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Coefficient determination problem in the system of integro-differential equation for visco-elastic porous medium

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Abstract. We consider a system of hyperbolic integro-differential equations for visco-elastic porous medium. In this paper we look for a Lame coefficient, for this we will pass a new variable then will use the method of separation of the singularities and have got the new initial-boundary problem. We apply the principle of contraction mappings to this system in the space of continuous functions with a ordinary norm. The theorem of local solvability of the classical solution to the problem is proved and the stability estimate of a solution to the inverse problem is obtained.

Keywords: integro-differential equation, inverse problem, Dirac's delta function, hyperbolic equation, Lame coefficient, Riccati equation, implicit function theorem, Gronuoll-Bellman lemma

MSC (2010): 35L20; 35R30; 35Q99

1 Problem Formulation

We consider a system of integro-differential hyperbolic equations in the domain $\mathbb{R}^2_+ := \{(z,t) : z \in \mathbb{R},\, t > 0\}$

$$\rho_s \frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial z}\Sigma(z,t) - \chi\rho_l^2\Big(\frac{\partial U}{\partial t} - \frac{\partial V}{\partial t}\Big), \tag{1.1}$$

$$\frac{\partial V}{\partial t} = \chi\rho_l\Big(U - V\Big). \tag{1.2}$$

with the initial and boundary conditions

$$U|_{t\le 0} = \frac{\partial U}{\partial t}\Big|_{t\le 0} \equiv 0, \quad V|_{t\le 0} \equiv 0, \quad \Sigma(+0,t) = \delta'(t), \tag{1.3}$$

here, $U(z,t)$ and $V(z,t)$ are the velocities of elastic porous body with a constant partial density ρ_s, and of the fluid with a constant partial density ρ_l, χ is coefficient of intercomponent friction (assumed to be constant and positive everywhere in this paper), and $\delta'(t)$ is the derivative of the Dirac delta function. The tension $\Sigma(z,t)$ is related to $U(z,t)$ by the formula

$$\Sigma(z,t) = \mu(z)\frac{\partial U}{\partial z} + \mu(z)\int\limits_0^t k(t-\tau)\frac{\partial U}{\partial z}(z,\tau)d\tau, \tag{1.4}$$

where $k : \mathbf{R} \to \mathbf{R}$ is a given function, characterizing viscosity of the medium and in $C^2[0,\infty)$ satisfying $k(t) = 0$ for any $t < 0$, $\mu(z)$ is the Lame coefficient.

The problem of determining U and V from (1.1)-(1.4) with given ρ_s, ρ_l, χ, $\mu(z)$, and $k(t)$ is called the direct problem for porous medium.

The following inverse problem can be formulated for (1.1)-(1.4): to find $\mu(z) \in C^2[0,\infty)$ in (1.1) through (1.4) if for the solution of the direct problem, we know that

$$U\Big|_{z=0} = f(t), \tag{1.5}$$

where $f(t)$ is a given function.

In other words, the task is to find $\mu(z)$ by known coefficients ρ_s, ρ_l, χ and function $k(t)$. Indeed, if one finds $\mu(z)$, then will find the functions U and V as a solution to (1.1)-(1.2) and (1.3) as well. This is why the inverse problem is often formulated as the problem of finding the functions $\mu(z)$, $U(z,t)$ and $V(z,t)$.

This work is continuation of the article [6]. Equations (1.1), (1.2), and (1.4) describe the propagation of elastic SH (transverse) waves in a porous medium with memory for the one-dimensional case (in terms of spatial variables). We note that the direct and inverse dynamical problems for these equations for $k(t) \equiv 0$ were studied in [4], [10]. We also note that in the absence of porosity $\rho_l = 0, V(z,t) = 0$, inverse problem $(1.1)-(1.5)$ becomes completely analogous to the problem studied in [8]. The main feature, appropriate to [9]-[7] and to the present work, is the use of a source localized on the boundary of the considered space domain; this source initiates the physical process of wave transmission.

Now, let us return to the main problem. We introduce a new variable x according to the equality

$$x = \tau(z) := \int_0^z \frac{d\xi}{c_t(\xi)}, \qquad c_t(z) := \sqrt{\frac{\mu(z)}{\rho_s}}, \quad \lambda := \frac{\chi\rho_l^2}{\rho_s}.$$

Let $\tau^{-1}(x)$ denote the function inverse to $\tau(z)$ and set:

$$\upsilon(x,t) := U(\tau^{-1}(x),t), \quad \vartheta(x,t) := V(\tau^{-1}(x),t), \quad \sigma(x) := \sqrt{\rho_s\mu(x)}.$$

Next, we are interested in the classical solution of the initial-boundary value problem (1.1)-(1.5), i.e., $\upsilon \in C^{2,2}(D_T)$, $\vartheta \in C^{0,1}(D_T)$, $D_T = \{(x,t) : 0 \le x \le t \le T - x\}$, where $C^{k,m}(D_T)$ is the space of k times continuously differentiable functions with respect to x, and m times continuously differentiable functions with respect to t.

