M. Mirsaburov

SINGULYAR KOEFFITSIYENTLI GELLERSTEDT TENGLAMASI UCHUN TRIKOMI MASALASI

(Ushbu o‘quv qo‘llanma Termiz davlat universiteti ilmiy kengashining 3.11.2022-yilda bo‘lib o‘tgan yig‘ilishi tomonidan 3-bayonnoma bilan nashrga tavsiya etilgan).

“Surxon-nashr”, 2022

UDK. 517. 956
UO‘K 519.642.7(075.8)
KBK 22.161.6ya73
M 53

Mirsaburov M. Singulyar koeffitsiyentli Gellerstedt tenglamasi uchun Trikomi masalasi: — Termiz: “Surxon-nashr”, 2022 – 236 bet.

O‘quv qo‘llanmada singulyar koeffitsiyentli buziluvchan elliptik va giperbolik turdagi tenglamalar hamda aralash turdagi tenglamalar uchun Dirixle, shakli o‘zgargan Xolmgren va shakli o‘zgargan Koshi hamda singulyar koeffitsientli aralash turdagi tenglama uchun Trikomi masalalari o‘rganilgan.

O‘quv qo‘llanma matematika yo‘nalishi magistrantlari, tayanch doktorantlar va ilmiy xodimlarga mo‘ljallangan.

Mas’ul muharrir:
Islomov B.– fizika-matematika fanlari doktori,professor

Taqrizchilar:

ZikirovO.S.–O‘zMU Matematika fakulteti dekani, fizika-matematika fanlaridoktori, professor.

RuziyevM.X.–O‘zbekiston FA V.I.Romanovskiy nomidagi Matematika instituti differensial tenglamalar va ularning tatbiqlari laboratoriyasi yetakchi ilmiy xodimi, fizika-matematika fanlari doktori.

ISBN:978-9943-8664-8-5

Qizim Shaxloning yorqin xotirasiga bag‘ishlayman
Muallif

KIRISH

Buziluvchan giperbolik, elliptik turdagi va aralash turdagi tenglamalar uchun chegaraviy masalalar nazariyasi zamonaviy xususiy hosilali differensial tenglamalar nazariyasining asosiy yo‘nalishlaridan biri hisoblanadi va u muhim amaliy masalalarni yechishda qo‘llaniladi. Buziluvchan giperbolik, elliptik va aralash turdagi tenglamalar nazariyasining rivojlanishi dastlab G. Darbu [38], F.Trikomi [40], E. Xolmgren [43] va S. Gellerstedtlarning [42] mos ravishda 1894, 1923, 1927 va 1938-yillarda e’lon qilingan fundamental ishlaridan boshlangan.

Aralash turdagi

$$T(u) = yu_{xx} + u_{yy} = 0 \qquad (1)$$

tenglama uchun birinchi fundamental tadqiqotlarni italyan matematigi Franchesko Trikomi bajargan. U hozirgi vaqtda uning nomi bilan ataluvchi quyidagi Trikomi masalasini ta’riflagan va yechgan: $z=x+iy$ kompleks tekisligining $y>0$ yarim tekisligida uchlari $A(0,0)$ va $B(1,0)$ nuqtalarda bo‘lgan Γ silliq Jordan chizig‘i bilan, $y<0$ yarim tekislikda esa (1) tenglamaning AC va BC xarakteristikalari bilan chegaralangan bir bog‘lamli Ω sohada (1) tenglamaning ushbu

$$u(x,y)=\varphi(x,y),\ (x,y)\in\Gamma, \qquad (2)$$

$$u(x,y)=\psi(x),\ (x,y)\in AC, \qquad (3)$$

$$\lim_{y\to -0} u_y = \lim_{y=+0} u_y, \qquad (4)$$

shartlarni qanoatlantiruvchi regulyar yechimi $u(x,y)$ topilsin. $u(x,y)$ funksiya regulyar yechim deyiladi, agarda u ushbu shartlarni qanoatlantirsa:

1) $u(x,y)$ - $\overline{\Omega}$ sohada uzluksiz;

2) birinchi tartibli hosilalar A va B nuqtalardan tashqari barcha $\overline{\Omega}$ sohada uzluksiz va bu nuqtalarda birdan kichik tartibda cheksizlikka aylanishi mumkin;

3) ikkinchi tartibli hosilalar Ω sohaning buzilish chizig‘idan tashqari barcha nuqtalarida uzluksiz, bu hosilalar buzilish chizig‘ida mavjud bo‘lmasligi ham mumkin;

4) $u(x,y)$ funksiya $\Omega \setminus AB$ sohaning barcha nuqtalarida (1) tenglamani qanoatlantiradi.

Bu ishlardan keyin buziluvchan va aralash turdagi tenglamalar uchun chegaraviy masalalar nazariyasi ko‘p yo‘nalishlarda o‘rganildi va rivojlantirildi. Xususan, Trikomi masalasi umumiyroq aralash turdagi tenglamalar uchun [4,5,6,11,13,14,15] ishlarda, Trikomi masalasining har xil modefikatsiyasi [3,10,17-

23,32-37,41,42] ishlarda spektral masalalar [1,28,30,37] ishlarda o‘rganildi. Eng muhim natijalar va adabiyotlar ro‘yxati A. V. Bitsadze [4], M. M. Smirnov [38], M. S. Salaxitdinov [31], T. D. Djurayev [11], A. M. Naxushev [27], E. I. Moiseyev [28], A. P. Soldatov [39], A.I.Kojanov [15] monografiyalarida keltirilgan.

Quyidagi singulyar koeffitsiyentli buziluvchan giperbolik tipdagi tenglamani $z = x + iy$, $\operatorname{Im} z < 0$ kompleks yarim tekislikdao‘rganamiz:

$$-(-y)^m u_{xx} + u_{yy} + \alpha_0 (-y)^{m/2-1} u_x + \beta_0 y^{-1} u_y = 0, \qquad (5)$$

bu yerda m , α_0 va β_0 - haqiqiy sonlar, hamda ular ushbu

$$-m/2 \le \beta_0 \le (m+4)/2, \, |\alpha_0| \le (m+2)/2,$$

shartlarni qanoatlantiradi. D_0 soha $z = x + iy$ kompleks tekislikning bir bog‘lamli sohasi bo‘lib, u (5) tenglamaning

$$AC: x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = -1,$$

$$BC: x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1,$$

xarakteristikalari hamda $y = 0$ o‘qining AB kesmasi bilan chegaralangan bir bog‘lamli sohasi bo‘lsin.

(5) tenglama shu narsa bilan e’tiborliki, birinchidan, bu tenglamaning kichik hadlari oldidagi koeffitsiyentlari singulyar maxsuslikka ega, ikkinchidan, bu yerda

$$K(y)h(x,y)u_{xx} + u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y) \qquad (6)$$

buziluvchan umumiy giperbolik tipdagi tenglama uchun Koshi masalasini normal yechilishining

$$\lim_{y \to -0} \frac{ya(x,y)}{\sqrt{-K(y)}} = 0, \qquad (7)$$

Protter sharti [38] buziladi, bu yerda $h(x,y) > 0$, $K(0) \equiv 0$, $K(y) < 0$, $y < 0$ da (7) shart bajarilmasligiga qaramasdan, agar $|\alpha_0| \le m/2$, $\beta_0 = 0$ bo‘lsa, (5) tenglama uchun Koshi masalasi korrekt qo‘yilgan [38].

Bundan, (5) tenglama uchun Koshi masalasini normal yechilishida (7) shart zaruriy shart emasligi kelib chiqadi.

Endi (5) tenglamada $\beta_0 = 0$, $\alpha_0 = -m/2$ bo‘lsin:

$$-(-y)^m u_{xx} + u_{yy} - (m/2)(-y)^{m/2-1} u_x = 0, \qquad (8)$$

(8) tenglama uchun Darbu masalasini ta'riflaymiz.

Darbuning ikkinchi masalasi: D_0 sohada (8) tenglamaning ushbu

$$u_y(x,0) = \nu(x),\ x \in I :\ u\big|_{BC} = \psi(x)\ ,\ x \in [0,1], \qquad (9)$$

shartlarni qanoatlantiruvchi regulyar $u(x,y) \in C(\overline{D}_0) \cap C^2(D_0)$ yechimi topilsin, bu yerda $\nu(x) \in C^2(I),\ \psi(x) \in C^1(\bar{I}) \cap C^2(I),\ I = (-1,1)$- $y = 0$ o'qining intervali.

1-teorema.*Darbuning ikkinchi masalasiga mos bir jinsli masala cheksiz ko'p chiziqli bog'liq bo'lmagan yechimlarga ega, bir jinsli bo'lmagan masala esa faqat va faqat*

$$\nu(2x-1) = ((m+2)/2)^{\beta} (1-x)^{\beta} \psi'(x)\ ,\ x \in (0,1),$$

shart bo'lgandagina yechimga ega bo'ladi, bu yerda $\beta = m/(m+2)$.

Bir jinsli Darbuning ikkinchi masalasining barcha notrivial yechimlari

$$u(x,y) = \tau_0\left(x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} \right) - \tau_0(1),$$

formula bilan beriladi, bu yerda $\tau_0(x) \in C(\bar{I}) \cap C^2(I)$ sinfdagi ixtiyoriy funksiya.

Endi (8) tenglama uchun (9) Darbu shartlarini ushbu

$$u_y(x,0) = \nu(x)\ ,\ x \in I\ ;\ u\big|_{AC} = \psi(x),\ x \in [-1,0] \qquad (10)$$

shaklda beramiz.

2-teorema.*(8), (10) masala yagona yechimga ega .*

1-va 2-teoremalardan ushbu xulosa kelib chiqadi: qat'iy giperbolik tenglamalar uchun qo'yilgan Koshi masalasining korrektligidan Darbu masalasining korrektligi kelib chiqadi. Buziluvchan giperbolik tenglamalarda esa, umuman olganda, Koshi masalasi korrektligidan Darbu masalasining korrektligi kelib chiqmaydi. Buning ustiga (8) buziluvchan giperbolik tenglama uchun umuman olganda xarakteristikalar, chegaraviy shartlarning ularda qo'yilishi ma'nosida teng huquqli emas.

(5) tenglamada $\alpha_0 = 0$ bo'lsin:

$$-(-y)^m u_{xx}+u_{yy}+(\beta_0/y)u_y=0. \qquad (11)$$

Bu tenglama juda ko‘p matematiklar tomonidan o‘rganilgan [6,12,20]. Umuman olganda, (11) tenglama uchun oddiy Koshi masalasi korrekt bo‘lmasligi mumkin. A. V. Bitsadze [6] (11) tenglama uchun boshlang‘ich shartlari bir jinsli bo‘lgan:

$$u(x,0)=0,\ x\in\bar{I};\quad \lim_{y\to-0}\frac{\partial u}{\partial y}=0,\ x\in I;$$

Koshi masalasi $\beta_0=-m/2$ bo‘lganda, ushbu

$$u_0(x,y)=\tau_0\left[x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}\right]-\tau_0\left[x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}\right],$$

ko‘rinishdagi notrivial yechimlarga ega ekanligini ko‘rsatgan, bu yerda $\tau_0(x)$ ikki marta uzluksiz hosilaga ega bo‘lgan ixtiyoriy funksiya. Shu holatdan kelib chiqib, A. V. Bitsadze boshlang‘ich shartlari

$$u(x,0)=\tau(x),\ x\in\bar{I};\ \lim_{y\to-0}(-y)^{\beta_0}\frac{\partial u}{\partial y}=\nu(x),\ x\in I, \qquad (12)$$

ko‘rinishda bo‘lgan shakli o‘zgargan Koshi masalasini o‘rgangan va uni korrekt ekanligini ko‘rsatgan, bu yerda

$$-(m/2)\le\beta_0<1.$$

Agar $\beta_0\ge1$ bo‘lsa, (11) tenglamaning yechimlari buzilish chizig‘i atrofida chegaralangan bo‘lmaydi. Haqiqatdan ham, ushbu

$$u_0(x,y)=\begin{cases}(-y)^{1-\beta_0}, & agar\ \beta_0\ne1\ bo'lsa,\\ \ln(-y), & agar\ \beta_0=1\ bo'lsa\end{cases}$$

xususiy yechimlar yuqoridagi fikrimizni tasdiqlaydi.

$\beta_0>1$ bo‘lganda Koshi masalasi korrekt bo‘lishi uchun boshlang‘ich shartlar

$$\lim_{y\to-0}(-y)^{\beta_0-1}u(x,y)=\tau(x);\ \lim_{y\to-0}(-y)^{2-\beta_0}\frac{\partial}{\partial y}\left((-y)^{\beta_0-1}u(x,y)\right)$$

ko‘rinishda bo‘lishi kerak;

$\beta_0 = 1$ bo‘lganda esa Koshi masalasi korrekt bo‘lishi uchun boshlang‘ich shartlar

$$\lim_{y \to -0} \frac{u(x,y)}{\ln(-y)^{(m+2)/2}} = \tau(x),$$

$$\lim_{y \to -0} (-y) \ln^2(-y)^{(m+2)/2} \frac{\partial}{\partial y}\left[\frac{u(x,y) - A(x,y)}{\ln(-y)^{(m+2)/2}}\right] = \nu(x),$$

ko‘rinishda bo‘lishi kerak, bu yerda $A(x,y)-$aniq ko‘rinishga ega bo‘lgan maxsus kiritilgan funksiya.

Shunday qilib, (5) tenglama yechimining tuzilishi va differensial xossalari uning kichik hadlari oldidagi koeffitsiyentlar α_0 va β_0 ga bog‘liqdir. (5) tenglama uchun masalalar $\alpha_0 O \beta_0$ parametrik tekislikda $P\,(\alpha_0, \beta_0)$ nuqtaning o‘zgarishiga qarab qo‘yiladi.

$y > 0$ yarim tekislikda

$$y^m u_{xx} + u_{yy} + (\beta_0 / y) u_y = 0 \tag{13}$$

tenglamani o‘rganamiz.

(13) tenglama shu bilan xarakterliki, uning uchun oddiy N masalasi korrekt emas. Haqiqatdan ham, Ω_0 -yuqori $y > 0$ yarim tekislikda yotuvchi va uchlari $A(-1,0)$, $B(1,0)$ nuqtada bo‘lgan (13) tenglamaning normal chizig‘i, $\sigma_0 : x^2 + 4(m+2)^{-2} y^{m+2} = 1$ chizig‘i hamda $y = 0$ o‘qining AB kesmasi bilan chegaralangan bir bog‘lamli bo‘lsin. Ushbu masalani ta’riflaymiz.

N **masalasi.** Ω_0 sohada (13) tenglamaning ushbu

$$u\Big|_{\sigma_0} = \varphi_0(x,y),\ (x,y) \in \sigma_0,$$

$$\frac{\partial u}{\partial y}\Big|_{y=0} = \nu(x),\ x \in I = (-1,1),$$

shartlarni qanoatlantiruvchi regulyar yechimi $u(x,y) \in C(\overline{\Omega}_0) \cap C^2(\Omega_0)$ topilsin.

Bevosita tekshirish yordamida ko‘rsatish mumkinki, ushbu

$$u(x,y)=\frac{1-x^2-\dfrac{4}{(m+2)^2}y^{m+2}}{\left(1-x+\dfrac{2}{m+2}y^{\frac{m+2}{2}}\right)^2+\left(1+x+\dfrac{2}{m+2}y^{\frac{m+2}{2}}\right)^2}$$

funksiya bir jinsli N masalaning notrivial yechimi bo‘ladi, ya’ni (13) tenglama uchun N masalasi korrekt emas. Shu munosabat bilan A.V.Bitsadze (13) tenglama uchun ushbu shakli o‘zgargan N masalasini o‘rgangan:

Ω_0 sohada (13) tenglamaning ushbu

$$u\Big|_{\sigma_0}=\varphi_0(x,y), \quad (x,y)\in\sigma_0,$$

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y}=\nu(x),\ x\in I=(-1,1)$$

shartlarni qanoatlantiruvchi regulyar yechimi topilsin.

Shakli o‘zgargan N masalasi korrekt qo‘yilgan.

Ushbu o‘quv qo‘llanmada asosan singulyar koeffitsiyentli

$$signy\,|y|^m u_{xx}+u_{yy}+\alpha_0|y|^{m/2-1}u_x+(\beta_0/y)u_y=0 \qquad (14)$$

tenglama o‘rganilgan. (14) tenglama $z=x+iy$, kompleks tekisligining $\operatorname{Im}z>0$ yuqori yarim tekisligida uchlari $A(-1,0)$ va $B(1,0)$ nuqtalarda va yuqori yarim tekislikda joylashgan Γ: $y=f(x)$ chizig‘i bilan, $\operatorname{Im}z<0$ pastki yarim tekislikda esa (14) tenglamaning AC va BC xarakteristikalari bilan chegaralangan bir bog‘lamli D sohada o‘rganildi.

Asosiy e’tibor (14) tenglama uchun $D^-=D\cap\{y<0\}$ sohada shakli o‘zgargan Koshi masalasini o‘rganishga, $D^+=D\cap\{y>0\}$ sohada Dirixle va shakli o‘zgargan N masalasini, aralash Dsohada esa Trikomi masalasini hamda Frankl turidagi nolokal masalalarni o‘rganishga qaratilgan.

I BOB. GIPERGEOMETRIK FUNKSIYA. IXTIYORIY TARTIBLI INTEGRO-DIFFERENSIAL OPERATORLAR. SHAKLI O‘ZGARGAN KOSHI MASALASI.

Ushbu bobda kitobxonga o‘quv qo‘llanmani o‘qishda qulaylik yaratish maqsadida biz keyinchalik foydalaniladigan asosiy tushuncha va formulalarni [2,31,38] adabiyotlardan keltiramiz.

1-§. Gamma va beta funksiyalari.

1. Gamma funksiyasi. $\Gamma(z)$ gamma funksiyasi:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt, \quad (\operatorname{Re} z>0) \tag{1.1}$$

Eylerning ikkinchi tur integrali yordamida aniqlanadi. (1.1) integralni ikki integral yig‘indisi orqali ifodalaymiz [2,38]:

$$\Gamma(z)=\int_0^1 e^{-t}t^{z-1}dt+\int_1^\infty e^{-t}t^{z-1}dt=P(z)+Q(z). \tag{1.2}$$

$P(z)$ funksiyani $\operatorname{Re} z>0$ yarim tekislikda regulyar funksiya ekanligini ko‘rsatish qiyin emas, $Q(z)-$butun funksiya (butun kompleks tekislikda golomorf). Shunday qilib, (1.1) formula $\operatorname{Re} z>0$ yarim tekislikda regulyar funksiyani aniqlaydi. $\Gamma(z)$ funksiyani butun kompleks tekislikka analitik davom ettirish mumkin. Haqiqatdan ham, e^{-t} funksiyani darajali qatorga yoyib, bu yoyilmani t^{z-1} ga ko‘paytirib va $[0,1]$ kesmada hadma-had integrallab, ushbu funksional qatorga kelamiz:

$$P(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{z+n}, \quad (\operatorname{Re} z>0). \tag{1.3}$$

(1.3) qatorning hadlari $z\neq 0,-1,-2,\ldots$ nuqtalardan tashqari barcha nuqtalarda regulyar funksiyalardir va (1.3) qator $|z+k|\geq\delta>0$ $(k=0,1,2,\ldots;\delta>0$-ixtiyoriy kichik son) sohada tekis yaqinlashuvchi. Veyershtrass teoremasiga ko‘ra, (1.3) qator yig‘indisi meromorf funksiyadir, $z=-n$ nuqtalar bu funksiyaning oddiy qutblari bo‘lib, bu qutblardagi funksiyaning chegirmalari $res(P(-n))=\frac{(-1)^n}{n!}$ ga tengdir. (1.3) funksiya $\operatorname{Re} z>0$ yarim tekislikda $P(z)$ integral bilan ustma-ust tushadi.

Demak, (1.3) qator $P(z)$ integralning analitik davomidan iboratdir. Shunday qilib, (1.2) dagi ikkinchi yig‘indi $Q(z)-$butun funksiya bo‘lgani uchun, $\Gamma(z)$ funksiya $z=-n$ nuqtalarda oddiy qutblarga, hamda bu nuqtalarda mos ravishda $\frac{(-1)^n}{n!}$ ga teng bo‘lgan chegirmalarga ega bo‘lgan meromorf funksiyadir. $\Gamma(z)$ uchun ushbu funksional munosabatlar o‘rinlidir:

$$\Gamma(1+z)=z\Gamma(z),$$

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},$$

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

$$\Gamma(z)\Gamma(-z)=-\frac{\pi}{z\sin\pi z}, \tag{1.4}$$

$$\Gamma(1)=1, \Gamma(n+1)=n!, \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

$\Gamma(z)$ funksiya butun kompleks tekislikda nollarga ega emas, demak, $\frac{1}{\Gamma(z)}-$ butun funksiyadir.

2. Beta-funksiyasi. $B(p,q)$ beta-funksiya:

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt \text{ , } \operatorname{Re} p>0 \text{ , } \operatorname{Re} q>0, \tag{1.5}$$

Eylerning birinchi tur integrali yordamida aniqlanadi. $B(p,q)$ funksiya $\Gamma(z)$ orqali

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{1.6}$$

formula yordamida ifodalanadi.

2-§. Gaussning gipergeometrik funksiyasi

1. Gauss tenglamasi. Buziladigan giperbolik va elliptik tipdagi tenglamalar nazariyasida ushbu

$$z(1-z)\omega''(z)+\left[c-(a+b+1)z\right]\omega'(z)-ab\omega(z)=0\,, \qquad (1.7)$$

Gauss tenglamasining yechimlari fundamental ahamiyatga ega, bu yerda a,b,c –parametrlar bo‘lib, ular ixtiyoriy kompleks yoki haqiqiy sonlar bo‘lishi mumkin. (1.7) tenglama uchta: 0, 1 , ∞ regulyar maxsus nuqtalarga ega.

O‘zgaruvchilarni maxsus almashtirish yordamida buziluvchan giperbolik va elliptik tipdagi tenglamalar (1.7) tenglamaga olib kelinishi mumkin va bu tenglamaning yechimlaridan mos ravishda Riman funksiyasini, Grin funksiyasini tuzishda fundamental ahamiyatga ega.

Dastlab, (1.7) tenglamaning yechimini $z=0$ nuqta atrofida topamiz. Yechimni

$$\omega_1(z)=\sum_{n=0}^{\infty}c_n z^n\;, \qquad (1.8)$$

darajali qator ko‘rinishida izlaymiz. Bu yerda c_n -hozircha noma’lum sonlar. (1.8) dan ushbu hosilalarni hisoblaymiz:

$$\omega_1'(z)=\sum_{n=0}^{\infty}c_n nz^{n-1}=\sum_{n=0}^{\infty}c_n nz^{n-1},$$

$$\omega_1''(z)=\sum_{n=0}^{\infty}c_n n(\mathrm{n}-1)z^{n-2}=\sum_{n=2}^{\infty}c_n n(n-1)z^{n-2}.$$

Endi bu hosilalarni (1.7) tenglamaga qo‘yib, quyidagi munosabatni hosil qilamiz:

$$\sum_{n=0}^{\infty}\left[c_{n+1}(n+1)(n+c)-c_n(n+a)(n+b)\right]z^n=0,$$

bu yerdan z^n oldidagi umumiy koeffitsiyentni nolga tenglashtirib, ushbu

$$c_{n+1}=\frac{(a+n)(b+n)}{(n+1)(c+n)}c_n,\quad (n=0,1,2...;c\neq -n)$$

rekurrent formulaga kelamiz.

(1.7) tenglamaning bir jinsli ekanligidan foydalanib, umumiyatlikni buzmasdan $c_0 = 1$ deb qabul qilamiz va

$$\omega_1(z) = F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n$$
$$1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a \cdot (a+1) b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \cdots, \qquad (1.9)$$

Gaussning gipergeometrik qatoriga kelamiz, bu yerda

$$(a)_0 = 1, (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

belgilashlar kiritilgan. Dalamber alomatiga ko‘ra, (1.9) darajali qatorning yaqinlashish radiusi $R = 1$ ekanligini ko‘rsatish qiyin emas. Demak, (1.9) darajali qator $|z| \le q < 1$ doirada absolyut va tekis yaqinlashadi. Raabe alomati yordamida (1.9) gipergeometrik qator uchun ushbu tasdiqlarni isbotlash qiyin emas:

agar $\mathrm{Re}(c-a-b) > 0$ bo‘lsa, (1.9) qator $|z| = 1$ aylanada tekisva absolyut yaqinlashadi;

agar $-1 \le \mathrm{Re}(c-a-b) \le 0$ bo‘lsa, (1.9) qator $|z| = 1$ aylananing $|1-z| < \delta$ ($\delta > 0$ yetarli kichik son) doiradan tashqarida yotgan bo‘lagida tekis va absolyut yaqinlashadi;

Agar $\mathrm{Re}(c-a-b) < -1$ bo‘lsa, (1.9) qator $|z| = 1$ aylanada uzoqlashuvchi bo‘ladi.

Gipergeometrik funksiyalarning sodda xossalarini keltiramiz, bu xossalar (1.9) darajali qatorning ko‘rinishidan bevosita kelib chiqadi.

1^0. Agar $a = -n$ yoki $b = -n$ bo‘lsa, bu yerda $n = 0,1,2,....$, (1.9) darajali qator uziladi, ya’ni $F(-n,b,c;z)$ yoki $F(a,-n,c;z)$ n-darajali ko‘phadga aylanadi;

2^0. $F(a,b,c;z)$ gipergeometrik funksiya a va b parametrlarga nisbatan simmetrikdir, ya’ni

$$F(a,b,c;z) = F(b,a,c;z)$$

3^0. $b = c$ bo‘lganda

$$F(a,b,b;z) = (1-z)^{-a} \qquad (1.10)$$

tenglikka ega bo‘lamiz.

(1.7) tenglamaning ikkinchi yechimini topish uchun $\omega(z)$ o‘rniga

$$\omega(z) = z^q u(z) \tag{1.11}$$

formula yordamida yangi funksiya kiritamiz, bu yerda q-hozircha ixtiyoriy noma'lum son. (1.11) tenglikni (1.7) tenglamaga qo'yib, ushbu tenglamaga ega bo'lamiz:

$$z(1-z)u'' + [2q + c - (2q + a + b + 1)z]u' - \left[-\frac{q(q-1+c)}{z} + q(q+a+b) + ab\right]u = 0.$$

Bu tenglamada $q = 1 - c$ deb olsak, u holda oxirgi tenglama

$$z(1-z)u''(z) + [c_1 - (a_1 + b_1 + 1)z]u'(z) - a_1 b_1 u = 0$$

tenglamaga aylanadi, bu yerdagi parametrlar

$$a_1 = a + 1 - c,$$
$$b_1 = b + 1 - c,$$
$$c_1 = 2 - c$$

tengliklar bilan aniqlanadi .

Shunday qilib, (1.9) va (1.11) ga asosan (1.7) tenglamaning ikkinchi yechimi

$$\omega_2(z) = z^{1-c} F(a+1-c, b+1-c, 2-c; z) \tag{1.12}$$

ko'rinishda bo'ladi, bu yerda $2 - c \neq 0, -1, -2, ...$

(1.7) tenglamaning (1.9) yechimida $c \neq -n$ shart bajarilishi kerak edi. Endi biz (1.12) ga asosan (1.7) tenglamaning yechimini $c = -n$ holida ham hosil qilishimiz mumkin:

$$\omega(z) = z^{n+1} \sum_{m=0}^{\infty} \frac{(a+n+1)_n (b+n+1)_n}{m!(2+n)_m} z^n = z^{n+1} F(a+n+1, b+n+1, n+2; z). \tag{1.13}$$

(1.7) tenglamaning topilgan $\omega_1(z)$ va $\omega_2(z)$ yechimlari chiziqli erkli, demak uning umumiy yechimi

$$\omega(z) = c_1 F(a,b,c;z) + c_2 z^{1-c} F(a+1-c, b+1-c, 2-c; z) \tag{1.14}$$

formula bilan beriladi, bu yerda c_1 va c_2 ixtiyoriy o'zgarmas sonlardir.

(1.7) tenglamaning yechimini $z=1$ maxsus nuqta atorofida hosil qilish uchun z ni $1-z$ ga almashtirish yetarlidir. Bu holda (1.7) tenglama parametrlari $a_1=a$, $b_1=b$, $c_1=a+b-c-1$ lardan iborat bo‘lgan gipergeometrik tenglamaga aylanadi. Bu holda (1.7) tenglamaning $z=1$ maxsus nuqta atrofida

$$\omega_3(z)=F(a,b,1+a+b-c;1-z),$$

$$\omega_4(z)=(1-z)^{c-a-b}F(c-b,c-a,1+c-a-b;1-z), \qquad (1.15)$$

chiziqli erkli yechimlarini hosil qilish qiyin emas, bu yerda $c-a-b$ butun sonlar bo‘lmasligi kerak.

Nihoyat, (1.7) tenglamaning yechimlarini cheksiz uzoqlashgan maxsus nuqta $z=\infty$ atrofida topish uchun erkli o‘zgaruvchi z va $\omega(z)$ funksiyani ushbu formulalar yordamida almashtiramiz:

$$\varsigma=\frac{1}{z},\ \omega(z)=\varsigma^a u(\varsigma),$$

bu holda (1.7) tenglama $u(\varsigma)$ funksiyaga nisbatan parametrlari $a_1=a$, $b_1=b+a-c$, $c_1=1+a-b$ bo‘lgan gipergeometrik funksiyaga aylanadi. Shunday qilib, (1.7) tenglamaning $z=\infty$ maxsus nuqta atrofidagi chiziqli erkli yechimlari ushbu ko‘rinishda bo‘ladi:

$$\begin{aligned}\omega_5(z)&=z^{-a}F\left(a,1+a-c,1+a-b;\frac{1}{z}\right),\\ \omega_6(z)&=z^{-b}F\left(b,1+b-c,1+b-a;\frac{1}{z}\right),\end{aligned} \qquad (1.16)$$

bu yerda $a-b$ butun sonlar bo‘lmasligi kerak.

Shunday qilib, biz (1.7) Gauss tenglamasining oltita asosiy yechimlarini gipergeometrik funksiyalar orqali ifodaladik.

2. Gipergeometrik funksiyani analitik davom ettirish. (1.7) tenglamaning $|z|<1$ doirada aniqlangan regulyar yechimi $\omega_1(z)=F(a,b,c;z)$ ni z o‘zgaruvchining butun kompleks tekisligiga analitik davom ettirish mumkin. $F(a,b,c;z)$ funksiyaning analitik davomini ham $F(a,b,c;z)$ simvol bilan belgilaymiz va u (1.9) qatorning $|z|<1$ doiradan tashqariga davom ettirilgan analitik funksiyasining bosh shoxchasini ifodalaydi.

$F(a,b,c;z)$ gipergeometrik funksiyani analitik davom ettirishni xususan Eylerning ushbu

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,dt = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F(a,b,c;z)\ , \qquad (1.17)$$

$$0<\operatorname{Re}a<\operatorname{Re}c\ ,\ |\arg(1-z)|<\pi\ ,$$

gipergeometrik integrali yordamida amalga oshirish mumkin.

(1.17) tenglikningchap tomonidagi integral $(1,\infty)$ kesimli butun kompleks tekislikda regulyar funksiyani beradi. (1.17) tenglikni isbotlash uchun analitik davom ettirish prinsipiga ko‘ra, uni $|z|<1$ doira ichida tekshirish yetarlidir. $(1-zt)^{-b}$ funksiyani zt $(|zt|<1)$ ning darajalari bo‘yicha binomial qatorga yoyamiz va bu yoyilmani $t^{a-1}(1-t)^{c-a-1}$ ga ko‘paytirib, t bo‘yicha $(0,1)$ oraliqda hadma-had integrallaymiz va (1.5) formulani qo‘llab, ushbu:

$$\int_0^1 t^{a-1}\left(1-t\right)^{c-a-1}\left(1-zt\right)^{-b}dt = \sum_{n=0}^{\infty}\frac{(b)_n z^n}{n!}\int_0^1 t^{n+a-1}\left(1-t\right)^{c-a-1}dt =$$

$$=\sum_{n=0}^{\infty}\frac{(b)_n z^n}{n!}B\left(n+a,c-a\right)=\sum_{n=0}^{\infty}\frac{(b)_n z^n}{n!}\frac{\Gamma\left(n+a\right)\Gamma\left(c-a\right)}{\Gamma\left(n+c\right)} \qquad (1.17)$$

$$=\frac{\Gamma\left(a\right)\Gamma\left(c-a\right)}{\Gamma\left(c\right)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n=\frac{\Gamma\left(a\right)\Gamma\left(c-a\right)}{\Gamma\left(c\right)}F(a,b,c;z)$$

formula $\operatorname{Re}(c-a-b)>0$, $c\neq 0,-1,-2,....$ bo‘lganda $F(a,b,c;1)$ ni hisoblashga imkon beradi:

$$F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \qquad (1.18)$$

(1.17) integralda

$$t=\frac{1-s}{1-zs}\ ,1-t=\frac{s(1-z)}{1-zs},\ 1-tz=\frac{1-z}{1-zs}\ ,\ \ dt=-\frac{(1-z)ds}{(1-zs)^2}\ ,$$

almashtirishni bajarib, ushbu

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F(a,b,c;z)=\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt=$$
$$=(1-z)^{c-a-b}\int_0^1 s^{c-a-1}(1-s)^{c-(c-a)-1}(1-zs)^{-(c-b)}ds=$$
$$=(1-z)^{c-a-b}\frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)}F(c-a,c-b,c;z),$$

ya'ni

$$F(a,b,c;z)=(1-z)^{c-a-b}F(c-a,c-b,c;z), \qquad (1.19)$$
$$|\arg(1-z)|<\pi$$

formulani hosil qilamiz.

(1.19) formula avtotransformatsiya formulasi deyiladi.

Argumentlari z va $1-z$ bo'lgan gipergeometrik funksiyalar o'rtasida funksional munosabatlarni keltirib chiqaramiz. $|z|<1$ va $|1-z|<1$ doiralarning kesishmasida (1.7) tenglama yechimi $\omega_1(z)=F(a,b,c;z)$ shu tenglamaning chiziqli erkli yechimlari $\omega_3(z)$ va $\omega_4(z)$ larning chiziqli kombinatsiyasi orqali ifodalanadi:

$$F(a,b,c;z)=AF(a,b,a+b-c+1;1-z)+$$
$$+B(1-z)^{c-a-b}F(c-a,c-b,c-a-b+1;1-z), \qquad (1.20)$$

$$c-a-b\neq 0,\pm1,\pm2,\pm3,...;\ |\arg(1-z)|<\pi .$$

$F(a,b,c;z)$ funksiya a,b,c parametrlarning analitik funksiyasidan iborat, demak (1.20) formuladagi A va B koeffitsientlar ham a,b,c parametlarning analitik funksiyasi bo'ladi. A va B koeffitsientlarni aniqlashda biz a,b,c parametlarga shunday «qulay» shartlarni qo'yamiz, ular analitik davom ettirish prinsipi yordamida shunday minimal shartlarga keltiriladiki, bu shartlarda oxirgi natija ma'noga ega bo'lmagan qiymatlarni qabul qilmaydi.

(1.20) tenglamaning o'ng tomoni $z=1$ nuqtada a,b,c $(c\neq -n)$ parametrlarining ixtiyoriy qiymatida ma'noga ega, chap tomonining chekliligi $\mathrm{Re}(c-a-b)$ ifodaning ishorasiga bog'liq.

1. $\mathrm{Re}(c-a-b)>0$ bo'lsin u holda (1.20) da $z=1$ deb hisoblab, ushbu

$$A = F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

tenglikka kelamiz.

2. $\mathrm{Re}(c-a-b) < 0$ bo‘lsin. (1.20) tenglikning chap tomoniga avtotransformatsiya formulasi (1.19) ni qo‘llab, hosil bo‘lgan tenglikni $(1-z)^{a+b-c}$ ifodaga ko‘paytirib va oxirgi natijada $z=1$ deb hisoblab, ushbu

$$B = F(c-a,c-b,c;1) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$

tenglikka kelamiz.

Aniqlangan A va B koeffitsientlarni (1.20) tenglikka qo‘yib, ushbu

$$F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,a+b-c+1;1-z) +$$
$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} \times F(c-a,c-b,c-a-b+1;1-z), \qquad (1.21)$$

$$c-a-b \neq 0,\pm1,\pm2,.... \quad |\arg(1-z| < \pi ,$$

Bols formulasini hosil qilamiz [1].

(1.21) formula $F(a,b,c;z)$ gipergeometrik funksiyani $|z|<1$ sohadan $|1-z|<1$, $|\arg(1-z)|<\pi$ sohaga analitik davomini beradi.

(1.17) formuladagi integralda integral o‘zgaruvchisini $t = 1-s$ formula bilan almashtirib, ushbu

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,dt = (1-z)^{-b}\int_0^1 s^{c-a-1}(1-s)^{a-1}(1-\frac{z}{z-1}s)^{-b}\,ds,$$

yoki (1.17) ni hisobga olib,

$$F(a,b,c;z) = (1-z)^{-b} F(c-a,b,c;\frac{z}{z-1}), \qquad (1.22)$$

formulaga kelamiz.

$\operatorname{Re} z < \frac{1}{2}$ bo‘lganda, $\left|\frac{z}{z-1}\right| < 1$ tengsizlik o‘rinli. Shunday qilib, (1.22) formula $F(a,b,c;z)$ funksiyani $|z|<1$ doiradan $\operatorname{Re} z < \frac{1}{2}$ yarim tekislikka analitik davomini beradi. (1.22) formulada z o‘rniga $1-z$ almashtirish bajarib, (1.22) formulani ushbu:

$$F(a,b,c;1-z) = z^{-b} F(c-a,b,c;\frac{z-1}{z}) \qquad (1.23)$$

ko‘rinishda yozib olamiz.

