Numerical methods for solving the two-dimensional boundary value problem of the elasticity theory

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Abstract--**Using the theory of elasticity and thermoelasticity of the space of deformations, coupled boundary-value problems are formulated. The theory of elasticity and thermoelasticity of the deformation space, unlike the existing one, allows us to formulate conjugate boundary-value problems for increments of displacement and temperature. Explicit and implicit finite-difference equations are constructed for the two - dimensional case of boundary value problems. The numerical solution of explicit finite-difference equations reduces to the use of recurrence formulas, while the implicit scheme reduces to the use of the elimination method.**

Keywords--boundary value problem, elasticity, numerical method, iterative method, boundary conditions.

I. INTRODUCTION

In the world in many areas of science and technology, scientific and practical research often comes down to solving related or unrelated problems of elasticity. Elastic processes and their mathematical models are the subject of applied mathematics, computational mathematics, mathematical modeling and object-oriented programming. In many cases, finite-difference methods are used to solve the problems of elasticity and determining reliability of structural elements numerically. Therefore, the study of the process of heat distribution taking into account stress, strain and temperature, as well as electromagnetic and piezoelectric properties, with the aim of constructing algorithms for numerical solution and creating a set of applied programs is one of the most important tasks.

World experience shows that, in the numerical solution of coupled and unrelated problems of elasticity, calculations using finite-difference methods, explicit and implicit difference schemes, are relevant problems of applied mathematics. In this case, finite-difference methods, as a means of calculating heat propagation causing the appearance of stresses, strains, and other fields in the body, acquire special significance.

In this regard, the urgent is the task of constructing numerical models of elastic problems for isotropic and anisotropic bodies, as well as the task of creating the appropriate complex of programs, focused on a wide range of users, having a convenient interface.

In recent years, the theory of elasticity has found wide application in solving engineering problems. There are many cases where the elementary methods of resistance of materials are unsuitable in order to provide satisfactory information on the distribution of stresses in engineering structures; then you have to resort to more advanced methods of elasticity theory. The elementary theory is insufficient to form an idea of local stresses near the zones of application of loads, taking into account various factors, such as temperature and electromagnetic fields of deformation.

In recent years, significant successes have been achieved in solving such practically important problems. In cases where it is difficult to obtain an exact solution, approximate methods have been developed. Therefore, to solve these problems, numerical methods are used. To discretize these problems of boundary value problems, the finite difference method [3], the finite element method [2], the variation difference method [7], and the boundary element method [1] are usually used by many other methods.

II. NUMERICAL METHOD FOR SOLVING PROBLEMS OF THE THEORY OF ELASTICITY

A numerical method for solving problems of the theory of elasticity is used to numerically solve the boundary value problem of thermoelasticity. Mathematical and numerical models of the two-dimensional boundary value elasticity problem for an isotropic rectangle with boundary conditions of the first and second types are considered. Finite-difference schemes have been constructed which, in combination with the iterative method, allow finding the desired nodal values of displacements [1].

Consider the elasticity problem for an isotropic material. It consists of equilibrium equations:

$$\sum_{j=1}^{3} \sigma_{ij,j} + X_i = 0, \qquad i = 1,3 \tag{1}$$

Hooke's law for isotropic materials

$$\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu \varepsilon_{ij} \tag{2}$$

Cauchy relations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{3}$$

and boundary conditions

$$u_i\big|_{\Sigma_1} = u_i^\circ, \qquad \sum_{j=1}^{3} \sigma_{ij} n_j \big|_{\Sigma_2} = S_i^\circ \tag{4}$$

where is σ_{ij} -the stress tensor, ε_{ij} --is the strain tensor, u_i - are the displacement components, X_i - -are the volume forces, λ, μ - are the Lame elastic constants, $\theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ -the spherical part of the strain tensor, σ_{ij} - is the Kronecker symbol, n_i -is the external normal to the surface, $\sum_2$, S_1, S_2, S_3 -are the components of the external load vector [1].