Equalities $(1.1)-(1.5)$ with respect to the new functions $\upsilon(x,t)$, $\vartheta(x,t)$ and the variable x is rewritten in the form

$$\frac{\partial^2 \upsilon}{\partial t^2} = \Big(\frac{\partial^2}{\partial x^2} + \frac{\sigma'(x)}{\sigma(x)}\frac{\partial}{\partial x}\Big)\Big[\upsilon(x,t) + \int_0^t k(t-\tau)\upsilon(x,\tau)d\tau\Big] - \lambda\frac{\partial}{\partial t}\Big(\upsilon - \vartheta\Big), \tag{1.6}$$

$$\frac{\partial \vartheta}{\partial t} = \chi\rho_l\big(\upsilon - \vartheta\big),\ x > 0, t > 0 \tag{1.7}$$

with the initial and boundary conditions

$$\upsilon|_{t\leq 0} = \frac{\partial \upsilon}{\partial t}\Big|_{t\leq 0} \equiv 0, \quad \vartheta|_{t\leq 0} \equiv 0, \tag{1.8}$$

$$\big[\sigma(x)\frac{\partial \upsilon}{\partial x} + \int\limits_0^t k(t-\tau)\sigma(x)\frac{\partial \upsilon}{\partial x}(x,\tau)d\tau\big]_{x=+0} = \delta^{'}(t), \tag{1.9}$$

$$\upsilon|_{x=+0} = f(t), \quad t > 0. \tag{1.10}$$

Now we transform integro-differential equation (1.6) such that, first, there are no derivatives of the function υ with respect to x in the integrand and, second, the coefficients of υ_x and υ_t in terms outside the integral are equal to zero. These requirements can be satisfied by introducing a new function W by the formula

$$\Big[\upsilon(x,t) + \int\limits_0^t k(t-\tau)\upsilon(x,\tau)d\tau\Big] \exp\Big(\frac{\lambda - k(0)}{2}t\Big)\sqrt{\sigma(x)} = W(x,t). \tag{1.11}$$

It is easy to verify by direct calculation that the function υ is then expressed in terms of W as

$$\upsilon(x,t) = \Big[\exp\Big(\frac{k(0)-\lambda}{2}t\Big)W(x,t) + \int\limits_0^t r(t-\tau)\exp\Big(\frac{k(0)-\lambda}{2}\tau\Big)W(x,\tau)d\tau\Big]\frac{1}{\sqrt{\sigma(x)}},$$

where

$$r(t) = -k(t) - \int\limits_0^t k(t-\tau)r(\tau)d\tau. \tag{1.12}$$

With the new functions $W(x,t)$ and $r(t)$, we rewrite Eqs. (1.6)-(1.10) in the form

$$\frac{\partial^2 W}{\partial t^2} = \frac{\partial^2 W}{\partial x^2} + H(x)W - \int\limits_0^t h(t-\tau)W(x,\tau)d\tau - \int\limits_0^t q(t-\tau)W(x,\tau)d\tau -$$

$$- \int\limits_0^t g(t-\tau)W(x,\tau)d\tau - \omega(x,t)\vartheta, \tag{1.13}$$

$$\vartheta(x,t) = \frac{\chi\rho_l}{\sqrt{\sigma(x)}}\int_0^t \exp\big(\chi\rho_l(\tau - t)\big)\Big[\exp((k(0)-\lambda)\tau/2)W(x,\tau) +$$

$$+\int_0^{\tau} r(\tau - s)\exp((k(0)-\lambda)s/2)W(x,s)ds\Big]d\tau, \qquad (1.14)$$

$$W|_{t\le 0}\equiv 0, \quad W|_{x=+0} = f_0(t) + \int_0^t k_0(t-\tau)f_0(\tau)d\tau, \qquad (1.15)$$

$$\frac{\partial W}{\partial x}\Big|_{x=+0} = \delta^{'}(t) - \frac{r(0)}{2}\delta(t) + \frac{\sigma^{'}(0)}{2}W(0,t), \qquad (1.16)$$

