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# ON THE APPROXIMATION OF PERIODIC FUNCTIONS OF MANY VARIABLES BY SUMS OF MARCINKIEWICZ TYPE

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## Abstract

In this article, we consider the deviations of periodic functions $f(x_1, x_2, \ldots, x_k)$ from a sum of Marcinkiewicz type and from Abelian means in space $L_p^{(k)}$ $(1 \leq p \leq \infty)$, depending on the best approximations of functions by trigonometric polynomials.

## Introduction

The norm of a function $f$ in $L_p^{(k)}$ $(1 \leq p \leq \infty)$, is defined by

$$\| f \|_p = \| f(x_1, x_2, \ldots, x_k) \|_p$$

$$= \left\{ \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} | f(x_1, x_2, \ldots x_k) |^p dx_1 \ldots dx_k \right\}^{\frac{1}{p}} < \infty, 1 \leq p < \infty$$

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and when $p = \infty$, by

$$\| f \|_{\infty} = \| f(x_1, x_2, \dots, x_k) \|_{\infty} = \sup_i | f(x_1, x_2, \dots, x_k) | < \infty.$$

Let $f(x_1, x_2, \dots, x_k) \in L_p^{(k)}$ and

$$\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_k=-\infty}^{\infty} C_{\nu_1, \nu_2, \dots, \nu_k} e^{i(\nu_1 x_1 + \cdots + \nu_k x_k)} \tag{1}$$

be its Fourier series. The coefficients are determined by

$$C_{\nu_1, \nu_2, \dots, \nu_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(t_1, t_2, \dots, t_k) e^{i(\nu_1 x_1 + \cdots + \nu_k x_k)} dt_1 \dots dt_k. \tag{2}$$

The partial sum of the series (1) is

$$S_n^k = S_{n_1, n_2, \dots, n_k}(x) = \sum_{\nu_1=-n_1}^{n_1} \cdots \sum_{\nu_k=-n_k}^{n_k} C_{\nu_1, \nu_2, \dots, \nu_k} e^{i(\nu_1 x_1 + \cdots + \nu_k x_k)} \tag{3}$$

or

$$S_n^{(k)} = S_{n_1, n_2, \dots, n_k}(x)$$

$$= \frac{1}{\pi^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1 + t_1, \dots, x_k + t_k) \prod_{m=1}^{k} D_{n_m}(t_m) dt_1 \dots dt_k, \tag{4}$$

where $D_n(t) = \dfrac{\sin \frac{2n+1}{2} t}{2 \sin \frac{t}{2}}$ is the Dirichlet kernel.

We denote the Fejer mean of the series (1) by

$$\sigma_n^{(k)} = \sigma_{n_1, n_2, \dots, n_k}(x) = \frac{1}{(n_1 + 1)} \cdots \frac{1}{(n_k + 1)} \sum_{\nu_1=-n_1}^{n_1} \cdots \sum_{\nu_k=-n_k}^{n_k} S_{\nu_1, \nu_2, \dots, \nu_k} \tag{5}$$

and Abelian middle of the series (1) by

$$f(r, x) = f(r_1, \ldots, r_k; x_1, \ldots, x_k)$$

$$= \sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_k=-\infty}^{\infty} c_{\nu_1,\ldots,\nu_k} r_1^{|\nu_1|} \ldots r_k^{|\nu_k|} e^{i(\nu_1 x_1 + \cdots + \nu_k x_k)},$$

where $0 < r_i < 1$, $i = 1, 2, \ldots, k$. (6)

Series (1) is called *summable* by the method $(C^{(k)}; 1) = (C; 1, 1, \ldots, 1)$, i.e., the Fejer method to a function $f(x_1, x_2, \ldots, x_k)$, if

$$\lim_{n\to\infty} \sigma_n^k(x) = f(x) = f(x_1, x_2, \ldots, x_k),$$

where $(n \to \infty)$ means $(n_1 \to \infty, n_2 \to \infty, \ldots, n_k \to \infty)$.

The series (1) is called *summable* by the Abel method to a function $f(x_1, x_2, \ldots, x_k)$ if

$$\lim_{r\to 1} f(r, x) = f(x) = f(x_1, x_2, \ldots, x_k),$$

where $(r \to 1)$ means $(r_1 \to 1, r_2 \to 1, \ldots, r_k \to 1)$.

The following statements are known [1, p. 463]:

(1) Let the problem be any function $\varphi(u), u \geq 0,$ positive, increasing, and of order $o(u \ln u)$ for $u \to \infty$.

Then there exists a periodic and integrable function $f(x_1, x_2, \ldots, x_k) \geq 0$ such that $\varphi(f)$ is integrable, and the series (1) cannot be summed anywhere by any of the methods $(C; 1, 1, \ldots, 1)$ and Abel.

(2) If $|f(x_1, x_2, \ldots, x_k)| \ln^+ |f(x_1, x_2, \ldots, x_k)|$ is integrable (in particular, if $f(x_1, x_2, \ldots, x_k) \in L_p^{(k)}$, $1 < p < \infty$, then we can sum the series (1) by the method $(C; 1, 1, \ldots, 1)$ for almost all points $(x_1, x_2, \ldots, x_k)$.

