

O`ZBEKISTON RESPUBLIKASI
OLY VA O`RTA MAXSUS TA`LIM VAZIRLIGI
FARG`ONA POLITEKNIKA INSTITUTI
“OLY MATEMATIKA” KAFEDRASI

**“OLY MATEMATIKA” FANINING
“KARRALI, EGRI CHIZIQLI VA SIRT INTEGRALLARI”
MAVZUSI BO`YICHA**

200000 – «Ijtimoiy soha, iqtisod va huquq»,
310000«Muhandislik va muhandislik ishi»
yo`nalishi bo`yicha bakalavriat talabalari uchun

USLUBIY QO`LLANMA

Farg`ona – 2019 y.

Mazkur uslubiy qo'llanma mustaqil ta'limni bajarish uchun talabalarga mo'ljallangan bo'lib, unda har bir topshiriq uchun kerakli nazariy ma'lumotlar va formulalar keltirilgan. Mustaqil ta'lim bo'yicha bir nechta na'munaviy topshiriqlar yechib ko'rsatilgan, talabalar mustaqil bajarishi uchun topshiriq variantlari berilgan va shu bilan birga talabalar o'z bilimlarini sinashi uchun nazorat ishlari variantlari keltirilgan.

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Mazkur uslubiy qo'llanma "Oliy matematika" kafedrasida ko'rib chiqilgan.
Bayonnoma № _____ 2019 y.
"Ishlab chiqarishda boshqaruv" fakulteti uslubiy kengashida muhokama qilingan va ma'qullagan.

Bayonnoma № _____ 2019 y.

FarPI uslubiy kengashi
tomonidan tasdiqlangan va
chop etilgan
" _____ " _____ 2019y

Kirish

Bozor iqtisodiyoti ilm-fan va texnikaning amalda tezroq va ko`proq qo`llanishini taqozo etadi. Bu masalani matematika sohasida hal etish o`ta muhim va dolzarbdir. Chunki matematika fani barcha fanlar uchun mustahkam poydevor va asosdir. Talabalar matematikadan nazariy va amaliy bilimlarini mustahkamlashi, ularning matematik fikrlashi uchun ko`proq mustaqil masalalarni yechishi va ularni amaliyotdagi ahamiyatini tushunishi ularni bilim samaradorligini yuqori bo`lishiga olib keladi.

Tavsiya etilayotgan uslubiy qo`llanma “Oliy matematika” fanining “Karrali, egri chiziqli va sirt integrallari” qismi bo`yicha bajarilgan bo`lib, oliy o`quv yurti talabalari auditoriyasi uchun mo`ljallangan.

Uslubiy qo`llanmada “Karrali, egri chiziqli va sirt integrallari” mavzulariga oid qisqacha nazariy qismining bayoni berilib, unga doir masalalar yechib ko`rsatilgan va mustaqil yechish uchun masalalar berilgan.

Uslubiy qo`llanmadan 200000 – «Ijtimoiy soha, iqtisod va xuquq», 310000«Muhandislik va muhandislik ishi» yo`nalishi bo`yicha bakalavriat talabalari foydalanishi mumkin.

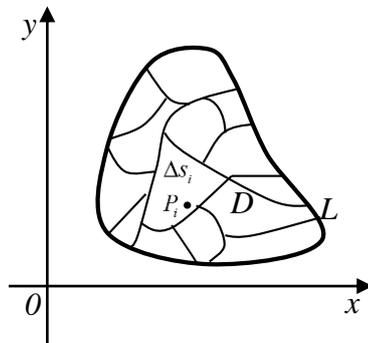
Karrali integrallar

Oxy tekislikda L chiziq bilan chegaralangan D yopiq sohani qaraymiz.

D sohada uzluksiz funksiya $z = f(x, y)$ berilgan bo'lsin. D sohani ixtiyoriy chiziqlar bilan n ta bo'lakka bo'lamiz: $\Delta s_1, \Delta s_2, \dots, \Delta s_n$

Ularni yuzachalar deb ataymiz. Yangi simvollar kiritmaslik maqsadida $\Delta s_1, \Delta s_2, \dots, \Delta s_n$ orqali bularning nomlarinigina emas, yuzalarni ham belgilaymiz. Δs_i yuzalarning har birida P_i nuqta olamiz (bu nuqta yuzaning ichida yoki chegarasida yotishining farqi yo'q), bunda n ta nuqta hosil bo'ladi: P_1, P_2, \dots, P_n .

Funksiyaning tanlangan nuqtalardagi qiymatlarini $f(P_1), f(P_2), \dots, f(P_n)$ bilan belgilaymiz va $f(P_i)\Delta s_i$ ko'rinishdagi ko'paytmalarning yig'indisini tuzamiz:



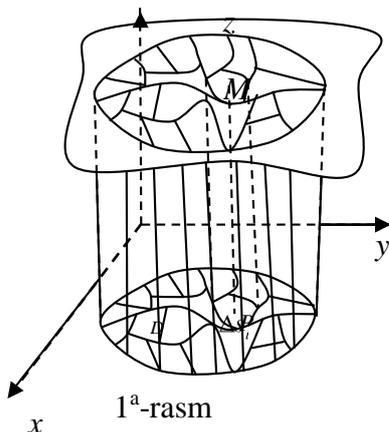
1-rasm

$$V_n = f(P_1)\Delta s_1 + f(P_2)\Delta s_2 + \dots + f(P_n)\Delta s_n = \sum_{i=1}^n f(P_i)\Delta s_i \quad (1)$$

bu yig'indi D sohada $f(x, y)$ funksiya uchun integral yig'indi deb ataladi.

Agar D sohada $f \geq 0$ bo'lsa, u holda har bir $f(P_i)\Delta s_i$ qo'shiluvchini, geometrik jihatdan asosi Δs_i ga, balandligi esa $f(P_i)$ gat eng bo'lgan silindrning hajmi deb qarash mumkin.

V_n yig'indi, ko'rsatilgan elementar silindrchalarning hajmlarining yig'indisi, ya'ni qandaydir "pog'onali" jismning hajmi bo'ladi.



1^a-rasm

Berilgan D soha uchun $f(x, y)$ funksiya yordami bilan tuzilgan integral yig'indilarning D sohani Δs_i bo'laklarga turli usullar bilan bo'lishdan hosil qilingan ixtiyoriy

$$V_{n_1}, V_{n_2}, \dots, V_{n_k}, \dots \quad (2)$$

ketma-ketlikni qaraymiz. $n_k \rightarrow \infty$ da Δs_i yuzalarning maksimal diametri nolga intiladi deb faraz etamiz.

1-teorema. Agar $f(x, y)$ funksiya yopiq D sohada uzluksiz va $n_k \rightarrow \infty$ da Δs_i yuzaning maksimal diametri nolga intilsa, (1) integral yig'indilardan hosil bo'lgan (2) ketma-ketlikning limiti mavjud bo'ladi. Bu limit (2) shakldagi har qanday ketma-ketlik uchun bir xildir, ya'ni u D sohani Δs_i yuzaga bo'lish usuliga va bu Δs_i yuza ichida P_i nuqtani tanlab olish usuliga bog'liq emas.

Bu limitni $f(x, y)$ funksiyaning D soha bo'yicha olingan ikki o'lchovli integrali deyiladi va quyidagicha belgilanadi:

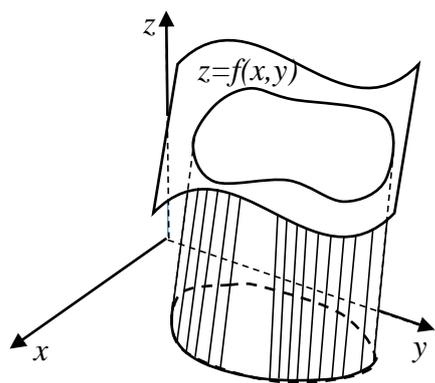
$$\iint_D f(P) ds \quad \text{yoki} \quad \iint_D f(x, y) dx dy$$

ya'ni:

$$\lim_{\text{diam} \Delta s_i \rightarrow 0} \sum_{i=1}^n f(P_i) \Delta s_i = \iint_D f(x, y) dx dy$$

Bu yerda D soha *integrallash sohasi* deyiladi.

Agar $f(x, y) \geq 0$ bo'lsa, $f(x, y)$ funksiyaning D soha bo'yicha olingan ikki o'lchovli integrali $z = f(x, y)$ sirt $z = 0$ tekislik va yasovchisi Oz o'qqa parallel, yo'naltiruvchisi esa D sohaning chegarasidan iborat bo'lgan silindrik sirt bilan chegaralangan jismning hajmiga tengdir.



1^b-rasm

Ikki karrali integrallarning xossalari

$$\iint_{\sigma} kf(x, y)d\sigma = k \iint_{\sigma} f(x, y)d\sigma .$$

$$2. \iint_{\sigma} [f(x, y) + \varphi(x, y)]d\sigma = \iint_{\sigma} f(x, y)d\sigma + \iint_{\sigma} \varphi(x, y)d\sigma .$$

3. Agar σ integrallash sohasida $f(x, y)$ va $\varphi(x, y)$ funksiyalar $f(x, y) \geq \varphi(x, y)$ tengsizlikni qanoatlantirsa, u holda

$$\iint_{\sigma} f(x, y)d\sigma \geq \iint_{\sigma} \varphi(x, y)d\sigma$$

4. O'rta qiymat haqidagi teorema. Aytaylik σ chegaralangan yopiq sohada $f(x, y)$ funksiya uzluksiz bo'lsa, u holda shu sohada Shunday $P_0(x_0, y_0)$ nuqta mavjudki, bu nuqtada $\iint_{\sigma} f(x, y)d\sigma = f(x_0, y_0)\sigma$ o'rinli bo'ladi.

Additivlik xossasi. Agar integrallash sohasini $\sigma_1, \sigma_2, \dots, \sigma_k$ kabi bir necha sohalarga bo'lish mumkin bo'lsa, quyidagini yozish mumkin

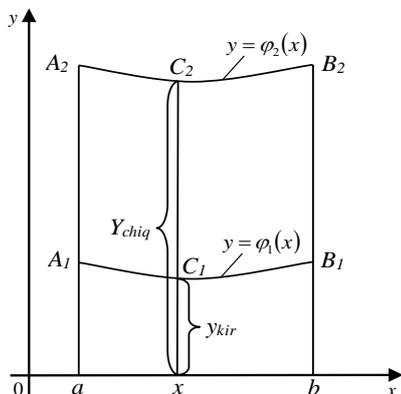
$$\iint_{\sigma} f(x, y)d\sigma = \iint_{\sigma_1} f(x, y)d\sigma + \iint_{\sigma_2} f(x, y)d\sigma + \dots + \iint_{\sigma_k} f(x, y)d\sigma .$$

Dekart koordinatalar sistemasida ikki karrali integrallarni hisoblash

$f(x, y)$ uzluksiz funksiyaning $\iint_{\sigma} f(x, y)d\sigma$ ikki karrali integralni hisoblash talab etilgan bo'lsin.

Aytaylik σ integrallash sohasi $y = \varphi_1(x)$ va $y = \varphi_2(x)$ egri chiziqlar va shu bilan birga $x=a$, $x=b$ to'g'ri chiziqlar bilan chegaralangan bo'lsin, bundan tashqari a va b oralig'idagi barcha x lar uchun $\varphi_2(x) \geq \varphi_1(x)$ shart bajarilsin.

$(x; 0)$ nuqtadan Oy o'qqa parallel to'g'ri chiziq o'tkazamiz. Bu to'g'ri chiziq egri chiziqlarni mos ravishda C_1 va C_2 nuqtalarda kesib o'tadi.

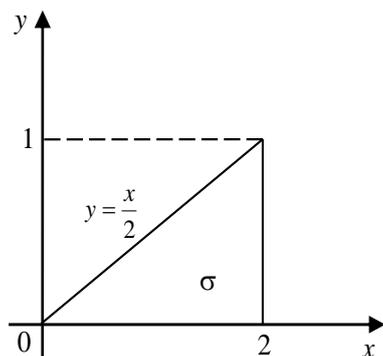


2-rasm

$$\iint_{\sigma} f(x, y)d\sigma = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy, \quad (3)$$

$$\text{yoki } \iint_{\sigma} f(x, y) d\sigma = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \quad (4)$$

Misol 1. $y=0$, $x=2$, $y=\frac{x}{2}$ uchburchak bilan chegaralangan sohada $\iint_{\sigma} (x^2 + y^2) d\sigma$ ikki karrali integral hisoblansin(3-rasm).



3-rasm

Yechish. Yuqoridagi (3) formula bo'yicha hisoblasak, u holda $y_{chiq} = \phi_1(x) = 0$, $y_{kir} = \phi(x) = \frac{x}{2}$, $a=0$, $b=2$.

$$\iint_{\sigma} (x^2 + y^2) d\sigma = \int_0^2 dx \int_0^{\frac{x}{2}} (x^2 + y^2) dy.$$

$$\int_0^{\frac{x}{2}} (x^2 + y^2) dy = \left(x^2 y + \frac{y^3}{3} \right) \Big|_0^{\frac{x}{2}} = x^2 \cdot \frac{x}{2} + \frac{\left(\frac{x}{2}\right)^3}{3} = \frac{13}{24} x^3.$$

$$\text{Demak, } \iint_{\sigma} (x^2 + y^2) d\sigma = \int_0^2 \frac{13}{24} x^3 dx = \frac{13x^4}{24 \cdot 4} \Big|_0^2 = \frac{13}{6}.$$

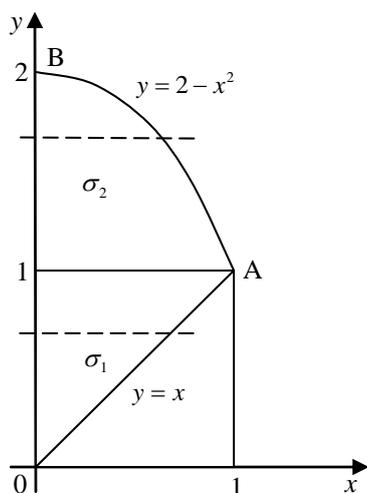
Shu kabi $\iint_{\sigma} (x^2 + y^2) d\sigma$ uchun (4) ni qo'llasak ham ushbu natijaga kelamiz. Bu

holda $x_{kir} = \phi_1(y) = 2y$, $x_{chiq} = \phi_2(y) = 2$, $c=0$, $d=1$ bo'lib, $\iint_{\sigma} (x^2 + y^2) d\sigma = \int_0^1 dy \int_{2y}^2 (x^2 + y^2) dx$ ga

ega bo'lamiz. $\int_{2y}^2 (x^2 + y^2) dx = \left(\frac{x^3}{3} + y^2 x \right) \Big|_{2y}^2 = \left(\frac{8}{3} + 2y^2 \right) - \left(\frac{8y^3}{3} + 2y^2 \right) = \frac{8}{3} + 2y^2 - \frac{14}{3} y^3$,

$$\text{va } \iint_{\sigma} (x^2 + y^2) d\sigma = \int_0^1 \left(\frac{8}{3} + 2y^2 - \frac{14}{3} y^3 \right) dy = \left(\frac{8}{3} y + \frac{2y^3}{3} - \frac{7y^4}{6} \right) \Big|_0^1 = \frac{13}{6}.$$

Misol 2. σ integrallash sohasi $x=0$, $y=x$, $y=2-x^2$ chiziqlar bilan chegaralangan. Quyidagi integral hisoblansin $\iint_{\sigma} xy^2 d\sigma$, (4-rasm).



4-rasm

Yechish. Ikki karrali integralni hioblash uchun (3) formulani qo'llaymiz. Bu yerda $y_{kir} = \phi_1(x) = x$, $y_{chiq} = \phi_2(x) = 2 - x^2$, $a=0$, $b=1$. Shuning uchun

$$\iint_{\sigma} xy^2 d\sigma = \int_0^1 dx \int_x^{2-x^2} xy^2 dy.$$

$$\int_x^{2-x^2} xy^2 dy = \frac{xy^3}{3} \Big|_x^{2-x^2} = \frac{x(2-x^2)^3}{3} - \frac{x^4}{3}.$$

$$\text{Demak, } \iint_{\sigma} xy^2 d\sigma = \int_0^1 \left[\frac{x(2-x^2)^3}{3} - \frac{x^4}{3} \right] dx =$$

$$= -\frac{1}{6} \int_0^1 (2-x^2)^3 d(2-x^2) - \frac{1}{3} \int_0^1 x^4 dx = -\frac{1}{6} \frac{(2-x^2)^4}{4} \Big|_0^1 - \frac{x^5}{15} \Big|_0^1 = -\frac{1}{24} + \frac{2^4}{24} - \frac{1}{15} = \frac{67}{120}.$$

Agar ushbu $\iint_{\sigma} xy^2 d\sigma$ integralni (4) formula bilan yechmoqchi bo'lsak, σ sohani ikkita σ_1 va σ_2 sohalarga ajratish kerak. Natijada integral quyidagi ko'rinishga keladi: $\iint_{\sigma} xy^2 d\sigma = \iint_{\sigma_1} xy^2 d\sigma + \iint_{\sigma_2} xy^2 d\sigma$.