We write equations (1)-(3) for the two-dimensional case: then the equilibrium equation.

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + X_1 = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + X_2 = 0 \qquad (5)$$

Hooke's law

$$\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{11} = (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22}$$
$$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{22} = (\lambda + 2\mu)\varepsilon_{22} + \lambda\varepsilon_{11}$$
$$\sigma_{12} = 2\mu\varepsilon_{12} \qquad (6)$$

and, the ratio of Cauchy

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$$
$$\varepsilon_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2}$$
$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \qquad (7)$$

and, boundary conditions at the borders $\Gamma_1 = (x_1 = 0, l_1 : 0 \le x_2 \le l_2)$ and $\Gamma_2 = (x_2 = 0, l_2 : 0 \le x_1 \le l_1)$ accordingly take the form [5, 9]:

$$u_1\big|_{\Gamma_1} = u_1^o , \quad u_2\big|_{\Gamma_1} = u_2^o ,$$
$$(\sigma_{11} n_1 + \sigma_{12} n_2)\big|_{\Gamma_2} = S_1 , \quad (\sigma_{21} n_1 + \sigma_{22} n_2)\big|_{\Gamma_2} = S_2 . \qquad (8)$$
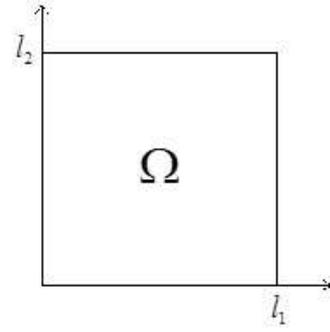

Fig 1. Rectangular area.

Substituting (7) into (6) and obtained in (5), we have the equations of equilibrium with respect to displacements:

$$(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x_1^2} + (\lambda + \mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \mu\frac{\partial^2 u_1}{\partial x_2^2} + X_1 = 0$$
$$(\lambda + 2\mu)\frac{\partial^2 u_2}{\partial x_2^2} + (\lambda + \mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \mu\frac{\partial^2 u_2}{\partial x_1^2} + X_2 = 0 . \qquad (9)$$

Bearing the following notation $u_1 = u$, $u_1 = u$, $u_2 = v$, $x_1 = x$, $x_2 = y$ equations (9) can be written in the following form

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} + \mu\frac{\partial^2 u}{\partial y^2} + X_1 = 0$$
$$(\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} + \mu\frac{\partial^2 v}{\partial x^2} + X_2 = 0. \qquad (10)$$

Equations (10), for simplicity, we consider in a rectangular region $\Omega = \{0 \le x \le l_1, 0 \le y \le l\}$ with the following boundary conditions:

$$u(x,y)\big|_{y=0} = 0 \quad , \quad v(x,y)\big|_{y=0} = \sin\frac{\pi x}{l_1},$$
$$u(x,y)\big|_{y=l_2} = 0 \quad , \quad v(x,y)\big|_{y=l_2} = -\sin\frac{\pi x}{l_1},$$
$$u(x,y)\big|_{x=0} = \sin\frac{\pi y}{l_2} \quad , \quad v(x,y)\big|_{x=0} = 0, \qquad (11)$$
$$u(x,y)\big|_{x=l_1} = -\sin\frac{\pi y}{l_2} \quad , \quad v(x,y)\big|_{x=l_1} = 0.$$

To construct a finite-difference scheme for the two-dimensional boundary value problem of the theory of elasticity (10-11), the lengths of the sides of the rectangular region l dividing into N_k can find that where $k=1.2...$ Then the nodal points have the form

$$x_i = h_1 \cdot i, \quad i = \overline{0, N_1},$$
$$y_j = h_2 \cdot j, \quad j = \overline{0, N_2}.$$