where

$$H(x) := k^{'}(0) - \frac{3}{4}\big(k(0)\big)^2 + \frac{\lambda k(0)}{2} - \frac{3}{4}\lambda^2 - \frac{1}{2}\frac{\sigma^{''}(x)}{\sigma(x)} + \frac{1}{4}\Big(\frac{\sigma^{'}(x)}{\sigma(x)}\Big)^2 + \lambda\cdot\chi\rho_l, \quad (1.17)$$

$$f_0(t) := \exp\Big(\frac{\lambda + r(0)}{2}t\Big)f(t), \;\; k_0(t) := \exp\Big(\frac{\lambda + r(0)}{2}t\Big)k(t),$$

$$h(t) := r^{''}(t)\exp\Big(\frac{\lambda + r(0)}{2}t\Big), \qquad q(t) := -\lambda\exp\Big(\frac{\lambda + r(0)}{2}t\Big)r^{'}(t),$$

$$g(t) := -\lambda\cdot\chi\rho_l\exp\Big(\frac{\lambda + r(0)}{2}t\Big)r(t), \qquad \omega(x,t) := \lambda\cdot\chi\rho_l\sqrt{\sigma(x)}\exp\Big(\frac{\lambda + r(0)}{2}t\Big).$$

In (1.16), we used the equality $k(0) = -r(0)$, which follows from (1.12).

Hyperbolic equation theory implies that the function $W(x,t)$ as the solution of direct problem (1.13)-(1.16) has the property $W\equiv 0, \; t<x, \; x>0$ and in the neighborhood of the characteristic $t=x$ has the structure:

$$W(x,t) = \alpha(x)\delta(t-x) + \theta(t-x)p(x,t), \qquad (1.18)$$

where $p(x,t)$ is the regular function.

For solvability of the inverse problem in accordance with (1.18) the function $f(t)$ must have the form

$$f(t) = -\delta(t) + \theta(t)f_{00}(t), \qquad (1.19)$$

where $f_{00}(t)$ is the given regular function.

We set $\beta(x) := p(x,x+0)$. Substituting function (1.18) in (1.13)-(1.16) and using the method of separation of the singularities [2], we obtain $2\alpha'(x) = 0, \; \alpha(0) = -1, \; -\alpha^{''}(x) + 2\beta'(x) - H(x)\alpha(x) = 0, \; \beta(0) = \alpha'(0) + \frac{r(0)}{2} + \frac{\sigma'(0)}{2}$. Solving these ordinary differential equations, we obtain

$$\alpha(x) = -1, \; \beta(x) = \beta(0) - \frac{1}{2}\int_0^x H(\xi)d\xi. \qquad (1.20)$$

It follows from the above that the function $p(x,t)$ in the domain $D := \{(x,t) : t > x > 0\}$ satisfies the equations

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} + H(x)p(x,t) + \eta(t-x) - \int\limits_x^t \eta(t-\tau)p(x,\tau)d\tau - \omega(x,t)\vartheta(x,t), \quad (1.21)$$

$$\vartheta(x,t) = \vartheta_0(x,t) + g_2(x)\int\limits_x^t e^{\chi\rho_l(\tau-t)}e^{(k(0)-\lambda)\tau/2}p(x,\tau)d\tau +$$

$$+g_2(x)\int\limits_0^t\int\limits_x^\tau r(\tau-s)e^{\chi\rho_l(\tau-t)}e^{(k(0)-\lambda)s/2}p(x,s)dsd\tau, \quad (1.22)$$

$$p\Big|_{t=x+0} = \beta(x), \quad (1.23)$$

$$p\Big|_{x=+0} = \tilde{f}_0(t) - k_0(t) + \int\limits_0^t k_0(t-\tau)\tilde{f}_0(\tau)d\tau =: \varrho(t), \quad (1.24)$$

$$\frac{\partial p}{\partial x}\Big|_{x=+0} = \frac{\sigma'(0)}{2}\varrho(t), \quad (1.25)$$

where $\eta(t) := h(t) + g(t) + q(t)$,

$$g_1(x) = -\frac{\chi\rho_l}{\sqrt{\sigma(x)}}e^{\frac{k(0)-\lambda}{2}x}, \ g_2(x) = \frac{\chi\rho_l}{\sqrt{\sigma(x)}}, \ \tilde{f}_0(t) := \exp((\lambda + r(0))t/2)f_0(t),$$

$$\vartheta_0(x,t) := g_1(x)e^{\chi\rho_l(x-t)} + g_1(x)\int\limits_0^t r(\tau-x)e^{\chi\rho_l(\tau-t)}d\tau.$$

We impose the continuity condition for the functions $p(x,t)$ $\frac{\partial p}{\partial x}(x,t)$ at $x = t = 0$. From relations (1.21)-(1.25), we can easily express $r(0)$ and $r'(0)$ in terms of the known data:

$$r(0) = \lambda + \sigma'(0) - 2\tilde{f}_0(0), \ r^{'}(0) = -\tilde{f}_0'(0) + \big(\tilde{f}_0(0) + \frac{1}{2}r(0)\big)r(0), \quad (1.26)$$

To obtain the last equality for $r'(0)$, we used the relation $k'(0) = -r'(0) + r^2(0)$, which follows from (1.12). Further, we assume that instead of $r(0)$ and $r'(0)$, their values are substituted in $H(x)$.