Gipergeometrik funksiyalar o‘rtasida boshqa funksional munosabatlarni keltirib chiqarilgan formulalarning kombinatsiyalari yordamida hosil qilish mumkin. Masalan, (1.22) va (1.21) formulalarni ketma-ket qo‘llab, ushbu

$$\begin{aligned} F(a,b,c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(1-z)^{-a} F(a,c-b,a-b+1;\frac{1}{1-z}) + \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(1-z)^{-b} F(c-a,b,b-a+1;\frac{1}{1-z}), \end{aligned} \qquad (1.24)$$

$$a-b \neq 0, \pm 1, \pm 2, \cdots; \; |\arg(1-z)| < \pi,$$

funksional munosabatni hosil qilamiz, bu formula $F(a,b,c;z)$ gipergeometrik funksiyani $|z|<1$ doiradan $|z-1|>1, |\arg(1-z)|<\pi$ sohaga analitik davom ettirish imkonini beradi. Yana (1.24) va (1.22) formulalarni ketma-ket qo‘llab, ushbu

$$\begin{aligned} F(a,b,c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a} F(a,a-c+1,a-b+1;\frac{1}{z}) + \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b} F(b,b-c+1,b-a+1;\frac{1}{z}) \end{aligned}, \qquad (1.25)$$

$$a-b \neq 0, \pm 1, \pm 2, \dots; \; |\arg(-z)| < \pi, \; |\arg(1-z)| < \pi .$$

formulaga ega bo‘lamiz, bu formula $F(a,b,c;z)$ gipergeometrik funksiyani $|z|<1$ doiradan $|z|>1, |\arg(z)|<\pi$ sohaga analitik davomini beradi.

Nihoyat, (1.21) va (1.23) formulalarni ketma-ket qo‘llab, ushbu

$$F(a,b,c;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}z^{-a}\times F(a,a-c+1,a+b-c+1;\frac{z-1}{z})+$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{a-c}(1-z)^{c-a-b}\times F(c-a,1-a,c-a-b+1;\frac{z-1}{z}),$$
(1.26)
$$c-a-b\neq 0,\pm1,\pm2,\ldots,\ |\arg z|<\pi,\quad |\arg(1-z)|<\pi$$

formulaga ega bo‘lamiz, bu formula $F(a,b,c;z)$ gipergeometrik funksiyani $|z|<1$ doiradan $\operatorname{Re} z>\frac{1}{2}$ sohaga analitik davom ettirish imkonini beradi.

3. Elementar munosabatlar. $F(a,b,c;z)$ gipergeometrik funksiyaning uning bevosita (1.9) qator bilan aniqlangan ifodasidan kelib chiqadigan xossalarini o‘rganamiz. Masalan, ushbu

$$cF(a,b-1,c;z)+(a-b)zF(a,b,c+1;z)=cF(a-1,b,c;z) \qquad (1.27)$$

formulani isbotlaymiz.

(1.27) ning chap tomonini (1.9) ga asosan darajali qatorlarga yoyamiz, bu yoyilmada z^n oldidagi koeffitsient ushbu:

$$c\frac{(a)_n(b-1)_n}{(c)_n n!}+(a-b)\frac{(a)_n(b)_n}{(c+1)_n(n-1)!}=\frac{(a)_{n-1}(b)_{n-1}}{(c+1)_{n-1}(n-1)!}\left[a-b+\frac{(b-1)(a+n-1)}{n}\right]=$$
$$=\frac{c(a)_{n-1}(b)_{n-1}}{(c)_n n!}(a-1)(b+n-1)=c\frac{(a-1)_{n-1}(b)_{n-1}}{(c)_n n!}$$

ko‘rinishga ega. Bu esa (1.27) ning chap tomonini qator yoyilmasidagi z^n oldidagi koeffitsientni beradi. Shunday qilib, (1.27) tenglik isbot bo‘ldi.

Ushbu oltita:

$$F(a\pm1,b,c;z),\quad F(a,b\pm1,c;z),\quad F(a,b,c\pm1;z)$$

funksiyalar $F(a,b,c;z)$ funksiya bilan yondosh funksiyalar deyiladi. $F(a,b,c;z)$ va unga ixtiyoriy ikkita yondosh funksiyalar o‘rtasida koeffitsientlari z ga nisbatan chiziqli funksiya bo‘lgan bog‘liqlik mavjud. Bu bog‘liqliklarning 15 turi mavjud bo‘lib, uni birinchi marta Gauss topgan. Quyida bu munosabatlarning to‘liq jadvalini keltiramiz, bu munosabatlarda yozuvni soddalashtirish maqsadida $F(a,b,c;z)$ ni F bilan, $F(a\pm1,b,c;z), F(a,b\pm1,c;z), F(a,b,c\pm1;z)$ larni esa mos ravishda $F(a\pm1), F(b\pm1), F(c\pm1)$ deb belgilaymiz.

$$[c-2a-(b-a)z]F+a(1-z)F(a+1)-(c-a)F(a-1)=0, \qquad (1.28)$$

$$(b-a)F+aF(a+1)-bF(b+1)=0, \tag{1.29}$$

$$(c-a-b)F+a(1-z)F(a+1)-(c-b)F(b-1)=0, \tag{1.30}$$

$$c[a-(c-b)z]F-ac(1-z)F(a+1)+(c-a)(c-b)zF(c+1)=0, \tag{1.31}$$

$$(c-a-1)F+aF(a+1)-(c-1)F(c-1)=0, \tag{1.32}$$

$$(c-a-b)F-(c-a)F(a-1)+b(1-z)F(b+1)=0, \tag{1.33}$$

$$(b-a)(1-z)F-(c-a)F(a-1)+(c-b)F(b-1)=0, \tag{1.34}$$

$$c(1-z)F-cF(a-1)+(c-b)zF(c+1)=0, \tag{1.35}$$

$$[a-1-(c-b-1)z]F+(c-a)F(a-1)-(c-1)(1-z)F(c-1)=0, \tag{1.36}$$

$$[c-2b+(b-a)z]F+b(1-z)F(b+1)-(c-b)F(b-1)=0, \tag{1.37}$$

$$c[b-(c-a)z]F-bc(1-z)F(b+1)+(c-a)(c-b)zF(c+1)=0, \tag{1.38}$$

$$(c-b-1)F+bF(b+1)-(c-1)F(c-1)=0, \tag{1.39}$$

$$c(1-z)F-cF(b-1)+(c-a)zF(c+1)=0, \tag{1.40}$$

$$[b-1-(c-a-1)z]F+(c-b)F(b-1)-(c-1)(1-z)F(c-1)=0, \tag{1.41}$$

$$c[c-1-(2c-a-b-1)z]F+(c-a)(c-b)zF(c+1)-c(c-1)(1-z)F(c-1)=0. \tag{1.42}$$

Agar yondosh gipergeometrik funksiyalarda ikkita parametr bir xil bo‘lsa, ular o‘rtasida ushbu munosabatlar o‘rinli:

$$(c-a)F(a-1,b,c;z)+(2a-c-az+bz)F(a,b,c;z)+a(z-1)F(a+1,b,c;z)=0 \tag{1.43}$$

$$(c-b)F(a,b-1,c;z)+(2b-c-bz+az)F(a,b,c;z)+b(z-1)F(a,b+1,c;z)=0. \tag{1.44}$$

$$c(c-1)(z-1)F(a,b,c-1;z)+c\left[c-1-(2c-a-b-1)z\right]F(a,b,c;z)+$$
$$+(c-a)(c-b)zF(a,b,c+1;z)=0. \tag{1.45}$$

4. Gipergeometrik funksiyalarni differensiallash qoidalari.

$z^{\lambda}(1-z)^{\mu}F(a,b,c;z)$ ifodani λ va μ parametrlarning ba'zi bir qiymatlarida differensiallash qoidalarini keltiramiz:

$$\frac{d^n}{dz^n}F(a,b,c;z)=\frac{(a)_n(b)_n}{(c)_n}F(a+n,b+n,c+n;z), \tag{1.46}$$

$$\frac{d^n}{dz^n}\left[z^{a+n-1}F(a,b,c;z)\right]=(a)_n z^{a-1}F(a+n,b,c;z), \tag{1.47}$$

$$\frac{d^n}{dz^n}\left[z^{c-1}F(a,b,c;z)\right]=(c-n)_n z^{c-1-n}F(a,b,c-n;z), \tag{1.48}$$

$$\frac{d^n}{dz^n}\left[z^{c-a+n-1}(1-z)^{a+b-c}F(a,b,c;z)\right]=(c-a)_n z^{c-a-1}(1-z)^{a+b-c-n}F(a-n,b,c;z), \tag{1.49}$$

$$\frac{d^n}{dz^n}\left[(1-z)^{a+b-c}F(a,b,c;z)\right]=\frac{(c-a)_n(c-b)_n}{(c)_n}(1-z)^{a+b-c-n}F(a,b,c+n;z), \tag{1.50}$$

$$\frac{d^n}{dz^n}\left[(1-z)^{a+n-1}F(a,b,c;z)\right]=\frac{(-1)^n(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}F(a+n,b,c+n;z), \tag{1.51}$$

$$\frac{d^n}{dz^n}\left[z^{c-1}(1-z)^{b-c+n}F(a,b;c;z)\right]=(c-n)_n z^{c-1-n}(1-z)^{b-c}F(a-n,b,c-n;z), \tag{1.52}$$

$$\frac{d^n}{dz^n}\left[z^{c-1}(1-z)^{a+b-c}F(a,b,c;z)\right]=(c-n)_n z^{c-1-n}(1-z)^{a+b-c-n}F(a-n,b-n,c-n;z). \tag{1.53}$$

(1.46) formulani matematik induksiya metodi yordamida isbotlaymiz. (1.46) formuladan $n=1$ da ushbu tenglikka ega bo‘lamiz:

$$\frac{d}{dz}F(a,b,c;z)=\frac{ab}{c}F(a+1,b+1,c+1;z). \qquad (1.54)$$

Haqiqatdan ham, gipergeometrik qator (1.9) ga ko‘ra

$$\frac{d}{dz}F(a,b,c;z)=\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}nz^{n-1}=$$

$$=\frac{ab}{c}\sum_{k=0}^{\infty}\frac{(a+1)_k(b+1)_k}{k!(c+1)_k}z^k=\frac{ab}{c}F(a+1,b+1,c+1;z)$$

tenglik o‘rinli bo‘ladi.

Shunday qilib, $n=1$ da (1.46) formula to‘g‘ri. Faraz qilaylik, bu formula $n=k$ da to‘g‘ri bo‘lsin, u holda

$$\frac{d^{k+1}}{dz^{k+1}}F(a,b,c;z)=\frac{d}{dz}\left[\frac{d^k}{dz^k}F(a,b,c;z)\right]=\frac{d}{dz}\frac{(a)_k(b)_k}{(c)_k}F(a+k,b+k,c+k;z)=$$

$$=\frac{(a)_{k+1}(b)_{k+1}}{(c)_{k+1}}F(a+k+1,b+k+1,c+k+1;z).$$

Bu yerda (1.54) tenglikni hisobga olib,

$$\frac{d^{k+1}}{dz^{k+1}}F(a,b,c;z)=\frac{(a)_{k+1}(b)_{k+1}}{(c)_{k+1}}F(a+k+1,b+k+1,c+k+1;z)$$

tenglikka kelamiz. Shu bilan (1.46) formula isbot bo‘ldi.

(1.47)-(1.53) formulalar ham yuqoridagi usulga o‘xshash isbotlanadi.

5. Ikki o‘zgaruvchili gipergeometrik funksiya. Ushbu ikkita ikkinchi tartibli xususiy hosilali differensial tenglamalarni o‘rganamiz:

$$x(1-x)\frac{\partial^2 z}{\partial x^2}+y(1-x)\frac{\partial^2 z}{\partial x\partial y}+\left[c-(a+b+1)x\right]\frac{\partial z}{\partial x}-by\frac{\partial z}{\partial y}-abz=0, \qquad (1.55)$$

$$y(1-y)\frac{\partial^2 z}{\partial y^2}+x(1-y)\frac{\partial^2 z}{\partial x\partial y}+\left[c-(a+b'+1)y\right]\frac{\partial z}{\partial y}-b'x\frac{\partial z}{\partial x}-ab'z=0 \qquad (1.56)$$

bu yerda $z=z(x,y)$ -noma’lum funksiya.

Ta'rif. *Ikki* x *va* y *o'zgaruvchili* $F_1(a,b,b,c;x,y)$ *gipergeometrik funksiya deb:*

$$F_1(a,b,b',c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_m}{(c)_{m+n}\,m!n!}x^m y^n \qquad (1.57)$$

qator bilan aniqlanuvchi funksiyaga aytiladi.

Bu funksiya birgalikda bo'lgan (1.55) va (1.56) tenglamalarning yechimi bo'ladi. (1.57) qator $|x|<1$, $|y|<1$ sohada absolyut va tekis yaqinlashuvchi bo'ladi.

Pikar ikki o'zgaruvchili gipergeometrik funksiya F_1 ushbu

$$F_1(a,b,b',c;x,y)=$$
$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\cdot\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}(1-yt)^{-b'}dt, \qquad (1.58)$$

aniq integral yordamida aniqlanishini isbotlagan, bu yerda $\mathrm{Re}\,a>0,\ \mathrm{Re}\,c>\mathrm{Re}\,a$.

$F_1(a,b,b',c;x,y)$-gipergeometrik funksiyaning asosiy xossalari:

1^0. Ushbu

$$F_1(a,b,b',c;x,y)=F_1(a,b',b,c;y,x) \quad (1.59)$$

tenglik o'rinli.

2^0. Rekurrent munosabatlar:

$$(c-b-b'-1)F_1(a,b,b',c;x,y)+bF_1(a,b+1,b',c;x,y)+b'F_1(a,b,b'+1,c;x,y)=$$
$$=(c-1)F_1(a,b,b',c-1;x,y), \qquad (1.60)$$

$$(c-a-1)F_1(a,b,b',c;x,y)+aF_1(a+1,b,b',c;x,y)=$$
$$=(c-1)F_1(a,b,b',c-1;x,y). \qquad (1.61)$$

3^0. Differensiallash formulalari:

$$\frac{\partial}{\partial x}F_1(a,b,b',c;x,y)=\frac{ab}{c}F_1(a+1,b+1,b',c+1;x,y), \quad (1.62)$$

$$\frac{\partial}{\partial y}F_1(a,b,b',c;x,y)=\frac{a\cdot b'}{c}F_1(a+1,b,b'+1,c+1;x,y), \qquad (1.63)$$

$$\frac{x}{b}\frac{\partial}{\partial x}F_1(a,b,b',c;x,y)=F_1(a,b+1,b',c;x,y)-F_1(a,b,b',c;x,y),$$

$$\frac{y}{b'}\frac{\partial}{\partial y}F_1(a,b,b',c;x,y)=F_1(a,b,b'+1,c;x,y)-F_1(a,b,b',c;x,y),$$

(1.64)

4⁰. Parametrlarning xususiy qiymatlarida bir o‘zgaruvchili gipergeometrik funksiya orqali ifodasi:

$$F_1(a,b,b',b+b';x,y)=(1-y)^{-a}F\left(a,b,b+b';\frac{x-y}{1-y}\right). \quad (1.65)$$

5⁰. O‘zgaruvchilarning xususiy qiymatlarida bir o‘zgaruvchili gipergeometrik funksiya orqali ifodasi:

$$F_1(a,b,b',c;x,1)=\frac{\Gamma(c)\Gamma(c-a-b')}{\Gamma(c-a)\Gamma(c-b')}F(a,b,c-b';x) \quad (1.66)$$

$$\mathrm{Re}(c-a-b')>0,$$

$$F_1(a,b,b',c;1,y)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b',c-b;y), \quad (1.67)$$

$$\mathrm{Re}(c-a-b)>0,$$

$$F(a,b,b',c;x,x)=F(a,b+b',c;x), \quad (1.68)$$

6⁰. Quyidagi tengliklar o‘rinli:

$$F_1(a,b,b',c;x,y)==(1-x)^{-b}(1-y)^{-b'}F_1\left(c-a,b,b',c;\frac{x}{x-1},\frac{y}{y-1}\right) \quad (1.69)$$

$$F_1(a,b,b',c;x,y)=(1-x)^{-a}F_1a,\left(c-b-b',b',c;\frac{x}{x-1},\frac{y-x}{1-x}\right) \quad (1.70)$$

$$F_1(a,b,b',c;x,y)=(1-y)^{-a}F_1\left(a,b,c-b-b',c;\frac{y-x}{y-1},\frac{x}{y-1}\right) \quad (1.71)$$

$$F_1(a,b,b',c;x,y)=(1-x)^{c-a-b}(1-y)^{-b'}F_1\left(c-a,c-b-b',b',c;x,\frac{x-y}{1-y}\right)\quad(1.72)$$

$$F_1(a,b,b',c;x,y)=(1-x)^{-b}(1-y)^{c-a-b'}F_1\left(c-a,b,c-b-b',c;\frac{x-y}{x-1},y\right)\quad(1.73)$$

3-§. Ixtiyoriy tartibli integro-differensial operatorlar

$f(x)$ -funksiya $L(a,b),\ a<b<\infty$ sinfga tegishli bo‘lganixtiyoriy funksiya bo‘lsin. Ushbu

$$D_{a,x}^{\ell}f(x)=\begin{cases}\dfrac{1}{\Gamma(-\ell)}\displaystyle\int_a^x\dfrac{f(t)dt}{(x-t)^{1+\ell}}, & agar\quad \ell<0,\\ \dfrac{d^{n+1}}{dx^{n+1}}D_{a,x}^{\ell-(n+1)}f(x), & agar\quad \ell>0,\end{cases}\quad(1.74)$$

$$D_{x,b}^{\ell}f(x)=\begin{cases}\dfrac{1}{\Gamma(-\ell)}\displaystyle\int_x^b\dfrac{f(t)}{(t-x)^{1+\ell}}dt, & agar\quad \ell<0,\\ (-1)^{n+1}\dfrac{d^{n+1}}{dx^{n+1}}D_{x,b}^{\ell-(n+1)}f(x), & agar\quad \ell>0,\end{cases}$$

belgilashlarni kiritamiz, bu yerda $D_{a,x}^{\ell}$ va $D_{x,b}^{\ell}$ ifodalar agar $\ell<0$ bo‘lsa, ℓ kasr tartibli integral operatorni, agar $\ell>0$ bo‘lsa, Liuvill ma’nosidagi umumlashgan hosilani beradi. $n=[\ell]-\ell$ sonining butun qismi.

Ta’rifga asosan:

$$D_{a,x}^{0}f(x)=D_{x,b}^{0}f(x)=f(x)\quad(1.75)$$

deb hisoblaymiz.

1. Kasr tartibli integral operatorlarning xossalari. Kasr tartibli integral operatorlar uchun ushbu teoremalar o‘rinlidir, qulaylik uchun $\ell=-\alpha<0$, $\alpha>0$ deb hisoblaymiz.

1.1 Teorema. *Agar* $f(x)\in L_p(a,b)$, $p>1$, $a<b<\infty$, $0<\alpha<\dfrac{1}{p}$, $q=p/(1-p\alpha)$, *bo‘lsa, u holda* $D_{a,x}^{-\alpha}f(x)\in L_q(a,b)$, *shu bilan birga*

$$\left(\int_a^b \left|D_{a,x}^{-\alpha} f(x)\right|^q dx\right)^{\frac{1}{q}} \le k\left(\int_a^b \left|f(x)\right|^p dx\right)^{\frac{1}{p}},$$

tengsizlik o‘rinli, bu yerda $k = k(p,\alpha)$ *faqat* p *va* α *ga bog‘liq.*

Natija. *Agar* $p > 1$ *va* $\alpha = \dfrac{1}{p}$ *bo‘lsa, u holda* $D_{a,x}^{-\alpha} f(x)$ operator chegaralangan.

Teorema. $f(x)$ *funksiya* (a,b) *oraliqda* $\alpha \in (0,1]$ *ko‘rsatkich va* M *Gyolder o‘zgarmasi bilan* (a,b) *intervalda Gyolder shartini qanoatlantiradi deyiladi, agar ixtiyoriy* $x_1, x_2 \in (a,b)$ *uchun*

$$\left|f(x_1) - f(x_2)\right| \le M\left|x_2 - x_1\right|^\alpha, \quad \left|x_1\right| < 1, \quad \left|x_2\right| < 1,$$

$$\left|f(x_1) - f(x_2)\right| < M\left|\frac{1}{x_1} - \frac{1}{x_2}\right|^\alpha, \quad \left|x_1\right| > 1, \quad \left|x_2\right| > 1, \qquad (1.76)$$

tengsizliklar bajarilsa. Qisqacha qilib bunday funksiyalar H *yoki* $H(\alpha)$ *shartini qanoatlantiradi deyiladi.*

Agar $\alpha > 1$ bo‘lsa, (1.76) dan ko‘rinib turibdiki, $f'(x) \equiv 0$ ya’ni $f(x) = const$.

$H(\alpha)$ shartni qanoatlantiruvchi funksiyalarning xossalarini keltiramiz:

1^0. Agar $f(x)$ funksiya (a,b) intervalda $\left|f'(x)\right| < M$ chekli hosilaga ega bo‘lsa, u holda $f(x)$ funksiya Gyolder shartini $\alpha = 1$ ko‘rsatkich bilan qanoatlantiradi (Lipshits sharti).

2^0. Agar $f(x)$ funksiya (a,b) chekli intervalda α ko‘rsatkich bilan Gyolder shartini qanoatlantirsa, u holda bu funksiya $\beta < \alpha$ ko‘rsatkich bilan ham Gyolder shartini qanoatlantiradi.

Shunday qilib, kichik α uchun kengroq funksiyalar sinfi mos keladi. Eng tor sinf bu Lipshits shartini qanoatlantiruvchi funksiyalar sinfidir.

3^0. Agar $f_1(x)$ va $f_2(x)$ (a,b) oraliqda mos ravishda $H(\alpha_1)$ va $H(\alpha_2)$ shartlarni qanoatlantirsa, u holda

$$f_1(x) + f_2(x), \quad f_1(x) \cdot f_2(x), \quad \frac{f_1(x)}{f_2(x)} \quad (f_2(x) \ne 0)$$

funksiyalar $\alpha = \min(\alpha_1, \alpha_2)$ shart bilan Gyolder shartini qanoatlantiradi.

1.2. Teorema. $p>1$, $\frac{1}{p}<\alpha<\frac{1}{p}+1$ *yoki* $p=1$, $1\le\alpha<2$ *bo'lib,* $f(x)\in L_p(a,b)$, *bo'lsa, u holda* $D_{a,x}^{-\alpha}f(x)$ (a,b) *oraliqda* $\alpha-\frac{1}{p}$ *ko'rsatkich bilan Gyolder shartini qanoatlantiradi.*

1.3. Teorema. *k va* α *sonlari uchun* $k\ge 0, \alpha>0, k+\alpha<1$ *shartlar bajarilsin. Agar* $f(x)$ *funksiya* (a,b) *oraliqda* k *ko'rsatkich bilan Gyolder shartlarini qanoatlantirsa va kichik* $x-a$ *lar uchun* $f(x)=O\left((x-a)^k\right)$ *bo'lsa, u holda* $D_{a.x}^{-\alpha}f(x)$ (a,b) *oraliqda* $k+\alpha$ *ko'rsatkich bilan Gyolder shartini qanoatlantiradi, shu bilan birga kichik* $x-a$ *lar uchun*

$$D_{a,x}^{-\alpha}f(x)=O\left((x-a)^{k+\alpha}\right),$$

bo'ladi.

1.4. Teorema. *Agar* $g(x)\in C^{(0,\lambda)}[a,b]$ *bo'lsa, u holda* $g(x)$*ni*

$$g(x)=g(a)+D_{a.x}^{-\alpha}f(x),$$

ko'rinishda ifodalash mumkin, bu yerda $f(x)\in C^{(0,\lambda-\alpha)}(a,b)$, $0<\alpha<\lambda\le 1$.

2. Kasr tartibli differensial operatorlar xossalari.

1^0. $\ell=n+1$ bo'lsin, u holda (1.74) va (1.75) munosabatlarga ko'ra, $D_{a,x}^{n+1}f(x)=\frac{d^{n+1}}{dx^{n+1}}f(x)$ va $n=0$ uchun $D_{a,x}^{1}f(x)=\frac{d}{dx}f(x)$ tengliklar o'rinli.

2^0. Kasr tartibli differensial operatorlar uchun ekstremum prinsipi.

Musbat, kamaymaydigan $\omega(t)$ funksiya hamda $f(t)$ funksiyalar $[a,b]$ kesmada uzluksiz bo'lsin. Agar $[a,b]$ kesmada $f(t)$ funksiya o'zining musbat maksimumiga (manfiy minimumiga) $t=x$ nuqtada erishsa, $a<x<b$ va bu nuqtaning ixtiyoriy kichik atrofida $\omega(t)f(t)$ ko'paytma $\gamma>\alpha$ ko'rsatkich bilan Gyolder shartini qanoatlantirsa, u holda

$$D_{a,x}^{\alpha}\omega(x)f(x)>0\ \left(D_{a,x}^{\alpha}\omega(x)f(x)<0\right). \qquad (1.77)$$

Haqiqatdan ham,

$$\Gamma(1-\alpha)D^{\alpha}_{a,x}\omega f=\Gamma(1-\alpha)\frac{d}{dx}\left[D^{\alpha-1}_{a,x}\omega(x)f(x)\right]=\frac{d}{dx}\int_a^x\frac{\omega(t)f(t)}{(x-t)^{\alpha}}dt=\frac{d}{dx}\int_a^x\frac{\omega(t)f(t)-\omega(x)f(x)}{(x-t)^{a}}dt+$$

$$+\frac{d}{dx}\int_a^x\frac{\omega(x)f(x)}{(x-t)^{\alpha}}dt=\lim_{\varepsilon\to 0}\left\{\frac{d}{dx}\int_a^{x-\varepsilon}\frac{\omega(t)f(t)-\omega(x)f(x)}{(x-t)^{a}}dt+\frac{d}{dx}\int_a^{x-\varepsilon}\frac{\omega(x)f(x)}{(x-t)^{\alpha}}dt\right\}=$$

$$=\lim_{\varepsilon\to 0}\left\{\frac{\omega(x-\varepsilon)f(x-\varepsilon)-\omega(x)f(x)}{\varepsilon^{\alpha}}-\int_a^{x-\varepsilon}\frac{(\omega(x)f(x))'_x}{(x-t)^{\alpha}}dt-\alpha\int_{\alpha}^{x-\varepsilon}\frac{\omega(t)f(t)-\omega(x)f(x)}{(x-t)^{1+\alpha}}dt+\right.$$

$$\left.+\int_a^{x-\varepsilon}\frac{(\omega(x)f(x))'_x}{(x-t)^{\alpha}}dt-\alpha\int_a^{x-\varepsilon}\frac{\omega(x)f(x)}{(x-t)^{1+\alpha}}dt+\frac{\omega(x)f(x)}{\varepsilon^{\alpha}}\right\}.$$

Bu yerda $\varepsilon\to 0$ da limitga o‘tib va ushbu

$$\int_a^{x-\varepsilon}\frac{\omega(x)f(x)dt}{(x-t)^{1+\alpha}}=\frac{\omega(x)f(x)}{\alpha\varepsilon^{\alpha}}-\frac{\omega(x)f(x)}{\alpha(x-a)^{\alpha}},$$

$$\lim_{\varepsilon\to 0}\frac{\omega(x-\varepsilon)f(x-\varepsilon)-\omega(x)f(x)}{\varepsilon^{\alpha}}=0,$$

tengliklarni e’tiborga olib, quyidagi tenglikka ega bo‘lamiz:

$$\Gamma(1-\alpha)D^{\alpha}_{\alpha,x}\omega(x)f(x)=\frac{\omega(x)f(x)}{(x-a)^{\alpha}}+\alpha\int_a^{\delta}\frac{\omega(x)f(x)-\omega(t)f(t)}{(x-t)^{1+\alpha}}dt+ \\ +\alpha\int_{\delta}^{x}\frac{\omega(x)f(x)-\omega(t)f(t)}{(x-t)^{1+\alpha}}dt. \qquad (1.78)$$

(1.78) munosabatdan (1.77) tengsizlik kelib chiqadi.

Agar $\omega(t)$-musbat o‘smaydigan funksiya bo‘lsa, yuqoridagiga o‘xshash natijani $D^{\alpha}_{x,b}$ operator uchun ham olish mumkin. Ixtiyoriy tartibli integro-differensial operatorlar kompozitsiyalari uchun o‘rinli bo‘lgan ba’zi bir munosabatlarni keltiramiz.

1^0. *Agar* $f(x)\in L(a,b)$ *bo‘lsa, ixtiyoriy* $\alpha>0$ *va deyarli barcha* $x\in(a,b)$ *uchun*

$$D_{a,x}^{\alpha}D_{a,x}^{-\alpha}f(x)=f(x) \tag{1.79}$$

tenglik o'rinli.

2^0. $D_{a,x}^{\alpha}f(x)\in L(a,b)$ *bo'lsin, u holda deyarli barcha* $x\in(a,b)$ uchun

$$D_{a,x}^{-\alpha}D_{a,x}^{\alpha}f(x)=f(x)-\sum_{k=1}^{n}\left[D_{a,x}^{\alpha-k}f(x)\right]_{x=a}\frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)},$$
$$n-1<\alpha\le n\ . \tag{1.80}$$

tenglik o'rinlidir.

(1.79) va (1.80) tengliklarning umumlashmalarini keltiramiz.

3^0. $f(x)\in L(a,b)$ *bo'lsin. U holda:*

1) *agar* $\beta\ge\alpha>0$ *bo'lsa, u holda*

$$D_{a,x}^{\alpha}D_{a,x}^{-\beta}f(x)=D_{a,x}^{-(\beta-\alpha)}f(x)\ \ ,\ x\in(a,b), \tag{1.81}$$

2) *agar* $\alpha>\beta\ge0$ *bo'lib,* $f(x)$ *funksiyaning* (a,b) *da* $D_{a,x}^{\alpha-\beta}f(x)$ hosilasi mavjud bo'lsa, u holda

$$D_{a,x}^{\alpha}D_{a,x}^{-\beta}f(x)=D_{a,x}^{\alpha-\beta}f(x),\ x\in(a,b) \tag{1.82}$$

tenglik o'rinlidir.

4^0. $f(x)\in L(a,b)$ bo'lib, uning kasr tartibli hosilasi $D_{a,x}^{\beta}f(x)\in L(a,b)$ bo'lsin, bu yerda $n-1<\beta\le n\ (n\ge1)$, u holda ixtiyoriy $\alpha>0$ son uchun

$$D_{a,x}^{-\alpha}D_{a,x}^{\beta}f(x)=D_{a,x}^{\beta-\alpha}f(x)-\sum_{k=1}^{n}\left[D_{a,x}^{\beta-k}f(x)\right]_{x=a}\frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} \tag{1.83}$$

tenglik o'rinlidir.

5^0. $f(x)\in C^{q-1}(a,b)$ va $f^{(q)}(x)\in L(a,b)$ bo'lsin $(q\ge1)$, u holda ixtiyoriy $\alpha\ (0<\alpha\le q)$ uchun $D_{a,x}^{\alpha}f(x)$ hosila mavjud, shu bilan birga $n-1<\alpha\le n$ bo'lsa, u holda deyarli barcha $x\in(a,b)$ uchun ushbu

$$D_{a,x}^{\alpha} f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}+D_{a,x}^{-(n-\alpha)} f^{(n)}(x) \quad (1.84)$$

tenglik o‘rinlidir.

Biz keyinchalik foydalanadigan ba’zi bir ayniyatlarni keltiramiz:

1.1 Lemma. *Agar* $0<\alpha,\beta<1$ *va* $x^{-\alpha} f(x),\ x^{-\beta} f(x)\in L(a,b)$ *bo‘lsa, u holda deyarli barcha* $x\in(a,b)$ *uchun*

$$D_{a,x}^{-\beta}(x-a)^{-\beta} D_{a,x}^{-\alpha}(x-a)^{-\alpha} f(x)=D_{a,x}^{-\alpha}(x-a)^{-\alpha} D_{a,x}^{-\beta}(x-a)^{-\beta} f(x) \quad (1.85)$$

ayniyat o‘rinlidir.

Isbot. Kasr tartibli integral operator (1.74) ga ko‘ra:

$$D_{a,x}^{-\beta}(x-a)^{-\beta} D_{a,x}^{-\alpha}(x-a)^{-\alpha} f(x)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{(t-a)^{-\beta}}{(x-t)^{1-\beta}}dt\int_a^t \frac{(s-a)^{-\alpha}}{(t-s)^{1-\alpha}} f(s)ds=$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(s)ds\int_s^x \frac{(t-a)^{-\beta}(s-a)^{-\alpha}dt}{(x-t)^{1-\beta}(t-s)^{1-\alpha}}$$

Ichki integralda integral o‘zgaruvchisini $t=s+(x-s)\sigma$ ko‘rinishda almashtirib va gipergeometrik funksiyaning integral ifodasi (1.17) ni hisobga olib, ushbu tenglikka kelamiz:

$$D_{a,x}^{-\beta}(x-a)^{-\beta} D_{a,x}^{-\alpha}(x-a)^{-\alpha} f(x)=$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{f(s)ds}{(s-a)^{\alpha+\beta}(x-s)^{1-\alpha-\beta}}\int_0^1 \sigma^{\alpha-1}(1-\sigma)^{\beta-1}\times$$

$$\times\left(1-\frac{s-x}{s-a}\sigma\right)^{-\beta} d\sigma=\frac{1}{\Gamma(\alpha+\beta)}\int_a^x \frac{F\left(\alpha,\beta,\alpha+\beta;\frac{s-x}{s-a}\right)}{(s-a)^{\alpha+\beta}(x-s)^{1-\alpha-\beta}}ds.$$

Bu yerdan Gauss gipergeometrik funksiyasining birinchi ikki parametriga nisbatan simmetriklik xossasidan foydalanib, hamda α va β larning o‘rinlarini almashtirib, (1.85) ayniyatning to‘g‘ri ekanligiga ishonch hosil qilamiz.

Lemma. *Agar* $0<2\alpha<1$ *va* $(x-a)^{-\alpha} f(x)$, $(b-x)^{-\alpha} f(x)\in L(a,b)$ *bo‘lsa, u holda deyarli barcha* $x\in(a,b)$ *uchun*

$$D_{a,x}^{\alpha}(x-a)^{2\alpha-1}D_{a,x}^{\alpha-1}(x-a)^{-\alpha}f(x)=(x-a)^{\alpha-1}D_{a,x}^{2\alpha-1}f(x), \tag{1.86}$$

$$D_{x,b}^{\alpha}(b-x)^{2\alpha-1}D_{x,b}^{\alpha-1}(b-x)^{-\alpha}f(x)=(b-x)^{\alpha-1}D_{x,b}^{2\alpha-1}f(x) \tag{1.87}$$

munosabatlar o'rinli bo'ladi.

Isbot. (1.74) ga ko'ra

$$J=D_{a,x}^{\alpha}(x-a)^{2\alpha-1}D_{a,x}^{\alpha-1}(x-a)^{-\alpha}f(x)=\frac{d}{dx}D_{a,x}^{\alpha-1}(x-a)^{2\alpha-1}D_{a,x}^{\alpha-1}(x-a)^{-\alpha}f(x)=$$

$$=\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_a^x\frac{(t-a)^{2\alpha-1}}{(x-t)^{\alpha}}dt\int_a^t\frac{(s-a)^{-\alpha}f(s)ds}{(t-s)^{\alpha}}=\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_a^x\frac{f(s)ds}{(s-a)^{\alpha}}\int_s^x\frac{(t-a)^{2\alpha-1}dt}{(x-t)^{\alpha}(t-s)^{\alpha}} \tag{1.88}$$

tenglikka ega bo'lamiz.

(1.88) tenglikning ichki integralida $t=x+(x-s)\sigma$ almashtirish bajaramiz va (1.17) dan ushbu

$$J=\frac{1}{\Gamma^2(1-\alpha)}\frac{d}{dx}\int_a^x\frac{f(s)}{(s-a)^{\alpha}}\left(\frac{x-s}{x-a}\right)^{1-2\alpha}ds\int_0^1\sigma^{-\alpha}(1-\sigma)^{-\alpha}\left(1-\frac{x-s}{x-a}\sigma\right)^{2\alpha-1}d\sigma=$$

$$=\frac{1}{\Gamma(2-2\alpha)}\frac{d}{dx}F\left(1-\alpha,1-2\alpha,2-2\alpha;\frac{x-s}{x-a}\right)ds,$$

tenglikka ega bo'lamiz, bu yerda (1.47) formulaga ko'ra differensiallash operatsiyasini bajaramiz va (1.10) formulani hisobga olib, ushbu

$$J=\frac{1-2\alpha}{\Gamma(2-2\alpha)}\int_a^x\frac{f(s)}{(s-a)^{\alpha}}\left(\frac{x-s}{x-a}\right)^{-2\alpha}F\left(1-\alpha,2-2\alpha,2-2\alpha;\frac{x-s}{x-a}\right)\cdot\frac{s-a}{(x-a)^2}ds=$$

$$=\frac{(x-a)^{a-1}}{\Gamma(1-2\alpha)}\int_0^x\frac{f(s)ds}{(x-s)^{2\alpha}}=(x-a)^{\alpha-1}D_{a,x}^{2\alpha-1}f(x)$$

yakuniy natijaga kelamiz

(1.86) ayniyat isbot bo'ldi.
(1.87) ayniyat yuqoridagidek isbotlanadi.

