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Global invariants of paths and curves for the group of all linear similarities in the two-dimensional Euclidean space

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Let E_2 be the 2-dimensional Euclidean space, LSim(2) be the group of all linear similarities of E_2 and $\mathrm{LSim}^+(2)$ be the group of all orientation-preserving linear similarities of E_2. The present paper is devoted to solutions of problems of global G-equivalence of paths and curves in E_2 for the groups $G = \mathrm{LSim}(2), \mathrm{LSim}^+(2)$. Complete systems of global G-invariants of a path and a curve in E_2 are obtained. Existence and uniqueness theorems are given. Evident forms of a path and a curve with the given global invariants are obtained.

Keywords: Curve; invariant; similarity; Euclidean geometry; similarity geometry.

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1. Introduction

Let E_n be the n-dimensional Euclidean space and $O(n)$ be the group of all orthogonal transformations of E_n. Put $O^+(n) = \{g \in O(n) \,|\, \det g = 1\}$, $M(n)=\{F : E_n \to E_n \,|\, Fx = gx + b,\ g \in O(n),\ b \in E_n\}$ and $M^+(n) = \{F \in M(n) \,|\, \det g = 1\}$.

Let $\mathrm{Sim}(n)$ be the group of all similarities of E_n, $\mathrm{Sim}^+(n)$ be the group of all orientation-preserving similarities of E_n, $\mathrm{LSim}(n)$ be the group of all linear similarities of E_n and $\mathrm{LSim}^+(n)$ be the group of all orientation-preserving linear similarities of E_n.

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Similarity has an important role in different areas of mathematics, physics, chemistry, biology and computer sciences [3–5, 7–10, 16, 17, 20–24].

In the classical differential geometry of curves in $E_n, n \geq 2$, using Frenet–Serret equations, curvature functions $k_i, i = 1, \ldots, n-1$, of the curve were obtained in [1, p. 172]. The function k_i is $M(n)$-invariant for $i = 1, \ldots n-2$. But k_{n-1} is not $M(n)$-invariant. It is $M^+(n)$-invariant. Thus, invariant theory of curves in the classical differential geometry was developed only for the group $M^+(n)$. In addition, the method of the orthogonal frame in the classical differential geometry give conditions only for the local $M^+(n)$-equivalence of curves.

In the similarity geometry of curves in $E_n, n \geq 2$, using curvatures $k_i, i = 1, \ldots, n-1$, of the curve $x(t)$ in the Euclidean space E_n, curvature functions $p_i(\sigma), i = 1, \ldots, n-1$, of the curve in the similarity geometry in E_n were obtained in [13]. The curvatures $p_i(\sigma), i = 1, \ldots, n-1$, has the following form $p_1(\sigma) = \frac{1}{k_1(\sigma)}\frac{dk_1(\sigma)}{d\sigma}, p_i(\sigma) = \frac{k_i(\sigma)}{k_1(\sigma)}, i = 2, \ldots, n-1$. Here, σ is a spherical arc length parameter of the curve $x(t)$. The function $p_i(\sigma)$ is $\mathrm{Sim}(n)$-invariant for $i = 1, \ldots, n-2$. But $p_{n-1}(\sigma)$ is not $\mathrm{Sim}(n)$-invariant. It is $\mathrm{Sim}^+(n)$-invariant. In the paper [13], the uniqueness and existence theorems for a curve obtained only for the group $\mathrm{Sim}^+(n)$. Thus, invariant theory of curves in the similarity geometry in E_n was developed only for the group $\mathrm{Sim}^+(n)$. In addition, the method of the orthogonal frame in the similarity geometry give conditions only for the local $\mathrm{Sim}^+(n)$-equivalence of curves. Similar results for similarity geometry of curves in the Minkowski n-space were obtained in the paper [17]. In the work [6], the theory of local invariants of curves in $E(2)$ for the group $\mathrm{Sim}(2)$ was given without of proofs.

It is shown that the geometric motions of some class of space curves in similarity and centro-affine geometries are closely related to integrable equations (see [9–12, 20–22, 24, 25]).

Invariant representations of a system for matching and pose estimation of 3D space curves under similarity transformation are investigated in [24].

In works [2, 14], by using invariant parametrization of curves, the problem of global G-equivalence of curves (that is nonparametrized curves) in E_n for the groups $G = M(n), M^+(n)$ is reduced to the problem of global G-equivalence of paths (that is parametrized curves). In these works also complete systems of global G-invariants of nondegenerate paths and nondegenerate curves in E_n were obtained for the groups $G = M(n), M^+(n)$. Systems of generators of differential field of G-invariant differential rational functions of a path in E_n for the groups $M(n)$ and $M^+(n)$ are obtained in the same papers. In papers [15, 18, 19], this approach was developed for curves in the pseudo-Euclidean spaces.

The important problem is to find simple but efficient method for the G-equivalence check of two regular paths (that is regular parametrized curves) and curves in E_n for the groups $G = \mathrm{Sim}(n), \mathrm{Sim}^+(n), \mathrm{LSim}(n), \mathrm{LSim}^+(n)$ in terms of **global** G-invariants of paths and curves.

The present paper is devoted to the conditions of the global G-equivalence of two regular paths and curves in similarity geometries of the groups $G = \mathrm{LSim}(2)$ and $\mathrm{LSim}^+(2)$. The uniqueness and existence theorems for a path and a curve are given. These theorems give evident form of regular paths and curves in terms of global invariants of paths and curves.

This paper is organized as follows. In Sec. 2, some known results on descriptions of the groups $G = \mathrm{LSim}(2)$ and $\mathrm{LSim}^+(2)$ in terms of complex numbers are given. In Sec. 3, fundamental G-invariants of a path in E_2 for the groups $G = \mathrm{LSim}(2), \mathrm{LSim}^+(2)$ are given. In Sec. 4, complete systems of global G-invariants of a regular path and uniqueness theorems for regular paths are given.

In Sec. 5, theorems on an existence for regular paths and evident forms of a path with given G-invariants are obtained. In Sec. 6, the type and invariant parametrizations of an L-nondegenerate curve are defined and investigated. In Sec. 7, complete systems of G-invariants of an L-nondegenerate curve and uniqueness theorems for L-nondegenerate curves are obtained.

2. Groups LSim(2) and LSim$^+$(2)

In this section, some known results on descriptions of the groups $\mathrm{LSim}(2)$ and $\mathrm{LSim}^+(2)$ in terms of complex numbers are given (see [3]).

Let R be the field of real numbers and Ω be the field of complex numbers. The multiplication in Ω is given by $(a_1 + ia_2)(b_1 + ib_2) = (a_1b_1 - a_2b_2) + i(a_1b_2 + a_2b_1)$. We will consider the element $a = a_1 + ia_2$ also in the form $a = \binom{a_1}{a_2}$. For $a = a_1 + ia_2$, denote by P_a the matrix $\left(\begin{smallmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{smallmatrix}\right)$ and consider P_a also as the transformation $P_a : \Omega \to \Omega$, defined by

$$P_a b = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 - a_2b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix} \quad \text{for } b = b_1 + ib_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Then we have the equality

$$ab = P_a b. \tag{1}$$

for all $a, b \in \Omega$. Let $P(\Omega)$ denote the set of all matrices P_a, where $a \in \Omega$. We consider on $P(\Omega)$ the following standard matrix operations: the componentwise addition, a scalar multiplication and the multiplication of matrices. Then $P(\Omega)$ is a field, where the unit element is the unit matrix. The following Propositions 1–5 are known.

Proposition 1. *The mapping $P : \Omega \to P(\Omega)$, where $P : a \to P_a$ for all $a \in \Omega$, is an isomorphism of fields.*

For vectors $a = a_1 + ia_2, b = b_1 + ib_2 \in \Omega$, we put $\langle a, b\rangle = a_1b_1 + a_2b_2$. Then $\langle a, b\rangle$ is a bilinear form on R^2 and $\langle a, a\rangle = a_1^2 + a_2^2$ is a quadratic form on R^2. Put $Q(a) = \langle a, a\rangle$. We consider the field Ω also as the two-dimensional Euclidean space E_2 with the scalar product $\langle a, b\rangle$. Then $\|a\| = |a| = \sqrt{Q(a)}, \forall a \in \Omega$.

Proposition 2. *Equalities* $Q(a) = \det(P_a)$, $Q(ab) = Q(a)Q(b)$ *and* $|ab| = |a||b|$ *hold for all* $a, b \in \Omega$.

An endomorphism ψ of a vector space Ω is called an involution of the field Ω if $\psi(\psi(a)) = a$ and $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \Omega$. For an element $a = a_1 + ia_2 \in \Omega$, we set $\overline{a} = a_1 - ia_2$.

Proposition 3. *The mapping* $a \to \overline{a}$ *is an involution of the field* Ω. *In addition, for an arbitrary element* $a = a_1 + ia_2 \in \Omega$, *equalities* $a + \overline{a} = 2a_1, \langle a, a\rangle = a\overline{a} = a_1^2 + a_2^2 \in R$ *hold.*

Proposition 4. *Let* $a \in \Omega$. *Then the element* a^{-1} *exists if and only if* $Q(a) \neq 0$. *In the case* $Q(a) \neq 0$, *equalities* $a^{-1} = \frac{\overline{a}}{Q(a)}$ *and* $Q(a^{-1}) = \frac{1}{Q(a)}$ *hold.*

Let $W = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We will use W also for the writing of the element $\overline{a}$ in the form $\overline{a} = Wa$.

Proposition 5. $Q(Wa) = Q(a)$ *for all* $a \in \Omega$ *and* $\langle Wa, Wb\rangle = \langle a, b\rangle$ *for all* $a, b \in \Omega$.

Let $a = a_1 + ia_2 \in \Omega^*$ that is $|a| \neq 0$. Put

$$P_a^+ = \begin{pmatrix} \dfrac{a_1}{|a|} & \dfrac{-a_2}{|a|} \\[2ex] \dfrac{a_2}{|a|} & \dfrac{a_1}{|a|} \end{pmatrix}.$$

Proposition 6. *Let* $a = (a_1 + ia_2) \in \Omega^*$. *Then the equality* $P_a = |a|P_a^+$ *holds, where* $P_a^+ \in O^+(2)$.

Proof. The equality $P_a = |a|P_a^+$ is obvious. Since $(\frac{a_1}{|a|})^2 + (\frac{a_2}{|a|})^2 = 1$, the implication $P_a^+ \in O^+(2)$ follows from [3, pp. 161–162]. □

Put $\Omega^* = \{a \in \Omega \,|\, Q(a) \neq 0\}$. Ω^* is a group with respect to the multiplication operation in the field Ω. Put $S(\Omega^*) = \{a \in \Omega \,|\, Q(a) = 1\}$, $P(\Omega^*) = \{P_a \,|\, a \in \Omega^*\}$ and $P(S(\Omega^*)) = \{P_a \,|\, a \in S(\Omega^*)\}$. $S(\Omega^*)$ is a subgroup of the group Ω^* and $S(\Omega^*) = \{e^{i\varphi} \,|\, \varphi \in R\}$. Denote by $P(\Omega^*)W$ the set of all matrices $\{gW \,|\, g \in P(\Omega^*)\}$, where gW is the multiplication of matrices g and W.