Proposition 1.1. *Definition of $H(x)$ is equivalent the definition of $\mu(x)$.*

Proof. From the relation of (1.17), we can rewrite the following form:

$$H(x) = H_0 + \frac{\sigma'^2(x) - 2\sigma(x)\sigma''(x)}{4\sigma^2(x)}, \tag{1.27}$$

where $H_0 = k'(0) - \frac{3}{4}k^2(0) + \frac{1}{2}\lambda k(0) - \frac{3}{4}\lambda^2 + \lambda\chi\rho_l$. Hence, $\mu(x)$ is in the class $C^2[0,\infty)$.

The last equality is equivalent to the following Cauchy problem for Riccati equation

$$\begin{cases} y' = -\frac{1}{4}y^2 + H_1(x) \\ y(0) = y_0, \end{cases}, \tag{1.28}$$

where $H_1(x) := 4(H_0 - H(x))$, $y(x) = \mu'(x)/\mu(x)$, $y_0 := 4\tilde{f}_0(0) - 2k(0)$. Generally speaking, the Riccati equation does not integrable the quadrature, but we can show the existence and of uniqueness solution in a small neighborhood of a point $(0, y_0)$ of (1.28) by the implicit function theorem [16] or the existence and uniqueness theorem of Cauchy [14].

Let, solution is represented by $\widetilde{y}(x)$ and here $\widetilde{y}(0) = y_0$, $\widetilde{y} \in C^2(0, y_0)$.

Then, from differential equation,

$$\frac{\mu'(x)}{\mu(x)} = \widetilde{y}(x)$$

we get

$$\mu(x) = \frac{1}{\rho_s} \exp\Big(\int_0^x \widetilde{y}(\xi)d\xi\Big)$$

here $\widetilde{y}(x)$ is related to $H(x)$ by integral form.

The inverse preposition was defined by (1.27), i.e., if $\mu(x)$ is given, than $H(x)$ will defined one-to-one.

So, define $H(x)$ is equivalent to define $\mu(x)$.

2 Reducing the inverse problem to an equivalent system of integral equation

Let $D_T = \{(x,t) : 0 \le x \le t \le T - x\}$.

Lemma 2.1. *Let $T > 0$ be sufficiently small number and the following conditions hold:*
i) the function $f(t)$ has the form (1.19);
ii) $f_{00}(t) \in C^1[0,T]$;

Then, problem (1.21)-(1.25) for $(x,t) \in D_T$ is equivalent to the following problem of finding functions $p(x,t)$, $H(x)$ and $\overline{\vartheta}(x,t)$ from the system of integral equations

$$p(x,t) = p_0(x,t) - \frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} \overline{\omega}(\tau)\overline{\vartheta}(\xi,\tau)d\tau d\xi + \frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} H(\xi)p(\xi,\tau)d\tau d\xi -$$

$$-\frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} \int_\xi^\tau \eta(\tau-l)p(\xi,l)dl d\tau d\xi, \tag{2.1}$$

$$\overline{\vartheta}(x,t) = \overline{\vartheta}_0(x,t) + \chi\rho_l \int_x^t e^{\chi\rho_l(\tau-t)}e^{(k(0)-\lambda)\tau/2}p(x,\tau)d\tau +$$

$$+\chi\rho_l \int_0^t \int_x^\tau r(\tau-s)e^{\chi\rho_l(\tau-t)}e^{(k(0)-\lambda)s/2}p(x,s)ds d\tau, \tag{2.2}$$

$$H(x) = -2\varrho'(2x) - \sigma'(0)\varrho(2x) + 2\int_0^x \eta(2x-2\xi)d\xi + 2\Big\{\int_0^x \overline{\omega}(2x-\xi)\overline{\vartheta}(\xi,2x-\xi)d\xi -$$

$$-\int_0^x [H(\xi)+H(2x-\xi)]p(\xi,2x-\xi)d\xi + 2\int_0^x \int_\xi^{2x-\xi} \eta(2x-\xi-l)p(\xi,l)dl d\xi\Big\}, \tag{2.3}$$

here $\overline{\omega}(t)\cdot\overline{\vartheta}(x,t) = \omega(x,t)\cdot\vartheta(x,t)$.