1.3 Lemma. *Agar* $0<2\beta<1$ *va* $(x-a)^{\beta-1}f(x),\,(b-x)^{\beta-1}f(x)\in L(a,b)$ *bo'lsa, u holda deyarli barcha* $x\in(a,b)$ *uchun ushbu*

$$D_{a,x}^{1-\beta}(x-a)^{1-2\beta}D_{a,x}^{-\beta}(x-a)^{\beta-1}f(x)=(x-a)^{-\beta}D_{a,x}^{1-2\beta}f(x),\quad (1.89)$$

$$D_{x,b}^{1-\beta}(b-x)^{1-2\beta}D_{x,b}^{-\beta}(b-x)^{\beta-1}f(x)=(b-x)^{-\beta}D_{x,b}^{1-2\beta}f(x),\quad (1.90)$$

ayniyatlar o'rinli.

Isbot. (1.89) tenglikning chap tomonini $g(x)$ orqali belgilaymiz va integro-differensial operator (1.74) ga ko'ra

$$g(x)=D_{a,x}^{1-\beta}(x-a)^{1-2\beta}D_{a,x}^{-\beta}(x-a)^{\beta-1}f(x)=\frac{1}{\Gamma^2(\beta)}\frac{d}{dx}\int_a^x\frac{f(s)ds}{(s-a)^{1-\beta}}\int_s^x\frac{(t-a)^{1-2\beta}dt}{(x-t)^{1-\beta}(t-s)^{1-\beta}}$$

ifodani hosil qilamiz. Bu tenglikning ichki integralida integral o'zgaruvchisini $t=s+(x-s)\sigma$ ko'rinishda almashtirib,

$$g(x)=\frac{1}{\Gamma(2\beta)}\frac{d}{dx}\int_a^x\frac{f(s)}{(s-a)^{1-\beta}}\left(\frac{x-s}{x-a}\right)^{2\beta-1}F\left(\beta,2\beta-1,2\beta;\frac{x-s}{x-a}\right)ds$$

tenglikka kelamiz. Bu yerda bevosita differensiallash operatsiyasini bajarib bo'lmaydi. Shuning uchun ushbu funksiyani kiritamiz:

$$g_\varepsilon(x)=\frac{1}{\Gamma(2\beta)}\frac{d}{dx}\int_a^{x-\varepsilon}\frac{f(s)}{(s-a)^{1-\beta}}\left(\frac{x-s}{x-a}\right)^{2\beta-1}F\left(\beta,2\beta-1,2\beta;\frac{x-s}{x-a}\right)ds\quad (1.91)$$

Endi (1.48) formulani hisobga olib, differensiallash operatsiyasini bajaramiz va (1.10) formulaga asosan, ushbu natijaga ega bo'lamiz:

$$g_\varepsilon(x)=\frac{1}{\Gamma(2\beta)}\cdot\frac{f(x-\varepsilon)}{(x-\varepsilon-a)^{1-\beta}}\left(\frac{\varepsilon}{x-a}\right)^{2\beta-1}F\left(\beta,2\beta-1,2\beta;\frac{\varepsilon}{x-a}\right)+$$
$$+\frac{(2\beta-1)(x-a)^{-\beta}}{\Gamma(2\beta)}\int_a^{x-\varepsilon}\frac{f(s)ds}{(x-s)^{2-2\beta}}\,.$$

Bu yerda ushbu

$$(2\beta-1)\int_a^{x-\varepsilon}\frac{f(s)ds}{(x-s)^{2-2\beta}}=\frac{d}{dx}\int_a^{x-\varepsilon}\frac{f(s)ds}{(x-s)^{1-2\beta}}-\varepsilon^{2\beta-1}f(x-\varepsilon),$$

tenglikni hisobga olib, quyidagi tenglikka kelamiz:

$$g_\varepsilon(x)=\frac{\varepsilon^{2\beta-1}(x-a)^{-\beta}}{\Gamma(2\beta)}\left[\left(\frac{x-a}{x-\varepsilon-a}\right)^{1-\beta}F\left(\beta,2\beta-1,2\beta;\frac{\varepsilon}{x-a}\right)-1\right]f(x-\varepsilon)+$$

$$+\frac{(x-a)^{-\beta}}{\Gamma(2\beta)}\frac{d}{dx}\int_a^{x-\varepsilon}\frac{f(s)ds}{(x-s)^{1-2\beta}}\quad. \tag{1.92}$$

(1.92) tenglikda $\varepsilon\to 0$ da limitga o'tib, ushbu

$$g(x)=\lim_{\varepsilon\to 0}g_\varepsilon(x)=(x-a)^{-\beta}D_{a,x}^{1-2\beta}f(x),$$

yakuniy natijaga ega bo'lamiz.

(1.89) ayniyat isbot bo'ldi.

(1.90) ayniyat ham yuqoridagidek isbotlanadi.

1.4 Lemma. *Agar $\varphi(x)\in C^{(0,\alpha)}(a,b)$ bo'lsa, u holda ushbu*

$$D_{x,b}^{-\alpha}\phi(x)=\cos\alpha\pi D_{a,x}^{-\alpha}\phi(x)+\frac{\sin\alpha\pi}{\pi}\int_a^b\left(\frac{b-x}{b-t}\right)^{\alpha}\frac{D_{a,t}^{-\alpha}\phi(t)}{t-x}dt\,, \tag{1.93}$$

$$D_{a,x}^{-\alpha}\phi(x)=\cos\alpha\pi D_{x,b}^{-\alpha}\phi(x)-\frac{\sin\alpha\pi}{\pi}\int_a^b\left(\frac{x-a}{t-a}\right)^{\alpha}\frac{D_{t,b}^{-\alpha}\phi(t)}{t-x}dt,\,, \tag{1.94}$$

$$D_{x,b}^{-\alpha}\phi(x)=\cos\alpha\pi D_{a,x}^{-\alpha}\phi(x)+\frac{\sin\alpha\pi}{\pi\Gamma(\alpha)}\int_a^x\frac{(x-t)^{\alpha-1}}{(t-a)^{\alpha}}dt\int_a^b\frac{(z-a)^{\alpha}\phi(z)dz}{z-t}\,, \tag{1.95}$$

$$D_{a,x}^{-\alpha}\phi(x)=\cos\alpha\pi D_{x,b}^{-\alpha}\phi(x)-\frac{\sin\alpha\pi}{\pi\Gamma(\alpha)}\int_x^b\frac{(t-x)^{\alpha-1}}{(b-t)^{\alpha}}dt\int_a^b\frac{(b-z)^{\alpha}\phi(z)}{z-t}dz\,, \tag{1.96}$$

ayniyatlar o'rinlidir $(0<\alpha<1)$:

Isbot. Ushbu integralni o'rganamiz:

$$\Phi(x)=\frac{1}{\pi\Gamma(\alpha)}\int_a^b\left(\frac{b-x}{b-t}\right)^{\alpha}\frac{dt}{t-x}\int_a^t(t-s)^{\alpha-1}\phi(s)ds\,.$$

Bu integralda integrallash tartibini o‘zgartiramiz:

$$\Phi(x)=\frac{(b-x)^{\alpha}}{\pi\Gamma(\alpha)}\int_a^b\phi(s)ds\int_s^b\frac{(b-t)^{-\alpha}(t-s)^{\alpha-1}}{t-x}dt. \qquad (1.97)$$

Ichki integralni hisoblaymiz:

$$I(x,s)=\int_s^b\frac{(b-t)^{-\alpha}(t-s)^{\alpha-1}}{t-x}dt. \qquad (1.98)$$

Agar $s<x<b$ bo‘lsa, bu integral $t=x$ nuqtada singulyar maxsuslikka ega, agar $x<s$ bo‘lsa, bu integralda singulyar maxsuslik yo‘q. Shuning uchun, (1.98) integralni hisoblashda ikki holni o‘rganamiz: $x>s$ va $x<s$.

a) $x<s$ bo‘lsin, u holda $t-x\neq 0$. (1.98) integralda $t=s+(b-s)\sigma$ almashtirish bajaramiz, hamda (1.17) va (1.10) formulalarga asosan, ushbu tenglikka kelamiz:

$$I(x,s)=\frac{\pi}{\sin\alpha\pi}(s-x)^{\alpha-1}(b-x)^{-\alpha}\,. \qquad (1.99)$$

b) $x>s$ bo‘lsin. (1.98) integralning Koshi ma’nosidagi bosh qiymatini hisoblaymiz. Buning uchun (1.98) ni ushbu ko‘rinishda yozib olamiz:

$$I(x,s)=\lim_{\delta\to 0}\left\{-\int_s^x\frac{(b-t)^{-\alpha}(t-s)^{\alpha-1}}{(x-t)^{1-\delta}}dt+\int_x^b\frac{(b-t)^{-\alpha}(t-s)^{\alpha-1}}{(t-x)^{1-\delta}}dt\right\},$$

bu yerda birinchi va ikkinchi integrallarda mos ravishda integral o‘zgaruvchilarini $t=s+(x-s)\sigma$ va $t=b+(x-b)\sigma$ ko‘rinishda almashtirib, keyin (1.17) gipergeometrik funksiyalarning integral ifodasidan va (1.19) avtotransformatsiya formulasidan foydalanib, ushbu

$$I(x,s)=\lim_{\delta\to0}\left\{-(b-s)^{-\alpha}(x-s)^{\alpha-1+\delta}\frac{\Gamma(\alpha)\Gamma(\delta)}{\Gamma(\alpha+\delta)}\left(\frac{b-x}{b-s}\right)^{\delta-\alpha}F\left(\delta,\delta,\alpha+\delta;\frac{x-s}{b-s}\right)+\right.$$

$$\left.+(b-x)^{\delta-\alpha}(b-s)^{\alpha-1}\frac{\Gamma(1-\alpha)\Gamma(\delta)}{\Gamma(1-\alpha+\delta)}\left(\frac{x-s}{b-s}\right)^{\delta+\alpha-1}F\left(\delta,\delta,1-\alpha+\delta;\frac{b-x}{b-s}\right)\right\} \qquad (1.100)$$

tenglikni hosil qilamiz. (1.100) tenglikning birinchi qo‘shiluvchisiga (1.21) Bols formulasini qo‘llab, uni ushbu ko‘rinishda yozib olamiz:

$$I(x,s)=\lim_{\delta\to0}\left[\left(\frac{1}{b-s}\right)^{\delta}(b-x)^{\delta-\alpha}(x-s)^{\alpha-1+\delta}\times\right.$$

$$\times\Gamma(\delta)\left(\frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+\delta)}-\frac{\Gamma(\alpha-\delta)}{\Gamma(\alpha)}\right)F\left(\delta,\delta,1-\alpha+\delta;\frac{b-x}{b-s}\right)-$$

$$\left.-(b-s)^{-\alpha}(x-s)^{\alpha-1+\delta}\frac{\Gamma(\alpha)\Gamma(\delta-\alpha)}{\Gamma(1+\delta)}\delta F\left(\alpha,\alpha,\delta-\alpha+1;\frac{b-x}{b-s}\right)\right] \qquad (1.101)$$

$\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin\pi\alpha}$ tenglikni hisobga olgan holda, ushbu

$$\lim_{\delta\to0}\Gamma(\delta)\left(\frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+\delta)}-\frac{\Gamma(\alpha-\delta)}{\Gamma(\alpha)}\right)=-\pi ctg\alpha\pi \qquad (1.102)$$

limitning to‘g‘riligiga ishonch hosil qilish qiyin emas.

Endi (1.101) da $\delta\to0$ da limitga o‘tib, ushbu

$$I(x,s)=-\pi ctg\alpha\pi(b-x)^{-\alpha}(x-s)^{\alpha-1} \qquad (1.103)$$

tenglikni hosil qilamiz

Shunday qilib, (1.99) va (1.103) ga asosan,

$$I(x,s)=\int_s^b \frac{(b-t)^{-\alpha}(t-s)^{\alpha-1}}{t-x}dt=$$

(1.104)

$$=\begin{cases} \dfrac{\pi}{\sin\alpha\pi}(s-x)^{\alpha-1}(b-x)^{-\alpha}\ , & x<s \\ -\pi ctg\alpha\pi(x-s)^{\alpha-1}(b-x)^{-\alpha}, & x>s \end{cases}$$

Nihoyat, (1.97) dan (1.104) ni hisobga olib,

$$\Phi(x)=\frac{(b-x)^{\alpha}}{\pi\Gamma(\alpha)}\left[\int_a^x \phi(s)I(x,s)ds+\int_x^b \phi(s)I(x,s)ds\right]=$$

$$=-\frac{1}{\sin\alpha\pi}\left[\cos\alpha\pi\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\phi(s)ds}{(x-s)^{1-\alpha}}-\frac{1}{\Gamma(\alpha)}\int_x^b\frac{\phi(s)ds}{(s-x)^{1-\alpha}}\right]= \quad (1.105)$$

$$=-\frac{1}{\sin\alpha\pi}\left[\cos\alpha\pi D_{a,x}^{-\alpha}\phi(x)-D_{x,b}^{-\alpha}\phi(x)\right],$$

tenglikka kelamiz, bu yerda (1.93) formula kelib chiqadi. Qolgan ayniyatlar yuqoridagidek isbotlanadi.

Natija. $\Phi(x)$ *funksiya* $D_{a.,x}^{-\alpha}\varphi(x)$ *Abel integrali yordamida ifodalansin, u holda ushbu ayniyatlar o'rinlidir:*

$$D_{a,x}^{-\alpha}D_{t,b}^{\alpha}\Phi(t)=\cos\alpha\pi\Phi(x)-\frac{\sin\alpha\pi}{\pi}\int_a^b\left(\frac{x-a}{t-a}\right)^{\alpha}\frac{\Phi(t)dt}{t-x}, \quad (1.106)$$

$$D_{x,b}^{-\alpha}D_{a,t}^{\alpha}\Phi(t)=\cos\alpha\pi\Phi(x)-\frac{\sin\alpha\pi}{\pi}\int_a^b\left(\frac{b-x}{b-t}\right)^{\alpha}\frac{\Phi(t)dt}{t-x}. \quad (1.107)$$

Lemma. $\varphi(x)\in C^{(0,\lambda)}(a,b)$ *bo'lsin, u holda ushbu ayniyatlar o'rinlidir:*

$$D_{a,x}^{\alpha}D_{t.,b}^{-\alpha}\varphi(t)=\cos\alpha\pi\varphi(x)+\frac{\sin\alpha\pi}{\pi}\int_a^b\left(\frac{t-a}{x-a}\right)^{-\alpha}\frac{\varphi(t)dt}{t-x}, \quad (1.108)$$

$$D_{x,b}^{\alpha} D_{a,t}^{-\alpha} \varphi(t) = \cos\alpha\pi\varphi(x) - \frac{\sin\alpha\pi}{\pi} \int_a^b \left(\frac{b-t}{b-x} \right)^{\alpha} \frac{\varphi(t)dt}{t-x} \quad (1.109)$$

Isbot. Integro-differensial operatorlarning ta'rifiga asosan, (1.74) formuladan ushbu tengliklarni hosil qilamiz:

$$F(x) = D_{a,x}^{\alpha} D_{t,b}^{-\alpha} \phi(t) = \frac{d}{dx} D_{a,x}^{\alpha-1} D_{t,b}^{-\alpha} \phi(t) = \frac{\sin\alpha\pi}{\pi} \frac{d}{dx} \int_a^x \frac{dt}{(x-t)^{\alpha}} \int_t^b \frac{\phi(s)ds}{(s-t)^{1-\alpha}} =$$

$$= \frac{\sin\alpha\pi}{\pi} \frac{d}{dx} \left[\int_a^x \phi(s)ds \int_a^s (x-t)^{-\alpha} (s-t)^{\alpha-1} dt + \int_x^b \phi(s)ds \int_a^x (x-t)^{-\alpha} (s-t)^{\alpha-1} dt \right] \quad (1.110)$$

(1.110) munosabatdagi ichki integrallarda integral o'zgaruvchisini ushbu ko'rinishda almashtiramiz:

$$z = \frac{s-t}{x-t}, \quad t = \frac{s-xz}{1-z}, \quad x-t = \frac{x-s}{1-z}, \quad s-t = \frac{z(x-s)}{1-z}, \quad (1.111)$$

(1.111) tengliklarni hisobga olib, (1.110) ni ushbu ko'rinishda yozib olamiz:

$$F(x) = \frac{\sin\alpha\pi}{\pi} \frac{d}{dx} \left[\int_a^x \varphi(s)ds \int_0^{(s-a)/(x-a)} \frac{z^{a-1}dz}{1-z} - \int_x^b \varphi(s)ds \int_{(s-a)/(x-a)}^{\infty} \frac{z^{a-1}dz}{1-z} \right]. \quad (1.112)$$

Quyidagi integralni kiritamiz:

$$f_{\varepsilon}(x) = \int_a^{x-\varepsilon} \varphi(s)ds \int_0^{(s-a)/(x-a)} \frac{z^{a-1}}{1-z} dz - \int_{x+\varepsilon}^{b} \varphi(s)ds \int_{(s-a)/(x-a)}^{\infty} \frac{z^{a-1}}{1-z} dz. \quad (1.113)$$

Bu integralda x bo'yicha differensiallash operatsiyasini bajaramiz:

$$\frac{df_{\varepsilon}(x)}{dx} = \phi(x-\varepsilon) \int_0^{(x-\varepsilon-a)/(x-a)} \frac{z^{\alpha-1}dz}{1-z} + \phi(x+\varepsilon) \int_{(x+\varepsilon-a)/(x-a)}^{\infty} \frac{z^{\alpha-1}dz}{1-z} +$$
$$+ \int_a^{x-\varepsilon} \left(\frac{s-a}{x-a} \right)^{\alpha} \frac{\phi(s)ds}{s-x} + \int_{x+\varepsilon}^{b} \left(\frac{s-a}{x-a} \right)^{\alpha} \frac{\phi(s)ds}{s-x} \quad (1.114)$$

Endi (1.114) da $\varepsilon \to 0$ da limitga o‘tib, (1.112) ifodadan ushbu tenglikni hosil qilamiz:

$$F(x) = \frac{\sin \alpha\pi}{\pi} \lim_{\varepsilon \to 0} \frac{df_\varepsilon(x)}{dx} = \frac{\sin \alpha\pi}{\pi} \left[\varphi(x) \int_0^\infty \frac{z^{\alpha-1} dz}{1-z} + \int_a^b \left(\frac{s-a}{x-a} \right)^\alpha \frac{\varphi(s) ds}{s-x} \right].$$

Quyidagi integralni hisoblaymiz:

$$\int_0^\infty \frac{z^{a-1}}{1-z} dz = \lim_{\varepsilon \to 0} \left[\int_0^1 \frac{z^{a-1} dz}{(1-z)^{1-\varepsilon}} - \int_1^\infty \frac{z^{a-1} dz}{(z-1)^{1-\varepsilon}} \right] = \lim_{\varepsilon \to 0} \left[B(\alpha, \varepsilon) - B(1-\alpha-\varepsilon, \varepsilon) \right] =$$

$$= \lim_{\varepsilon \to 0} \Gamma(\varepsilon) \left[\frac{\Gamma(a)}{\Gamma(a+\varepsilon)} - \frac{\Gamma(1-a-\varepsilon)}{\Gamma(1-a)} \right] = \pi ctg \alpha\pi,$$

demak,

$$F(x) = \frac{\sin \alpha\pi}{\pi} \left[\pi ctg \alpha\pi \phi(x) + \int_a^b \left(\frac{s-a}{x-a} \right)^\alpha \frac{\phi(s) ds}{s-x} \right] = \cos \alpha\pi \phi(x) + \frac{\sin \alpha\pi}{\pi} \int_a^b \left(\frac{s-a}{x-a} \right)^\alpha \frac{\phi(s) ds}{s-x}.$$

Shunday qilib, (1.108) ayniyat isbotlandi. (1.109) ayniyat ham yuqoridagidek isbotlanadi.

1.5 Lemma isbotlandi.

4-§. Shakli o‘zgargan Koshi masalasi.

Ushbu

$$-(-y)^m u_{xx} + u_{yy} + \left(\alpha_0 / (-y)^{1-m/2} \right) u_x + (\beta_0 / y) u_y = 0 \quad (1.115)$$

tenglamani $z = x + iy$, $\operatorname{Im} z < 0$ kompleks yarim tekisligining chekli bir bog‘lamli D sohasida o‘rganamiz. D soha (1.115) tenglamaning

$$AC: x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = -1, \qquad BC: x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1,$$

xarakteristikalari, hamda $y = 0$ o‘qining AB kesmasi bilan chegaralangan, bu yerda $A(-1,0)$, $B(1,0)$.

(1.115) tenglamada m, α_0 va β_0-o‘zgarmas sonlar bo‘lib, ular ushbu shartlarni qanoatlantiradi:

$$m > 0, \; -m/2 < \beta_0 < (m+4)/2, \; |\alpha_0| < (m+2)/2.$$

(1.115) tenglama yechimining tuzilishi va differensial xossalari uning kichik hadlari oldidagi α_0 va β_0 koeffitsientlariga qat’iy bog‘liqdir.

Bu bog‘liqlikni oydinlashtirish maqsadida, α_0 va β_0 parametrlar tekisligida

$$A_0D_0 : \beta_0 - \alpha_0 = (m+4)/2, \;\; D_0B_0 : \beta_0 + \alpha_0 = (m+4)/2 \;,$$

$$B_0C_0 : \beta_0 - \alpha_0 = -m/2, \quad A_0C_0 : \beta_0 + \alpha_0 = -m/2 \;,$$

$A_0 D_0 B_0 C_0$ kvadratni kiritamiz va $P(\alpha_0, \beta_0)$ nuqtaning bu kvadratda o‘zgarishiga qarab, (1.115) tenglama uchun masalalar qo‘yamiz.

(1.115) tenglamaning regulyar yechimi deganda, D sohada ikkinchi tartibli uzluksiz hosilalarga ega bo‘lgan va bu tenglamani qanoatlantiradigan $u(x, y)$ funksiya tushuniladi.

1. $P(\alpha_0, \beta_0) \in \Delta A_0B_0C_0 \cup A_0C_0 \cup B_0C_0 \cup \{C_0\}$ **shakli o‘zgargan Koshi masalasi.**

$P(\alpha_0, \beta_0) \in \Delta A_0B_0C_0 \cup A_0C_0 \cup B_0C_0 \cup \{C_0\}$ bo‘lsin.

Shakli o‘zgargan Koshi masalasi. D sohada (1) tenglamaning ushbu

$$u(x,0) = \tau(x), \; x \in \bar{J} \;, \tag{1.116}$$

$$\lim_{y \to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} = \nu(x), \;\; x \in J \;, \tag{1.117}$$

boshlang‘ich shartlarni qanoatlantiruvchi $u(x, y) \in C(\bar{D}) \cap C^2(D)$ regulyar yechimi topilsin, bu yerda $\tau(x) \in C(\bar{J}) \cap C^2(J)$, $\nu(x) \in C^2(J)$-berilgan funksiyalar, $J = (-1,1)$ $y = 0$ o‘qining intervali.

Shakli o‘zgargan Koshi masalasini Riman metodi yordamida yechamiz. (1.115) tenglama quyidagi xarakteristik koordinatalarga nisbatan:

$$\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \quad \eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \tag{1.118}$$

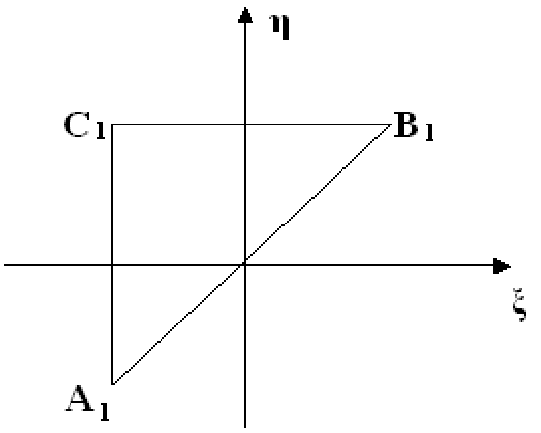
$$\eta - \xi = \frac{4}{m+2}(-y)^{\frac{m+2}{2}} \geq 0, \ \eta \geq \xi$$

ushbu

$$L(u) = u_{\xi\eta} + \frac{\beta}{\eta - \xi}u_{\xi} - \frac{\alpha}{\eta - \xi}u_{\eta} = 0, \tag{1.119}$$

Eyler-Puasson-Darbu tenglamasiga o‘tadi, bu yerda

$$\alpha = \frac{m + 2(\beta_0 + \alpha_0)}{2(m+2)}, \ \beta = \frac{m + 2(\beta_0 - \alpha_0)}{2(m+2)}$$



(1.118) akslantirishda $x0y$ tekislikdagi D soha $\xi 0 \eta$ tekislikdagi $\Delta = A_1, B_1, C_1$ uchburchakka akslanadi: bu uchburchakning uchlari $A_1 = A_1(-1,-1)$, $B_1 = B_1(1,1)$ va $C_1 = C_1(-1,1)$ nuqtalarda bo‘lib, tomonlari $\xi = -1$, $-1 \leq \eta \leq 1$; $\eta = 1$, $-1 \leq \xi \leq 1$; $\eta - \xi = 0$ kesmalardan iborat, bu almashtirishda (1.116) va (1.117) shartlar ushbu

$$\lim_{\eta - \xi \to 0} u(\xi, \eta) = \tau(\xi), -1 \leq \xi \leq 1, \tag{1.120}$$

$$\lim_{\eta - \xi \to 0} \left(\frac{m+2}{4}(\eta - \xi) \right)^{\alpha + \beta} \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) = \nu(\xi), \quad -1 < \xi < 1. \tag{1.121}$$

ko‘rinishda bo‘ladi.
(1.119) tenglama bilan birgalikda unga qo‘shma

$$M(\xi)=\frac{\partial^2\vartheta}{\partial\xi\partial\eta}-\frac{\partial}{\partial\xi}\left(\frac{\beta}{\eta-\xi}\vartheta\right)+\frac{\partial}{\partial\eta}\left(\frac{\alpha}{\eta-\xi}\vartheta\right)=0. \quad (1.122)$$

tenglamani ham o‘rganamiz.

Giperbolik tenglamalar nazariyasida $R(\xi,\eta;\xi_0,\eta_0)$-Riman funksiyasi fundamental ahamiyatga ega va u (1.119) tenglama uchun quyidagicha aniqlanadi:

1) $R(\xi,\eta;\xi_0,\eta_0)$ - $(\xi\ ,\eta)$ o‘zgaruvchi bo‘yicha (1.122) qo‘shma tenglamaning yechimi bo‘ladi;

2) $\xi=\xi_0$, $\eta=\eta_0$ xarakteristikalarda $R(\xi,\eta;\xi_0,\eta_0)$ funksiya ushbu

$$R(\xi_0,\eta;\xi_0,\eta_0)=\exp\left(\int\limits_{\eta_0}^{\eta}\frac{\beta dt}{t-\xi_0}\right)=\exp\left(\beta\ln\frac{\eta-\xi_0}{\eta_0-\xi_0}\right)=\left(\frac{\eta-\xi_0}{\eta_0-\xi_0}\right)^{\beta} \quad (1.123)$$

$$R(\xi,\eta_0;\xi_0,\eta_0)=\exp\left(-\int\limits_{\xi_0}^{\xi}\frac{\alpha dt}{\eta_0-t}\right)=\exp\left(\alpha\ln\frac{\eta_0-\xi}{\eta_0-\xi_0}\right)=\left(\frac{\eta_0-\xi}{\eta_0-\xi_0}\right)^{\alpha} \quad (1.124)$$

qiymatlarni qabul qiladi, bu yerda $\eta\le\eta_0,\ \xi\ge\xi_0,\ \eta\ge\xi$.

(1.119) tenglama uchun Riman funksiyasi ushbu

$$R(\xi,\eta;\xi_0,\eta_0)=\frac{(\eta-\xi)^{\alpha+\beta}}{(\eta_0-\xi)^{\beta}(\eta-\xi_0)^{\alpha}}F(\beta,\alpha,1;\sigma), \quad (1.125)$$

ko‘rinishga ega, bu yerda $F(\cdots)$ Gaussning gipergeometrik funksiyasi va

$$\sigma=\frac{(\xi-\xi_0)(\eta-\eta_0)}{(\xi-\eta_0)(\eta-\xi_0)}.$$

Δ_ε orqali P_1P: $\xi=\xi_0$, $PP_2:\eta=\eta_0$ va $\eta=\xi+\varepsilon$ to‘g‘ri chiziqning P_1P_2 kesmasi bilan chegaralangan uchburchakni belgilaymiz, bu yerda

$$P=P(\xi_0,\eta_0)\ ,\ \ P_1=P_1(\xi_0,\xi_0+\xi)\ ,\ P_2=P_2(\eta_0-\varepsilon,\eta_0)$$

$R(\xi,\eta;\xi_0,\eta_0)$ va $u(\xi,\eta)$ funksiyalar uchun ushbu

$$2(R(\xi,\eta;\xi_0,\eta_0)Lu(\xi,\eta)-u(\xi,\eta)MR(\xi,\eta;\xi_0,\eta_0))=$$

$$=\frac{\partial}{\partial\xi}\left(R\frac{\partial u}{\partial\eta}-u\frac{\partial R}{\partial\eta}+\frac{2\beta}{\eta-\xi}uR\right)+\frac{\partial}{\partial\eta}\left(R\frac{\partial u}{\partial\xi}-u\frac{\partial R}{\partial\xi}-\frac{2\alpha}{\eta-\xi}uR\right) \tag{1.126}$$

ayniyat o‘rinli, (1.126) tenglikni Δ_ε soha bo‘yicha integrallab, keyin esa Gauss-Ostrogradskiy formulasini qo‘llab va $Lu=0$, $MR=0$ ayniyatlarni hisobga olib, ushbu

$$0=\int\limits_{\partial\Delta_\varepsilon}\left(R\frac{\partial u}{\partial\eta}-u\frac{\partial R}{\partial\eta}+\frac{2\beta}{\eta-\xi}uR\right)d\eta-\left(R\frac{\partial u}{\partial\xi}-u\frac{\partial R}{\partial\xi}-\frac{2\alpha}{\eta-\xi}uR\right)d\xi$$

tenglikka kelamiz, bu yerda $\partial\Delta_\varepsilon=P_1P_2\cup P_2P\cup PP_1$ - Δ_ε soha chegarasi. Endi P_2P da: $d\eta=0$, PP_1 da: $d\xi=0$ ekanligini e’tiborga olib, oxirgi tenglikni ushbu:

$$\int\limits_{P_1P_2}\left\{\left(R\frac{\partial u}{\partial\eta}-u\frac{\partial R}{\partial\eta}+\frac{2\beta}{\eta-\xi}uR\right)d\eta-\left(R\frac{\partial u}{\partial\xi}-u\frac{\partial R}{\partial\xi}-\frac{2\alpha}{\eta-\xi}uR\right)d\xi\right\}-$$

$$-\int\limits_{P_2P}\left(R\frac{\partial u}{\partial\xi}-u\frac{\partial R}{\partial\xi}-\frac{2\alpha}{\eta-\xi}uR\right)d\xi+\int\limits_{PP_1}\left(R\frac{\partial u}{\partial\eta}-u\frac{\partial R}{\partial\eta}+\frac{2\beta}{\eta-\xi}uR\right)d\eta \tag{1.127}$$

ko‘rinishda yozib olamiz.

(1.127) tenglikning oxirgi ikki integralini, aniqrog‘i $u(\xi,\eta)$ ning hosilalari qatnashgan hadlarini bo‘laklab integrallab, ushbu

$$\int\limits_{P_2P}\left(R\frac{\partial u}{\partial\xi}-u\frac{\partial R}{\partial\xi}-\frac{\alpha R}{\eta_0-\xi}uR\right)d\xi=u(P)R(P,P)-u(P_2)R(P_2,P)-2\int\limits_{P_2P}u\left(\frac{\partial R}{\partial\xi}+\frac{2\alpha}{\eta_0-\xi}\right)d\xi \tag{1.128}$$

$$\int\limits_{PP_1}\left(R\frac{\partial u}{\partial\eta}-u\frac{\partial R}{\partial\xi}+\frac{2\beta}{\eta-\xi_0}uR\right)d\eta=u(P_1)R(P_1,P)-u(P)R(P,P) \tag{1.129}$$

ifodalarni hosil qilamiz. (1.123) va (1.124) tengliklarga ko‘ra, (1.128) va (1.129) tengliklarning o‘ng tomonidagi integrallar nolga, $R(P,P)=1$ ga tengdir.

Shunday qilib, yuqorida aytilganlarni hisobga olib, (1.127) tenglikni ushbu:

$$u(\xi_0,\eta_0)=\frac{(uR)_{P_1}+(uR)_{P_2}}{2}+\int\limits_{\xi_0}^{\eta_0-\varepsilon}\left\{\left[\frac{\alpha+\beta}{\eta-\xi}R+\frac{1}{2}\left(\frac{\partial R}{\partial\xi}-\frac{\partial R}{\partial\eta}\right)\right]u\right\}\Bigg|_{\eta=\xi+\varepsilon}d\xi+$$

$$+\frac{1}{2}\int\limits_{\xi_0}^{\eta_0-\varepsilon}\left[\left(\frac{\partial u}{\partial\eta}-\frac{\partial u}{\partial\xi}\right)R\right]\Bigg|_{\eta=\xi+\varepsilon}d\xi \tag{1.130}$$

ko‘rinishda yozib olamiz.

(1.130) munosabatga Riman formulasi deyiladi.

Endi $P(\alpha_0,\beta_0)$ nuqtaning $A_0C_0B_0D_0$ kvadratda joylashishiga qarab, (1.115) tenglama uchun (1.116) va (1.117) boshlang‘ich shartlarni qanoatlantiruvchi shakli o‘zgargan Koshi masalasi yechimini beruvchi formulalarni keltirib chiqaramiz.

A. $P(\alpha_0,\beta_0)\in\Delta A_0B_0C_0$ bo‘lsin. Bu holda $\alpha>0$, $\beta>0$, $\alpha+\beta<1$. (1.121) boshlang‘ich shartga asosan, (1.21) Bols formulasini hisobga olgan holda, ushbu:

$$\lim_{\varepsilon\to 0}\left(\frac{\partial u}{\partial\eta}-\frac{\partial u}{\partial\xi}\right)R(\xi,\eta;\xi_0,\eta_0)\Bigg|_{\eta=\xi+\varepsilon}=-\left(\frac{4}{m+2}\right)^{\alpha+\beta}\frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left(\eta_0-\xi\right)^{-\beta}(\xi-\xi_0)^{-\alpha}\nu(\xi) \tag{1.131}$$

tenglikning to‘g‘riligini tekshirish qiyin emas.

Endi quyidagi limitni hisoblaymiz:

$$I=\lim_{\varepsilon\to 0}\left[\frac{\alpha+\beta}{\eta-\xi}R+\frac{1}{2}\left(\frac{\partial R}{\partial\xi}-\frac{\partial R}{\partial\eta}\right)\right]\Bigg|_{\eta=\xi+\varepsilon}u\left(\xi,\xi+\varepsilon\right)=$$

$$=\lim_{\varepsilon\to 0}\frac{(\eta-\xi)^{\alpha+\beta}}{2(\eta_0-\xi)^{\beta}(\eta-\xi_0)^{\alpha}}\left[\frac{\beta}{\eta_0-\xi}F(\beta,\alpha,1;\sigma)+\frac{\partial F(\beta,\alpha,1;\sigma)}{\partial\sigma}\frac{\partial\sigma}{\partial\xi}+\right.$$

$$\left.+\frac{\alpha}{\eta-\xi_0}F(\beta,\alpha,1;\sigma)+\frac{\partial F(\beta,\alpha,1;\sigma)}{\partial\sigma}\frac{\partial\sigma}{\partial\eta}\right]\Bigg|_{\eta=\xi+\varepsilon}u(\xi,\xi+\varepsilon)$$

bu yerdan ushbu:

$$1-\sigma=\frac{(\eta_0-\xi_0)(\eta-\xi)}{(\eta_0-\xi)(\eta-\xi_0)},\qquad \sigma_\xi=\frac{\eta-\eta_0}{\eta-\xi_0}\cdot\frac{\xi_0-\eta_0}{(\xi-\eta_0)^2},$$

$$\sigma_\eta = \frac{\xi-\xi_0}{\xi-\eta_0}\cdot\frac{\eta_0-\xi_0}{(\eta-\xi_0)^2}$$

tengliklarni va (1.46), (1.19) formulalarni hisobga olib, ushbu

$$I=\lim_{\varepsilon\to 0}\frac{(\eta-\xi)^{\alpha+\beta}}{2(\eta_0-\xi)^\beta(\eta-\xi_0)^\alpha}\left\{\frac{\beta}{\eta_0-\xi}F(\beta,\alpha,1;\sigma)+\frac{\alpha}{\eta-\xi_0}F(\beta,\alpha,1;\sigma)+\alpha\beta\left(\frac{(\eta_0-\xi)(\eta-\xi_0)}{(\eta_0-\xi_0)(\eta-\xi)}\right)^{\alpha+\beta}\times\right.$$

$$\left.\times\left[\frac{(\eta-\eta_0)(\xi_0-\eta_0)}{(\eta-\xi_0)(\xi-\eta_0)^2}+\frac{(\xi-\xi_0)(\xi_0-\eta_0)}{(\xi-\eta_0)(\eta-\xi_0)^2}\right]F(1-\alpha,1-\beta,2;\sigma)\right\}\Bigg|_{\eta=\xi+\varepsilon} u(\xi,\xi+\varepsilon)$$

(1.132)

tenglikni hosil qilamiz. Oxirgi tenglikda $\varepsilon\to 0$ da limitga o‘tib,

$$I=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{(\eta_0-\xi_0)^{1-\alpha-\beta}}{(\eta_0-\xi)^{1-\alpha}(\xi-\xi_0)^{1-\beta}}\tau(\xi) \qquad (1.133)$$

natijaga kelamiz.

