int_{\mathbb{R}^n} e^{-\alpha|x|} f(x)dx = V.p.\int_{\mathbb{R}^n} f(x)\,dx$$

tenglik bajariladi.

Isbot. Sferik koordinatalarni kiritib, radiusi *r* va markazi koordinatalar boshida bo'lgan sfera bo'yicha

$$\psi(r) = \int\limits_{|x|=r} f(x)\, d\sigma(x)$$

sirt integralini qaraylik.

Ravshanki,

$$\int\limits_{|x|\le R} f(x)\, dx = \int\limits_0^R \psi(r)\, dr$$

va demak, (1.3.1) integralning bosh qiymati uchun

$$V.p. \int\limits_{\mathbb{R}^n} f(x)\, dx = \int\limits_0^\infty \psi(r)\, dr \qquad (1.3.3)$$

tenglik o'rinli.

Bundan chiqdi, (1.3.2) integralni quyidagi

$$A(\alpha) = \int\limits_0^\infty e^{-\alpha r}\, \psi(r) dr \qquad (1.3.4)$$

ko'rinishda yozish mumkin.

Isbotni yakunlash uchun Abelning teoremasini qo'llash yetarli. Bu teoramaga ko'ra, (1.3.4) tenglikning o'ng tarafidagi integral $\alpha \to 0$ da (1.3.3) ning o'ng tarafidagi integralga yaqinlashadi.

Natija. *Faraz qilqylik, $f(x)$ funksiya $\mathbb{R}^n$ fazoda uzluksiz bo'lsin. Agar (1.3.1) xosmas integral yaqinlashsa, u holda quyidagi*

$$\lim_{\alpha\to 0+0} \int\limits_{\mathbb{R}^n} e^{-\alpha|x|} f(x) dx = \int\limits_{\mathbb{R}^n} f(x)\, dx$$

tenglik bajariladi.

II BOB. Parametrga bog'liq karrali xosmas integrallarni asimptotik hisoblashning Laplas usuli.

2.1-§ O'zgarmas amplitudali ikki karrali Laplas integralining asimptotikasi.

Biz Laplas usulining dastlab sodda holini, ya'ni

$$\iint_{\Omega} e^{-\lambda g(t_1,t_2)}\, dt_1 dt_2 \qquad (2.1.1)$$

ko'rinishdagi integralning $\lambda \to +\infty$ dagi asimptotikasini topish uchun ishlatiladigan holni keltiramiz.

Agar biror yagona $(t_1^0, t_2^0) \in \Omega$ nuqtada $\nabla g(t_1^0, t_2^0) = 0$ va

$$det g''(t_1^0, t_2^0) = \begin{vmatrix} \frac{\partial^2 g}{\partial t_1^2} & \frac{\partial^2 g}{\partial t_1 \partial t_2} \\ \frac{\partial^2 g}{\partial t_2 \partial t_1} & \frac{\partial^2 g}{\partial t_2^2} \end{vmatrix}_{(t_1^0, t_2^0)} > 0$$

bo'lib, Ω soha ichida g lokal minimumga erishsa, (Odatda bunday shartlarni qanoatlantiruvchi nuqta g funksiyaning xosmas kritik nuqtasi deyiladi) u holda (2.1.1) integralda o'zgaruvchilarni shunday almashtirish mumkinki, bunda integral

$$I(\lambda) = \iint_{\mathbb{R}^2} e^{-\lambda(x_1^2+x_2^2)} f(x_1, x_2) dx_1 dx_2 \qquad (2.1.2)$$

ko'rinishga keladi.

Faraz qilaylik $f(x_1, x_2)$ shunday funksiya bo'lsinki, quyidagi

$$\iint_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)} |f(x_1, x_2)| dx_1 dx_2 < +\infty \qquad (2.1.3)$$

xosmas integral yaqinlashsin.

E'tibor bersak (2.1.2) integral ostidagi ifoda eksponenta (0,0) nuqtada 1 ga teng bo'lib, boshqa nuqtalarda 1 dan qat'iy kichik. Shuning uchun, $\lambda \to +\infty$ da integral ostidagi ifoda noldan tashqari barcha nuqtalarda nolga intiladi. Ravshanki, bunda (2.1.2) integral ham nolga intiladi va bizning maqsadimiz mana shu intilishni baholashdan iborat.

Laplas usuli g'oyasi quyidagiga asoslangan: (2.1.2) integralga asosiy qiymatni (0,0) nuqtaning istalgan kichik atrofi bo'yicha olingan integral beradi va bu atrofda esa, integral asimptotikasini, f funksiyani uning Teylor qatori bilan almashtirib, topish mumkin.

Masalan: $f(x_1, x_2) \equiv 1$ bo'lsa,

$$\iint_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)}\, dx_1 dx_2 = \frac{\pi}{\lambda} \quad (2.1.4)$$

Haqiqatdan ham, (2.1.4) ning o'rinli ekanligini quyidagi hisoblashlarda ko'ramiz.

(2.1.4) tenglikning chap tomonini quyidagi limitik ko'rinishda aniqlashimiz mumkin.

$$\iint_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)}\, dx_1 dx_2 = \lim_{R\to+\infty} \iint_{B[0,R]} e^{-(x_1^2+x_2^2)}\, dx_1 dx_2 \quad (2.1.5)$$

Bunda, $B[0,R] = \{(x_1, x_2) \epsilon \mathbb{R}^2, x_1^2 + x_2^2 \leq R^2\}$. Odatda $B[0,R]$ to`plam $\mathbb{R}^2$ tekislikni qamrab oluvchi doiralar ketma-ketligi deb ataladi.

Dastlab quyidagi integralni hisoblab olamiz.

$$\iint_{B[0,R]} e^{-(x_1^2+x_2^2)}\, dx_1 dx_2$$

Yuqoridagi integralni hisoblash uchun Dekart koordinatalar sistemasidan qutb koordinatalar sistemasiga o'tamiz. Buning uchun quyidagi almashtirish orqali o'tiladi.

$$\begin{cases} x_1 = \rho cos\varphi \\ x_2 = \rho sin\varphi \end{cases} \quad 0 \leq \varphi < 2\pi, 0 < \rho \leq R$$

Bu almashtirishni yakobiyani uchun quyidagi tenglik o`rinli

$$\frac{D(x_1, x_2)}{D(\rho, \varphi)} = \rho.$$

Demak,

$$\iint_{B[0,R]} e^{-(x_1^2+x_2^2)}\, dx_1 dx_2 = \int_0^{2\pi} d\varphi \int_0^R e^{-\lambda\rho^2} \rho\, d\rho = \pi\left(\frac{1 - e^{-\lambda R^2}}{\lambda}\right)$$

Endi, (2.1.5) tenglikning o'ng tomoniga yuqoridagi tenglikni qo`yib limitga o'tsak (2.1.4) tenglik o'rinli ekanligini ko'ramiz.

(2.1.4) tenglikning chap tomonidagi integral parametrga teskari proporsional ravishda nolga intiladi.

Xuddi shu natijani $\mathbb{R}^2$ tekislik bo'yicha emas, balki istalgan kichik $U_\delta(0,0)$ doira bo'yicha integrallanganda ham olamiz; integralni bunday almashtirishdagi xato eksponensial kichik bo’ladi.

Laplas usulini qo'llash uchun (2.1.2) integral ostidagi funksiyadan faqat koordinatalar boshining biror atrofidagina silliqlik talab qilinadi, boshqa nuqtalarda esa, funksiya ixtiyoriy lokal integrallanuvchi bo'lib, (2.1.3) shartni qanoatlantirishi yetarli.

Biz (2.1.2) integralga asosiy qiymatni (0,0) nuqtaning istalgancha kichik atrofi berishi haqidagi tasdiqni isbotlashdan boshlaymiz.

2.1.1-tasdiq: *Faraz qilaylik, f funksiya (2.1.3) shartni qanoatlantirsin. U holda istalgan $\delta > 0$ uchun (2.1.2) integral*

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 =$$

$$= \iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 + O(1)e^{-\lambda\delta^2}, \qquad \lambda \geq 1 \quad (2.1.6)$$

bahoni qanoatlantiradi.

Isbot: Quyidagi belgilashlarni kiritamiz.

$$I_1(\lambda) = \iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2$$

$$I_2(\lambda) = \iint\limits_{\mathbb{R}^2\setminus B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2$$

U holda,

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 = I_1(\lambda) + I_2(\lambda) \qquad (2.1.7)$$

ga ega bo'lamiz.

$I_2(\lambda)$ integralni baholaymiz. Ravshanki, $\lambda \geq 1$ va $B[0,\delta]$ doiradan tashqaridagi nuqtalar uchun

$$e^{-\lambda(x_1^2+x_2^2)} = e^{-(\lambda-1)(x_1^2+x_2^2)}e^{-(x_1^2+x_2^2)} \le e^{-(\lambda-1)\delta^2}e^{-(x_1^2+x_2^2)}$$

tengsizlik o'rinli.

Shuning uchun, (2.1.3) shartga ko'ra

$$|I_2(\lambda)| \le e^{-(\lambda-1)\delta^2} \iint\limits_{\mathbb{R}^2\backslash B[0,\delta]} e^{-(x_1^2+x_2^2)}|f(x_1,x_2)|\, dx_1dx_2 = C(\delta)e^{-\lambda\delta^2}$$

Natija sifatida (2.1.4) tenglikning umumiy ko'rinishini olishimiz mumkin.

2.1.2-tasdiq: *Har qanday* $\delta > 0$ *va ixtiyoriy* $k \geq 0,\ l \geq 0$ *uchun*

$$\iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)}\, x_1^{2k}x_2^{2l}dx_1dx_2 = \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\lambda^{k+l+1}} + O\left(e^{-\lambda\delta^2}\right) \quad (2.1.8)$$

baho o'rinli.

Isbot: (2.1.6) bahoda $f(x_1,x_2) = x_1^{2k}x_2^{2l}$ deb olsak,

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda(x_1^2+x_2^2)}\, x_1^{2k}x_2^{2l}dx_1dx_2 =$$

$$= \iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)}\, x_1^{2k}x_2^{2l}dx_1dx_2 + O\left(e^{-\lambda\delta^2}\right) \quad (2.1.9)$$

tenglikka ega bo'lamiz.

(2.1.9) tenglikning chap tomonini hisoblaymiz.