Proof. Using by formula d'Alembert, we obtain the following solution

$$p_0(x,t) = \frac{1}{2}\big(\varrho(t+x)+\varrho(t-x)\big) + \frac{\sigma'(0)}{4}\int_{t-x}^{t+x} \varrho(\tau)d\tau + \frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} \eta(\tau-\xi)d\tau d\xi,$$

for this problem

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\Big)p_0 = \eta(t-x), \tag{2.4}$$

$$p_0\Big|_{x=+0} = \varrho(t), \quad \frac{\partial p_0}{\partial x}\Big|_{x=+0} = \frac{\sigma'(0)}{2}\varrho(t), \tag{2.5}$$

Exactly the same way, rewrite the (1.20) equation and (1.23), (1.24) conditions and using by d'Alembert formula, we will get the integral equation (2.1). Moreover (2.2) is obtains in simplest way by using $\overline{\omega}(t)\cdot\overline{\vartheta}(x,t) = \omega(x,t)\cdot\vartheta(x,t)$.

Differentiating the condition (1.23) by x and letting $t = x$, after that using the equality (1.20) instead of $\beta(x)$, we obtain

$$\varrho'(2x) + \frac{\sigma'(0)}{2}\varrho(2x) + \int_0^x \eta(2x-2\xi)d\xi - \int_0^x \varpi(2x-\xi)\overline{\vartheta}(\xi, 2x-\xi)d\xi +$$

$$+ \int_0^x [H(\xi)+H(2x-\xi)]p(\xi, 2x-\xi)d\xi - 2\int_0^x \int_\xi^{2x-\xi} \eta(2x-\xi-\alpha)p(\xi,\alpha)d\alpha d\xi = -\frac{1}{2}H(x)$$

and here we came to equation (2.3).

We state the equivalence of system of integral equations (2.1)-(2.3) to the problem (1.21)-(1.25) in the ordinary way [6]. The lemma is proved.

3 Main results and their proofs

Consider in the domain D_T equalities (2.1), (2.2) and (2.3) which determine the system of nonliner integral second-order equations with respect to the functions p, $\overline{\vartheta}$ and H. This system of equations is of a small parameter, which is expressed through the measure of the integration domain in them. Due to the presence of this small parameter, one can apply to the system, in a sufficiently small domain, the principle of contracted mapping. Indeed, write the system of equations as an operator equation

$$\varphi = A\varphi, \tag{3.1}$$

where φ is the vector function of two variables x, t with the components φ_1, φ_2, and φ_3, in which case

$$\varphi = [\varphi_1(x,t), \varphi_2(x,t), \varphi_3(x)] = [p(x,t), \overline{\vartheta}(x,t), H(x)]$$

and the operator A is determined in the set of functions $\varphi \in C[D_T]$ and, in line with equalities (2.1), (2.2) and (2.3), has the form $A = (A_1, A_2, A_3)$:

$$A_1\varphi = \varphi_{01} - \frac{1}{2}\Big\{ \int_0^x \int_{t-x+\xi}^{t+x-\xi} \varpi(\tau)\varphi_2(\xi,\tau)d\tau d\xi - \int_0^x \int_{t-x+\xi}^{t+x-\xi} \varphi_3(\xi)\varphi_1(\xi,\tau)d\tau d\xi +$$

$$+ \int_0^x \int_{t-x+\xi}^{t+x-\xi} \int_\xi^\tau \eta(\tau-l)\varphi_1(\xi,l)dl d\tau d\xi \Big\}, \tag{3.2}$$

$$A_2\varphi = \varphi_{02} + \chi\rho_l[\int_x^t e^{\chi\rho_l(\tau-t)} e^{(k(0)-\lambda)\tau/2}\varphi_1(x,\tau)d\tau +$$

$$+\int_0^t \int_x^\tau r(\tau-s)e^{\chi\rho_l(\tau-t)}e^{(k(0)-\lambda)s/2}\varphi_1(x,s)dsd\tau], \tag{3.3}$$

$$A_3\varphi=\varphi_{03}+2\Big\{\int_0^x \overline{\omega}(2x-\xi)\varphi_2(\xi,2x-\xi)d\xi-\int_0^x[\varphi_3(\xi)+\varphi_3(2x-\xi)]\varphi_1(\xi,2x-\xi)d\xi+$$

$$+2\int_0^x \int_\xi^{2x-\xi}\eta(2x-\xi-l)\varphi_1(\xi,l)dld\xi\Big\}. \tag{3.4}$$