Har bir soha uchun integralni hisoblasak: $\iint_{\sigma_1} xy^2 d\sigma = \int_0^1 dy \int_0^y xy^2 dx$, bu yerda

$$x = \phi_1(y) = 0, \quad x = \phi_2(y) = y, \quad c=0, \quad d=1.$$

Demak:

$$\int_0^y xy^2 dx = \frac{x^2 y^2}{2} \Big|_0^y = \frac{y^4}{2}.$$

$$\text{So'ng, } \iint_{\sigma_1} xy^2 d\sigma = \int_0^1 \frac{y^4}{2} dy = \frac{1}{10}.$$

Huddi shu kabi quyidagini hisoblaymiz $\iint_{\sigma_2} xy^2 = \int_1^2 dy \int_0^{\sqrt{2-y}} xy^2 dx$, bu yerda $x=0$,

$$x = \sqrt{2-y}, c=1, d=2.$$

$$\int_0^{\sqrt{2-y}} xy^2 dx = \frac{x^2 y^2}{2} \Big|_0^{\sqrt{2-y}} = \frac{(2-y)y^2}{2} = y^2 - \frac{y^3}{2}.$$

$$\text{Demak, } \iint_{\sigma_2} xy^2 d\sigma = \int_1^2 (y^2 - \frac{y^3}{2}) dy = \frac{y^3}{3} - \frac{y^4}{8} \Big|_1^2 = \frac{11}{24}.$$

$$\text{va nihoyat, } \iint_{\sigma} xy^2 d\sigma = \frac{1}{10} + \frac{11}{24} = \frac{67}{120}.$$

Mustaqil hisoblang:

	Misol	javobi
1.	$\iint_D (x^2 + 2y) dx dy$, bu yerda $D - (0 \leq x \leq 1, 0 \leq y \leq 2)$ to'g'ri to'rtburchak.	$\frac{14}{3}$.
2.	$\iint_D \frac{dx dz}{(x+z)^2}$, bu yerda $D - (3 \leq x \leq 4, 1 \leq z \leq 2)$ to'g'ri to'rtburchak .	$\ln \frac{25}{24}$.
3.	$\iint_D xy dx dy$, bu yerda $D -$ uchlari $O(0;0), A(0;1), B(1;0)$ bo'lgan uchburchak.	$\frac{1}{24}$.
4.	$\iint_D 2y dx dy$, bu yerda $D - y = \sqrt{x}$ parabola va $y=0, x+y=2$ to'g'ri chiziqlar bilan chegaralangan.	$\frac{5}{6}$.
5.	$\iint_D y^2 z^{-2} dy dz$, bu yerda $D - y=2, z=y$ to'g'ri chiziq va $z = \frac{1}{y}$ giperbola bilan chegaralangan.	$\frac{9}{4}$.
6.	$\iint_D (x+y) dx dy$, bu yerda $D - y=2x, y=0$ va $y=4-2x$ to'g'ri chiziqlar bilan chegaralangan.	$3\frac{1}{3}$
7.	$\iint_D (x^2 + y) dx dy$, bu yerda $D - y=x^2, y=0$ va $x=2$ chiziqlar bilan chegaralangan.	$\frac{48}{5}$
8.	$\iint_D xy^2 dx dy$, bu yerda $D - y=2x, y=-x$ va $x=2$ chiziqlar bilan chegaralangan.	19,2

9.	$\iint_D \frac{dx dy}{\sin x}$, bu yerda D $x = \frac{\pi}{2}$, $x = \pi$, $y = 0$ va $y = \cos x$ chiziqlar bilan chegaralangan.	$\ln 2$
10.	$\iint_D x^2 e^{-x} dx dy$, bu yerda D $x = 2$, $x = 3$, $y = 0$ va $y = e^x$ chiziqlar bilan chegaralangan.	$\frac{19}{3}$
11.	$\iint_D dx dy$, bu yerda D $y = 2$, $y = 4$, $y = \ln x$ va $x = 0$ chiziqlar bilan chegaralangan.	$e^4 - e^2$
12.	$\iint_D y \cos xy dx dy$, bu yerda D $y = \pi$, $y = 3\pi$, $x = \frac{1}{2}$ va $x = 1$ chiziqlar bilan chegaralangan.	0
13.	$\iint_D \left(6x^2 y^2 + \frac{25}{3} x^4 y^4 \right) dx dy$, bu yerda D $x = 1$, $y = x^2$, $y = -\sqrt{x}$ chiziqlar bilan chegaralangan.	1

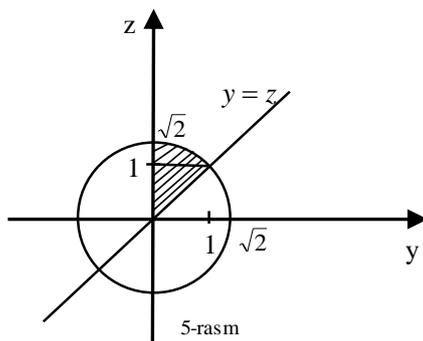
Karrali integrallarda integrallash tartibini o'zgartirish

Misol 3. Integrallash tartibini o'zgartiring

$$\int_0^1 dy \int_y^{\sqrt{2-y^2}} f(y, z) dz.$$

Yechish.

D soha yOz tekisligida $y = 0$ va $y = 1$ to'g'ri chiziqlar orasida joylashgan, uning chegaralari $z = y$, yuqorisi: $z = \sqrt{2-y^2}$. D sohani Oz o'qqa proyeksiyalaylik. Natijada $[0, \sqrt{2}]$ oraliqqa ega bo'lamiz. D sohaning chap chegarasi $y = 0$ to'g'ri



chiziq,

chap chegarasi $[0,1]$ oraliqda - $y = z$ tog'ri chiziq, $[1, \sqrt{2}]$ oraliqda - $y = \sqrt{2 - z^2}$ aylana yoyi. Shuning uchun D sohani ikki qismga ajratamiz (D_1 va D_2), integralning o'zi ham ikkita integral yig'indisidan iborat bo'ladi:

$$\int_0^1 dy \int_y^{\sqrt{2-y^2}} f(y, z) dz = \int_0^1 dz \int_0^z f(y, z) dy + \int_1^{\sqrt{2}} dz \int_0^{\sqrt{2-z^2}} f(y, z) dy.$$

Mustaqil hisoblang (integrallash chegarasini o'zgartiring):

	Misol	javobi
1	$\int_0^1 dx \int_{x^3}^{x^{-2}} f(x, y) dy.$	$\int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} f(x, y) dx.$
2	$\int_0^2 dx \int_x^{2x} f(x, z) dz.$	$\int_0^2 dy \int_{\frac{y}{2}}^y f(x, y) dx + \int_2^4 dy \int_{\frac{y}{2}}^2 f(x, y) dx.$
3	$\int_0^1 dy \int_{e^{-y}}^{e^y} f(y, z) dz.$	$\int_{\frac{1}{e}}^1 dz \int_{\ln \frac{1}{z}}^1 f(y, z) dy + \int_1^e dz \int_{\ln z}^1 f(y, z) dy.$
4	$\int_0^1 dy \int_y^{2-y} f(x, y) dx.$	$\int_0^1 dx \int_0^x f(x, y) dy + \int_1^2 dx \int_0^{2-x} f(x, y) dy.$
5	$\int_1^4 dx \int_0^{\sqrt{x}} f dy.$	$\int_0^1 dy \int_1^4 f dx + \int_1^2 dy \int_{y^2}^4 f dx.$
6	$\int_{-2}^{-\sqrt{3}} dx \int_0^{\sqrt{4-x^2}} f dy + \int_{-\sqrt{3}}^0 dx \int_0^{2-\sqrt{4-x^2}} f dy.$	$\int_0^1 dy \int_{-\sqrt{4-y^2}}^{-\sqrt{4-(y-2)^2}} f dx.$

Yassi soha yuzasini hisoblash

xOy tekislikda joylashgan D sohaning S yuzasi

$$S = \iint_D dx dy. \quad (4)$$

formula yordamida hisoblanadi

Misol 4.

$y = 2$ to'g'ri chiziq va $y = x^2 - 1$ parabola bilan chegaralangan D yassi sohaning yuzasi hisoblansin

Yechish.

D sohani Ox va Oy o'qlarga proyeksiyalash mumkin; Oy o'qqa proyeksiyalaylik. D soha Oy o'qqa nisbatan simmetrik, shuning uchun D sohaning o'ng tomonda turgan yuzini hisoblab, ikkiga ko'paytirish mumkin. D sohaning o'ng tomonini Oy o'qqa $[-1, 2]$ kesmaga proyeksiyalasak, quyidagilarga ega bo'lamiz. Chap tomondan $x = 0$ to'g'ri chiziq bilan, o'ng tomondan $y = x^2 - 1$, yoki $x = \sqrt{y+1}$ chiziq bilan chegaralangan. Natijada:

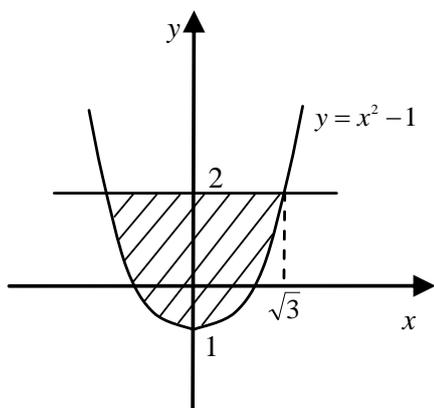
$$\frac{S}{2} = \int_{-1}^2 dy \int_0^{\sqrt{y+1}} dx = \int_{-1}^2 x \Big|_0^{\sqrt{y+1}} dy = \int_{-1}^2 \sqrt{y+1} dy = \frac{2}{3} (y+1)^{\frac{3}{2}} \Big|_{-1}^2 = 2\sqrt{3},$$

$$\text{va } S = 4\sqrt{3}.$$

$$\text{Yoki } \frac{S}{2} = \int_0^{\sqrt{3}} dx \int_{x^2-1}^2 dy = \int_0^{\sqrt{3}} (y|_{x^2-1}^2) dx = \int_0^{\sqrt{3}} (2 - x^2 + 1) dx =$$

$$= \int_0^{\sqrt{3}} (3 - x^2) dx = 3x \Big|_0^{\sqrt{3}} - \frac{x^3}{3} \Big|_0^{\sqrt{3}} = 3\sqrt{3} - \frac{3\sqrt{3}}{3} = 2\sqrt{3} \Rightarrow$$

$$\Rightarrow S = 4\sqrt{3} \text{ (kv. birlik)}$$



6-rasm

Mustaqil hisoblang:

Quyidagi chiziqlar bilan chegaralangan yassi figura yuzini hisoblang:

№	Misol	javobi (kv.birlik.)
1.	$y = 0, x = 1$ to'g'ri chiziqlar va $y = x^3$.	$\frac{1}{4}$
2.	$x = 0, z = 0, x = 2$ to'g'ri chiziqlar va $z = e^x$.	$e^2 - 1$

3.	$y=0, z=1, z=3$ to'gri chiziqlar va $z=\frac{1}{y}$ giperbola.	$\ln 3$
4.	$z^2 = x+2$ parabola va $x=2$ to'gri chiziq .	$10\frac{2}{3}$.
5.	$y = \sin x, y = 0, x = \frac{\pi}{4}, x = \pi$.	$1 + \frac{\sqrt{2}}{2}$
6.	$y = \operatorname{tg}x, y = \operatorname{ctg}x, x = 0, x = \frac{\pi}{2}, y = 0$.	$\ln 2$

Jism hajmini hisoblash.

Asosi xOy tekislikda D sohadan iborat bo'lgan vertical silindrik jism yuqoridan $z = f(x, y)$ sirt bilan chegaralangan bo'lsa, uning hajmi

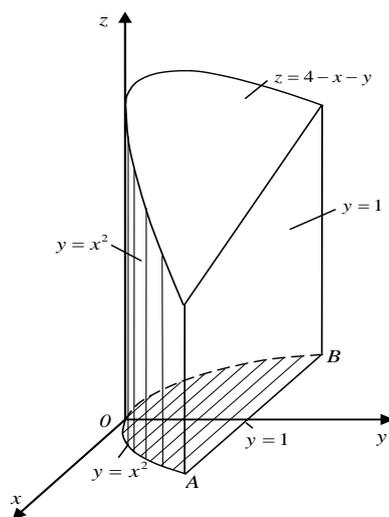
$$V = \iint_D z \, dx \, dy. \quad (5)$$

integral yordamida hisoblanadi.

Murakkabroq formaga ega bo'lgan jismlarning hajmini hisoblash uchun bu jismlarni bir necha vertikal silindrik jismlarning yig'indisi sifatida hisoblash mumkin.

Misol 5. Quyidagi sirtlar bilan chegaralangan jism hajmini hisoblang $y = x^2$, $y = 1$, $x + y + z = 4$, $z = 0$.

Yechish. Berilgan jism (7-rasm) yuqoridan $z = 4 - x - y$ tekislik bilan, quyidan xOy tekislik bilan chegaralangan, yon tomondan $y = x^2$ parabola va $y = 1$ to'g'ri chiziq bilan chegaralangan vertikal silindrdan iborat.



7-rasm

(5) formulaga asosan bu jism hajmi

$$V = \iint_{0AB} z \, dx \, dy = \int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} (4-x-y) \, dx =$$

$$= \int_0^1 \left[(4-y)x - \frac{x^2}{2} \right]_{x=-\sqrt{y}}^{x=\sqrt{y}} dy = 2 \int_0^1 (4-y)\sqrt{y} \, dy = \frac{68}{15}$$

ga teng.

Boshqacha tartibda integrallasak

$$V = \int_{-1}^1 dx \int_{x^2}^1 (4-x-y) \, dy = \dots = \frac{68}{15}.$$

Mustaqil hisoblang:

Quyidagi sirtlar bilan chegaralangan jism hajmini hisoblang:

№	Misol	javobi (kub.birlik.)
1.	$x^2 + y^2 = 4, z = 0, z = 3.$	12π
2.	$x + y + z = 2, z = 1, x = 0, y = 0.$	$\frac{2}{3}$
3.	$z = 0, z = 6, y^2 = x, x = 9$	216

Uch karrali integral

Fazoda S yopiq sirt bilan chegaralangan biror V soha berilgan bo'lsin. V soha va uning chegarasida biror $f(x, y, z)$ uzluksiz funksiya aniqlangan bo'lsin, bunda x, y, z soha nuqtasining to'g'ri burchakli koordinatalari. Aniqlik uchun $f(x, y, z) \geq 0$ bo'lgan holda biz funktsiyani qandaydir bir moddaning V sohaga taqsimlanish zichligi deb hisoblashimiz mumkin. Δv_i simvol bilan sohaning o'zigina emas, balki uning hajmini ham belgilab, V sohani ixtiyoriy ravishda Δv_i sohalarga bo'lamiz. Har bir Δv_i sohada ixtiyoriy P_i nuqtani tanlab olamiz va f funksiyaning bu nuqtadagi qiymatini $f(P_i)$ bilan belgilaymiz. Integral yig'indini

$$\sum f(P_i) \Delta v_i$$

ko'rinishda tuzamiz va bunda Δv_i ning eng kata diametrini nolga intiladigan qilib, Δv_i sohalarning sonini cheksiz orttirib boramiz. Agar $f(x, y, z)$ funksiya

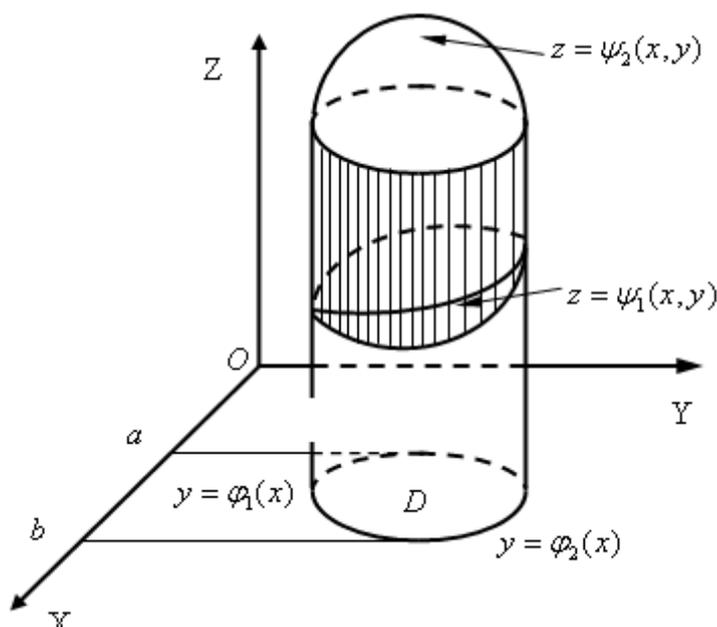
uzluksiz bo'lsa, $\sum f(P_i)\Delta v_i$ shakldagi integral yig'indining limiti mavjud bo'ladi, bunda integral yig'indining limiti ikki o'lchovli integralni ta'riflagandagi ma'noda tushuniladi. V sohani bo'lish usuliga ham, P_i nuqtani tanlab olish usuliga ham bog'liq bo'lmagan bu limit $\iiint_V f(P)dv$ simvol bilan belgilanadi va *uch o'lchovli integral* deb ataladi. Shunday qilib, ta'rifga ko'ra:

$$\lim_{\text{diam}\Delta v_i \rightarrow 0} \sum_{i=1}^n f(P_i)\Delta v_i = \iiint_V f(P)dv$$

yoki

$$\iiint_V f(P)dv = \iiint_V f(x, y, z)dx dy dz$$

Agar V soha yuqoridan $z = \psi_2(x, y)$ sirt, quyidan $z = \psi_1(x, y)$ sirt bilan, yonidan Oz o'qiga parallel bo'lgan silindrik sirt dan iborat (8-rasm),



(8-rasm)

$\psi_1(x, y) \leq \psi_2(x, y)$ shart uchun $\forall (x, y) \in D$ uch karrali integral:

$$\iiint_V f(x, y, z)dx dy dz = \iint_D dx dy \int_{\psi_1(x, y)}^{\psi_2(x, y)} f(x, y, z)dz \quad (7)$$

(7) dagi

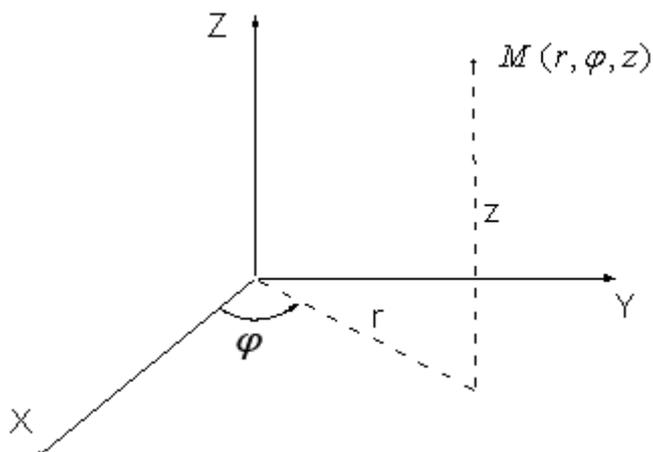
$$\int_{\psi_1(x, y)}^{\psi_2(x, y)} f(x, y, z)dz$$

ichki integral z bo'yicha $\psi_1(x, y)$ dan $\psi_2(x, y)$ gacha oraliqda hisoblanadi.

$$\iiint_V f(x, y, z) dx dy dz = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) dz. \quad (8)$$

$\varphi_1(x)$ dan $\varphi_2(x)$ gacha bo'lgan funksiyalar $\forall x \in [a, b]$ kesmada va $\psi_1(x, y)$, $\psi_2(x, y)$, lar $\forall (x, y) \in D$ da uzluksiz.