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda(x_1^2+x_2^2)}\, x_1^{2k}x_2^{2l}dx_1dx_2 = \begin{bmatrix} x_1 = \rho cos\varphi & 0 \le \varphi < 2\pi, 0 < \rho < +\infty \\ x_2 = \rho sin\varphi & \\ & \dfrac{D(x_1,x_2)}{D(\rho,\varphi)} = \rho \end{bmatrix} =$$

$$\int\limits_0^{2\pi} d\varphi \int\limits_0^{+\infty} e^{-\lambda\rho^2}\, \rho^{2(k+l)+1} sin^{2l}\varphi cos^{2k}\varphi d\rho = \int\limits_0^{2\pi} sin^{2l}\varphi cos^{2k}\varphi d\varphi \times$$

$$\times \int\limits_0^{+\infty} e^{-\lambda\rho^2}\, \rho^{2(k+l)+1} d\rho$$

Yuqoridagi integrallarni alohida-alohida hisoblab olamiz.

$$\int_0^{2\pi} sin^{2l}\varphi cos^{2k}\varphi d\varphi = 4\int_0^{\frac{\pi}{2}} sin^{2l}\varphi cos^{2k}\varphi d\varphi = \left[tg\varphi = \sqrt{y}\right] =$$

$$= 2\int_0^{+\infty} \frac{y^{l-\frac{1}{2}}}{(1+y)^{k+l+1}}dy = 2B\left(k+\frac{1}{2}, l+\frac{1}{2}\right)$$

Endi quyidagi integralni hisoblaymiz.

$$\int_0^{+\infty} e^{-\lambda\rho^2}\rho^{2(k+l)+1}d\rho = [\lambda\rho^2 = y] = \frac{1}{2\lambda^{k+l+1}}\int_0^{+\infty} e^{-y}y^{k+l}\,dy = \frac{\Gamma(k+l+1)}{2\lambda^{k+l+1}}$$

Yuqorida hisoblangan ikki integrallarning natijasi sifatida quyidagini yozamiz.

$$\iint_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)}x_1^{2k}x_2^{2l}dx_1dx_2 = B\left(k+\frac{1}{2}, l+\frac{1}{2}\right)\times\frac{\Gamma(k+l+1)}{\lambda^{k+l+1}} =$$

$$= \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\lambda^{k+l+1}} \qquad (2.1.10)$$

Ko`rinib turibdiki (2.1.9) tenglikdan (2.1.8) tenglik kelib chiqadi.

Keyingi mulohazalar uchun zarur bo'lgan quyidagi integralga e'tibor qilamiz.

$$\iint_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)}x_1^i x_2^j dx_1dx_2 \quad i,j\in\mathbb{N}\cup\{0\} \qquad (2.1.11)$$

Xususan, *i* va *j* sonlari juft bo'lganda (2.1.10) tenglik o'rinli. Agar *i* va *j* sonlaridan kamida biri toq bo'lsa, u holda (2.1.11) integralni qiymati nolga teng bo'ladi. Chunki, koordinatalar boshiga nisbatan simmetrik bo'lgan toq funksiyadan olingan integral nolga teng.

Navbatdagi tasdiq yuqoridagi mulohazalarning natijasidir.

2.1.1-teorema: *Faraz qilaylik, f funksiya (2.1.3) shartni qanoatlantirib, koordinatalar boshining biror atrofida* 2m *marta uzluksiz xususiy hosilalarga ega bo'lsin. U holda* $\lambda\to+\infty$ *da*

$$\iint_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)}f(x_1,x_2)dx_1dx_2 = \lambda^{-1}\left[\sum_{i=0}^{m-1}a_i\lambda^{-i}+\frac{O(1)}{\lambda^m}\right] \qquad (2.1.12)$$

asimptotik tenglik o'rinli.

Bunda a_i koeffisiyent quyidagicha aniqlanadi:

$$a_i = \frac{\sum_{j=0}^{i} \frac{\partial^{(2i)} f(0,0)}{\partial x_1^{2(i-j)} \partial x_2^{2j}} C_{2i}^{2j} \Gamma\left(i-j+\frac{1}{2}\right) \Gamma\left(j+\frac{1}{2}\right)}{(2i)!}$$

Isbot: Aytaylik, f funksiya *2m* marta uzluksiz xususiy hosilalarga ega bo'lgan doira $B[0,\delta]$ bo'lsin.

Agar $e^{-\lambda\delta^2}$ eksponenta $\lambda \to +\infty$ da λ^{-m-1} darajadan tezroq nolga intilishini hisobga olsak 2.1.1-tasdiqqa ko'ra, quyidagi

$$\iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)} f(x_1,x_2) dx_1 dx_2 =$$

$$= \sum_{i=0}^{m-1} \frac{\sum_{j=0}^{i} \frac{\partial^{(2i)} f(0,0)}{\partial x_1^{2(i-j)} \partial x_2^{2j}} C_{2i}^{2j} \Gamma\left(i-j+\frac{1}{2}\right) \Gamma\left(j+\frac{1}{2}\right)}{(2i)!\, \lambda^{i+1}} + \frac{O(1)}{\lambda^{m+1}} \quad (2.1.13)$$

formulani isbotlash yetarli.

Berilgan f funksiyani $B[0,\delta]$ doirada Teylor formulasi bilan almashtiramiz:

$$f(x_1,x_2) = \sum_{i=0}^{2m-1} \frac{d^i f(0,0)}{i!} + \frac{d^{2m} f(\theta x_1, \theta x_2)}{(2m)!}, (0 < \theta < 1, (x_1,x_2) \in U_\delta(0,0))$$

Bu formulani (2.1.13) ning chap qismidagi integralga qo'yib, integral chegaralari simmetrik ekanligini hisobga olsak, toq *i* larga mos keladigan integrallar nolga teng bo'ladi. Juft *i* larga mos keladigan integrallarga 2.1.2-tasdiqni qo'llasak, ya'ni

$$\iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)} f(x_1,x_2) dx_1 dx_2 =$$

$$= \iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)} \left[\sum_{i=0}^{2m-1} \frac{d^i f(0,0)}{i!} + \frac{d^{2m} f(\theta x_1, \theta x_2)}{(2m)!}\right] dx_1 dx_2 \quad (2.1.14)$$

Yuqorida aytilganidek, toq *i* larga mos kelgan integrallarni qiymati nol ekanligidan (2.1.14) ni quyidagicha yozamiz:

$$\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1,x_2)dx_1dx_2 = \iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} \times$$

$$\times\left[\sum_{i=0}^{m-1}\frac{d^{2i}f(0,0)}{(2i)!}+\frac{d^{2m}f(\theta x_1,\theta x_2)}{(2m)!}\right]dx_1dx_2, \quad (2.1.15)$$

Formal ravishda:

$$d^{2i}f(0,0) = \left(\frac{\partial}{\partial x_1}x_1 + \frac{\partial}{\partial x_2}x_2\right)^{2i} f(0,0)$$

deb qabul qilsak,

$$d^{2i}f(0,0) = \sum_{j=0}^{2i} C_{2i}^{j}\frac{\partial^{(2i)}f(0,0)}{\partial x_1^{2i-j}\partial x_2^{j}}x_1^{2i-j}x_2^{j} \qquad (2.1.16)$$

ga ega bo'lamiz.

(2.1.16) ni (2.1.15) ga qo'ysak,

$$\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)}\left[\sum_{i=0}^{m-1}\frac{d^{2i}f(0,0)}{(2i)!}+\frac{d^{2m}f(\theta x_1,\theta x_2)}{(2m)!}\right]dx_1dx_2 =$$

$$=\sum_{i=0}^{m-1}\frac{1}{(2i)!}\sum_{j=0}^{2i}C_{2i}^{j}\frac{\partial^{(2i)}f(0,0)}{\partial x_1^{2i-j}\partial x_2^{j}}\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)}x_1^{2i-j}x_2^{j}dx_1dx_2 +$$

$$+\frac{1}{(2m)!}\sum_{j=0}^{2m}C_{2m}^{j}\frac{\partial^{(2m)}f(\theta x_1^*,\theta x_2^*)}{\partial x_1^{2m-j}\partial x_2^{j}}\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)}x_1^{2m-j}x_2^{j}dx_1dx_2 \quad (2.1.17)$$

ga kelamiz.

(2.1.17) tenglikning tarkibidagi quyidagi

$$\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)}x_1^{2i-j}x_2^{j}dx_1dx_2$$

integral barcha toq j sonlar uchun nolga teng ekanligini hisobga olsak, (2.1.17) ni quyidagicha yozishimiz mumkin.

$$\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} \left[\sum_{i=0}^{m-1} \frac{d^{2i} f(0,0)}{(2i)!} + \frac{d^{2m} f(\theta x_1, \theta x_2)}{(2m)!}\right] dx_1 dx_2 =$$

$$= \sum_{i=0}^{m-1} \frac{1}{(2i)!} \sum_{j=0}^{2i} C_{2i}^{2j} \frac{\partial^{(2i)} f(0,0)}{\partial x_1^{2(i-j)} \partial x_2^{2j}} \iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} x_1^{2(i-j)} x_2^{2j} dx_1 dx_2 +$$

$$+ \frac{1}{(2m)!} \sum_{j=0}^{2m} C_{2m}^{2j} \frac{\partial^{(2m)} f(\theta x_1^*, \theta x_2^*)}{\partial x_1^{2(m-j)} \partial x_2^{2j}} \iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)} x_1^{2(m-j)} x_2^{2j} dx_1 dx_2 \quad (2.1.18)$$

Endi (2.1.18) tenglikdagi integrallarga 2.1.2-tasdiqni qo'llasak hamda $e^{-\lambda\delta^2}$ eksponenta $\lambda \to +\infty$ da λ^{-m-1} darajadan tezroq nolga intilishini hisobga olsak talab qilingan (2.1.12) tenglikka ega bo'lamiz.

2.1.1-natija: *Koordinatalar boshining biror atrofida ikki marta uzluksiz xususiy hosilalarga ega bo'lgan (2.1.3) shartni qanoatlantiruvchi f funksiya uchun*

$$\iint\limits_{\mathbb{R}^2} e^{-\lambda\left(x_1^2+x_2^2\right)} f(x_1, x_2) dx_1 dx_2 = \frac{\pi}{\lambda}\left[f(0,0) + \frac{O(1)}{\lambda}\right], \lambda \to +\infty \ (2.1.19)$$

asimptotik tenglik o'rinli.

Kelgusida biz uchun kerak bo'ladigan bir lemmani qaraymiz:

2.1.1-lemma: *Faraz qilaylik $h(y_1, y_2)$ funksiya $(y_1, y_2) = (0,0)$ nuqtaning biror atrofida yetarlicha silliq bo'lib, quyidagi shartlarni qanoatlantirsin:*

$1^\circ. h(0,0) = 0$

$2^\circ. \nabla h(0,0) = 0$

$$3^\circ \begin{cases} \dfrac{\partial^2 h(0,0)}{\partial y_1^2} > 0 \\ \left.\begin{vmatrix} \dfrac{\partial^2 h}{\partial y_1^2} & \dfrac{\partial^2 h}{\partial y_1 \partial y_2} \\ \dfrac{\partial^2 h}{\partial y_2 \partial y_1} & \dfrac{\partial^2 h}{\partial y_2^2} \end{vmatrix}\right|_{(0,0)} > 0 \end{cases}$$

U holda $(y_1, y_2) = (0,0)$ nuqtaning shunday U, $(x_1, x_2) = (0,0)$ nuqtaning biror V atrofi hamda $\varphi: V \to U$ diffemorf akslantirish topiladiki,

$h\big(\varphi(\mathrm{x})\big) = x_1^2 + x_2^2$ *tenglik o'rinli bo'ladi.*

Isbot: $h(y_1, y_2)$ funksiyani $U_\delta(0,0)$ atrofda quyidagicha yozib olamiz:

$$h(y_1,y_2)=\frac{1}{2}\left(\frac{\partial^2\, h(\theta y_1,\theta y_2)}{\partial y_1^2}y_1^2+2\frac{\partial^2\, h(\theta y_1,\theta y_2)}{\partial y_1\partial y_2}y_1y_2+\frac{\partial^2\, h(\theta y_1,\theta y_2)}{\partial y_2^2}y_2^2\right);\quad (0<\theta<1)\qquad (2.1.20)$$

Quyidagi belgilashlarni kiritamiz:

$$a_{11}=\frac{\partial^2\, h(0,0)}{\partial y_1^2}+\alpha_{11}(y_1,y_2);$$

$$a_{12}=\frac{\partial^2\, h(0,0)}{\partial y_1\partial y_2}+\alpha_{12}(y_1,y_2);$$

$$a_{22}=\frac{\partial^2\, h(0,0)}{\partial y_2^2}+\alpha_{22}(y_1,y_2);$$

Yuqoridagi α_{11}, α_{12} va α_{22} lar $(y_1, y_2)\to 0$ intilganda nolga intiluvchi cheksiz kichik miqdorlardir.