Saks [2] showed that there is a non-negative $2\pi$ - periodic function in each of the variables $f(x_1, x_2, \ldots, x_k) \in L_1^{(k)}$, for which almost everywhere on $Q = [-\pi, \pi; \ldots, -\pi, \pi]$

$$\lim_{n\to\infty} \sigma_n^{(k)} = \lim \sigma_{n_1}, \ldots, \sigma_{n_k} = \infty. \tag{7}$$

Note that Saks in [2] proved (7) when $k = 2$, but this is true for any $k \geq 3$ (see [1, pp. 463-464]). Such reasoning holds for (6), i.e., $f(r, x)$ is unlimited at $r \to 1$ (see [1, p. 464]).

In connection with this (i.e., by relation (7)), Marcinkiewicz first considered the sum (see [3]):

$$\sigma_{n, n, \ldots, n}(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} S_{\nu, \nu, \ldots, \nu}(x_1, x_2, \ldots, x_k);$$

and proved that

(1) If $f(x_1, x_2, \ldots, x_k)$ is a continuous and $2\pi$ - periodic function in each of the variables $x_\nu$ $(\nu = 1, 2, \ldots, k)$, then

$$\sigma_{n, n, \ldots, n}(x) \to f(x_1, x_2, \ldots, x_k);$$

uniformly.

(2) If a $2\pi$ - periodic function $f(x_1, x_2, \ldots, x_k) \in L \ln^+ L$, i.e., the function $f(x_1, x_2, \ldots, x_k) \ln| f(x_1, x_2, \ldots, x_k)|$ is integrable, then

$$\lim \sigma_{n, n, \ldots, n}(x) = f(x_1, x_2, \ldots, x_k)$$

almost everywhere on $Q = [-\pi, \pi; \ldots, -\pi, \pi]$.

Note that these statements were proved by Marcinkiewicz for $k = 2$, but also hold for $k \geq 3$.

**Main Result**

In this paper, we consider deviations of periodic functions $f(x_1, x_2, \ldots, x_k)$ from the Marcinkiewicz sum $\sigma_{n, n, \ldots, n}(x)$ and the following sums in the space $L_p^{(k)}$ $(1 \leq p \leq \infty)$:

$$\sigma_{n, 2n}(x) = \frac{1}{n} \sum_{\nu=n+1}^{2n} S_{\nu, \nu, \ldots, \nu}(x_1, x_2, \ldots, x_k)$$

(like Valle-Poussin);

$$\sigma(r, x) = \sigma(r; x_1, x_2, \ldots, x_k)$$

$$= (1-r)\sum_{\nu=0}^{\infty} r^{\nu} S_{\nu, \nu, \ldots, \nu}(x_1, x_2, \ldots, x_k)\ (0 \leq r < 1)$$

(like Abel).

Note that Abelian mean $\sigma(r, x)$ is a harmonic function in each pair of variables $(r_\nu, x_\nu)(\nu = 1, 2, \ldots, k)$, i.e., satisfies the Laplace equation, and the sum $\sigma_{n, n, \ldots, n}(x)$ is a trigonometric polynomial of order $n$ in each variable $x_\nu$ $(\nu = 1, 2, \ldots, k)$.

The theorem proved below is a generalization of our work in [6].

**Theorem 1.** *If* $f(x_1, x_2, \ldots, x_k) \in L_p^{(k)}, 1 \leq p \leq \infty,$ *then*

$$R_1(f)_p = \| f(x_1, x_2, \ldots, x_k) - \sigma_{n, 2n}(x)\|_p \leq C_1 E_{n, n, \ldots, n}(f)_p, \tag{8}$$

*where* $C_1$ *is a constant, independent of f,* $p$, $n$, $E_{n, n, \ldots, n}(f)_p$ *is the best approximation of function* $f(x_1, x_2, \ldots, x_k)$ *for each variable by trigonometric polynomials of order* $\leq n$ *in the space* $L_p^{(k)}$, *i.e.*,

$$E_{n_1, n_2, \ldots, n_k}(f)_p = \inf \| f(x_1, x_2, \ldots, x_k) - T_{n_1, n_2, \ldots, n_k}(x_1, x_2, \ldots, x_k)\|_p.$$

**Theorem 2.** *If* $f(x_1, x_2, \ldots, x_k) \in L_p^{(k)}$, $1 \leq p \leq \infty$, *then*

$$R_2(f)_p \| f(x_1, x_2, \ldots, x_k) - \sigma_n(x) \|_p \leq \frac{C_2}{n+1} \sum_{\nu=0}^{n+1} E_{\nu, \nu, \ldots, \nu}(f)_p, \quad (9)$$

*where* $C_2$ *is a constant independent of* $f, p, n$.

**Theorem 3.** *If* $f(x_1, x_2, \ldots, x_k) \in L_p^{(k)}$, $1 \leq p \leq \infty$, *then*

$$R_3(f)_p \| f(x_1, x_2, \ldots, x_k) - \sigma(r; x_1, x_2, \ldots, x_k) \|_p$$

$$\leq C_3(1-r) \sum_{\nu=0}^{\infty} r^{\nu} E_{\nu, \nu, \ldots, \nu}(f)_p \quad (0 < r < 1), \quad (10)$$

*where* $C_3$ *is a constant independent of* $f, p, n$.