Further, replacing the derivatives in equations (10) with the corresponding difference relations, we can find the following finite-difference equations [3, 13]

$$(\lambda + 2\mu)\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_1^2} + (\lambda + \mu)\frac{v_{i+1,j+1} - v_{i-1,j+1} - v_{i+1,j-1} + v_{i-1,j-1}}{4h_1 h_2} +$$
$$+\mu\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_2^2} + X_1 = 0$$

$$(\lambda + 2\mu)\frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h_2^2} + (\lambda + \mu)\frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4h_1 h_2} +$$
$$+\mu\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h_1^2} + X_2 = 0. \tag{12}$$

The boundary conditions (11) with respect to the nodal points have the form i.e.

$$u_{i0} = 0 \quad , \quad v_{i0} = \sin\frac{\pi x_i}{l_1}$$
$$u_{iN_2} = 0 \quad , \quad v_{iN_2} = -\sin\frac{\pi x_i}{l_1} \tag{3}$$
$$u_{0j} = \sin\frac{\pi y_j}{l_2} \quad , \quad v_{0j} = 0$$
$$u_{N_1 j} = -\sin\frac{\pi y_j}{l_2} \quad , \quad v_{N_1 j} = 0$$

Where $i = \overline{1, N_1}, j = \overline{0, N_2}$.

Note that the approximation order of the finite-difference scheme is $O(h^2)$ [6]. According to [8, 10], difference equation (12) can be considered as a linear, symmetric, and positive definite operator. Then, by the Lax-Milgram theorem, the convergence of the difference scheme [4] is ensured.

The scheme of difference equations (12) is solvable with respect to displacements u_{ij} and v_{ij} i.e.

$$u_{i,j} = (4h_2^2(\lambda + 2\mu)(u_{i+1,j} + u_{i-1,j}) + 4h_1^2\mu(u_{i,j+1} + u_{i,j-1}) + h_1 h_2 (\lambda + \mu)*$$
$$*(v_{i+1,j+1} - v_{i-1,j+1} - v_{i+1,j-1} + v_{i-1,j-1}) + X_1)/(8h_2^2(\lambda + 2\mu) + 8h_1^2\mu)$$
$$v_{i,j} = (4h_1^2(\lambda + 2\mu)(v_{i,j+1} + v_{i,j-1}) + 4h_2^2\mu(v_{i+1,j} + v_{i-1,j}) + h_1 h_2 (\lambda + \mu)*$$
$$*(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}) + X_2)/(8h_1^2(\lambda + 2\mu) + 8h_2^2\mu). \tag{14}$$

.

Further, based on relations (14), we organize the following iterative process by index $k = 0, 1, 2, ...$

$$u_{i,j}^{(k+1)} = (4h_2^2(\lambda + 2\mu)(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)}) + 4h_1^2\mu(u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}) + h_1 h_2 (\lambda +$$
$$+\mu)*(v_{i+1,j+1}^{(k)} - v_{i-1,j+1}^{(k)} - v_{i+1,j-1}^{(k)} + v_{i-1,j-1}^{(k)}) + X_1)/(8h_2^2(\lambda + 2\mu) + 8h_1^2\mu)$$

$$v_{i,j}^{(k+1)} = (4h_1^2(\lambda + 2\mu)(v_{i,j+1}^{(k)} + v_{i,j-1}^{(k)}) + 4h_2^2\mu(v_{i+1,j}^{(k)} + v_{i-1,j}^{(k)}) + h_1 h_2 (\lambda +$$
$$+\mu)*(u_{i+1,j+1}^{(k)} - u_{i-1,j+1}^{(k)} - u_{i+1,j-1}^{(k)} + u_{i-1,j-1}^{(k)}) + X_2)/(8h_1^2(\lambda + 2\mu) + 8h_2^2\mu)$$
$$\tag{15}$$

taking into account boundary conditions (13)

$$u_{i0}^{(0)} = 0 \quad , \quad v_{i0}^{(0)} = \sin\frac{\pi x_i}{l_1},$$
$$u_{iN_2}^{(0)} = 0 \quad , \quad v_{iN_2}^{(0)} = -\sin\frac{\pi x_i}{l_1},$$
$$\tag{16}$$
$$u_{0j}^{(0)} = \sin\frac{\pi y_j}{l_2} \quad , \quad v_{0j}^{(0)} = 0,$$
$$u_{N_1 j}^{(0)} = -\sin\frac{\pi y_j}{l_2} \quad , \quad v_{N_1 j}^{(0)} = 0.$$