Yuqoridagi belgilashlarga ko'ra (2.1.20) quyidagi ko'rinishga keladi.

$$h(y_1,y_2)=\frac{1}{2}(a_{11}y_1^2+2a_{12}y_1y_2+a_{22}y_2^2)\qquad (2.1.21)$$

(2.1.21) tenglikni o'ng tomonidagi kvadratik formani kanonik shakilga keltisak, quyidagiga ega bo'lamiz:

$$h(y_1,y_2)=\left(\sqrt{\frac{a_{11}}{2}}\left(y_1+\frac{a_{12}}{a_{11}}y_2\right)\right)^2+\left(\sqrt{\frac{a_{11}a_{22}-a_{12}^2}{2a_{11}}}y_2\right)^2\quad (2.1.22)$$

Agar

$$\begin{cases} x_1=\sqrt{\dfrac{a_{11}}{2}}\left(y_1+\dfrac{a_{12}}{a_{11}}y_2\right)\\ x_2=\sqrt{\dfrac{a_{11}a_{22}-a_{12}^2}{2a_{11}}}y_2\end{cases}\qquad (2.1.23)$$

almashtirish olsak, bu almashtirish $(y_1, y_2) = (0,0)$ nuqtaning yetarli kichik atrofida o'zaro bir qiymatlidir.

Haqiqatdan ham,

$$\begin{vmatrix} \sqrt{\frac{a_{11}}{2}} & \frac{a_{12}}{\sqrt{2a_{11}}} \\ 0 & \sqrt{\frac{a_{11}a_{22}-a_{12}^2}{2a_{11}}} \end{vmatrix}_{(0,0)} = \frac{\sqrt{\begin{vmatrix} \frac{\partial^2 h}{\partial y_1^2} & \frac{\partial^2 h}{\partial y_1 \partial y_2} \\ \frac{\partial^2 h}{\partial y_2 \partial y_1} & \frac{\partial^2 h}{\partial y_2^2} \end{vmatrix}_{(0,0)}}}{2} \neq 0$$

Demak, $(y_1, y_2) = (0,0)$nuqtaning yetarli kichik atrofida (2.1.23) ga teskari bir qiymatli almashtirish mavjud.

2.1.2-natija: *2.1.1- lemmani shartlarini qanoatlantiruvchi φ akslantirishning yakobiani uchun*

$$J(\varphi(0,0)) = \frac{D(y_1, y_2)}{D(x_1, x_2)}\bigg|_{(0,0)} = \frac{2}{\sqrt{\begin{vmatrix} \frac{\partial^2 h}{\partial y_1^2} & \frac{\partial^2 h}{\partial y_1 \partial y_2} \\ \frac{\partial^2 h}{\partial y_2 \partial y_1} & \frac{\partial^2 h}{\partial y_2^2} \end{vmatrix}_{(0,0)}}} \qquad (2.1.24)$$

tenglik o'rinli.

Endi, (2.1.1) integralni (2.1.2) integralga olib kelish masalasini qaraymiz. $g(t_1, t_2)$ funksiyani xosmas kritik nuqtasi bo'lgan (t_1^0, t_2^0) nuqtani

$$\begin{cases} t_1 - t_1^0 = y_1 \\ t_2 - t_2^0 = y_2 \end{cases} \qquad (2.1.25)$$

chiziqli almashtirish yordamida koordinatalar boshiga olib kelamiz.

Quyidagi yordamchi funksiya tuzib olamiz:

$$h(y_1, y_2) = g(y_1 + t_1^0, y_2 + t_2^0) - g(t_1^0, t_2^0) \qquad (2.1.26)$$

u holda

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\, dt_1 dt_2 = \begin{bmatrix} t_1 - t_1^0 = y_1 \\ t_2 - t_2^0 = y_2 \end{bmatrix} = \iint\limits_{\Omega`} e^{-\lambda g\left(y_1 + t_1^0, y_2 + t_2^0\right)}\, dy_1 dy_2 =$$

$$= \iint\limits_{\Omega`} e^{-\lambda\left(g\left(y_1+t_1^0, y_2+t_2^0\right) - g\left(t_1^0, t_2^0\right)\right)}\, e^{-\lambda g\left(t_1^0, t_2^0\right)} dy_1 dy_2 \qquad (2.1.27)$$

(2.1.26) tenglikka ko'ra, (2.1.27) dagi oxirgi tenglikni quyidagicha yozamiz:

$$\iint\limits_{\Omega`} e^{-\lambda\left(g(y_1+t_1^0,y_2+t_2^0)-g(t_1^0,t_2^0)\right)} e^{-\lambda g(t_1^0,t_2^0)} dy_1 dy_2 =$$

$$= e^{-\lambda g(t_1^0,t_2^0)} \iint\limits_{\Omega`} e^{-\lambda h(y_1,y_2)} dy_1 dy_2 \qquad (2.1.28)$$

Ma'lumki, Ω` soha uchun (0,0) nuqta ichki nuqta, u holda 1- lemmani shartlarini qanoatlantiruvchi $h(y_1, y_2)$ funksiya uchun shunday $U_{\delta_1}(0,0)$ atrof topiladiki, bu atrof Ω` sohaga tegishli bo'ladi.

Faraz qilaylik bu atrof $B[0, \delta_1]$ doira bo'lsin. U holda Ω` soha quyidagi ikki sohaga ajraladi:

$$\Omega` = B[0, \delta_1] \cup \Omega`\backslash B[0, \delta_1]$$

(2.1.28) tenglikdagi oxirgi integralni quyidagicha yozib olamiz:

$$\iint\limits_{\Omega`} e^{-\lambda h(y_1,y_2)} dy_1 dy_2 =$$

$$= \iint\limits_{B[0,\delta_1]} e^{-\lambda h(y_1,y_2)} dy_1 dy_2 + \iint\limits_{\Omega`\backslash B[0,\delta_1]} e^{-\lambda h(y_1,y_2)} dy_1 dy_2 \quad (2.1.29)$$

Yuqoridagi tenglikni o'ng tomonidagi ikkinchi integralni baholaymiz:

$$\left| \iint\limits_{\Omega`\backslash B[0,\delta_1]} e^{-\lambda h(y_1,y_2)} dy_1 dy_2 \right| \leq e^{-\lambda m} |\Omega`|; \quad \left(m = \min_{\Omega`\backslash B[0,\delta_1]} h(y_1, y_2) \right) \quad (2.1.30)$$

$h(y_1, y_2)$ funksiya 1-lemma shartlarini qanoatlantirgani uchun (2.1.29) tenglikning o'ng tomonidagi birinchi integralni quyidagi ko'rinishda yozamiz:

$$\iint\limits_{B[0,\delta_1]} e^{-\lambda h(y_1,y_2)} dy_1 dy_2 = \iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)} \left| \frac{D(y_1, y_2)}{D(x_1, x_2)} \right| dx_1 dx_2 \qquad (2.1.31)$$

Agar $f(x_1, x_2) = \left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right|$ deb olsak, (2.1.31) tenglikni o'ng tomonidagi integral

$\lambda \to +\infty$ da eksponensial kichik xatolik bilan (2.1.2) integral ekanligini payqash mumkin.

Yuqoridagi (2.1.30) va (2.1.31) larga asosan (2.1.29) ni quyidagi ko'rinishda yozamiz:

$$\iint\limits_{\Omega`} e^{-\lambda h(y_1,y_2)}\,dy_1dy_2 =$$

$$= \iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)}\left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right| dx_1dx_2 + O(1)e^{-\lambda m} \quad (2.1.32)$$

Endi, (2.1.27), (2.1.28) va (2.1.32) tengliklarga ko'ra quyidagi natijaga kelamiz:

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,dt_1dt_2 = e^{-\lambda g\left(t_1^0,t_2^0\right)} \times$$

$$\times\left[\iint\limits_{B[0,\delta]} e^{-\lambda\left(x_1^2+x_2^2\right)}\left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right| dx_1dx_2 + O(1)e^{-\lambda m}\right] \qquad (2.1.33)$$

2.1.2-natijaga va (2.1.26) tenglikka asosan quyidagini yozamiz:

$$J\big(\varphi(0,0)\big) = \left.\frac{D(y_1,y_2)}{D(x_1,x_2)}\right|_{(0,0)} = \frac{2}{\sqrt{\begin{vmatrix}\dfrac{\partial^2 h}{\partial y_1^2} & \dfrac{\partial^2 h}{\partial y_1\partial y_2}\\ \dfrac{\partial^2 h}{\partial y_2\partial y_1} & \dfrac{\partial^2 h}{\partial y_2^2}\end{vmatrix}_{(0,0)}}} =$$

$$= \frac{2}{\sqrt{\begin{vmatrix}\dfrac{\partial^2 g}{\partial t_1^2} & \dfrac{\partial^2 g}{\partial t_1\partial t_2}\\ \dfrac{\partial^2 g}{\partial t_2\partial t_1} & \dfrac{\partial^2 g}{\partial t_2^2}\end{vmatrix}_{\left(t_1^0,t_2^0\right)}}} = \frac{2}{\sqrt{\det(g''(t_1^0,t_2^0)}} \qquad (2.1.34)$$

2.1.1-natijaga (2.1.34) va (2.1.33) tengliklarga asosan (2.1.1) integral uchun quyidagi natijani olamiz:

2.1.3-natija: *Agar biror yagona* $(t_1^0,t_2^0)\in\Omega$ *da* $\nabla g(t_1^0,t_2^0)=0$ *va*

$$det g''(t_1^0,t_2^0) = \begin{vmatrix}\dfrac{\partial^2 g}{\partial t_1^2} & \dfrac{\partial^2 g}{\partial t_1\partial t_2}\\ \dfrac{\partial^2 g}{\partial t_2\partial t_1} & \dfrac{\partial^2 g}{\partial t_2^2}\end{vmatrix}_{\left(t_1^0,t_2^0\right)} > 0$$

bo'lib bu nuqta minimum nuqtasi bo'lsa, u holda quyidagi

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,dt_1dt_2 =$$

$$= e^{-\lambda g\left(t_1^0,t_2^0\right)}\frac{\pi}{\lambda}\left[\frac{2}{\sqrt{\det(g''(t_1^0,t_2^0)}}+\frac{O(1)}{\lambda}\right], \lambda \to +\infty \quad (2.1.35)$$

asimptotik tenglik o'rinli.