The set of all linear similarities of the two-dimensional Euclidean space E_2 is denoted by LSim(2). The set of all orientation-preserving (respectively orientation-reversing) linear similarities of E_2 is denoted by $\mathrm{LSim}^+(2)$ (respectively $\mathrm{LSim}^-(2)$). We note that $\mathrm{LSim}^+(2) \cap \mathrm{LSim}^-(2) = \emptyset$. The set LSim(2) is a group with respect to the composition of similarities and the set $\mathrm{LSim}^+(2)$ is a subgroup of LSim(2).

Theorem 7 (see [3, p. 229]). *The following equalities are hold*:

(i) $\mathrm{LSim}^+(2) = \{P_a : E_2 \to E_2 \,|\, a \in \Omega^*\}$.
(ii) $\mathrm{LSim}^-(2) = \{P_aW : E_2 \to E_2 \,|\, a \in \Omega^*\}$.
(iii) $\mathrm{LSim}(2) = \mathrm{LSim}^+(2) \cup \mathrm{LSim}^-(2)$.

3. Fundamental G-Invariants of a Path in E_2 for the Groups $G = \mathrm{LSim}(2), \mathrm{LSim}^+(2)$

In this section, the definition of an L-regular path is given. Moreover, fundamental G-invariants of an L-regular path in E_2 for the groups $G = \mathrm{LSim}(2), \mathrm{LSim}^+(2)$ are given.

Let $T = (a, b)$ be an open interval of R and $x : T \to E_2$ be a $C^{(1)}$-mapping.

Definition 8 (see [2]). A $C^{(1)}$-mapping $x : T \to E_2$ is called a T-path (parametrized curve) in E_2.

If $x(t)$ is a T-path then $Fx(t)$ is a T-path in E_2 for any $F \in \mathrm{LSim}(2)$. Let G be a one of the groups $\mathrm{LSim}(2)$ and $\mathrm{LSim}^+(2)$.

Definition 9. T-paths $x(t)$ and $y(t)$ in E_2 are called G-equivalent if there exists $F \in G$ such that $y(t) = Fx(t)$. In this case, we write $x(t) \overset{G}{\sim} y(t)$.

Definition 10. A function $f(x(t), y(t), \dots, z(t))$ of a finite number of T-paths $x(t), y(t), \dots, z(t)$ is called G-invariant if $f(Fx(t), Fy(t), \dots, Fz(t)) = f(x(t), y(t), \dots, z(t))$ for all $F \in G$, all T-paths $x(t), y(t), \dots, z(t)$ and all $t \in T$.

Let $x(t) = (x_1(t), x_2(t))$ be a T-path in E_2 and $x'(t) = (x_1'(t), x_2'(t))$ is its first derivative. For the vectors $x(t), x'(t)$, the determinant

$$\begin{vmatrix} x_1(t) & x_1'(t) \\ x_2(t) & x_2'(t) \end{vmatrix}$$

will be denoted by $[x(t)x'(t)]$.

Definition 11. A T-path $x(t)$ in E_2 is called L-regular if $x(t) \neq 0$ for all $t \in T$.

It is obvious that a T-path $x(t)$ in E_2 is L-regular if and only if $Q(x(t)) \neq 0$ for all $t \in T$.

Example 12. Consider the T-path $x(t) = (t^2, e^t)$ in E_2, where $T = R$. Then, $x(t) \neq 0$ for all $t \in R$. That is, $x(t)$ is an L-regular T-path.

Proposition 13.

(i) *The function* $\frac{\langle x(t), x'(t) \rangle}{Q(x(t))}$ *is* $\mathrm{LSim}(2)$*-invariant on the set of all* L*-regular* T*-paths* $x(t)$ *in* E_2.

(ii) *The function* $\frac{[x(t)x'(t)]}{Q(x(t))}$ *is* $\mathrm{LSim}^+(2)$*-invariant on the set of all* L*-regular* T*-paths* $x(t)$ *in* E_2.

(iii) *The function* $\frac{[x(t)x'(t)]^2}{Q^2(x(t))}$ *is* $\mathrm{LSim}(2)$*-invariant on the set of all* L*-regular* T*-paths* $x(t)$ *in* E_2.

(iv) *The function* $\frac{\langle x'(t), x'(t) \rangle}{Q(x(t))}$ *is* $\mathrm{LSim}(2)$*-invariant on the set of all* L*-regular* T*-paths* $x(t)$ *in* E_2.

Proof. (i) Let $F \in \mathrm{LSim}^+(2)$. Then, by Theorem 7(i) and Proposition 6, F has the following form $F(x) = P_a x = |a|P_a^+ x$, where $a \in \Omega^*$ for all $x \in E_2$ and $P_a^+ \in O^+(2)$. By (1) and the equality $P_a = |a|P_a^+$, we have $F(x(t)) = P_a x(t) = |a|P_a^+ x(t)$ for all $t \in T$. Using this equality and $P_a^+ \in O^+(2)$, we obtain

$$\frac{\langle Fx(t), (Fx)'(t)\rangle}{Q(Fx(t))} = \frac{\langle |a|P_a^+ x(t), |a|P_a^+ x'(t)\rangle}{Q(|a|P_a^+ x(t))} = \frac{|a|^2\langle P_a^+ x(t), P_a^+ x'(t)\rangle}{|a|^2 Q(P_a^+ x(t))} = \frac{\langle x(t), x'(t)\rangle}{Q(x(t))}.$$

Hence, $\frac{\langle x(t),x'(t)\rangle}{Q(x(t))}$ is $\mathrm{LSim}^+(2)$-invariant. Using Proposition 5, we obtain that

$$\frac{\langle Wx(t), (Wx)'(t)\rangle}{Q(Wx(t))} = \frac{\langle x(t), x'(t)\rangle}{Q(x(t))}.$$

Thus, $\frac{\langle x(t),x'(t)\rangle}{Q(x(t))}$ is $\mathrm{LSim}^+(2)$-invariant and W-invariant. Hence, it is $\mathrm{LSim}(2)$-invariant. Proofs of other statements are similar. □

4. Complete Systems of Invariants of an L-Regular Path and Uniqueness Theorems for L-Regular Paths

In this section, definitions of completely L-degenerate and L-nondegenerate T-paths are given. The conditions of the global G-equivalence of L-regular paths, completely L-degenerate paths and L-nondegenerate paths for the groups $G = \mathrm{LSim}(2), \mathrm{LSim}^+(2)$ are obtained.

Let $x(t)$ and $y(t)$ be two regular T-paths such that their the fundamental G-invariants are equal. Then, we obtain that there exists the unique $g \in G$ such that $y(t) = gx(t)$ for all $t \in T$ and for the groups $G = \mathrm{LSim}(2), \mathrm{LSim}^+(2)$. Moreover, the evident form of $g \in G$ is determined in terms of T-paths $x(t)$ and $y(t)$.

Proposition 14. *Let $u, v \in \Omega$. Assume that $Q(u) \neq 0$. Then the element vu^{-1} exists, the following equalities hold*:

$$vu^{-1} = \frac{\langle u, v\rangle}{Q(u)} + i\frac{[uv]}{Q(u)}$$

and

$$P_{vu^{-1}} = \begin{pmatrix} \dfrac{\langle u, v\rangle}{Q(u)} & -\dfrac{[uv]}{Q(u)} \\ \dfrac{[uv]}{Q(u)} & \dfrac{\langle u, v\rangle}{Q(u)} \end{pmatrix}. \qquad (2)$$

Proof. Let $u = u_1 + iu_2, v = v_1 + iv_2$. Since $Q(u) \neq 0$, element u^{-1} exists by Proposition 4. Hence vu^{-1} exists. By Proposition 4, $u^{-1} = \frac{\overline{u}}{Q(u)}$. Using $\overline{u} = u_1 - iu_2$

and the multiplication in the field Ω, we obtain the equality $vu^{-1} = \frac{\langle u,v\rangle}{Q(u)} + i\frac{[uv]}{Q(u)}$. This equality and the notation of P_a imply (2). □

Theorem 15. (i) *Let $x(t)$ and $y(t)$ be L-regular T-paths in E_2 such that $x(t) \overset{\mathrm{LSim}^+(2)}{\sim} y(t)$. Then, for all $t \in T$, the following equalities hold*:

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = \dfrac{\langle y(t), y'(t)\rangle}{Q(y(t))} \\ \dfrac{[x(t)x'(t)]}{Q(x(t))} = \dfrac{[y(t)y'(t)]}{Q(y(t))}. \end{cases} \tag{3}$$

(ii) *Conversely, assume that $x(t)$ and $y(t)$ be L-regular T-paths in E_2 such that equalities* (3) *hold. Then $x(t) \overset{\mathrm{LSim}^+(2)}{\sim} y(t)$. Moreover, there exists the unique $F \in \mathrm{LSim}^+(2)$ such that $y(t) = Fx(t)$ for all $t \in T$. In this case, $F = U$, where $U \in \mathrm{LSim}^+(2)$ and it has the following form*

$$U = \begin{pmatrix} \dfrac{\langle x(t), y(t)\rangle}{Q(x(t))} & -\dfrac{[x(t)y(t)]}{Q(x(t))} \\ \dfrac{[x(t)y(t)]}{Q(x(t))} & \dfrac{\langle x(t), y(t)\rangle}{Q(x(t))} \end{pmatrix}, \tag{4}$$

and U does not depend on $t \in T$.

Proof. (i) Assume that $x(t) \overset{\mathrm{LSim}^+(2)}{\sim} y(t)$. Then, by Theorem 7(i), $a \in \Omega^*$ exists such that $y(t) = ax(t)$ for all $t \in T$. This equality implies that $y'(t) = ax'(t)$ for all $t \in T$. Since $x(t)$ and $y(t)$ are L-regular paths, we have $Q(x(t)) \neq 0$ and $Q(y(t)) \neq 0$ for all $t \in T$. Then, by Proposition 4, $(x(t))^{-1}$ and $(y(t))^{-1}$ exist for all $t \in T$. We put $u = y(t), v = y'(t)$ in Proposition 14. Then we obtain

$$\frac{y'(t)}{y(t)} = \frac{\langle y(t), y'(t)\rangle + i[y(t)y'(t)]}{Q(y(t))}. \tag{5}$$

Similarly, we obtain

$$\frac{x'(t)}{x(t)} = \frac{\langle x(t), x'(t)\rangle + i[x(t)x'(t)]}{Q(x(t))}. \tag{6}$$

Equalities $y(t) = ax(t)$ and $y'(t) = ax'(t)$ imply $\frac{y'(t)}{y(t)} = \frac{x'(t)}{x(t)}$. This equality, (5) and (6) imply equalities (3).

(ii) Conversely, assume that equalities (3) hold. Equalities (5), (6) and (3) imply the equality $y'(t)(y(t))^{-1} = x'(t)(x(t))^{-1}$. Hence, $y'(t)(y(t))^{-1} - x'(t)(x(t))^{-1} = 0$ for all $t \in T$. Using this equality, we obtain

$$\begin{aligned} \frac{d(y(t)(x(t))^{-1})}{dt} &= y'(t)(x(t))^{-1} - y(t)x'(t)(x(t))^{-2} \\ &= y(t)(y'(t)(y(t))^{-1} - x'(t)(x(t))^{-1})(x(t))^{-1} = 0 \end{aligned}$$

for all $t \in T$. This means that the function $y(t)(x(t))^{-1}$ is constant on T. Put $g = y(t)(x(t))^{-1}$. Since $Q(x(t)) \neq 0$ and $Q(y(t)) \neq 0$ for all $t \in T$, we have $g \neq 0$.