Uch karrali integralni dekart koordinatalar sistemasida hisoblash qiyinchilik tug'dirganda yangi o'zgaruvchilarga o'tish yaxshi samara beradi. Xususiyl holda quyidagi almashtirishdan foydalanish mumkin $x = r \cos \varphi$, $y = r \sin \varphi$, $z = z$, bu yerda $(0 \leq r < +\infty, 0 \leq \varphi \leq 2\pi, -\infty < z < +\infty)$, (r, φ, z) koordinatlar silindrik koordinatlar deyiladi (9-rasm).



(9-rasm)

Bu holda integralning ko'rinishi:

$$\iiint_V f(x, y, z) dx dy dz = \iiint_{V'} f(r \cos \varphi, r \sin \varphi, z) r dr d\varphi dz \quad (9)$$

Agar V $x=a, x=b, y=c, y=d, z=p, z=q$ tekisliklar bilan chegaralangan bo'lsa:

$$\iiint_V f(x, y, z) dx dy dz = \int_a^b dx \int_c^d dy \int_p^q f(x, y, z) dz,$$

Agar V soha $\alpha \leq \varphi \leq \beta$, $r_1(\varphi) \leq r \leq r_2(\varphi)$, $z_1(r, \varphi) \leq z \leq z_2(r, \varphi)$,

tengsizliklar bilan chegaralangan bo'lsa, integral:

$$\int_{\alpha}^{\beta} d\varphi \int_{r_1(\varphi)}^{r_2(\varphi)} r dr \int_{z_1(r, \varphi)}^{z_2(r, \varphi)} f(r \cos \varphi, r \sin \varphi, z) dz.$$

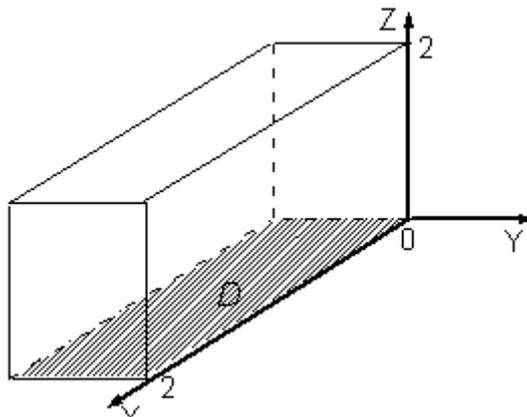
ko'rinishga keladi

6-misol. Integralni hisoblang

$$\iiint_V 8y^2 z e^{-xyz} dx dy dz;$$

$V: x=2, x=0, y=-1, y=0, z=2, z=0.$

Yechish: integrallash sohasini chizmada ko'rsataylik (10-rasm).



Shartga ko'ra: $0 \leq x \leq 2, -1 \leq y \leq 0, 0 \leq z \leq 2.$

demak:

$$\begin{aligned}
 J &= 8 \int_{-1}^0 y^2 dy \int_0^2 z dz \int_0^2 e^{-xyz} dx = \\
 &= 8 \int_{-1}^0 y^2 dy \int_0^2 z dz \cdot \left(-\frac{1}{yz}\right) \int_0^2 e^{-xyz} d(-xyz) = \\
 &= -8 \int_{-1}^0 y dy \int_0^2 dz \cdot e^{-xyz} \Big|_{x=0}^{x=2} = \\
 &= -8 \int_{-1}^0 y dy \int_0^2 (e^{-2yz} - 1) dz = \\
 &= -8 \int_{-1}^0 y dy \left(-\frac{1}{2y} \cdot \int_0^2 e^{-2yz} d(-2yz) - \int_0^2 dz \right) = 8 \int_{-1}^0 y \left(-\frac{1}{2y} \cdot e^{-2yz} - z \right) \Big|_{z=0}^{z=2} = \\
 &= 8 \int_{-1}^0 y \left(\frac{1}{2y} e^{-4y} + 2 - \frac{1}{2y} \right) dy = 8 \int_{-1}^0 \left(\frac{1}{2} e^{-4y} + 2y - \frac{1}{2} \right) dy = \\
 &= 8 \left(-\frac{1}{8} e^{-4y} + y^2 - \frac{1}{2} y \right) \Big|_{-1}^0 = \left(-e^{-4y} + 8y^2 - 4y \right) \Big|_{-1}^0 = e^4 - 13.
 \end{aligned}$$

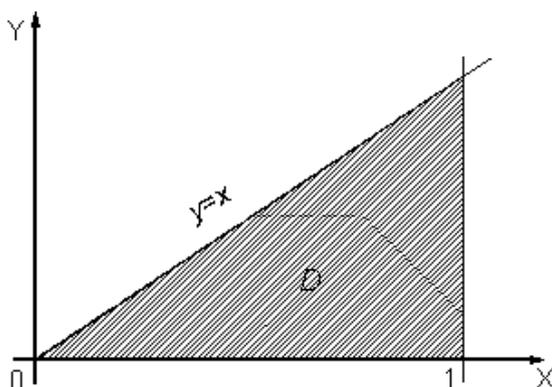
javob: $e^4 - 13 \approx 41,583.$

7-misol. Integralni hisoblang

$$\iiint_V \left(5x + \frac{3z}{2} \right) dx dy dz;$$

$V: y=x, y=0, x=1, z=x^2+15y^2, z=0.$

Yechish: $y = x$, $y = 0$, $x = 1$ lar fazoda tekisliklarni ifodalaydi: $y=0 -xOz$ tekislik ; $x=1 -yOz$ tekislikka parallel tekislik; $y=x - Oz$ o'q orqali o'tuvchi tekislik. xOy tekisligida quyidagi sohani belgilaydi (11-rasm).



(11-rasm)

$z=x^2+15y^2$ elliptik paraboloid $z=0$ tekisligi bilan $O(0,0,0)$ umumiy nuqtaga ega. V jismning xOy tekisligiga proyeksiyasi D sohadan I borat (11-rasm).

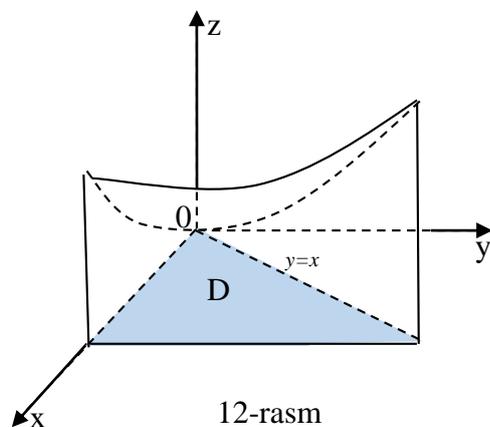
$$\text{Bu sohada } \begin{cases} 0 \leq x \leq 1, \\ 0 \leq y \leq x. \end{cases}$$

$\forall (x, y) \in D$ uchun $z = 0 \leq z \leq x^2 + 15y^2$ tengsizlikni qanoatlantiradi,

bundan:

$$\begin{aligned} J &= \int_0^1 dx \int_0^x dy \int_0^{x^2+15y^2} \left(5x + \frac{3}{2}z \right) dz = \int_0^1 dx \int_0^x dy \left(5xz + \frac{3z^2}{4} \right) \Big|_0^{x^2+15y^2} = \\ &= \int_0^1 dx \int_0^x \left(5x(x^2 + 15y^2) + \frac{3}{4}(x^2 + 15y^2)^2 \right) dy = \\ &= \int_0^1 dx \int_0^x \left(5x^3 + 75xy^2 + \frac{3}{4}x^4 + \frac{45}{2}x^2y^2 + \frac{675}{4}y^4 \right) dy = \\ &= \int_0^1 dx \left(5x^3y + 25xy^3 + \frac{3}{4}x^4y + \frac{15}{2}x^2y^3 + \frac{135}{4}y^5 \right) \Big|_0^x = \\ &= \int_0^1 \left(5x^4 + 25x^4 + \frac{3}{4}x^5 + \frac{15}{2}x^5 + \frac{135}{4}x^5 \right) dx = \int_0^1 (30x^4 + 42x^5) dx = \\ &= 6 \int_0^1 (5x^4 + 7x^5) dx = 6 \left(x^5 + \frac{7}{6}x^6 \right) \Big|_0^1 = 6 \left(1 + \frac{7}{6} \right) = 13. \end{aligned}$$

javob: 13.



12-rasm

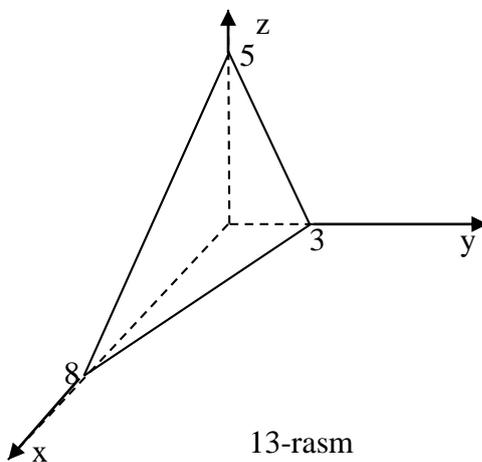
8-misol. Integralni hisoblang

$$\iiint_V \frac{dx dy dz}{\left(1 + \frac{x}{8} + \frac{y}{3} + \frac{z}{5}\right)^6};$$

$$V: \frac{x}{8} + \frac{y}{3} + \frac{z}{5} = 1, \quad x=0, y=0, z=0.$$

Yechish : V soha $x=0, y=0, z=0$ va $\frac{x}{8} + \frac{y}{3} + \frac{z}{5} = 1$ tekisliklar bilan

chegaralangan. $\frac{x}{8} + \frac{y}{3} + \frac{z}{5} = 1$ tekislik Ox, Oy, Oz o'qlarda mos ravishda $a=8, b=3, c=5$ kesmalar ajratadi. 13-rasm (uchburchakli piramida).



13-rasm

D soha V ni xOy tekisligiga proyeksiyalashdan hosil bo'ladi. Soha Ox , Oy o'qlar bilan $\frac{x}{8} + \frac{y}{3} = 1$ to'g'ri chiziqlar bilan chegaralangan.

$$\frac{x}{8} + \frac{y}{3} = 1 \quad \text{tenglamadan } y = 3\left(1 - \frac{x}{8}\right) \text{ yozish mumkin}$$

$$\frac{x}{8} + \frac{y}{3} + \frac{z}{5} = 1 \Rightarrow z = 5\left(1 - \frac{x}{8} - \frac{y}{3}\right). \text{ Natijada integral chegaralariga ega bo'ldik:}$$

$$0 \leq x \leq 8, \quad 0 \leq y \leq 3\left(1 - \frac{x}{8}\right), \quad 0 \leq z \leq 5\left(1 - \frac{x}{8} - \frac{y}{3}\right).$$

Shunday qilib:

$$\begin{aligned} J &= \int_0^8 dx \int_0^{3(1-x/8)} dy \int_0^{5(1-x/8-y/3)} \frac{dz}{\left(1 + \frac{x}{8} + \frac{y}{3} + \frac{z}{5}\right)^6} = \\ &= 5 \int_0^8 dx \int_0^{3(1-x/8)} dy \int_0^{5(1-x/8-y/3)} \left(1 + \frac{x}{8} + \frac{y}{3} + \frac{z}{5}\right)^{-6} d\left(1 + \frac{x}{8} + \frac{y}{3} + \frac{z}{5}\right) = \\ &= \int_0^8 dx \int_0^{3(1-x/8)} dy \left(1 + \frac{x}{8} + \frac{y}{3} + \frac{z}{5}\right)^{-5} \Big|_0^{5(1-x/8-y/3)} = \\ &= \int_0^8 dx \int_0^{3(1-x/8)} \left(1 + \frac{x}{8} + \frac{y}{3} + 1 - \frac{x}{8} - \frac{y}{3}\right)^{-5} dy = -2 \int_0^8 dx \int_0^{3(1-x/8)} dy = -2 \int_0^8 dx \cdot y \Big|_0^{3(1-x/8)} = \\ &= -6 \int_0^8 \left(1 - \frac{x}{8}\right) dx = -6 \left(x - \frac{x^2}{16}\right) \Big|_0^8 = -6(8 - 4) = -24. \end{aligned}$$

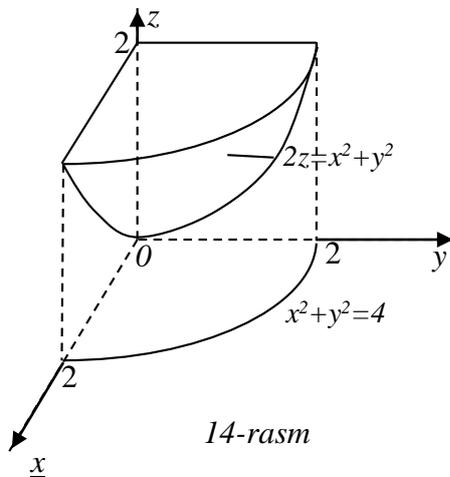
javob: -24.

9-misol. Integralni hisoblang

$$\iiint_V (x^2 + y^2) dx dy dz;$$

V : $x=0$, $y=0$, $z=2$ tekisliklar va birinchi oktantada joylashgan $2z = x^2 + y^2$ paraboloid.

Yechish: jism quyidagi ko'rinishga ega (14-rasm).



14-rasm

Jismni xOy tekislikka proyeksiyalasak, D soha aylananing to'rtidan biri.

Haqiqatdan,

$$\begin{cases} 2z = x^2 + y^2, \\ z = 2. \end{cases} \quad \text{sistemaga ko'ra}$$

$$x^2 + y^2 = 4$$

Integralni yechish uchun silindrik koordinatlarga o'tamiz D sohada φ qutb burchagi

$$0 \leq \varphi \leq \frac{\pi}{2}$$

oraligida o'zgaradi.

Qutb koordinatiga o'tsak: $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$.

D sohada $0 \leq r \leq 2$. O'z navbatida paraboloid: $r^2 = 2z$ yoki $z = \frac{r^2}{2}$. V da

$$\frac{r^2}{2} \leq z \leq 2$$

Formulaga ko'ra:

$$J = \iiint_V r^3 dr d\varphi dz = \int_0^{\pi/2} d\varphi \int_0^2 r^3 dr \int_{r^2/2}^2 dz =$$

$$\int_0^{\pi/2} d\varphi \int_0^2 r^3 dr \cdot z \Big|_{r^2/2}^2 = \int_0^{\pi/2} d\varphi \int_0^2 r^3 \left(2 - \frac{r^2}{2}\right) dr =$$

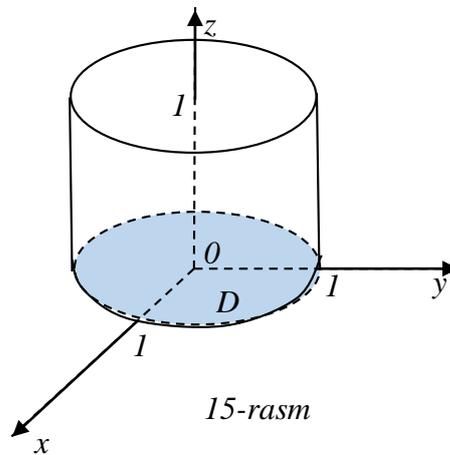
$$= \int_0^{\pi/2} d\varphi \int_0^2 (2r^3 - \frac{1}{2}r^5) dr = \varphi \Big|_0^{\pi/2} \cdot \left(\frac{r^4}{2} - \frac{1}{12}r^6 \right) \Big|_0^2 = \frac{\pi}{2} \cdot \left(8 - \frac{16}{3} \right) = \frac{4\pi}{3}$$

javob: $\frac{4\pi}{3} \approx 4,187.$

10-misol. Integralni hisoblang

$$\iiint_V (x^2 + y^2 + z)^3 dx dy dz$$

V : $x^2 + y^2 = 1$ aylanma silindr va $z=0, z=1$ tekisliklar bilan chegaralangan (15-rasm).



Yechish: $x^2 + y^2 = r^2$ desak, $dx dy = r dr d\varphi$ bo'lib:

$$J = \iiint_V (r^2 + z)^3 r dr d\varphi dz$$

D soha $x^2 + y^2 = 1$ aylana $r^2 = 1, r = 1$ demak $0 \leq \varphi \leq 2\pi$ U holda:

$$\begin{aligned} J &= \int_0^{2\pi} d\varphi \int_0^1 r dr \int_0^1 (r^2 + z)^3 dz = \int_0^{2\pi} d\varphi \int_0^1 r dr \frac{(r^2 + z)^4}{4} \Big|_{z=0}^{z=1} = \\ &= \frac{1}{4} \int_0^{2\pi} d\varphi \int_0^1 r ((r^2 + 1)^4 - r^8) dr = \\ &= \frac{1}{4} \int_0^{2\pi} d\varphi \left(\frac{1}{10} (r^2 + 1)^5 \Big|_0^1 - \frac{1}{10} \right) = \frac{1}{4} \varphi \Big|_0^{2\pi} \left(\frac{1}{10} (32 - 1) - \frac{1}{10} \right) = \\ &= \frac{1}{40} 2\pi (30) = \frac{3\pi}{2}. \end{aligned}$$

javob: $\frac{3\pi}{2} \approx 4,71$.

Uch karrali integrallarning tadbiqlari

V sohani egallagan v jism hajmi:

$$v = \iiint_V dx dy dz \quad (10)$$

Integral yordamida hisoblanadi

$$v = \iiint_V dx dy dz \text{ zichlikka ega bo'lgan jism massasi}$$

$$m = \iiint_V \gamma(x, y, z) dx dy dz. \quad (11)$$

Formula bilan hisoblanadi.