Shunday qilib, (1.130) formuladan (1.131) va (1.133) tengliklarga ko‘ra, ushbu

$$u(\xi_0,\eta_0)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{\xi_0}^{\eta_0}\frac{(\eta_0-\xi_0)^{1-\alpha-\beta}\tau(\xi)d\xi}{(\eta_0-\xi)^{1-\alpha}(\xi-\xi_0)^{1-\beta}}-$$

$$-\left(\frac{4}{m+2}\right)^{\alpha+\beta}\frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)\Gamma(1-\beta)}\int_{\xi_0}^{\eta_0}\frac{\nu(\xi)d\xi}{(\eta_0-\xi)^\beta(\xi-\xi_0)^\alpha} \qquad (1.134)$$

tenglikka ega bo‘lamiz. Bu yerda $\xi=\xi_0+(\eta_0-\xi_0)\dfrac{1+\sigma}{2}$ almashtirish bajarib va eski x, y o‘zgaruvchilarga o‘tib, ushbu

$$u(x,y)=\gamma_1\int_{-2}^{1}\tau\left(x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right)(1-t)^{\alpha-1}(1+t)^{\beta-1}dt+$$

$$+\gamma_2(-y)^{1-\beta_0}\int_{-1}^{1} v\left(x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right)(1-t)^{-\beta}(1+t)^{-\alpha}\,dt \qquad (1.135)$$

formulaga ega bo‘lamiz, bu yerda

$$\gamma_1=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}2^{1-\alpha-\beta},\quad \gamma_2=-\frac{\Gamma(2-\alpha-\beta)2^{\alpha+\beta-1}}{(1-\beta_0)\Gamma(1-\alpha)\Gamma(1-\beta)}.$$

(1.135) formula shakli o‘zgargan Koshi masalasining yechimini beruvchi Darbu formulasi deyiladi.

(1.135) formulaning tuzilishidan ko‘rinib turibdiki, agar $\tau(x)$, $\nu(x)\in C^2((x_1,x_2))$ sinflarga tegishli bo‘lib, ular (x_1,x_2) intervalning chap chegarasi x_1 nuqtada mos ravishda β va $1-\alpha$ dan kichik, o‘ng chegarasi x_2 nuqtada mos ravishda α va $1-\beta$ dan kichik tartibda cheksizlikka aylansa, u holda, $u(x,y)$ funksiya $[x_1,x_2]$ kesma va

$$x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=x_1,\quad x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=x_2$$

xarakteristikalar bilan chegaralangan D sohada ikkinchi tartibli uzluksiz hosilalarga ega bo‘ladi. Bevosita hisoblashlar yordamida (1.135) formula (1.115) tenglamaning yechimi bo‘lishini va bu yechim (1.116), (1.117) boshlang‘ich shartlarni qanoatlantirishini tekshirib ko‘rish qiyin emas. (1.135) formulani hosil qilish usulining o‘zidan (1.115)-(1.117) shakli o‘zgargan Koshi masalasi yechimi yagona (Riman funksiyasi Volterra integral tenglamasinining yechimidan iborat) va u boshlang‘ich shartlarga uzluksiz bog‘liq ekanligi kelib chiqadi.

Agar $\tau(x)$ va $\nu(x)$ funksiyalar (x_1,x_2) intervalda uzluksiz bo‘lsa, (1.135) ifodaga (1.115) tenglamaning umumlashgan yechimi deyiladi. Umumlashgan $u(x,y)$ yechim u yoki bu aniq bir silliqlikka ega bo‘lishi uchun, $\tau(x)$ va $\nu(x)$ funksiyalarning o‘zi ma’lum bir silliqliklarga ega bo‘lishi zarur. Keyinchalik umumlashgan yechimni $A_1B_1C_1$ xarakteristik uchburchakda o‘rganamiz, bu xarakteristik uchburchak $\xi=-1$ xarakteristikaning A_1C_1 kesmasi, $\eta=1$ xarakteristikaning C_1B_1 kesmasi va buzilish chizig‘i $\eta=\xi$ ning ABkesmasi bilan chegaralangan.

2. (1. 115) tenglamaning R_1 sinfga tegishli umumlashgan yechimlari.

Shakli o‘zgargan Koshi masalasi uchun K. I. Babenko [5] tomonidan kiritilgan quyidagi umumlashgan yechimlar sinfini kiritamiz.

Ta’rif. (1.115) tenglamaning (1.135) umumlashgan yechimi R_1 sinfga tegishli deyiladi, agarda $\tau(t)$ funksiya $-1 \le t < 1$ oraliqda $\alpha_1 > 1-\beta$, $\nu(t)$ funksiya esa $-1 \le t < 1$ oraliqda $\alpha_2 > \alpha$ ko‘rsatkich bilan Gyolder shartini qanoatlantirsa.

1.6 Lemma. *Agar (1.115) tenglamaning $u(x,y)$ -umumlashgan yechimi R_1 sinfga tegishli bo‘lsa, u holda u_x va u_y lar ABC uchburchakda uzluksiz, $(-y)^{\beta_0} u_y$ esa $y=0$ buzilish chizig‘igacha uzluksiz va ushbu*

$$\lim_{y\to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} = \nu(x), \quad -1 < x < 1$$

tenglik o‘rinlidir.

Isbot. $\tau(x) \in C^{(0,\alpha_1)}[-1,1]$ va $\nu(x) \in C^{(0,\alpha_2)}[-1,1)$ bo‘lgani uchun ularni quyidagi ko‘rinishda ifodalash mumkin:

$$\begin{aligned} \tau(t) &= \tau(-1) + \int_{-1}^{t} (t-s)^{-\beta+\varepsilon} \varphi(s)ds, \\ \nu(t) &= \nu(-1) + \int_{-1}^{t} (t-s)^{\alpha-1+\varepsilon} \psi(s)ds, \end{aligned} \tag{1.136}$$

bu yerda $\varepsilon > 0$-yetarli kichik son, $\varphi(s)$ va $\psi(s)$ esa $-1 \le s \le 1$ kesmada uzluksiz.

(1.136) ni (1.134) formulaga qo‘yib, ushbu:

$$u(\xi,\eta) = \tau(-1) - \left(\frac{4}{m+2}\right)^{\alpha+\beta} \frac{(\eta-\xi)^{1-\alpha-\beta}}{2(1-\alpha-\beta)} \nu(-1) +$$

$$+\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_{\xi}^{\eta} \frac{(\eta-\xi)^{1-\alpha-\beta}\, dt}{(\eta-t)^{1-\alpha}(t-\xi)^{1-\beta}} \int_{-1}^{t} (t-s)^{-\beta+\varepsilon} \phi(s)ds -$$

$$-\left(\frac{4}{m+2}\right)^{\alpha+\beta} \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)\Gamma(1-\beta)} \int_{\xi}^{\eta} \frac{dt}{(\eta-t)^{\beta}(t-\xi)^{\alpha}} \int_{-1}^{t} (t-s)^{\alpha-1+\varepsilon} \psi(s)ds$$

tenglikka ega bo‘lamiz. Bu yerda integrallash tartibini o‘zgartirib, $u(\xi,\eta)$ni quyidagi ko‘rinishda yozib olamiz.

$$u(\xi,\eta)=\tau\left(-1\right)-\left(\frac{4}{m+2}\right)^{\alpha+\beta}\frac{\left(\eta-\xi\right)^{1-\alpha-\beta}}{2(1-\alpha-\beta)}\nu(-1)+$$

$$+\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{-1}^{\xi}\phi(s)ds\int_{\xi}^{\eta}\frac{\left(\eta-\xi\right)^{1-\alpha-\beta}(t-s)^{-\beta+\varepsilon}dt}{\left(\eta-t\right)^{1-\alpha}(t-\xi)^{1-\beta}}+$$

$$+\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{\xi}^{\eta}\phi(s)ds\int_{s}^{\eta}\frac{(\eta-\xi)^{1-\alpha-\beta}(t-s)^{-\beta+\varepsilon}dt}{(\eta-t)^{1-\alpha}(t-\xi)^{1-\beta}}-$$

$$-\left(\frac{4}{m+2}\right)^{\alpha+\beta}\frac{\Gamma(1-\alpha-\beta)}{2\Gamma\left(1-\alpha\right)\Gamma\left(1-\beta\right)}\int_{-1}^{\xi}\psi\left(s\right)ds\int_{\xi}^{\eta}\frac{\left(t-s\right)^{\alpha-1+\varepsilon}dt}{\left(\eta-t\right)^{\beta}\left(t-\xi\right)^{\alpha}}-$$

$$-\left(\frac{4}{m+2}\right)^{\alpha+\beta}\frac{\Gamma\left(1-\alpha-\beta\right)}{2\Gamma\left(1-\alpha\right)\Gamma\left(1-\beta\right)}\int_{\xi}^{\eta}\psi\left(s\right)ds\int_{s}^{\eta}\frac{\left(t-s\right)^{\alpha-1+\varepsilon}dt}{\left(\eta-t\right)^{\beta}(t-\xi)^{\alpha}} \qquad (1.137)$$

(1.137) ifodaning birinchi va uchinchi integrallarining ichki integrallarida $t=\eta-(\eta-\xi)\sigma$ ko‘rinishda, ikkinchi va to‘rtinchi integrallarning ichki integrallarida $t=\eta-\left(\eta-s\right)\sigma$ shaklda integral o‘zgaruvchilarini almashtirib va gipergeometrik funksiyalarning integral ifodasidan foydalanib, $u(\xi,\eta)$ ni quyidagi ko‘rinishda yozib olamiz:

$$u(\xi,\eta)=\tau(-1)-\left(\frac{4}{m+2}\right)^{\alpha+\beta-1}\frac{(\eta-\xi)^{1-\alpha-\beta}}{1-\beta_0}\nu(-1)+$$

$$+\int_{-1}^{\xi}\phi_{11}(s)(\eta-s)^{-\beta+\varepsilon}F\left(\alpha,\beta-\varepsilon,\alpha+\beta;\frac{\eta-\xi}{\eta-s}\right)ds+$$

$$+\int_{\xi}^{\eta}\phi_{22}(s)(\eta-s)^{\alpha-\beta+\varepsilon}(\eta-\xi)^{-\alpha}F\left(\alpha,1-\beta,1+\alpha-\beta+\varepsilon;\frac{\eta-s}{\eta-\xi}\right)ds+$$

$$+\int_{-1}^{\xi}\psi_{11}(s)(\eta-\xi)^{1-\alpha-\beta}(\eta-s)^{\alpha-1+\varepsilon}F\left(1-\beta,1-\alpha-\varepsilon,2-\alpha-\beta;\frac{\eta-\xi}{\eta-s}\right)ds+$$

$$+\int_{\xi}^{\eta}\psi_{22}(s)(\eta-\xi)^{-\alpha}(\eta-s)^{\alpha-\beta+\varepsilon}\times F\left(1-\beta,\alpha,1-\beta+\alpha+\varepsilon;\frac{\eta-s}{\eta-\xi}\right)ds \quad (1.138)$$

bu yerda

$$\varphi_{11}(s)=\varphi(s),$$

$$\varphi_{22}(s)=\frac{\Gamma(\alpha+\beta)\Gamma(1-\beta+\varepsilon)}{\Gamma(\beta)\Gamma(1+\alpha-\beta+\varepsilon)}\varphi(s),$$

$$\psi_{11}(s)=-\left(\frac{4}{m+2}\right)^{\alpha+\beta}\frac{\Gamma(1-\alpha-\beta)}{2\Gamma(2-\alpha-\beta)}\psi(s),$$

$$\psi_{22}(s)=-\left(\frac{4}{m+2}\right)^{\alpha+\beta}\frac{\Gamma(1-\alpha-\beta)\Gamma(\alpha+\varepsilon)}{2\Gamma(1-\alpha)\Gamma(1-\beta+\alpha+\varepsilon)}\psi(s) \quad (1.139)$$

(1.138) tenglikning o‘ng tomonidagi uchinchi integral ostidagi ifodani (1.21) Bols formulasiga ko‘ra quyidagi ko‘rinishda yozib olamiz:

$$(1-z)^{1-\alpha-\beta}F(1-\beta,1-\alpha-\varepsilon,2-\alpha-\beta;1-z)=\frac{\Gamma(\alpha)\Gamma(\beta-\varepsilon)}{\Gamma(1-\varepsilon)\Gamma(\alpha+\beta-1)}$$

$$\left[F(\beta-\varepsilon,\alpha,1-\varepsilon;z)-\frac{\Gamma(1-\varepsilon)\Gamma(1-\alpha-\beta)}{\Gamma(1-\beta)\Gamma(1-\alpha-\varepsilon)}F(\beta-\varepsilon,\alpha,\alpha+\beta;1-z)\right] \quad (1.140)$$

bu yerda $1-z=\dfrac{\eta-\xi}{\eta-s}$. Endi (1.138) formula (1.140) tenglikka ko‘ra, ushbu:

$$u(\xi,\eta)=\tau(-1)-\left(\frac{4}{m+2}\right)^{\alpha+\beta-1}\frac{(\eta-\xi)^{1-\alpha-\beta}}{1-\beta_0}\nu(-1)+$$

$$+\int_{-1}^{\xi}\phi_1(s)(\eta-s)^{-\beta+\varepsilon}F\left(\alpha,\beta-\varepsilon,\alpha+\beta;\frac{\eta-\xi}{\eta-s}\right)ds+$$

$$+\int_{\xi}^{\eta}\phi_2(s)(\eta-\xi)^{-\alpha}(\eta-s)^{\alpha-\beta+\varepsilon}F\left(\alpha,1-\beta,1+\alpha-\beta+\varepsilon:\frac{\eta-s}{\eta-\xi}\right)ds+$$

$$+\int_{-1}^{\xi}\psi_1(s)(\eta-s)^{-\beta+\varepsilon}F\left(\beta-\varepsilon,\alpha,1-\varepsilon;\frac{\xi-s}{\eta-s}\right)ds, \quad (1.141)$$

ko‘rinishni oladi, bu yerda

$$\varphi_1(s)=\varphi_{11}(s)-\frac{\Gamma(1-\alpha-\varepsilon)\Gamma(\beta-\varepsilon)\Gamma(\alpha)}{\Gamma(1-\beta)\Gamma(1-\alpha-\beta)\Gamma(\alpha+\beta-1)}\psi_{11}(s),$$

$$\varphi_2(s)=\varphi_{22}(s)+\psi_{22}(s),$$

$$\psi_1(s)=\frac{\Gamma(\beta-\varepsilon)\Gamma(\alpha)}{\Gamma(1-\varepsilon)\Gamma(\alpha+\beta-1)}\psi_{11}(s)$$

(1.140) formuladan ko‘rinib turibdiki, u_ξ va u_η hosilalar ABC uchburchakda mavjud va ular uchun

$$\frac{\partial u}{\partial \xi}=O\left((\eta-\xi)^{-\alpha-\beta}\right),\qquad \frac{\partial u}{\partial \eta}=O\left((\eta-\xi)^{-\alpha-\beta}\right)$$

tengliklar o‘rinli.

(1.141) formuladagi birinchi va ikkinchi integrallardan olingan birinchi tartibli hosilalar $c(\eta-\zeta)^{-\beta+\varepsilon}$ miqdor bilan chegaralangan. Yuqorida keltirilgan mulohazalarga asosan, ushbu:

$$\left(\frac{m+2}{4}\right)^{\alpha+\beta}(\eta-\xi)^{\alpha+\beta}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)=$$

$$=\nu(-1)+2\left(\frac{m+2}{4}\right)^{\alpha+\beta}\frac{\alpha(\beta-\varepsilon)}{1-\varepsilon}\int_{-1}^{\xi}\psi_1(s)(\eta-s)^{\alpha-1+\varepsilon}\times \tag{1.142}$$

$$\times F\left(1-\beta,1-\alpha-\varepsilon,2-\varepsilon;\frac{\xi-s}{\eta-s}\right)ds+O\left((\eta-\xi)^{\alpha+\varepsilon}\right)$$

tenglikka ega bo‘lamiz. Shunday qilib, (1.142) tenglikda ifodalarni hisobga olgan holda $\eta-\xi\to 0$ da limitga o‘tib, ushbu tenglikka kelamiz:

$$\lim_{\eta-\xi\to 0}\left(\frac{m+2}{4}\right)^{\alpha+\beta}(\eta-\xi)^{\alpha+\beta}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)=$$

$$=\nu(-1)+\int_{-1}^{\xi}\psi(s)(\xi-s)^{\alpha-1+\varepsilon}\,ds=\nu(\xi) \tag{1.143}$$

1.6 Lemma isbot bo‘ldi.

V . $P(\alpha_0,\beta_0)\in A_0C_0$ bo‘lsin. Bu holda $\alpha=0$, $0<\beta<1$ va Riman funksiyasi quyidagi ko‘rinishda bo‘ladi:

$$R(\xi,\eta;\xi_0,\eta_0)=\left(\frac{\eta-\xi}{\eta_0-\xi}\right)^{\beta} \tag{1.144}$$

(1.144) ga asosan, (1.130) formula ushbu ko‘rinishda bo‘ladi:

$$u(\xi_0,\eta_0)=u(\eta_0-\varepsilon,\eta_0)+\frac{\varepsilon^\beta}{2(\eta_0-\xi_0)^\beta}\left(u(\xi_0,\xi_0+\varepsilon)-u(\eta_0-\varepsilon,\eta_0)\right)+$$

$$+\frac{\beta\varepsilon^\beta}{2}\int_{\xi_0}^{\eta_0-\varepsilon}\frac{u(\xi,\xi+\varepsilon)-u(\eta_0-\varepsilon,\eta_0)}{(\eta_0-\xi)^{\beta+1}}d\xi-\frac{1}{2}\left(\frac{4}{m+2}\right)^\beta\int_{\xi_0}^{\eta_0-\varepsilon}\frac{v(\xi)d\xi}{(\eta_0-\xi)^\beta} \quad (1.145)$$

Endi (1.145) da $\varepsilon\to 0$ da limitga o‘tib, keyin dastlabki x, y o‘zgaruvchilarga qaytib, ushbu:

$$u(x,y)=\tau\left(x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}\right)-\frac{2^\beta(-y)^{1-\beta_0}}{m+2}\int_{-1}^{1}v\left(x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right)(1-t)^{-\beta}dt \quad (1.146)$$

yechimga ega bo‘lamiz.

S. $P(\alpha_0,\beta_0)\in B_0C_0$ bo‘lsin. Bu holda $\beta=0$, $0<\alpha<1$ va Riman funksiyasi ushbu:

$$R(\xi,\eta;\xi_0,\eta_0)=\left(\frac{\eta-\xi}{\eta-\xi_0}\right)^\alpha \quad (1.147)$$

ko‘rinishda bo‘ladi.

(1.147) tenglikka asosan, (1.130) formulani quyidagi ko‘rinishda yozib olamiz:

$$u(\xi_0,\eta_0)=\frac{u(\xi_0,\xi_0+\varepsilon)+u(\xi_0+\varepsilon,\xi_0)}{2}+$$

$$+\frac{\varepsilon^\alpha}{2(\eta_0-\xi_0)^\alpha}\left(u(\eta_0-\varepsilon,\eta_0)-u(\xi_0+\varepsilon,\xi_0)\right)+ \quad (1.148)$$

$$+\frac{\varepsilon^\alpha}{2}\int_{\xi_0}^{\eta_0-\varepsilon}\frac{u(\xi,\xi+\varepsilon)-u(\xi_0+\varepsilon,\xi_0)}{(\xi+\varepsilon-\xi_0)^{1+\alpha}}d\xi-\frac{1}{2}\left(\frac{4}{m+2}\right)^\alpha\int_{\xi_0}^{\eta_0-\varepsilon}\frac{v(\xi)d\xi}{(\xi+\varepsilon-\xi_0)^\alpha}$$

Endi (1.148) da $\varepsilon \to 0$ da limitga o‘tib va eski x, y o‘zgaruvchilarga qaytib, ushbu:

$$u(x,y)=\tau\left(x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}\right)-\frac{2^{\alpha}(-y)^{1-\beta_0}}{m+2}\int_{-1}^{1}\nu\left(x-\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right)(1-t)^{-\alpha}dt \quad (1.149)$$

yechimga ega bo‘lamiz.

D. $P(\alpha_0,\beta_0)=C_0(0,-m/2)$ bo‘lsin. Bu holda $\alpha=\beta=0$ va $R(\xi,\eta;\xi_0,\eta_0)\equiv 1$.
Bu yerdan, (1.130) Riman formulasiga ko‘ra, ushbu:

$$u(\xi_0,\eta_0)=\frac{u(\xi_0,\xi_0+\varepsilon)+u(\eta_0-\varepsilon,\eta_0)}{2}+\frac{1}{2}\int_{\xi_0}^{\eta_0}\left(\frac{\partial u}{\partial \eta}-\frac{\partial u}{\partial \xi}\right)d\xi \quad (1.150)$$

tenglikka kelamiz. (1.150) da $\varepsilon \to 0$ da limitga o‘tib va eski x, y o‘zgaruvchilarga qaytib, ushbu:

$$u(x,y)=\frac{\tau\left(x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}\right)+\tau\left(x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}\right)}{2}-$$
$$-\frac{(-y)^{\frac{m+2}{2}}}{m+2}\int_{-1}^{1}\nu\left(x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right)dt \quad (1.151)$$

Dalamber formulasini hosil qilamiz.

II BOB. ELLIPTIK TURDAGI BUZILUVCHAN TENGLAMALARNING BIR SINFI UCHUN DIRIXLE VA SHAKLI O‘ZGARGAN XOLMGREN MASALALARI

Bu bobda singulyar koeffitsientli, buziluvchan elliptik turdagi

$$E(u) = y^m u_{xx} + u_{yy} + (\beta_0 / y) u_y = 0, \quad y > 0$$

tenglama uchun Dirixle va shakli o‘zgargan Xolmgren masalalari o‘rganiladi.

1-§. Dirixle va shakli o‘zgargan Xolgmren masalalarining qo‘yilishi va yechimning yagonaligi.

$z = x + iy$ kompleks tekisligining yuqori $\operatorname{Im} z > 0$ yarim tekisligida

$$y^m u_{xx} + u_{yy} + (\beta_0 / y) u_y = 0, \qquad (2.1)$$

tenglamani o‘rganamiz, bu yerda m, β_0–o‘zgarmas sonlar bo‘lib, $m > 0$, $-(m/2) < \beta_0 < 1$ shartlarni qanoatlantiradi.

Ω – chekli bir bog‘lamli soha bo‘lib, uchlari $A(-a,0)$ va $B(a,0)$ nuqtalarda bo‘lgan va $y > 0$ yarim tekislikda yotuvchi silliq Jordan chizig‘i $\Gamma : x = x(s), y = y(s)$ bu yerda s– parametr $\overset{\cup}{MB}$–yoy uzunligi, hamda $y = 0$ o‘qining AB kesmasi bilan chegaralangan bo‘lsin. (2.1) tenglama uchun Ω sohada Dirixle va shakli o‘zgargan Xolmgren masalalarini o‘rganamiz.

Dirixle masalasi. Ω sohada (2.1) tenglamaning ushbu

$$u\big|_{\Gamma} = \varphi(s) \quad 0 \le s \le l\,;\ u(x,0) = \tau(x),\ x \in I, \qquad (2.2)$$

shartlarni qanoatlantiruvchi regulyar yechimi $u(x,y) \in C(\overline{\Omega}) \cap C^2(\Omega)$ topilsin, bu yerda s - Γ chiziqning BM yoyi uzunligi,

l–butun Γ chiziq yoyi uzunligi: $\varphi(s)$ va $\tau(x)$–berilgan uzluksiz funksiyalar, shu bilan birga $\tau(-a) = \varphi(l), \tau(a) = \varphi(0)$, $I = (-a,a), y = 0$ o‘qining intervali.

Ekstremum prinsipi Ω sohada (2.1) tenglamaning $u(x,y)$ regulyaryechimi hech bir $(x,y) \in \Omega$ nuqtada o‘zining musbat maksimumiga va manfiy minimumiga erishmaydi.

Isboti. Ushbu

$$\upsilon(x,y)=u(x,y)/A(y) \tag{2.3}$$

funksiyani qaraymiz, bu yerda

$$A(y)=e^{d^{1-\beta_0}}-\varepsilon\, e^{y^{1-\beta_0}},$$

d – bu Ω soha diametri, $0<\varepsilon<1$. Bevosita hisoblashlar yordamida

$$E(u)=A(y)E_1(\upsilon)$$

tenglikning to‘g‘riligiga ishonch hosil qilish qiyin emas, bu yerda

$$E_1(\upsilon)=y^m\upsilon_{xx}+\upsilon_{yy}+\frac{1}{y}\left(\beta_0+2yA_y\right)\upsilon_y+\frac{1}{A}\left(\frac{\beta_0}{y}A_y+A_{yy}\right)\upsilon, \tag{2.4}$$

$$A_y=-\varepsilon\left(1-\beta_0\right)e^{y^{1-\beta_0}}y^{-\beta_0},$$

$$A_{yy}=-\varepsilon\left(1-\beta_0\right)^2e^{y^{1-\beta_0}}y^{-2\beta_0}+\varepsilon\beta_0\left(1-\beta_0\right)e^{y^{1-\beta_0}}y^{-\beta_0-1},$$

$$\frac{1}{A}\left(\frac{\beta_0}{y}A_y+A_{yy}\right)=-\frac{\varepsilon}{A}\left(1-\beta_0\right)^2e^{y^{1-\beta_0}}y^{-2\beta_0}<0. \tag{2.5}$$

(2.5) tengsizlikka asosan, (2.4) tenglama yechimi $\vartheta(x,y)$ Ω soha ichidagi hech bir (x_0,y_0) nuqtada o‘zining musbat maksimumiga erishmaydi. Haqiqatdan ham, teskarisini faraz qilaylik, (x_0,y_0) nuqtada $\vartheta(x,y)$ funksiya o‘zining musbat maksimumiga erishsin, u holda bu nuqtada

$$\frac{\partial\vartheta(x_0,y_0)}{\partial x}=0,\ \frac{\partial\vartheta(x_0,y_0)}{\partial y}=0,\ \frac{\partial^2\vartheta(x_0,y_0)}{\partial x^2}\le 0,\ \frac{\partial^2\vartheta(x_0,y_0)}{\partial y^2}\le 0$$

bo‘lgani uchun, (2.3) dan $E_1(\vartheta)<0$. Bu esa $E_1(\vartheta)=0$ tenglikka ziddir. Aynan shu mulohazalarni takrorlab, $\vartheta(x,y)$ funksiya Ω sohaning hech bir ichki nuqtasida o‘zining manfiy minimumga erishmasligini ko‘rsatish mumkin.

Shunday qilib, (2.3) ga asosan, (2.1) tenglamaning regulyar yechimi $u(x,y)$ o‘zining musbat maksimumi va manfiy minimumini Ω sohaning ichki nuqtalarida qabul qilmaydi.

2.1-teorema. Ω *sohada (2.1) tenglama uchun qo'yilgan Dirixle masalasining yechimi mavjud bo'lsa, u yagonadir.*

Isboti. Faraz qilaylik, qo'yilgan masala ikkita u_1 va u_2 yechimlarga ega bo'lsin, u holda berilgan tenglama va chegaraviy shartlar chiziqli bo'lgani uchun $w = u_1 - u_2$ funksiya(2.1) tenglamani va bir jinsli

$$w|_\Gamma = (u_1 - u_2)|_\Gamma = \varphi(s) - \varphi(s) = 0;$$

$$w(x,0) = u_1(x,0) - u_2(x,0) = \tau(x) - \tau(x) = 0, \qquad (2.6)$$

shartlarni qanoatlantiradi. Ekstremum prinsipiga ko'ra, $\overline{\Omega}$ sohada uzluksiz $w(x,y)$ funksiya o'zining ekstremumlarini faqat $\partial\overline{\Omega} = \Gamma \cup AB$ da qabul qiladi, ya'ni

$$0 = \min_{(x,y)\in\partial\overline{\Omega}} w(x,y) \le \underset{(x,y)\in\overline{\Omega}}{w}(x,y) \le \max_{(x,y)\in\partial\overline{\Omega}} w(x,y) = 0.$$

Bundan esa $w(x,y) \equiv 0$, $(x,y) \in \overline{\Omega}$.

2.1- teorema isbot bo'ldi.

Shakli o'zgargan Xolmgren masalasi. Ω sohada (2.1) tenglamaning ushbu

$$u|_\Gamma = \varphi(s) \quad 0 \le s \le l; \ \lim_{y\to+0} y^{\beta_0} \frac{\partial u}{\partial y} = \nu(x), \ x \in I, \qquad (2.7)$$

shartlarni qanoatlantiruvchi yechimi $u(x,y) \in C(\overline{\Omega}) \cap C^2(\Omega)$ topilsin, bu yerda $\varphi(s)$ funksiya $0 \le s \le l$ da uzluksiz, $\nu(x)$ funksiya esa I intervalda uzluksiz bo'lib, bu intervalning chegaraviy nuqtalarida $1-2\beta$ dan kichik tartibda cheksizlikka intilishi mumkin, $\beta = (m + 2\beta_0)/2(m+2)$.

2.1-lemma. *Agar* Ω *sohada* $u(x,y)$ *funksiya:*

1) $u(x,y) \in C(\overline{\Omega}) \cap C^2(\Omega)$ $E(u) \ge 0 (\le 0)$ *shartlarni qanoatlantirsa va* Ω *sohada o'zining eng katta musbat (eng kichik manfiy) qiymatini* $(x_0, 0)$, $x_0 \in I$ *nuqtada qabul qilsa,*

2) $u(x,y)$ *ning* Γ *chiziqdagi qiymati* $u(x_0, 0)$ *qiymatdan kichik (katta) bo'lsa, u holda*

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y}<0, \quad (>0), \tag{2.8}$$

tengsizlik, (bu limitni mavjud bo'lishi sharti bilan) o'rinlidir.

Isboti. Musbat maksimum holini o'rganib chiqamiz. Bevosita hosila ta'rifidan,

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial u(x_0,y)}{\partial y}>0$$

tengsizlikning bajarilishi mumkin emas. Faraz qilaylik,

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial u(x_0,y)}{\partial y}=0 \tag{2.9}$$

bo'lsin. d orqali Ω sohaning diametrini belgilaymiz. Umumiylikni buzmasdan, $u(x_0,0)=1$ deb olishimiz mumkin. Lemma shartiga ko'ra, $\max\limits_{(x,y)\in\Gamma} u(x,y)\le 1-\varepsilon$, bu yerda $0<\varepsilon<1$. Ushbu

$$\upsilon(x,y)=u(x,y)/A(y),$$

funksiyani kiritamiz [33], bu yerda

$$A(y)=e^{d^{1-\beta_0}}-\varepsilon^{y^{1-\beta_0}}.$$

Γ chiziqda

$$\upsilon(x,y)\le\frac{1-\varepsilon}{e^{d^{1-\beta_0}}-\varepsilon e^{y^{1-\beta_0}}}\le\frac{1-\varepsilon}{e^{d^{1-\beta_0}}(1-\varepsilon)}<\frac{1}{e^{d^{1-\beta_0}}-\varepsilon},$$

$[-1,1]$ kesmada esa

$$\upsilon(x,0)\le\frac{1}{e^{d^{1-\beta_0}}-\varepsilon}, \quad \upsilon(x_0,0)=\frac{1}{e^{d^{1-\beta_0}}-\varepsilon}.$$

Ushbu

$$E(u)=A(y)E_1(\upsilon), \tag{2.10}$$

tenglikning to‘g‘riligiga ishonch hosil qilish qiyin emas, bu yerda $E_1(\vartheta)$ (2.4) tenglik bilan aniqlanuvchi operator. Lemma shartiga ko‘ra, (2.10) tenglikdan $E_1(\vartheta)\geq 0$ tengsizlik kelib chiqadi. Bundan tashqari, (2.9) tenglikka ko‘ra,

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial\upsilon(x_0,y)}{\partial y}=\frac{\varepsilon(1-\beta_0)}{\left(e^{d^{1-\beta_0}}-\varepsilon\right)^2}>0 .$$

Bundan esa $(x_0,0)$ nuqtaning shunday kichik atrofi borki $(y>0)$, bu atrofda $\frac{\partial\upsilon}{\partial y}>0$ ya’ni $\upsilon(x,y)$ funksiya bu atrofda $x=x_0$ chizig‘ida o‘suvchi bo‘ladi va o‘zining eng katta qiymatini soha ichida qabul qiladi, buning esa (2.5) tengsizlikka ko‘ra bo‘lishi mumkin emas. Yuqoridagi mulohazalarni takrorlab, manfiy minimum holini ham o‘rganish mumkin.

2.1-lemma isbot bo‘ldi.

2.2-teorema.*Shakli o‘zgargan Xolmgren masalasi (2.7) shartlarga mos bir jinsli shartlarda faqat aynan nolga teng bo‘lgan yechimga ega.*

Isboti. (2.1) tenglama uchun ekstremum prinsipi va 2.1-lemmaga ko‘ra, shakli o‘zgargan bir jinsli Xolmgren masalasining $u(x,y)\in C(\overline{\Omega})$ yechimi o‘zining ekstremumlarini Γ da qabul qiladi, ya’ni

$$0=\min_{(x,y)\in\Gamma}u(x,y)\leq u(x,y)\leq\max_{(x,y)\in\Gamma}u(x,y)=0 .$$

Bu yerdan

$$u(x,y)\equiv 0\,,\ (x,y)\in\overline{\Omega}\,.$$

2.2-teorema isbot bo‘ldi.

2-§. Potensiallar nazariyasi.

Elliptik turdagi

$$y^m u_{xx}+u_{yy}=0,\quad m>0 \qquad (2.11)$$

buziluvchan tenglama uchun potensiallar nazariyasi S. Gellerstedt tomonidan qurilgan va bu nazariya asosida elliptik turdagi tenglamalar uchun qo‘yilgan asosiy chegaraviy masalalar yechimlarining integral ifodasi aralash turdagi tenglamalar uchun chegaraviy masalalarni o‘rganishda juda qulay hisoblanadi. S. Gellerstedtning potensiallar nazariyasi aralash turdagi tenglamalar nazariyasining yanada rivojlanishida juda katta ahamiyatga ega.

(2.1) tenglama uchun potensiallar nazariyasi (2.11) tenglama uchun qurilgan potensiallar nazariyasi bilan katta bog'liqlikda bo'lsada, ayrim muhim farqlarga ega.

(2.1) tenglama bilan birgalikda unga qo'shma

$$E^*(\upsilon)= y^m\upsilon_{xx} +\upsilon_{yy} -(\beta_0 / y)\upsilon_y +(\beta_0 / y^2)\upsilon = 0 ,\quad(2.12)$$

tenglamani ham o'rganamiz.

Ushbu

$$E^*\left(y^{\beta_0}u\right)= y^{\beta_0}E(u), \quad (2.13)$$

munosabatni bevosita tekshirib ko'rish qiyin emas.

1. **Grin formulasi**. Ushbu ayniyatni o'rganamiz:

$$uE^*(\upsilon)-\upsilon E(u)=\frac{\partial}{\partial x}\left[y^m\left(u\upsilon_x-\upsilon u_x\right)\right]+\frac{\partial}{\partial y}\left[u\upsilon_y-\upsilon u_y-\frac{\beta_0}{y}u\upsilon\right].$$

Bu ayniyatning ikkala tomonini Ω soha bo'yicha integrallab va Gauss-Ostrogradskiy formulasini qo'llab [33], quyidagi Grin formulasini hosil qilamiz:

$$\iint_{\Omega}\left[uE^*(\upsilon)-\upsilon E(u)\right]dxdy=-\int_{\gamma}\left(u\upsilon_y-\upsilon u_y-\frac{\beta_0}{y}u\upsilon\right)dx+y^m\left(u\upsilon_x-\upsilon u_x\right)dy,$$

bu yerda γ - kontur Ω soha chegarasi.

Agar $u(x,y)$ va $\upsilon(x,y)$ funksiyalar mos ravishda (2.1) va (2.12) tenglamaning yechimi bo'lsa, u holda oxirgi formuladan ushbu tenglikni hosil qilamiz:

$$\int_{\gamma}\left\{uA_s[\upsilon]-\upsilon A_s[u]+\frac{\beta_0}{y}u\upsilon\frac{dx}{ds}\right\}ds=0, \quad (2.14)$$

Bu yerda $A_s[\ ] = y^m \dfrac{dy}{ds}\dfrac{\partial}{\partial x} - \dfrac{dx}{ds}\dfrac{\partial}{\partial y}$ –konormal hosila, Γ chiziqda s yoy uzunligining sanoq boshi $B(1,0)$ nuqta hisoblanadi. Shuning uchun $\dfrac{dy}{ds} = \cos(n,x)$, $\dfrac{dx}{ds} = -\cos(n,y)$, $\vec{n} - \gamma$ chiziqqa o‘tkazilgan tashqi normal. (2.14) formulada $\upsilon = y^{\beta_0}$ deb hisoblab, ushbu

$$\int\limits_{\gamma} y^{\beta_0} A_s\left[u\right] ds = 0 \tag{2.15}$$

tenglikka kelamiz, ya’ni (2.1) tenglama yechimining konormal hosilasini y^{β_0} ga ko‘paytmasidan γ kontur bo‘yicha olingan integral nolga tengdir.