Since $y(t) = y(t)(x(t))^{-1}x(t)) = (y(t)(x(t))^{-1})x(t)$, we have $y(t) = gx(t)$. Using (1), we obtain $y(t) = gx(t) = P_g x(t)$. By $g = y(t)(x(t))^{-1} = \frac{\langle x(t), y(t)\rangle}{Q(x(t))} + i\frac{[x(t)y(t)]}{Q(x(t))}$ and Proposition 14, P_g has the form (4) and $P_g = U$. Since g is a constant, U does not depend on $t \in T$. Then $y(t) = Ux(t) = P_g x(t) = Fx(t)$. By Theorem 7(i), $F \in \mathrm{LSim}^+(2)$. Hence $x(t) \overset{\mathrm{LSim}^+(2)}{\sim} y(t)$.

Prove the uniqueness of $U \in \mathrm{LSim}^+(2)$ satisfying the condition $y(t) = Ux(t)$, assume that $H \in \mathrm{Sim}^+(2)$ exists such that $y(t) = Hx(t)$. Then by (1), Proposition 1 and Theorem 7(i), there exists the unique $a \in \Omega^*$ such that $H = P_a$. Hence, we have $y(t) = P_a x(t)$. By (1), we obtain $y(t) = ax(t)$. Since $Q(x(t)) \neq 0$, $y(t) = ax(t)$ implies that $a = y(t)(x(t))^{-1} = g$. Hence, $P_a = P_g = U$. The uniqueness of U is proved. □

Example 16. Consider L-regular T-paths $x(t) = (\frac{t^4+1}{t^2+1}, \frac{t^2}{t^2+1})$ and $y(t) = (\frac{1-3t^2+t^4}{t^2+1}, \frac{3+t^2+3t^4}{t^2+1})$, where $T = R$. It is easy to see that equalities in (3) hold for these paths $x(t)$ and $y(t)$. Then, by Theorem 15(ii), $x(t) \overset{\mathrm{LSim}^+(2)}{\sim} y(t)$. Moreover, by Theorem 15(ii), we obtain that $U = (\begin{smallmatrix} 1 & -3 \\ 3 & 1 \end{smallmatrix})$.

Definition 17. (i) A T-path $x(t)$ in E_2 is called completely L-degenerate if $[x(t)x'(t)] = 0$ for all $t \in T$.

(ii) A T-path $x(t)$ in E_2 is called L-nondegenerate if $[x(t)x'(t)] \neq 0$ for all $t \in T$.

It is obvious that every L-nondegenerate path is L-regular. Assume that $x(t)$ and $y(t)$ be T-paths in E_2 such that $x(t)$ is completely L-degenerate and $x \overset{\mathrm{LSim}^+(2)}{\sim} y$, then $y(t)$ is also completely L-degenerate. Similarly, it is obvious that if $x(t)$ is L-nondegenerate and $x \overset{\mathrm{LSim}(2)}{\sim} y$, then $y(t)$ is also an L-nondegenerate.

Example 18. Consider the T-path $x(t) = (t^2, \frac{t^2}{2})$ in E_2, where $T = (0, 1)$. Then, $[x(t)x'(t)] = 0$ for all $t \in T$. Hence, $x(t)$ is a completely L-degenerate T-path. Moreover, since $t^2 \neq 0$ for all $t \in T$, we obtain that $x(t) = (t^2, \frac{t^2}{2})$ is a completely L-degenerate L-regular T-path.

Example 19. Consider the T-path $x(t) = (\frac{-1}{t}, \frac{-1}{t-1})$ in E_2, where $T = (2, \infty)$. Then, $[x(t)x'(t)] \neq 0$ for all $t \in T$. Hence, $x(t)$ is an L-nondegenerate T-path.

Example 20. Consider the T-path $x(t) = (t^2 + 1, \frac{1}{t^2+1})$ in E_2, where $T = R$. Then, $[x(t)x'(t)] = 0$ for $t = 0$, but $x(t) \neq 0$ for all $t \in T$ Hence, $x(t)$ is not an L-nondegenerate T-path, but $x(t)$ is an L-regular T-path.

Theorem 21. (i) *Let $x(t), y(t)$ be completely L-degenerate L-regular T-paths in E_2 such that $x(t) \stackrel{\mathrm{LSim}(2)}{\sim} y(t)$. Then, the following equality holds*:

$$\frac{\langle x(t), x'(t)\rangle}{Q(x(t))} = \frac{\langle y(t), y'(t)\rangle}{Q(y(t))}, \tag{7}$$

for all $t \in T$.

(ii) *Conversely, assume that $x(t), y(t)$ be completely L-degenerate L-regular T-paths in E_2 such that* (7) *holds. Then $x(t) \stackrel{\mathrm{LSim}(2)}{\sim} y(t)$. Moreover, only two elements $F = F_1, F_2 \in \mathrm{LSim}(2)$ exist such that $y(t) = Fx(t)$ for all $t \in T$. Here $F_1 = U_1 x(t), F_2 = (U_2 W)x(t)$, where $U_1, U_2 \in \mathrm{LSim}^+(2)$, U_1 has the form* (4) *and U_2 has the form*

$$U_2 = \begin{pmatrix} \frac{\langle Wx(t), y(t)\rangle}{Q(Wx(t))} & -\frac{[Wx(t)y(t)]}{Q(Wx(t))} \\ \frac{[Wx(t)y(t)]}{Q(Wx(t))} & \frac{\langle Wx(t), y(t)\rangle}{Q(Wx(t))} \end{pmatrix}. \tag{8}$$

Matrices U_1 and U_2 do not depend on $t \in T$.

Proof. (i) Assume that $x(t) \stackrel{\mathrm{LSim}(2)}{\sim} y(t)$. The function $\frac{\langle x(t), x'(t)\rangle}{Q(x(t))}$ is LSim(2)-invariant, so the equality (7) holds.

(ii) Conversely, assume that (7) holds. Since $x(t)$ and $y(t)$ are completely L-degenerate paths, we have

$$[x(t)x'(t)] = [y(t)y'(t)] = 0. \tag{9}$$

for all $t \in T$. This equality and (7) imply equalities (3). Then, by Theorem 15(ii), the unique $F \in \mathrm{LSim}^+(2)$ exists such that $y(t) = Fx(t)$ for all $t \in T$. This means that the unique $U_1 \in \mathrm{LSim}^+(2)$ exists such that $y(t) = U_1 x(t)$ for all $t \in T$. Here, U_1 has the form (4). By Theorem 15, U_1 does not depend on $t \in T$.

Consider the T-path $Wx(t)$. Since functions $\langle x(t), x(t)\rangle$ and $\langle x(t), x'(t)\rangle$ are W-invariant, we have $\langle Wx(t), Wx(t)\rangle = \langle x(t), x(t)\rangle$ and $\langle Wx(t), Wx'(t)\rangle = \langle x(t), x'(t)\rangle$ for all $t \in T$. Similarly we obtain $\langle Wy(t), Wy(t)\rangle = \langle y(t), y(t)\rangle$ and $\langle Wy(t), Wy'(t)\rangle = \langle y(t), y'(t)\rangle$ for all $t \in T$. Using $\det(W) = -1$ and (9) for all $t \in T$, we obtain

$$[Wx(t)(Wx)'(t)] = \det(W)[x(t)x'(t)] = -[x(t)x'(t)] = [y(t)y'(t)] = 0$$

Using equalities $\langle Wx(t), Wx(t)\rangle = \langle x(t), x(t)\rangle$, $\langle Wx(t), Wx'(t)\rangle = \langle x(t), x'(t)\rangle$, $[Wx(t)(Wx)'(t)] = [y(t)y'(t)] = 0$ and equalities (9), we obtain the equalities:

$$\begin{cases} \dfrac{\langle Wx(t), (Wx)'(t)\rangle}{Q(Wx(t))} = \dfrac{\langle y(t), y'(t)\rangle}{Q(y(t))} \\ \dfrac{[Wx(t)(Wx)'(t)]}{Q(Wx(t))} = \dfrac{[y(t)y'(t)]}{Q(y(t))}. \end{cases}$$

for all $t \in T$. Then, by Theorem 15(ii), the unique $U_2 \in \mathrm{LSim}^+(2)$ exists such that $y(t) = U_2(Wx(t)) = (U_2W)x(t)$ for all $t \in T$. Here, U_2 has the form (8). By Theorem 15, U_2 does not depend on $t \in T$.

Let $F \in \mathrm{LSim}(2)$ such that $y(t) = Fx(t)$. We will prove that $Fx(t) = U_1x(t)$ or $Fx(t) = (U_2W)x(t)$. Since $F \in \mathrm{LSim}(2)$, we obtain $F \in \mathrm{LSim}^+(2)$ or $F \in \mathrm{LSim}^-(2) = \mathrm{LSim}^+(2)W$. Assume that $F \in \mathrm{LSim}^+(2)$. Then, by the uniqueness in Theorem 15, $F = U_1$. Assume that $F \in \mathrm{LSim}^+(2)W$. Then F has the form $F = DW$, where $D \in \mathrm{LSim}^+(2)$. We have $y(t) = (DW)x(t) = D(Wx(t))$. Hence, paths $y(t)$ and $Wx(t)$ are $\mathrm{Sim}^+(2)$-equivalent. By the uniqueness in Theorem 15, $D = U_2$. □

Theorem 22. (i) *Let $x(t)$ and $y(t)$ be L-nondegenerate T-paths in E_2 such that $x \overset{\mathrm{LSim}(2)}{\sim} y$. Then, for all $t \in T$, the following equalities hold*

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = \dfrac{\langle y(t), y'(t)\rangle}{Q(y(t))} \\ \dfrac{[x(t)x'(t)]^2}{Q^2(x(t))} = \dfrac{[y(t)y'(t)]^2}{Q^2(y(t))}. \end{cases} \tag{10}$$

(ii) *Conversely, assume that $x(t)$ and $y(t)$ be L-nondegenerate T-paths in E_2 such that* (10) *hold, then $x \overset{\mathrm{LSim}(2)}{\sim} y$. Moreover, the unique $F \in \mathrm{LSim}(2)$ exists such that $y(t) = Fx(t)$ for all $t \in J$. Then, only the following cases exist*:

(a) $[x(t)x'(t)] > 0$ *and* $[y(t)y'(t)] > 0$ *for all* $t \in T$.
(b) $[x(t)x'(t)] > 0$ *and* $[y(t)y'(t)] < 0$ *for all* $t \in T$.
(c) $[x(t)x'(t)] < 0$ *and* $[y(t)y'(t)] > 0$ *for all* $t \in T$.
(d) $[x(t)x'(t)] < 0$ *and* $[y(t)y'(t)] < 0$ *for all* $t \in T$.

In cases (a) *and* (d), $F = U_1$, *where* $U_1 \in \mathrm{LSim}^+(2)$ *and has the form* (4).
In cases (b) *and* (c), $F = (U_2W)$, *where* $U_2 \in \mathrm{LSim}^+(2)$ *and it has the form* (8).

Proof. (i) Let $x \overset{\mathrm{LSim}(2)}{\sim} y$. The functions $\frac{\langle x(t),x'(t)\rangle}{Q(x(t))}$ and $\frac{[x(t)x'(t)]^2}{Q^2(x(t))}$ are LSim(2)-invariant for all $t \in T$. So equalities (10) hold.