Zhizhivshvili [4] (for $k = 2$, that is, for the function of two variables) obtained the following estimates for $1 \leq p \leq \infty$:

$$R_2(f)_p \leq c(p)\left\{\omega^{(1)}\left(f; \frac{\ln n}{n}\right)_{L_p} + \omega^{(2)}\left(f; \frac{\ln n}{n}\right)_{L_p}\right\}, \quad (11)$$

where $c(p)$ is a constant independent of $p$, and

$$\omega^{(1)}(f; \delta)_{L_p} = \sup_{0<n\leq\delta} \| f(x+h, y) - f(x, y) \|_{L_p} \quad (0 < \sigma < 1),$$

$$\omega^{(2)}(f; \delta)_{L_p} = \sup_{0<n\leq\delta} \| f(x, y+h) - f(x, y) \|_{L_p}$$

are partial moduli of continuity of the function $f(x, y)$.

For the case of a continuous functions $f(x, y)$, Taberski [5] (for $k = 2$, that is, for the function of two variables) established that if $f(x, y) \in \wedge$, then at $p = \infty$:

$$R_3(f)_p \le C\begin{cases}(1-r)|\ln(1-r)|, & 1 \le \alpha, \beta \le 2,\\ (1-r)^{\beta}, & 1 \le \alpha \le 2,\ 0 < \beta < 1,\\ (1-r)^{\gamma}, & \gamma = \min(\alpha, \beta),\ 0 < \alpha < 1,\ 0 < \beta \le 2,\end{cases}$$

where the constant $C$ does not depend on $f$ and $r$; $\wedge(\alpha, \beta)$ class of function $f(x, y)$ for which

$$\Omega_{1,1}(f; s, t) = \sup_{\substack{|u|\le s\\ |v|\le t}} \| f(x+u, y+v) + f(x+u, y-u) + f(x-u, y+v)$$

$$+ f(x-u, y-v) - 4f(x, y)\|_{L_p}$$

$$\le 4(s^{\alpha} + t^{\alpha}).$$

Note that for cases when $\omega^{(1)}(f; \delta)_{L_p} \le C\delta^{\alpha}$, $\omega^{(2)}(f; \delta)_{L_p} \le C\delta^{\alpha}$ $(0 < \alpha < 1)$ from the result of Zhizhivshvili [4], it follows that

$$R_2(f)_p \le C\left(\frac{\ln n}{n}\right)^{\alpha} \quad (p = 1,\ p = \infty),$$

where from Theorem 2 for the case under consideration, we obtain (see (8), (9)) that

$$R_2(f)_p \le \frac{C}{n^{\alpha}}\ (1 \le p \le \infty), \quad E_{n,n}(f)_p \le C \cdot \omega^{(k)}\left(f; \frac{1}{n}\right)_p, \quad k = 1, 2.$$

Other than that, which holds (see, e.g., [7, p. 288]), is

$$\frac{1}{n+1}\sum_{k=0}^{n} E_{k,k}(f)_{L_p} \le C\left\{\omega^{(1)}\left(f; \frac{\ln n}{n}\right)_{L_p} + \omega^{(2)}\left(f; \frac{\ln n}{n}\right)_{L_p}\right\}.$$

Thus, inequality (9) is in order more accurate than inequality (11) obtained by Zhizhivshvili in [4]. The estimate (9) is indicated in [4] for $1 < p < \infty$.

We also note that Theorem 3 for the class $\wedge(\alpha, \beta)$ with $1 < \alpha, \beta \le 2$ gives a more accurate order of magnitude estimate for $R_3(f)_p (p = \infty)$ than the estimate obtained by Taberski in [5].

Indeed, since (see, e.g., [7, p. 126 and p. 288]),

$$E_{m,n}(f)_{L_p} \le C\left\{\omega_2^{(1)}\left(f;\frac{1}{m}\right)_{L_p} + \omega_2^{(2)}\left(f;\frac{1}{n}\right)_{L_p}\right\} \Leftrightarrow \Omega_{1,1}\left(f;\frac{1}{m},\frac{1}{n}\right)$$

for the class $\wedge(\alpha, \beta)$, from inequality (10), we obtain

$$R_3(f)_p \le C\begin{cases}(1-r), & \alpha = 1,\, 1<\beta\le 2 \text{ или } \beta = 1,\, 1<\alpha\le 2,\\ (1-r)^{\gamma}, & \gamma = \min(\alpha, \beta),\, 0<\alpha<1,\, 0<\beta\le 2,\\ (1-r), & 1<\alpha, \beta \le 2,\end{cases}$$

where

$$\omega_2^{(1)}(f;\delta)_{L_p} = \sup_{0<n\le\delta}\| f(x+h, y) + f(x-h, y) - 2f(x, y)\|_{L_p},$$

$$\omega_2^{(2)}(f;\delta)_{L_p} = \sup_{0<n\le\delta}\| f(x, y+h) + f(x, y-h) - 2f(x, y)\|_{L_p}$$

are type moduli of smoothness of the second order (see [7, pp. 115-126]).