Kerak bo'lganda (10) va (11) formulalarda silindrik koordinatlarga o'tish mumkin. U holda (10) formula:

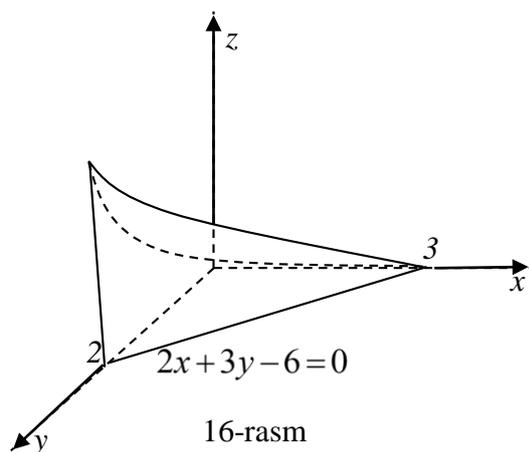
$$v = \iiint_V r dr d\varphi dz \quad (12)$$

ko'rinishga keladi.

11-misol. Quyidagi sirtlar bilan chegaralangan jism hajmini toping

$$2x + 3y - 6 = 0, z = y^2, x = 0, y = 0, z = 0.$$

Yechish: jism koordinata tekisliklari, $2x + 3y - 6 = 0$ tekislik $z = y^2$ parabolic silindr bilan chegaralangan(16-rasm).



x, y, z lar quyidagi oraliqda o'zgaradi:

$$0 \leq y \leq 2, \quad 0 \leq x \leq \frac{1}{2}(6 - 3y), \quad 0 \leq z \leq y^2.$$

Bundan:

$$v = \int_0^2 dy \int_0^{\frac{3}{2}(2-y)} dx \int_0^{y^2} dz = \int_0^2 dy \int_0^{\frac{3}{2}(2-y)} dx \cdot z \Big|_0^{y^2} = \int_0^2 y^2 dy \int_0^{\frac{3}{2}(2-y)} dx = \int_0^2 y^2 dy \cdot x \Big|_0^{\frac{3}{2}(2-y)} =$$

$$= \frac{3}{2} \int_0^2 y^2 (2-y) dy = \frac{3}{2} \int_0^2 (2y^2 - y^3) dy = \frac{3}{2} \left(\frac{2}{3} y^3 - \frac{y^4}{4} \right) \Big|_0^2 = \frac{3}{2} \left(\frac{16}{3} - \frac{16}{4} \right) = \frac{24}{12} = 2$$

javob. 2 kub birlik.

12-misol. Quyidagi sirtlar bilan chegaralangan jism hajmini toping

$$x = 4y^2 + 2, x = 6, z = x^2 + 4y^2 + y + 1, z = x^2 + 4y^2 + y + 4.$$

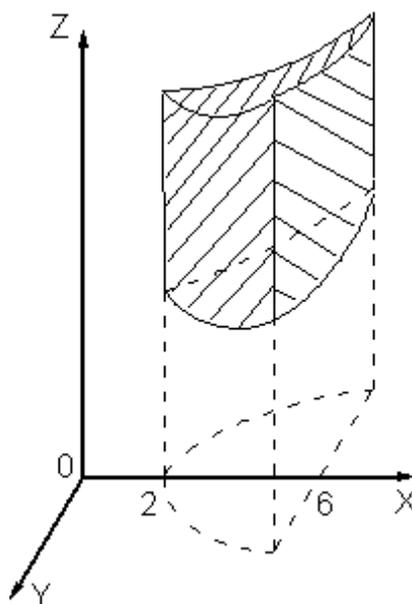
Yechish: jism yuqoridn va quyidan mos ravishda elliptik paraboloidlar bilan

chegaralangan: uchi $(0; -\frac{1}{8}; \frac{15}{16})$ $z = x^2 + 4y^2 + y + 1 \Rightarrow z = x^2 + 4(y + \frac{1}{8})^2 + \frac{15}{16}$

va uchi $(0; -\frac{1}{8}; \frac{63}{16})$

$$z = x^2 + 4y^2 + y + 4 \Rightarrow z = x^2 + 4(y + \frac{1}{8})^2 + \frac{63}{16} ;$$

Yon tomondan $x = 4y^2 + 2$ parabolic silindr va yOz ga parallel $x=6$ tekislik (17-rasm).



(17-rasm)

Jismni xOy tekislikka proyeksiyalasak, $x = 4y^2 + 2$ parabola va $x=6$ to'g'ri chiziq bilan chegaralangan. Kesishish nuqtasi: $4y^2 + 2 = 6 \Rightarrow y^2 = 1, y = \pm 1$.

Shunday qilib, jism hajmi:

$$v = \iiint_V dx dy dz = \int_{-1}^1 dy \int_{4y^2+2}^6 dx \int_{x^2+4y^2+y+1}^{x^2+4y^2+y+4} dz = \int_{-1}^1 dy \int_{4y^2+2}^6 dx \cdot z \Big|_{x^2+4y^2+y+1}^{x^2+4y^2+y+4} =$$

$$= \int_{-1}^1 dy \int_{4y^2+2}^6 (x^2 + 4y^2 + y + 4 - x^2 - 4y^2 - y - 1) dx = 3 \int_{-1}^1 dy \int_{4y^2+2}^6 dx = 3 \int_{-1}^1 dy \cdot x \Big|_{4y^2+2}^6 =$$

$$= 12 \int_{-1}^1 (1 - y^2) dy = 12 \left(y - \frac{y^3}{3} \right) \Big|_{-1}^1 = 12 \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = 16.$$

javob: 16.

Egri chiziqli va sirt integrallari

Oxy tekislikda ikki o'zgaruvchili uzluksiz $L = AB$ egri chiziq berilgan bo'lsin. Egri chiziqni ixtiyoriy n bo'llaka bo'lamiz. $A = M_0, M_1, M_2, \dots, M_n = B$ hosil bo'lgan $M_{i-1}M_i$ qismida *ihiyoriy* $\overline{M_i(x_i, y_i)}$ nuqta ni tanlab, quyidagi yig'indini hosil qilamiz.

$$S_n = \sum_{i=1}^n f(x_i, y_i) \Delta l$$

$\Delta l_i = M_{i-1}M_i - \overline{M_{i-1}M_i}$ yoy uzunligi. Olingan yig'indi L egri chiziqda berilgan egri chiziqda berilgan $f(x, y)$ funksiya uchun I- tur integral yig'indi deyiladi.

D orqali $\overline{M_{i-1}M_i}$ yoylarning eng uzunini belgilaylik

$$d = \max_i \Delta l_i;$$

Agar $d \rightarrow 0$ S_n integral yig'indining limiti mavjud bo'lsa, u holda uni I- tur egri chiziqli integral deyiladi va u quyidagicha belgilanadi.

$$\int_L f(x, y) dl$$

Yoki

$$\int_{AB} f(x, y) dl$$

Agar $f(x, y)$ funksiya uzluksiz bo'lsa

$$\int_L f(x, y) dl$$

mavjud.

I- tur integral uchun quyidagi tasdiq o'rinli

$$\int_{AB} f(x, y) dl = \int_{BA} f(x, y) dl$$

ya'ni, chiziqli integral yo'nalishga bog'liq emas.

I- tur egri chiziqli integrallarni hisoblash usullari.

Bu kabi integrallarni hisoblash aniq integralni hisoblashga keltiriladi.

1. Agar L $y = y(x)$ $x \in [a; b]$ uzluksiz funksiya orqali berilgan bo'lsa,

$$\int_L f(x, y) dl = \int_a^b f(x, y(x)) \sqrt{1 + (y'(x))^2} dx$$

$dl = \sqrt{1 + (y'(x))^2} dx$ yoy uzunligining differentsiali deyiladi.

2. Agar L parametrik tenglama orqali berilsa, $x = x(t)$, $y = y(t)$, $z = z(t)$ $t \in [a; b]$

$f(x, y, z)$ – L egri chiziq ustidagi funksiya.

$$\int_L f(x, y, z) dl = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

3. Agar L $r = r(\varphi)$, $\varphi \in [\alpha, \beta]$ qutb koordinatlar sistemasi orqali berilgan bo'lsa.

$$\int_L f(x, y) dl = \int_\alpha^\beta f(r \cos \varphi, r \sin \varphi) \sqrt{r^2 + (r')^2} d\varphi$$

Egri chiziqli integralning tadbig'i.

1. Agar integral ostidagi funksiya birga teng bo'lsa,

$$\int_L dl$$

integral L egri chiziqning uzunligiga teng.

$$S = \int_L dl$$

2. Oxy tekislikda $Z=f(x,y) \geq 0$ funksiya L egri chiziqda berilgan bo'lsin. OZ ga parallel, L yo'naltiruvchi silindrik sirtni yasash mumkin. U L va $Z=f(x,y)$ soha bilan chegaralangan Bunday silindrik sirt yuzasi quyidagi formula orqali topiladi:

$$S = \int_L f(x, y) dl$$

3. Agar $L=AB - \rho = \rho(x, u)$ zichlikka ega egri chiziq bo'lsa, uning massasi

$$m = \int_{AB} \rho(x, y) dl$$

(1 tur egri chiziq integralining fizik ma'nosi)

1-misol. Egri chizikli integralni hisoblang.

$$\int_L \frac{x}{y} dl$$

L –(2;2) va (8;4) nuqtalar orasidagi $y^2 = 2x$ parabola yoyi.

Yechish : $u = \sqrt{2x}$,

$$u' = \frac{1}{\sqrt{2x}} ;$$

$$dl = \sqrt{1 + (u')^2} dx = \sqrt{1 + \frac{1}{2x}} dx$$

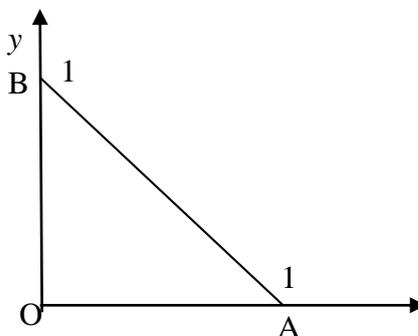
$$\begin{aligned} \int_L \frac{x}{y} dl &= \int_2^8 \frac{x}{\sqrt{2x}} \sqrt{1 + \frac{1}{2x}} dx = \int_2^8 \frac{x\sqrt{1+2x}}{2x} dx = \frac{1}{2} \int_2^8 \sqrt{1+2x} dx \\ &= \frac{1}{2 \cdot 3} (1+2x)^{\frac{3}{2}} \Big|_2^8 = \frac{1}{6} (17\sqrt{17} - 5\sqrt{5}) \end{aligned}$$

2-misol. Egri chizikli integralni hisoblash.

$$\int_L (x^2 + y^3) dl$$

L- A(1;0) , B (0;1) , O(0;0) nuqtadan o'tuvchi Δ

Yechish :



$$\int_L (x^2 + y^3) dl = \int_{AB} (x^2 + y^3) dl + \int_{BO} (x^2 + y^3) dl + \int_{OA} (x^2 + y^3) dl$$

1. (AB). AB to'g'ri chiziq $y = 1 - x$ tenglama bilan berilgan $dl = \sqrt{1 + (y')^2} dx = \sqrt{2} dx$ $0 \leq x \leq 1$ ekanligini inobatga olsak

$$\begin{aligned} \int_{AB} (x^2 + y^3) dl &= \int_0^1 [x^2 + (1-x)^3] \sqrt{2} dx = \sqrt{2} \left(\frac{x^3}{3} - \frac{(1-x)^4}{4} \right) \Big|_0^1 \\ &= \sqrt{2} \left(\frac{1}{3} + \frac{1}{4} \right) = \frac{7\sqrt{2}}{12} \end{aligned}$$

2. (BO): $x = 0$, $0 \leq y \leq 1$, $dl = dy$.

$$\int_{BO} (x^2 + y^3) dl = \int_0^1 y^3 dy = \frac{1}{4}$$

3. (OA): $y = 0$, $0 \leq x \leq 1$, $dl = dx$

$$\int_{OA} (x^2 + y^3) dl = \int_0^1 x^2 dx = \frac{1}{3}$$

va nihoyat

$$\int_L (x^2 + y^3) dl = \frac{7\sqrt{2}}{12} + \frac{1}{4} + \frac{1}{3} = \frac{7\sqrt{2} + 7}{12} = \frac{7(\sqrt{2} + 1)}{12}$$

3- misol. Egri chizikli integralni hisoblang

$$\int_L \sqrt{x^2 + y^2} dl$$

bu yerda $L - x^2 + y^2 = ax$ ($a > 0$) aylana.

Yechish:

Qutb koordinatalar sistemasiga o'tamiz.

$x = r \cos \varphi$, $y = r \sin \varphi$. U holda $x^2 + y^2 = r^2$

$r^2 = ar \cos \varphi$ ya'ni, $r = a \cos \varphi$

$dl = \sqrt{r^2 + (r')^2} d\varphi = \sqrt{a^2 \cos^2 \varphi + a^2 \sin^2 \varphi} d\varphi = a d\varphi$

$$\varphi \in \left[-\frac{\pi}{2}; \frac{\pi}{2} \right]$$

Demak,

$$\int_L \sqrt{x^2 + y^2} dl = a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a \cos \varphi d\varphi = 2a^2$$

4-misol. 3 o'zgaruvchili I- tur egri chiziqli integralni hisoblang.

$$\int_L (5z - 2\sqrt{x^2 + y^2}) dl$$

Bu yerda $L: x = t \cos t, \quad y = t \sin t, \quad z = t,$

$0 \leq t \leq \pi$ parametrik funksiyalar orqali berilgan egri chiziq yoyi

Yechish:

t o'zgaruvchiga o'tib olamiz.

$$5z - 2\sqrt{x^2 + y^2} = 5t - 2\sqrt{t^2(\cos^2 t + \sin^2 t)} = 3t$$

dl ni t orqali ifodalaymiz.

$$\begin{aligned} dl &= \sqrt{(x')^2 + (y')^2 + (z')^2} dt = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} dl \\ &= \sqrt{(\cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t) + (\sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t) + 1} dt \\ &= \sqrt{(\cos^2 t + \sin^2 t) + t^2(\sin^2 t + \cos^2 t) + 1} = \sqrt{2 + t^2} dt \end{aligned}$$

Shunday qilib,

$$\begin{aligned} \int_L (5z - 2\sqrt{x^2 + y^2}) dl &= \int_0^\pi 3t \sqrt{2 + t^2} dt = \int_0^\pi \frac{3}{2} \sqrt{2 + t^2} d(2 + t^2) = (2 + t^2)^{\frac{3}{2}} \Big|_0^\pi \\ &= \sqrt{(2 + \pi^2)^3} - 2\sqrt{2} \end{aligned}$$

Egri chiziqli integrallarni hisoblang 5-14

5.

$$\int_L (xy) dl, \quad L - |x| + |y| = a$$

kvadrat konturi.

6.

$$\int_L \frac{dl}{\sqrt{x^2 + y^2 + 4}};$$

$L - OA, O(0; 0), A(1; 2)$ kesma

7.

$$\int_L \frac{dl}{x+y}$$

$L - AB$ kesma, $A(2;4)$, $B(1;3)$

8.

$$\int_L \frac{dl}{x-y}$$

$L - MN$ kesma, $M(0;-2)$, $N(4;0)$

9.

$$\int_L y^2 dl, \quad ,$$

$L - x = a(t - \sin t)$, $y = a(1 - \cos t)$ $0 \leq t \leq 2\pi$ sikloida

yoyi.

10.

$$\int_L (x^2 + y^2 + z^2) dl.$$

$L - x = a \cos t$, $y = a \sin t$, $z = bt$ zanjir chizig'i. $0 \leq t \leq 2\pi$

11.

$$\int_L (x+y) dl,$$

$L - r^2 = a^2 \cos 2\varphi$ lemniskataning o'ng yaprog'i

12.

$$\int_L (x^2 + y^2) dl,$$

$L - x^2 + y^2 = a^2$ aylana

13

$$\int_L xy dl,$$

$L - \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $x \geq 0$, $y \geq 0$ ellipsning chorak qismi

14.

$$\int_L y dl,$$

$L - x^2 = 2py$ parabola bilan kesilgan $y^2 = 2px$ parabola yoyi.

15. $x^2 + y^2 = R^2$ tsilindirning quyidan Oxy , yuqoridan $f(x, y) = R + \frac{x^2}{R}$ tekisliklar bilan chegaralangan qismi yon yog'i yuzasi hisoblansin.

Yechish :

$$S = \int_L \left(R + \frac{x^2}{R}\right) dl$$

Silindir sirti $f(x, y) = R + \frac{x^2}{R}$ sirtlar Oxz va Oyz koordinata tekisliklariga nisbatan simmetrik, shuning uchun $y \geq 0, x \geq 0$ integralni 4 dan biri uchun hisoblab, 4ga ko'paytirish kifoya.

Quyidagilarga egamiz:

$$y = \sqrt{R^2 - x^2}, \quad y' = -\frac{x}{\sqrt{R^2 - x^2}}$$

$$dl = \sqrt{1 + (y')^2} dx = \sqrt{1 + \frac{x^2}{R^2 - x^2}} dx = \frac{R dx}{\sqrt{R^2 - x^2}}$$

Demak

$$S = 4 \int_0^R \left(R + \frac{x^2}{R}\right) \frac{R dx}{\sqrt{R^2 - x^2}} = 4 \int_0^R \frac{R^2 + x^2}{\sqrt{R^2 - x^2}} dx$$

$$x = R \sin \varphi \quad dx = R \cos \varphi d\varphi \quad 0 \leq \varphi \leq \frac{\pi}{2}$$

$$\sqrt{R^2 - x^2} = R \cos \varphi$$

$$S = 4 \int_0^{\frac{\pi}{2}} \frac{R^2 + (R \sin \varphi)^2}{R \cos \varphi} \cdot R \cos \varphi d\varphi = 4 \int_0^{\frac{\pi}{2}} (R^2 + R^2 \sin^2 \varphi) d\varphi =$$

$$4R^2 \int_0^{\frac{\pi}{2}} \left(1 + \frac{1 - \cos 2\varphi}{2}\right) d\varphi = 3\pi R^2$$

16. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ellipsni 1- chorakdagi qismi massasi hisoblansin, bu yerda har bir nuqtadagi zichligi k koeffitsient bilan ordinataga proporsional.