Agar $f(x_1,x_2) = \left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right|$ deb olsak, (2.1.35) tenglikni yanada umumiy ko'rinishini yozishimiz mumkin.

Ya'ni,

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,dt_1dt_2 = e^{-\lambda g\left(t_1^0,t_2^0\right)} \times$$

$$\times\left[\sum_{i=0}^{m-1}\frac{\sum_{j=0}^{i}\frac{\partial^{(2i)}f(0,0)}{\partial x_1^{2(i-j)}\partial x_2^{2j}}C_{2i}^{2j}\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!\,\lambda^{i+1}}+\frac{O(1)}{\lambda^{m+1}}\right]$$

Yoki yuqoridagi tenglikni soddaroq quyidagicha

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,dt_1dt_2 = e^{-\lambda g\left(t_1^0,t_2^0\right)}\lambda^{-1}\left[\sum_{i=0}^{m-1}a_i\lambda^{-i}+\frac{O(1)}{\lambda^m}\right], \lambda \to +\infty \quad 2.1.36$$

yozishimiz mumkin, bunda a_i koeffisiyent quyidagicha aniqlanadi.

$$a_i = \frac{\sum_{j=0}^{i}\frac{\partial^{(2i)}f(0,0)}{\partial x_1^{2(i-j)}\partial x_2^{2j}}C_{2i}^{2j}\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!} \quad 2.1.37$$

Yuqoridagi mulohazalarni yakuni sifatida quyidagi teoremani keltiramiz:

2.1.2-teorema: *Faraz qilaylik, $g(t_1,t_2)$ funksiya Ω sohada yotuvchi yagona (t_1^0,t_2^0) nuqtada xosmas kritik nuqtaga ega bo'lsin. Agar shu (t_1^0,t_2^0) nuqta $g(t_1,t_2)$ funksiya uchun lokal minimum nuqta bo'lsa, u holda , $\lambda \to +\infty$ da quyidagi asimptotik yoyilma o'rinli bo'ladi:*

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,dt_1dt_2 \approx e^{-\lambda g\left(t_1^0,t_2^0\right)}\lambda^{-1}\sum_{i=0}^{\infty}a_i\lambda^{-i} \quad 2.1.38$$

bunda $a_i = \dfrac{\sum_{j=0}^{i} \frac{\partial^{(2i)} f(0,0)}{\partial x_1^{2(i-j)} \partial x_2^{2j}} C_{2i}^{2j} \Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!}$ *kabi aniqlanadi.*

Oxirgi asimptotik yoyilma quyidagi ma'noda tushuniladi: istalgan nomanfiy butun r va N sonlari uchun quyidagi munosabat o'rinli:

$$\left(\frac{d}{d\lambda}\right)^r \left[\iint_{\Omega} e^{-\lambda g(t_1,t_2)}\, dt_1 dt_2 - e^{-\lambda g\left(t_1^0,t_2^0\right)} \lambda^{-1} \sum_{i=0}^{N} a_i \lambda^{-i}\right] =$$

$$= O\left(\lambda^{-r-(N+1)}\right)\, \lambda \to +\infty$$

2.2-§ O'zgaruvchan amplitudali ikki karrali Laplas integralining asimptotik yoyilmasi.

Biz bu paragrafda umumiy holda $\lambda \to +\infty$ da quyidagi integralning asimptotikasini qaraymiz:

$$I(\lambda) = \iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)} \psi(t_1,t_2) dt_1 dt_2 \qquad (2.2.1)$$

2.2.1-teorema: *Faraz qilaylik Ω sohada yotuvchi yagona (t_1^0, t_2^0) nuqta faza funksiyasi uchun xosmas kritik nuqta , aniqrog`i minimum nuqtasi bo'lib $\psi(t_1, t_2)$ funksiyaning tashuvchisi (t_1^0, t_2^0) nuqtaning atrofida bo'lsin , u holda $\lambda \to +\infty$ da quyidagi asimptotik yoyilma o'rinli bo'ladi:*

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)} \psi(t_1,t_2) dt_1 dt_2 \approx e^{-\lambda g(t_1^0,t_2^0)} \lambda^{-1} \sum_{i=0}^{\infty} a_i \lambda^{-i} \qquad (2.2.2)$$

Eslatma: Yuqoridagi asimptotik yoyilma quyidagi ma'noda tushuniladi: istalgan nomanfiy butun r va N sonlari uchun quyidagi munosabat o'rinli:

$$\left(\frac{d}{d\lambda}\right)^r \left[I(\lambda) - e^{-\lambda g(t_1^0,t_2^0)} \lambda^{-1} \sum_{i=0}^{N} a_i \lambda^{-i} \right] =$$

$$= O\left(\lambda^{-r-(N+1)}\right) \lambda \to +\infty$$

Isbot: Faza funksiyasining kritik nuqtasini siljitish orqali koordinatalar boshiga olib kelish va ba'zi yordamchi funksiyalar tuzish bilan (2.2.1) integralni soddalashtiramiz ya'ni:

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)} \psi(t_1,t_2) dt_1 dt_2 = \begin{bmatrix} t_1 - t_1^0 = y_1 \\ t_2 - t_2^0 = y_2 \end{bmatrix} =$$

$$= \iint\limits_{\Omega`} e^{-\lambda g(y_1 + t_1^0, y_2 + t_2^0)} \psi(y_1 + t_1^0, y_2 + t_2^0) dy_1 dy_2 =$$

$$= \iint\limits_{\Omega`} e^{-\lambda\left(g(y_1+t_1^0, y_2+t_2^0)-g(t_1^0,t_2^0)\right)}\, e^{-\lambda g(t_1^0,t_2^0)}\psi(y_1+t_1^0, y_2+t_2^0)dy_1dy_2 =$$

$$= \ [h(y_1,y_2) = g(y_1+t_1^0, y_2+t_2^0) - g(t_1^0,t_2^0);\ \omega(y_1,y_2) =$$
$$= \psi(y_1+t_1^0, y_2+t_2^0)\ \ ] =$$

$$= e^{-\lambda g(t_1^0,t_2^0)} \iint\limits_{\Omega`} e^{-\lambda h(y_1,y_2)}\, \omega(y_1,y_2)dy_1dy_2$$

Yuqoridagi tenglikda oxirgi integralni qaraymiz. Chunki, shu integralni o'rganishimiz osonroq. Buni quyida ko'ramiz:

$$\iint\limits_{\Omega`} e^{-\lambda h(y_1,y_2)}\, \omega(y_1,y_2)dy_1dy_2 \qquad (2.2.3)$$

yuqorida integral ostidagi $h(y_1,y_2)$ funksiya 2.1.1-lemma shartlarini qanoatlantirishi teorema shartidan ma'lum ekanligini hisobga olsak u holda (2.2.3) integralning integrallash sohasini shunday ikki qismga ajratamizki, bu ajratilgan sohalardan birida $h(y_1,y_2)$ funksiya 2.1.1-lemma shartlarini qanoatlantiradi, ikkinchisida esa (2.2.3) integral $\lambda \to +\infty$ da cheksiz kichik bo'ladi. Bularni quyida batafsil ko'rib chiqamiz.

Faraz qilaylik $B[0,\delta_1]$ $h(y_1,y_2)$ funksiya 2.1.1-lemma shartlarini qanoatlantirdigan soha bo’lsin, u holda $\Omega`$ sohani quyidagi sohalarga ajratishimiz mumkin:

$$\Omega` = B[0,\delta_1] \cup \Omega`\backslash B[0,\delta_1]$$

Endi (2.2.3) integralni quyidagicha yozamiz:

$$\iint\limits_{\Omega`} e^{-\lambda h(y_1,y_2)}\, \omega(y_1,y_2)dy_1dy_2 =$$

$$= \iint\limits_{B[0,\delta_1]} e^{-\lambda h(y_1,y_2)}\, \omega(y_1,y_2)dy_1dy_2 +$$

$$+ \iint\limits_{\Omega`\backslash B[0,\delta_1]} e^{-\lambda h(y_1,y_2)}\, \omega(y_1,y_2)dy_1dy_2 \qquad (2.2.4)$$

Yuqoridagi tenglikning o'ng tomonidagi ikkinchi integralni baholaymiz:

$$\left|\iint\limits_{\Omega`\backslash B[0,\delta_1]} e^{-\lambda h(y_1,y_2)}\,\omega(y_1,y_2)dy_1dy_2\right| \le e^{-\lambda m}M|\Omega`|; \quad (2.2.5)$$

$$\left(m = \min_{\Omega`\backslash B[0,\delta_1]} h(y_1,y_2)\ ; M = \max_{\Omega`\backslash B[0,\delta_1]}|\omega(y_1,y_2)|\right)$$

$h(y_1,y_2)$ funksiya 2.1.1-lemmaning shartlarini qanoatlantirgani uchun (2.2.4) tenglikning o'ng tomonidagi birinchi integralni quyidagi ko'rinishda yozamiz:

$$\iint\limits_{B[0,\delta_1]} e^{-\lambda h(y_1,y_2)}\,\omega(y_1,y_2)dy_1dy_2 =$$

$$= \iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)}\,\omega(y_1(x_1,x_2),y_2(x_1,x_2))\left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right| dx_1dx_2 \quad (2.2.6)$$

Agar $f(x_1,x_2) = \omega(y_1(x_1,x_2),y_2(x_1,x_2))\left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right|$ deb olsak, (2.2.6) tenglikning o'ng tomonidagi integral λ→ +∞ da eksponensial kichik xatolik bilan (2.1.2) integral ekanligini payqash mumkin.

Yuqoridagi (2.2.5) va (2.2.6) larga asosan (2.2.4) ni quyidagi ko'rinishda yozamiz:

$$\iint\limits_{\Omega`} e^{-\lambda h(y_1,y_2)}\,\omega(y_1,y_2)dy_1dy_2 =$$

$$= \iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)}\,\omega\big(y_1(x_1,x_2),y_2(x_1,x_2)\big)\left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right| dx_1dx_2 +$$

$$+O(1)e^{-\lambda m} \qquad (2.2.7)$$

Demak, yuqoridagilarga asosan λ→ +∞ da (2.2.1) integral quyidagiga teng ekan:

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,\psi(t_1,t_2)dt_1dt_2 = e^{-\lambda g\left(t_1^0,t_2^0\right)} \times$$

$$\times\left[\iint\limits_{B[0,\delta]} e^{-\lambda(x_1^2+x_2^2)}\,\omega\big(y_1(x_1,x_2),y_2(x_1,x_2)\big)\left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right| dx_1dx_2 + O(1)e^{-\lambda m}\right]$$

Agar 2.1.1-teoremadagi $f(x_1,x_2)$ funksiya sifatida quyidagini olsak,

$$f(x_1,x_2)=\omega(y_1(x_1,x_2),y_2(x_1,x_2))\left|\frac{D(y_1,y_2)}{D(x_1,x_2)}\right|,$$

(2.2.1) integral uchun λ→ +∞ da quyidagini olamiz:

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,\psi(t_1,t_2)dt_1dt_2=e^{-\lambda g(t_1^0,t_2^0)}\lambda^{-1}\left[\sum_{i=0}^{m-1}a_i\lambda^{-i}+\frac{O(1)}{\lambda^m}\right]$$

bunda a_i koeffisiyent quyidagicha aniqlanadi:

$$a_i=\frac{\sum_{j=0}^{i}\frac{\partial^{(2i)}f(0,0)}{\partial x_1^{2(i-j)}\partial x_2^{2j}}C_{2i}^{2j}\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!}$$

Bundan esa teoremaning isboti kelib chiqadi.