(ii) Conversely, assume that equalities (10) hold. Since $x(t)$ and $y(t)$ are L-nondegenerate T-paths, $[x(t)x'(t)] \neq 0$ and $[y(t)y'(t)] \neq 0$ for all $t \in T$. Hence only the above cases (a), (b), (c), (d) exist.

By (10), we have the equality $\frac{[x(t)x'(t)]^2}{Q^2(x(t))} = \frac{[y(t)y'(t)]^2}{Q^2(y(t))}$. In cases (a) and (d) this equality and the inequality $Q(x(t)) > 0$ imply the equality $\frac{[x(t)x'(t)]}{Q(x(t))} = \frac{[y(t)y'(t)]}{Q(y(t))}$. This equality and equalities (10) imply (3). Then, by Theorem 15, we obtain that there exists the unique $U_1 \in \mathrm{LSim}^+(2)$ such that $y(t) = U_1x(t)$ for all $t \in T$, where the matrix U_1 has the form (4).

In cases (b) and (c), the equality $\frac{[x(t)x'(t)]^2}{Q^2(x(t))} = \frac{[y(t)y'(t)]^2}{Q^2(y(t))}$ implies the following equality $-\frac{[x(t)x'(t)]}{Q(x(t))} = \frac{[y(t)y'(t)]}{Q(y(t))}$. Using this equality, the equality $Q(Wx)(t)) = Q(x(t))$, the equality $\det(W) = -1$ and the inequality $Q(x(t)) > 0$, we obtain

$$\frac{[Wx(t)(Wx)'(t)]}{Q(Wx(t))} = \frac{\det(W)[x(t)x'(t)]}{Q(x(t))} = -\frac{[x(t)x'(t)]}{Q(x(t))} = \frac{[y(t)y'(t)]}{Q(y(t))}$$

for all $t \in T$. This equality and the equality $\frac{\langle Wx(t),(Wx)'(t)\rangle}{Q(Wx(t))} = \frac{\langle x(t),x'(t)\rangle}{Q(x(t))} = \frac{\langle y(t),y'(t)\rangle}{Q(y(t))}$ imply the equalities (3) for the paths $Wx(t)$ and $y(t)$. By Theorem 15(ii), the unique $U_2 \in \mathrm{LSim}^+(2)$ exists such that $y(t) = U_2(Wx(t))$, where U_2 has the form (8). Since $U_2 \in \mathrm{LSim}^+(2)$, we have $U_2W \in \mathrm{LSim}^+(2)W \subset \mathrm{LSim}(2)$. □

Lemma 23. *For all vectors y_1, y_2, z_1, z_2 in E_2, the equality $[y_1y_2][z_1z_2] = \det\|\langle y_i, z_k\rangle\|_{i,k=1,2}$ holds.*

Proof. A proof of this lemma is given in [15, Lemma 13]. □

Theorem 24. (i) *Let $x(t), y(t)$ be L-nondegenerate T-paths in E_2 such that $x \overset{\mathrm{LSim}(2)}{\sim} y$. Then, for all $t \in T$, the following equalities hold*

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = \dfrac{\langle y(t), y'(t)\rangle}{Q(y(t))}, \\ \dfrac{\langle x'(t), x'(t)\rangle}{Q(x(t))} = \dfrac{\langle y'(t), y'(t)\rangle}{Q(y(t))}. \end{cases} \tag{11}$$

(ii) *Conversely, assume that $x(t), y(t)$ be L-nondegenerate T-paths in E_2 such that equalities* (11) *hold, then $x \overset{\mathrm{LSim}(2)}{\sim} y$. Moreover, the unique $F \in \mathrm{LSim}(2)$ exists such that $y(t) = Fx(t)$ for all $t \in T$. Then, only the cases* (a), (b), (c), (d) *in Theorem* 22 *exist.*

In cases (a) and (d), $F = U_1$, where $U_1 \in \mathrm{LSim}^+(2)$ and it has the form (4).
In cases (b) and (c), $F = (U_2W)$, where $U_2 \in \mathrm{LSim}^+(2)$ and it has the form (8).

Proof. (i) Let $x \overset{\mathrm{LSim}(2)}{\sim} y$. Applying Lemma 23 to vectors $y_1 = z_1 = x(t), y_2 = z_2 = x'(t)$, we obtain the following equality

$$[x(t)x'(t)]^2 = \langle x(t), x(t)\rangle\langle x'(t), x'(t)\rangle - \langle x(t), x'(t)\rangle^2.$$

This equality and the equality $Q(x(t)) = \langle x(t), x(t)\rangle$ imply the following equality

$$\frac{[x(t)x'(t)]^2}{Q^2(x(t))} = \frac{\langle x'(t), x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t), x'(t)\rangle^2}{Q^2(x(t))}. \tag{12}$$

This equality implies the following equality

$$\frac{\langle x'(t), x'(t)\rangle}{Q(x(t))} = \frac{\langle x(t), x'(t)\rangle^2}{Q^2(x(t))} + \frac{[x(t)x'(t)]^2}{Q^2(x(t))}. \tag{13}$$

Since functions $\frac{\langle x(t),x'(t)\rangle^2}{Q^2(x(t))}$ and $\frac{[x(t)x'(t)]^2}{Q^2(x(t))}$ are LSim(2)-invariants, the function $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))}$ is LSim(2)-invariant by the equality (13). Since functions $\frac{\langle x(t),x'(t)\rangle}{Q(x(t))}$ and $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))}$ are LSim(2)-invariants, so equalities (11) hold.

(ii) Conversely, assume that the equalities (11) hold. Equalities (11) and (12) imply equalities (10). Then the present theorem follows from Theorem 22. □

Proposition 25. *Let $x(t)$ be an L-regular T-path in E_2. Then*

(i) *$x(t)$ is an L-nondegenerate T-path in E_2 if and only if $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t),x'(t)\rangle^2}{Q^2(x(t))} > 0$ for all $t \in T$.*

(ii) *$x(t)$ is a completely L-degenerate T-path in E_2 if and only if $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t),x'(t)\rangle^2}{Q^2(x(t))} = 0$ for all $t \in T$.*

Proof. (i) Assume that $x(t)$ is an L-nondegenerate T-path in E_2. Then $[x(t)x'(t)] \neq 0$ for all $t \in T$. This inequality implies $[x(t)x'(t)]^2 > 0$ for all $t \in T$. This inequality and equality (12) imply the inequality $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t),x'(t)\rangle^2}{Q^2(x(t))} > 0$ for all $t \in T$.

Assume that the inequality $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t),x'(t)\rangle^2}{Q^2(x(t))} > 0$ holds for all $t \in T$. Then this inequality and equality (12) imply that $[x(t)x'(t)]^2 > 0$ for all $t \in T$. Hence $x(t)$ is an L-nondegenerate T-path in E_2.

(ii) Assume that $x(t)$ is a completely L-degenerate T-path in E_2. Then $[x(t)x'(t)] = 0$ for all $t \in T$. This equality implies $[x(t)x'(t)]^2 = 0$ for all $t \in T$. This equality and equality (12) imply the equality $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t),x'(t)\rangle^2}{Q^2(x(t))} = 0$ for all $t \in T$.

Assume that the equality $\frac{\langle x'(t),x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t),x'(t)\rangle^2}{Q^2(x(t))} = 0$ holds for all $t \in T$. Then this equality and the equality (12) imply that $[x(t)x'(t)]^2 = 0$ for all $t \in T$. Hence, $x(t)$ is a completely L-degenerate T-path in E_2. □

5. Theorems on an Existence for L-Regular Paths

In this section, we prove that for arbitrary real continuous functions $a(t)$ and $b(t)$ on T there exists an L-regular T-path $x(t)$ in R^2 such that $a(t) = \frac{\langle x(t),x'(t)\rangle}{Q(x(t))}$ and $b(t) = \frac{[x(t)x'(t)]}{Q(x(t))}$ for all $t \in T$. Moreover, we give the general evident form of all L-regular T-paths $x(t)$ such that $a(t) = \frac{\langle x(t),x'(t)\rangle}{Q(x(t))}$ and $b(t) = \frac{[x(t)x'(t)]}{Q(x(t))}$ for all $t \in T$.

Similar results will be given below for completely L-degenerate L-regular paths and L-nondegenerate paths.

Theorem 26. *Let $a(t)$ and $b(t)$ be arbitrary real continuous functions on T. Then every an L-regular T-path $x(t)$, satisfying to the following system of equations*

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t) \\ \dfrac{[x(t)x'(t)]}{Q(x(t))} = b(t), \end{cases} \tag{14}$$

has the following form

$$x(t) = Ke^{\int_{t_0}^{t}(a(u)+ib(u))du}, \tag{15}$$

where K is an arbitrary element of Ω^* and $t_0 \in T$.

Conversely, every path of the form (15) *in E_2 is an L-regular path such that equalities* (14) *hold.*

Proof. Let $a(t)$ and $b(t)$ be arbitrary real continuous functions on T. Assume that $x(t)$ is an L-regular T-path in E_2 such that equalities (14) hold. The identity $vu^{-1} = \frac{(u,v)}{Q(u)} + i\frac{[uv]}{Q(u)}$ in Proposition 14 implies the following identity for the path $x(t)$:

$$x'(t)(x(t))^{-1} = \frac{\langle x(t), x'(t)\rangle}{Q(x(t))} + i\frac{[x(t)x'(t)]}{Q(x(t))}.$$

This identity and equalities (14) imply the following equality for the T-path $x(t)$ in Ω:

$$x'(t) = (a(t) + ib(t))x(t).$$

This differential equation has the following general solution:

$$x(t) = Ke^{\int_{t_0}^{t}(a(u)+ib(u))du},$$

where $K \in \Omega$ and $t_0 \in T$. Since $x(t) \neq 0$ for all $t \in T$, this equality implies $K \neq 0$.

Conversely, assume that an L-regular path $x(t)$ has the form (15). Then it is easy to see that equalities (14) hold. □

Example 27. Let $T = (0, \infty)$. Consider the following real continuous functions on T: $a(t) = \frac{1+2t^2}{t+t^3}$ and $b(t) = \frac{1}{1+t^2}$. In this case, the system of equalities (14) has the following general solution:

$$x(t) = Ke^{\int_1^t(a(u)+ib(u))du} = Ke^{\int_1^t a(u)du}\left(\cos\left(\int_1^t b(u)du\right) + i\sin\left(\int_1^t b(u)du\right)\right)$$

$$= \frac{K}{\sqrt{2}}t\sqrt{t^2+1}\left(\frac{t+1}{\sqrt{2}\sqrt{t^2+1}} + i\frac{t-1}{\sqrt{2}\sqrt{t^2+1}}\right) = \frac{K}{2}((t^2+t) + i(t^2-t)),$$

where $K \in \Omega^*$.

Corollary 28. *Let $a(t)$ be an arbitrary real continuous function on T. Then every completely L-degenerate L-regular T-path $x(t)$ in E_2, satisfying to the following equation*

$$\frac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t), \tag{16}$$

has the following form

$$x(t) = Ke^{\int_{t_0}^{t} a(u)du}, \tag{17}$$

where K is an arbitrary element of Ω^* and $t_0 \in T$.