Yechish:

$$\rho(x, y) = k y$$

$$m = \int_L ky dl$$

$x \geq 0, y \geq 0$ I – chorak uchun

$x = a \cos t, y = b \sin t$ fokuslar orasidagi masofa $c = \sqrt{a^2 - b^2}$
ekstsentrisset $\frac{c}{a} = \xi$

$$\begin{aligned} dl &= \sqrt{(x')^2 + (y')^2} dt = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt = \\ &= \sqrt{a^2(1 - \cos^2 t) + b^2 \cos^2 t} dt = \sqrt{a^2 - (a^2 - b^2) \cos^2 t} dt = \\ &= \sqrt{a^2 \left(1 - \frac{a^2 - b^2}{a^2} \cos^2 t\right)} = a \sqrt{1 - \xi^2 \cos^2 t} dt \end{aligned}$$

m massani hisoblaymiz

$$\begin{aligned} m &= kab \int_0^{\frac{\pi}{2}} \sin t \sqrt{1 - \xi^2 \cos^2 t} dt = \\ &= -\frac{kab}{\xi} \int_0^{\frac{\pi}{2}} \sqrt{1 - (\xi \cos t)^2} d(\xi \cos t) \\ &= \int \sqrt{1 - u^2} du = \frac{1}{2} (u \sqrt{1 - u^2} + \arcsin u) \end{aligned}$$

formuladan foydalansak, $u = \xi \cos t$

$$\begin{aligned} m &= -\frac{kab}{\xi} \cdot \frac{1}{2} (\xi \cos t \sqrt{1 - \xi^2 \cos^2 t} + \arcsin(\xi \cos t)) \Big|_0^{\frac{\pi}{2}} \\ &= -\frac{kab}{2\xi} (-\xi \sqrt{1 - \xi^2} - \arcsin \xi) \end{aligned}$$

$\xi = \frac{\sqrt{a^2 - b^2}}{a}$; $\sqrt{1 - \xi^2} = \frac{b}{a}$ ekanini hisobga olsak

$$m = \frac{ka^2b}{2\sqrt{a^2 - b^2}} \left(\frac{b}{a^2} \sqrt{a^2 - b^2} + \arcsin \frac{\sqrt{a^2 - b^2}}{a} \right)$$

17-misol.

$x = a(t - \sin t), y = a(1 - \cos t), 0 \leq t \leq 2\pi$ bir jinsli sikloida yoyi massasini va og'irlik markazi koordinatalarini toping.

Yechish: $x_c = \frac{M_y}{m}, y_c = \frac{M_x}{m}$

$$m = \int_L dl, \quad M_y = \int_L x dl, \quad M_x = \int_L y dl$$

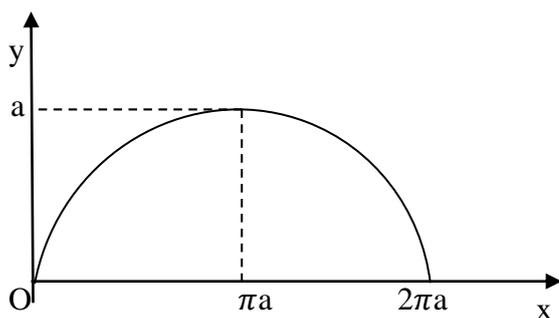
x', y' va dl ni topamiz.

$$x' = a(1 - \cos t), \quad y' = a \sin t$$

$$\begin{aligned} dl &= \sqrt{a^2(1 - 2 \cos t + \cos^2 t) + a^2 \sin^2 t} dt = \\ &= a \sqrt{1 - 2 \cos t + (\cos^2 t + \sin^2 t)} dt = a \sqrt{2(1 - \cos t)} \\ &= a \sqrt{2 \cdot 2 \sin^2 \frac{t}{2}} = 2a \sin \frac{t}{2} \end{aligned}$$

Demak,

$$m = \int_L dl = 2a \int_0^{2\pi} \sin \frac{t}{2} dt = -4a \cos \frac{t}{2} \Big|_0^{2\pi} = 8a$$



Rasmdan ko'rinib turibdiki, tsikloida $x = \pi a$ to'g'ri chiziqqa nisbatan simmetrik, shuning uchun $x_s = \pi a$

Demak: M_u ni hisoblamasa ham bo'ladi. $x_s = \frac{M_y}{m}$

$M_y = 8\pi a^2$. (Buni o'zingiz hisoblashinigiz mumkin.) M_x ni hisoblaylik.

$$\begin{aligned}
M_x &= \int_L y dl = \int_0^{2\pi} a(1 - \cos t) \cdot 2a \sin \frac{t}{2} dt \\
&= 2a^2 \int_0^{2\pi} 2 \sin^2 \frac{t}{2} \cdot a \sin \frac{t}{2} dt \\
&= 4a^2 \int_0^{2\pi} \sin^3 \frac{t}{2} dt \\
&= -8a^2 \int_0^{2\pi} \left(1 - \cos^2 \frac{t}{2}\right) d\left(\cos \frac{t}{2}\right) = -8a^2 \left(\cos \frac{t}{2} - \frac{1}{3} \cos^3 \frac{t}{2}\right) \Big|_0^{2\pi} \\
&= \frac{32}{3} a^2
\end{aligned}$$

Nihoyat, $m = 8a$, $M_x = \frac{32}{3} a^2$, $M_y = 8\pi a^2$, $x_c = \pi a$, $y_c = \frac{4}{3} a$

18-misol.

$x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leq t \leq \pi$
sikloida yoyining og'irlik markazini toping.

19-misol.

$x^2 + y^2$, $x \geq 0$, $y \geq 0$ aylana chorak qismining koordinata o'qlari va koordinata boshiga nisbatan inersiya momentlarini toping.

Yoy zichligi taqsimoti doimiy va k ga teng.

Yechish:

Berilgan egri chiziq (chorak aylana) I-chorakda $y = x$ ga nisbatan simmetrik demak, J_x va J_y bir xil

$$J_x = J_y = \int_L y^2 dl = \int_L x^2 dl$$

Aylaning tenglamasini parametrik ko'rinishini yozsak, $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq \frac{\pi}{2}$, $dl = a dt$ quyidagiga, ega bo'lamiz.

$$\int_L x^2 dl = a^3 \int_0^{\frac{\pi}{2}} \cos^2 t dt = \frac{a^3}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t) dt = \frac{a^3}{2} \left(t + \frac{\sin 2t}{2}\right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi a^3}{4}$$

Shunday qilib, $I_x = I_y = \frac{\pi a^3}{4}$; $I_0 = I_x + I_y = \frac{\pi a^3}{2}$;

20- misol.

$x = 5 \cos t, y = 4 \sin t$ tenglama bilan berilgan ellipsning chorak qismi (I chorakdagi qismi) massasini toping. Zichligi $\rho = y$ deb oling.

$$21. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Ellips konturining massasini toping.

Uning chiziqli zichligi har bir $M(x, y)$ nuqtada $|u|$ ga teng.

22. $x = a \cos t, y = a \sin t, z = bt$ 1-vint chizig'ining Oz o'qqa nisbatan inersiya momentini toping.

L yo'naltiruvchi ma'lum bo'lganda, quyidan Oxu, yuqoridan $z = f(x, y)$ tekisliklar bilan chegaralangan silindrik sohaning yuzalari hisoblansin. 23-26

$$23. f(x, y) = \sqrt{2x - 4x^2}, y^2 = 2x$$

$$24. f(x, y) = \frac{xy}{2R}, x^2 + y^2 = R^2$$

$$25. f(x, y) = 2 - \sqrt{x}, y^2 = \frac{4}{9}(x - 1)^3$$

$$26. f(x, y) = x, y = \frac{3}{8}x^2, (x \in [0; 4])$$

I-tur egri chiziqli integraldan foydalanib, berilgan yoy uzunliklarini hisoblang. 27-30.

$$27. ay^2 = x^3, 0 \leq x \leq 5a$$

$$28. y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right), 0 \leq x \leq 4$$

$$29. y = 1 - \ln \cos x, 0 \leq x \leq \frac{\pi}{4}$$

$$30. ch = a \sin^3 \frac{4}{3};$$

I-tur egri chiziqli integraldan foydalanib, egri chiziqlarning og'irlik markazini toping. 31-34

$$31. y^2 = ax^3 - x^4$$

$$32. \sqrt{x} + \sqrt{y} = \sqrt{a}, (0 \leq x \leq a)$$

$$33. y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) (-a \leq x \leq a)$$

$$34. x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}, y \geq 0.$$

Mustaqil yechish uchun misollar.

I-tur egri chiziqli integrallarni hisoblash.

35.

$$\int_L \sqrt{x^2 + y^2} dl$$

$$L - x = a(\cos t + t \sin t), y = a(\sin t - t \cos t), 0 \leq t \leq 2\pi$$

36.

$$\int_L \frac{dl}{x^2 + y^2 + z^2}$$

$L - x = a \cos t, \quad y = a \sin t, \quad z = bt, \quad 0 \leq t \leq 2\pi$

37.

$$\int_L (x + z) dl,$$

$L - x = t, \quad y = \frac{3t^2}{\sqrt{2}}, \quad z = t^3, \quad 0 \leq t \leq 1$

38. $O(0; 0; 0)$ va $A(a; 0; a)$ nuqta lar bilan chegaralangan $x = ae^t \cos t, y = ae^t \sin t, z = ae^t$ chiziq uzunligini toping.

39. $r = a(1 + \cos \varphi)$ kardioida bilan chegaralangan figuraning og'irlik markazi koordinatalarini toping.

40. $r = ae^\varphi, \quad \varphi_1 = \frac{\pi}{2}, \quad \varphi_2 = \pi$ oraliqdagi logarifmik spiral yoyining og'irlik markazi koordinatalarini toping.

41. Hisoblang.

$$\int_L |x + y| dl.$$

L-uchlari $A(0; 0) \quad B(1; 0) \quad C(0; 1)$ nuqtalarda bo'lgan uchburchak konturi.

42. Integralni hisoblang.

$$\int_L \sqrt{2y^2 + z^2} dl,$$

L- $\begin{cases} x^2 + y^2 + z^2 = a^2 \\ x = y \end{cases}$ aylana.

43. $y = x^2$ parabolik tsilindrning, $z = 0, z = 2x, x = 0, x = 1$ tekisliklar bilan chegaralangan qismi yon sirtini toping.

44. $\rho(x, y) = e^{-x}y$ zichlikka ega bo'lgan $x = \ln(1 + t^2), y = 2 \arctg t$ ($t = 0$ dan $t = 1$ gacha) egri chiziqning massasini toping.

45. $\rho(x, y) = xy$ zichlikka ega bo'lgan, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning $x \geq 0$, $y \leq 0$ qismi massasini toping.
Hisoblang. 46.48
46.

$$\int_L (x - y) dl$$

L: $x^2 + y^2 = ax$
47.

$$\int_L x \sqrt{x^2 - y^2} dl$$

bu yerda L- $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ $x \geq 0$ (Lemniskata yarmi)

48.

$$\int_L \arctg \frac{y}{x} dl$$

L- $r = 2\varphi$ Arximed spiralinin radiusi R ga teng markazi koordinata boshida bo'lgan aylana bilan chegaralangan qismi.

Aytaylik, L=AB- silliq egri chiziq, $P(x, y)$ L egri chiziqning nuqtalarida aniqlangan qandaydir funksiya. L ni nta ixtiyoriy qismlarga bo'lamiz:

$A = M_0, M_1, M_2, \dots, M_n = B$. hosil bo'lgan yoylarda (M_{i-1}, M_i) yoylarda) ixtiyoriy $\overline{M_1}(\overline{x}_i, \overline{y}_i)$ nuqta ni tanlaymiz va quyidagi ko'paytmani hosil qilamiz

$P(\overline{x}_i, \overline{y}_i) \cdot \Delta x_i$ ularni qo'shib quyidagi yig'indiga yega bo'lamiz.

$$S_{n,x} = \sum_{i=0}^n P(\overline{x}_i, \overline{y}_i) \Delta x_i$$

Bu yig'indi x koordinata bo'yicha $P(x, y)$ uchun II – tur egri chizikli integral deyiladi va quyidagicha belgilanadi:

$$\int_L P(x, y) dx$$

SHu kabi u koordinata bo'yicha II – tur egri chizikli integralni ifodalash mumkin.

$$\int_L Q(x, y)dy$$

$Q(x, y)$ – uzluksiz funksiya.

Egri chizikli integralning yig'indisi to'la II – tur egri chizikli integral deyiladi va quyidagicha ifodalaniladi:

$$\int_L P(x, y)dx + Q(x, y)dy$$

II – tur egri chizikli integral shuningdek koordinatalar bo'yicha egri chizikli integral deb ham nomlanadi.

Xususan,

$$\int_{BA} P(x, y)dx + Q(x, y)dy = - \int_{AB} P(x, y)dx + Q(x, y)dy$$

ya'ni egri chizikli integral yo'nalishi o'zgarganda ishora almashadi.

II – tur egri chizikli integrallar

II – tur egri chizikli integralni hisoblash.

Aytaylik $L - y = y(x), x \in [a; b]$ uzluksiz differentsiallanuvchi funksiya orqali berilgan bo'lsin; u holda

$$\int_L P(x, y)dx + Q(x, y)dy = \int_a^b [P(x, y(x)) + Q(x, y(x))y'(x)]dx$$

Agar $L - x = x(t), y = y(t), t \in [a, b]$ parametrik funksiyalar orqali berilgan bo'lsa,

$$\int_L P(x, y)dx + Q(x, y)dy = \int_a^b [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)] dt$$

Bu tenglikni fazoviy holat uchun ham tadbqiq qilish mumkin. (P, Q, R funksiyalar (x, y, z) argument uchun qisqartirib yozildi)

$$\int_L Pdx + Qdy + Rdz = \int_a^b (P, x'(t) + Qy'(t) + R \cdot z'(t))dt$$

Bu yerda L egri chiziq $x = x(t), y = y(t), z = z(t)$ parametrik formulalar bilan berilgan.

II – tur egri chiziqli integralni tadbiqlari.

$$\int_L Pdx + Qdy$$

Integralni $I = Pj + Qj$ va $ds = idx + jdy$ vektorlar ko'paytmasi sifatida tassavur qilish mumkin.

$$\int_L Pdx + Qdy = \int_L F(x,y)ds$$

U holda

$$\int_L Fds$$

Integral $F = Pj + Qj$ o'zgaruvchi kuchning $M = M(x,y)$ nuqtaning $L=AB$ egri chiziq bo'ylab A nuqtadan dan B nuqtaga o'tganda bajargan ishni ifodalaydi.

Agar $A=B$ bo'lib L yopiq egri chiziq bo'lsa, integral

$$\oint Pdx + Qdy$$

ko'rinishda belgilanadi.

Aytaylik Oxu tekislikda bir xil tarqalgan (ya'ni teshiklari bo'lmagan) D soha mavjud b-sin, u $L = \partial p$ egri chiziq bilan chegaralangan ($\partial p - D$ sohasining chegarasi), shu bilan birga $P(x,y)$ va $Q(x,y)$ D sohada va uning chegarasida xususiy hosilalari bilan uzluksiz funksiyalar.

Teorema 1. A va B lar -D sohaning ixtiyoriy nuqtasi bo'lsin, A_mB va A_nB -shu 2 nuqtani birlashtiruvchi qandaydir silliq egri chiziq. U holda quyidagi shartlar teng kuchli:

$$1. \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad (\text{Grin sharti})$$

2.

$$\int_{A_mB} Pdx + Qdy = \int_{A_nB} Pdx + Qdy$$

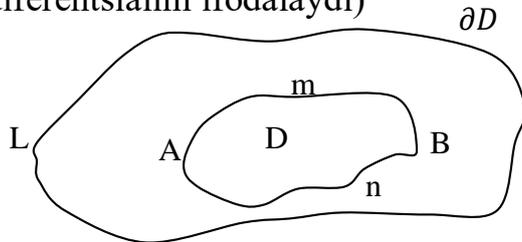
(ya'ni egri chiziqli integral integrallash yo'liga bog'liq emas).

3.

$$\int_{AnBmA} Pdx + Qdy = 0$$

(ixtiyoriy yopiq yo'l bo'yicha olingan integral nolga teng)

4. $Pdx + Qdy = dU$ ($Pdx + Qdy$ qandaydir $U = U(x, y)$ funksiyaning to'la differentsialini ifodalaydi)



Yuqori keltirilgan teorema shartlaridan *ixtiyoriy* 1 tasi bajarilsa, integralni Nyuton-Leybnits formulasi orqali hisoblash mumkin (D sohasidagi (x_0, u_0) va (x_1, u_1) nuqtalarni birlashtiruvchi egri chiziq bo'yicha integral)

$$\int_{(x_0, y_0)}^{(x_1, y_1)} p dx + Q dy = U(x, y) \Big|_{(x_0, y_0)}^{(x_1, y_1)} = U(x_1, y_1) - U(x_0, y_0)$$

bu yerda $U(x, y)$ $Pdx + Qdy$ uchun boshlang'ich funksiya.

Boshqa tomondan $U(x, y)$ boshlang'ich funksiya egri chizikli integral yordamida hisolanishi mumkin.

$$U(x, y) = \int_{(x_0, y_0)}^{(x_1, y_1)} P dx + Q dy$$

SHu shartlar bo'yicha yopiq kontur bo'yicha olingan egri chizikli integralni ikki karrali integralni hisoblashga o'tkazadigan Grin formulasi o'rinalidir.