To prove Theorem 1, we need the following lemma. Theorems 2 and 3 are proved based on Theorem 1.

**Lemma.** *If* $D_k(u) = \frac{1}{2} + \sum_{\nu=1}^{k}\cos\nu u,$ *then*

$$\int_0^{\pi}\cdots\int_0^{\pi}\left|\frac{1}{n}\sum_{m=n+1}^{2n} D_m(x_1)\, D_m(x_2)\cdots D_m(x_k)\right| dx_1 dx_2\cdots dx_k \le C,$$

*where C is a constant independent of* $n$ $(n \ge k)$.

**Proof.** Due to the complexity of the calculation, we prove the lemma for the case $k = 2.$ For $k \ge 3,$ the method of proof is similar.

Using elementary formulas and elementary transformations, it is easy to obtain that

$$K_n(x, y) = \sum_{k=0}^{n} D_k(x) D_k(y)$$

$$= \frac{1}{16 \sin \frac{x}{2} \sin \frac{y}{2}} \left\{ \frac{\sin(n+1)(x-y)}{\sin \frac{x-y}{2}} - \frac{\sin(n+1)(x+y)}{\sin \frac{x+y}{2}} \right\}. \quad (12)$$

We divide the square $[0, \pi; 0, \pi]$ into the following parts:

$$D_1 = \left\{ 0 \le x, \ y \le \frac{2\pi}{n} \right\},$$

$$D_2 = \left\{ \frac{2\pi}{n} < x \le \pi, \ x - \frac{\pi}{n} < y \le x \right\},$$

$$D_3 = \left\{ \frac{2\pi}{n} < x \le \pi, \ 0 < y \le \frac{\pi}{n} \right\},$$

$$D_4 = \left\{ \frac{2\pi}{n} < y \le \pi, \ y - \frac{\pi}{n} < x \le y \right\},$$

$$D_5 = \left\{ \frac{2\pi}{n} < y \le \pi, \ 0 < x \le \frac{\pi}{n} \right\},$$

$$D_6 = \left\{ \frac{2\pi}{n} < x \le \pi, \ \frac{\pi}{n} < y \le x - \frac{\pi}{n} \right\},$$

$$D_7 = \left\{ \frac{2\pi}{n} < y \le \pi, \ \frac{\pi}{n} < x \le y - \frac{\pi}{n} \right\}.$$

Then

$$\int_0^{\pi} \int_0^{\pi} \left| \sum_{k=n+1}^{2n} D_k(x) D_k(y) \right| dxdy$$

$$= \sum_{i=1}^{7} \int_{D_i} \int \left| \sum_{k=n+1}^{2n} D_k(x) D_k(y) \right| dxdy = \sum_{i=1}^{7} J_i. \quad (13)$$

To prove the lemma, it suffices to estimate $J_i$ $(i = 1, 2, 3, 4, 5, 6)$.

Note that

$$\left|\sum_{k=n+1}^{2n} D_k(x)D_k(y)\right| \leq |K_{2n}(x, y)| + |K_n(x, y)|$$

$$\leq \frac{1}{8\sin\frac{x}{2}\sin\frac{y}{n}}\left\{\frac{1}{\left|\sin\frac{x-y}{2}\right|} + \frac{1}{\left|\sin\frac{x+y}{2}\right|}\right\} \quad (14)$$

when $0 < x \pm y \leq 2\pi - \delta,\ \delta > 0,\ 0 < x,\ y \leq x.$

On account of the fact that

$$|D_k(u)| \leq K, \quad 0 < u \leq \pi, \quad (15)$$

we get

$$J_1 = \int_0^{\frac{2\pi}{n}}\int_0^{\frac{2\pi}{n}}\left|\sum_{k=n+1}^{2n} D_k(x)D_k(y)\right| dxdy \leq \int_0^{\frac{2\pi}{n}}\int_0^{\frac{2\pi}{n}}\sum_{k=n+1}^{2n} k^2 dxdy \leq 16\pi^2 n. \quad (16)$$

In what follows, we need the following identity:

$$\frac{1}{\pi(n+1)}\int_{-\pi}^{\pi}\sum_{k=0}^{n} D_k(u)du = \frac{1}{\pi(n+1)}\int_{-\pi}^{\pi}\left(\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}\right)^2 du = 1. \quad (17)$$