2.2.1-natija: *(2.2.2) yoyilma uchun quyidagi formula o'rinli:*

$$\iint\limits_{\Omega} e^{-\lambda g(t_1,t_2)}\,\psi(t_1,t_2)dt_1dt_2=$$

$$=e^{-\lambda g(t_1^0,t_2^0)}\frac{2\pi}{\lambda}\left[\frac{\psi(t_1^0,t_2^0)}{\sqrt{\det(g''(t_1^0,t_2^0)}}+\frac{O(1)}{\lambda}\right],\lambda\to+\infty$$

2.3-§ $\mathbb{R}^n$ fazosida Laplas integralining asimptotikasi.

Biz bu paragrafda $\mathbb{R}^n$ fazosida Laplas integralining asimptotikasini qaraymiz. Dastlab $\lambda \to +\infty$ da quyidagi integralni qaraymiz:

$$F(\lambda) = \int_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})d\mathrm{x}, \qquad (\mathrm{x} = (x_1, x_2, \dots, x_n)) \qquad (2.3.1)$$

Faraz qilaylik $f(\mathrm{x})$ shunday funksiya bo'lsinki, quyidagi

$$\int_{\mathbb{R}^n} e^{-|\mathrm{x}|^2} |f(\mathrm{x})| d\mathrm{x} < +\infty \qquad (2.3.2)$$

xosmas integral yaqinlashuvchi bo'lsin.

2.3.1-tasdiq: *Faraz qilaylik, f funksiya (2.3.2) shartni qanoatlantirsin. U holda istalgan $\delta > 0$ uchun (2.3.1) integral uchun*

$$\int_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})d\mathrm{x} = \int_{B[0,\delta]} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})d\mathrm{x} + O(1)e^{-\lambda\delta^2}, \qquad \lambda \geq 1 \quad (2.3.3)$$

baho o`rinli bo`ladi.

Isbot: 2.1.1-tasdiq isboti kabi isbotlanadi.

2.3.2-tasdiq: *Har qanday $\delta > 0$ va ixtiyoriy $k_1 \geq 0, k_2 \geq 0, \dots, k_n \geq 0$ uchun*

$$\int_{B[0,\delta]} e^{-\lambda|\mathrm{x}|^2} x_1^{2k_1} x_2^{2k_2} \dots x_n^{2k_n} d\mathrm{x} =$$

$$= \frac{\Gamma\left(k_1 + \frac{1}{2}\right)\Gamma\left(k_1 + \frac{1}{2}\right)\dots\Gamma\left(k_1 + \frac{1}{2}\right)}{\lambda^{k_1+k_2+\dots+k_n+\frac{n}{2}}} + O\left(e^{-\lambda\delta^2}\right) \quad (2.3.4)$$

baho o'rinli.

Isbot: 2.1.2-tasdiq isboti kabi isbotlanadi.

2.3.3-tasdiq: *Hech bo'lmaganda biri toq bo'lgan $k_1 > 0, k_2 > 0, \dots, k_n > 0$ sonlar uchun quyidagi tenglik o'rinli.*

$$\int_{B[0,\delta]} e^{-\lambda|\mathrm{x}|^2} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} d\mathrm{x} = 0 \qquad (2.3.5)$$

Isboti Toq funksiyani koordinatalar boshiga nisbatan simmetrik soha bo'yicha olingan integrali nolga teng ekanligidan kelib chiqadi.

Yuqoridagi mulohazalarning natijasi sifatida quyidagi teoremaga kelamiz.

2.3.1-teorema: *Faraz qilaylik, f funksiya (2.3.2) shartni qanoatlantirib, koordinatalar boshining biror atrofida* 2m *marta uzluksiz xususiy hosilalarga ega bo'lsin. U holda* $\lambda \to +\infty$ *da*

$$\int\limits_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})\mathrm{dx} = \lambda^{-\frac{n}{2}}\left[\sum_{i=0}^{m-1} a_i\lambda^{-i} + \frac{O(1)}{\lambda^m}\right] \qquad (2.3.6)$$

asimptotik tenglik o'rinli.

Bu yerda a_i *koeffisiyent quyidagicha aniqlanadi:*

$$a_i = \frac{\sum_{j_1=0}^{i}\sum_{j_2=0}^{j_1}\dots\sum_{j_{n-1}=0}^{j_{n-2}} \frac{\partial^{2i} f(0)}{\partial x_1^{2(i-j_1)}\partial x_2^{2(j_1-j_2)}\dots\partial x_{n-1}^{2(j_{n-2}-j_{n-1})}\partial x_n^{2j_{n-1}}} c(i, j_1, j_2, .., j_{n-1})}{(2i)!}$$

bunda $c(i, j_1, j_2, .., j_{n-1})$ *quyidagicha aniqlanadi:*

$$c(i, j_1, j_2, .., j_{n-1}) = C_{2i}^{2j_1} C_{2j_1}^{2j_2} \dots C_{2j_{n-2}}^{2j_{n-1}} \Gamma\left(i - j_1 + \frac{1}{2}\right)\Gamma\left(j_1 - j_2 + \frac{1}{2}\right)\dots\Gamma(j_{n-1} + \frac{1}{2})$$

Isbot: 2.1.1-teorema isboti kabi isbotlanadi.

2.3.1-natija: *Koordinatalar boshining biror atrofida ikki marta uzluksiz xususiy hosilalarga ega bo'lgan (2) shartni qanoatlantiruvchi f funksiya uchun*

$$\int\limits_{\mathbb{R}^n} e^{-\lambda|\mathrm{x}|^2} f(\mathrm{x})\mathrm{dx} = \left(\frac{\pi}{\lambda}\right)^{\frac{n}{2}}\left[f(0) + \frac{O(1)}{\lambda}\right], \lambda \to +\infty \ (2.3.7)$$

asimptotik tenglik o'rinli.

Kelgusida kerak bo'ladigan ba'zi tushunchalarni keltiramiz.

Faraz qilaylik Φ funksiya IR^n fazoning biror U to'plamida aniqlangan cheksiz silliq funksiya bo'lib, uning uchun $\mathrm{x} = \mathrm{x}^0 \in U$ kritik nuqta bo'lsin.

Ta'rif. *Agar quyidagi* $n \times n$ *simmetrik matritsa*

$$\left[\frac{\partial^2\Phi}{\partial x_k \partial x_j}\right](\mathrm{x}^0)$$

teskarilanuvchi bo'lsa, u holda $\mathrm{x} = \mathrm{x}^0$ *kritik nuqta xosmas kritik nuqta deyiladi.*

Teskari funksiya haqidagi teoremadan foydalanib, osongina ko'rsatish mumkinki, agar $x = x^0$ xosmas kritik nuqta bo'lsa, u holda u yakkalangan bo'ladi.

Agar kritik nuqta xosmas bo'lsa, u holda $y = \nabla\Phi$ akslantirishning lokal teskarilanuvchi bo'lishini isbotlash qiyin emas va bundan kritik nuqtaning yakkalangan bo'lishi kelib chiqadi.

Avvalo xosmas kritik nuqtaga ega bo'lgan funksiyani shu nuqta atrofida normal shaklga keltirish haqidagi Mors lemmasini isbotlaymiz.

2.3.2- teorema (*Mors lemmasi*). *Faraz qilaylik, $x = x^0$ xosmas kritik nuqta bo'lsin. U holda shu nuqtaning shunday U, $y = 0$ nuqtaning biror V atrofi hamda $\varphi : V \mapsto U$ diffeomorf akslantirish topiladiki, biror $0 \le m \le n$ butun son uchun*

$$\Phi(\varphi(y)) = \sum_{j=1}^{m} y_j^2 - \sum_{j=m+1}^{n} y_j^2,$$

tenglik o'rinli bo'ladi. Bu yerda m soni

$$\left[\frac{\partial^2\Phi}{\partial x_k \partial x_j}\right](x^0)$$

matrisaning musbat xos qiymatlari soni bilan ustma-ust tushadi.

Isbot. Mors lemmasini isbotlash uchun biz umumiylikka ziyon keltirmasdan $x^0 = 0$ deb olishimiz mumkin. Aks holda, siljitish orqali biz kritik nuqtani koordinatalar boshiga keltirishimiz mumkin. Endi biz Φ funksiyani quyidagi shaklda yozamiz:

$$\Phi(x) = \sum_{k,j} x_k x_j \Phi_{kj}(x)$$

bu yerdagi $\Phi_{kj}(x)(k, j = \overline{1,n})$ silliq funksiyalar bo'lib, barcha indekslar uchun $\Phi_{kj}(x) = \Phi_{jk}(x)$ shart bajariladi. Haqiqatan ham, $\Phi(0) = 0, \nabla\Phi(0) = 0$ bo'lgani uchun bo'lakalb integrallash formulasidan foydalanib, quyidagi tengliklarga ega bo'lamiz:

$$\Phi(x) = \int_0^1 \frac{d}{dt}[\Phi(tx)]dt = \int_0^1 (1-t)\frac{d^2}{dt^2}[\Phi(tx)]dt.$$

Endi 2-integralda t parametr bo'yicha hosila olishdan

$$\Phi_{kj}(x) = \int_0^1 (1-t)\frac{\partial^2}{\partial x_k \partial x_j}\Phi(tx)dt, \quad k, j = \overline{1,n}$$

tengliklarga kelamiz. Natijada silliq funksiyalarda xususiy hosilalarning tartibini o'zgartirish hosila qiymatini o'zgartirmasligidan yuqoridagi tengliklarga kelamiz. Shuni ta'kidlaymizki,

$$\Phi_{kj}(0)=\frac{1}{2}\frac{\partial^2}{\partial x_k \partial x_j}\Phi(0) \quad i,j=\overline{1,n}$$

tengliklar bajariladi.

Shuni qayd etish kerakki, chiziqli almashtirishlar, xususan ortogonal almashtirishlar Φ funksiyaning bir jinsli, xususan kvadratik qismlarini o'zgartirmaydi. Aniqrog'i, bir jinsli qism bir jinsli qismga o'tadi. Shuning uchun biz IR_x^n fazoda ortogonal almashtirishlarni qo'llab, $\Phi_{kj}(0)=\lambda_j\delta_{kj}$ tengliklarga ega bo'lamiz. Bu yerda δ_{kj} - Kroneker simvoli.