Denote

$$\|\varphi\|(T)=\max_{1\le k\le 3}\max_{(x,t)\in D_T}|\varphi_k(x,t)|$$

$$\varphi_0(x,t)=[\varphi_{01},\varphi_{02},\varphi_{03}]:=$$

$$=\Big[p_0(x,t),\overline{\vartheta}_0(x,t),-2\varrho'(2x)-\sigma'(0)\varrho(2x)+2\int_0^x\eta(2x-2\xi)d\xi\Big], \tag{3.5}$$

and consider in the space $C(D_{T_0})$, $0<T_0\le T$ the set $B(\varphi_0,\|\varphi_0\|)$ of functions $\varphi(x,t)$, which can obey the inequality

$$\|\varphi-\varphi_0\|(T_0)\le\|\varphi_0\|(T)$$

It can be demonstrated that at sufficiently small T_0 the operator A realizes contracted mapping of the set $B(\varphi_0,\|\varphi_0\|)$ onto itself. Indeed, for $\varphi\in B(\varphi_0,\|\varphi_0\|)$ the inequality

$$\|\varphi\|(T_0)\le 2\|\varphi_0\|(T)$$

is valid. On the other hand, by way of estimating the integrals in (3.2)-(3.4) one gets

$$\|A_1\varphi-\varphi_{01}\|=\max_{(x,t)\in D_T}|A_1\varphi-\varphi_{01}|\le\|\varphi_0\|T^2\Big[\frac{\overline{\omega}_0}{4}+\frac{\|\varphi_0\|}{2}+\frac{4}{3}\eta_0T\Big]:=\gamma_1\|\varphi_0\|,$$

$$\|A_2\varphi-\varphi_{02}\|=\max_{(x,t)\in D_T}|A_2\varphi-\varphi_{02}|\le\|\varphi_0\|\Big[2e^{|k(0)-\lambda|T/2}\big(e^{\chi\rho_lT}-1\big)(1+r_0)\Big]:=\gamma_2\|\varphi_0\|,$$

$$\|A_3\varphi-\varphi_{03}\|=\max_{x\in[0,T/2]}|A_3\varphi-\varphi_{03}|\le\|\varphi_0\|\|\varphi_0\|[2\overline{\omega}_0T+8\|\varphi_0\|T+2\eta_0T^2]:=\gamma_3\|\varphi_0\|$$

where $\overline{\omega}_0:=\max_{t\in[0,T]}|\overline{\omega}(t)|$, $\eta_0:=\max_{t\in[0,T]}|\eta(t)|$, $r_0:=\max_{t\in[0,T]}|r(t)|$.

The operator A self-maps a ball to the same ball, i.e., $A\varphi\in B(\varphi_0,\|\varphi_0\|)$, if for all T the following inequality holds:

$$\gamma:=\max_{1\le k\le 3}\gamma_i<1$$

Now let φ^1 and φ^2 be two arbitrary elements in $A\varphi \in B(\varphi_0, \|\varphi_0\|)$. In this case, using the obvious inequalities

$$|\varphi_k^{(1)}\varphi_s^{(1)} - \varphi_k^{(2)}\varphi_s^{(2)}| \leq |\varphi_k^{(1)} - \varphi_k^{(2)}||\varphi_s^{(1)}| + |\varphi_k^{(2)}||\varphi_s^{(1)} - \varphi_s^{(2)}| \leq 4\|\varphi_0\|\|\varphi^{(1)} - \varphi^{(2)}\|$$

after some easy estimations, we find that for $(x,t) \in D_T$,

$$\|A\varphi^1 - A\varphi^2\| \leq \frac{\gamma}{2} \cdot \|\varphi^1 - \varphi^2\|$$

hold. Therefore, the operator A is contracting on $B(\varphi_0, \|\varphi_0\|)$. By the Banach fixed-point theorem, Eq. (3.1) is then solvable and has a unique solution in $B(\varphi_0, \|\varphi_0\|)$ for sufficiently small $T > 0$.

The following theorem is valid.

Theorem 3.1. *Assume that the function $f(t)$ is represented in the form*

$$f(t) = -\delta(t) + \theta(t) f_{00}(t),$$

where $f_{00}(t) \in C[0,T]$, and $\theta(t)$ is the Heaviside function. Furthermore, let $k(t) \in C^2[0,T]$. Then there exists an unique solution of inverse problem (1.1)-(1.5), in the class $\mu(z) \in C^2[0, \tau^{-1}(T_0/2)]$ at sufficiently small $T_0 > 0$, where $T_0 \in (0, T^*)$, $T^* = \min\{T, y_0\}$ and y_0 is given bellow.