At a zero approximation i.e. for $k=0$ nodal values of the sought quantities, $u_{ij}^{(0)}$, $v_{ij}^{(0)}$ -at the boundary of a rectangular region, Ω they are known from the boundary conditions (16). In internal nodes, the values of displacements in the zeroth approximation *(k=0)* are considered trivial[13, 11].

Further, continuing the iterative process, one can find the desired u_{ij}, v_{ij} ,displacement values with a given accuracy ε . The convergence of the iterative method is achieved according to the generalized theorem on the convergence of iterative methods [2].

Note that the following functions

$$u = \cos\frac{\pi x}{l_1}\sin\frac{\pi y}{l_2}, \quad v = \sin\frac{\pi x}{l_1}\cos\frac{\pi y}{l_2} \tag{17}$$

satisfy the boundary conditions (11).

Substituting (17) into equations (10), we can verify that for the following values of volume forces

$$X_1 = -(\lambda+2\mu)\frac{\pi^2}{l_1^2}\cos\frac{\pi x_i}{l_1}\sin\frac{\pi y_j}{l_2} - (\lambda+\mu)\frac{\pi^2}{l_1 l_2}\cos\frac{\pi x_i}{l_1}\sin\frac{\pi y_j}{l_2} -$$

$$-\mu\frac{\pi^2}{l_2^2}\cos\frac{\pi x_i}{l_1}\sin\frac{\pi y_j}{l_2}$$

$$X_2 = -(\lambda+2\mu)\frac{\pi^2}{l_2^2}\sin\frac{\pi x_i}{l_1}\cos\frac{\pi y_j}{l_2} - (\lambda+\mu)\frac{\pi^2}{l_1 l_2}\sin\frac{\pi x_i}{l_1}\cos\frac{\pi y_j}{l_2} -$$

$$-\mu\frac{\pi^2}{l_1^2}\sin\frac{\pi x_i}{l_1}\cos\frac{\pi y_j}{l_2} \qquad (18)$$

functions u, v satisfy the equations (10).

Thus, the functions, u, v (17) are an exact solution to the boundary-value problem (10-11) for bulk forces (18).

The iterative process (15-16) continues until the following condition is satisfied, $\left\| u^{\rightarrow(k+1)} - u^{\rightarrow(k)} \right\| \le \varepsilon$, where $\|.\|$ the quadratic norm of successive approximations of the difference of the displacement vector in space L_2 [18, 19].

The problem was solved with the following parameter values $\mu = 0.5$, $\lambda = 0.8$, $l_1 = l_2 = 1$, $N_1 = N_2 = 10$.

TABLE I. Function Values u(x,y) at $\varepsilon = 0.01$.

	x=0	x=0.1	x=0.2	x=0.3	x=0.4
y=0	0	0	0	0	0
y=0.1	0.30902	0.27424	0.22276	0.15742	0.08149
y=0.2	0.58779	0.53354	0.43485	0.30553	0.15732
y=0.3	0.80902	0.74704	0.61436	0.43307	0.22294
y=0.4	0.95106	0.88750	0.73525	0.52011	0.26821
y=0.5	1	0.93653	0.77780	0.55107	0.28434
y=0.6	0.95106	0.88750	0.73525	0.52011	0.26821
y=0.7	0.80902	0.74704	0.61436	0.43307	0.22294
y=0.8	0.58779	0.53354	0.43485	0.30553	0.15732
y=0.9	0.30902	0.27424	0.22276	0.15742	0.08149
y=1	0	0	0	0	0