Shunday qilib, biz yetarli kichik x lar uchun $\Phi_{kk}(\mathrm{x})\neq 0 (k=1,\ldots,n)$ deb hisoblashimiz mumkin. Endi kvadrat uchhaddan to'la kvadrat ajratish usulidan foydalanamiz:

$$\Phi(\mathrm{x})=\Phi_{11}(\mathrm{x})\left(x_1^2+2x_1\left(x_2\frac{\Phi_{12}}{\Phi_{11}}+\ldots+x_n\frac{\Phi_{1n}}{\Phi_{11}}\right)\right)+\sum_{k,j\geq 2}x_k x_j\Phi_{kj}(\mathrm{x})=$$

$$=\Phi_{11}(\mathrm{x})\left(x_1+x_2\frac{\Phi_{12}}{\Phi_{11}}+\ldots+x_n\frac{\Phi_{1n}}{\Phi_{11}}\right)^2+\sum_{k,j\geq 2}x_k x_j\Phi_{kj}^1(\mathrm{x}).$$

Oxirgi tenglikda $\Phi_{kj}^1(\mathrm{x})(k,j=\overline{2,n})$ silliq funksiyalar bo'lib, $\Phi_{kj}^1(0)=\Phi_{kj}(0)=\lambda_k\delta_{kj}(k,j=\overline{2,n})$ tengliklarni qanoatlantiradi. Xususan, $\Phi_{kk}^1(0)=\lambda_k\neq 0(k=\overline{2,n})$ shart bajariladi. Quyidagi

$$\xi_1(\mathrm{x})=x_1+x_2\frac{\Phi_{12}}{\Phi_{11}}+\ldots+x_n\frac{\Phi_{1n}}{\Phi_{11}}$$

belgilashni kiritamiz.

Shuni ta'kidlash kerakki, $\xi_1(\mathrm{x})$ funksiya uchun $\dfrac{\partial\xi_1(0)}{\partial x_j}=\delta_{1j}$ shart bajariladi. Yuqoridagi mulohazalarni takrorlab, Φ funksiya uchun $(n-1)$ - qadamdan keyin quyidagi tenglikni hosil qilamiz:

$$\Phi(\mathrm{x})=\Phi_{11}(\mathrm{x})\xi_1^2(\mathrm{x})+\Phi_{22}^1(\mathrm{x})\xi_2^2(\mathrm{x})+\ldots+\Phi_{nn}^{(n-1)}(\mathrm{x})\xi_n^2(\mathrm{x}),$$

bu yerda $\Phi_{kk}^{(k-1)}(0)=\lambda_k\neq 0(k=1,\ldots,n)$ va $\dfrac{\partial\xi_k(0)}{\partial x_j}=\delta_{kj}$ shartlar bajariladi. Shuning uchun

$$y_k=|\Phi_{kk}^{(k-1)}(\mathrm{x})|^{1/2}\ \xi_k(x), \quad k=1,\ldots,n$$

almashtirish silliq va bu akslantirish yakobianining koordinatalar boshidagi qiymati noldan farqli. Demak, bu akslantirishning silliq teskarisi mavjud. Endi y koordinatalarda Φ funksiyani yozsak:

$$\Phi(x(\mathrm{y})) := \Phi(0) \pm y_1^2 + \ldots \pm y_n^2$$

tenglikka ega bo'lamiz. *2.3.2-teorema isbotlandi*.

2.3.2-natija. *2.3.2-teorema shartlarini qanoatlantiruvchi φ akslantirishning yakobiani uchun*

$$J\varphi(0) = \left| det \frac{\partial^2 \Phi(0)}{\partial x_k \partial x_j} \right|^{1/2}$$

tenglik o'rinli.

Endi quyidagi integralning asinptotik yoyilmasini qaraymiz:

$$I(\lambda) = \int\limits_{\mathbb{R}^n} e^{-\lambda g(\mathrm{t})} \psi(\mathrm{t}) \mathrm{dt} \qquad (2.3.8)$$

2.3.3-teorema: *Faraz qilaylik, $g(\mathrm{t})$ funksiya t^0 nuqtada xosmas kritik nuqtaga ega bo'lib bu nuqta minimum nuqtasi bo'lsin. Agar $\psi(\mathrm{t})$ funksiyaning tashuvchisi t^0 nuqtaning yetarli kichik atrofida joylashgan bo'lsa, u holda $\lambda \to +\infty$ da quyidagi asimptotik formula o'rinli bo'ladi:*

$$I(\lambda) \sim e^{-\lambda g(\mathrm{t}^0)} \lambda^{-\frac{n}{2}} \sum_{i=0}^{\infty} a_i \lambda^{-i}, \qquad (2.3.9)$$

bu yerda a_i koeffisiyentlar quyidagicha aniqlanadi:

$$a_i = \frac{\sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} \ldots \sum_{j_{n-1}=0}^{j_{n-2}} \frac{\partial^{2i} f(0)}{\partial y_1^{2(i-j_1)} \partial y_2^{2(j_1-j_2)} \ldots \partial y_{n-1}^{2(j_{n-2}-j_{n-1})} \partial y_n^{2j_{n-1}}} c(i, j_1, j_2, \ldots, j_{n-1})}{(2i)!}$$

bunda o'z navbatida $c(i, j_1, j_2, \ldots, j_{n-1})$ koeffisiyentlar va f(y) funksiya quyidagicha aniqlanadi:

$$c(i, j_1, j_2, \ldots, j_{n-1}) = C_{2i}^{2j_1} C_{2j_1}^{2j_2} \ldots C_{2j_{n-2}}^{2j_{n-1}} \Gamma\left(i - j_1 + \frac{1}{2}\right) \Gamma\left(j_1 - j_2 + \frac{1}{2}\right) \ldots \Gamma(j_{n-1} + \frac{1}{2})$$

$$f(\mathrm{y}) = 2^{\frac{n}{2}} \frac{\psi(\mathrm{t}^0 + \varphi(\mathrm{y}))}{\sqrt{det \Phi''_{\mathrm{xx}}(\varphi(\mathrm{y}))}}$$

bu yerda $\Phi(\mathrm{x}) = g(\mathrm{x} + \mathrm{t}^0) - g(\mathrm{t}^0)$, φ esa 2.3.2-teoramadagi diffeomorf akslantirish.

Eslatma. a_0 koeffitiyent uchun quyidagi formula o'rinli:

$$a_0 = \frac{(2\pi)^{\frac{n}{2}}\psi(\mathrm{t}^0)e^{-\lambda g(\mathrm{t}^0)}}{\sqrt{|det g''_{\mathrm{tt}}(\mathrm{t}^0)|}}$$

Isbot: (2.3.8) integralni quyidagicha yozib olamiz:

$$\int_{\mathbb{R}^n} e^{-\lambda g(\mathrm{t})}\psi(\mathrm{t})\mathrm{dt} = e^{-\lambda g(\mathrm{t}^0)} \int_{\mathbb{R}^n} e^{-\lambda(g(\mathrm{t})-g(\mathrm{t}^0))}\psi(\mathrm{t})\mathrm{dt} \qquad (2.3.10)$$

Xosmas kritik nuqtani koordinatalar boshiga olib kelish uchun quyidagi chiziqli almashtirish $\mathrm{t}-\mathrm{t}^0=\mathrm{x}$ orqali (2.3.10) tenglikni o'ng tomoni quyidagi ko'rinishga keladi.

$$e^{-\lambda g(\mathrm{t}^0)} \int_{\mathbb{R}^n} e^{-\lambda(g(\mathrm{t})-g(\mathrm{t}^0))}\psi(\mathrm{t})\mathrm{dt} = e^{-\lambda g(\mathrm{t}^0)} \int_{\mathbb{R}^n} e^{-\lambda(g(\mathrm{x}+\mathrm{t}^0)-g(\mathrm{t}^0))}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt}$$

Agar $\Phi(\mathrm{x}) = g(\mathrm{x}+\mathrm{t}^0) - g(\mathrm{t}^0)$ deb olsak, yuqoridagi tenglikni o'ng tomoni quyidagi ko'rinishga keladi.

$$e^{-\lambda g(\mathrm{t}^0)} \int_{\mathbb{R}^n} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} \qquad (2.3.11)$$

Teoremaning shartiga ko'ra $\Phi(\mathrm{x})$ funksiya uchun 2.3.2-teorema shartlari bajariladi. U holda x=0 nuqtaning shunday U, $\mathrm{y}=0$ nuqtaning biror V atrofi hamda $\varphi : V \mapsto U$ diffeomorf akslantirish topiladiki,

$$\Phi(\varphi(\mathrm{y})) = \sum_{j=0}^{n} y_j^2 \qquad (2.3.12)$$

tenglik o'rinli bo'ladi.

Faraz qilaylik $B[0,\delta_1]$ shar shu U atrofga tegishli bo'lsin. U holda (2.3.11) integralni shunday ikki qismga ajratamizki, ulardan birida $\lambda \to +\infty$ da integral asosiy qiymatiga erishadi ikkinchisida integral cheksiz kichik bo'ladi. Bularni quyida batafsil ko'ramiz.

(2.3.11) dagi integralning integrallash sohasini $B[0,\delta_1]$ va $\mathbb{R}^n \backslash B[0,\delta_1]$ qismlarga bo'lib quyidagiga ega bo'lamiz:

$$\int\limits_{\mathbb{R}^n} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} = \int\limits_{B[0,\delta_1]} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} +$$

$$+ \int\limits_{\mathbb{R}^n\setminus B[0,\delta_1]} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} \qquad (2.3.13)$$

Yuqoridagi tenglikni o'ng tomonidagi ikkinchi integralni baholaymiz:

$$\left| \int\limits_{\mathbb{R}^n\setminus B[0,\delta_1]} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} \right| \le e^{-\lambda m} M \left| supp\big(\psi(\mathrm{x}+\mathrm{t}^0)\big) \right| \quad (2.3.14)$$

bu yerda $m = \min\limits_{\mathbb{R}^n\setminus B[0,\delta_1]} \Phi(\mathrm{x})\,, \quad M = \max\limits_{supp(\psi(\mathrm{x}+\mathrm{t}^0))} \psi(\mathrm{x}+\mathrm{t}^0)$.

Ma'lumki, $B[0,\delta_1]$ sharga $\varphi{:}V \mapsto U$ diffeomorf akslantiri mavjudki, (2.3.12) tenglik bajariladi. U holda (2.3.12) tenglikni o'ng tomonidagi birinchi integralni quyidagicha yozishimiz mumkin:

$$\int\limits_{B[0,\delta_1]} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} = \int\limits_{B[0,\delta]} e^{-\lambda|\mathrm{y}|^2}\psi(\varphi(\mathrm{y})+\mathrm{t}^0)J\big(\varphi(\mathrm{y})\big)\mathrm{dy} \quad (2.3.15)$$

Agar *f*(y) funksiya sifatida quyidagini olsak:

$$f(\mathrm{y}) = \psi(\varphi(\mathrm{y})+\mathrm{t}^0)J\big(\varphi(\mathrm{y})\big)$$

u holda (2.3.15) tenglik quyidagi ko'rinishga keladi:

$$\int\limits_{B[0,\delta_1]} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} = \int\limits_{B[0,\delta]} e^{-\lambda|\mathrm{y}|^2} f(\mathrm{y})\mathrm{dy}$$

Demak, (2.3.13) tenglikni quyidagicha yozish mumkin:

$$\int\limits_{\mathbb{R}^n} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} = \int\limits_{B[0,\delta]} e^{-\lambda|\mathrm{y}|^2} f(\mathrm{y})\mathrm{dy} + O(1)e^{-\lambda m} \quad (2.3.16)$$

yuqoridagi tenglikni o'ng tomonidagi ikkinchi integral uchun 2.3.1-tasdiqni va 2.3.1-teoremani qo'llasak hamda $\lambda\to+\infty$ da $e^{-\lambda m}$ eksponenta $\lambda^{-m-\frac{n}{2}}$ darajadan tezroq nolga intilishini hisobga olsak, (2.3.16) quyidagi ko'rinishga keladi:

$$\int\limits_{\mathbb{R}^n} e^{-\lambda\Phi(\mathrm{x})}\psi(\mathrm{x}+\mathrm{t}^0)\mathrm{dt} = \lambda^{-\frac{n}{2}}\left[\sum_{i=0}^{m-1} a_i\lambda^{-i} + \frac{O(1)}{\lambda^m}\right]$$

yuqoridagi tenglikni bilvosita (2.3.10) ga olib borib qo`ysak, quyidagiga ega bo'lamiz:

$$\int\limits_{\mathbb{R}^n} e^{-\lambda g(\mathrm{t})}\psi(\mathrm{t})\mathrm{dt} = e^{-\lambda g(\mathrm{t}^0)}\lambda^{-\frac{n}{2}}\left[\sum_{i=0}^{m-1} a_i\lambda^{-i} + \frac{O(1)}{\lambda^m}\right], \lambda \to +\infty$$

bu esa teoremaning isbotini yakunlaydi.