Conversely, every path of the form (17) *in E_2 is a completely L-degenerate L-regular path such that the equality* (16) *holds.*

Proof. For every completely L-degenerate L-regular T-path $x(t)$ in E_2, we have $[x(t)x'(t)] = 0$ for all $t \in T$. Therefore this theorem is a particular case of Theorem 26. ☐

Let $x(t)$ be an L-nondegenerate T-path in E_2. Then $[x(t)x'(t)] > 0$ for all $t \in T$ or $[x(t)x'(t)] < 0$ for all $t \in T$.

Corollary 29. *Let $a(t)$ and $b(t)$ be real continuous functions on T. Assume that $x(t)$ be an L-nondegenerate T-path in E_2 such that*

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t), \\ \dfrac{[x(t)x'(t)]^2}{Q^2(x(t))} = b^2(t). \end{cases} \tag{18}$$

Then,

(i) *$b(t) \neq 0$ for all $t \in T$.*

(ii) *In the case $[x(t)x'(t)] > 0$ for all $t \in T$, it has the following form*

$$x(t) = Ke^{\int_{t_0}^{t} (a(u)+ib(u))du}, \tag{19}$$

where K is an arbitrary element of Ω^* and $t_0 \in T$.

(iii) *In the case $[x(t)x'(t)] < 0$ for all $t \in T$, it has the following form*

$$x(t) = Ke^{\int_{t_0}^{t} (a(u)-ib(u))du}, \tag{20}$$

where K is an arbitrary element of Ω^* and $t_0 \in T$.

Conversely, in the case $b(t) \neq 0$ for all $t \in T$, every T-path of the forms (19) *and* (20) *in E_2 is an L-nondegenerate T-path such that equalities* (18) *hold.*

Proof. (i) Let $x(t)$ be an L-nondegenerate T-path in E_2. Then, we have $[x(t)x'(t)] \neq 0$ for all $t \in T$. Since $b^2(t) = \frac{[x(t)x'(t)]^2}{Q^2(x(t))}$, it is easy see that $b(t) \neq 0$ for all $t \in T$.

Now we prove the statements (ii) and (iii), There are only two cases:

(j) $[x(t)x'(t)] > 0$ for all $t \in T$.
(jj) $[x(t)x'(t)] < 0$ for all $t \in T$.

It is obvious that, in the case (j), the equality

$$\frac{[x(t)x'(t)]^2}{Q^2(x(t))} = b^2(t) \tag{21}$$

is equivalent to the equality

$$\frac{[x(t)x'(t)]}{Q(x(t))} = b(t).$$

Then the system of equalities (18) is equivalent to the following system of equalities

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t), \\ \dfrac{[x(t)x'(t)]}{Q(x(t))} = b(t). \end{cases}$$

By Theorem 26, a general solution of this system is (19). So, the proof of (ii) is completed.

Similarly, in the case (jj), the equality (21) is equivalent to the equality

$$\frac{[x(t)x'(t)]}{Q(x(t))} = -b(t).$$

Then the system of equalities (18) is equivalent to the following system of equalities

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t), \\ \dfrac{[x(t)x'(t)]}{Q(x(t))} = -b(t). \end{cases}$$

By Theorem 26, a general solution of this system is (20). So, the proof of (ii) is completed.

Conversely, assume that $b(t) \neq 0$ for all $t \in T$. Assume that a path $x(t)$ has the form (19). Then it is easy to see that equalities (18) hold and $x(t)$ is an L-nondegenerate path. Similarly, assume that a path $x(t)$ has the form (20). Then it is easy to see that equalities (18) hold and $x(t)$ is an L-nondegenerate path. □

Proposition 30. *Let $x(t)$ be an L-nondegenerate T-path in E_2 such that*

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t), \\ \dfrac{\langle x'(t), x'(t)\rangle}{Q(x(t))} = c(t). \end{cases} \tag{22}$$

Then $c(t) - a^2(t) > 0$ for all $t \in T$.

Proof. Assume that an L-nondegenerate path $x(t)$ in E_2 such that equalities (22) hold. Since $x(t)$ is an L-nondegenerate, $[x(t)x'(t)]^2 > 0$ for all $t \in T$. Using equalities (12) and (22), we obtain

$$\frac{[x(t)x'(t)]^2}{Q^2(x(t))} = \frac{\langle x'(t), x'(t)\rangle}{Q(x(t))} - \frac{\langle x(t), x'(t)\rangle^2}{Q^2(x(t))} = c(t) - a^2(t).$$

Hence $c(t) - a^2(t) > 0$ for all $t \in T$. □

Corollary 31. *Let $a(t)$ and $c(t)$ be real continuous functions on T such that $c(t) - a^2(t) > 0$ for all $t \in T$. Assume that $x(t)$ is an L-nondegenerate T-path in E_2 such that equalities* (22) *hold. Then,*

(a) *In the case $[x(t)x'(t)] > 0$ for all $t \in T$, it has the following form*

$$x(t) = Ke^{\int_{t_0}^{v}(a(u)+i\sqrt{c(u)-a^2(u)})du}, \tag{23}$$

where K is an arbitrary element of Ω^* and $t_0 \in T$.

(b) *In the case $[x(t)x'(t)] < 0$ for all $t \in T$, it has the following form*

$$x(t) = Ke^{\int_{t_0}^{v}(a(u)-i\sqrt{c(u)-a^2(u)})du}, \tag{24}$$

where K is an arbitrary element of Ω^* and $t_0 \in T$.

Conversely, every path of forms (23) *and* (24) *in E_2 is an L-nondegenerate path such that equalities* (22) *hold.*

Proof. Assume that an L-nondegenerate path $x(t)$ in E_2 such that equalities (22) hold. Since $x(t)$ is an L-nondegenerate T-path in E_2, there are only two cases:

(i) $[x(t)x'(t)] > 0$ for all $t \in T$;
(ii) $[x(t)x'(t)] < 0$ for all $t \in T$.

In the case (i), the system (22) of equalities and the equality (12) imply that

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t) \\ \dfrac{[x(t)x'(t)]}{Q(x(t))} = \sqrt{c(t) - a^2(t)}. \end{cases}$$

By Theorem 26, this system of equalities implies the equality (23).

In the case (ii), the system (22) of equalities and the equality (12) imply that

$$\begin{cases} \dfrac{\langle x(t), x'(t)\rangle}{Q(x(t))} = a(t) \\ \dfrac{[x(t)x'(t)]}{Q(x(t))} = -\sqrt{c(t) - a^2(t)}. \end{cases}$$

By Theorem 26, this system of equalities imply the equality (24). □

6. The Type and Invariant Parametrizations of an L-Nondegenerate Curve

In this section, definitions of the type of an L-nondegenerate curve, the invariant parametrizations of L-nondegenerate curve and the G-equivalence of curves are introduced. We obtain that there exist only four LSim(2)-types $(0, l)$ of an L-nondegenerate path $x(t)$, where $l < +\infty$, $(0, +\infty)$, $(-\infty, 0)$, and $(-\infty, +\infty)$. All possible invariant parametrizations of an L-nondegenerate curve with a fixed type are described. The problem of the G-equivalence of L-nondegenerate curves is reduced to that of L-nondegenerate paths for the groups $G = \mathrm{LSim}(2), \mathrm{LSim}^{+}(2)$.

Let $J_1 = (a, b) \subseteq R$ and $J_2 = (c, d) \subseteq R$.

Definition 32 (see [2]). A J_1-path $x(t)$ and a J_2-path $y(r)$ in E_2 are called D-equivalent if a $C^{(1)}$-diffeomorphism $\varphi : J_2 \to J_1$ exists such that $\varphi^{'}(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in J_2$. A class of D-equivalent paths in E_2 is called a curve (nonparametrized curve) in E_2. We will denote it by α. A path $x \in \alpha$ is called a parametrization of a curve α.

Let $\alpha = \{h_\tau, \tau \in \Gamma\}$ be a curve in E_2, where h_τ is a parametrization of α. Then $F\alpha = \{Fh_\tau, \tau \in \Gamma\}$ is a curve in E_2 for any $F \in \mathrm{LSim}(2)$.

Definition 33 (see [2]). Two curves α and β in E_2 are called G-equivalent if $\beta = F\alpha$ for some $F \in G$ and denoted by $\alpha \overset{G}{\sim} \beta$.

Definition 34. A curve α is called L-regular if it contains an L-regular path.

Proposition 35. *Let α be an L-regular curve. Then every parametrization $x \in \alpha$ is an L-regular path.*

Proof. Let α be an L-regular curve. Then there exists a parametrization $x \in \alpha$ such that $x(t)$ is an L-regular J-path, where $J = (a, b) \subseteq R$. Let $y \in \alpha$ be an arbitrary parametrization and $y(u)$ is an U-path, where $U = (c, d) \subseteq R$. Therefore, there exists a $C^{(1)}$-diffeomorphism $\varphi : U \to J$ such that $\varphi^{'}(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in U$. Since $x(t)$ is an L-regular J-path, $\langle x(t), x(t)\rangle \neq 0$ for all $t \in J$. Therefore $\langle y(r), y(r)\rangle = \langle x(\varphi(r)), x(\varphi(r))\rangle \neq 0$ for all $r \in U$. Hence $y(r)$ is an L-regular path. □

Definition 36. A curve α is called L-nondegenerate if it contains an L-nondegenerate path.

Proposition 37. *Let α be an L-nondegenerate curve. Then every parametrization $x \in \alpha$ is an L-nondegenerate path.*

Proof. Let α be an L-nondegenerate curve. Then there exists a parametrization $x \in \alpha$ such that $x(t)$ is an L-nondegenerate J-path, where $J = (a, b) \subseteq R$. Let $y \in \alpha$ be an arbitrary parametrization and $y(u)$ is an U-path, where $U = (c, d) \subseteq R$.

Then there exists a $C^{(1)}$-diffeomorphism $\varphi : U \to J$ such that $\varphi^{'}(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in U$. Since $x(t)$ is an L-nondegenerate J-path, $[x(t)x'(t)] \neq 0$ for all $t \in J$. We have

$$\begin{aligned}[y(r)y'(r)] &= \left[x(\varphi(r))\frac{dx(\varphi(r))}{dr}\right] \\ &= \left[x(\varphi(r))\left(\varphi'(r)\frac{dx(\varphi(r))}{d\varphi}\right)\right] \\ &= \varphi'(r)\left[x(\varphi(r))\frac{dx(\varphi(r))}{d\varphi}\right].\end{aligned}$$

This equality and the inequality $[x(t)x'(t)] \neq 0$ for all $t \in J$ imply the inequality $[x(\varphi(r))\frac{dx(\varphi(r))}{d\varphi}] \neq 0$ for all $r \in U$. Since $\varphi'(r) > 0$ for all $r \in U$ and $[x(\varphi(r))\frac{dx(\varphi(r))}{d\varphi}] \neq 0 \ \forall r \in U$, the above equality implies $[y(r)y'(r)] \neq 0, \forall r \in U$. Hence, $y(r)$ is an L-nondegenerate path. □

Now we define invariant parametrizations of L-nondegenerate curves in E_2. Let $x(t)$ be an L-nondegenerate J-path in E_2, where $J = (a, b) \subseteq R, a < b$. For $c, d \in R$ such that $a < c < d < b$, we put $l_x(c, d) = \int_c^d \frac{|[x(t)x'(t)]|}{Q(x(t))} dt$. It is easy to see that the limits $l_x(a, d) = \lim_{c\to a} l_x(c, d) \leq +\infty$ and $l_x(c, b) = \lim_{d\to b} l_x(c, d) \leq +\infty$ exist. There are only four possibilities:

(γ_1) $0 < l_x(a, d) < +\infty, 0 < l_x(c, b) < +\infty$.
(γ_2) $0 < l_x(a, d) < +\infty, l_x(c, b) = +\infty$.
(γ_3) $l_x(a, d) = +\infty, 0 < l_x(c, b) < +\infty$.
(γ_4) $l_x(a, d) = +\infty, l_x(c, b) = +\infty$.