x=0.5	x=0.6	x=0.7	x=0.8	x=0.9	x=1
0	0	0	0	0	0
0	-0.08149	-0.15742	-0.22276	-0.27424	-0.30902
0	-0.15732	-0.30553	-0.43485	-0.53354	-0.58779
0	-0.22294	-0.43307	-0.61436	-0.74704	-0.80902
0	-0.26821	-0.52011	-0.73525	-0.88750	-0.95106
0	-0.28434	-0.55107	-0.77780	-0.93653	-1
0	-0.26821	-0.52011	-0.73525	-0.88750	-0.95106
0	-0.22294	-0.43307	-0.61436	-0.74704	-0.80902
0	-0.15732	-0.30553	-0.43485	-0.53354	-0.58779
0	-0.08149	-0.15742	-0.22276	-0.27424	-0.30902
0	0	0	0	0	0

TABLE II. Function Values u(x,y) at $\varepsilon = 0.001$.

	x=0	x=0.1	x=0.2	x=0.3	x=0.4
y=0	0	0	0	0	0
y=0.1	0.30902	0.29397	0.24997	0.18162	0.09551
y=0.2	0.58779	0.55939	0.47549	0.34514	0.18132
y=0.3	0.80902	0.77069	0.65545	0.47574	0.24989
y=0.4	0.95106	0.90681	0.77172	0.56032	0.29433
y=0.5	1	0.95380	0.81195	0.58961	0.30974
y=0.6	0.95106	0.90681	0.77172	0.56032	0.29433
y=0.7	0.80902	0.77069	0.65545	0.47574	0.24989
y=0.8	0.58779	0.55939	0.47549	0.34514	0.18132
y=0.9	0.30902	0.29397	0.24997	0.18162	0.09551
y=1	0	0	0	0	0

x=0.5	x=0.6	x=0.7	x=0.8	x=0.9	x=1
0	0	0	0	0	0
0	-0.09551	-0.18162	-0.24997	-0.29397	-0.30902
0	-0.18132	-0.34514	-0.47549	-0.55939	-0.58779
0	-0.24989	-0.47574	-0.65545	-0.77069	-0.80902
0	-0.29433	-0.56032	-0.77172	-0.90681	-0.95106
0	-0.30974	-0.58961	-0.81195	-0.95380	-1
0	-0.29433	-0.56032	-0.77172	-0.90681	-0.95106
0	-0.24989	-0.47574	-0.65545	-0.77069	-0.80902
0	-0.18132	-0.34514	-0.47549	-0.55939	-0.58779
0	-0.09551	-0.18162	-0.24997	-0.29397	-0.30902
0	0	0	0	0	0

TABLE III. Exact Solution Values u(x,y).

	x=0	x=0.1	x=0.2	x=0.3	x=0.4
y=0	0	0	0	0	0
y=0.1	0.30902	0.29389	0.25000	0.18164	0.09549
y=0.2	0.58779	0.55902	0.47553	0.34549	0.18164
y=0.3	0.80902	0.76942	0.65451	0.47553	0.25000
y=0.4	0.95106	0.90451	0.76942	0.55902	0.29389
y=0.5	1	0.95106	0.80902	0.58779	0.30902
y=0.6	0.95106	0.90451	0.76942	0.55902	0.29389
y=0.7	0.80902	0.76942	0.65451	0.47553	0.25000
y=0.8	0.58779	0.55902	0.47553	0.34549	0.18164
y=0.9	0.30902	0.29389	0.25000	0.18164	0.09549
y=1	0	0	0	0	0