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Aytish joizki, $S=S(D)$ D sohaning yuzasi II tur egri chizikli integral bo'yicha hisoblanishi mumkin.

$$S = \frac{1}{2} \oint_{\partial D} x dy - y dx$$

Bu Grin formulasidan kelib chiqadi

$$P = -\frac{1}{2}y, \quad Q = \frac{1}{2}y$$

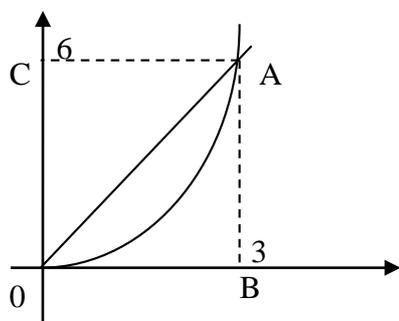
49. $P(x, y) = 8x + 4y + 2$, $Q(x, y) = 8y + 2$ funksiyalar va $A(3:6)$, $B(3:0)$, $C(0:6)$ nuqtalar berilgan.

$$\int_L (8x + 4y + 2)dx + (8y + 2)dy$$

Egri chiziqdilar byicha hisoblansin.

- 1) L- OA kesma
- 2) L- OBA siniq chiziq
- 3) L- OCA siniq chiziq
- 4) L- Oy o'qqa nisbatan simetrik va O va A nuqtalardan o'tuvchi parabola
- 5) Grin shartlari bajarilishini tekshiring.

Yechish: Integrallash yo'li quyidagi chizmada ko'rsatilgan:



- 1) OA kesma $y=2x$, $x \in [0; 3]$ orqali ifodalanadi; U holda $dy=2dx$ va

$$\begin{aligned} \int_{OA} Pdx + Qdy &= \int_0^3 [8x + 4 \cdot 2x + 2]dx + (8 \cdot 2x + 2) \cdot 2dx \\ &= \int_0^3 (48x + 6)dx = 24x^2 + 6x \Big|_0^3 = 234 \end{aligned}$$

- 2) Alohida OB va BA bo'yicha integralni olib qo'shamiz

a) OB uchun $y=0$, $0 \leq x \leq 3$, ya'ni $dy=0$ bundan

$$\int_{OB} (8x + 4y + 2)dx + (8y + 2)dy = \int_0^3 (8x + 2)dx = (4x^2 + 2x) \Big|_0^3 = 42$$

b) BA uchun $x=3$, $0 \leq y \leq 6$, $dx=0$

$$\int_{BA} (8x + 4y + 2)dx + (8y + 2)dy = \int_0^6 (8y + 2)dy = (4y^2 + 2y) \Big|_0^6 = 156$$

Nihoyat

$$\int_{OBA} (8x + 4y + 2)dx + (8y + 2)dy = 42 + 156 = 198$$

3) Bu integralni yuqoridagi kabi hisoblaymiz.

a) OC: $x=0$: (ya'ni $dx=0$), $0 \leq y \leq 6$

$$\int_{OC} (8x + 4y + 2)dx + (8y + 2)dy = \int_0^6 (8y + 2)dy = 156$$

b) CA: $0 \leq x \leq 3$, $y=6$, $dy=0$, demak,

$$\int_{CA} (8x + 4y + 2)dx + (8y + 2)dy = \int_0^3 (8x + 26)dx = 114$$

Nihoyat

$$\int_{OCA} (8x + 4y + 2)dx + (8y + 2)dy = 114 + 156 = 270$$

4) $y = ax^2$ ga A(3:6) nuqtalarni qo'yib parabola tenglamasini topamiz

$y = \frac{2x^2}{3}$, bunda $0 \leq x \leq 3$ va $dy = \frac{4}{3}x dx$

$$\begin{aligned} \int_{OA} (8x + 4y + 2)dx + (8y + 2)dy \\ &= \int_0^3 \left[\left(8x + \frac{8x^2}{3} + 2 \right) dx + \left(\frac{16x^2}{3} + 2 \right) \frac{4}{3} x dx \right] \\ &= \int_0^3 \left(\frac{64}{9}x^3 + \frac{8}{3}x^2 + \frac{32}{3}x + 2 \right) dx \\ &= \left(\frac{16}{9}x^4 + \frac{8}{9}x^3 + \frac{16}{3}x^2 + 2x \right) \Big|_0^3 = 222 \end{aligned}$$

5. $\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}(8x + 4y + 2) = 4$; $\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}(8y + 2) = 0$ ya'ni Grin sharti

bajarilmayapti.

50-misol.

$P(x, y) = y + 3$, $Q(x, y) = 8x + 7y + 6$ va A(9;4) B(9;0) C(0;4)

nuqtalar berilgan. Egri chizikli integral hisoblansin.

$$\int_L (y + 3)dx + (8x + 7y + 6)dy$$

bu yerda:

- 1) L – OA kesma
- 2) L – OBA siniq chiziq
- 3) L – OCA siniq chiziq
- 4) L – Oy o'qqa nisbatan simmetrik parabolaning O (0;0) A (9;4) nuqta larni birlashtiruvchi qismi.

5. Grin shartini tekshiring.

51. Misol:

$$\int_L (4y + 4)dx + (3x + 3y + 4)dy$$

O(0;0) , A (2;6) , B (2;0) , C (0;6) xar xil yo'llar orqali:

- 1) L= OA
- 2) L= OCA
- 3) L= OBA
- 4) L- $y = \frac{3}{2}x^2$ parabolaning OA yoyi

52- misol.

$$\int_L (2xy)dx - x^2 dy$$

O (0;0) , A (2;1) , B (2;0) , C (6;1) xar xil yo'llar bo'yicha egri chiziqli integral hisoblansin.

- 1) L- OA kesma
- 2) L- O va A nuqta dan o'tuvchi Oy o'qqa simetrik parabola
- 3) L- Oxga simmetrik va OA nuqta lardan o'tuvchi parabola
- 4) L- OBA siniq chiziq
- 5) L- OCA siniq chiziq.

53- misol.

Integralni hisoblang.

$$\int_L y^2 dx + x^2 dy$$

L- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Elipsning soat strelkasi yo'nalishidan yuqori qismi.

Yechish:

Elipsning parametrik tenglamalaridan foydalanamiz. $x = a \cos t$,
 $y = b \sin t, t \in [0; \pi]$ ya'ni: $dx = -a \sin t dt$ $dy = b \cos t dt$

Integralga qo'ysak. ($t = \pi$ dan 0 ga qadar o'zgaradi).

$$\begin{aligned}
\int_L y^2 dx + x^2 dy &= \int_{\pi}^0 (-b^2 \sin^2 t \cdot a \sin t + a^2 \cos^2 t \cdot b \cos t) dt \\
&= \int_{\pi}^0 ab^2 \sin^2 t \sin t dt - \int_{\pi}^0 a^2 b \cos^2 t \cdot \cos t dt \\
&= -ab^2 \int_{\pi}^0 (1 - \cos^2 t) d(\cos t) - a^2 b \int_{\pi}^0 (1 - \sin^2 t) d(\sin t) \\
&= -ab^2 \left(\cos t - \frac{\cos^3 t}{3} \right) \Big|_{\pi}^0 - a^2 b \left(\sin t - \frac{\sin^3 t}{3} \right) \Big|_{\pi}^0 = \frac{4}{3} ab^2
\end{aligned}$$

54-misol

Hisoblang.

$$\int_L \frac{x^2 dy - y^2 dx}{x^{\frac{5}{3}} + y^{\frac{5}{3}}}$$

L- A (R;0) dan B (0;R) ga o'tuvchi $X=R \cos^3 t$ $y=R \sin^3 t$ parametrik tenglamalar bilan berilgan egri chiziq yoyi.

55-misol.

Hisoblang.

$$\int_L xy dx$$

L- $y = \sin x$ (0;0) dan $(\pi; 0)$ nuqta ga o'tuvchi sinusonda yoyi.

56-misol.

$$\int_L x dy$$

L- $\frac{x}{a} + \frac{y}{b} = 1$ t/chiziqning A (a; 0) va V (0; b) kesmasi.

57-misol.

$$\int_L (x^2 - y^2) dx + (x^2 + y^2) dy$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning (soat strelkasiga teskari) musbat yo'nalishi bo'yicha integralni hisoblang.

58-misol.

$$\int_L yzdx + xzdy + xydz$$

t- 0 dan 2π gacha $x = a \cos t$, $y = a \sin t$, $z = bt$ vint chizig'i yoyi bo'yicha hisoblang.

Yechish:

Avval o'zgaruvchilar differentsialini topamiz. $dx = -a \sin t$, $dy = a \cos t dt$, $dz = bdt$. Integral ostidagi ifodani t orqali belgilaymiz.

$$\begin{aligned} \int_L yzdx + xzdy + xydz &= \int_0^{2\pi} (-a^2 b t \sin^2 t + a^2 b t \cos^2 t + ba^2 \sin t \cos t) dt \\ &= a^2 b \int_0^{2\pi} \left(t \cos 2t + \frac{\sin 2t}{2} \right) dt \\ &= a^2 b \left(t \cdot \frac{\sin 2t}{2} \Big|_0^{2\pi} - \frac{1}{2} \int_0^{2\pi} \sin 2t dt - \frac{\cos 2t}{4} \Big|_0^{2\pi} \right) = 0 \end{aligned}$$

59-misol.

Hisoblang.

$$\int_L xdy - ydx$$

$$L- x = 2\sqrt{5} \cos^3 t, \quad y = 4\sqrt{5} \sin^3 t \quad t \in [0; 2\pi]$$

Tenglamalar bilan berilgan.

60. L- (0;0), A(2;8) nuqtalarni birlashtiruvchi $y = ax^2$ parabola yoyi bo'yicha integralni hisoblang.

$$\int_L (x^2 - 2xy^2 + 3)dx + (y^2 - 2x^2y + 3)dy.$$

61. Hisoblang.

$$\int_L ydx + zdy + xdz$$

$L-x = acost, y = asint, z = bt, 0 \leq t \leq 2\pi$ parametrning kamayish yo'nalishida

62.

$$\int_{(0,0)}^{(10,10)} (x+y)dy + (x-y)dy$$

Integral integrallash yo'nalishiga bog'liq emasligini ko'rsating.

Yechish: Aytaylik $a = x + y, a = x - y, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 1$

Sharti o'rinli bo'ldi. OB kesma $y=x, x \in [0;10] dy=dx$

$$\int_{(0,0)}^{(10,10)} (x+y)dx + (x-y)dy = \int_0^{10} (x+x)dx = x^2 \Big|_0^{10} = 100$$

63. Hisoblang.

$$\int_{(0,0)}^{(1,1)} (3x^2 - 3y)dx + (3y^2 - 3x)dy$$

64. Hisoblang.

$$\int_{(1,1)}^{(2,0)} (3x^2 + 6xy^2)dx - (6x^2y + 4y^3)dy$$

65. Hisoblang.

$$\int_{(0,0)}^{(2,3)} (x^3 - 3xy^2 + 2) - (3x^2y - y^2)dy$$

66. Hisoblang.

$$\int_L (x+1)dx + xyzdy + y^2zdz$$

$L-C(2;3;-1)$ va $D(3,-2,0)$ nuqtalarni tutashtiruvchi to'g'ri chiziq kesmasi.

Yechish: Bu kesma uchun to'g'ri chiziq tenglamasini tuzamiz.

$$\frac{x-2}{1} = \frac{y-3}{-5} = \frac{z+1}{1}$$

$$x = 2 + t, y = 3 - 5t, z = -1 + t, t \in [0; 1]$$

$$dx = dt, dy = -5dt, dz = dt$$

Javob:

$$\int_0^1 [(3+t)dt - 5(2+t)(3-5t)(-1+t)dt + (3-5t)^2(-1+t)dt]$$
$$= \int_0^1 (24 - 25t - 45t^2 + 50t^3)dt = 9$$

67. Hisoblang.

$$\int_L xydx - ydy$$

L-C(4;0), D(0;2) nuqtyalarni tutashtiruvchi egri chiziq.

CD- To'g'ri chiziq kesmasi

CD- Ox ga nisbatan simmetrik parabola

CD- OY o'qqa nisbatan simmetrik parabola

CD- markazi koordinata boshida bo'lgan ellips yoyi.

To'la differensialdan iborat integralni hisoblang (68-71).

68.

$$\int_{(-1,2)}^{(2,3)} xdy + ydx$$

69.

$$\int_{(0,1)}^{(3,4)} xdx + ydy$$

70.

$$\int_{(0,0)}^{(1,1)} (x+y)(dx+dy)$$

71.

$$\int_{(1,2)}^{(2,1)} \frac{ydx - xdy}{y^2},$$
$$y \neq 0$$

72.

$$(3x^2y + \frac{1}{y})dx + (x^3 - \frac{x}{y^2})dy$$

Ifoda to'la differensialliginia tekshiring va U(x,y) funksiyani toping.

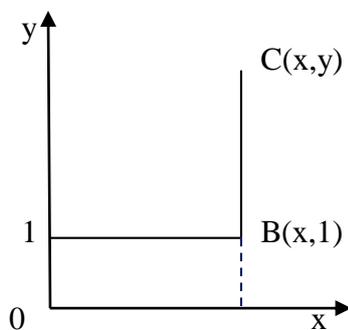
$$\text{Yeshish: } P=3x^2y+\frac{1}{y}, \quad Q=x^3-\frac{x}{y^2}$$

$$\frac{\partial P}{\partial y}=3x^2-\frac{1}{y^2}, \quad \frac{\partial Q}{\partial x}=3x^2-\frac{1}{y^2} \quad y \neq 0$$

Tenglik sharti bajarilmoqda ($\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$) Demak yuqoridagi ifoda qandaydir $U(x,y)$ funksiyadan olingan to'la differensial va uni egri chiziqli integral sifatida topamiz.

$$\int_{(x_0,y_0)}^{(x,y)} (3x^2y + \frac{1}{y})dx + (x^3 - \frac{x}{y^2})dy$$

Bu yerda $(x_0,y_0) = (0,1)$ tekislikdagi Ox o'qda yotmaydigan tayin nuqta Masalan $(x_0,y_0) = (0,1)$ $L=ABC$ sifatida rasmda ko'rsatilgan yo'lni tanlaymiz. U holda qisqacha quyidagicha yozishimiz mumkin.



$$U(x,y) = \int_{(x_0,y_0)}^{(x,y)} = \int_{ABC} = \int_{AB} + \int_{BC}$$

1) (A,B): $y=1$ ya'ni $dy = 0$

$$\int_{AB} (3x^2 - y + \frac{1}{y})dx + (x^3 - \frac{x}{y^2})dy = \int_{(0,1)}^{(x,1)} (3x^2 + 1)dx = x^3 + x$$

2) (BC) : x -tayin nuqta, $dx=0$

$$\int_{BC} (3x^2y + \frac{1}{y})dx + (x^3 - \frac{x}{y^2})dy = \int_{(x,1)}^{(x,y)} (x^3 - \frac{x}{y^2})dy = \left(x^3y + \frac{x}{y}\right) \Big|_{(x,1)}^{(x,y)} = x^3y + \frac{x}{y} - x^3 - x$$

3) Demak $u(x,y) = x^3 + x + x^3y + \frac{x}{y} - x^3 - x = x^3y + \frac{x}{y}$

Tekshirsak $dU = d(x^3 + \frac{x}{y}) = (3x^2y + \frac{1}{y})dx + (x^3 + \frac{x}{y^2})dy$

Integral ostidagi funksiyaning boshlang'ich funksiyasini topish orqali egri chiziqli integralni hisoblash (73-76)

73.

$$\int_{(-2,-1)}^{(3,0)} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy$$

74.

$$\int_{(0,-1)}^{(1,0)} \frac{xdy - ydx}{(x - y)^2}$$

$y \neq x$

75.

$$\int_{(1,1)}^{(3,1)} \frac{(x + 2y)dx + ydy}{(x + y)^2}$$

, $(x+y) \neq 0$

76.

$$\int_{(3,4)}^{(12,5)} \left(\frac{x}{\sqrt{x^2 + y^2}} + y \right) dx + \left(\frac{y}{\sqrt{x^2 + y^2}} + x \right) dy$$

Fazodagi egri chiziqlar konturidan olingan egri chiziqli integrallarni hisoblang. (77-78)

77.

$$\int_L (y - z)dx + (z - x)dy + (x - y)dz$$

, bu yerda $L - x = acost, y = asint, z = bt, 0 \leq t \leq 2\pi$ vint chizig'i o'rami.

78.

$$\oint_L ydx + zdy + xdz,$$

$L - x = Rcosacost, y = Rcosasint, z = Rsina(\alpha - cost)$ tenglama bilan berilgan aylana

to'la differensialdan olingan egri chiziqli integral hisoblansin (oldindan boshlang'ichini topib hisoblang).

79.

$$\int_{(1,0,-3)}^{(6,4,8)} xdx + ydy - zdz$$

80.

$$\int_{(1,1,1)}^{(a,b,c)} yzdx + xzdy + xydz$$

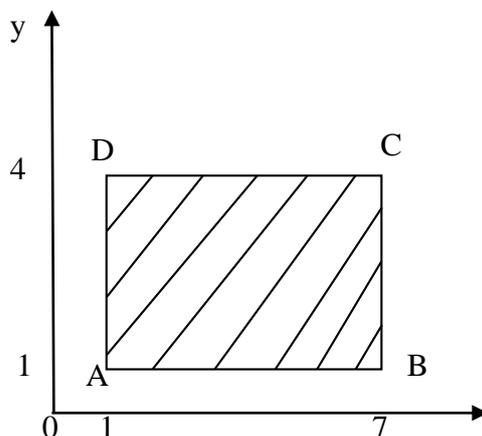
81.

$$\int_{(0,0,0)}^{(3,4,5)} \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}}$$

82. Grin formulasi yordamida egri chiziqli integralni ikki karraliga o'tkazib hisoblang.

$$\oint_L (\sqrt{x^2 + y^2})dx + y[xy + \ln(x + \sqrt{x^2 + y^2})]dy$$

L-ABCD to'g'ri burchakli to'rt burchak, bu yerda A(1,1), B(7,1), C(7,4), D(1,4)



Yechish: $P = \sqrt{x^2 + y^2}$, $Q = y[xy + \ln(x + \sqrt{x^2 + y^2})]$

$$\frac{\partial Q}{\partial x} = y \left[\left(y + \frac{1}{x + \sqrt{x^2 + y^2}} \right) \left(1 + \frac{x}{\sqrt{x^2 + y^2}} \right) \right] = y \left[y + \frac{1}{\sqrt{x^2 + y^2}} \right] = y \frac{y\sqrt{x^2 + y^2} + 1}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial P}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + \frac{y}{\sqrt{x^2 + y^2}} - \frac{y}{\sqrt{x^2 + y^2}} = y^2$$

Demak Grin formulasi yordamida egri chiziqli integral y^2 bo'yicha ABCD to'g'ri burchakli to'rt burchak orqali olingan ikki karrali integralga teng.

$$\begin{aligned} \oint_L Pdx + Qdy &= \iint_{ABCD} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \iint_{ABCD} y^2 dxdy = \int_1^7 dx \int_1^4 y^2 dy = x \Big|_1^7 \quad \frac{y^3}{3} \Big|_1^4 \\ &= 7 \frac{63}{3} = 147 \end{aligned}$$

83. Grin formulasi yordamida hisoblang.

$$\int_L 2(x^2 + y^2)dx + (x + y)^2 dy$$

Bu yerda $L-\Delta ABCD$ konturi $A(1,1)$, $B(2,2)$, $C(1,3)$ olingan natijani egri chiziqli integralni hisoblash yordamida tekshirish.