Endi (2.14) formulada $\upsilon = y^{\beta_0}$, u ni u^2 ga almashtirib, ushbu:

$$\iint\limits_{\Omega} y^{\beta_0}\left[y^m \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right] dx\,dy = \int\limits_{\gamma} y^{\beta_0} u\, A_s[u] ds \tag{2.16}$$

formulaga ega bo‘lamiz, bu yerda $u(x,y)$ (2.1) tenglamaning yechimi.

2. (2.1) tenglamaning fundamental yechimlari. (2.1) tenglamaning yechimini

$$u = \left(r_1^2\right)^{-\beta} \omega(\sigma) \tag{2.17}$$

ko‘rinishda izlaymiz, bu yerda

$$\sigma = \frac{r^2}{r_1^2}, \quad \beta = \frac{m + 2\beta_0}{2(m+2)},$$

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\} = \left(x - x_0\right)^2 + \frac{4}{(m+2)^2}\left(y^{\frac{m+2}{2}} \mp y_0^{\frac{m+2}{2}} \right)^2,$$

$\omega(\sigma)$-noma'lum funksiya (2.17) ni (2.1) tenglamaga qo'yib va $\omega(\sigma)$ ga nisbatan ba'zi bir hisoblashlarni bajarib, ushbu

$$E\left[\left(r_1^2\right)^{-\beta}\omega(\sigma)\right]=\sigma(1-\sigma)\frac{d^2\omega}{d\sigma^2}+\left[1-(1+2\beta)\sigma\right]\frac{d\omega}{d\sigma}-\beta^2\omega=0$$

Gauss tenglamasiga kelamiz.

Bu tenglama $\sigma=1$ nuqta atrofida quyidagi ikkita chiziqli erkli yechimga ega:

$$\omega_1(\sigma)=F(\beta,\beta,2\beta;1-\sigma)$$

$$\omega_2(\sigma)=(1-\sigma)^{1-2\beta}F(1-\beta,1-\beta,2-2\beta;1-\sigma) \quad (2.18)$$

(2.18) ni (2.17) tenglikka qo'yib, ushbu:

$$q_1(x,y;x_0,y_0)=k_1\left(r_1^2\right)^{-\beta}F(\beta,\beta,2\beta;1-\sigma)$$

$$q_2(x,y,x_0,y_0)=k_2\left(\frac{4}{m+2}\right)^{4\beta-2}\left(r_1^2\right)^{-\beta}(1-\sigma)^{1-2\beta}\times F(1-\beta,1-\beta,2-2\beta;1-\sigma) \quad (2.19)$$

yechimlarga kelamiz, bu yerda

$$k_1=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2\beta}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)},\quad k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)}. \quad (2.20)$$

Bu funksiyalar (x,y) o'zgaruvchilarga nisbatan (2.1) tenglamaning yechimidan iborat, shu bilan birga yaxshi ma'lum [81]

$$F\left(a,b,a+b;1-\sigma\right)=-\frac{\Gamma\left(a+b\right)}{\Gamma\left(a\right)\Gamma\left(b\right)}F\left(a,b,1;\sigma\right)\ln\sigma+$$

$$+\frac{\Gamma\left(a+b\right)}{\Gamma^{2}\left(a\right)\Gamma^{2}\left(b\right)}\sum_{k=0}^{\infty}\frac{\Gamma\left(a+k\right)\Gamma\left(b+k\right)}{\left(k!\right)^{2}}\left[2\frac{\Gamma'\left(1+k\right)}{\Gamma\left(1+k\right)}-\right. \tag{2.21}$$

$$\left.-\frac{\Gamma'\left(a+k\right)}{\Gamma\left(a+k\right)}-\frac{\Gamma'\left(b+k\right)}{\Gamma\left(b+k\right)}\right]\sigma^{k}$$

formulaga ko‘ra bu yechimlar $r\to 0$ ga, ya’ni $\sigma\to 0$ ga intilganda logarifmik maxsuslikka ega. Demak, (2.19) yechimlar (2.1) tenglamaning fundamental yechimlari ekan.

Bevosita hisoblashlar yordamida (2.19) formulalardan

$$\lim_{y\to 0}y^{\beta_0}\frac{\partial q_1\left(x,y;x_0,y_0\right)}{\partial y}=0, \tag{2.22}$$

$$q_2\left(x,0;x_0,y_0\right)=0, \tag{2.23}$$

tengliklarni to‘g‘ri ekanligini ko‘rsatish mumkin.

3. Ikkilangan qatlam potensiali. $W^{(1)}(x,y)$ Γ chiziqning parametrik tenglamasi $x=x(s)$, $y=y(s)$ bo‘lsin, bu yerda s $B(a,0)$-hisob boshidan $M(x(s),y(s))\in\Gamma$ nuqtagacha bo‘lgan chiziqning yoy uzunligi, Γ chiziq quyidagi shartlarni qanoatlantirsin:

1) $x(s)$, $y(s)$ funksiyalar $[0,l]$ kesmada birinchi tartibliuzluksiz hosilaga ega va bu hosilalar bir vaqtda nolga aylanmaydi.

2) $x''(s)$, $y''(s)$ hosilalar $[0,l]$ da Gyolder shartini qanoatlantiradi.

3) Γ chiziqning A va B nuqtalari atrofida

$$\left|\frac{dx}{ds}\right|\le cy^{m+1}(s), \tag{2.24}$$

shart bajariladi, bu yerda c – o‘zgarmas son, Γ chiziqning o‘zgaruvchi nuqtalarining koordinatalarini (ξ,η) orqali belgilaymiz.

Ushbu

$$W^{(1)}(x,y)=\int_0^l \mu_1(t)\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x,y)\right]dt\,, \qquad (2.25)$$

integralni o‘rganamiz, bu yerda $\mu_1(t)\in C[0,l]$,

$$A_t=[q_1(\xi,\eta;x,y)]=\eta^m\frac{d\eta}{dt}\cdot\frac{\partial q_1}{\partial\xi}-\frac{d\xi}{dt}\cdot\frac{\partial q_1}{\partial\eta}.$$

(2.25) tenglama bilan aniqlangan integralni ikki qatlam potensiali deb ataymiz. Ravshanki, $W^{(1)}(x,y)$-funksiya $y>0$ yarim tekislikning, Γ chiziq hamda Ox o‘qi nuqtalarini o‘z ichiga olmagan ixtiyoriy sohada (2.1) tenglamaning regulyar yechimi bo‘ladi.

2.2-lemma. $\mu_1(t)\equiv 1$ *bo‘lganda ikkilangan qatlam potensiali* $W^{(1)}(x,y)=W_1^{(1)}(x,y)$ *uchun ushbu:*

$$W_1^{(1)}(x,y)=\begin{cases}0\ , \text{ agar } (x,y) \text{ nuqta } o\chi \text{ o'qidan} \\ \quad \text{yuqorida joylashib, } (x,y)\notin\overline{\Omega} \text{ bo'lsa;} \\ -1\ , \text{ agar } (x,y)\in\Omega\cup(-a,a) \text{ bo'lsa;} \\ -\dfrac{1}{2}\ , \text{ agar } (x,y)\in\Gamma\cup\{(-a,0);(a,0)\} \text{ bo'lsa.}\end{cases} \qquad (2.26)$$

munosabatlar o‘rinlidir.

Isboti.

a) (x,y) nuqta Ω sohadan tashqaridan hamda Ox o‘qidan yuqorida joylashgan bo‘lsin (2.15) va (2.22) tengliklarga asosan

$$W_1^{(1)}(x,y)=\int_0^l \eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,y\right)\right]dt+$$

$$+\int_{-1}^{1}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,y\right)\right]\Big|_{\eta=0}dt=$$

$$=\int_{\partial\Omega}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,y\right)\right]dt=0\,,\ (x,y)\notin\Omega\,,\ y>0$$

b) (x,y) nuqta Ω soha ichida yotsin. Ω sohadan markazi (x,y) nuqtada bo‘lgan va radiusi ρ ga teng bo‘lgan doirani ajratib olamiz va Ω ning qolgan qismini Ω_ρ orqali belgilaymiz. C_ρ orqali ajratib olingan doiraning aylanasini belgilaymiz.

Ω_ρ sohada $q_1(\xi(t),\eta(t);x,y)$ funksiya (2.1) tenglamaning regulyar yechimidan iborat bo‘ladi. (2.15) tenglikka ko‘ra:

$$\int_{\partial\Omega_\rho}\eta^{\beta_0}A_t\left[q_1(\xi(t),\eta(t);x,y)\right]dt=0,$$

bu yerda $\partial\Omega_\rho=\Gamma\cup AB\cup C_p^-$, bu yerda C_ρ^-–integrallash yo‘nalishi soat strelkasi harakati bo‘yicha ekanligini anglatadi. Bu yerdan (2.22) tenglikni hisobga olib, ushbu:

$$\int_\Gamma\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x,y)\right]dt=\int_{C_\rho}\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x,y)\right]dt,$$

tenglikni hosil qilamiz, demak

$$W_1^{(1)}(x,y)=\int_{C_\rho}\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x,y)\right]dt. \tag{2.27}$$

Bevosita hisoblashlar yordamida ushbu:

$$A_t\left[q_1\left(\xi(t),\eta(t);x,y\right)\right]=-\frac{\beta k_1}{r_1^{2\beta}}F\left(\beta,\beta-1,2\beta;1-\sigma\right)A_t\left[\ln r^2\right]+$$

$$+\frac{4\beta k_1}{m+2}\frac{y^{\frac{m+2}{2}}\eta^{\frac{m}{2}}}{r_1^{2(1+\beta)}}F\left(\beta+1,\beta,2\beta+1;1-\sigma\right)\frac{d\xi(t)}{dt}=Q(s;x,y)+P(s;x,y) \tag{2.28}$$

tenglikni to‘g‘ri ekanligini tekshirib ko‘rish qiyin emas.

Konormal hosilaning bu qiymatini (2.27) formulaga qo‘yib, quyidagi tenglikni hosil qilamiz:

$$W_1^{(1)}(x,y)=\int_{C_\rho}\eta^{\beta_0}(t)\left\{-\frac{\beta k_1}{r_1^{2\beta}}F\left(\beta,\beta-1,2\beta;1-\sigma\right)\times\right.$$

$$\left.\times A_t\left[\ln r^2\right]+\frac{4\beta k_1}{m+2}\frac{y^{\frac{m+2}{2}}\eta^{\frac{m}{2}}}{r_1^{2(\beta+1)}}F\left(\beta+1,\beta,2\beta+1;1-\sigma\right)\frac{d\xi(t)}{dt}\right\}dt. \tag{2.29}$$

Markazi (x,y) nuqtada bo‘lgan ushbu:

$$\xi=x+\rho\cos\varphi,\qquad \eta=y+\rho\sin\varphi\ . \tag{2.30}$$

qutb koordinatalar sistemasini kiritamiz. Kiritilgan qutb koordinatalar sistemasida

$$A_t\left[\ln r^2\right]=\eta^m\frac{2(\xi-x)}{r^2}\frac{d\eta}{dt}-\frac{\frac{4}{m+2}\left(\eta^{\frac{m+2}{2}}-y^{\frac{m+2}{2}}\right)\eta^{\frac{m}{2}}}{r^2}\frac{d\xi}{dt}, \tag{2.31}$$

bu yerda

$$r^2=(\xi-x)^2+\frac{4}{(m+2)^2}\left(\eta^{\frac{m+2}{2}}-y^{\frac{m+2}{2}}\right)^2=\rho^2\cos^2\varphi+$$

$$+\frac{4}{(m+2)^2}\left[(y+\rho\sin\varphi)^{\frac{m+2}{2}}-y^{\frac{m+2}{2}}\right]^2 .$$

Darajali ifodaning quyidagi

$$(a+b)^{\mu}=a^{\mu}+\mu a^{\mu-1}b+\frac{\mu(\mu-1)}{2!}a^{\mu-2}b^2+\ldots ,$$

yoyilmasini $(y+\sin\varphi)^{\frac{m+2}{2}}$ ifodaga qo‘llab, ushbu:

$$r^2=\rho^2\cos^2\phi+\frac{4}{(m+2)^2}\times$$

$$\times\left[y^{\frac{m+2}{2}}+\frac{m+2}{2}y^{\frac{m}{2}}\rho\sin\phi+O\left(\rho^2\right)-y^{\frac{m+2}{2}}\right]^2=$$

$$=\rho^2\cos^2\phi+y^m\rho^2\sin^2\phi+O\left(\rho^3\right)$$

tenglikka ega bo‘lamiz, bu yerda $O(\rho^2)$, $O(\rho^3)$ – mos ravishda ρ^2, ρ^3 tartibli cheksiz kichik miqdorlar.

Shunday qilib,

$$A_t\left[\ln r^2\right]dt=\frac{2(y+\rho\sin\phi)^m\cos^2\phi+2y^{\frac{m}{2}}\sin^2\phi(y+\rho\sin\phi)^{\frac{m}{2}}+O(\rho)}{\cos^2\phi+y^m\sin^2\phi+O(\rho)}d\phi . \tag{2.32}$$

(2.32) tenglikda $\rho\to 0$ limitga o‘tib, ushbu:

$$\lim_{\rho\to 0} A_t\left[\ln r^2\right]dt = \frac{2y^m}{\cos^2\varphi + y^m\sin^2\varphi}d\varphi \tag{2.33}$$

tenglikni hosil qilamiz. (2.30) tenglikdan $\rho\to 0$ da $\xi\to x$, $\eta\to y$, demak $\sigma\to 0$.

Endi $\lim\limits_{\rho\to 0}\rho\ln\rho = 0$ tenglikni hisobga olib,

$$\lim_{\rho\to 0}\rho F(1+\beta,\beta,2\beta+1;1-\sigma)=0. \tag{2.34}$$

tenglikni hosil qilamiz. (2.29) tenglikda $\rho\to 0$ da limitga o'tib hamda (2.33) va (2.34) tengliklarni hisobga olib, ushbu:

$$W_1^{(1)}(x,y)=\lim_{\rho\to 0}\int_{C_\rho}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,y\right)\right]dt=$$

$$=-2\beta k_1\left(\frac{4}{m+2}\right)^{-2\beta}F(\beta,\beta-1,2\beta;1)\int_0^{2\pi}\frac{d\left(y^{\frac{m}{2}}tg\phi\right)}{1+\left(y^{\frac{m}{2}}tg\phi\right)^2}. \tag{2.35}$$

tenglikni hosil qilamiz. Xosmas integralning ushbu

$$\int_0^{2\pi}\frac{d\left(y^{\frac{m}{2}}tg\phi\right)}{1+\left(y^{\frac{m}{2}}tg\phi\right)^2}=2\pi$$

qiymatini va $F(\beta,\beta-1,2\beta;1)=\Gamma(2\beta)/\beta\Gamma^2(\beta)$ tenglikni e'tiborga olib, (2.35) tenglikni quyidagi ko'rinishda yozib olamiz:

$$W_1^{(1)}(x,y)=-4\pi\left(\frac{4}{m+2}\right)^{-2\beta}\frac{\Gamma(2\beta)}{\Gamma^2(\beta)}k_1,$$

yoki k_1 ning (2.20) dagi qiymatiga asosan, $W_1^{(1)}(x,y)=-1,\ (x,y)\in\Omega$ tenglikka kelamiz

s) (x,y) nuqta Γ chiziqda yotuvchi $M_0(x(s),y(s))$ $(0<s<l)$ nuqta bilan ustma-ust tushsin, bu yerda s – berilgan nuqtaning yoy absissasi. $W_1^{(1)}(x,y)$ potensialning to‘g‘ri qiymatini topamiz.

Markazi M_0 nuqtada bo‘lgan ρ radiusli C_ρ aylana yasaymiz. Γ chiziqning bu aylana ichidagi qismini Γ_ρ orqali, qolgan qismini $\Gamma\setminus\Gamma_\rho$ orqali belgilaymiz. C_ρ aylananing Ω soha ichida yotgan qismini C'_ρ orqali belgilaymiz va $\Gamma\setminus\Gamma_\rho$, C'_ρ, hamda Ox o‘qining [-1,1] kesmasi bilan chegaralangan sohani Ω_ρ orqali belgilaymiz.

$W_1^{(1)}(x,y)$-ikki qatlam potensialining to‘g‘ri qiymatini ushbu:

$$W_1^{(1)}(x(s),y(s))=\int_0^l \eta^{\beta_0}(t)A_t\left[q_1(x(t),y(t);x(s),y(s))\right]dt, \quad (2.36)$$

orqali belgilaymiz, (2.36) integralda $\xi(t)$, $\eta(t)$ –o‘zgaruvchi koordinata, $x(s),y(s)$ –fiksirlangan koordinata.

(2.28), (2.31) tengliklardan ko‘rinib turibdiki, (2.36) tenglikdagi integral ostidagi funksiya $t=s$ bo‘lganda singulyar maxsuslikka ega, ya’ni integral xosmas integraldir. Shuning uchun, (2.36) ni xosmas integralning ta’rifidan foydalanib hisoblaymiz:

$$W_1^{(1)}(x(s),y(s))=\lim_{\rho\to 0}\int_{\Gamma\setminus\Gamma_\rho} \eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x(s),y(s))\right]dt. \quad (2.37)$$

(2.22) tenglikni hisobga olib, (2.37) ning o‘ng tomonini quyidagicha almashtiramiz:

$$W_1^{(1)}\left(x(s),y(s)\right)=\lim_{\rho\to 0}\int_{\Gamma\setminus\Gamma_\rho} \eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t)\ ;x(s),y(s)\right)\right]dt+$$

$$+\int_{AB}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x(s),y(s)\right)\right]\Big|_{\eta(t)=0}dt+$$

$$+\int_{-C'_\rho}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x(s),y(s)\right)\right]dt-$$

$$-\int_{-C'_\rho}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x(s),y(s)\right)\right]dt,$$

bu yerda $-C'_\rho$ integrallash yo‘nalishi soat strelkasi harakati bo‘yicha bajarilishini bildiradi.

Shunday qilib,

$$W_1^{(1)}\left(x(s),y(s)\right)=\lim_{\rho\to 0}\int_{\partial\Omega_\rho}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x(s),y(s)\right)\right]dt+$$

$$+\lim_{\rho\to 0}\int_{C'_\rho}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x(s),y(s)\right)\right]dt. \tag{2.38}$$

$M_0(x(s),y(s))\notin\Omega_\rho$ bo‘lgani uchun, $q_1(\xi(t),\eta(t);x(s),y(s))$ Ω_ρ sohada (2.1) tenglamaning regulyar yechimi bo‘ladi, bu yerdan (2.15) ni e’tiborga olib, ushbu:

$$W_1^{(1)}(x(s),y(s))=\lim_{\rho\to 0}\int_{C'_\rho}\eta^{\beta_0}(t)A_t[q_1(\xi(t),\eta(t);x(s),y(s))]dt, \tag{2.39}$$

tenglikka kelamiz.

Markazi $M_0(x(s),y(s))$ bo‘lgan

$$\xi(t)=x(s)+\rho\cos\varphi,\ \eta(t)=y(s)+\rho\sin\varphi,$$

qutb koordinatalar sistemasini kiritamiz: $(\xi(t),\eta(t))\in\Gamma$ bo‘lib, $\xi'(t),\eta'(t)$ uzluksiz bo‘lgan uchun, (2.28) , (2.32) , (2.33) ni e’tiborga olib, (2.39) dan ushbu:

$$W_1^{(1)}\left(x(s),y(s)\right)=-2k_1\beta\left(\frac{4}{m+2}\right)^{-2\beta}\times F\left(\beta,\beta-1,2\beta;1\right)\int_{\phi_1}^{\phi_2}\frac{y^{\frac{m}{2}}d\phi}{\cos^2\phi+y^m\sin^2\phi},$$

tenglikka kelamiz, bu yerda φ_1, $(\xi(t),\eta(t))\in\Gamma$ nuqtada Γ chiziqqa o‘tkazilgan urinma bilan x o‘qi orasidagi burchak $\varphi_2=\varphi_1+\pi$. Bu yerdan (2.20) ga ko‘ra, ushbu

$$W_1^{(1)}(x,y)=-\frac{1}{2},\quad (x,y)\in\Gamma, \tag{2.40}$$

tenglikka kelamiz

d) Endi (x,y) nuqta absissalar o‘qida joylashgan holnio‘rganamiz.

$y=\delta$ to‘g‘ri chiziqni o‘tkazamiz, bu yerda δ-juda kichik musbat son. Ω_δ orqali Ω sohaning $y=\delta$ to‘g‘ri chiziqdan yuqorida joylashgan qismini belgilaymiz, $A_0(x_1,\delta)$ va $B_0(x_2,\delta)$ orqali esa $y=\delta$ to‘g‘ri chiziqning Γ chiziq bilan kesishish nuqtalarini belgilaymiz. $W_1^{(1)}(x,0)$ ikki qatlam potensialini quyidagicha yozib olamiz:

$$W_1^{(1)}(x,0)=\left(\int_0^{\varepsilon_2}+\int_{\varepsilon_2}^{l-\varepsilon_1}+\int_{l-\varepsilon_1}^{l}+\int_{A_0B_0}-\int_{A_0B_0}\right)\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,0\right)\right]dt=$$

$$=\left(\int_0^{\varepsilon_2}+\int_{l-\varepsilon_1}^{l}-\int_{A_0B_0}\right)\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,0\right)\right]dt+\int_{\partial\Omega_\delta}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,0\right)\right]dt$$

(2.41)

bu yerda $\Gamma=\overset{\cup}{BB_0}\cup\overset{\cup}{B_0A_0}\cup\overset{\cup}{A_0A}$; $A_0B_0=\{y=\delta,x_1\le x\le x_2\}$. Shunday qilib, bu yerda ε_1 va ε_2 mos ravishda $\overset{\cup}{AA_0}$ va $\overset{\cup}{BB_0}$ yoylar uzunligi. (2.41) ning oxirgi integrali (2.15) tenglikka ko‘ra nolga teng. Shunday qilib,

$$W_1^{(1)}(x,0)=\int_0^{\varepsilon_2}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,0\right)\right]dt+\int_{l-\varepsilon_1}^{\varepsilon_l}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t);x,0\right)\right]dt-$$

$$-\int_{x_1}^{x_2}\eta^{\beta_0}(t)A_t\left[q_1\left(\xi(t),\eta(t)x,0\right)\right]\Big|_{\eta=\delta}dt$$

Bu yerda $x_1=x_1(l-\varepsilon_1)$, $x_2=x_2(\varepsilon_2)$ lar mos ravishda A_0 va B_0 nuqtalarning absissalari $q_1(\xi,\eta;x,y)$ dan η bo‘yicha hosilani hisoblaymiz, dastlab uni quyidagicha yozib olamiz:

$$q_1(\xi,\eta;x,y)=k_1\left(r_1^2\right)^{-\beta}F(\beta,\beta,2\beta;1-\sigma)=\frac{k_1}{\left(r_1^2-r^2\right)^{\beta}}\left[(1-\sigma)^{\beta}F(\beta,\beta,2\beta;1-\sigma)\right]$$

$$\frac{\partial q_1(\xi,\eta;x,y)}{\partial\eta}=\frac{-8\beta k_1}{m+2}\cdot\frac{y^{(m+2)/2}\eta^{m/2}}{\left(r_1^2-r^2\right)^{\beta+1}}\times(1-\sigma)^{\beta}F(\beta,\beta,2\beta;1-\sigma)-$$

$$-\frac{k_1}{\left(r_1^2-r^2\right)^{\beta}}\beta(1-\sigma)^{\beta-1}F(\beta+1,\beta,2\beta;1-\sigma)\times\left[\frac{-4}{m+2}\left(y^{(m+2)/2}-\eta^{(m+2)/2}\right)\eta^{m/2}\frac{(1-\sigma)+\sigma}{r_1^2}-\right.$$

$$\left.-\frac{4}{m+2}\frac{r^2}{r_1^4}\left(y^{(m+2)/2}+\eta^{(m+2)/2}\right)\eta^{m/2}\right]=\frac{4\beta k_1}{(m+2)r^2r_1^{2\beta}}\eta^{m/2}\left(y^{(m+2)/2}-\eta^{(m+2)/2}\right)F(\beta-1,\beta,2\beta;1-\sigma)-$$

$$-\frac{8\beta k_1 y^{(m+2)/2}\eta^{m/2}}{(m+2)r_1^{2(\beta+1)}(1-\sigma)}\left[F(\beta,\beta,2\beta;1-\sigma)-\sigma F(\beta+1,\beta,2\beta;1-\sigma)\right].$$

(2.42)

Bu yerda (1.35) va

$$(c-b)zF(a,b,c+1;z)=cF(a-1,b,c;z)-c(1-z)F(a,b,c;z)\ ,$$

formulani qo‘llab, (2.42) ni ushbu ko‘rinishda yozib olamiz:

$$\frac{\partial q_1(\xi,\eta;x,y)}{\partial\eta}=\frac{4k_1\beta}{(m+2)r^2r_1^{2\beta}}\times\eta^{m/2}\left(y^{(m+2)/2}-\eta^{(m+2)/2}\right)F(\beta,\beta-1,2\beta;1-\sigma)-$$

$$-\frac{4\beta k_1}{m+2}r_1^{-2(\beta+1)}y^{(m+2)/2}\eta^{m/2}F(\beta+1,\beta,2\beta+1;1-\sigma)\quad,\ (2.43)$$

(2.43) da $y=0$, $\eta(t)=\delta$ deb, uni ushbu:

$$\frac{\partial q_1(\xi(t),\delta;x,0)}{\partial\eta}=-\frac{4\beta k_1}{m+2}\delta^{m+1}\left[(\xi-x)^2+\frac{4\delta^{m+2}}{(m+2)^2}\right]^{-1-\beta},\quad(2.44)$$

ko‘rinishda yozib olamiz.

Shunday qilib,

$$W_1^{(1)}(x,0)=A_1(\varepsilon_1)+A_2(\varepsilon_2)-\frac{4\beta k_1}{m+2}\int_{x_1}^{x_2}\delta^{m+1+\beta_0}\times$$

$$\times\left[(\xi-x)^2+\frac{4\delta^{m+2}}{(m+2)^2}\right]^{-1-\beta}d\xi=A_1(\varepsilon_1)+A_2(\varepsilon_2)-\frac{4\beta k_1}{m+2}\left(\frac{4}{(m+2)^2}\right)^{-1-\beta}\times \tag{2.45}$$

$$\times\delta^{-(m+2)(\beta+1)+m+1+\beta_0}\times\int_{x_1}^{x_2}\left[1+\left(\frac{m+2}{2}\delta^{-\frac{m+2}{2}}(\xi-x)\right)^2\right]^{-1-\beta}d\xi$$

bu yerda

$$A_1(\varepsilon_1)=\int_{l-\varepsilon_1}^{l}\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x,0)\right]dt,$$

$$A_2(\varepsilon_2)=\int_{0}^{\varepsilon_2}\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x,0)\right]dt.$$

(2.45) tenglikdagi integralda integral o‘zgaruvchisini ushbu

$$t=(m+2)(\xi-x)/2\delta^{(m+2)/2},$$

ko‘rinishda almashtirib, (2.45) ni quyidagicha yozib olamiz:

$$W_1^{(1)}(x,0)=A_1(\varepsilon_1)+A_2(\varepsilon_2)-2\beta k_1\times\left(\frac{2}{m+2}\right)^{-2\beta}\int_{\alpha_1}^{\alpha_2}\left(1+t^2\right)^{-1-\beta}dt, \tag{2.46}$$

bu yerda

$$\alpha_1=\frac{(m+2)(x_1-x)}{2\delta^{(m+2)/2}},\ \alpha_2=\frac{(m+2)(x_2-x)}{2\delta^{(m+2)/2}}.$$

(2.46) integralni ushbu hollarda hisoblaymiz.

1) $(x,0)$ nuqta $(-a,a)$ interval ichida joylashgan bo‘lsin, bu holda $\delta \to 0$ da x_1 va x_2 mos ravishda $-a$ va a ga intiladi va $\lim_{\delta\to 0} A_k(\varepsilon_k)=0$, $k=1,2$. Shunday qilib, $\delta \to 0$ da ikki qatlam potensiali ushbu

$$W_1^{(1)}(x,0)= -2\beta k_1\left(\frac{2}{m+2}\right)^{-2\beta}\int_{-\infty}^{\infty}\left(1+t^2\right)^{-1-\beta}dt \,. \qquad (2.47)$$

ko‘rinishni oladi.

Ushbu ma’lum:

$$\int_{-\infty}^{\infty}\left(1+t^2\right)^{-1-\beta}dt=\frac{\pi\Gamma(2\beta)}{2^{2\beta-1}\beta\Gamma^2(\beta)},$$

tenglikka asoslanib, (2.47) dan

$$W_1^{(1)}(x,0)=-4\pi k_1\left(\frac{4}{m+2}\right)^{-2\beta}\frac{\Gamma(2\beta)}{\Gamma^2(\beta)}\,. \qquad (2.48)$$

tenglikka kelamiz. Endi (2.48) ga (2.20) dan k_1 ning qiymatini qo‘yib, yakuniy natija

$$W_1^{(1)}(x,0)=-1 \qquad (2.49)$$

tenglikka kelamiz.

2. $(x,0)$ nuqta $(-a,\ 0)$ nuqta bilan ustma-ust tushsin, u holda:

$$\alpha_1=\frac{(m+2)(x_1+a)}{2\delta^{(m+2)/2}},\ \alpha_2=\frac{(m+2)(x_2+a)}{2\delta^{(m+2)/2}}.$$

Γ chiziqqa qo‘yilgan (2.24) shartga ko‘ra, $(-a,\ 0)$ nuqta atrofida $(x_1+a)<c\delta^{m+1}$ tengsizlikka ega bo‘lamiz, bu yerdan δ ning nolga intilishidan α_1 ning ham nolga intilishi kelib chiqadi, ya’ni $\delta\to 0$, $\alpha_1\to 0$. Shunday qilib, (2.46) da $\delta\to 0$ da ushbu:

$$W_1^{(1)}(-1,0)=-2\beta k_1\left(\frac{2}{m+2}\right)^{-2\beta}\int_{0}^{\infty}\left(1+t^2\right)^{-\beta-1}dt=-\frac{1}{2}\,. \qquad (2.50)$$

tenglikni hosil qilamiz.

3. $(x,0)$ nuqta endi (1,0) nuqta bilan ustma-ust tushsin, bu yerda ham 2) dagi mulohazalarni takrorlab,

$$W_1^{(1)}(1,0) = -\frac{1}{2}$$

tenglikka kelamiz.

Shunday qilib,

$$W_1^{(1)}(x,0) = \begin{cases} 0\ , & агар \quad |x| > a\ , \\ -1\ , & агар \quad |x| < a\ , \\ -\frac{1}{2}\ , & агар \quad |x| = a\ . \end{cases} \qquad (2.51).$$

Lemma 2.2 isbot bo‘ldi.

2.3-lemma. *Agar Γ chiziq 59-betda qo‘yilgan barcha shartlarni qanoatlantirsa, u holda shunday V soni mavjudki, uning uchun*

$$\int_0^l \left| \eta^{\beta_0}(t) A_t \left[q_1\left(\xi(t), \eta(t); x, y\right) \right] \right| dt \le B$$

tengsizlik yuqori yarim tekislikda yotuvchi ixtiyoriy (x,y) nuqta uchun o‘rinli bo‘ladi.

Isboti. (2.28) formuladan foydalanamiz. $P(t;x,y)$ funksiyaning aniqlanishidan, u faqat Γ chiziq nuqtalarida faqat logarifmik maxsuslikka ega bo‘lishi mumkin, shuning uchun ham

$$\int_{\varepsilon}^{l-\varepsilon} \left| P(t;x,y) \right| dt \le c_1 \, (\varepsilon \ge 0), \qquad (2.53)$$

bu yerda $c_1 - x, y$ o‘zgaruvchilarga bog‘liq emas.

$$\int_0^{\varepsilon} \left|P(t;x,y)\right| dt \quad va \quad \int_{l-\varepsilon}^{l} \left|P(t;x,y)\right| dt$$

integrallar bir xil baholanadi. (2.24) tengsizlikka asosan, yuqoridagi integrallardan birinchisi uchun

$$\left| \frac{4\beta\, k_1}{m+2} \frac{y^{\frac{m+2}{2}} \cdot \eta^{\frac{m}{2}+\beta_0}}{r_1^{2\beta+1}} \right| \leq C_2$$

tengsizlikni hisobga olib, ushbu

$$\int_0^{\varepsilon} \left|P(t;x,y)\right| dt \leq C_2 \int_0^{\varepsilon} F(1+\beta,\beta_1 1+2\beta;1-\sigma) \frac{\eta^{m+1}(t)}{r_1} dt \leq C_3 \qquad (2.54)$$

tengsizlikka kelamiz.

Shunday qilib, (2.53) va (2,54) tengsizliklarga asosan,

$$\int_0^{\varepsilon} \left|P(t;x,y)\right| dt \leq C_4 \;. \qquad (2.55)$$

Endi

$$\int_0^{\varepsilon} \left|Q(t;x,y)\right| dt = \int_0^{\tilde{l}} \frac{\beta k_1}{r_1^{2\beta}}\, \eta^{\beta_0}(t) \times \left|F(\beta,\beta-1,2\beta;1-\sigma)\right| \left|A_t\left[\ln r^2\right]\right| dt \qquad (2.56)$$

integralni baholaymiz.

(2,53) da

$$\tilde{\eta} = \frac{2}{m+2} \eta^{\frac{2}{m+2}}, \xi = \xi ,$$

almashtirish bajarib va $(\xi, \tilde{\eta})$ tekislikda $\tilde{\Gamma}$ uzunligini $\tilde{S}$ deb belgilab olamiz.

Ushbu hisoblashlarni bajaramiz:

$$\left|A_t\left[\ln r^2\right]\right| = \eta^m \frac{d\eta}{dt}\frac{\partial \ln r^2}{\partial t} - \frac{d\xi}{dt}\frac{\partial \ln r^2}{\partial \eta} =$$

$$= \frac{2\eta^m(\xi - x)}{r^2}\frac{d\eta}{dt} - \frac{2}{r^2}\left(\frac{2}{m+2}\eta^{\frac{m+2}{2}} - \frac{2}{m+2}y^{\frac{m+2}{2}}\right)\eta^{\frac{m}{2}}\frac{d\xi}{dt}, \quad (2.57)$$

bu yerda ushbu

$$\tilde{\eta} = \frac{2}{m+2}\eta^{\frac{m+2}{2}},\ d\tilde{\eta} = \eta^{\frac{m}{2}}d\eta,\ \tilde{y} = \frac{2}{m+2}y^{\frac{m+2}{2}},\ \tilde{r} = (\xi - x)^2 + (\tilde{\eta} - \tilde{y})^2$$

almashtirishni bajaramiz, u holda (2,57) ushbu

$$A_s\left[\ln \tilde{r}^2\right] = \left(\frac{m+2}{2}\tilde{\eta}\right)^{\frac{2}{m+2}}\left(\frac{2(\xi - x)}{\tilde{r}^2}\frac{d\tilde{\eta}}{d\tilde{t}} - \frac{2(\tilde{\eta} - \tilde{y})}{\tilde{r}^2}\frac{d\xi}{d\tilde{t}}\right)\cdot\frac{d\tilde{t}}{d\tilde{t}}, \quad (2.58)$$

ko‘rinishni oladi.

(2.58) da $\frac{d\tilde{\eta}}{d\tilde{t}} = \cos(\xi, \tilde{n}),\ \frac{d\xi}{d\tilde{t}} = -\cos(\tilde{\eta}, \tilde{n})$ tenglikni hisobga olib, (2,58) ni ushbu ko‘rinishda yozib olamiz

$$A_t\left[\ln \tilde{r}^2\right] = \left(\frac{m+2}{2}\tilde{\eta}\right)^{\frac{2}{m+2}}\left(\frac{\partial \ln \tilde{r}^2}{\partial \xi}\cdot\cos(\xi, \tilde{n}) + \frac{\partial \ln \tilde{r}^2}{\partial \tilde{n}}\cos(\tilde{\eta}, \tilde{n})\right)\frac{d\tilde{t}}{dt} =$$

$$= \left(\frac{m+2}{2}\tilde{\eta}\right)^{\frac{2}{m+2}}\frac{\partial \ln \tilde{r}^2}{\partial \tilde{n}}\frac{d\tilde{t}}{dt} = \left(\frac{m+2}{2}\tilde{\eta}\right)^{\frac{2}{m+2}}\frac{\cos\phi}{\tilde{r}}\frac{d\tilde{t}}{dt}, \quad (2.59)$$

bu yerda

$$\cos\varphi = \frac{\xi - x}{\tilde{r}}\cos(\xi, \tilde{n}) + \frac{\tilde{\eta} - \tilde{y}}{\tilde{r}}\cos(\tilde{\eta}, \tilde{n}) =$$

$$= \cos(\xi, \tilde{r})\cdot\cos(\xi, \tilde{n}) + \cos(\tilde{\eta}, r)\cos(\tilde{\eta}, \tilde{n}), \quad (2.60)$$

φ esa$(\xi,\tilde{\eta})$ nuqtaning radius vektori $\tilde{r}$ va shu nuqtaga o‘tkazilgan tashqi normal $\tilde{n}$ orasidagi burchak. Shunday qilib, (2.56) tenglikdan, (2.59) ni hisobga olib, ushbu tengsizlikka kelamiz:

$$\int_0^{\varepsilon}\left|Q(t;x,y)\right|dt \le C_5\int_0^{\tilde{l}}\frac{\tilde{\eta}^{2\beta}}{r_1^{2\beta}}\left|F(\beta,\beta-1,2\beta;1-\sigma)\right|\times\left|\frac{\partial}{\partial\tilde{n}}\ln r^2\right|d\tilde{t} \le C_6\int_0^{\tilde{l}}\frac{|\cos\phi|}{\tilde{r}}d\tilde{t}. \tag{2.61}$$

Logarifmik potensiallar nazariyasiga asosan,

$$\int_0^{\tilde{l}}\frac{|\cos\varphi|}{\tilde{r}}d\tilde{t} \le C_7. \tag{2.62}$$

Bu yerdan isbotlangan (2.59), (2.62) tengsizliklarga asosan 2.3-lemmaning isboti kelib chiqadi.