| x=0.5	x=0.6	x=0.7	x=0.8	x=0.9	x=1
0	0	0	0	0	0
0	-0.09549	-0.18164	-0.25000	-0.29389	-0.30902
0	-0.18164	-0.34549	-0.47553	-0.55902	-0.58779
0	-0.25000	-0.47553	-0.65451	-0.76942	-0.80902
0	-0.29389	-0.55902	-0.76942	-0.90451	-0.95106
0	-0.30902	-0.58779	-0.80902	-0.95106	-1
0	-0.29389	-0.55902	-0.76942	-0.90451	-0.95106
0	-0.25000	-0.47553	-0.65451	-0.76942	-0.80902
0	-0.18164	-0.34549	-0.47553	-0.55902	-0.58779
0	-0.09549	-0.18164	-0.25000	-0.29389	-0.30902
0	0	0	0	0	0
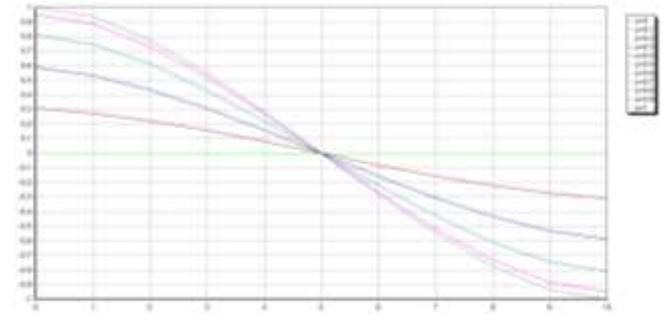
Fig 2. The graph of the distribution of the values of the function $u(x,y)$ along the OX axis at.
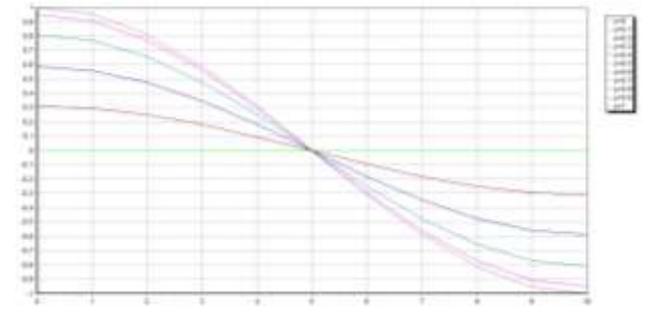

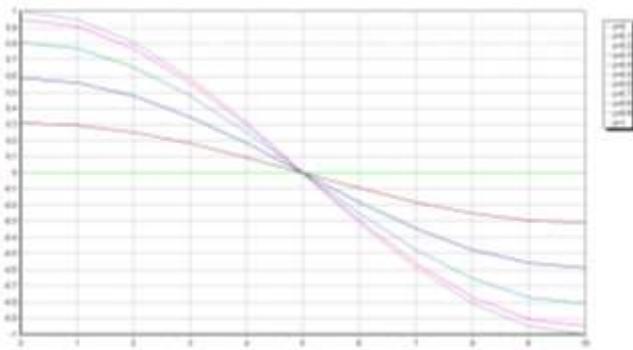Fig 3. The graph of the distribution of the values of the function $u(x,y)$ along the OX axis at.

Fig 4. The graph of the distribution of the values of the function *u(x,y)* along the OX axis according to the exact solution.

Comparison of the numerical results of the boundary value problem (10-11) with the exact solution (17) is shown in Fig. 3. and Fig. 4. It can be seen from the graphs that the displacement components are close enough, which ensures the reliability of the results and the validity of the proposed numerical solution method[14, 16].

CONCLUSIONS

A new numerical approach to solving the first static boundary value problem of the theory of elasticity is proposed. The boundary conditions are specified with respect to displacements. The finite-difference equations solved with respect to the sought quantities in combination with the iterative method give results that coincide with the exact solution and the solutions obtained by other methods[11]. Based on the iterative method under consideration, a number of boundary value problems of the theory of elasticity are solved under various boundary conditions.

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