Denote

$$\mathbf{U}(x) = \max_{(\xi,\tau)\in D_T} |\widetilde{p}(\xi,\tau)|,\ \mathbf{V}(x) = \max_{(\xi,\tau)\in D_T} |\widetilde{\overline{\vartheta}}(\xi,\tau)|,$$

$$\mathbf{Q}(x) = \max(|\widetilde{H}(x)|, |\widetilde{H}(2x - T/2)|),\ \|u\|_k = \sum_{|\alpha|\leq k} \max_{(x,t)\in D_T} |L^\alpha u(x,t)|,\ i = 0,1,2,3$$

$$L^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial t^{\alpha_2}},\ \alpha = (\alpha_1, \alpha_2),\ |\alpha| = \alpha_1 + \alpha_2.$$

Consider one further theorem characterizing the estimation of conventional stability of the inverse problem solution. Such an estimate can be obtain if we set a certain class of data $M(K,T)$ for the functions p_0, $\overline{\vartheta}_0$ and a class $Q(M,T)$ for the function $H(x)$. Assume that p_0 and $\overline{\vartheta}_0$ belong to the class $M(K,T)$ if they obey the inequalities

$$\|p_0\|_{C^{2,2}(D_T)} \leq K,\ \|\overline{\vartheta}_0\|_{C^{2,3}(D_T)} \leq K$$

having universal for the whole class positive constant K. Analogously, $H(x) \in Q(M,T)$ if

$$\|H\|_{C^2[0,\tau^{-1}(T_0/2)]} \leq M.$$

Theorem 3.2. *Let $H^{(1)}(x)$, $H^{(2)}(x)$ be two solutions to the inverse problem (1.1)-(1.5) with the data $\varrho^{(1)}(t)$ and $\varrho^{(2)}(t)$, respectively. Also, $H^{(1)}$, $H^{(2)} \in Q(M,T)$ and $\varrho^{(1)}$, $\varrho^{(2)} \in M(K,T)$. Then, the estimate*

$$\|H^{(1)} - H^{(2)}\|_{C^2[0,\tau^{-1}(T^*/2)]} \leq C\|\varrho^{(1)} - \varrho^{(2)}\|_{C^2[0,T]}, \tag{3.6}$$

is valid; the constant C depending here only on the choice of classes $M(K,T)$, $Q(K,T)$.

Proof. Let $p^{(1)}$, $p^{(2)}$ denote the solutions to (1.1)-(1.5) corresponding to the functions $H^{(1)}(x)$, $H^{(2)}(x)$. If the difference between two functions, whose only difference in notation is the over bar, is denoted by the same latter with a tilde ($\sim$), for instance, $\widetilde{p} = p^{(1)} - p^{(2)}$, $\widetilde{H} = H^{(1)} - H^{(2)}$ and etc., then equations (2.1), (2.2) and (2.3) give the following system of equalities

$$\widetilde{p}(x,t) = \widetilde{p}_0(x,t) - \frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{t+x-\xi} [\widetilde{\overline{\omega}}(\tau)\overline{\vartheta}^{(1)}(\xi,\tau) + \overline{\omega}^{(2)}(\tau)\widetilde{\overline{\vartheta}}(\xi,\tau)]d\tau d\xi +$$

$$+\frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{t+x-\xi} [\widetilde{H}(\xi)p^{(1)}(\xi,\tau) + H^{(2)}(\xi)\widetilde{p}(\xi,\tau)]d\tau d\xi -$$

$$-\frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{t+x-\xi} \int\limits_{\xi}^{\tau} [\widetilde{\eta}(\tau - l)p^{(1)}(\xi,l) + \eta^{(2)}(\tau - l)\widetilde{p}(\xi,l)]dl d\tau d\xi,$$

$$\widetilde{\overline{\vartheta}}(x,t) = \widetilde{\overline{\vartheta}}_0(x,t) + \chi\rho_l \int\limits_x^t e^{\chi\rho_l(\tau-t)}e^{(k(0)-\lambda)\tau/2}\widetilde{p}(x,\tau)d\tau +$$

$$+\chi\rho_l \int\limits_0^t \int\limits_x^{\tau} e^{\chi\rho_l(\tau-t)}e^{(k(0)-\lambda)s/2}[\widetilde{r}(\tau - s)p^{(1)}(x,s) + r^{(2)}(\tau - s)\widetilde{p}(x,s)]ds d\tau, \tag{3.7}$$

$$\widetilde{H}(x) = -2\widetilde{\varrho}'(2x) - \sigma'(0)\widetilde{\varrho}(2x) + 2\int\limits_0^x \widetilde{\eta}(2x - 2\xi)d\xi + 2\Big\{\int\limits_0^x [\widetilde{\overline{\omega}}(2x-\xi)\overline{\vartheta}^{(1)}(\xi, 2x-\xi) +$$