Applying the Abel transform and using (15) and (17), we obtain

$$J_2 = \int_{\frac{2\pi}{m}}^{\pi} dx \int_{x-\frac{\pi}{n}}^{x}\left|\sum_{k=n+1}^{2n} D_k(x)D_k(y)\right| dxdy$$

$$\leq \int_{\frac{2\pi}{n}}^{\pi} dx \int_{x-\frac{\pi}{n}}^{x}\left|\sum_{k=n+1}^{2n}[D_k(y) - D_{k+1}(y)]\sum_{i=0}^{k} D_i(x)\right| dy$$

$$+\int_{\frac{2\pi}{m}}^{\pi} dx \int_{x-\frac{\pi}{n}}^{x} \left| D_{2n}(y) \sum_{i=0}^{2n} D_i(x) \right| dy$$

$$+\int_{\frac{2\pi}{m}}^{\pi} dx \int_{x-\frac{\pi}{n}}^{x} \left| D_{n+1}(y) \sum_{i=0}^{n} D_i(x) \right| dy$$

$$= \int_{\frac{2\pi}{m}}^{\pi} dx \int_{x-\frac{\pi}{n}}^{x} \sum_{k=n+1}^{2n} |\cos ky| \pi(k+1) \frac{1}{\pi(k+1)} \sum_{i=0}^{k} D_i(x) dy$$

$$+\int_{\frac{2\pi}{m}}^{\pi} dx \int_{x-\frac{\pi}{n}}^{x} |D_{2n}(y)| \pi(2\pi+1) \cdot \frac{1}{\pi(2\pi+1)} \sum_{i=0}^{2n} D_i(x) dy$$

$$+\int_{\frac{2\pi}{n}}^{\pi} dx \int_{x-\frac{\pi}{n}}^{x} |D_{n+1}(y)| \pi(n+1) \cdot \frac{1}{\pi(n+1)} \sum_{i=0}^{n} D_i(x) dy$$

$$\leq \frac{\pi^2}{n} \sum_{k=n+1}^{2n} (k+1) + 2\pi^2(2n+1) + \frac{\pi^2(n+1)^2}{n} \leq 13\pi^2 n. \tag{18}$$

Applying the Abel transform and using (15), (17), we obtain

$$J_3 = \int_0^{\frac{\pi}{2}} dy \int_{\frac{2\pi}{n}}^{\pi} \left| \sum_{k=n+1}^{2n} D_k(x) D_k(y) \right| dx$$

$$\leq \int_0^{\frac{\pi}{n}} dy \int_{\frac{2\pi}{n}}^{\pi} \left| \sum_{k=n+1}^{2n} [D_k(x) - D_{k+1}(y)] \sum_{i=0}^{k} D_i(x) \right| dx$$

$$+\int_0^{\frac{\pi}{n}} dy \int_{\frac{2\pi}{n}}^{\pi} \left| D_{2n}(y) \sum_{i=0}^{2n} D_i(x) \right| dx + \int_0^{\frac{\pi}{n}} dy \int_{\frac{2\pi}{n}}^{\pi} \left| D_{n+1}(y) \sum_{i=0}^{n} D_i(x) \right| dx$$

$$\leq \int_0^{\frac{\pi}{n}} dy \int_{\frac{2\pi}{n}}^{\pi} \sum_{k=n+1}^{2n} |\cos ky| \pi(k+1) \frac{1}{\pi(k+1)} \sum_{i=0}^{k} D_i(x) dx$$

$$+\int_0^{\frac{\pi}{n}} dy \int_{\frac{2\pi}{n}}^{\pi} |D_{2n}(y)| \pi(2n+1) \frac{1}{\pi(2n+1)} \sum_{i=0}^{2n} D_i(x) dx$$

$$+\int_0^{\frac{\pi}{n}} dy \int_{\frac{2\pi}{n}}^{\pi} | D_{n+1}(y) | \pi(n+1) \cdot \frac{1}{\pi(n+1)} \sum_{i=0}^{n} D_i(x) dx$$

$$\leq \pi \cdot \frac{\pi}{n} \sum_{k=n+1}^{2n} (k+1) + 2\pi^2(2n+1) + \pi(n+1)^2 \cdot \frac{\pi}{n} \leq 13\pi^2 n. \tag{19}$$

By virtue of (14),

$$J_6 = \int_{\frac{2\pi}{n}}^{\pi} dx \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \left| \sum_{k=n+1}^{2n} D_k(x) D_k(y) \right| dxdy$$

$$\leq \frac{1}{8} \int_{\frac{2\pi}{n}}^{\pi} \frac{dx}{\sin \frac{x}{2}} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{\sin \frac{y}{n} \sin \frac{x-y}{2}}$$

$$+ \frac{1}{8} \int_{\frac{2\pi}{n}}^{\pi} \frac{dx}{\sin \frac{x}{2}} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{\sin \frac{y}{n} \sin \frac{x+y}{2}} = J_6^{(1)} + J_6^{(2)}. \tag{20}$$

Since $\sin t \geq \frac{2}{\pi} t,\ 0 < t \leq \frac{\pi}{2}$,

$$J_6^{(1)} \leq \frac{\pi^3}{8} \int_{\frac{2\pi}{n}}^{\pi} \frac{dx}{x} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{x(x-y)}$$

$$= \frac{\pi^3}{8} \int_{\frac{2\pi}{n}}^{\pi} \frac{1}{x^2} \left[ -\ln \frac{\pi}{n} + \ln\left( x - \frac{\pi}{n} \right) + \ln\left( x - \frac{\pi}{n} \right) - \ln \frac{\pi}{n} \right] dx$$

$$= J_6^{(2)} = \frac{\pi^3}{4} \int_{\frac{2\pi}{n}}^{\pi} \ln \frac{nx - \pi}{\pi} dx = \frac{\pi^3}{4} \int_1^{n-1} \frac{\ln t}{\left( \frac{\pi}{n} \right)^2 (t+1)^2} \cdot \frac{\pi}{n} dt$$

$$\leq \frac{\pi^2}{4} n \int_1^{\infty} \frac{\ln t}{(t+1)^2} dt \leq C \cdot n. \tag{21}$$