Suppose that the case (γ_1) or (γ_2) holds for some $c, d \in J$. Then $l = l_x(a, d) + l_x(c, b) - l_x(c, d)$, where $0 \leq l \leq +\infty$, does not depend on $c, d \in J$. In this case, we say that x belongs to the LSim(2)-type of $(0, l)$. The cases (γ_3) and (γ_4) do not depend on c, d. In these cases, we say that x belongs to the LSim(2)-types of $(-\infty, 0)$ and $(-\infty, +\infty)$, respectively. There exist paths of all LSim(2)-types $(0, l)$, where $l < +\infty$, $(0, +\infty)$, $(-\infty, 0)$, and $(-\infty, +\infty)$. The LSim(2)-type of a path x will be denoted by $TL(x)$.

Example 38. Let $T = (a, b)$, $a, b \in R, a < b$. Consider a T-path $x(t) = (\cos t, \sin t)$ in E_2. For $c, d \in T, c < d$, we obtain $l_x(c, d) = \int_c^d \frac{|[x(t)x'(t)]|}{Q(x(t))} dt = \int_c^d dt = d - c$.

(i) Let $T = (0, l)$, where $0 < l < +\infty$. We have $l_x(0, d) = \lim_{c\to 0} l_x(c, d) = \lim_{c\to 0}(d - c) = d$ and $l_x(c, l) = \lim_{d\to l} l_x(c, d) = \lim_{d\to l}(d - c) = l - c$. Since $l_x(0, d) < +\infty$ and $l_x(c, l) < +\infty$, the type of the path is $(0, l)$, where $l < +\infty$.

(ii) Let $T = (0, +\infty)$. We have $l_x(0, d) = \lim_{c\to 0} l_x(c, d) = \lim_{c\to 0}(d - c) = d$ and $l_x(c, +\infty) = \lim_{d\to+\infty} l_x(c, d) = \lim_{d\to+\infty}(d - c) = +\infty$. Since $l_x(0, d) < +\infty$ and $l_x(c, +\infty) = +\infty$, the type of the path is $(0, +\infty)$.

(iii) Let $T=(-\infty,0)$, we have $l_x(-\infty,d)=\lim_{c\to-\infty} l_x(c,d)=\lim_{c\to-\infty}(d-c)=+\infty$ and $l_x(c,0)=\lim_{d\to 0} l_x(c,d)=\lim_{d\to 0}(d-c)=-c$. Since $l_x(-\infty,d)=+\infty$ and $l_x(c,0)<+\infty$, the type of the path is $(-\infty,0)$.

(iv) For $T=(-\infty,+\infty)$, we have $l_x(-\infty,d)=\lim_{c\to-\infty} l_x(c,d)=\lim_{c\to-\infty}(d-c)=+\infty$ and $l_x(c,+\infty)=\lim_{d\to+\infty} l_x(c,d)=\lim_{d\to+\infty}(d-c)=+\infty$. Since $l_x(-\infty,d)=+\infty$ and $l_x(c,+\infty)=+\infty$, the type of the path is $(-\infty,+\infty)$.

Proposition 39. (i) *Let x and y be L-nondegenerate paths in E_2. If $x \overset{\mathrm{LSim}(2)}{\sim} y$, then $TL(x)=TL(y)$.*

(ii) *If x and y are parametrizations of an L-nondegenerate curve α, then $TL(x)=TL(y)$.*

Proof. It is easy and omitted. □

The LSim(2)-type of a path $x\in\alpha$ is called the LSim(2)-type of the nondegenerate curve α and denoted by $TL(\alpha)$.

Proposition 40. *Let α and β be nondegenerate curves in E_2. If $\alpha \overset{\mathrm{LSim}(2)}{\sim} \beta$, then $TL(\alpha)=TL(\beta)$.*

Proof. It is easy and omitted. □

Now we define an invariant parametrization of an L-nondegenerate curve in E_2. Let $J=(a,b)$ and $x(t)$ be an L-nondegenerate J-path in E_2. We define the arc length function $s_x(t)$ for each LSim(2)-type as follows. We put $s_x(t)=l_x(a,t)$ for the case $TL(x)=(0,l)$, where $l\leq+\infty$, and $s_x(t)=-l_x(t,b)$ for the case $TL(x)=(-\infty,0)$.

Assume that $TL(x)=(-\infty,+\infty)$. In order to define the function $s_x(t)$ in this case, we choose a fixed point in every interval $J=(a,b)$ of R and denote it by a_J. Put $a_J=0$ for $J=(-\infty,+\infty)$. For the J-path $x(t)$, we set $s_x(t)=l_x(a_J,t)$.

Since $s'_x(t)>0$ for all $t\in J$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $TL(x)$ and $t'_x(s)>0$ for all $s\in TL(x)$.

Proposition 41. *Let $U=(a,b)$ and x be an L-nondegenerate U-path in E_2 and $J=(c,d)\subseteq R$. Then*:

(i) $s_{Fx}(t)=s_x(t),\forall t\in U$, *and* $t_{Fx}(s)=t_x(s),\forall\, s\in TL(x)$, *for all* $F\in\mathrm{LSim}(2)$.

(ii) *the equalities* $s_{x(\varphi)}(r)=s_x(\varphi(r))+s_0,\forall r\in J$, *and* $\varphi(t_{x(\varphi)}(s+s_0))=t_x(s),\forall s\in TL(x)$, *hold for any* $C^{(1)}$*-diffeomorphism* $\varphi:J=(c,d)\to U$ *such that* $\varphi'(r)>0$ *for all* $r\in J$, *where* $s_0=0$ *for* $TL(x)\neq(-\infty,+\infty)$ *and* $s_0=l_x(\varphi(a_J),a_U)$ *for* $TL(x)=(-\infty,+\infty)$.

Proof. It is similar to the proof of [2, Proposition 2]. □

Let α be an L-nondegenerate curve, $x \in \alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 42 (see [2]). The parametrization $x(t_x(s))$ of an L-nondegenerate curve α is called an invariant parametrization of α.

We denote the set of all invariant parametrizations of an L-nondegenerate α by $Ip(\alpha)$. Every $y \in Ip(\alpha)$ is a J-path, where $J = TL(\alpha)$.

Proposition 43. *Let α be an L-nondegenerate curve, $x \in \alpha$ and x be a J-path, where $J = TL(\alpha)$. Then the following conditions are equivalent*:

(i) *x is an invariant parametrization of α.*
(ii) $\frac{|[x(s)x'(s)]|}{Q(x(s))} = 1$ *for all $s \in TL(\alpha)$.*
(iii) $s_x(s) = s$ *for all $s \in TL(\alpha)$.*

Proof. It is similar to the proof of [2, Proposition 3]. □

Suppose $J \subseteq R$ is one of the sets $(0, l), l < +\infty$; $(0, +\infty)$, $(-\infty, 0)$ or $(-\infty, +\infty)$.

Theorem 44. *Let α be an L-nondegenerate curve in E_2 and $x(s) \in Ip(\alpha)$. Then, $x(s)$ has the following form*

$$x(s) = Ke^{\int_{s_0}^{s}(a(u)+i)du} \tag{25}$$

or the following form

$$x(s) = Ke^{\int_{s_0}^{s}(a(u)-i)du}, \tag{26}$$

where K is an arbitrary element of Ω^*, $s_0 \in J$ and $a(u)$ is a real continuous functions on J.

Conversely, every path $x(t)$ of the form (25) *or* (26) *is an invariant parametrization of an L-nondegenerate curve in E_2.*

Proof. Let α be an L-nondegenerate curve in E_2 and $x(s) \in Ip(\alpha)$. Then, by Proposition 43(ii), $\frac{|[x(s)x'(s)]|}{Q(x(s))} = 1, \forall s \in J$. Put $b(s) = \frac{[x(s)x'(s)]}{Q(x(s))}$ and $a(s) = \frac{(x(s),x'(s))}{Q(x(s))}$. Then, by Theorem 26, $x(s)$ has the form (15), where $|b(s)| = 1, \forall\, s \in J$. In the case $b(s) = 1, \forall s \in J$, the form (15) has the form (25). In the case $b(s) = -1, \forall\, s \in J$, the form (15) has the form (26).

Conversely, assume that $x(s)$ has the form (25) or (26). Then it has the form (15), where $|b(s)| = 1, \forall s \in J$, Then, by Theorem 26, the equalities (14) hold. These equalities and $|b(s)| = 1, \forall s \in J$ imply that $\frac{|[x(s)x'(s)]|}{Q(x(s))} = 1, \forall s \in J$. Then, by Proposition 43(ii), the path $x(s)$ is an invariant parametrization. □

Proposition 45. *Let α be an L-nondegenerate curve and $TL(\alpha) \neq (-\infty, +\infty)$. Then there exists a unique invariant parametrization of α.*

Proof. It is similar to the proof of [2, Proposition 4]. □

Example 46. Let $J = (0, 2\pi)$. Consider the J-path $x(t) = (\cos t, \sin t)$ in E_2. Let α be a curve such that $x(t) \in \alpha$. By Example 38, the type of the path $x(t)$ is $(0, 2\pi)$. Hence, the type $TL(\alpha)$ of the curve α is $J = (0, 2\pi)$. Since $[x(t)x'(t)] = 1 \neq 0$ for all $t \in J$, $x(t)$ is an L-nondegenerate path. Hence, the curve α is an L-nondegenerate curve. Since $\frac{|[x(t)x'(t)]|}{Q(x(t))} = 1$ for all $t \in TL(\alpha) = (0, 2\pi)$, by Proposition 43, t is the invariant parameter of the curve α and $x(t)$ is an invariant parametrization of α. Since $TL(\alpha) = (0, 2\pi) \neq (-\infty, +\infty)$, by Proposition 45, the curve α has the unique invariant parametrization and it is $x(t)$.

Let $I = (0, \pi)$. Consider the I-path $y(u) = (\cos 2u, \sin 2u)$ in E_2. Then $y(u) = x(\varphi(u))$ for all $u \in I$, where $\varphi : I \to J = (0, 2\pi)$ is the homeomorphism $\varphi(u) = 2u$ such that $\varphi'(u) > 0$ for all $u \in I$. By Definition 32, paths $x(t)$ and $y(u)$ are D-equivalent. Hence, by this definition, $y(u) \in \alpha$.

Let $J = (0, 2\pi)$. Consider J-paths $x(t) = (\cos t, \sin t)$ and $y(t) = (\cos 2t, \sin 2t)$ in E_2. It is easy to see that $x(t)$ and $y(t)$ are not D-equivalent. Hence, by Definition 32, $y(t)$ is not a parametrization of the curve α.

Let α be an L-nondegenerate curve and $TL(\alpha) = (-\infty, +\infty)$. Then it is easy to see that the set $Ip(\alpha)$ is infinite and it is not countable.