$$+\overline{\omega}^{(2)}(2x - \xi)\widetilde{\overline{\vartheta}}(\xi, 2x - \xi)]d\xi - \int\limits_0^x [\widetilde{H}(\xi)p^{(1)}(\xi, 2x - \xi) + H^{(2)}(\xi)\widetilde{p}(\xi, 2x - \xi)]d\xi -$$

$$-\int\limits_0^x [\widetilde{H}(2x - \xi)p^{(1)}(\xi, 2x - \xi) + H^{(2)}(2x - \xi)\widetilde{p}(\xi, 2x - \xi)]d\xi +$$

$$+2\int_0^x \int_\xi^{2x-\xi} [\widetilde{\eta}(2x-\xi-l)p^{(1)}(\xi,l) + +\eta^{(2)}(2x-\xi-l)\widetilde{p}(\xi,l)]dld\xi\Big\}.$$

Consider this system with respect to the functions $\widetilde{p}(x,t)$, $\widetilde{\overline{\vartheta}}(x,t)$ and $\widetilde{H}$. Note that the functions $\widetilde{p}_0$, $\overline{\vartheta}^{(2)}$, $\widetilde{\overline{\vartheta}}_0$, $H(x)$ and $H^{(2)}(x)$ included in in can be estimated on the basis of a *priori* information on the problem data. Indeed, there is the obvious estimate

$$\|\widetilde{p}\|_2 \leq C_1\|\widetilde{\varrho}\|_2, \tag{3.8}$$

and

$$\|\widetilde{\overline{\vartheta}}_0\|_2 \leq C_2[\chi\rho_l \exp(|k(0)-\lambda|T/4)] := C_2K.$$

By the same way, we get the following estimates:

$$\|p^{(2)}\|_0 \leq C_3K, \quad \|\overline{\vartheta}^{(2)}\|_0 \leq C_4K. \tag{3.9}$$

Here the constants C_i, $i = 1,2,3,4$, are only $T-$dependent.

Then, from relations (3.7) making use of inequality (3.9), and one gets

$$\mathbf{U}(x) \leq \|\widetilde{p}_0\|_0+\frac{1}{4}T^2K\|\widetilde{\omega}\|_0+\frac{2}{3}T^3K\|\widetilde{\eta}_0\|+\frac{1}{4}T^2K\mathbf{V}(x)+\frac{1}{4}T^2K\mathbf{Q}(x)+\frac{1}{2}KT^2\int_0^x \mathbf{U}(\xi)d\xi,$$

$$\mathbf{V}(x) \leq \|\widetilde{\overline{\vartheta}}\|_0+\|\widetilde{r}\|_2TKe^{|k(0)-\lambda|T/2}[e^{\xi\rho_lT}-1]+(KT+1)e^{|k(0)-\lambda|T/2}[e^{\xi\rho_lT}-1]\mathbf{U}(x)$$

$$\mathbf{Q}(x) \leq \|\widetilde{p'}_0\|_0 + \frac{1}{2}T^2K\|\widetilde{\eta}\|_0 + TK\mathbf{V}(x) + (2TM+\frac{1}{2}T^2K)\mathbf{U}(x) + 2K\int_0^x \mathbf{Q}(\xi)d\xi,$$

To demonstrate that these inequalities result is estimate (3.6) let

$$\mathbf{W}(x) = \max\Big(\mathbf{U}(x),\ \mathbf{V}(x),\ \mathbf{Q}(x)\Big).$$

Then one has

$$\mathbf{U}(x) \leq C_{00} + \frac{1}{2}KT^2\int_0^x \mathbf{W}(\xi)d\xi,$$

$$\mathbf{Q}(x) \leq C_{01} + \Big(T^3M + \frac{1}{4}T^4K + 2\Big)K\int_0^x \mathbf{W}(\xi)d\xi,$$

here C_{00}, $C_{01}-$are related to the given constants.

Hence,

$$\mathbf{W}(x) \leq W_0 + \lambda\int_0^x W(\xi)d\xi, \tag{3.10}$$

where,

$$W_0 = \max\{C_{00},\, C_{01}\},\ \lambda = \max\left\{\frac{1}{2}KT^2,\ \left(T^3M + \frac{1}{4}T^4K + 2\right)K\right\}$$

By Gronuoll-Bellman lemma the inequality (3.10) implies $\mathbf{W}(x) \leq W_0 e^{\lambda x}$. Now taking into account the inequality (3.8) we obtain the estimate (3.6). $\square$

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