Now, consider $J_6^{(2)}$. We have

$$J_6^{(2)} = \frac{1}{8}\int_{\frac{2\pi}{n}}^{\pi} \frac{dx}{\sin\frac{x}{2}} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{\sin\frac{y}{2}\sin\frac{x+y}{2}}$$

$$+ \frac{1}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi} \frac{dx}{\sin\frac{x}{2}} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{\sin\frac{y}{2}\sin\frac{x+y}{2}} = j_1 + j_2. \tag{22}$$

Since $\sin t \geq \frac{2}{\pi}x$ for $0 < t \leq \frac{\pi}{2}$,

$$j_1 \leq \frac{\pi^3}{8}\int_{\frac{2\pi}{n}}^{\frac{\pi}{2}+\frac{\pi}{2n}} \frac{dx}{x} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{y(y+x)}$$

$$\leq \frac{\pi^3}{8}\int_{\frac{2\pi}{n}}^{\frac{\pi}{2}+\frac{\pi}{2n}} \frac{dx}{x} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{2y\sqrt{xy}} = \frac{\pi^3}{16}\int_{\frac{2\pi}{n}}^{\frac{\pi}{2}+\frac{\pi}{2n}} x^{-\frac{3}{2}}dx \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} y^{-\frac{3}{2}}dy$$

$$\leq \frac{\pi^3}{16}\int_{\frac{2\pi}{n}}^{\pi} x^{-\frac{3}{2}}dx \int_{\frac{\pi}{n}}^{\pi} y^{-\frac{3}{2}}dy = \frac{\pi^3}{16}\left(-\frac{1}{\sqrt{\pi}} + \frac{\sqrt{n}}{\sqrt{2\pi}}\right)^2 \leq \frac{\pi^2}{16\sqrt{2}} \cdot n. \tag{23}$$

For the second term in (22), we have

$$j_2 \leq \frac{\pi^2}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi} \frac{dx}{x} \int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}} \frac{dy}{y\sin\frac{x+y}{2}}. \tag{24}$$

Since, if $\frac{\pi}{2} \leq x \leq \pi$, $\frac{\pi}{n} \leq y \leq x - \frac{\pi}{n}$ $(n \geq 3)$, we have

$$\frac{\pi}{2} < \frac{x+y}{2} \leq \frac{\pi+y}{2} \leq \frac{\pi + x - \frac{\pi}{n}}{2} \leq \frac{2\pi - \frac{\pi}{n}}{2} = \pi - \frac{\pi}{2n}.$$

It follows that

$$\sin\frac{x+y}{2} \geq \sin\frac{\pi+y}{2}, \quad \frac{\pi}{2} \leq x \leq \pi, \quad \frac{\pi}{n} \leq y \leq x - \frac{\pi}{n}. \tag{25}$$

By virtue of (25), from (24), we obtain

$$j_2 \le \frac{\pi^2}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi}\int_{\frac{\pi}{n}}^{\pi}\frac{dx}{x}\int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}}\frac{dy}{y\sin\frac{\pi+y}{2}}$$

$$= \frac{\pi^2}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi}\frac{dx}{x}\int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}}\frac{dy}{y\sin\left(\pi-\frac{\pi+y}{2}\right)}$$

$$= \frac{\pi^2}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi}\frac{dx}{x}\int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}}\frac{dy}{y\sin\frac{\pi-y}{2}} \le \frac{\pi^3}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi}\frac{dx}{x}\int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}}\frac{dy}{y(\pi-y)}$$

$$= \frac{\pi^3}{8\pi}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi}\frac{dx}{x}\int_{\frac{\pi}{n}}^{x-\frac{\pi}{n}}\left(\frac{1}{y}+\frac{1}{\pi-y}\right)dy$$

$$= \frac{\pi^2}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi}\frac{dx}{x}\left[\ln\frac{xn-\pi}{\pi}-\ln\left(\pi-x+\frac{\pi}{n}\right)+\ln\left(\pi-\frac{\pi}{n}\right)\right]$$

$$= \frac{\pi^2}{8}\int_{\frac{\pi}{2}+\frac{\pi}{2n}}^{\pi}\frac{1}{x}\ln\frac{(xn-\pi)\left(1-\frac{1}{n}\right)}{\pi-x+\frac{\pi}{n}}dx$$

$$\le \frac{\pi^2}{8}\ln\frac{(n-1)\frac{\pi}{2}\left(1-\frac{1}{n}\right)}{\frac{\pi}{n}}\int_{\frac{\pi}{2}}^{\pi}\frac{dx}{x}$$

$$= \frac{\pi^2}{8}\ln(n-1)^2\ln 2 \le \frac{\pi^2\ln 2}{4}\cdot n = c\cdot n. \tag{26}$$

From (22), (23) and (26), it follows that

$$J_6^{(2)} \le \frac{\pi^2 n}{16\sqrt{2}} + \frac{\pi^2\ln 2}{4}n \le Cn. \tag{27}$$

Now, it follows from (13), (16), (18), (19), (20), (21) and (27) that

$$\sum_{i=1}^{7} J_i \leq Cn,$$

$$\frac{1}{n}\sum_{i=1}^{7} J_i = \int_0^{\pi}\int_0^{\pi}\left|\frac{1}{n}\sum_{k=n+1}^{2n} D_k(x)D_k(y)\right| dxdy \leq C.$$

The lemma is proved.