2.3.3-natija: *Faraz qilaylik,* $g(\mathrm{t})$ *funksiya* t^0 *nuqtada xosmas kritik nuqtaga ega bo'lib bu nuqta minimum nuqtasi bo'lsin. Agar* $\psi(\mathrm{t})$ *funksiyaning tashuvchisi* t^0 *nuqtaning yetarli kichik atrofida joylashgan bo'lsa, u holda* $\lambda \to +\infty$ *da quyidagi asimptotik tenglik o'rinli bo'ladi:*

$$\int\limits_{\mathbb{R}^n} e^{-\lambda g(\mathrm{t})}\psi(\mathrm{t})\mathrm{dt} = e^{-\lambda g(\mathrm{t}^0)}\left(\frac{2\pi}{\lambda}\right)^{-\frac{n}{2}}\left[\frac{\psi(\mathrm{t}^0)}{\sqrt{|det g''_{\mathrm{tt}}(\mathrm{t}^0)|}} + \frac{O(1)}{\lambda}\right]$$

III BOB. Karrali Laplas integrali asimptotik yoyilmasining ba'zi tadbiqlari.

3.1-§ Plastinkada issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimining asimptotikasi.

Biz bu paragrafda ikki karrali Laplas integralining asimptotik yoyilmasining tadbiqi sifatida plastinkada issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasining asimptotik yechimini qaraymiz.

Plastinkada issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi quyidagicha aniqlanadi:

$$\begin{cases} \dfrac{\partial u(x_1,x_2,t)}{\partial t} = a^2\left(\dfrac{\partial^2 u(x_1,x_2,t)}{\partial x_1^2} + \dfrac{\partial^2 u(x_1,x_2,t)}{\partial x_2^2}\right) & (3.1.1) \\ u(x_1,x_2,t)|_{t=+0} = \psi(x_1,x_2) & (3.1.2) \end{cases}$$

(3.1.2) shartni qanoatlantiruvchi (3.1.1) tenglamaning yechimi ushbu

$$u(x_1,x_2,t) = \frac{1}{4a^2\pi t}\iint\limits_{\mathbb{R}^2} \psi(\xi_1,\xi_2)e^{-\frac{r^2}{4a^2t}}d\xi_1 d\xi_2; \quad (3.1.3)$$

$$(r = |(x_1,x_2) - (\xi_1,\xi_2)|)$$

formula bilan aniqlanadi.

3.1.1-teorema: *Faraz qilaylik $\psi(x_1,x_2)$ funksiya $C_0^\infty(\mathbb{R}^2)$ sinfga tegishli bo'lsin. U holda (3.1.2) shartni qanoatlantiruvchi (3.1.1) tenglamaning yechimi uchun $t\to 0$ da quyidagi yoyilma o'rinli:*

$$u(x_1,x_2,t) \approx \sum_{i=0}^{\infty} b_i t^i \qquad (3.1.4)$$

bu yerda b_i koeffisiyent quyidagicha aniqlanadi:

$$b_i = \frac{(4a^2)^i}{(2i)!\,\pi}\sum_{j=0}^{i}\frac{\partial^{2i}\psi(x_1,x_2)}{\partial x_1^{2(i-j)}\partial x_2^{2j}}C_{2i}^{2j}\,\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)$$

Isbot: Yuqoridagi (3.1.3) tenglikning o'ng tomonidagi integralni quyuidagi ko'rinishda yozib olamiz:

$$\iint\limits_{\mathbb{R}^2} \psi(\xi_1,\xi_2)e^{-\frac{((x_1-\xi_1)^2+(x_2-\xi_2)^2)}{4a^2t}}d\xi_1 d\xi_2 \qquad (3.1.4)$$

va bu integralda quyidagicha almashtirish olamiz:

$$\begin{cases} x_1 - \xi_1 = 2ay_1 \\ x_2 - \xi_2 = 2ay^2 \end{cases} \qquad (3.1.5)$$

Bunday almashtirish natijasida (3.1.4) integral quyidagi ko'rinishga keladi:

$$4a^2 \iint\limits_{\mathbb{R}^2} \psi(x_1 - 2ay_1, x_2 - 2ay_2) e^{-\frac{(y_1^2+y_2^2)}{t}} dy_1 dy_2 \qquad (3.1.6)$$

Agar (3.1.6) integralda $t = \frac{1}{\lambda}$ belgilash olsak u holda quyidagiga ega bo'lamiz.

$$4a^2 \iint\limits_{\mathbb{R}^2} \psi(x_1 - 2ay_1, x_2 - 2ay_2) e^{-\lambda(y_1^2+y_2^2)} dy_1 dy_2 \qquad (3.1.7)$$

(3.1.7) integral uchun 2.1.1-teoremani tadbiq qilsak quyidagiga ega bo'lamiz.

$$4a^2 \iint\limits_{\mathbb{R}^2} e^{-\lambda(y_1^2+y_2^2)} \psi(x_1 - 2ay_1, x_2 - 2ay_2) dx_1 dx_2 =$$

$$= 4a^2 \lambda^{-1} \left[\sum_{i=0}^{m-1} a_i \lambda^{-i} + \frac{O(1)}{\lambda^m} \right] \quad \lambda \to +\infty \quad (3.1.8)$$

bunda a_i koeffisiyent quyidagicha aniqlanadi:

$$a_i = \frac{\sum_{j=0}^{i} \frac{\partial^{(2i)} \psi(x_1, x_2)}{\partial y_1^{2(i-j)} \partial y_2^{2j}} C_{2i}^{2j} \Gamma\left(i - j + \frac{1}{2}\right) \Gamma\left(j + \frac{1}{2}\right)}{(2i)!}$$

Agar bir qancha hisoblashlarga asosan quyidagi

$$\frac{\partial^{(2i)} \psi(x_1, x_2)}{\partial y_1^{2(i-j)} \partial y_2^{2j}} = (4a^2)^i \frac{\partial^{(2i)} \psi(x_1, x_2)}{\partial x_1^{2(i-j)} \partial x_2^{2j}}$$

formula o'rinli ekanligini hisobga olsak, u holda a_i koeffisiyent uchun quyidagiga ega bo'lamiz:

$$a_i = \frac{\sum_{j=0}^{i} (4a^2)^i \frac{\partial^{(2i)} \psi(x_1, x_2)}{\partial x_1^{2(i-j)} \partial x_2^{2j}} C_{2i}^{2j} \Gamma\left(i - j + \frac{1}{2}\right) \Gamma\left(j + \frac{1}{2}\right)}{(2i)!}$$

Yuqoridagi formulani (3.1.8) ga olib borib qo'ysak $\lambda \to +\infty$ quyidagiga ega bo'lamiz:

$$4a^2\iint\limits_{\mathbb{R}^2} e^{-\lambda(y_1^2+y_2^2)}\,\psi(x_1-2ay_1,x_2-2ay_2)dx_1dx_2 =$$

$$= 4a^2\lambda^{-1}\left[\sum_{i=0}^{m-1}\frac{\sum_{j=0}^{i}(4a^2)^i\,\frac{\partial^{(2i)}\psi(x_1,x_2)}{\partial x_1^{2(i-j)}\partial x_2^{2j}}\,C_{2i}^{2j}\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!}\lambda^{-i}+\frac{O(1)}{\lambda^m}\right]$$

(3.1.4), (3.1.5), (3.1.6) tengliklarni hamda $t=\frac{1}{\lambda}$ ekanligini hisobga olib

yuqoridagi tenglikni (3.1.3) integralga olib borib qo'ysak $t\to 0$ quyidagi formulaga ega bo'lamiz:

$$u(x_1,x_2,t)=\sum_{i=0}^{m-1}\frac{\sum_{j=0}^{i}(4a^2)^i\,\frac{\partial^{(2i)}\psi(x_1,x_2)}{\partial x_1^{2(i-j)}\partial x_2^{2j}}\,C_{2i}^{2j}\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!\,\pi}t^i+$$
$$+O(1)t^m$$

Agar

$$\frac{(4a^2)^i}{(2i)!\,\pi}\sum_{j=0}^{i}\frac{\partial^{2i}\psi(x_1,x_2)}{\partial x_1^{2(i-j)}\partial x_2^{2j}}C_{2i}^{2j}\Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)=b_i$$

deb olsak, teoremaning isboti yakunlanadi.

3.1.1-natija: *Faraz qilaylik $\psi(x_1,x_2)$ funksiya $C_0^\infty(\mathbb{R}^2)$ sinfga tegishli bo'lsin. U holda (3.1.2) shartni qanoatlantiruvchi (3.1.1) tenglamaning yechimi uchun $t\to 0$ da*

$$u(x_1,x_2,t)=\psi(x_1,x_2)+a^2\left(\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}\right)t+\frac{a^4}{2}\left(\frac{\partial^4\psi}{\partial x_1^4}+2\frac{\partial^2\psi}{\partial x_1^2\partial x_2^2}+\frac{\partial^4\psi}{\partial x_2^4}\right)t^2$$
$$+\frac{a^6}{6}\left(\frac{\partial^6\psi}{\partial x_1^6}+3\frac{\partial^6\psi}{\partial x_1^4\partial x_2^2}+3\frac{\partial^6\psi}{\partial x_1^2\partial x_2^4}+\frac{\partial^6\psi}{\partial x_2^6}\right)t^3+O(1)t^4$$

asimptotik tenglik o'rinli.

3.2-§ $\mathbb{R}^n$fazosida issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimining asimptotikasi.

Bu paragrafda *n* karrali Laplas integrali asimptotik yoyilmasining tadbiqi sifatida $\mathbb{R}^n$fazosida issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasining asimptotik yechimini qaraymiz.