2.4-lemma. *Agar* (x,y) *nuqta* Γ *chiziqda yotsa, u holda ushbu:*

$$\eta^{\beta_0}\left|A_t(\xi,\eta;x,y)\right| \le \beta_1\frac{\eta^{\frac{m}{2}+\beta_0}}{r_1^{2\beta}}\left(\ln\frac{1}{\sigma}+1\right) \tag{2.63}$$

tengsizlik o‘rinlidir.

Bu lemmaning isboti bevosita (2.28) tengsizlikdan kelib chiqadi.

(2.25) formula ikki qatlam potensiali zichligi $\mu_1(t)\equiv 1$ bo‘lsa, (x,y) nuqta Γ chiziqdan o‘tganda uzilishga ega bo‘lishini ko‘rsatadi.

Agar ikki qatlam potensialida $\mu_1(t)$ ixtiyoriy uzluksiz funksiya bo‘lsa, uning uchun ushbu teorema o‘rinlidir:

2.2-teorema. *Agar* (x,y) *nuqta* Γ *chiziq nuqtasi* $(\xi(s),\eta(s))$ *ga* Ω *soha ichidan yoki tashqarisidan intilgandagi limit qiymatlarini mos ravishda* $W_i^{(1)}(s)$ *yoki* $W_\ell^{(1)}(s)$ *orqali belgilasak, u holda ushbu formulalar o‘rinlidir:*

$$W_i^{(1)}(s) = -\frac{1}{2}\mu_1(s)+\int_0^l\mu_1(t)K_1(s,t)dt$$

$$W_e^{(1)}(s) = \frac{1}{2}\mu_1(s)+\int_0^l\mu_1(t)K_1(s,t)dt, \tag{2.64}$$

bu yerda

$$K_1(s,t)=\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x(s),y(s))\right]. \tag{2.65}$$

t esa $(\xi(t),\eta(t))\in\Gamma$ o'zgaruvchi nuqtaning yoy absissasi, s esa $(x(s),y(s))\in\Gamma$ fiksirlangan nuqtaning yoy absissasi.

Isboti: Dastlab ikki qatlam potensialini Γ chiziq nuqtalarida mavjud ekanligini ko'rsatamiz.

(2.28) ifodani uzluksizlikka tekshiramiz, buning uchun (2.31) ni o'rganamiz va uni ushbu ko'rinishda yozib olamiz:

$$A_t\left[\ln r^2\right]=-\eta^{\frac{m}{2}}(t)\frac{\partial}{\partial t}arctg\frac{\xi(t)-x(s)}{\frac{2}{m+2}\left(\eta^{\frac{m}{2}}(t)-y^{\frac{m+2}{2}}(s)\right)}. \tag{2.66}$$

(2,66) ifoda t va s o'zgaruvchilar bo'yicha G chiziqda uzluksizdir. Haqiqatdan ham, ushbu:

$$\alpha=\alpha(s,t)=\frac{\xi(t)-x(s)}{t-s},\beta=\beta(s,t)=\frac{\frac{2}{m+2}\left(\eta^{\frac{m+2}{2}}(t)-y^{\frac{m+2}{2}}(s)\right)}{t-s}$$

belgilashlarni kiritib, (2.66) ni ushbu:

$$A_t\left[\ln r^2\right]=-\eta^{\frac{m}{2}}(t)\frac{\partial}{\partial t}arctg\frac{\alpha}{\beta}=-\eta^{\frac{m}{2}}\frac{\alpha'\beta-\beta'\alpha}{\alpha^2+\beta^2}$$

ko'rinishda yozib olamiz va bu yerda $t\to s$ da limitga o'tib, quyidagi tenglikni hosil qilamiz:

$$\lim_{t\to s}A_t\left[\ln r^2\right]=-y^{\frac{m}{2}}(s)\frac{x''(s)y^{\frac{m}{2}}(s)y'(s)-x'(s)y^{\frac{m}{2}}(s)y''(s)}{\left(x'(s)\right)^2+y^m\left(y'(s)\right)^2}=$$

$$=y^m(s)\frac{x'(s)y''(s)-x''(s)y'(s)}{\left(x'(s)\right)^2+y^m\left(y'(s)\right)^2} \tag{2.67}$$

$A_t\left[\ln r^2\right]$ning $t=s$ bo‘lgandagi qiymatini (2.67) o‘ng tomonidagi qiymatiga teng qilib olib, $A_t\left[\ln r^2\right]$ ni Γ chiziqda uzluksizligini ta’minlaymiz. Shunday qilib, yuqoridagi mulohazalardan, (2.28) ifodaga asosan ikki qatlam potensialini chiziqdagi qiymati mavjud.

Endi (2.64) formulalarni isbotlaymiz. Γ chiziqda $(x_0(s_0), y_0(s_0))$ nuqta olamiz va $w^{(1)}(x,y)$ ni ushbu ko‘rinishda yozamiz:

$$w^{(1)}(x,y)=\int_0^l(\mu_1(t)-\mu_1(s_0))\eta^{\beta_0}(t)A_t[q_1(\xi(t),\eta(t));x,y]dt+\mu_1(s_0)\times$$
$$\times\int_0^l\eta^{\beta_0}(t)A_t[q_1(\xi(t),\eta(t));x,y]dt=\tilde{w}^{(1)}(x,y)+\mu_1(s_0)w_1^{(1)}(x,y). \tag{2.68}$$

$\tilde{w}^{(1)}(x,y)$ potensialning $(x_0(s_0), y_0(s_0))$ nuqtada uzluksiz ekanligini ko‘rsatamiz. Shu maqsadda, $(x_0(s_0), y_0(s_0))$ nuqtani markaz qilib, ρ radiusli C_ρ aylana chizamiz. Γ chiziqning bu aylana ichidagi qismini Γ_ρ, aylana tashqarisidagi qismini esa Γ/Γ_ρ orqali belgilab olamiz, u holda:

$$\tilde{w}^{(1)}(x,y)=\int_{\Gamma_\rho}(\mu_1(t)-\mu_1(s_0))\eta^{\beta_0}(t)A_t[q_1(\xi(t),\eta(t));x,y]dt+$$
$$+\int_{\Gamma\backslash\Gamma_\rho}(\mu_1(t)-\mu_1(s_0))\eta^{\beta_0}(t)A_t[q_1(\xi(t),\eta(t));x,y]dt= \tag{2.69}$$
$$=\tilde{w}_1^{(1)}(x,y)+\tilde{w}_2^{(1)}(x,y).$$

Bu yerdan,

$$\left|\tilde{w}^{(1)}(x,y)-\tilde{w}^{(1)}(x_0,y_0)\right|=\left|\tilde{w}_1^{(1)}(x,y)\right|+$$
$$+\left|\tilde{w}_1^{(1)}(x_0,y_0)\right|+\left|\tilde{w}_2^{(1)}(x,y)-\tilde{w}_2^{(1)}(x_0,y_0)\right| \tag{2.70}$$

$\mu(\xi)$ funksiya uzluksiz bo‘lgani uchun ρ ni shunday tanlash mumkinki,

$$\sqrt{|\xi(t)-x(s_0)|^2+|\eta(t)-y(s_0)|^2}<\rho$$

bo‘lganda

$$\left|\mu_1(t) - \mu_1(s_0)\right| < \frac{\varepsilon}{3c}$$

bo‘ladi, bu yerda

$$\int_0^l A_t \left|q_1(\xi(t), \eta(t)); x, y\right| dt \le c.$$

Bu yerdan ixtiyoriy $x \in R_2$ uchun

$$\begin{gathered}\left|\widetilde{W}_1^{(1)}(x,y)\right| \le \int_{\Gamma_\rho} \left|\mu_1(t) - \mu_1(s_0)\right| \left|\eta^{\beta_0}(t) A_t\left[q_1(\xi(t),\eta(t)); x, y\right]\right| dt \le \frac{\varepsilon}{3c} \cdot \\ \cdot \int_{\Gamma_\rho} \left|\eta^{\beta_0}(t) A_t\left[q_1(\xi(t),\eta(t)); x, y\right]\right| dt \le \frac{\varepsilon}{3}.\end{gathered} \tag{2.71}$$

Xususiy holda:

$$\left|\widetilde{W}_1^{(1)}(x_0, y_0)\right| \le \frac{\varepsilon}{3}. \tag{2.72}$$

$\widetilde{W}_2^{(1)}(x,y)$ potensialda integral $\Gamma \setminus \Gamma_\rho$ chiziq bo‘yicha bajarilyapti, (x_0, y_0) nuqta esa Γ_ρ da joylashgan. Shuning uchun, $\widetilde{W}_2^{(1)}(x,y)$ uzluksizdir, ya’ni shunday $\delta > 0$ son mavjudki, $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ bo‘lganda,

$$\left|\widetilde{W}_2^{(1)}(x,y)\right| < \frac{\varepsilon}{3}$$

bo‘ladi. Shunday qilib, $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \min(\delta, \rho)$ bo‘lganda,

$$\left|\widetilde{W}^{(1)}(x,y) - \widetilde{W}^{(1)}(x_0,y_0)\right| < \varepsilon \tag{2.73}$$

bo‘ladi, ya’ni (x_0, y_0) nuqtada $\widetilde{W}^{(1)}(x,y)$ potensial uzluksiz. Shunday ekan, $\widetilde{W}^{(1)}(x,y)$ potensialning limit qiymatlari va to‘g‘ri qiymati (x_0, y_0) nuqtada ustma-ust tushadi, ya’ni

$$\widetilde{W}_i^{(1)}(x_0,y_0)=\widetilde{W}_e^{(1)}(x_0,y_0)=\widetilde{W}_0^{(1)}(x_0,y_0). \qquad (2.74)$$

(2.26) formulaga asosan,

$$\widetilde{W}_{1i}^{(1)}(x_0,y_0)=-1;\ \widetilde{W}_{1e}^{(1)}(x_0,y_0)=0;\ \widetilde{W}_{10}^{(1)}(x_0,y_0)=-\frac{1}{2} \qquad (2.75).$$

Endi (2.68) formuladan (2.74) va (2.75) tengliklarni e'tiborga olib, quyidagi munosabatlarni hosil qilamiz:

$$W_i^{(1)}(x_0,y_0)=\widetilde{W}_i^{(1)}(x_0,y_0)+\mu_1(s_0)W_{1i}^{(1)}(x_0,y_0)=\widetilde{W}_0^{(1)}(x_0,y_0)-\mu_1(s_0)\ . \qquad (2.76)$$

$$W_e^{(1)}(x_0,y_0)=\widetilde{W}_e^{(1)}(x_0,y_0)+\mu_1(s_0)W_{1e}^{(1)}(x_0,y_0)=\widetilde{w}_0^{(1)}(x_0,y_0). \qquad (2.77)$$

Lekin

$$\begin{aligned}\widetilde{W}_0^{(1)}(x_0,y_0)&=\int_\Gamma(\mu_1(t)-\mu_1(s_0))\eta^{\beta_0}(t)A_t[q_1(\xi(s),\eta(s));x_0,y_0]dt=\\&=W_0^{1}(x_0,y_0)-\mu_1(s_0)W_{10}^{(1)}(x_0,y_0)=W_0^{(1)}(x_0,y_0)+\frac{1}{2}\mu_1(s_0)\end{aligned} \qquad (2.78)$$

Shunday qilib, (2.76), (2.77) formulalardan (2.78) ga asoslanib, (2.64) formulalarni hosil qilamiz.

2.2-teorema isbot bo'ldi.

$W^{(1)}(x,y)$ – ikki qatlam potensiali Γ chiziq nuqtalari to'plamida uzluksiz. Haqiqatdan ham, Γ chiziqda (x_0,y_0) fiksirlangan va (x,y)o'zgaruvchi nuqtalarni olaylik va (x,y) ni Γ chiziq bo'yicha (x_0,y_0) ga intiltiraylik. (2.68) tenglikdan:

$$W^{(1)}(x,y)=\widetilde{W}^{(1)}(x,y)-\frac{1}{2}\mu_1(s_0).\quad (x,y)\in\Gamma.$$

Bu yerda $\widetilde{w}^{(1)}(x,y)$ uzluksiz funksiya bo'lgani uchun

$$\lim_{(x,y)\to(x_0,y_0)}W^{(1)}(x,y)=\widetilde{W}^{(1)}(x_0,y_0)-\frac{1}{2}\mu_1(s_0)=W^{(1)}(x_0,y_0)\ ,$$

ya'ni $W^{(1)}(x,y)$ funksiya Γ chiziqda uzluksiz ekan.

4. Ikkilangan qatlam potensiali. $W^2(x,y)$ (2.1) tenglamaning ikkinchi fundamental yechimi

$$q_2(x,y;x_0,y_0)=k_2\left(\frac{4}{m+2}\right)^{4\beta-2}\left(r_1^2\right)^{-\beta}(1-\sigma)^{1-2\beta}\times F(1-\beta,1\ 1-\beta,\ 2-2\beta;\ 1-\sigma)$$

(2.79)

dan foydalanib, ushbu

$$W^{(2)}(x,y)=\int_0^l \mu_2(t)\eta^{\beta_0}(t)A_t[q_1(\xi(t),\eta(t);x,y)]dt$$

formula yordamida $W^{(2)}(x,y)-$ikkilangan qatlam potensialini aniqlaymiz, bu yerda $\mu_2(t)\in C[0,l]$,

$$A_t=[q_1(\xi,\eta;x,y)]=\eta^m\frac{d\eta}{dt}\cdot\frac{\partial q_1}{\partial\xi}-\frac{d\xi}{dt}\cdot\frac{\partial q_1}{\partial\eta}, \tag{2.80}$$

$W^2(x,y)$funksiya yuqori yarim tekislikning Γ chiziq va OX o‘qi nuqtalari bilan ustma-ust tushmaydigan barcha nuqtalarda (2.1) tenglamaning regulyar yechimi bo‘ladi. Ikki qatlam potensiali yuqori yarim tekislikning barcha nuqtalarida aniqlangan va

$$k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)}. \tag{2.81}$$

2.5-lemma. *Ikki qatlam potensiali zichligi* $\mu_2(t)\equiv 1$ *bo‘lganda ushbu munosabatlar o‘rinlidir:*

$$\int_0^l A_t[q_2(\xi,\eta;x,y)]ds=\begin{cases} i(x,y)-1 \quad agar \quad (x,y)\in\Omega\cup(-a,a) \quad bo'lsa; \\ i(x,y)-\dfrac{1}{2} \quad agar \quad (x,y)\in\Gamma\cup\{(-a,a)\ ;\ (a,0)\} \quad bo'lsa; \\ i(x,y) \quad agar \quad (x,y)\notin\overline{\Omega} \quad bo'lsa; \end{cases}$$

(2.82)

Bu yerda

$$i(x,y)=\int_{-a}^{a}\eta^{\beta_0}\left.\frac{\partial q_2(\xi,0;x,y)}{\partial\eta}\right|_{\eta=0}d\xi$$

2.5-lemma isboti 2.4-lemma isbotidek bajariladi.

2.6-lemma.*Agar Γ chiziq 59-betda qo'yilgan shartlarni qanoatlantirsa, u holda yuqori yarim tekislikda yotuvchi ixtiyoriy (x, y) nuqta uchun shunday B_2 soni mavjudki, uning uchun ushbu*

$$\int_0^l \eta^{\beta_0}(t) A_t\left[q_2(\xi(t), \eta(t); x, y)\right] dt \le B_2 \tag{2.83}$$

tengsizlik bajariladi.

2.7-lemma.*Agar (x, y) nuqta Γ chiziqda yotsa, u holda ushbu*

$$\eta^{\beta_0}(t) A_t\left[q_2(\xi(t), \eta(t); x(s), y(s))\right] \le \overline{B}_2 \frac{\eta^{\frac{m}{2}+\beta_0}(t)}{\left(r_1^2\right)^{\beta}}\left(\ln\frac{1}{\sigma}+1\right) \tag{2.84}$$

tengsizlik o'rinli bo'ladi.

2.6 va 2.7-lemmalarning isboti bevosita ushbu

$$\eta^{\beta_0}(t) A_t\left[q_2(\xi(t), \eta(t); x(s), y(s))\right] = \frac{k_2 y^{1-\beta_0}(s)}{r_1^{2(1-\beta)}} F(1-\beta, -\beta, 1-2\beta; 1-\sigma)\frac{d\xi}{dt} -$$

$$-\frac{k_2(1-\beta)\eta(t) y^{1-\beta_0}(s)}{r_1^{2(1-\beta)}} F(1-\beta, -\beta, 2-2\beta; 1-\sigma) A_t\left[\ln r^2\right] \tag{2.85}$$

formuladan kelib chiqadi.

2.3-teorema.*Agar Γ chiziq 59-betdagi shartlarniqanoatlantirsa, hamda $\mu_2(t)$ funksiya uzluksiz bo'lsa, u holdaushbu formulalar o'rinlidir:*

$$\begin{aligned} W_i^{(2)}(s) &= -\frac{1}{2}\mu_2(s) + \int_0^l \mu_2(t) K_2(s,t) dt, \\ W_e^{(2)}(s) &= \frac{1}{2}\mu_2(s) + \int_0^l \mu_2(t) K_2(s,t) dt, \end{aligned} \tag{2.86}$$

bu yerda

$$K_2(s,t)=\eta^{\beta_0}(t)A_t[q_2(\xi(t),\eta(t);x(s),y(s))] \qquad ,(2.87)$$

$(\xi(t),\eta(t))$ *va* $(x(s),y(s))$ *nuqtalar* Γ *chiziqda yotibdi.*

2.3-teoremaning isboti bevosita 2.5-lemmadan kelib chiqadi.

2.6-lemma. *Agar* Γ chiziq *59*-betda *keltirilgan shartlarniqanoatlantirsa va* $\mu_1(t)$ *chegaralangan funksiya bo'lsa, u holda* $\mu(s)=\int\limits_0^l \mu_1(t)K_1(s,t)dt$ *funksiya qandaydir* α *ko'rsatkich bilan Gyolder shartini qanoatlantiradi.*

5. Oddiy qatlam potensiali. $V_1(x,y)$. Γ chiziq 59-betda keltirilgan shartlarni qanoatlantirsin,ushbu

$$V_1(x,y)=\int\limits_0^l \rho_1(t)q_1(\xi(t),\eta(t);x,y)dt, \qquad (2.88)$$

funksiyaga oddiy qatlam potensiali deyiladi, bu yerda $\rho_1(t)$- zichlik $0\le t\le l$ oraliqda uzluksiz funksiya bo'lib, Γ chiziqning uchlarida $\eta^m(t)$ tartibda nolga aylanadi, $q_1(\xi,\eta;x,y)$ esa (2.1) tenglamaning fundamental yechimi. Oddiy qatlam potensiali (2.88) yuqori yarim tekislikda aniqlangan va u Γ chiziqni kesib o'tganda uzluksiz. Oddiy qatlam potensiali yuqori yarim tekislikning, Γ va Ox o'qining nuqtalaridan tashqari barcha nuqtalarida (2.1) tenglamaning regulyar yechimidan iborat. $q_1(\xi,\eta;x,y)$-fundamental yechimning (2.19) ifodasidan (x,y) nuqta cheksizlikka intilganda, $V_1(x,y)$ oddiy qatlam potensialini nolga intilishi kelib chiqadi. Haqiqatdan ham, (x,y) nuqta tenglamasi

$$x^2+\frac{4}{(m+2)^2}y^{m+2}=R^2$$

dan iborat bo'lgan normal chiziqda yotsin, u holda (2.19) ga asosan,

$$|V_1(x,y)|\le\int\limits_0^l|\rho_1(t)||q_1(\xi,\eta;x,y)|dt\le M\cdot R^{-2\beta}, \qquad (2.89)$$

bu yerda $R\ge R_0$, R_0,M –o'zgarmas sonlar.

6. Oddiy qatlam potensialining konormal hosilasi. Γ chiziqda ixtiyoriy $N(x(s),y(s))$ nuqta olamiz va bu nuqtada $\vec{n}$-normal o'tkazamiz. Bu normalda yotuvchi hamda Γ chiziqqa tegishli bo‘lmagan ixtiyoriy $M(x,y)$ nuqtani olamiz va bu nuqtada (2.88) funksiyaning konormal hosilasini hisoblaymiz:

$$A_s[V_1(x,y)] = \int_0^l \rho_1(t) A_s[q_1(\xi,\eta;x,y)]dt\,, \tag{2.90}$$

bu yerda

$$A_s[\ ] = y^m \frac{dy}{ds}\frac{\partial}{\partial x} - \frac{dx}{ds}\frac{\partial}{\partial y},\quad \frac{dy}{ds} = \cos(n,x),$$

$$\frac{dx}{ds} = -\cos(n,y),\ \cos(n,x),\ \cos(n,y)$$

Γ chiziqning $N(x(s),y(s))$ nuqtasiga o‘tkazilgan tashqi normalning yo‘naltiruvchi kosinuslari. (2.90) integral agar $M(x,y)$nuqta $N(x(s),y(s))\in\Gamma$ nuqta bilan ustma-ust tushganda, y^{β_0} salmoq bilan mavjud, ya’ni

$$y^{\beta}(s) A_s[V_1(x(s),y(s))] = \int_0^l \rho_1(t) y^{\beta_0}(s) A_s[q_1(\rho,\eta;x(s),y(s))]dt$$

ifoda aniq qiymatga ega.

$$y^{\beta_0}(s) A_s[V_1(x(s),y(s))]_i \text{ va } y^{\beta_0}(s) A_s[V_1(x(s),y(s))]_e$$

orqali $V_1(x,y)$ – oddiy qatlam potensialining konormal hosilasini y^{β_0} salmoq bilan, $M(x,y)$ nuqta $N(x(s),y(s))$ nuqtaga mos ravishda Ω soha ichidan va Ω soha tashqarisidan intilganda limit qiymatini belgilaymiz.

2.4-teorema.*Agar* Γ *chiziq 59-betdagi shartlarni qanoatlantirsa va* $\rho(t)\in C[0,l]$ *bo‘lsa, u holda*

$$y^{\beta_0}(s) A_s[V_1(x(s),y(s))]_i = \frac{1}{2}\rho_1(s) + \int_0^l \rho_1(t) K_1(t,s)dt$$

$$y^{\beta_0}(s) A_s[V_1(x(s),y(s))]_e = -\frac{1}{2}\rho_1(s) + \int_0^l \rho_1(t) K_1(t,s)dt \tag{2.91}$$

formulalar o‘rinlidir, bu yerda

$$K_1(t,s) = y^{\beta_0}(s) A_s\left[q_1(\xi(t),\eta(t);x(s),y(s))\right].$$

Bu teorema 2.3-lemmadan foydalanib, 2.3-teorema kabi isbotlanadi.

Bu formulalar bevosita oddiy qatlam potensiali konormal hosilasining qiymatini (x,y) nuqta Γ chiziqdan o‘tganda sakrash kattaligini hisoblash imkonini beradi, ya’ni

$$A_s\left[V_1(x,y)\right]_i - A_s\left[V(x,y)\right]_e = \rho_1(s). \tag{2.92}$$

Ikki qatlam potensialidek, ushbu baholarni olish mumkin:

$$\left|A_s\left[\vartheta_1(x,y)\right]\right| \le M y^m R^{-2\beta-1}, \tag{2.93}$$

bu yerda M, R_0 - o‘zgarmas sonlar.

Oddiy qatlam potensiali uchun $V_1(x,y)$ (2.16) Grin formulasi o‘rinlidir:

$$\iint_D y^{\beta_0}\left[y^m\left(\frac{\partial V_1}{\partial x}\right)^2 + \left(\frac{\partial V_1}{\partial y}\right)^2\right]dxdy = \int_0^l y^{\beta_0} V_1\, A_s\left[V_1\right]_i ds. \tag{2.94}$$

Ω_1' orqali Γ chiziq, Ox o‘qining $[-R,-a]$ *ва* $[a,R]$ kesmasi va C_R normal chiziqlar bilan chegaralangan sohani belgilaymiz. $\Omega \subset \Omega', \Omega_1'$ soha uchun (2.93) formulani qo‘llab, hamda $\eta^{\beta_0}\dfrac{\partial q_1}{\partial\eta}\Big|_{\eta=0} = 0$ ekanligini, hamda (2.89) va (2.93) tengsizliklarni e’tiborga olib va $R \to \infty$ da limitga o‘tib, ushbu

$$\iint_{\Omega'} y^{\beta_0}\left[y^m\left(\frac{\partial V_1}{\partial x}\right)^2 + \left(\frac{\partial V_1}{\partial y}\right)^2\right]dxdy = -\int_0^l y^{\beta_0} V_1\, A_s\left[V_1\right]_e ds, \tag{2.95}$$

tenglikka kelamiz, bu yerda Ω' orqali Ω' sohaning $y>0$ yarim tekislikdagi tashqi sohasi belgilangan.

7. Oddiy qatlam potensiali. $V_2(x,y)$(2.1) tenglamaning ikkinchi fundamental yechimi $q_2(x,y;x_0,y_0)$ dan foydalanib, $V_2(x,y)$ oddiy qatlam potensialini quyidagicha aniqlaymiz

$$V_2(x,y)=\int_0^l \rho_2(t)q_2(\xi(t),\eta(t);x,y)dt\,, \qquad (2.96)$$

bu yerda $\rho_2(t)\in C[0,l]$. $V_2(x,y)$ – oddiy qatlam potensiali yuqori yarim tekislikda uzluksiz va yuqori yarim tekislikda yotuvchi hamda Γ va Ox o‘qi nuqtalari bilan umumiy nuqtaga ega bo‘lmagan har qanday sohada (2.1) tenglamaning regulyar yechimi.

$V_2(x,y)$ – oddiy qatlam potensialining y^{β_0} salmoq bilan konormal hosilasi ushbu ko‘rinishda bo‘ladi

$$y^{\beta_0}(s)A_s[V_2(x,y)]=\int_0^l \rho_2(t)y^{\beta_0}(s)A_s[q_2(\xi(t),\eta(t);x,y)]dt\,, \qquad (2.97)$$

bu yerda

$$A_s[\ ]=y^m\frac{dy}{ds}\cdot\frac{\partial}{\partial x}-\frac{dx}{ds}\frac{\partial}{\partial y},\quad \frac{dy}{ds}=\cos(n,x),\quad \frac{dx}{ds}=-\cos(n,y),$$

ya’ni

$$\cos(n,x),\ \cos(n,y),\ \ N(x(s),y(s))$$

nuqtaga o‘tkazilgan tashqi normalning yo‘naltiruvchi kosinuslari.

(2.97) integral $M(x,y)$nuqta $N(x(s),y(s))$ nuqta bilan ustma-ust tushganda ham aniq qiymatga ega.

2.5-teorema.*Agar* Γ *chiziq 59-betdagi shartlarni qanoatlantirsa va* $\rho_2(t)$*-uzluksiz funksiya bo‘lsa, u holda ushbu:*

$$y^{\beta_0}(s)A_s[V_2(x,y)]_i=\frac{1}{2}\rho_2(s)+\int_0^l \rho_2(t)K_2(t,s)dt$$

$$y^{\beta_0}(s)A_s[V_2(x,y)]_e=-\frac{1}{2}\rho_2(s)+\int_0^l \rho_2(t)K_2(t,s)dt \qquad (2.98)$$

formulalar o‘rinlidir, bu yerda

$$K_2(t,s)=y^{\beta}(s)A_s[q_2(\xi(t),\eta(t);x(s),y(s))]. \qquad (2.99)$$

Bu formulalardan ushbu

$$y^{\beta_0}(s)A_s[V_2(x,y)]_i - y^{\beta_0}(s)A_s[V_2(x,y)]_l = \rho_2(s) \qquad (2.100)$$

tenglikni hosil qilamiz.

Agar (x,y) nuqta $C_R : x^2 + \dfrac{4}{(m+2)^2} y^{m+2} = R^2$ normal chiziqda bo'lsa, (x,y) cheksizlikka intilganda

$$|V_2(x,y)| \le \frac{M}{R}, \quad y^{\beta_0}|A_s[V_2(x,y)]| \le MR^{2(\beta-1)} \qquad (2.101)$$

baholar o'rinlidir, bu yerda M va R_0 o'zgarmas sonlar.

$V_2(x,y)$ – oddiy qatlam potensiali uchun (2.94) va (2.95) formulalar o'rinlidir.

8. Potensiallar zichliklari uchun integral tenglamalar. (2.64), (2.86) va (2.91), (2.98) formulalarni ushbu ko'rinishda yozib olamiz:

$$\mu_j(s) - \lambda\int_0^l K_j(s,t)\mu_j(t)dt = f_j(s), \qquad (j=1,2), \quad (2.102)$$

$$\rho_j(s) - \lambda\int_0^l K_j(t,s)\rho_j(t) = g_j(s), \quad (j=1,2), \qquad (2.103)$$

bu yerda

$$\lambda = 2, \quad f_j(s) = -2w_e^{(j)}(s), \quad g_j(s) = -2y^{\beta_0}(s)A_s[V_j(x(s),y(s))]_e,$$

$$\lambda = -2, \quad f_j(s) = 2w_e^{(j)}(s), \quad g_j(s) = 2y^{\beta_0}A_s[V_j(x(s),y(s))]_i.$$

(2.102) va (2.103) tenglamalar o'zaro qo'shma tenglamalar bo'lib, (2.63), (2.84) tengsizliklarga ko'ra, ularga Fredgolm teoremalarini qo'llash mumkin.

$\lambda = 2$, $K_1(s,t)$yadroningxarakteristik soni bo'lmasligini ko'rsatamiz. Bu tasdiq ushbu

$$\rho(s) - 2\int_0^l K_1(t,s)\rho(t)dt = 0 \qquad (2.104)$$

bir jinsli integral tenglamaning notrivial yechimga ega emasligiga ekvivalentdir.

Faraz qilaylik, (2.104) tenglama uzluksiz $\rho_0(s) \neq 0$ yechimga ega $\rho_0(t)$ zichlikka ega bo‘lgan $V_1^0(x, y)$ – oddiy qatlam potensialini tuzamiz, bu potensial Ω va Ω' sohada (2.1) tenglamaning regulyar yechimi bo‘lib, uning konormal hosilasining Γ chiziqdagi limit qiymati (2.91) va (2.104) tengliklarga asosan, nolga teng, ya’ni:

$$y^{\beta_0} A_s\left[V^0{}_1(x, y)\right]_i = 0 \tag{2.105}$$

$V_1^0(x, y)$ – oddiy qatlam potensialiga (2.95) Grin formulasini qo‘llab, Ω' sohada $V_1^0(x, y) = const$ tenglikka kelamiz. Cheksizlikda oddiy qatlam potensiali (2.89) bahoga ko‘ra, 0 ga aylanadi. Bundan, Ω' va Γ da $V_1^0(x, y) \equiv 0$ degan xulosaga kelamiz. (2.94) formulada $V_1^0(x, y)|_\Gamma = 0$ bo‘lgani uchun Ω -da $V_1^0(x, y) \equiv 0$. Bundan $y^{\beta_0} A_s\left[V^0{}_1(x, y)\right]_i \equiv 0$ va (2.100) formulaga ko‘ra, $\rho_0(t) \equiv 0$ tenglikka kelamiz. Shunday qilib, (2.104) bir jinsli tenglama faqat $\rho_0(t) \equiv 0$ trivial yechimga ega bo‘ladi: demak, $\lambda = 2$

Ushbu

$$\mu(s) - \lambda \int_0^l K_1(s, t)\mu(t)dt = 0 \tag{2.106}$$

bir jinsli tenglama uchun $\lambda = -2$ xarakteristik son haqiqatdan ham (2.26) tenglikka asosan, ixtiyoriy $\mu(t) = const$ (2.106) tenglamaning yechimi bo‘lishi kelib chiqadi.

9. Shakli o‘zgargan N masalasining Grin funksiyasi. Ω sohada (2.1) tenglama uchun shakli o‘zgargan N masalasining Grin funksiyasi deb, quyidagi shartlarni qanoatlantiruvchi $G_1(x, y; x_0\, y_0)$ funksiyaga aytiladi [2,13]:

1⁰) $y^{\beta_0} G_1(x, y; x_0\, y_0)$ – funksiya Ω sohaning $(x, y) \neq (x_0, y_0)$ nuqtasidan tashqari barcha nuqtalarida, (x_0, y_0) bo‘yicha (2.1) tenglamaning, (x, y) bo‘yicha esa (2.12) tenglamaning regulyar yechimi bo‘ladi;

2⁰)

$$G_1(x, y; x_0\, y_0)|_\Gamma = 0,\ y^{\beta_0} \frac{\partial G_1}{\partial y}|_{y=0} = 0,\ (x_0\, y_0) \in \Omega \tag{2.107}$$

chegaraviy shartlarni qanoatlantiradi.

3[0]) $G_1(x,y;x_0\,y_0)$ funksiyani

$$G_1(x,y;x_0\,y_0)=q_1(x,y;x_0\,y_0)+\vartheta_1(x,y;x_0\,y_0)\ , \qquad (2.108)$$

shaklda ifodalash mumkin, bu yerda $q_1(x,y;x_0\,y_0)=k_1 r_1^{-2\beta}F(\beta,\beta;2\beta;1-\sigma)$ (2.1) tenglamaning fundamental yechimi.

$\vartheta_1(x,y;x_0\,y_0)$ esa Ω sohaning barcha nuqtalarida (2.1) tenglamaning regulyar yechimi.

Shakli o'zgargan N masalasining Grin funksiyasini tuzish,uning regulyar qismi $\vartheta_1(x,y;x_0\,y_0)$ ni topishga olib kelinadi va u (2.107), (2.208) ifodalardan, (2.22) ni hisobga olgan holda, ushbu:

$$\vartheta_1(x,y;x_0\,y_0)\big|_\Gamma=-q_1(x,y;x_0\,y_0),\quad (x_0\,y_0)\in\Omega, \qquad (2.109)$$

$$y^{\beta_0}\frac{\partial\vartheta_1(x,y;x_0\,y_0)}{\partial y}\Big|_{y=0}=0, \qquad (2.110)$$

chegaraviy shartlarni qanoatlantirishi kerak.

$\vartheta_1(x,y;x_0\,y_0)$ funksiyani ikki qatlam potensiali ko'rinishida izlaymiz:

$$\vartheta_1(x,y;x_0\,y_0)=\int_0^l \mu_1(t;x_0,y_0)\,\eta^{\beta_0}(t)A_t\left[q_1,(\xi(t),\eta(t);x,y)\right]dt. \qquad (2.111)$$

(2.64) tengliklardan birinchisi va (2.109) chegaraviy shartni e'tiborga olib, $\mu_1(t;x_0\,y_0)$ zichlik uchun ushbu

$$\mu_1(s;x_0,y_0)-2\int_0^l K_1(s,t)\mu_1(t;x_0,y_0)dt=2q_1(x(s),y(s);x_0,y_0), \qquad (2.112)$$

integral tenglamaga ega bo'lamiz.

(2.112) tenglamaning o'ng tomoni $(x_0\,y_0)\in\Omega$ bo'lgani uchun, s o'zgaruvchining uzluksiz funksiyasidir. §2 ning 9-bandida $\lambda=2$ soni $K_1(s,t)$ yadroning xarakteristik soni bo'lmasligi isbotlangan edi, demak, (2.112) tenglamaning yechimi mavjud va uning uzluksiz yechimini ushbu

$$\mu_1(s;x_0,y_0)=2q_1(x,y;x_0,y_0)+4\int_0^l R_1(s,t;2)\,q_1(\xi(t),y(t);x_0,y_0)\,dt$$

(2.113)

ko‘rinishda yozish mumkin, bu yerda $R_1(s,t;\lambda)$, $K_1(s,t)$ yadroning rezolventasi, $(x(s),\,y(s))\in\Gamma$. (2.113) ni (2.111) ga qo‘yib, quyidagi tenglikni hosil qilamiz:

$$\vartheta_1(x,y;x_0\ y_0)=2\int_0^l q_1(\xi(t),\eta(t);x_0,y_0)\cdot\eta^{\beta_0}(t)\times A_t\left[q_1(\xi(t),\eta(t);x,y)\right]dt+$$

$$+4\int_0^l\int_0^l \eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x,y)\right]R_1(t,s;2)\times q_1(x(s),y(s);x_0,y_0)\,dtds.$$

(2.114)

Quyidagi funksiyani kiritamiz:

$$g(x,y)=\begin{cases}\vartheta_1(x,y;x_0,y_0), & (x,y)\in\Omega\ ,\\ -q_1(x,y;x_0,y_0), & (x,y)\in\Omega'\ .\end{cases}$$

$g(x,y)$ funksiya (2.1) tenglamaning Ω va Ω' sohalarning har birida regulyar yechimi bo‘ladi. $(x_0,y_0)\notin\Omega'$ bo‘lgani uchun, $g(x,y)$ funksiya Ω' sohada ixtiyoriy tartibli uzluksiz hosilaga ega va bu hosilalar $\Omega'\cup\Gamma$ da uzluksiz. Ω' sohada $g(x,y)=-q_1(x,y;x_0,y_0)$ funksiyani (2.1) tenglamaning

$$y^{\beta_0}(s)A_s[g(x,y)]|_\Gamma=-y^{\beta_0}(s)A_s[q_1(x(s),y(s);x_0,y_0)],\quad (2.115)$$

$$y^{\beta_0}\frac{\partial g(x,y)}{\partial y}\Big|_{y=0}=0,\quad (2.116)$$

chegaraviy shartlarni qanoatlantiruvchi yechimi deb qabul qilish mumkin.