To'la differensiallar bo'yicha funksiyani topping (84-88).

$$84. du = x^2 dx + y^2 dy$$

$$85. du = u(x^2 - y^2)(x dx - y dy)$$

$$86. du = \frac{(x+2y)}{(x+y)^2} dx + y dy$$

$$87. du = \frac{x}{y\sqrt{x^2+y^2}} dx - \frac{x^2+\sqrt{x^2+y^2}}{y^2\sqrt{x^2+y^2}} dy$$

$$88. du = \left(\frac{x-2y}{(y-x)^2} + x\right) dx + \left(\frac{y}{(y-x)^2} - y^2\right) dy$$

Grin formulasi yirdamida egri chiziqli integralni hisoblang. (89-92)

89.

$$\oint \frac{x dy + y dx}{x^2 + y^2}$$

$L : (x-1)^2 + (y-1)^2 = 1$ aylana bo'yicha soat strelkasiga teskari yo'nalish.

90.

$$\oint (xy + y + x) dx + (yx - y + x) dy$$

, $L - \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellips

91.

$$\oint (xy + x + y) dx + (xy + x - y) dy$$

, $L - x^2 + y^2 = ax$ aylana

92.

$$\oint (x^4 + 4xy^3) dx + (6x^2y^2 - 5y^4) dy$$

, $L - A(-a; \sqrt{2b})$ va $B(a, \sqrt{2b})$ nuqtalarni tutashtiruvchi $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ giperbola yuqori qismli yoyi va AB kesmani tutashtiruvchi to'g'ri chiziq qismi.

93. Egri chiziqli integral yordamida ellips yuzasini hisoblang.

Yechish: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsni parametric ko'rinishda yozamiz: $x = acost$, $y = bsint$, $0 \leq t \leq 2\pi$.

$$S = \frac{1}{2} \oint_{\partial D} x dy - y dx = \frac{1}{2} \int_0^{2\pi} (acost bcost + bsint asint) dt = \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab$$

Quyidagi chiziqlar bilan chegeralangan fugura yuzalarini hisoblang (94-98)

$$94. \text{Astroida } x = acos^3 t, y = asin^3 t$$

$$95. \text{Kardioida } x = a(2cost - cos2t), y = a(2sint - sin2t)$$

$$96. \text{Dekart yaprog'i tuguni } x^3 + y^3 - 3axy = 0 \quad (a > 0)$$

$$97. \text{Tugun } (x+y)^4 = x^2 y$$

$$98. \text{Bernulli lemniskatasi } (x^2 + y^2)^2 = a^2(x^2 - y^2)$$

99. C(a,0) dan B(-a,0) nuqtaga $\frac{x^2}{a^2} + \frac{y^2}{b^2}$

Ellips orqali o'tishda $F = yi - xj$ kuchli maydonda material nuqtaning bajargan ishini hisoblang.

Yechish: $F = Pi + aj$ kuchli maydonda M nuqtaning CB chiziq bo'yicha bajargan A ishi

$$\int_{CB} Pdx + Qdy$$

ga teng. CB ellips yoyini quyidagicha yozamiz $x = acost, y = bsint, t \in [0; \pi]$ $dx = -asintdt, dy = bcostdt$ va

$$A = \int_{CB} ydx - xdy = \int_0^{\pi} (-absin^2t - abcos^2t)dt = -ab \int_0^{\pi} dt = \pi ab$$

100. $F = -yi + (3y - 8x)j$ kuch berilgan. Material nuqtaning A(9,4), B(-9,4), C(-9,-4), D(9,-4) uchli to'g'ri to'rtburchak konturi bo'yicha harakatlangandagi A ishini hisoblang.

101. $F = -yi + (3y - 8x)j$ kuchning $\frac{x^2}{81} + \frac{y^2}{16} = 1$ ellips bo'yicha harakatlangandagi A ishini hisoblang.

102. $F = 4yi + (4y - 3x)j$ kuchning uchlari A (2,-6), B(2,6), C(-2,6), D(-2,-6) bo'lgan to'g'ri to'rtburchak bo'yicha bajargan ishni toping.

103. $F = 4yi + (4y - 3x)j$ kuchning $\frac{x^2}{4} + \frac{y^2}{36} = 1$ ellips bo'yicha bajargan ishni toping.

Sirt integrali

I-tur sirt integrallarining ta'rifi va ularni hisoblash.

Aytaylik $f(x,y,z)$ funksiya qandaydir S silliq sirtga berilgan bo'lsin. S sohani S_1, \dots, S_n qismlarga ajratamiz, qism yuzlari mos ravishda $\Delta\sigma_1 \dots \Delta\sigma_n$ va diametrlari d_1, \dots, d_n . Har bir S_i qismda $M_i(x_i, y_i, z_i)$ nuqtani tanlab quyidagi yig'indini hosil qilamiz.

$$\sum_{i=1}^n f(x_i, y_i, z_i) \Delta\sigma$$

Bu yig'indi $f(x, y, z)$ funksiya uchun I-tur integral yig'indi deyiladi. Agar $d \rightarrow 0$ ($d = \max(d_i)$) integral yig'indi mavjud bo'lsa (y S ni qismlarga ajratish usuliga va M_i nuqta tanlanishiga bog'liq bo'lmaydi) u holda bu yig'indi I-tur sirt integrali deb ataladi va quyidagicha ifodalanadi.

$$\iint_S f(x, y, z) d\sigma$$

Agar $f(x, y, z)$ uzluksiz bo'lsa

$$\iint_S f(x, y, z) d\sigma$$

integral mavjud.

I-tur sirt integrallari I tur egri chiziqli integral kabi aniqlanadi. Ularning xossalari ham o'xshashdir.

Agar S sirt D sohaning o'xy tekislikga $z = z(x, y)$ funksiya orqali berilsa, shu bilan birga $z(x, y)$ funksiya $z'_x = z'_x(x, y)$ va $z'_y = z'_y(x, y)$ hosilalari bilan uzluksiz bo'lsa, sirt integrali quyidagi ikki karrali integralni hisoblashga keltiriladi.

$$\iint_S f(x, y, z) d\sigma = \int_D f(x, y, z(x, y)) \sqrt{1 + (z'_x)^2 + (z'_y)^2} dx dy$$

Agar S parametrik ko'rinishda $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ orqali berilsa $xyz - \sigma$ sohaning Ouv tekislikda uzluksiz differensiallanuvchi bo'lsa, u holda

$$\iint_S f(x, y, z) d\sigma = \iint_{\sigma} [x(u, v), y(u, v), z(u, v)] \sqrt{EH - F^2} duv$$

bu yerda $E = \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$, $H = \left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2 + \left(\frac{dz}{dv}\right)^2$

$$F = \frac{dx}{du} \frac{dx}{dv} + \frac{dy}{du} \frac{dy}{dv} + \frac{dz}{du} \frac{dz}{dv}$$

I-tur integralining tadbiqlari

Aytaylik $S - \rho = \rho(x, y, z)$ zichlikka ega bo'lgan material sirt. U holda sirt integrali yordamida quyidagilarni hisoblash mumkin:

Bu sirtning tekisliklariga nisbatan statistik momentlarini

$$M_{xy} = \iint_S z \rho d\sigma$$

$$M_{yz} = \iint_S x \rho d\sigma$$

$$M_{xz} = \iint_S y \rho d\sigma$$

Sirtning og'irlik markazi koordinatalarini topish.

$$x_c = \frac{M_{yz}}{m}; \quad y_c = \frac{M_{xz}}{m}; \quad z_c = \frac{M_{xy}}{m}$$

Bu yerda

$$m = \iint_S \rho d\sigma$$

Koordinata o'qlari va koordinata boshiga nisbatan inersiya momentini topish.

$$J_x = \iint_S (y^2 + z^2) \rho d\sigma,$$

$$J_y = \iint_S (x^2 + z^2) \rho d\sigma,$$

$$J_z = \iint_S (x^2 + y^2) \rho d\sigma,$$

$$J_0 = \iint_S (x^2 + y^2 + z^2) \rho d\sigma,$$

II- tur sirt integralini aniqlanishi va hisoblash.

S sirtning yuzasini quyidagi formula yordamida hisoblash mumkin;

$$\iint_S d\sigma = S_{yuz}$$

Agar $\rho(x, y, z)$ – S sirtning sirt zichligi bo'lsa, u holda m massa

$$m = \int_S \rho(x, y, z) d\sigma$$

Aytaylik S-silliq orientirlangan sirt bo'lsin, unda $R(x, y, z)$ sirtida berilgan uzluksiz funksiya, har bir M nuqta $\vec{n}(M)$ musbat yo'naltirilgan normal ($\vec{n}(M)$ -uzluksiz vektor funksiya) S^+ sifatida S sirtning \vec{n} normalning birligi va Oz o'qning orasidagi burchak o'tkir bo'lgan qismi tanlanadi. Endi S sirtni $S_1 \dots S_n$, diametrlari $d_1 \dots d_n$, qismlarga ajratamiz. $\Delta p_1 \dots \Delta p_n$ orqali $S_1 \dots S_n$ qismlarning Oxy tekisliklarga o'tkazilgan proyeksiyalarini belgilaymiz ($d = \max d_i$) har bir S qismdan $M_i(x_i, y_i, z_i)$ nuqtani tanlab quyidagi yig'indini tuzamiz:

$$\sum_{i=1}^n R(x_i, y_i, z_i) \Delta p_i$$

Bu yig'indi $R(x, y, z)$ funksiya uchun II tur integral yig'indi deyiladi. $d \rightarrow 0$ integral yig'indi limiti II tur sirt integrali deyiladi va

$$\iint_{S^+} R(x, y, z) dx dy$$

orqali ifodalanadi.

Shu kabi II tur sirt integrallari aniqlanadi:

$$\iint_{S^+} P(x, y, z) dy dz$$

va

$$\iint_{S^+} Q(x, y, z) dx dz$$

3 ta ko'rsatilgan sirt integrallarining yig'indisi to'la II tur sirt integrali deyiladi:

$$\iint_{S^+} P dy dz + Q dx dz + R dx dy$$

Agar S sirt $z = z(x, y)$, $(x, y) \in D$ COxy ko'rinishiga ega bo'lsa, u holda II tur sirt integral ikki karrali integralni hisoblashga keltiriladi.

$$\iint_{S^+} R(x, y, z) dx dy = \iint_D R(x, y, z(x, y)) dx dy$$

Agar S sirtning qarama-qarshi S tomoni tanlansa

$$\iint_{S^-} R(x, y, z) dx dy = - \iint_D R(x, y, z(x, y)) dx dy$$

Shu kabi qolgan sirt integrali hisoblanadi.

$$\iint_{S^+} P(x, y, z) dy dz$$

va

$$\iint_{S^+} Q(x, y, z) dx dz$$

I va II tur sirt integrallarining aloqasi.

Agar α, β, γ -n normalning bilish bilan Ox, Oy, Oz o'qlarning musbat yo'nalishi orasidagi burchak bo'lsa, ularning aloqasi quyidagicha aniqlanadi:

$$\iint_{S^+} P dy dz + Q dx dz + R dx dy = \iint_{S^+} (P \cos \alpha + Q \cos \beta + R \cos \gamma) d\sigma$$

$n = \{\cos \alpha, \cos \beta, \cos \gamma\}$ ekanligidan I-tur sirt integralni vector ko'rinishida yozish mumkin:

$$\iint_S F n d\sigma$$

$F = \{P, Q, R\}$ - S ga orientlangan vektor maydon.

F vektor maydon sifatida S sirtida oqib o'tayotgan suyuqlik tezligi maydoni sifatida ko'rilsa, integral

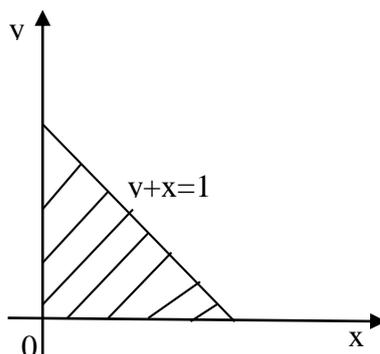
$$\iint_S Fnd\sigma$$

S sirtida musbat yo'nalishda 1 birlik vaqt mobaynida oqib o'tayotgan suyuqlik miqdori sifatida tushuntirish mumkin. Shuning uchun bu integral S sirtidagi F vector maydonning oqimi deyiladi.

104. I-tur sirt integrali hisoblansin:

$$\iint_{\sigma} \frac{d\sigma}{(1+x+z)^3}$$

Bu yerda σ - $x+y+z=1$ tekislikning 1 oktontadagi qismi.



Yechish: σ sirtini quyidagicha ifodalaymiz: $z=1-x-y$, $(x,y) \in D$, bu yerda D - $x=0$, $y=0$ va $x+y=1$ chiziqlar bilan chegaralangan Δ .

Shu bilan birga $d\sigma = \sqrt{1 + (z'_x)^2 + (z'_y)^2} dx dy = \sqrt{3} dx dy$. Bu integral ikki karrali integralga keltiriladi:

$$\begin{aligned} \iint_{\sigma} \frac{d\sigma}{(1+x+z)^3} &= \iint_{\sigma} \frac{\sqrt{3} dx dy}{(2-y)^3} = \sqrt{3} \int_0^1 dx \int_0^{1-x} \frac{dy}{(2-y)^3} = \sqrt{3} \int_0^1 dx \left(\frac{1}{(2-y)^2} \Big|_0^{1-x} \right) \\ &= \frac{\sqrt{3}}{2} \left(-\frac{1}{1+x} - \frac{1}{4}x \right) \Big|_0^1 = \frac{\sqrt{3}}{2} \frac{1}{4} = \frac{\sqrt{3}}{8} \end{aligned}$$

I-tur sirt integralini hisoblang (105-106)

105.

$$\iint_{\sigma} \frac{d\sigma}{(1+x+z)^2}$$

, bu yerda σ - $x+y+z=1$ tekislikning $x \geq 0, y \geq 0, z \geq 0$ qismi.

106.

$$\iint_{\sigma} \left(z + 2x + \frac{4}{3}y \right) d\sigma, \sigma - 6x + 4y + 3z = 11$$

tekislikning I oktontadagi qismi.

107. I-tur sirt integralini hisoblang.

$\sigma-x^2+y^2+z^2=R^2$ sfera.

Yechish: σ sirtning koordinata tekisliklariga nisbatan simmetrik ekanini e'tiborga olsak, $x \geq 0, y \geq 0, z \geq 0$ uchun hisoblab, natijani 8 ga ko'paytiramiz.

Sferik koordinatalaridan foydalanib, sferaning parametrik tenglamasini yozamiz. $x = R \sin \varphi \cos \theta$,

$y = R \sin \varphi \sin \theta, z = R \cos \varphi$ $u = \theta, v = \varphi$ desak

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = (-R \sin \varphi \sin \theta)^2 + (R \sin \varphi \cos \theta)^2 + 0 = R^2 \sin^2 \varphi$$

$$= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = (R \cos \varphi \cos \theta)^2 + (-R \cos \varphi \sin \theta)^2 + (-R \sin \varphi)^2$$

$$= R^2$$

$$F = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} = (-R \sin \varphi \sin \theta)(R \cos \varphi \cos \theta) + (R \sin \varphi \cos \theta)(-R \cos \varphi \sin \theta)$$

$$= 0$$

$$\sqrt{EG - F^2} = R^2 \sin \varphi$$

Integral soha $x^2+y^2 \leq R^2$ aylana chorak qismi (uni B orqali ifodalaymiz) parametrik formada quyidagi ko'rinishga ega.

$$R^2 \sin^2 \varphi \cos^2 \theta + R^2 \sin^2 \varphi \sin^2 \theta \leq R^2, 0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}$$

Integral ostidagi funksiyani parametric ifodalaymiz $f(x,y) = x^2+y^2$
 $x^2+y^2+z^2=R^2$ sferada $f(x,y) = R^2-z^2 = R^2+R^2 \cos^2 \varphi = R^2(1-\cos^2 \varphi)$.

Berilgan integral quyidagiga teng.

$$8 \int_B R^2 (1 - \cos^2 \varphi) R^2 \sin^2 \varphi d\theta d\varphi = -8R^2 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} (1 - \cos^2 \varphi) d\cos \varphi$$

$$= -8R^4 \frac{\pi}{2} \left(\cos \varphi - \frac{\cos^3 \varphi}{3} \right) \Big|_0^{\frac{\pi}{2}} = \frac{8}{2} R^4$$

108.

$$\iint_S \sqrt{x^2 + y^2}$$

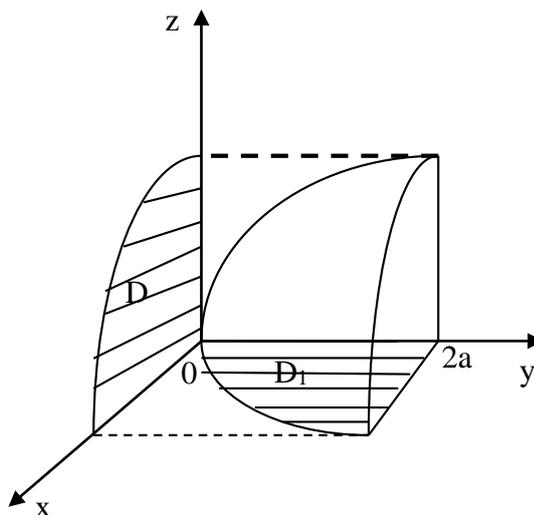
dS integrani hisoblang.

$$S - \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \text{ konusning yon sirti } (0 \leq z \leq b)$$

109. $ay = x^2 + y^2$ paraboloidning I oktantdagi $y=2a$ tekislik bilan chegaralangan qismi yuzasini hisoblang.