Remark 47. We note that if $x(t)$ is a periodic L-nondegenerate path then $TL(x) = (-\infty, +\infty)$.

Proposition 48. *Let α be an L-nondegenerate curve, $TL(\alpha) = (-\infty, +\infty)$ and $x \in Ip(\alpha)$. Then $Ip(\alpha) = \{y : y(s) = x(s + c), c \in (-\infty, +\infty)\}$.*

Proof. It is similar to the proof of [2, Proposition 5]. □

Example 49. Let $J = R$. Consider a J-path $x(t) = (\cos t, \sin t)$ in E_2. Since $[x(t)x'(t)] = 1 \neq 0$ for all $t \in J$, $x(t)$ is an L-nondegenerate path. Let α is a curve such that $x(t) \in \alpha$. By Definition 36, α is an L-nondegenerate curve and $TL(\alpha) = (-\infty, +\infty)$. Since $[x(t)x'(t)] = 1$ for all $t \in J$, by Proposition 43, we have $x(t) \in Ip(\alpha)$. Let $c \in R$. Consider the J-path $y(t) = x(t + c)$. The mapping $\varphi : J \to J$, where $\varphi(t) = t + c$ for all $t \in J$, is a homeomorphism such that $\varphi'(t) = 1 > 0$ for all $t \in J$. This implies that the J-paths $x(t)$ and $y(t)$ are D-equivalent. Hence, $y(t) = x(t + c) \in \alpha$ for all $c \in R$. By Proposition 48, $x(t + c) \in Ip(\alpha)$ for $c \in R$.

Let H be one of the groups LSim(2), $\mathrm{LSim}^+(2)$.

Theorem 50. *Let α, β be L-nondegenerate curves and $x \in Ip(\alpha), y \in Ip(\beta)$. Then*

(i) *for $TL(\alpha) = TL(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{H}{\sim} \beta$ if and only if $x \overset{H}{\sim} y$.*
(ii) *for $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$, $\alpha \overset{H}{\sim} \beta$ if and only if $x \overset{H}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c(s) = s + c$.*

Proof. A proof of this theorem for the group H is similar to the proof of [2, Theorem 1] and it is omitted. □

Theorem 50 reduces the problem of the H-equivalence of L-nondegenerate curves for the groups LSim(2), $\mathrm{LSim}^+(2)$ to that of paths only for the case $TL(\alpha) = TL(\beta) \neq (-\infty, +\infty)$.

Definition 51. Let $J = (-\infty, +\infty)$. J-paths $x(t)$ and $y(t)$ are called $[H, (-\infty, +\infty)]$-equivalent, if there exist $h \in H$ and $d \in (-\infty, +\infty)$ such that $y(t) = hx(t + d)$ for all $t \in J$.

Let α, β be L-nondegenerate curves such that $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$. In this case, Theorem 50 reduces the problem of H-equivalence of these curves to the problem of $[H, (-\infty, +\infty)]$-equivalence of paths.

7. Complete Systems of Invariants of an *L*-Nondegenerate Curve and Uniqueness Theorems for *L*-Nondegenerate Curves

In this section, we will give the conditions of the global G-equivalence of L-nondegenerate curves in terms of the type and the differential invariants of an L-nondegenerate curve for the groups $G = \mathrm{LSim}(2)$, $\mathrm{LSim}^+(2)$.

By Theorem 50, G-equivalence and uniqueness problems for curves are reduced to the same problems for invariant parametrizations of curves only for the case $TL(\alpha) = TL(\beta) \neq (-\infty, +\infty)$. Below we use this reduction.

Theorem 52. *Let α, β be L-nondegenerate curves in E_2, $TL(\alpha) \neq (-\infty, +\infty)$, $TL(\beta) \neq (-\infty, +\infty)$ and $x \in Ip(\alpha), y \in Ip(\beta)$.*

(i) *Assume that $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$. Then, for all $s \in TL(\alpha)$ the following equalities hold*

$$\begin{cases} TL(\alpha) = TL(\beta) \\ \dfrac{\langle x(s), x'(s)\rangle}{Q(x(s))} = \dfrac{\langle y(s), y'(s)\rangle}{Q(y(s))} \\ \mathrm{Sgn}([x(s)x'(s)]) = \mathrm{Sgn}([y(s)y'(s)]). \end{cases} \tag{27}$$

(ii) *Conversely, assume that equalities* (27) *hold. Then there exists the unique $F \in \mathrm{LSim}^+(2)$ such that $\beta = F\alpha$. In this case, $F\alpha = U\alpha$, where $U \in \mathrm{LSim}^+(2)$ and it has the following form*

$$U = \begin{pmatrix} \dfrac{\langle x(s), y(s)\rangle}{Q(x(s))} & -\dfrac{[x(s)y(s)]}{Q(x(s))} \\ \dfrac{[x(s)y(s)]}{Q(x(s))} & \dfrac{\langle x(s), y(s)\rangle}{Q(x(s))} \end{pmatrix}, \tag{28}$$

and U does not depend on $s \in TL(\alpha)$.

Proof. (i) Let $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$. By Proposition 40, we have $TL(\alpha) = TL(\beta)$. This equality, $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$ and Theorem 50(i) imply $x \overset{\mathrm{LSim}^+(2)}{\sim} y$. By Theorem 15,

$x \overset{\mathrm{LSim}^+(2)}{\sim} y$ implies (3). By (3), the equality $\frac{[x(s)x'(s)]}{Q(x(s))} = \frac{[y(s)y'(s)]}{Q(y(s))}$ holds for all $s \in TL(\alpha)$. Since $x(s)$ is an L-nondegenerate, we have $[x(s)x'(s)] > 0$ for all $s \in TL(\alpha)$ or $[x(s)x'(s)] < 0$ for all $s \in TL(\alpha)$. Similarly, since $y(s)$ is an L-nondegenerate, $[y(s)y'(s)] > 0$ for all $s \in TL(\alpha)$ or $[y(s)y'(s)] < 0$ for all $s \in TL(\alpha)$. By $Q(x(s)) > 0$ for all $s \in TL(\alpha)$ and $Q(y(s)) > 0$ for all $s \in TL(\alpha)$, the equality $\frac{[x(s)x'(s)]}{Q(x(s))} = \frac{[y(s)y'(s)]}{Q(y(s))}$ implies the equality $\mathrm{Sgn}([x(s)x'(s)]) = \mathrm{Sgn}([y(s)y'(s)])$ for all $s \in TL(\alpha)$. Hence equalities (27) hold.

(ii) Conversely, assume that equalities (27) hold. Since $x(s), y(s)$ are invariant parametrizations, by Proposition 43(ii), we have $\frac{|[x(s)x'(s)]|}{Q(x(s))} = 1$ and $\frac{|[y(s)y'(s)]|}{Q(y(s))} = 1$ for all $s \in TL(\alpha)$. These equalities imply $\frac{|[x(s)x'(s)]|}{Q(x(s))} = \frac{|[y(s)y'(s)]|}{Q(y(s))}$ for all $s \in TL(\alpha)$. This equality and the equality $\mathrm{Sgn}([x(s)x'(s)]) = \mathrm{Sgn}([y(s)y'(s)])$ imply the equality $\frac{[x(s)x'(s)]}{Q(x(s))} = \frac{[y(s)y'(s)]}{Q(y(s))}$ for all $s \in TL(\alpha)$. This equality and equalities (27) imply equalities (3). By Theorem 15, these equalities imply $x \overset{\mathrm{LSim}^+(2)}{\sim} y$ and there exists the unique $F \in \mathrm{LSim}^+(2)$ such that $y(s) = Fx(s)$. In this case, $F = U$, where $U \in \mathrm{LSim}^+(2)$ and it has the form (28). Here, U does not depend on $s \in TL(\alpha)$. Using $x \in Ip(\alpha), y \in Ip(\beta)$, Theorem 50(i) and $y(s) = Fx(s)$ for all $s \in TL(\alpha)$, we obtain that $\beta = F\alpha$. □

Remark 53. By Theorem 52, the system $\{TL(\alpha), \frac{\langle x(s), x'(s)\rangle}{Q(x(s))}, \mathrm{Sgn}([x(s)x'(s)])\}$, where $x(s)$ is the invariant parametrization of the curve α, is a complete system of invariants of the curve α for the case $TL(\alpha) \neq (-\infty, +\infty)$. But they are not invariants of a curve α, in the case $TL(\alpha) = (-\infty, +\infty)$.

Theorem 54. *Let α, β be L-nondegenerate curves in E_2, $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$ and $x \in Ip(\alpha), y \in Ip(\beta)$.*

(i) *Assume that $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$. Then there exists $s_1 \in (-\infty, +\infty)$ such that the following equalities hold for all $s \in (-\infty, +\infty)$*

$$\begin{cases} TL(\alpha) = TL(\beta) \\ \dfrac{\langle x(s+s_1), x'(s+s_1)\rangle}{Q(x(s+s_1))} = \dfrac{\langle y(s), y'(s)\rangle}{Q(y(s))} \\ \mathrm{Sgn}([x(s)x'(s)]) = \mathrm{Sgn}([y(s)y'(s)]). \end{cases} \tag{29}$$

(ii) *Conversely, assume that equalities* (29) *hold for some $s_1 \in (-\infty, +\infty)$ and all $s \in (-\infty, +\infty)$. Then there exists the unique $F \in \mathrm{LSim}^+(2)$ such that $\beta = F\alpha$.*

In this case, $F = U$, where $U \in \mathrm{LSim}^+(2)$ and it has the following form

$$U = \begin{pmatrix} \dfrac{\langle x(s+s_1), y(s)\rangle}{Q(x(s+s_1))} & -\dfrac{[x(s+s_1)y(s)]}{Q(x(s+s_1))} \\ \dfrac{[x(s+s_1)y(s)]}{Q(x(s+s_1))} & \dfrac{\langle x(s+s_1), y(s)\rangle}{Q(x(s+s_1))} \end{pmatrix}, \tag{30}$$

and U does not depend on $s \in L(\alpha)$.

Proof. (i) Let $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$. By Proposition 40, we have $TL(\alpha) = TL(\beta)$. By this equality, $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$ and Theorem 50(ii), there exits $s_1 \in (-\infty, +\infty)$ such that $x(s+s_1) \overset{\mathrm{LSim}^+(2)}{\sim} y(s)$. By Theorem 15(i), $x(s+s_1) \overset{\mathrm{LSim}^+(2)}{\sim} y(s)$ implies (3) for $x(s+s_1)$ and $y(s)$ that is:

$$\begin{cases} \dfrac{\langle x(s+s_1), x'(s+s_1)\rangle}{Q(x(s+s_1))} = \dfrac{\langle y(s), y'(s)\rangle}{Q(y(s))}, \\ \dfrac{[x(s+s_1)x'(s+s_1)]}{Q(x(s+s_1))} = \dfrac{[y(s)y'(s)]}{Q(y(s))}. \end{cases}$$

Hence the equality $\frac{\langle x(s+s_1), x'(s+s_1)\rangle}{Q(x(s+s_1))} = \frac{\langle y(s), y'(s)\rangle}{Q(y(s))}$ holds for all $s \in TL(\alpha)$. Since $x(s)$ is an L-nondegenerate, $[x(s+s_1)x'(s+s_1)] > 0$ for all $s, s_1 \in TL(\alpha)$ or $[x(s+s_1)x'(s+s_1)] < 0$ for all $s, s_1 \in TL(\alpha)$. Similarly, since $y(s)$ is an L-nondegenerate, $[y(s)y'(s)] > 0$ for all $s \in TL(\alpha)$ or $[y(s)y'(s)] < 0$ for all $s \in TL(\alpha)$. By $Q(x(s+s_1)) > 0$ for all $s, s_1 \in TL(\alpha)$ and $Q(y(s)) > 0$ for all $s \in TL(\alpha)$, the equality $\frac{[x(s+s_1)x'(s+s_1)]}{Q(x(s+s_1))} = \frac{[y(s)y'(s)]}{Q(y(s))}$ implies the equality $\mathrm{Sgn}([x(s)x'(s)]) = \mathrm{Sgn}([x(s+s_1)x'(s+s_1)]) = \mathrm{Sgn}([y(s)y'(s)])$ for all $s \in TL(\alpha)$. Hence equalities (29) hold.