Bu yechim $g(x,y)=-q_1(x,y;x_0,y_0)$ ko‘rinishda bo‘lsa ham, uni boshqa formada yozishga harakat qilamiz, ya’ni bu yechimni

$$g(x,y)=\int_0^l \rho_1(t,x_0,y_0)q_1(\xi,(t),\eta(t);x,y)dt\ ,\quad (x,y)\in\Omega',\quad (2.117)$$

oddiy qatlam potensiali ko‘rinishida ifodalaymiz. Bu yerda $\rho_1(t,x_0,y_0)$ noma’lum zichlik. (2.91) formulalarning ikkinchisidan foydalanib va (1.115) tenglikni hisobga olib, $\rho_1(s;x_0,y_0)$ zichlik uchun ushbu

$$\begin{aligned} &\rho_1(s,x_0,y_0)-2\int_0^l K_1(t,s)\rho_1(t,x_0,y_0)\,dt= \\ &2y^{\beta_0}(s)A_s\left[q_1\left(x(s),y(s);x_0,y_0\right)\right],\ \left(x(s),y(s)\right)\in\Gamma \end{aligned} \tag{2.118}$$

integral tenglama hosil qilamiz.

(2.118) integral tenglama (2.112) integral tenglamaga qo‘shmadir. (2.118) ning o‘ng tomoni s ning uzluksiz funksiyasidir, chunki $(x,y)\in\Omega$. Shunday qilib, (2.118) tenglama yagona uzluksiz yechimga egadir

$$\begin{aligned} &\rho_1(s;x_0\,y_0)=2\,y^{\beta_0}(s)A_s\left[q_1\left(x(s),y(s);x_0,y_0\right)\right]+ \\ &+4\int_0^l R_1(t,s;2)\eta^{\beta_0}(t)\cdot A_t\left[q_1\left(\xi(t),\eta(t);x_0,y_0\right)\right]dt \end{aligned} \tag{2.119}$$

Endi (2.117) formula bilan aniqlangan $g(x,y)$ni $(x,y)\in\Omega$ sohada o‘rganamiz.

Oddiy qatlam potensiali $g(x,y)$ ning Γ chiziqdagi qiymati $q_1(x,y;x_0\,y_0)$ ga teng, ya’ni Grin funksiyasining regulyar qismi $\vartheta_1(x,y;x_0\,y_0)$ ning Γ dagi qiymatiga teng, buning ustiga $y^{\beta_0}\dfrac{\partial g(x,y)}{\partial y}\Big|_{y=0}=0$, ya’ni $g(x,y)$ funksiya Ω sohada (2.1) tenglamaning N masalasi

$$u|_\Gamma=-g(x,y;x_0,y_0),\quad (x,y)\in\Gamma,$$

$$y^{\beta_0}\frac{\partial g(x,y)}{\partial y}\Big|_{y=0}=0$$

yechimidan iborat ekan. Lekin Grin funksiyasining regulyar qismi $\vartheta_1(x,y;x_0\,y_0)$ ham (2.109) va (2.110) shartlarga asosan, shakli o‘zgargan N masalasining yechimidir. Bu masalaning yechimini yagonaligiga ko‘ra, $(x,y)\in D$ sohada $g(x,y)=\vartheta_1(x,y;x_0,y_0)$, $(x,y)\in\Omega$, ya’ni

$$\vartheta_1(x,y;x_0\,y_0)=\int_0^l \rho_1(t;x_0,y_0)q_1(\xi(t),\eta(t);x,y). \qquad (2.120)$$

Shunday qilib, Grin funksiyasining regulyar qismi oddiy qatlam potensiali ko‘rinishida ifodalandi.

(2.91) formulalardan birinchisini (2.120) oddiy qatlam potensialiga qo‘llab, ushbu:

$$2y^{\beta_0}(s)A_s\left[\vartheta_1\left(x,(s),y(s);x_0,y_0\right)\right]_i=$$

$$\rho_1\left(s;x_0,y_0\right)+2\int_0^l K_1\left(t,s\right)\rho_1\left(t;x_0,y_0\right)dt \quad ,(2.121)$$

tenglikni hosil qilamiz.

$$y^{\beta_0}(s)A_s\left[q_1\left(x\,(s),y(s);x_0,y_0\right)\right]=y^{\beta_0}(s)A_s\left[q_2\left(x\,(s),y(s);x_0,y_0\right)\right]_i=$$

$$=y^{\beta_0}(s)A_s\left[q_1\left(x\,(s),y(s);x_0,y_0\right)\right]_e \quad \left((x_0,y_0)\in\Omega\right)$$

,

bo‘lgani uchun (2.118) munosabatga asosan,

$$2y^{\beta_0}(s)A_s\left[q_1\left(x,(s),y(s);x_0,y_0\right)\right]_i=$$

$$\rho_1\left(s;x_0,y_0\right)-2\int_0^l K_1\left(t,s\right)\rho_1\left(t;x_0,y_0\right)dt\,. \qquad (2.122)$$

(2.121) va (2.122) tengliklarni hadma-had qo‘shib va (2.108) munosabatni hisobga olib, ushbu tenglikni hosil qilamiz

$$\vartheta_1\left(x,y;x_0\,y_0\right)=\int_0^l \eta^{\beta_0}\left(t\right)A_t\left[G_1\left(\xi\left(t\right),\eta\left(t\right);x_0,y_0\right)\right]\times q_1\left(\xi\left(t\right),\eta\left(t\right);x,y\right)dt$$

(2.123)

Shunday qilib, (2.123) tenglikka asosan, (2.120) formulani quyidagicha yozamiz

$$\vartheta_1\left(x,y;x_0\,y_0\right)=\int_0^l \eta^{\beta_0}\left(t\right)A_t\left[G_1\left(\xi\left(t\right),\eta\left(t\right);x_0,y_0\right)\right]\times q_1\left(\xi\left(t\right),\eta\left(t\right);x,y\right)dt.$$

(2.124)

(2.119) tenglikning ikki tomonini $q_1(x(s),y(s);x,y)$ ga ko‘paytirib va hosil bo‘lgan tenglikni $[0,l]$kesmada s bo‘yicha integrallab, ushbu tenglikka kelamiz

$$\int_0^l \rho_1(s;x_0,y_0)\,q_1(x(s),y(s);x,y)\,ds=$$

$$=2\int_0^l q_1(x(s),y(s);x,y)\eta^{\beta_0}(s)A_s\left[q_1(x(s),y(s);x_0,y_0)\right]ds+$$

$$+4\int_0^l\int_0^1 R_1(t;s;2)\,q_1(x(s),y(s);x_0,y_0)\eta^{\beta_0}(t)\times A_t\left[q_1(\xi(t),\eta(t);x_0,y_0)\right]dt\,ds$$

(2.125)

Bu tenglikning chap tomoni (2.120) ga ko‘ra $\vartheta_1(x,y;x_0\,y_0)$ ga teng, o‘ng tomoni esa (2.114) ga ko‘ra $\vartheta_1(x,y;x_0\,y_0)$ ga teng, ya’ni

$$\vartheta_1(x,y;x_0\,y_0)=\vartheta_1(x_0\,y_0;x,y). \qquad (2.126)$$

Demak, $\vartheta_1(x,y;x_0\,y_0)$ funksiya D sohaning (x,y) va $(x_0\,y_0)$ nuqtalariga nisbatan simmetrikdir.

2.10-lemma. *Agar (x,y) va $(x_0\,y_0)$ nuqtalar D sohada bo‘lsa, $G_1(x,y;x_0\,y_0)$ Grin funksiyasi bu nuqtalarga nisbatan simmetrikdir.*

Lemmaning isboti Grin funksiyasining (2.108) ifodasi va (2.126) tenglikdan kelib chiqadi. $[-a,a]$ kesma va normalchiziq C_a :

$$x^2+\frac{4}{(m+2)^2}y^{m+2}=a^2$$

bilan chegaralangan Ω_0 normal soha uchun shakli o‘zgargan N masalasining Grin funksiyasi ushbu ko‘rinishga ega

$$G_{01}(x,y;x_0\,y_0)=q_1(x,y;x_0\,y_0)-\left(\frac{a}{R}\right)^{2\beta}q_1(x,y;\bar{x}_0\,\bar{y}_0), \qquad (2.127)$$

$$R^2 = x_0 + \frac{4}{(m+2)^2} y_0^{m+2}, \quad \bar{x}_0 = \frac{a^2}{R^2} x_0, \ \bar{y}_0^{\frac{(m+2)}{2}} = \frac{a^2}{R^2} y^{\frac{(m+2)}{2}},$$

$$\vartheta_{01}(x, y; x_0\, y_0) = -\left(\frac{a}{R}\right)^{2\beta} q_1(x, y; \bar{x}_0\, \bar{y}_0),$$

funksiyani

$$\vartheta_{01}(x, y; x_0\, y_0) = -\int_0^l \rho_1(s; x, y) \vartheta_{01}(x(s), y(s); x_0\, y_0) ds, \qquad (2.128)$$

ko‘rinishda ifodalash mumkin ekanligini isbotlaymiz, bu yerda $\rho_1(s; x, y)$ - (2.118) tenglamaning yechimidir.

Haqiqatdan ham, $(x_0, y_0)\ \Omega$ sohaning ixtiyoriy nuqtasi bo‘lsin. Ushbu

$$u(x, y; x_0, y_0) = -\int_0^l \rho_1(s; x, y) \vartheta_{01}(x(s), y(s); x_0\, y_0) ds, \qquad (2.129)$$

funksiyani o‘rganamiz, bu funksiya (2.1) tenglamani qanoatlantiradi, chunki (2.117) tenglikka ko‘ra $\rho_1(s; x, y)$ funksiya (2.1) tenglamani qanoatlantiradi. (2.129) tenglikda $\rho_1(s; x, y)$o‘rniga uning ifodasi (2.119) ni qo‘yib, ushbu

$$u(x, y; x_0, y_0) = -\int_0^l \vartheta_{01}(x(s), y(s); x_0\, y_0) \times \Big\{ 2y^{\beta_0}(s) \cdot A_s\big[q_1(x(s), y(s); x, y)\big] +$$

$$+ 4\int_0^l R_1(t, s; 2) \eta^{\beta_0}(t) A_t\big[q_1(\xi(t), \eta(t); x, y)\big] dt \Big\} ds = -\int_0^l y^{\beta_0}(s) \cdot A_s\big[q_1(x(s), y(s); x, y)\big]$$

$$\Big\{ 2\vartheta_{01}(x(s), y(s); x_0\, y_0) + 4\int_0^l R_1(s, t; 2) \vartheta_{01}(x(t), y(t); x_0, y_0) dt \Big\} ds =$$

$$= -\int_0^l \psi(s; x_0, y_0) \cdot A_s\big[q_1(x(s), \eta(s); x, y)\big] ds$$

(2.130)

bu yerda

$$\psi(s; x_0, y_0) = 2\vartheta_{01}(x(s), y(s); x_0, y_0) + 4\int_0^l R_1(s, t; 2) \vartheta_{01}(\xi(t), \eta(s); x_0, y_0) dt$$

(2.131)

ya'ni $\psi(s;x_0,y_0)$ – quyidagi integral tenglamaning yechimidir:

$$\psi(s;x_0,y_0)-2\int_0^l K_1(s,t)\psi(t;x_0,y_0)dt=2\vartheta_{01}(x,y;x_0,y_0). \quad (2.132)$$

(2.130) tenglikning o'ng tomonidagi ikki qatlam potensialiga (2.64) formulaning birinchisini qo'llab, ushbu tenglikka ega bo'lamiz:

$$u_i(x(s),y(s);x_0,y_0)=\frac{1}{2}\psi(s;x_0,y_0)-\int_0^l K_1(s,t)\psi(t;x_0,y_0)dt$$

Bu yerdan (2.132) ga ko'ra, ushbu tenglikka ega bo'lamiz:

$$u_i(x(s),y(s);x_0,y_0)=\vartheta_{01}(x(s),y(s);x_0,y_0),\ \ (x(s),y(s))\in\Gamma \quad (2.133)$$

$$y^{\beta_0}\left.\frac{\partial\rho_1(s;x,y)}{\partial y}\right|_{y=0}=0,$$

tenglikni hisobga olib, ushbu

$$y^{\beta_0}\left.\frac{\partial u(x,y;x_0,y_0)}{\partial y}\right|_{y=0}=0,\ y^{\beta_0}\left.\frac{\partial\vartheta_{01}(x,y;x_0,y_0)}{\partial y}\right|_{y=0}=0, \quad (2.134)$$

tengliklarni to'g'riligiga ishonch hosil qilish qiyin emas. (2.133) va (2.134) shartlarga ko'ra, $u(x,y;x_0,y_0)$ va $\vartheta_{01}(x,y;x_0,y_0)$ funksiyalar (2.1) tenglamani va bir xil chegaraviy shartlarni qanoatlantiradi, demak, shakli o'zgargan N masalasi yechimining yagonaligiga ko'ra:

$$u(x,y;x_0,y_0)=\vartheta_{01}(x,y;x_0,y_0).$$

Shunday qilib, (2.128) formula isbot bo'ldi.

Endi (2.108) tenglikdan (2.127) tenglikni ayirib, quyidagi ifodani hosil qilamiz:

$$H_1(x,y;x_0,y_0)=G_1(x,y;x_0,y_0)-G_{01}(x,y;x_0,y_0)=\vartheta_1(x,y;x_0,y_0)-\vartheta_{01}(x,y;x_0,y_0) \quad (2.135)$$

yoki (2.126) tenglikka ko'ra,

$$H_1(x,y;x_0,y_0)=\vartheta_1(x_0,y_0;x,y)-\vartheta_{01}(x,y;x_0,y_0).$$

Endi (2.120) va (2.128) tenglikka asosan, $H_1(x,y;x_0,y_0)$ ni quyidagicha almashtiramiz:

$$H_1(x,y;x_0,y_0)=\int_0^l \rho_1(t;x,y)q_1(\xi(t),\eta(t);x_0,y_0)d\ +\int_0^l \rho_1(t;x,y)\vartheta_{01}(\xi(t),\eta(t);x_0,y_0)dt=$$

$$=\int_0^l \rho_1(t;x,y)q_1(\xi(t),\eta(t);x_0,y_0)dt-\int_0^l \rho_1(t;x,y)\left(\frac{a}{R}\right)^{2\beta} q_1(\xi(t),\eta(t);\overline{x}_0,\overline{y}_0)dt$$

yoki

$$H_1(x,y;x_0,y_0)=\int_0^l \rho_1(t;x,y)G_{01}(\xi(t),\eta(t);x,y)dt\,. \quad (2.136)$$

10. Shakli o'zgargan N masalasining yechimi (x_0,y_0) nuqta Ω sohaga tegishli bo'lsin $(x_0,y_0)\in\Omega.$ Γ chiziqqa parallel Γ_ε va $y=\delta>\varepsilon$ to'g'ri chiziq bilan chegaralangan sohani $\Omega_{\varepsilon,\delta}$ orqali belgilaymiz. ε va δ ni shunday kichik tanlaymizki, $(x_0,y_0)\in\Omega_{\varepsilon,\delta}$ bo'lsin. $\Omega_{\varepsilon,\delta}$ sohadan markazi (x_0,y_0) nuqta, radiusi ρ ga teng bulgan doirani qirqib olamiz va $\Omega_{\varepsilon,\delta}$ ning qolgan qismini $\Omega_{\varepsilon,\delta}^{\rho}$ orqali belgilaymiz. $\Omega_{\varepsilon,\delta}^{\rho}$ sohada $G_1(x,y;x_0,y_0)$ – Grin funksiyasi (2.1) tenglamaning regulyar yechimidan iborat.

$u(x,y)$ funksiya sohada (2.1) tenglamaning ushbu:

$$u|_\Gamma=\varphi(s),\ 0\le s\le l;\qquad \lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y}=\nu(x),\ -a<x<a,$$

shartlarni qanoatlantiruvchi shakli o'zgargan N masalasining yechimi bo'lsin, bu yerda $\varphi(s)$, $0\le s\le l$ kesmada uzluksiz funksiya, $\nu(x)$ funksiya esa $(-a,a)$ intervalda uzluksiz va $x\to\pm a$ da $1-2\beta$ dan kichik tartibda cheksizlikka intilishi mumkin. (2,14) formulada $\vartheta=y^{\beta_0}G_1(x,y;x_0,y_0)$, $u(x,y)$ esa o'zgargan N masalasining yechimi deb qabul qilib, ushbu

$$\int_{\gamma_\varepsilon} y^{\beta_0}\{uA_s[G_1]-G_1A_s[u]\}ds=0,$$

yoki

$$\int_{\Gamma_\varepsilon} y^{\beta_0}\left\{\left[G_1 A_s[u]\right] - uA_s\left[G_1\right]\right\} ds + \int_{x_1}^{x_2} y^{\beta_0}\left(u\frac{\partial G_1}{\partial y} - G_1\frac{\partial u}{\partial y}\right) dx =$$

$$= \int_{C_\rho} y^{\beta}\left\{\left[G_1 A[u]\right] - uA_s\left[G_1\right]\right\} ds$$

tenglikka kelamiz, bu yerda $\gamma_\varepsilon = \partial\Omega^p_{\varepsilon,\delta}$, x_1, x_2 - Γ_ε chiziqning $y=\delta$ chizig'i bilan kesishgan nuqtasining absissasi, C_ρ kesib olingan doira aylanasi. Oxirgi tenglikda $p \to 0$ da limitga o'tib, hamda shakli o'zgargan N masalasining chegaraviy shartlarini hisobga olib, Ω sohada (2,1) tenglama uchun shakli o'zgargan N masalasi yechimini beruvchi

$$u(x_0, y_0) = -\int_{-a}^{a} v(t)G_1(t,0;x_0,y_0)dt - \int_0^l \phi(s)\eta^{\beta_0}(s)A_s\left[G_1(\xi(s),\eta(s);x_0,y_0)\right]ds \tag{2.137}$$

formulaga kelamiz. (2.137) formula shakli o'zgargan N masalasining yechimini berishini isbotlaymiz. (2.137) formuladagi birinchi qo'shiluvchini $J_1(x_0, y_0)$ orqali belgilab olamiz:

$$J_1(x_0, y_0) = -\int_{a}^{-a} v(t)G_1(t,0;x_0,y_0)dt \tag{2.138}$$

Ω sohada $J_1(x_0,y_0) \in C(\overline{\Omega})$ va u (2.1) tenglamaning regulyar yechimi ekanligi (2.138) formuladan ko'rinib turibdi. Quyidagi belgilashni kiritamiz:

$$\omega(x_0, y_0) = -\int_{-a}^{a} v(t)q_1(t,0;t_0,y_0)dt = -k_1\int_{-a}^{a} v(t)\left[(t-x_0)^2 + \frac{4}{(m+2)^2}y_0^{m+2}\right]^{-\beta} dt \tag{2.139}$$

$\omega(x_0, y_0)$ $\overline{\Omega}$ uzluksiz funksiyadir. $J_1(x_0, y_0)$ funksiyani (2.139), (2.114) tengliklar va $\vartheta_1(x, y; x_0, y_0)$ funksiyaning simmetrikligiga ko'ra, quyidagi ko'rinishda yozib olish mumkin:

$$J_1(x_0,y_0)=\omega(x_0,y_0)-\int\limits_{-a}^{a}v(t)\vartheta_1(t,0;x_0,y_0)dt=\omega(x_0,y_0)-\int\limits_{-a}^{a}v(t)\vartheta(x_0,y_0;t)dt=$$

$$=\omega(x_0,y_0)-\int\limits_{-a}^{a}v(x)\left\{2\int\limits_0^l q_1(\xi(t),\eta(t);x,0)\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x_0,y_0)\right]dt+\right.$$

$$\left.+4\int\limits_0^l\int\limits_0^l\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x_0,y_0)\right]R_1(t,s,2)q_1(x(s),y(s);x,0)dtds\right\}dx$$

ya'ni

$$J_1(x_0,y_0)=\omega(x_0,y_0)+2\int\limits_0^l\omega(\xi(t),\eta(t)),\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x_0,y_0)\right]dt+$$

$$+4\int\limits_0^l\int\limits_0^l\omega(\xi(t_1),\eta(t_1))\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x_0,y_0)\right]R_1(t,t_1;2)dtdt_1 \tag{2.140}$$

(2.140) formuladagi integrallar ikki qatlam potensiallaridan iborat. $(x_0(s),y_0(s))\in\Gamma$ bo'lganda, (2.64) formulaning birinchisini e'tiborga olib, quyidagi ifodaga kelamiz:

$$J_1(x_0(s),y_0(s))\big|_{\Gamma}=\omega(x_0(s),y_0(s))+$$

$$+2\left[-\frac{1}{2}\omega(x_0(s),y_0(s))+\int\limits_0^l\omega(\xi(t),\eta(t))K_1(s,t)dt\right]+$$

$$+4\int\limits_0^l\left[-\frac{1}{2}R_1(s,t_1;2)+\int\limits_0^l R_1(t,t_1;2)K(s,t)dt\right]\omega(\xi(t_1),\eta(t_1))dt_1=$$

$$= \omega(x_0(s), y_0(s)) - \omega(x_0(s), y_0(s)) +$$

$$+2\int_0^l \omega(\xi(t),\eta(t))K_1(s,t)dt - 2\int_0^l R_1(s,t_1;2)\omega(\xi(t_1),\eta(t_1))dt_1$$

$$+$$

$$+4\int_0^l\int_0^l R_1(t,t_1;2)K_1(s,t)\omega(\xi(t_1),\eta(t_1))dtdt_1 \qquad (2.141)$$

bu yerdagi ikki karrali integralda $t_1 \to t$, $t \to t_1$ almashtirish bajarib,

$$= -2\int_0^l R_1(s,t_1;2)\omega(\xi(t_1),\eta(t_1))dt_1 + 2\int_0^l \omega(\xi(t),\eta(t))\left[K_1(s,t) + 2\int_0^l K_1(s,t_1)R_1(t_1,t;2)dt_1\right]dt$$

tengliknihosil qilamiz. Endi bu yerda rezolventa uchun ushbu

$$R_1(s,t;2) = K_1(s,t) + 2\int_0^l K_1(s,t_1)R_1(t_1,t;s)dt_1 .$$

integral tenglamani hisobga olib, (2.141) dan ushbu tengliknihosil qilamiz:

$$J_1(x_0(s), y_0(s))\big|_{\Gamma} = -2\int_0^l \omega(\xi(t_1),\eta(t_1))R_1(s,t_1;2)dt_1 + 2\int_0^l \omega(\xi(t),\eta(t))R_1(s,t;2)dt = 0 . \qquad (2.142)$$

Endi

$$\lim_{y_0 \to 0} y_0^{\beta_0} \frac{\partial J_1(x_0, y_0)}{\partial y_0} = \nu(x_0) \ , \ -a < x < a , \qquad (2.143)$$

tenglikni isbotlaymiz. $J_1(x_0,y_0)$ integralni $\vartheta_1(x,y;x_0,y_0)$ ning simmetrikligiga, (2.139) va (2.120) formulaga ko‘ra, ushbu ko‘rinishda yozib olamiz:

$$J_1(x_0,y_0)=\int_{-a}^{a}\nu(x)G_1(x,0;x_0,y_0)dx=\omega(x_0,y_0)-$$

$$-\int_{-a}^{a}\nu(x)dx\int_{-a}^{a}\rho(t,x;0)q_1(\xi,\eta;x_0,y_0)dt. \tag{2.144}$$

Bu ifodaning ikkinchi qo‘shiluvchisidan y_0 bo‘yicha olingan hosila $y_0^{\beta_0}$ salmoq bilan $y_0\to 0$ da 0 ga teng, ya’ni

$$\lim_{y_0\to 0} y_0^{\beta_0}\frac{\partial q_1}{\partial y_0}=0. \tag{2.145}$$

Shunday qilib,

$$\lim_{y_0\to 0} y_0^{\beta_0}\frac{\partial\omega}{\partial y_0}=\nu(x_0),\ -a<x_0<a, \tag{2.146}$$

tenglikni isbotlash qoldi.

(2.139) tenglikni y_0 bo‘yicha differensiallab, quyidagi tenglikka ega bo‘lamiz:

$$\frac{\partial\omega}{\partial y_0}=\frac{4\beta k_1}{m+2}y_0^{m+1}\int_{-a}^{a}\nu(\xi)\left[(\xi-x_0)^2+\frac{4}{(m+2)}y_0^{m+2}\right]^{-\beta-1}d\xi \tag{2.147}$$

Ushbu

$$A(x_0,y_0)=\frac{4\beta k_1}{m+2}y_0^{m+1}\times\int_{-a}^{a}\left[(\xi-x_0)^2+\frac{4}{(m+2)}y_0^{m+2}\right]^{-\beta-1}d\xi \tag{2.148}$$

funksiyani kiritamiz va

$$\lim_{y_0\to 0} y^{\beta_0}A(x_0,y_0)=1 \tag{2.149}$$

tenglikni isbotlaymiz. Haqiqatdan ham, (2.148) tenglikda integral o‘zgaruvchisini $\xi=x_0+\frac{2}{m+2}y_0{}^{(m+2)/2}t$ ko‘rinishda almashtiramiz, u holda

$$A(x_0,y_0)=2\beta\left(\frac{2}{m+2}\right)^{-2\beta}k_1 y_0^{-\beta_0}\int_{\alpha_1}^{\alpha_2}\left(1+t^2\right)^{-\beta-1}dt, \qquad (1.150)$$

bu yerda

$$\alpha_1=\frac{(m+2)(a+x_0)}{2y_0^{(m+2)\backslash 2}},\ \ \alpha_2=\frac{(m+2)(a-x_0)}{2y_0^{(m+2)\backslash 2}}\ k_1=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2\beta}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)}.$$

(2.150) da $y_0\to 0$ da limitga o‘tib, ushbu

$$\lim_{y_0\to 0} y^{\beta_0}A(x_0,y_0)=\frac{2^{2\beta-1}\beta}{\pi}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)}\int_{-\infty}^{\infty}\left(1+t^2\right)^{-\beta-1}dt.$$

tenglikni olamiz, bu yerda

$$\int_{-\infty}^{\infty}\left(1+t^2\right)^{-\beta-1}dt=\frac{\pi\,\Gamma(2\beta)}{2^{2\beta-1}\beta\Gamma^2(\beta)},$$

tengliknihisobga olib,

$$\lim_{l_n\to 0}\nu^{\beta_0}A(x_0y_0)=\frac{2^{2\beta-1}\beta}{\pi}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)}\frac{\pi\Gamma(2\beta)}{2^{2\beta-1}\beta\Gamma^2(\beta)}=1, \qquad (2.151)$$

tenglikka kelamiz.

Shunday qilib,

$$\frac{4\beta k_1}{m+2}y_0^{m+1+\beta_0}\int_{-a}^{a}\left[(\xi-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]^{-\beta-1}d\xi=1. \qquad (2.152)$$

tenglikni isbotladik.

(2.152) tenglikni $\nu(\bar{x}_0)$ ga ko‘paytirib, ushbu tenglikni hosil qilamiz:

$$\nu(\bar{x}_0)=\frac{4\beta R_1}{m+2}y_0^{m+1+\beta_0}\int_{-\alpha}^{\alpha}\nu(\bar{x}_0)\left[(\xi-x_0)^2+\frac{4}{(m+2)}y_0^{m+2}\right]^{-\beta-1}d\xi.$$

Bu yerdan (2.147) tenglikni e’tiborga olib, ushbu

$$\lim_{\substack{x_0 \to \overline{x_0} \\ y_0 \to 0}} y_0^{\beta_0} \frac{\partial\omega(x_0,y_0)}{\partial y_0} = \nu(\overline{x_0}) + \frac{4\beta k_1}{m+2} \lim_{\substack{x \to \bar{x}_0 \\ y_0 \to 0}} y_0^{m+1+\beta_0} \int_{-a}^{a} \left[\nu(\xi) - \nu(\overline{x_0})\right] \times$$

$$\times \left[(\xi - x_0)^2 \frac{4}{(m+2)^2} y_0^{m+2} \right]^{-\beta-1} d\xi \tag{2.153}$$

munosabatga ega bo'lamiz. Endi quyidagi integralni

$$\frac{4\beta k_1}{(m+2)} y_0^{m+2+\beta_0} \int_{-a}^{a} \left[\nu(\xi) - \nu(\overline{x_0})\right] \left[(\xi - x_0)^2 + \frac{4}{(m+2)^2} y_0^{m+2} \right]^{-\beta-1} d\xi =$$

$$= \frac{4\beta k_1}{(m+2)} y_0^{m+1} \left\{ \int_{-a}^{\overline{x_0}-\delta} + \int_{\overline{x_0}-\delta}^{\overline{x_0}+\delta} + \int_{\overline{x_0}+\delta}^{a} \right\} = J_1 + J_2 + J_3 \tag{2.154}$$

$y_0 \to 0$, $x_0 \to \bar{x}_0$ $(-a < x < a)$ da nolga intilishini ko'rsatamiz. $\nu(x)$ funksiyamiz $(-a,a)$ intervalda uzluksiz. Shuning uchun oldindan berilgan ixtiyoriy $\varepsilon > 0$ soni uchun shunday $\delta > 0$ sonini ko'rsatish mumkinki, agar ξ va $\overline{x_0}$ uchun $|\xi - \bar{x}_0| < \delta$ tengsizlik bajarilsa, $|\nu(\xi) - \nu(\bar{x}_0)| < \varepsilon$ tengsizlik bajariladi.

Demak,

$$|J_2| < \frac{\varepsilon}{2} \frac{4\beta k_1}{m+2} y_0^{m+1+\beta_0} \int_{\overline{x_0}-\delta}^{\overline{x_0}+\delta} \left[(\xi - x_0)^2 \frac{4}{(m+2)^2} y_0^{m+2} \right]^{-\beta-1} d\xi =$$

$$= \frac{\varepsilon}{3} \frac{2^{2\beta-1}\beta}{\pi} \frac{\Gamma^2(\beta)}{\Gamma(2\beta)} \int_{\alpha_1}^{\alpha_2} \left(1+t^2\right)^{-\beta-1} dt < \frac{\varepsilon}{3} \times \frac{2^{2\beta-1}\beta}{\pi} \frac{\Gamma^2(\beta)}{\Gamma(2\beta)} \int_{-\infty}^{\infty} \left(1+t^2\right)^{-\beta-1} dt = \frac{\varepsilon}{3}, \tag{2.155}$$

bu yerda

$$\alpha_{12} = \frac{-(m+2)(x_0 - \bar{x}_0 + \delta)}{2y_0^{(m+2)\backslash 2}}, \quad \alpha_{1\delta} = \frac{(m+2)(x_0 - \overline{x_0} - \delta)}{2y_0^{(m+2)\backslash 2}}.$$

Endi J_1 integralni baholaymiz.Bu integralda $-a<\xi<\overline{x}_0-\delta$ va $\left|\xi-\overline{x_0}\right|\geq\delta$ shu bilan birga, $\left|x_0-\overline{x_0}\right|<\frac{\delta}{2}$ bo‘lganda, $\left|\xi-x_0\right|>\frac{\delta}{2}$ bo‘ladi va

$$(\xi-x_0)^2+\frac{4}{(m+2)}y_0^{m+2}>\frac{\delta^2}{4}.$$

Shunday qilib,

$$\left|J_1\right|\leq\frac{4\beta k_1}{m+2}y_0^{m+1+\beta_0}\left(\frac{4}{\delta^2}\right)^{\beta+1}\times\int\limits_{-a}^{x_0-\delta}\left[\nu(\xi)-\nu(\overline{x}0)\right]d\xi\leq Cy_0^{m+1+\beta_0}<\frac{\varepsilon}{3},$$

yetarli kichik $y_0>0$ uchun shu usul bilan uchinchi integral J_3 uchun ham $\left|J_3\right|<\frac{\varepsilon}{3}$ bahoni isbotlash mumkin. J_1, J_2, J_3 integrallar uchun olingan baholardan $\varepsilon>0$ sonning ixtiyoriyligidan foydalanib, $y_0\to 0$, $x_0\to\overline{x}_0$ $(-a<x<a)$ ga intilganda, (2.154) integralning nolga intilishi kelib chiqadi, bundan (2.153) ga asosan,

$$\lim_{\substack{x_0\to\overline{x_0}\\ y_0\to 0}} y^{\beta_0}\frac{\partial\omega(x_0,y_0)}{\partial y_0}=\nu(\overline{x}_0),\quad(-a<\overline{x_0}<a),$$

tenglikka kelamiz, ya’ni (2.143) formula isbot bo‘ldi.

Endi (2.137) formuladagi $J_2(x_0,y_0)$ni o‘rganamiz. Bu integralni (2.123), (2,119) formulalarga ko‘ra, quyidagi ko‘rinishda yozib olamiz:

$$J_2(x_0,y_0)=-\int\limits_0^l\phi(s)A_s\left[G_1(\xi,\eta;x_0,y_0)\right]ds=\int\limits_0^l\phi(s)\rho_1(s;x_0,y_0)ds=$$

$$-\int\limits_0^l\phi(s)\Big\{2y^{\beta_0}(s)\,A_s\left[q_1(x(s),y(s);x_0,y_0)\right]+$$

$$+4\int\limits_0^l R_1(t,s;2)\eta^{\beta_0}(t)A_t\left[q_1(\xi(t),\eta(t);x_0,y_0)\right]dt\Big\}\,ds=-\int\limits_0^l\theta(t)\eta^{\beta_0}(t)A_t\left[q_1(\xi,\eta;x_0,y_0)\right]ds$$

(2.156)

bu yerda

$$\theta(t) = 2\varphi(t) + 4\int_0^l R_1(t,s;2)\varphi(s)ds\,, \qquad (2.157)$$

ya'ni, $\theta(s)$ ushbu

$$\theta(s) - 2\int_0^l K_1(s,t)\theta(t)dt = 2\varphi(s)\,, \qquad (2.158)$$

integral tenglamaning yechimidir. $\theta(s)$ uzluksiz funksiya bo'lgani uchun $J_2(x_0, y_0)$ Ω sohada (2.1) tenglamaning regulyar yechimi va u $\overline{\Omega}$ sohada uzluksiz. $J_2(x_0, y_0)$ funksiya (2.156) tenglikka ko'ra, ikki qatlam potensialidan iborat, demak, $(x_0, y_0) \in \Gamma$ bo'lganda, uning uchun (2.64) tenglikning birinchisi o'rinlidir:

$$= \frac{1}{2}\theta(s) - \int_0^l \theta(t)K_1(s,t)dt = \frac{1}{2}\left[2\phi(s) + 4\int_0^l R_1(s,t;2)\phi(t)dt\right] -$$

$$-\int_0^l \left[\frac{1}{2}\phi(t) + 4\int_0^l R_1(t,t;2)\phi(t_1)dt\right]K_1(s,t)dt =$$

$$= \phi(s) + 2\int_0^l R_1(s,t_1;2)\phi(t_1)dt_1 - 2\int_0^l \phi(t)R_1(s,t)dt -$$

$$-4\int_0^l\int_0^l \phi(t_1)R_1(t,t_1;2)K_1(s,t)dtdt_1 = \phi(s) +$$

$$+2\int_0^l \phi(t_1)\left[R_1(s,t_1;2) - 2\int_0^l R_1(t,t_1;2)K_1(s,t)dt\right]\times dt_1 - 2\int_0^l \phi(t)K_1(s,t)dt =$$

$$= \phi(s) + 2\int_0^l R_1(s,t_1;2)\phi(t_1)dt_1 - 2\int_0^l \phi(t)\left[K_1(s,t) + \int_0^e K_1(s,t_1)R_1(t_1,t;2)dt_1\right]dt.$$

Bu yerda $R_1(s,t,2)$ rezolventa uchun

$$R_1(s,t,2) = K_1(s,t) + \int_0^l K_1(s,t_1)R_1(t_1,t;2)dt_1$$

tenglamani hisobga olib, ushbu

$$J(x_0,y_0)+2\int_0^l R_1(s,t;2)\varphi(t)dt-2\int_0^l R_1(s,t;2)\varphi(t)dt=\varphi(s), \quad (2.159)$$

tenglikni hosil qilamiz.

Bevosita hisoblash yordamida

$$\lim_{y_0\to 0} y^{\beta_0}\frac{\partial J_2(x_0,y_0)}{\partial y_0}=0, \qquad -a<x_0<a, \qquad (2.160)$$

tenglikka kelamiz.

Shunday qilib, (2.137) formula shakli o‘zgargan N masalasining yechimini berar ekan.