Yechish: 1-usul yuzani $y = \frac{x^2+y^2}{a}$ funksiya sifatida xva z o'zgaruvchilar orqali hisoblaymiz.

Demak, yuzani hisoblash quyidagi formulaga keladi.



$$S = \iint_D \sqrt{1 + (y'_x)^2 + (y'_z)^2} dx dz$$

D- sirtning Oxyz tekislikdagi proyeksiyasi $ay=x^2+z^2$ parabolaning $y=2a$ tekislik bilan $ay=x^2+z^2$ radiusi $a\sqrt{2}$ ga teng aylana orqali kesishadi. Demak, D- $x^2 + z^2 \leq 2a^2$ ($x \geq 0, z \geq 0$) aylana chorak qismi integral ostidagi funktsiyani aniqlaymiz $y'_x = \frac{2x}{a}$; $y'_z = \frac{2z}{a}$

$$1 + (y'_x)^2 + (y'_z)^2 = 1 + \frac{4x^2}{a^2} + \frac{4z^2}{a^2} = \frac{a^2 + 4(x^2 + z^2)}{a^2}$$

Shunday qilib

$$S = \frac{1}{a} \iint_D \sqrt{a^2 + 4(x^2 + z^2)} dx dz$$

Qutb koordinatalarga o'tib, quyidagiga ega bo'lamiz :

$$S = \frac{1}{a} \int_0^{\frac{\pi}{2}} d\varphi \int_0^{a\sqrt{2}} \sqrt{a^2 + 4r^2} r dr = \frac{1}{a} \varphi \left| \frac{1}{4 \cdot 3} (a^2 + 4r^2)^{3/2} \right|_0^{a\sqrt{2}} = \frac{13}{12} \pi a^3$$

2-usul Paraboloid sirtini $z = \sqrt{ay - x^2}$ sifatida ko'ramiz. U holda yuza

$$S = \iint_{D_1} \sqrt{1 + (z'_x)^2 + (z'_y)^2} dx dy$$

D_1 - sirtning Oxy tekislikdagi proyeksiyasi D_1 soha Oy o'qi, $x = \sqrt{ay}$, $y = 2a$ chiziqlar bilan chegaralangan integral ostidagi funktsiyani

$$z'_x = -\frac{x}{\sqrt{ay-x^2}}, \quad z'_y = \frac{a}{2\sqrt{ay-x^2}}, \quad 1 + (z'_x)^2 + (z'_y)^2 = \frac{4ay+a^2}{4(ay-x^2)}$$

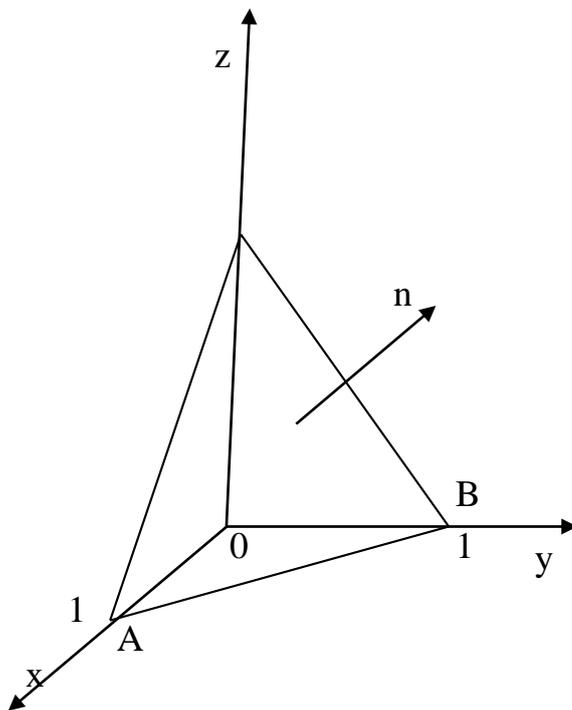
$$\begin{aligned}
S &= \frac{1}{2} \iint_{D_1} \frac{\sqrt{4ay + a^2}}{\sqrt{ay - x^2}} dx dy \\
&= \frac{1}{2} \int_0^{2a} \sqrt{4ay + a^2} dy \int_0^{\sqrt{ay}} \frac{dx}{\sqrt{ay - x^2}} = \frac{1}{2} \int_0^{2a} \sqrt{4ay + a^2} dy \cdot \arcsin \frac{x}{\sqrt{ay}} \Big|_0^{\sqrt{ay}} \\
&= \frac{1}{2} \cdot \frac{\pi}{2} \frac{(a^2 + 4ay)^{3/2}}{6} \Big|_0^{2a} = \frac{13}{12} \pi a^2
\end{aligned}$$

110. $4z = x^2 + y^2$ paraboloidning $y^2 = z$ silind va $z=3$ tekislik bilan chegaralangan qismini yuzini toping.

111. II tur sirt integralni hisoblang.

$$\iint_{\sigma} z dx dy + y dx dz + x dy dz$$

Bu yerda σ $-x+y+z=1$ tekislikning koordinata tekisliklari bilan chegaralangan qismi. Yechish: σ ni koordinata tekisliklariga proyeksiyalab hisoblaymiz.



$$\iint_{\sigma} z dx dy ni$$

hisoblaymiz z ni x va y orqali ifodalab ikki karrali integralni ΔAOB orqali hisoblaymiz.

$$z = 1 - x - y, 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x$$

$$\begin{aligned} \iint_{\sigma} z dx dy &= \iint_{OAB} (1-x-y) dx dy = \int_0^1 dx \int_0^{1-x} (1-x-y) dy = \int_0^1 dx \left[(1-x)y - \frac{y^2}{2} \right] \Big|_0^{1-x} \\ &= \int_0^1 \frac{(1-x)^2}{2} dx = -\frac{(1-x)^3}{6} \Big|_0^1 = \frac{1}{6} \end{aligned}$$

Qolgan

$$\iint_{\sigma} y dx dz \text{ va } \iint_{\sigma} x dy dz$$

integral ham shu natijaga keladi. Demak izlangan integral $3 \cdot \frac{1}{6} = \frac{1}{2}$ ga teng.

II-tur sirt integrallarini hisoblang. (112-116)

112.

$$\iint_{\sigma} yz dy dz + xz dx dz + xy dx dy, \quad \sigma - x + y + z = a,$$

$X=0$, $y=0$, $z=0$ tekislik bilan chegaralangan tetraedrning tashqi tomoni.

113.

$$\iint_S z dx dy, \quad S - \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

ellipsoidning tashqi tomoni.

114.

$$\iint_{\sigma} x^2 dy dz + y^2 dx dz + z^2 dx dy,$$

$\sigma - x^2 + y^2 + z^2 = a^2$ yarim sfera sirtining tashqi tomoni.

115.

$$\iint_{\sigma} x^3 dy dz + y^3 dx dz + z^3 dx dy$$

$\sigma - x^2 + y^2 + z^2 = a$ sferaning tashqi qismi.

116.

$$\iint_{\sigma} (x-y) dx dy + (z-x) dx dz + (y-z) dy dz,$$

$\sigma - x^2 + y^2 = z^2$ ($0 \leq z \leq h$) sirtning tashqi tomoni.

117. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoid sirtidagi $F(x, y, z) = xi + yj + zk$ vector maydonning I oktantada normali yo'nalishidagi oqimi topilsin.

Yechish: Izlanayotgan oqim quyidagiga teng.

$$\iint_{\sigma} F nd\sigma = \iint_{\sigma} (x \cos \alpha + y \cos \beta + z \cos \gamma) d\sigma$$

Oxirgi integral II tur sirt integralini hisoblashga keltiriladi.

$$\iint_{D_1} x dy dz + \iint_{D_2} y dx dz + \iint_{D_3} z dx dy$$

D_1, D_2, D_3 – ellipsoidning mos koordinata tekisliklariga tushirilgan proyeksiyasi.

Masalan:

$$\iint_{D_3} z dx dy$$

ni ko'raylik.

z ni x va y orqali ifodalaymiz D_3 – ellipsning ichki qismi choragi.

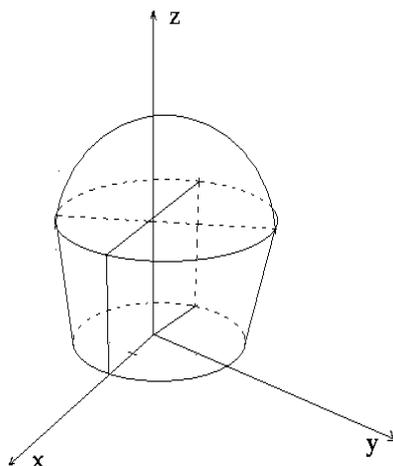
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, x \geq 0, y \geq 0$$

$$\iint_{D_3} z dx dy$$

ellipsoid hajmining sakkizdan biriga teng $\frac{1}{8} \cdot \frac{4}{3} \pi abc$, Qolgan soha integrallari uchun ham bu natija o'rinli bo'lib I-tur integral, ya'ni vector maydon oqimi

$$3 \cdot \frac{1}{8} \cdot \frac{4}{3} \pi abc = \frac{\pi abc}{2} \text{ ga teng.}$$

118. $F = x^2 i - y^2 j + z^2 k$ vektorning $x^2 + y^2 + z^2 = 3R^2$ va Oxy tekislik va $x^2 + y^2 + z^2 = R^2$ giperbolaoid bilan chegaralangan qismi jism sirtiga o'tgan oqimi topilsin.



Yechish: Quyidagicha

yozamiz ;

$$\begin{aligned} \iint_{\sigma} F nd\sigma &= \iint_{\sigma} (x^2 \cos\alpha - y^2 \cos\beta + z^2 \cos\gamma) d\sigma \\ &= \iint_{\sigma} x^2 \cos\alpha d\sigma - \iint_{\sigma} y^2 \cos\beta d\sigma + \iint_{\sigma} z^2 \cos\gamma d\sigma \end{aligned}$$

Oxz va Oyz tekisliklarga σ sirt 2 marta proyeksiyalanadi (ikki tomondan), shu bilan birga σ bu tekisliklarga nisbatan simmetrik. Shuning uchun mos integral 0 ga teng.

$$\iint_{\sigma} x^2 \cos\gamma d\sigma = \iint_{\sigma} y^2 \cos\beta d\sigma = 0$$

Endi

$$\iint_{\sigma} z^2 \cos y d\sigma$$

ni hisoblaymiz.

σ sirt 3 qismdan iborat.

$z = \sqrt{3R^2 - x^2 - y^2}$ sfera segmentining $\cos y > 0$ holati. Oxy ga bu segmentning proyeksiyasi $x^2 + y^2 \leq 2R^2$ doira $x^2 + y^2 + z^2 = 3R^2$ sfera segmenti $x^2 + y^2 - z^2 = R^2$ giperboloid bilan quyidagi chiziqlar orqali kesishadi.

$$\begin{cases} x^2 + y^2 + z^2 = 3R^2 \\ x^2 + y^2 - z^2 = R^2 \end{cases} \Rightarrow \begin{cases} x^2 + y^2 = 2R^2 \\ z = R \end{cases}$$

$\sqrt{2}R$ radiusli aylana

Paraboloid segmenti Oxy ga $R^2 \leq x^2 + y^2 \leq 2R^2, z = x^2 + y^2 - R^2$ xalqaga proyeksiyalanadi.

Nihoyat, 3 qism- bu $x^2 + y^2 \leq R^2$ doira, bu yerda $z=0$ shuning uchun

$$\begin{aligned} \iint_{\sigma} Fnd\sigma &= \iint_{\sigma} z^2 \cos y d\sigma \\ &= \iint_{x^2+y^2 \leq 2R^2} (3R^2 - x^2 - y^2) dx dy - \iint_{R^2 \leq x^2+y^2 \leq 2R^2} (x^2 + y^2 - R^2) dx dy \\ &= \frac{7\pi R^4}{2} \end{aligned}$$

119. $F = (x^2i + y^2j + z^2k)$ vektorning $\frac{H}{R}\sqrt{x^2 + y^2} \leq z \leq H$ tashqi normali yo'nalishi bo'yicha jism sirtidan o'tgan oqimni toping.

120. $F = 2xi - yj$ vektorning $x^2 + y^2 = R^2, x \geq 0, y \geq 0, 0 \leq z \leq H$ silindr sirtidan tashqi normal yo'nalishi bo'yicha o'tgan oqimni toping.

121. $x^2 + y^2 + z^2 = R^2, z \geq 0$ yarim sferaning zichligi $\rho = \sqrt{x^2 + y^2}$ ga teng bo'lsa, uning massasini toping.

Yechish.

$$m = \iint_{\sigma} \sqrt{x^2 + y^2} d\sigma$$

$$\text{Bu yerda } z = \sqrt{R^2 - x^2 - y^2}, \sqrt{1 + (z'_x)^2 + (z'_y)^2} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}$$

Demak,

$$m = \iint_{x^2+y^2 \leq R^2} R \frac{\sqrt{x^2 + y^2}}{\sqrt{R^2 - x^2 - y^2}} dx dy$$

Qutb koordinatalariga o'tsak,

$$m = R \iint_{r \leq R} \frac{r^2 dr d\varphi}{\sqrt{R^2 - r^2}} = R \int_0^{2\pi} d\varphi \int_0^R \frac{r^2}{\sqrt{R^2 - r^2}} dr = \frac{\pi^2 R^3}{3}$$

122. $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ kubning massasini aniqlang. Bunda sirtidagi har bir $M(x,y,z)$ nuqtaga $\rho(x,y,z) = xyz$ zichlik mos keladi.

123. $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ kubning massasini toping. Bunda sirtidagi koordinata $M(x,y,z)$ nuqtaga $\rho(x,y,z) = xyz$ zichlik mos keladi.

124. $az = x^2 + y^2$ ($0 \leq z \leq a$) bir jinsli paraboloidning og'irlik markazi koordinatalari aniqlansin.

125. $z = \sqrt{x^2 + y^2}$ ($0 \leq z \leq h$) Oz o'qqa nisbatan konus yonsirt uning inersiya momentini toping.

126. $\iint_{\sigma} d\sigma$ ni hisoblang.

$\sigma - z = x^2 + y^2$ paraboloidning $(x^2 + y^2) = x^2 - y^2$

Silindr bilan kesilgan qismi.

NAZORAT ISHI.

Variante 1

1. Hisoblang.

$$\int_L \frac{y^2}{x} dl$$

Bu yerda L- $y^2 = 2x$ paraboloning $(1, \sqrt{2})$ va $(2, 2)$ nuqtalar orasidagi qismi.

2. Hisoblang.

$$\int_L (4y + 4)dx + (3x + 3y + 4)dy$$

L- $x=0, y=0, 2x+3y=6$ uchburchak kontri, natijani Grin formulasi yordamida tekshiring.

3. Hisoblang.

$$\int_{(1,1)}^{(4,9)} \left(3x^2 - 3y^2 - \frac{1}{2} \sqrt{\frac{y}{x^3}} \right) dx + \left(-6xy + \frac{1}{2\sqrt{xy}} \right) dy$$

4. I-tur sirt integralini hisoblang.

$$\iint_S x^2 dS$$

Bu yerda S- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$, $0 \leq z \leq h$ kosinusning yon sirti.

5.II-tur sirt integralini hisoblang.

$$\iint_{\sigma} y^2 dx dz$$

σ - $x^2 + y^2 + z^2 = R^2$, $y \geq 0$

yarim sferaning ichki tomoni.

Variant 2

1.Hisoblang.

$$\int_L (x + y) dL$$

L- uchlari A(1,-1), B(-3,-1), C(-3,2) nuqtalarda bo'lgan ABC uchburchak konturi.

2.I-tur egri chiziqli integralni hisoblang.

$$\int_L (x + 1) dx + xyz dy + y^2 z dz$$

Bu yerda L- M(2,-1,3) bilan N(7,4,11) nuqtani tutashtiruvchi kesmasi.

3.Hisoblang.

$$\int_{(1,1)}^{(2,2)} \left(6x - 3y + \frac{1}{y} \right) dx + \left(-3x + \frac{x}{y^2} \right) dy$$

4.II-tur sirt integralini hisoblang.

$$\iint_S z^4 dS$$

S- $4(x^2 + y^2) = z^2$ $0 \leq z \leq 2$ kosinusni yon sirti.

5.II-tur sirt integralini hisoblang.

$$\iint_{\sigma} z^3 dx dy$$

σ - $x + y + z = 10$ tekislikning tashqi qismi $x \geq 0, y \geq 0, z \geq 0$ I-oktantadagi qismi.

Variante 3

1. Hisoblang.

$$\int_L (x^2 + y^2 - z) dl$$

L- $x = a \cos t, y = a \sin t, z = bt, 0 \leq t \leq \pi$ zanjir chizig'ining yoyi.

2. II-tur egri chizikli integral yordamida $x=8\cos^3 t, y=8\sin^3 t, 0 \leq t \leq 2\pi$ egri chiziqlar bilan chegaralangan fugura yuzini toping.

3. Hisoblang.

$$\int_{(1,1)}^{(3,3)} \left(3x\sqrt{x^2 + 3y^2} + \frac{1}{y} \right) dx + \left(9y\sqrt{x^2 + 3y^2} - \frac{x}{y^2} \right) dy$$

4. I-tur sirt integralini hisoblang.

$$\iint_S z^3 dS$$

S- $x^2 + y^2 + z^2 = R^2, z \geq 0$ yechim sfera yuqori qismi.

5. II-tur sirt integralini hisoblang.

$$\iint_{\sigma} z^4 dx dy$$

$\sigma - x^2 + y^2 + z^2 = R^2, z \geq 0,$

Yarim sfera sirtining ichki qismi.

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