(ii) Conversely, assume that equalities (29) hold for some $s_1 \in (-\infty, +\infty)$ and all $s \in (-\infty, +\infty)$. Since $x(s), y(s)$ are invariant parametrizations, by Proposition 43(ii), we have $\frac{|[x(s)x'(s)]|}{Q(x(s))} = 1$ and $\frac{|[y(s)y'(s)]|}{Q(y(s))} = 1$ for all $s \in TL(\alpha)$. These equalities imply $\frac{|[x(s+s_1)x'(s+s_1)]|}{Q(x(s+s_1))} = \frac{|[x(s)x'(s)]|}{Q(x(s))} = \frac{|[y(s)y'(s)]|}{Q(y(s))}$ for all $s \in TL(\alpha)$. These equalities and $\mathrm{Sgn}([x(s)x'(s)]) = \mathrm{Sgn}([y(s)y'(s)])$ imply the equality $\frac{[x(s+s_1)x'(s+s_1)]}{Q(x(s+s_1))} = \frac{[y(s)y'(s)]}{Q(y(s))}$ for all $s \in TL(\alpha)$. This equality and equalities (29) imply the equalities

$$\begin{cases} \dfrac{\langle x(s+s_1), x'(s+s_1)\rangle}{Q(x(s+s_1))} = \dfrac{\langle y(s), y'(s))}{Q(y(s)\rangle} \\ \dfrac{[x(s+s_1)x'(s+s_1)]}{Q(x(s+s_1))} = \dfrac{[y(s)y'(s)]}{Q(y(s))} \end{cases}$$

for all $s \in TL(\alpha)$. By Theorem 15, $x(s+s_1) \overset{\mathrm{LSim}^+(2)}{\sim} y(s)$ and there exists the unique $F \in \mathrm{LSim}^+(2)$ such that $y(s) = Fx(s+s_1)$ for all $s \in TL(\alpha)$.

In this case, $Fx(s+s_1) = Ux(s+s_1)$, where $U \in \mathrm{LSim}^+(2)$ and it has the form (30). Here, U does not depend on $s \in TL(\alpha)$. Using $x \in Ip(\alpha), y \in Ip(\beta)$, Theorem 50(ii) and $y(s) = Fx(s+s_1)$ for all $s \in TL(\alpha)$, we obtain that $\beta = F\alpha$. □

Example 55. (i) Let $T = (0, l)$, where $l \leq \infty$. Consider two T-paths $x(t) = (\cos t, \sin t)$ and $y(t) = (\sin t, \cos t)$. Since $[x(t)x'(t)] \neq 0$ and $[y(t)y'(t)] \neq 0$ for all $t \in T$, $x(t)$ and $y(t)$ are nondegenerate T-paths. Let α is a curve such that $x(t) \in \alpha$ and β is a curve such that $y(t) \in \beta$. By Definition 36, α and β are nondegenerate curves. By Proposition 43, since $\frac{|[x(t)x'(t)]|}{Q(x(t))} = 1$, $x(t)$ is an invariant parametrizations of α. By Proposition 43, since $\frac{|[y(t)y'(t)]|}{Q(y(t))} = 1$, $y(t)$ is an invariant parametrizations of β. Since $\frac{\langle x(t),x'(t)\rangle}{Q(x(t))} = \frac{\langle y(t),y'(t)\rangle}{Q(y(t))}$ and $\mathrm{Sgn}([x(t)x'(t)]) \neq \mathrm{Sgn}([y(t)y'(t)])$, we obtain $x(t) \overset{\mathrm{LSim}^+(2)}{\nsim} y(t)$.

By Theorem 52, this implies that $\alpha \overset{\mathrm{LSim}^+(2)}{\nsim} \beta$.

(ii) Let $T = (0, l)$, where $l \leq \infty$. Consider two T-paths $x(t) = (\cos t, \sin t)$ and $y(t) = (-\cos t, -\sin t)$. Since $[x(t)x'(t)] \neq 0$ and $[y(t)y'(t)] \neq 0$ for all $t \in T$, $x(t)$ and $y(t)$ are nondegenerate T-paths. Let α is a curve such that $x(t) \in \alpha$ and β is a curve such that $y(t) \in \beta$. By Definition 36, α and β are nondegenerate curves. By Proposition 43, since $\frac{|[x(t)x'(t)]|}{Q(x(t))} = 1$, $x(s)$ is an invariant parametrizations of α. By Proposition 43, since $\frac{|[y(t)y'(t)]|}{Q(y(t))} = 1$, $y(t)$ is an invariant parametrizations of β. Since $\frac{\langle x(t),x'(t)\rangle}{Q(x(t))} = \frac{\langle y(t),y'(t)\rangle}{Q(y(t))}$ and $\mathrm{Sgn}([x(t)x'(t)]) = \mathrm{Sgn}([y(t)y'(t)])$, we obtain $x(t) \overset{\mathrm{LSim}^+(2)}{\sim} y(t)$.

By Theorem 52, this implies that $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$.

(iii) Let $T = (-\infty, \infty)$. Consider two T-paths $x(t) = (\cos(t+c), \sin(t+c))$ and $y(t) = (\cos t, \sin t)$ for $c \in T$. Since $[x(t)x'(t)] \neq 0$ and $[y(t)y'(t)] \neq 0$ for all $t \in T$, $x(t)$ and $y(t)$ are nondegenerate T-paths. Let α is a curve such that $x(t) \in \alpha$ and β is a curve such that $y(t) \in \beta$. By Definition 36, α and β are nondegenerate curves. By Proposition 43, since $\frac{|[x(t)x'(t)]|}{Q(x(t))} = 1$, $x(t)$ is an invariant parametrizations of α. By Proposition 43, since $\frac{|[y(t)y'(t)]|}{Q(y(t))} = 1$, $y(t)$ is an invariant parametrizations of β. Since $\frac{\langle x(t+c),x'(t+c)\rangle}{Q(x(t))} = \frac{\langle y(t),y'(t)\rangle}{Q(y(t))}$ and $\mathrm{Sgn}([x(t)x'(t)]) = \mathrm{Sgn}([y(t)y'(t)])$, we obtain $x(t) \overset{\mathrm{LSim}^+(2)}{\sim} y(t)$.

By Theorem 54, this implies that $\alpha \overset{\mathrm{LSim}^+(2)}{\sim} \beta$.

Theorem 56. *Let α, β be L-nondegenerate curves in E_2, $TL(\alpha) \neq (-\infty, +\infty)$, $TL(\beta) \neq (-\infty, +\infty)$ and $x \in Ip(\alpha), y \in Ip(\beta)$.*

(i) *Assume that* $\alpha \overset{\mathrm{LSim}(2)}{\sim} \beta$. *Then we have*

$$\begin{cases} TL(\alpha) = TL(\beta) \\ \dfrac{\langle x(s), x'(s)\rangle}{Q(x(s))} = \dfrac{\langle y(s), y'(s)\rangle}{Q(y(s))} \end{cases} \tag{31}$$

for all $s \in TL(\alpha)$.

(ii) *Conversely, assume that equalities* (31) *hold. Then* $\alpha \overset{\mathrm{LSim}(2)}{\sim} \beta$. *Moreover, the unique* $F \in \mathrm{LSim}(2)$ *exists such that* $\beta = F\alpha$ *and only two following cases exist*:

(ii.1) $\mathrm{Sgn}([x(s)x'(s)]) = \mathrm{Sgn}([y(s), y'(s)])$.
(ii.2) $\mathrm{Sgn}([x(s)x'(s)]) = -\mathrm{Sgn}([y(s), y'(s)])$.

In this case (ii.1), $F = U_1$, *where* U_1 *has the form* (30).

In this case (ii.2), $F = (U_2W)$, *where* U_2 *has the following form*

$$U_2 = \begin{pmatrix} \dfrac{\langle Wx(s), y(s)\rangle}{Q(Wx(s))} & -\dfrac{[Wx(s)y(s)]}{Q(Wx(s))} \\ \dfrac{[Wx(s)y(s)]}{Q(Wx(s))} & \dfrac{\langle Wx(s), y(s)\rangle}{Q(Wx(s))} \end{pmatrix},$$

and it does not depend on $s \in TL(\alpha)$.

Proof. It follows easily from Theorems 22, 50(i) and 52. □

Theorem 57. *Let* α, β *be L-nondegenerate curves in* E_2, $TL(\alpha) = TL(\beta) = (-\infty, +\infty)$ *and* $x \in Ip(\alpha), y \in Ip(\beta)$.

(i) *Assume that* $\alpha \overset{\mathrm{LSim}(2)}{\sim} \beta$. *Then there exists* $s_1 \in (-\infty, +\infty)$ *such that the following equality*

$$\frac{\langle x(s+s_1), x'(s+s_1)\rangle}{Q(x(s+s_1))} = \frac{\langle y(s), y'(s)\rangle}{Q(y(s))} \tag{32}$$

holds for all $s \in (-\infty, +\infty)$.

(ii) *Conversely, assume that there exists* $s_1 \in (-\infty, +\infty)$ *such that the equality* (32) *holds for all* $s \in (-\infty, +\infty)$. *Then* $\alpha \overset{\mathrm{LSim}(2)}{\sim} \beta$. *Moreover,* $F \in \mathrm{LSim}(2)$ *exists such that* $\beta = F\alpha$ *and only two following cases exist*:

(ii.1) $[x(s+s_1)x'(s+s_1)] = [y(s)y'(s)]$;
(ii.2) $[x(s+s_1)x'(s+s_1)] = -[y(s)y'(s)]$.

In the case (ii.1), $F = U_1$, *where* U_1 *has the form* (30).

In the case (ii.2), $F = (U_2W,)$, *where* U_2 *has the following form*

$$U_2 = \begin{pmatrix} \dfrac{\langle Wx(s+s_1), y(s)\rangle}{Q(Wx(s+s_1))} & -\dfrac{[Wx(s+s_1)y(s)]}{Q(Wx(s+s_1))} \\ \dfrac{[Wx(s+s_1)y(s)]}{Q(Wx(s+s_1))} & \dfrac{\langle Wx(s+s_1), y(s)\rangle}{Q(Wx(s+s_1))}. \end{pmatrix},$$

Proof. It follows easily from Theorems 22, 50(ii) and 54. □

8. Conclusion

Methods, developed in the present work, will be useful in the theory of global invariants of curves in two-dimensional geometries and physics. This approach is also will be useful in the theory of global invariants of vector fields and a system of curves in two-dimensional geometries and physics.

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