**Proof of Theorem 1**

We prove the theorem for case $k = 2$. For $k \geq 3$, the method of the proof is similar to that for the case when $k = 2$.

Let $T_{m,n}(x, y)$ be a trigonometric polynomial that implements the best approximation of a function $f(x, y)$ of order *m* by *x*, order *n* by *y*.

Then, we have

$$f(x, y) - \frac{1}{n}\sum_{k=n+1}^{2n} S_{k,k}(f; x, y)$$

$$= \frac{1}{n}\sum_{k=n+1}^{2n} [f(x, y) - S_{k,k}(f; x, y)]$$

$$= \frac{1}{n}\sum_{k=n+1}^{2n} \left\{ f(x, y) - \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s, y+t)D_k(s)D_k(t)dsdt \right\}.$$

Since

$$T_{n,n}(x, y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} T_{n,n}(x+s, y+t)D_k(s)D_k(t)dsdt,$$

by virtue of $\frac{1}{\pi}\int_{-\pi}^{\pi} D_k(u)du = 1,$ we find

$$\frac{1}{n}\sum_{k=n+1}^{2n}\Big\{f(x, y) - \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x+s, y+t)$$

$$- T_{n+1, n+1}(x+s, y+t)]D_k(s)D_k(t)dsdt - T_{n+1, n+1}(x, y)\Big\}$$

$$= \frac{1}{n}\sum_{k=n+1}^{2n}\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{[f(x, y) - T_{n+1, n+1}(x, y)]$$

$$- [f(x+s, y+t) - T_{n+1, n+1}(x, y)]\}D_k(s)D_k(t)dsdt$$

$$= \frac{1}{n}\sum_{k=n+1}^{2n}\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{4[f(x, y) - T_{n+1, n+1}(x, y)]$$

$$- [f(x+s, y+t) - T_{n+1, n+1}(x+s, y+t) + f(x-s, y+t)]$$

$$- T_{n+1, n+1}(x+s, y+t) + f(x+s, y-t)$$

$$- T_{n+1, n+1}(x+s, y-t) + f(x-s, y-t)$$

$$- T_{n+1, n+1}(x-s, y-t)]\}D_k(s)D_k(t)dsdt$$

$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\varphi_{n+1, n+1}(x, y, s, t)\frac{1}{n}\sum_{k=n+1}^{2n}D_k(s)D_k(t)dsdt, \tag{28}$$

where

$$\varphi_{n+1, n+1}(x, y, s, t) = 4[f(x, y) - T_{n+1, n+1}(x, y)]$$

$$- [f(x+s, y+t) - T_{n+1, n+1}(x+s, y+t)$$

$$+ f(x-s, y+t) - T_{n+1, n+1}(x-s, y+t)$$

$$+ f(x+s, y-t) - T_{n+1, n+1}(x+s, y-t)$$

$$+ f(x-s, y-t) - T_{n+1, n+1}(x-s, y-t)].$$

Applying the generalized Minkowski inequality, we obtain (see (28)):

$$R_1(f)_p = \left\| f(x,\ y) - \frac{1}{n} \sum_{k=n+1}^{2n} S_{k,k}(f;\ x,\ y) \right\|_{L_p}$$

$$\leq \frac{1}{\pi^2} \int_0^{\pi}\int_0^{\pi} \left| \frac{1}{n} \sum_{k=n+1}^{2n} D_k(s) D_k(t) \right| \|\varphi_{n+1,\,n+1}(x,\ y,\ s,\ t)\|_{L_p} dsdt. \quad (29)$$

But

$$\| \varphi_{n+1,\,n+1}(x,\ y,\ s,\ t) \|_{L_p} \leq 8E_{n+1,\,n+1}(f)_{L_p}. \quad (30)$$

From (29), (30) and by virtue of the lemma, we obtain

$$R(f)_{L_p} \leq \frac{1}{\pi^2} C \cdot 8E_{n+1,\,n+1}(f)_{LP} \leq CE_{n,\,n}(f)_{LP}.$$

This proves the theorem.

**Proof of Theorem 2**

The proof is carried out for case $k = 2$. For $k \geq 3$, the method of proof is similar.