$\mathbb{R}^n$fazosida issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi quyidagicha aniqlanadi:

$$\begin{cases} \dfrac{\partial u(\mathrm{x},t)}{\partial t} = a^2\left(\dfrac{\partial^2 u(\mathrm{x},t)}{\partial x_1^2} + \dfrac{\partial^2 u(\mathrm{x},t)}{\partial x_2^2} + \cdots + \dfrac{\partial^2 u(\mathrm{x},t)}{\partial x_n^2}\right) & (3.2.1) \\ u(\mathrm{x},t)|_{t=+0} = \psi(\mathrm{x}) & (3.2.2) \end{cases}$$

(3.2.2) shartni qanoatlantiruvchi (3.2.1) tenglamani yechimi ushbu

$$u(\mathrm{x},t) = \frac{1}{(2a\sqrt{\pi t})^n}\iint\limits_{\mathbb{R}^n} \psi(\xi) e^{-\frac{r^2}{4a^2 t}} d\xi; \quad (3.2.3)$$

$$(r = |\mathrm{x} - \xi|)$$

formula bilan aniqlanadi.

3.2.1-teorema: *Faraz qilaylik $\psi(\mathrm{x})$ funksiya $C_0^\infty(\mathbb{R}^n)$ sinfga tegishli bo'lsin. U holda (3.2.2) shartni qanoatlantiruvchi (3.2.1) tenglamani yechimi uchun $t \to 0$ da quyidagi yoyilma o'rinli:*

$$u(\mathrm{x},t) \approx \sum_{i=0}^{\infty} b_i t^i \qquad (3.2.4)$$

bu yerda b_i koeffisiyent quyidagicha aniqlanadi:

$$b_i = \frac{(4a^2)^i}{(2i)!\,\pi^{\frac{n}{2}}}\sum_{j=0}^{i}\sum_{j_1=0}^{i}\sum_{j_2=0}^{j_1}\dots\sum_{j_{n-1}=0}^{j_{n-2}} \frac{\partial^{2i}\psi(\mathrm{x})}{\partial x_1^{2(i-j_1)}\partial x_2^{2(j_1-j_2)}\dots\partial x_{n-1}^{2(j_{n-2}-j_{n-1})}\partial x_n^{2j_{n-1}}} c(i,j_1,j_2,..,j_{n-1})$$

bunda $c(i,j_1,j_2,..,j_{n-1})$ quyidagicha aniqlanadi:

$$c(i,j_1,j_2,..,j_{n-1}) = C_{2i}^{2j_1} C_{2j_1}^{2j_2}\dots C_{2j_{n-2}}^{2j_{n-1}}\Gamma\left(i - j_1 + \frac{1}{2}\right)\Gamma\left(j_1 - j_2 + \frac{1}{2}\right)\dots\Gamma(j_{n-1} + \frac{1}{2})$$

Isbot: Yuqoridagi (3.2.3) tenglikni o'ng tomonidagi integralni quyuidagi ko'rinishda yozib olamiz:

$$\int_{\mathbb{R}^n} \psi(\xi) e^{-\frac{((x_1-\xi_1)^2+(x_2-\xi_2)^2+\ldots+(x_n-\xi_n)^2)}{4a^2 t}} d\xi \qquad (3.2.4)$$

va bu integralda quyidagicha almashtirish olamiz:

$$\mathrm{x} - \xi = 2a\mathrm{y} \quad (3.2.5)$$

Bunday almashtirish natijasida (3.2.4) integral quyidagi ko'rinishga keladi:

$$(2a)^n \int_{\mathbb{R}^n} \psi(\mathrm{x} - 2a\mathrm{y}) e^{-\frac{|\mathrm{y}|^2}{t}} \, d\mathrm{y} \quad (3.2.6)$$

Agar (3.2.6) integralda $t = \frac{1}{\lambda}$ belgilash olsak u holda quyidagiga ega bo'lamiz.

$$(2a)^n \int_{\mathbb{R}^n} \psi(\mathrm{x} - 2a\mathrm{y}) e^{-\lambda|\mathrm{y}|^2} \, d\mathrm{y} \quad (3.2.7)$$

(3.2.7) integral uchun 2.3.1-teoremani tadbiq qilsak quyidagiga ega bo'lamiz.

$$(2a)^n \int_{\mathbb{R}^n} \psi(\mathrm{x} - 2a\mathrm{y}) e^{-\lambda|\mathrm{y}|^2} \, d\mathrm{y} = (2a)^n \lambda^{-\frac{n}{2}} \left[\sum_{i=0}^{m-1} a_i \lambda^{-i} + \frac{O(1)}{\lambda^m} \right],$$
$$\lambda \to +\infty \quad (3.1.8)$$

bunda a_i koeffisiyent quyidagicha aniqlanadi:

$$a_i = \frac{\sum_{j=0}^{i} \sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} \ldots \sum_{j_{n-1}=0}^{j_{n-2}} \frac{\partial^{2i}\psi(\mathrm{x})}{\partial y_1^{2(i-j_1)} \partial y_2^{2(j_1-j_2)} \ldots \partial y_{n-1}^{2(j_{n-2}-j_{n-1})} \partial y_n^{2j_{n-1}}} c(i, j_1, j_2, \ldots, j_{n-1})}{(2i)!}$$

bu yerda $c(i, j_1, j_2, \ldots, j_{n-1})$ koeffisiyent quyidagicha aniqlanadi:

$$c(i, j_1, j_2, \ldots, j_{n-1}) = C_{2i}^{2j_1} C_{2j_1}^{2j_2} \ldots C_{2j_{n-2}}^{2j_{n-1}} \Gamma\left(i - j_1 + \frac{1}{2}\right) \Gamma\left(j_1 - j_2 + \frac{1}{2}\right) \ldots \Gamma(j_{n-1} + \frac{1}{2})$$

Agar bir qancha hisoblashlarga asosan quyidagi

$$\frac{\partial^{2i}\psi(\mathrm{x})}{\partial y_1^{2(i-j_1)} \partial y_2^{2(j_1-j_2)} \ldots \partial y_{n-1}^{2(j_{n-2}-j_{n-1})} \partial y_n^{2j_{n-1}}} =$$
$$= (4a^2)^i \frac{\partial^{2i}\psi(\mathrm{x})}{\partial x_1^{2(i-j_1)} \partial x_2^{2(j_1-j_2)} \ldots \partial x_{n-1}^{2(j_{n-2}-j_{n-1})} \partial x_n^{2j_{n-1}}}$$

formula o'rinli ekanligini hisobga olsak, u holda a_i koeffisiyent uchun quyidagiga ega bo'lamiz:

$$a_i = \frac{(4a^2)^i \sum_{j=0}^{i} \sum_{j_1=0}^{i} \sum_{j_2=0}^{j_1} \dots \sum_{j_{n-1}=0}^{j_{n-2}} \frac{\partial^{2i}\psi(\mathrm{x})}{\partial y_1^{2(i-j_1)} \partial y_2^{2(j_1-j_2)} \dots \partial y_{n-1}^{2(j_{n-2}-j_{n-1})} \partial y_n^{2j_{n-1}}} c(i, j_1, j_2, \dots, j_{n-1})}{(2i)!}$$

Yuqoridagi formulani (3.2.8) ga olib borib qo'ysak $\lambda \to +\infty$ quyidagiga ega bo'lamiz:

$$(2a)^n \int\limits_{\mathbb{R}^n} \psi(\mathrm{x} - 2a\mathrm{y}) e^{-\lambda|\mathrm{y}|^2}\, d\mathrm{y} = (2a)^n \lambda^{-\frac{n}{2}} \left[\sum_{i=0}^{m-1} a_i \lambda^{-i} + \frac{O(1)}{\lambda^m} \right]$$

(3.2.4), (3.2.5), (3.2.6) tengliklarni hamda $t = \frac{1}{\lambda}$ ekanligini hisobga olib

yuqoridagi tenglikni (3.2.3) integralga olib borib qo'ysak $t \to 0$ quyidagi formulaga ega bo'lamiz:

$$u(\mathrm{x}, t) = \sum_{i=0}^{m-1} \frac{a_i}{\pi^{\frac{n}{2}}} t^i + O(1) t^m$$

Agar $\frac{a_i}{\pi^{\frac{n}{2}}} = b_i$ deb olsak, teoremaning isboti yakunlanadi.

3.2.1-natija: *Faraz qilaylik $\psi(\mathrm{x})$ funksiya $C_0^\infty(\mathbb{R}^3)$ sinfga tegishli bo'lsin. U holda (3.2.2) shartni qanoatlantiruvchi (3.2.1) tenglamaning yechimi uchun $t \to 0$ da*

$$u(\mathrm{x}, t) = \psi(\mathrm{x}) + a_1 t + a_2 t^2 + a_3 t^3 + O(1) t^4$$

asimptotik tenglik o'rinli bo'lib, bunda a_1, a_2 va a_3 koeffisiyentlar quyidagicha aniqlanadi:

$$a_1 = a^2 \left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} \right)$$

$$a_2 = \frac{a^4}{2} \left(\frac{\partial^4 \psi}{\partial x_1^4} + \frac{\partial^4 \psi}{\partial x_2^4} + \frac{\partial^4 \psi}{\partial x_3^4} + 2 \frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + 2 \frac{\partial^4 \psi}{\partial x_1^2 \partial x_3^2} + 2 \frac{\partial^4 \psi}{\partial x_2^2 \partial x_3^2} \right)$$

$$a_3 = \frac{a^6}{6} \Big(\frac{\partial^6 \psi}{\partial x_1^6} + \frac{\partial^6 \psi}{\partial x_2^6} + \frac{\partial^6 \psi}{\partial x_3^6} + 3 \Big(\frac{\partial^6 \psi}{\partial x_1^4 \partial x_2^2} + \frac{\partial^6 \psi}{\partial x_1^4 \partial x_3^2} + \frac{\partial^6 \psi}{\partial x_2^4 \partial x_3^2} +$$

$$+ \frac{\partial^6 \psi}{\partial x_2^4 \partial x_1^2} + \frac{\partial^6 \psi}{\partial x_3^4 \partial x_1^2} + \frac{\partial^6 \psi}{\partial x_3^4 \partial x_2^2} \Big) + 15 \frac{\partial^6 \psi}{\partial x_1^2 \partial x_2^2 \partial x_3^2} \Big)$$

Xulosa

Magistrlik dissertatsiyasida ko'rilgan (o'rganilgan) natijalardan kelib chiqib quyidagi xulosalarga kelishimiz mumkin.

1. Parametrga bog'liq integrallarni asimptotik hisoblashning Laplas usulini o'rganilgan.
2. Amplitudasi cheksiz silliq, tashuvchisi kompakt bo‘lgan holda Laplas integralning cheksizdagi xarakteri faza funksiyasi kritik nuqtalari bilan aniqlanishi tadqiq qilingan.
3. Agar sohada faza funksiyasining xosmas kirtik nuqtalari bir qancha chekli bo'lsa, bu har bir nuqtalar uchun alohida yoyilmalar olingan.
4. Ikki (3.1.1-teorema) va uch o'lchovli(3.2.1-natija) hamda umumiy holda (3.2.1-teorema) $\mathbb{R}^n$ fazosida issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimini vaqtning cheksiz kichik qiymatlari uchun asimptotik yechimi uchun yoyilma olingan.

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