Let $2^m \leq n < 2^{m+1}$. Then

$$R_2(f)_p = \left\| f(x,\ y) - \frac{1}{n+1} \sum_{k=0}^{n} S_{k,k}(f;\ x,\ y) \right\|_{L_p}$$

$$= \left\| \frac{1}{n+1} \sum_{k=0}^{n} [f(x,\ y) - S_{k,k}(f;\ x,\ y) \right\|_{L_p}$$

$$= \left\| \frac{1}{n+1} \left\{ \sum_{\nu=0}^{m} \sum_{k=2^{\nu}}^{2^{\nu+1}-1} [f(x,\ y) - S_{k,k}(f;\ x,\ y)] + f(x,\ y) \right.\right.$$

$$\left.\left. - S_{0,0}(f;\ x,\ y) + \sum_{k=2^m}^{n} [f(x,\ y) - S_{k,k}(f;\ x,\ y)] \right\} \right\|_{LP}$$

$$\leq \left\| \frac{1}{n+1} \left\{ \sum_{\nu=0}^{m} 2^{\nu} \cdot \frac{1}{2^{\nu}} \sum_{k=2^{\nu}}^{2^{\nu+1}-1} [f(x, y) - S_{k,k}(f; x, y)] \right\} \right\|_{L_p}$$

$$+ \frac{1}{n+1} \| f(x, y) - S_{0,0}(f; x, y) \|_{L_P}$$

$$+ \frac{1}{n+1} \left\| \sum_{k=2^m}^{n} [f(x, y) - S_{k,k}(f; x, y)] \right\|_{L_p}. \tag{31}$$

By virtue of Theorem 1,

$$\left\| \frac{1}{2^{\nu}} \sum_{k=2^{\nu}}^{2^{\nu+1}-1} [f(x, y) - S_{k,k}(f; x, y)] \right\|_{L_p} \leq C E_{2^{\nu}, 2^{\nu}}(f)_{L_p} \tag{32}$$

and

$$\left\| \sum_{k=2^{\nu}}^{n} [f(x, y) - S_{k,k}(f; x, y)] \right\|_{L_p} \leq C(n - 2^m) E_{2^m, 2^m}(f)_{L_p}. \tag{33}$$

From (31), (32) and (33), we obtain

$$\rho_n(f)_{L_P} \leq \frac{C}{n+1} \sum_{\nu=0}^{m} 2^{\nu} E_{2^{\nu}, 2^{\nu}}(f)_{L_p}$$

$$+ \frac{1}{n+1} E_{0,0}(f)_{L_p} + \frac{C(n - 2^m)}{n+1} E_{2^m, 2^m}(f)_{L_p}$$

$$\leq \frac{C}{n+1} \sum_{k=1}^{2^m} E_{k,k}(f)_{L_p} + \frac{1}{n+1} E_{0,0}(f)_{L_p}$$

$$\leq \frac{C}{n+1} \sum_{k=1}^{2^m} E_{k,k}(f)_{L_p} \leq \frac{C}{n+1} \sum_{k=0}^{n} E_{k,k}(f)_{L_p}.$$

The theorem is thus proved.

**Proof of Theorem 3**

It is enough to prove for $k = 2$. For $k \geq 3$, the method of the proof is similar to that for the case when $k = 2$.

Using the Abel transform, we have (see, e.g., [8, p. 15]):

$$f(x, y) - (1-r)\sum_{k=0}^{\infty} r^k S_{k,k}(f; x, y)$$

$$= f(x, y) - (1-r)^2 \sum_{k=0}^{\infty} r^k \sum_{\nu=0}^{k} S_{k,k}(f; x, y)$$

$$= f(x, y) - (1-r)^2 \sum_{k=0}^{\infty} r^k (k+1) \cdot \frac{1}{k+1} \sum_{\nu=0}^{k} S_{\nu,\nu}(f; x, y)$$

$$= f(x, y) - (1-r)^2 \sum_{k=0}^{\infty} r^k (k+1)\sigma_k(f; x, y)$$

$$= (1-r)^2 \sum_{k=0}^{\infty} r^k (k+1)[f(x, y) - \sigma_k(f; x, y)].$$

Hence

$$R_3(f)_p = \| f(x, y) - \sigma(f; r; x, y) \|_{L_p}$$

$$= \left\| f(x, y) - (1-r)\sum_{k=0}^{\infty} r^k S_{k,k}(f; x, y) \right\|_{L_p}$$

$$\leq (1-r)^2 \sum_{k=0}^{\infty} r^k (k+1) \| f(x, y) - \sigma_k(f; x, y) \|_{L_p}.$$

By Theorem 2, we obtain

$$R_3(f)_p = (1-r)^2 \sum_{k=0}^{\infty} r^k (k+1) \frac{C}{k+1} \sum_{\nu=0}^{k} E_{\nu,\nu}(f)_{L_p}$$

$$= C(1-r)^2 \sum_{k=0}^{\infty} E_{\nu,\nu}(f)_{L_p} \sum_{k=\nu}^{\infty} r^k = C(1-r) \sum_{k=0}^{\infty} r^{\nu} E_{\nu,\nu}(f)_{L_p}.$$

The theorem is thus proved.

Other questions of approximation of functions of one variable in spaces $L_p(-\infty; \infty)(0 < p \leq \infty)$ and $H_p(-\infty; \infty)$ were considered by us in [9, 10].

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