Izoh: (2.137), (2.136), (2.127) formulalardan foydalanib, shakli o‘zgargan N masalasining yechimi ushbu ko‘rinishda yozib olish mumkin:

$$\begin{aligned} u(x_0,y_0)=&-\int_{-a}^{a}\nu(x)\left[G_{01}(x,0;x_0,y_0)+H_1(x,0;x_0,y_0)\right]dx-\\ &-\int_0^l\varphi(s)\left\{A_s\left[G_{01}(\xi,\eta;x_0,y_0)\right]+A_s\left[\mathrm{H}_1(\xi,\eta;x_0,y_0)\right]\right\}ds, \end{aligned} \qquad (2.161)$$

bu yerda

$$G_{01}(x,0;x_0,y_0)=k_1\times$$

$$\times\left\{\left[(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]^{-\beta}-\left[\left(a-\frac{x_0x}{a}\right)^2+\frac{4x^2}{a^2(m+2)^2}y_0^{m+2}\right]^{-\beta}\right\}$$

$$H_1(x,y;x_0,y_0)=\int_0^l\rho_2(t;x_0,y_0)G_{01}(\xi,\eta;x_0,y_0)dt, \qquad (2.161)$$

yechimni shaklda yozib olish aralash turdagi tenglama uchun Trikom masalasini yechishda juda qulaylik yaratadi. Normal Ω_0 soha uchun $H_1(x,y;x_0,y_0)\equiv 0$, bu holda (2.161) yechim juda sodda ko‘rinishda bo‘ladi [21].

$$u(x,y)=k_1\int_{-a}^{a}\nu(t)\times$$

$$\times\left\{\left[(x_0-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{-\beta}-\left[\left(a-\frac{xt}{a}\right)^2+\frac{4t^2}{a^2(m+2)^2}y^{m+2}\right]^{-\beta}\right\}dt-$$

$$-\eta_1\beta(m+2)(a^2-R^2)\int_0^l \eta^{\beta_0-1}(s)\varphi(\xi(s))(r_1^2)^{-\beta-1}F(\beta,\beta+1,2\beta;1-\sigma)\frac{d\xi(s)}{ds}ds$$

. (2.162)

11. Dirixle masalasining Grin funksiyasi. Ω sohada (2.1) tenglama uchun Dirixle masalasining Grin funksiyasi deb, quyidagi shartlarni qanoatlantiruvchi $G_2(x,y;x_0,y_0)$ funksiyaga aytiladi:

1) Ω sohaning (x_0,y_0) nuqtasidan tashqari barcha nuqtalarida (2.1) tenglamaning regulyar yechimi;

2)

$$G_2(x,y;x_0,y_0)\big|_{\Gamma\cup\overline{A}B}=0, \qquad (2.163)$$

chegaraviy shartni qanoatlantiradi;

1) uni

$$G_2(x,y;x_0,y_0)=q_2(x,y;x_0,y_0)+\vartheta_2(x,y;x_0,y_0), \qquad (2.164)$$

shakldа ifodalash mumkin, bu yerda

$$q_2(x,y;x_0,y_0)=k_2\left(\frac{4}{m+2}\right)^{4\beta-2}\times$$
$$\times r_1^{-2\beta}(1-\sigma)^{1-2\beta}F(1-\beta,1-\beta,2-2\beta;1-\sigma), \qquad (2.165)$$

(2.1) tenglamaning fundamental yechimi $\vartheta_1(x,y;x_0,y_0)$ Ω sohaning barcha nuqtalarida (2.1) tenglamaning regulyar yechimi.

Grin funksiyasini tuzish, uning regulyar qismi $\vartheta_2(x,y;x_0,y_0)$ ni topishga olib kelinadi, $\vartheta_2(x,y;x_0,y_0)$ funksiya (2.163) va (2.164) tengliklarga asosan,

$$\vartheta_2(x,y;x_0,y_0)\big|_{\Gamma}=-q_2(x,y;x_0,y_0)\big|_{\Gamma} \qquad (2.166)$$

$$\vartheta_2(x,0;x_0,y_0)=0 \qquad (2.167)$$

chegaraviy shartlarni qanoatlantirishi kerak.

$\vartheta_2(x,y;x_0,y_0)$ funksiyani ikki qatlam potensiali shaklida izlaymiz:

$$\vartheta_2(x,y;x_0,y_0)=\int_0^l \mu_2(t;x_0,y_0)\eta^{\beta_0}(t)A_t[\xi(t),\eta(t);x,y]dt. \quad (2.168)$$

Ikki qatlam potensiali $W^{(2)}(x,y)$ uchun o‘rinli bo‘lgan (2.86) munosabatlarning birinchisidan, hamda (2.166) chegaraviy shartlardan foydalanib, $\mu_2(t;x_0,y_0)$ zichlik uchun quyidagi integral tenglamani hosil qilamiz:

$$\mu_2(s;x_0,y_0)-2\int_0^l K_2(s,t)\,\mu_2(t;x_0,y_0)dt=2q_2(x(s),y(s);x_0,y_0) \quad (2.169)$$

(2.169) tenglamaning o‘ng tomoni s argumentning uzluksiz funksiyasidir $((x_0,y_0)\in\Omega\ ,\ (x(s),y(s)\in\Gamma)$. 2.7 lemmaga asosan, (2.169) integral tenglamaga Fredgolm nazariyasini qo‘llash mumkin.

1§ ning 9-bandida $\lambda=2,\ K_2(s,t)$ yadroning xarakteristik soni bo‘lmasligi isbotlangan edi, demak, (2.169) tenglama yechimi mavjud va uning uzluksiz yechimini:

$$\mu_2(s;x_0,y_0)=2q_2(x(s),y(s);x_0,y_0)+4\int_0^l R_2(s,t;2)\,q_2(\xi(t),\eta(t);x_0,y_0)dt, \quad (2.170)$$

shaklda ifodalash mumkin, bu yerda $R_2(s,t;2)$ orqali $K_2(s,t)$ yadroning rezolventasi belgilangan $(x(s),y(s))\in\Gamma$. (2.170) dan $\mu_2(s;x_0,y_0)$ ni (2.168) ga qo‘yib, ushbu

$$\vartheta_2(x,y;x_0,y_0)=2\int_0^l q_2(\xi,\eta;x_0,y_0)\eta^{\beta_0}(t)A_t[\xi(t),\eta(t);x,y]dt+$$
$$+4\int_0^l\int_0^l \eta^{\beta_0}(t)A_t[q_2(\xi(t),\eta(t);x,y)]R_2(t,s;2)q_2(x(s),y(s);x_0,y_0)dtds$$

tenglikka kelamiz.

Shakli o‘zgargan N masalasi uchun Grin funksiyasini tuzishdagi mulohazalarni takrorlab, Dirixle masalasi uchun Grin funksiyasining regulyar qismi $\vartheta_2(x,y;x_0,y_0)$ ni oddiy qatlam potensiali shaklida ifodalash mumkin:

$$\vartheta_2(x,y;x_0,y_0)=\int_0^l \rho_2(t;x_0,y_0)\,q_2(\xi(t),\eta(t);x,y)dt, \quad (2.172)$$

bu yerda

$$\rho_2(s;x_0,y_0)=2y^{\beta_0}(s)A_s\left[q_2(x(s),y(s));x_0,y_0\right]+4\int_0^l R_2(t,s;2)\,A_t\left[q_2(\xi(t),\eta(t));x_0,y_0\right]dt$$

(2.173)

ya'ni, $\rho_2(s;x_0,y_0)$ ushbu

$$\rho_2(s;x_0,y_0)-2\int_0^l K_2(t,s)\,\rho_2(t;x_0,y_0)\,dt=2y^{\beta_0}(s)A_s\left[q_2(x(s),y(s));x_0,y_0\right] \quad (2.174)$$

integral tenglamaning yechimi.

(2.172)ga oddiy qatlam potensiali konormal hosilasi uchun o'rinli bo'lgan (2.91) formulalardan birinchisini qo'llab, quyidagi tenglikni hosil qilamiz:

$$2y^{\beta_0}(s)A_s\left[\rho_2(x(s),y(s));x_0,y_0\right]\Big|_i=\rho_2(s;x_0,y_0)+2\int_0^l K_2(t,s)\,\rho_2(t;x_0,y_0)\,dt \quad . (2.175)$$

Endi (2.174) va (2.175) tenglamalardan

$$y^{\beta_0}(s)A_s\left[G_2(x(s),y(s));x_0,y_0\right]=\rho_2(s;x_0,y_0), \quad (2.176)$$

tenglikka kelamiz va bu tenglikka asosan, (2.172) formulani ushbu

$$\vartheta_2(x,y;x_0,y_0)=\int_0^l q_2(\xi(t),\eta(t);x,y)A_t\left[G_2\xi(t),\eta(t);x_0,y_0\right]dt, \quad (2.177)$$

ko'rinishda yozish mumkin.

2.11-lemma.*Agar* (x_0,y_0) *nuqta* Ω *soha ichida joylashgan bo'lsa, u holda* $G_2(x,y;x_0,y_0)$ *Grin funksiyasi* (x,y) *va* (x_0,y_0) *nuqtalarga nisbatan simmetrikdir.*

2.11 lemmaning isboti 2.10 lemmaning isbotiga o'xshash bajariladi.

Ω_0 normal soha uchun Grin funksiyasi quyidagi ko'rinishda bo'ladi:

$$G_{02}(x,y;x_0,y_0)=q_2(x,y;x_0,y_0)-\left(\frac{a}{R}\right)^{2\beta}q_2(x,y;\bar{x}_0,\bar{y}_0), \quad (2.178)$$

bu yerda

$$R^2=x_0{}^2+\frac{4}{(m+2)^2}y_0{}^{m+2},\ \bar{x}_0=\frac{a^2}{R^2}x_0,\bar{y}_0^{(m+2)/2}=\frac{a^2}{R^2}y_0^{(m+2)/2}$$

Grin funksiyasi $G_{02}(x,y;x_0,y_0)$ ning regulyar qismi

3-§. (2.1) tenglamada parametr $\beta_0 = -m/2$ bo'lgan holda Dirixle va shakli o'zgargan Xolmgren (N) masalalarining yechimi.

(2.1) tenglamani parametr $\beta_0 = -m/2$ bo'lgan holda o'rganamiz

$$y^m u_{xx} + u_{yy} - (m/2)\, u_y = 0 \qquad (2.185)$$

va bu tenglama uchun normal soha Ω_0 da Dirixle va shakli o'zgargan N masalalarini yechimini beruvchi formulalarni keltirib chiqaramiz.

1. (2.185) tenglama uchun shakli o'zgargan N masalasining yechimi. (2.162) formulada parametr $\beta_0 \to -m/2$, ya'ni $\beta \to 0$ limitga o'tib va ushbu limitlarni hisobga olib,

a) $\lim\limits_{\beta\to 0} F(\beta, \beta 1, 2\beta; 1-\sigma) = \dfrac{1+\sigma}{2\sigma}$;

b) $\lim\limits_{\beta\to 0} k_1 \beta = \dfrac{1}{2\pi}$;

c)

$$\lim_{\beta\to 0} k_1 \left\{ \left[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2} \right]^{-\beta} - \left[(1-xt)^2 + \frac{4t^2}{(m+2)^2} y^{m+2} \right]^{-\beta} \right\} =$$

$$= -\frac{1}{2\pi} \left\{ \ln\left[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2} \right] - \ln\left[(1-xt)^2 + \frac{4t^2}{(m+2)^2} y^{m+2} \right] \right\},$$

Ω_0 sohada shakli o'zgargan N masalasining

$$\lim_{y\to +0} y^{-m/2} \frac{\partial u}{\partial y} = \nu(x), x \in (-1,1); \quad u(x,y)\big|_{\sigma_0} = \varphi(x(s)) \;\; 0 \le s \le l \qquad (2.186)$$

yechimni beruvchi ushbu formulaga ega bo'lamiz:

$$u(x,y) = \frac{1}{2\pi} \int_{-1}^{1} \nu(t) \left\{ \ln\left[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2} \right] - \ln\left[(1-xt)^2 + \frac{4t^2}{(m+2)^2} y^{m+2} \right] \right\} dt -$$

$$- \frac{(m+2)(1-R^2)}{4\pi} \int_0^l \varphi(\xi(s)) \eta^{-(m+2)/2}(s) \left(r^{-2} + r_1^{-2} \right) d\xi(s).$$

(2.187)

2. (2.185) tenglama uchun Dirixle masalasining yechimi. (2.184) formulada parametr $\beta_0 \to -m/2$, ya'ni $\beta \to 0$ limitga o'tib, bevosita (2.184) formuladan Ω_0 normal sohada Dirixle masalasi

$$u(x,0) = \tau(x),\ x \in (-1,1); \quad u(x,y)\Big|_{\sigma_0} = \varphi(x(s)),\ 0 \le s \le l \quad (2.188)$$

yechimini beruvchi ushbu formulaga ega bo'lamiz:

$$u(x,y) = k_2 \frac{m+2}{2} y^{\frac{m+2}{2}} \int_{-1}^{1} \tau(t) \left\{ \left[(x-t)^2 + \frac{4}{(m+2)^2} y^{\frac{m+2}{2}} \right]^{-1} - \right.$$

$$\left. - \left[(1-xt)^2 + \frac{4t^2}{(m+2)^2} y^{m+2} \right]^{-1} \right\} dt - \quad (2.189)$$

$$- k_2 \frac{(m+2)^3}{16} \left(1 - R^2\right) \int_0^l \varphi(\xi(s)) \eta^{\frac{m+2}{2}}(s)(r^{-2} - r_1^{-2}) d\xi(s)$$

bu yerda $k_2 = \dfrac{1}{4\pi} \left(\dfrac{4}{m+2} \right)^2$.

III BOB. BUZILUVCHAN ELLIPTIK TURDAGI SINGULYAR KOEFFITSIENTLI TENGLAMALARNING BIR SINFI UCHUN YARIM TEKISLIKDA DIRIXLEVA SHAKLI O‘ZGARGAN N MASALALARI

Ushbu bobda $y \geq 0$ yarim tekislikda

$$L(\alpha_0, u) = y^m u_{xx} + u_{yy} + \alpha_0 y^{m/2-1} u_x + \beta_0 y^{-1} u_y = 0, \qquad y > 0, \quad (3.1)$$

tenglama uchun Dirixle va shakli o‘zgargan N masalasi o‘rganiladi, bu yerda m, α_0, β_0 – o‘zgarmas sonlar.

1-§. Dirixle va shakli o‘zgargan N masalalarning qo‘yilishi va yechimning yagonaligi.

Dirixle masalasi. $y > 0$ yarim tekislikda (3.1) tenglamaning ushbu

$$\lim_{y \to +0} u(x, y) = \tau(x), \qquad x \in (-\infty, +\infty) \quad (3.2)$$

shartni qanoatlantiruvchi va cheksizlikda nolga aylanuvchi regulyar yechimi $u(x, y) \in C(y \geq 0) \cap C^2(y > 0)$ topilsin.

Bu yerda $\tau(x) \in C(-\infty, +\infty)$ berilgan funksiya va u yetarli katta $|x|$ lar uchun

$$|\tau(x)| \leq M_0 / |x|^{\delta} \quad (3.3)$$

tengsizlikni qanoatlantiradi, bu yerda $M_0 > 0$ va $\delta > 0$ o‘zgarmas sonlar.

Shakli o‘zgargan N masalasi. $y > 0$ yarim tekislikda (3.1) tenglamaning ushbu

$$\lim_{y \to +0} y^{\beta_0} \frac{\partial u}{\partial y} = \nu(x), \qquad x \in (-\infty, +\infty) \quad (3.4)$$

shartni qanoatlantiruvchi va cheksizlikda nolga aylanuvchi regulyar yechimi $u(x, y) \in C(y \geq 0) \cap C^2(y > 0)$, hamda $y^{\beta_0} u_y \in C(y \geq 0)$ topilsin, bu yerda $\nu(x) \in C(-\infty, +\infty)$ va yetarli katta $|x|$ lar uchun:

$$|\nu(x)| \leq M_1 / |x|^{1-2a+\varepsilon} , \quad (3.5)$$

bu yerda $M_1 > 0$ va $\varepsilon > 0$ –o‘zgarmas sonlar, $2a = \alpha + \beta$, $0 < \alpha + \beta < 1$.

3.1-teorema. *Dirixle masalasi bittadan ortiq yechimga ega emas.*

Isbot. $\tau(x) \equiv 0$ bo‘lsin. $y \geq 0$ yarim tekislikda $L_R : x^2 + 4(m+2)^{-2} y^{m+2} = R^2$ normal chiziq hamda $y = 0$ o‘qining $[-R, R]$ kesmasi bilan chegaralangan Ω_R sohani tuzamiz. Ixtiyoriy $\varepsilon > 0$ sonini olamiz. $u(x, y)$ funksiyamiz cheksizlikda nolga aylanganligi sababli yetarli katta R uchun $|u(x, y)| < \varepsilon$, $(x, y) \in L_R$ tengsizlik o‘rinli bo‘ladi. Endi $y > 0$ tekislikning ixtiyoriy (x_0, y_0) nuqtasini olamiz. Yetarli katta R lar uchun (x_0, y_0) nuqta Ω_R soha ichiga tushadi va ekstremum prinsipiga ko‘ra $|u(x_0, y_0)| < \varepsilon$ tengsizlik o‘rinli bo‘ladi. ε – sonining ixtiyoriyligidan, $u(x_0, y_0) \equiv 0$ tenglik kelib chiqadi, bundan esa $y \geq 0$ yarim tekislikda $u(x, y) \equiv 0$ bo‘ladi.

3.1-teorema isbot bo‘ldi.

3.2-teorema. *Shakli o‘zgargan N masalasi bittadan ortiq yechimga ega emas.*

3.2-teorema 2.1-lemmani hisobga olgan holda 3.1-teorema kabi isbotlanadi.

2 -§. $L(\alpha_0, u) = 0$ tenglamaning fundamental yechimlari.

(3.1) tenglama $y > 0$ yarim tekislikda elliptik tipdagi tenglama bo‘lib, $y = 0$ o‘qida parabolik tipdagi tenglamaga aylanadi, bundan tashqari u_y oldidagi koeffitsient $y = 0$ o‘qida birinchi tartibli maxsuslikka, u_x oldidagi koeffitsient esa $1 - \frac{m}{2}$ $(0 < m < 2)$, tartibli maxsuslikka ega.

(3.1) tenglamaning fundamental yechimini

$$u(x, y) = k_1 \left(r_1^2\right)^{-a} \exp\{\varphi(\alpha_0; x, y; x_0, y_0)\} \omega(\sigma) \tag{3.6}$$

ko‘rinishda izlaymiz, bu yerda

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\} = (x - x_0)^2 + \frac{4}{(m+2)^2} \left(y^{\frac{m+2}{2}} \mp y_0^{\frac{m+2}{2}} \right)^2 ,$$

$$\sigma = \frac{r^2}{r_1^2}, \varphi(\alpha_0; x, y; x_0, y_0) = 2b \arcsin \frac{x - x_0}{r_1}, ,$$

$$a=\frac{2\beta_0+m}{2(m+2)},\quad b=-\frac{\alpha_0}{m+2},$$

$\omega(\sigma)$- yangi noma'lum funksiya. (3.6) ifodani (3.1) ga qo'yib, $\omega(\sigma)$ ga nisbatan ushbu tenglamaga kelamiz:

$$A(x,y)\omega''(\sigma)+B(x,y)\omega'(\sigma)+C(x,y)\omega(\sigma)=0, \qquad (3.7)$$

bu yerda

$$A(x,y)=y^m\left(\sigma'_x\right)^2+\left(\sigma^1_y\right)^2=\frac{4}{r_1^2}y^{\frac{m-2}{2}}y_0^{\frac{m+2}{2}}\sigma(1-\sigma),$$

$$B(x,y)=\frac{\alpha_0}{y^{1-m/2}}\sigma'_x-2ay^m\frac{\left(r_1^2\right)'_x}{r_1^2}\sigma'_x+2y^m\varphi'_x(\alpha_0;x,y;x_0,y_0)\sigma'_x+$$

$$+y^m\sigma''_{xx}+\frac{\beta_0}{y}\sigma'_y-2a\frac{\left(r_1^2\right)'_y}{r_1^2}\sigma'_y+$$

$$+2\varphi_y(\alpha_0;x,y;x_0,y_0)\sigma'_y+\sigma''_{yy}=\frac{4}{r_1^2}y^{\frac{m-2}{2}}y_0^{\frac{m+2}{2}}(1-(1+2a)\sigma),$$

$$\widetilde{N}(x,y)=-\frac{a\alpha_0}{y^{1-m/2}}\frac{(r_1^2)'_x}{r_1^2}+\frac{a_0}{y^{1-m/2}}\varphi'_x(\alpha_0;x,y;x_0,y_0)+$$

$$+a(a+1)y^m\frac{\left[(r_1^2)''_{yy}\right]}{r_1^2}-ay^m\frac{(r_1^2)''_{xx}}{r_1^2}-2a\frac{(r_1^2)'_x}{r_1^2}\varphi'_x(\alpha_0;x,y;x_0,y_0)+$$

$$+y^m\left[\varphi'_x(\alpha_0;x,y;x_0,y_0)\right]^2+y^m\varphi''_x(\alpha_0;x,y;x_0,y_0)-$$

$$-\frac{a\beta_0}{y}\frac{(r_1^2)'_y}{r_1^2}+\frac{\beta_0}{y}\varphi'_y(\alpha_0;x,y;x_0,y_0)+a(a+1)\frac{(r_1^2)'_y}{r_1^2}-$$

$$-a\frac{(r_1^2)''_{yy}}{r_1^2}-2a\frac{(r_1^2)'_y}{r_1^2}\varphi'_y(\alpha_0;x,y;x_0,y_0)+\left[\varphi'_y(\alpha_0;x,y;x_0,y_0)\right]^2$$

$$+\varphi''_{yy}(\alpha_0;x,y;x_0,y_0)=-\frac{4}{r_1^2}y^{\frac{m-2}{2}}y_0^{\frac{m+2}{2}}(a^2+b^2).$$

Endi $A(x,y), B(x,y), C(x,y)$ lar uchun keltirilgan ifodalarni hisobga olib, (3.7) tenglamani ushbu ko‘rinishga keltiramiz:

$$\sigma(1-\sigma)\omega''(\sigma)+[1-(2a+1)\sigma]\omega'(\sigma)-(a^2+b^2)\omega(\sigma)=0$$

yoki

$$\sigma(1-\sigma)\omega''(\sigma)+\left[1+(1+\lambda+\bar{\lambda})\sigma\right]\omega'(\sigma)-\lambda\bar{\lambda}\,\omega(\sigma)=0, \tag{3.8}$$

bu yerda $\lambda=a+ib$, $\bar{\lambda}=a-ib$. Bu tenglama Gauss tenglamasi bo‘lib, u $\sigma=1$ nuqta atrofida ikkita chiziqli erkli yechimga ega:

$$\omega_1(\sigma)=F(\lambda,\bar{\lambda}\lambda+\bar{\lambda};1-\sigma) \tag{3.9}$$

$$\omega_2(\sigma)=(1-\sigma)^{1-(\lambda+\bar{\lambda})}F(1-\lambda,1-\bar{\lambda};2-(\lambda+\bar{\lambda});1-\sigma). \tag{3.10}$$

(3.9) va (3.10) tenglamalardan $\omega_1(\sigma)$ va $\omega_2(\sigma)$ ning qiymatlarini (3.6) formulaga qo‘yib, (3.1) tenglamaning quyidagi ikkita fundamental yechimini hosil qilamiz:

$$q_1(\alpha_0;x,y;x_0,y_0))=k_1\left(r_1^2\right)^{-a}\times$$
$$\times\exp\{\varphi(\alpha_0;x,y;x_0,y_0)\}F(\lambda,\bar{\lambda};\lambda+\bar{\lambda};1-\sigma) \tag{3.11}$$

$$q_2(\alpha_0;x,y;x_0,y_0))=k_2\left(\frac{4}{m+2}\right)^{4a-2}(r_1^2)^{-a}\exp\{\varphi(\alpha_0;x,y;x_0,y_0)\}\times$$
$$\times(1-\sigma)^{1-2a}F(1-\lambda,1-\bar{\lambda},2-(\lambda+\bar{\lambda});1-\sigma), \tag{3.13}$$

bu yerda

$$k_1=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2a}\frac{\Gamma(\lambda)\Gamma(\bar{\lambda})}{\Gamma(\lambda+\bar{\lambda})},\quad k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\alpha}\frac{\Gamma(1-\lambda)\Gamma(1-\bar{\lambda})}{\Gamma(2-\lambda-\bar{\lambda})}.$$

$q_1(\alpha_0;x,y;x_0,y_0)$ va $q_2(\alpha_0;x,y;x_0,y_0)$ funksiyalar (x,y) o‘zgaruvchilar bo‘yicha $L(\alpha_0,u)=0$ tenglamaning, (x_0,y_0)o‘zgaruvchilar bo‘yicha esa $L(-\alpha_0,u)=0$ tenglamaning fundamental yechimlaridir.Bevosita hisoblashlar yordamida

$$\lim_{y\to -0} y^{\beta_0}\frac{\partial q_1(\alpha_0;x,y;x_0,y_0)}{\partial y}=0, \tag{3.13}$$

$$q_2(\alpha_0;x,0;x_0,y_0)=0, \tag{3.14}$$

tengliklarni isbotlash mumkin.

Endi $F(\alpha,\beta,\alpha+\beta;1-\sigma)$ – gipergeometrik funksiyaning maxsus ifodasi:

$$\begin{gathered} F(\alpha,\beta,\alpha+\beta;1-\sigma)= \\ =R_0(\alpha,\beta;x,y;x_0,y_0)\ln r^2+Q_0(\alpha,\beta;x,y;x_0,y_0) \end{gathered} \tag{3.15}$$

dan foydalanamiz, bu yerda

$$R_0(\alpha,\beta;x,y;x_0,y_0)=-\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}F(\alpha,\beta,1;\sigma), \tag{3.16}$$

$$Q_0(\alpha,\beta;x,y;x_0,y_0)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}F(\alpha,\beta,1;\sigma)\ln r_1^2+\frac{\Gamma(\alpha+\beta)}{\Gamma^2(\alpha)\Gamma^2(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{(1!)^2}\times$$

$$\times\left[2\frac{\Gamma'(1+k)}{\Gamma(1+k)}-\frac{\Gamma'(\alpha+k)}{\Gamma(\alpha+k)}-\frac{\Gamma'(\beta+k)}{\Gamma(\beta+k)}\right]\sigma^k. \tag{3.17}$$

(3.15) tenglikka asosan, (3.11) fundamental yechimni ushbu

$$q_1(\alpha_0;x,y;x_0,y_0)=q_1(\alpha_0;M,M_0)=R_1(\alpha_0;M,M_0)\ln r^2+Q_1(\alpha_0;M,M_0)$$

ko‘rinishda yozib olamiz, bu yerda

$$R_1(\alpha_0;M,M_0)=k_1\left(r_1^2\right)^{-a}\exp\{\varphi(\alpha_0;M,M_0)\}R_0(\lambda,\overline{\lambda};M,M_0) \tag{3.18}$$

$$Q_1(\alpha_0;M,M_0)=k_1\left(r_1^2\right)^{-a}\exp\{\varphi(\alpha_0;M,M_0)\}Q_0(\lambda,\bar{\lambda};M,M_0). \quad (3.19)$$

(3.1) tenglamaga Lagranj ma’nosidagi qo‘shma tenglama quyidagi ko‘rinishga ega:

$$L^*(\alpha_0,\upsilon)=y^m\upsilon_{xx}+\upsilon_{yy}-\frac{\alpha_0}{y^{1-m/2}}\upsilon_x-\frac{\beta_0}{y}\upsilon_y+\frac{\beta_0}{y^2}\upsilon=0. \quad (3.20)$$

Bevosita hisoblashlar yordamida ushbu

$$L^*(\alpha_0,y^{\beta_0}u)=y^{\beta_0}L(-\alpha_0,u) \quad (3.21)$$

ayniyatni to‘g‘riligiga ishonch hosil qilish mumkin.

Shunday qilib, agar $q_1(\alpha_0;x,y;x_0,y_0)$ funksiya $M(x,y)$ o‘zgaruvchi bo‘yicha (3.1) tenglamaning yechimi bo‘lsa, u holda

$$p_1(\alpha_0;M;M_0)=y^{\beta_0}q_1(-\alpha_0;M;M_0) \quad (3.22)$$

funksiya $M=M(x,y)$ o‘zgaruvchi bo‘yicha (3.20) qo‘shma tenglamaning yechimi, $M_0=M_0(x_0,y_0)$ o‘zgaruvchilar bo‘yicha (3.1) tenglamaning yechimi bo‘lishligi kelib chiqadi.

3- §.Chekli sohada $L(\alpha_0,u)=0$ tenglama yechimining integralifodasi.

Ushbu ayniyatni qaraymiz:

$$\vartheta L(\alpha_0,u)-uL^*(\alpha_0,\vartheta)=\frac{\partial}{\partial x}\left[y^m\left(\vartheta\frac{\partial u}{\partial x}-u\frac{\partial\vartheta}{\partial x}\right)+u\vartheta\frac{\alpha_0}{y^{1-m/2}}\right]+ \\ +\frac{\partial}{\partial y}\left[\left(\vartheta\frac{\partial u}{\partial y}-u\frac{\partial\vartheta}{\partial y}\right)+u\vartheta\frac{\beta_0}{y}\right]. \quad (3.23)$$

$(x_0,y_0)\in D$ bo‘lsin.

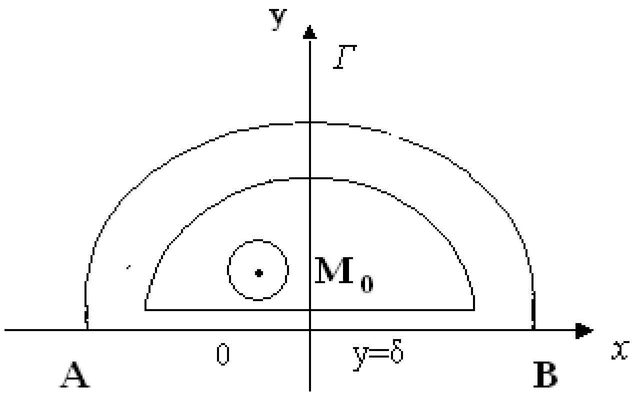
D-uchlari $A(-c,0)$ va $B(c,0)$ nuqtalarda bo‘lgan, $y>0$ yarim tekislikda yotuvchi Γ silliq Jordan chizig‘i hamda $y=0$ o‘qining AB kesmasi bilan chegaralangan bir bog‘lamli soha bo‘lsin.

Γ egri chiziqqa parallel Γ_ε egri chiziq va $y=\delta>\varepsilon$ to‘g‘ri chiziq kesmasi bilan chegaralangan $D_{\varepsilon,\delta}\subset D$ sohani qaraymiz.

ε va δ larni shunchalik kichik tanlaymizki, $M_0 = M_0(x_0, y_0)$ nuqta $D_{\varepsilon,\delta}$ sohaning ichida joylashsin.

$D_{\varepsilon,\delta}$ sohadan markazi $M_0 = M_0(x_0, y_0)$ nuqtada bo‘lgan kichik ρ radiusli Q_ρ doira ajratamiz va $D_{\varepsilon,\delta}$ sohaning qolgan qismini $D^{\rho}_{\varepsilon,\delta}$ bilan belgilaymiz.

$D^{\rho}_{\varepsilon,\delta}$ sohada $p_1(\alpha_0; M; M_0)$ funksiya (3.20) tenglamaning regulyar yechimi bo‘ladi.



$D^{\rho}_{\varepsilon,\delta}$ soha bo‘yicha (3.23) ayniyatni integrallab, so‘ngra unga ushbu

$$\int\limits_{\partial D} P(x,y)dx + Q(x,y)dx = \int\limits_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Grin formulasini qo‘llasak,

$$\iint\limits_{D^{\rho}_{\varepsilon,\delta}} \left[\vartheta L(\alpha_0, u) - u L^*(\alpha_0, \vartheta) \right] dxdy =$$

$$= \int\limits_{\partial D^{\rho}_{\varepsilon,\delta}} \left\{ -\left(\vartheta \frac{\partial u}{\partial y} - u \frac{\partial \vartheta}{\partial y} + u \vartheta \frac{\beta_0}{Y} \right) dx + \right. \tag{3.24}$$

$$\left. + \left[y^m \left(\vartheta \frac{\partial u}{\partial x} - u \frac{\partial \vartheta}{\partial y} \right) + u \vartheta \frac{\alpha_0}{y^{1-m/2}} \right] dy \right\}$$

ayniyat hosil bo‘ladi, bu yerda

$$\partial D^{\rho}_{\varepsilon,\delta} = \Gamma_{\varepsilon} \cup \{y=\delta\} \cup C_{\rho}. \qquad (3.25)$$

Faraz qilaylik, $u(x,y)$ funksiya $L(\alpha_0;u)=0$ tenglamaning regulyar yechimi bo‘lsin,

$$\vartheta = p_1(\alpha_0;M;M_0) \qquad \text{esa} \qquad L^*(\alpha_0,\vartheta)=0$$

tenglamaning fundamental yechimi bo‘lsin, u holda (3.24) ni quyidagicha yozib olamiz:

$$\int\limits_{\partial D^{\rho}_{\varepsilon,\delta}} \{\vartheta A_s[u] - uA_s[\vartheta] + u\vartheta A_{0s}(\alpha_0,\beta_0)\}ds = 0, \qquad (3.26)$$

$$\int\limits_{\partial D^{\rho}_{\varepsilon,\delta}} \{p_1(\alpha_0;M;M_0)A_S[u(M)] - u(M)A_S[p_1(\alpha_0;M;M_0)] +$$
$$+u(M)p_1(\alpha_0;M;M_0)A_{0S}(\alpha_0,\beta_0)\}ds = 0, \qquad (3.27)$$

bu yerda

$$A_S[u] = y^m \frac{dy}{ds}\frac{\partial u}{\partial x} - \frac{dx}{ds}\frac{\partial u}{\partial y}, \; A_{0S}(\alpha_0,\beta_0) = \frac{\alpha_0}{y^{1-m/2}}\frac{dy}{ds} - \frac{\beta_0}{y}\frac{dx}{ds}. \qquad (3.28)$$

$A_S[u]$ operatorga $u(x,y)$ funksiyaning konormal hosilasi deyiladi. Quyidagi hisoblashlarni bajaramiz:

$$A_S[P_1(\alpha_0;M;M_0)] = A_S\left[y^{\beta_0} q_1(-\alpha_0;M;M_0)\right] =$$
$$= A_S\left[y^{\beta_0}\right] q_1(-\alpha_0;M;M_0) + y^{\beta_0} A_S[q_1(-\alpha_0;M;M_0)] =$$
$$= -\frac{dx}{ds}\beta_0 y^{\beta_0 -1} q_1(-\alpha_0;M;M_0) + y^{\beta_0} A_S[q_1(-\alpha_0;M;M_0)] = \qquad (3.29)$$
$$= y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(0;\beta_0) + y^{\beta_0} A_S[q_1(-\alpha_0;M;M_0)].$$

Endi (3.29) ni va ushbu $A_{0S}(\alpha_0,\beta_0) = A_{0S}(\alpha_0,0) + A_{0S}(0,\beta_0)$ formulani hisobga olib, (3.27) tenglikni quyidagi ko‘rinishda yozamiz:

$$\int_{\partial D^{\rho}_{\varepsilon.\delta}} \Big\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) A_S[u(M)] - y^{\beta_0} u(M) A_S[q_1(-\alpha_0;M;M_0)] + \\ + u(M) y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(\alpha_0;0) \Big\} ds = 0. \tag{3.30}$$

Egri chiziqli integralning additivlik xossasiga ko‘ra, (3.25)ga binoan, (3.30) tenglikni

$$\int_{C_\rho} \Big\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) A_S[u(M)] - y^{\beta_0} u(M) A_S[q_1(-\alpha_0;M;M_0)] + \\ + u(M) y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(\alpha_0;0) \Big\} ds = \int_{x_1}^{x_2} \Big\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) A_S[u(M)] - \\ - y^{\beta_0} u(M) \times A_S[q_1(-\alpha_0;M;M_0)] + u(M) y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(\alpha_0,0) \Big\}\Big|_{y=\delta} ds + \\ + \int_{\Gamma_\varepsilon} \Big\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) \times A_S[u(M)] - y^{\beta_0} u(M) A_S[q_1(-\alpha_0;M;M_0)] + \\ + u(M) y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(\alpha_0,0) \Big\} ds \tag{3.31}$$

ko‘rinishda yozamiz, bu yerda $x_1, x_2 - \Gamma_\varepsilon$ egri chiziqning $y=\delta$ to‘g‘ri chiziq bilan kesishish nuqtalarining absissalari. C_ρ esa ajratilgan Q_ρ doiraning chegarasi.

$\rho \to 0$, ya’ni $M(x,y) \to M_0(x_0,y_0)$ bo‘lganda, (3,31) tenglamaning chap qismining limitini hisoblaymiz. Dastlab ba’zi soddalashtirishlar bajaramiz:

$$J(\rho;M_0) = \int_{C_\rho} \Big\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) A_S[u(M)] - \\ - y^{\beta_0} u(M) A_S[q_1(-\alpha_0;M;M_0)] + \\ + u(M) y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(\alpha_0,0) \Big\} ds. \tag{3.32}$$

Hisoblaymiz:

$$A_S[q_1(-\alpha_0;M;M_0)] = R_1(-\alpha_0;M;M_0) A_S[\ln r^2] + \\ + \ln r^2 A_S[R_1(-\alpha_0;M;M_0)] + A_S[Q(-\alpha_0;M;M_0)], \tag{3.33}$$

bu yerda

$$A_S\left[\ln r^2\right]=y^m\frac{dy}{ds}\frac{\partial\ln r^2}{\partial x}-\frac{dx}{ds}\frac{\partial\ln r^2}{\partial y}=y^m\frac{2(x-x_0)}{r^2}\frac{dy}{ds}-$$

$$-\frac{4}{m+2}\left(y^{\frac{m+2}{2}}-y_0^{\frac{m+2}{2}}\right)\frac{y^{\frac{m}{2}}}{r^2}\frac{dx}{ds}. \qquad (3.34)$$

(3.33) va (3.34) tengliklarga asosan, (3.32) tenglikni quyidagi ko‘rinishda yozamiz:

$$J(\rho;M_0)=\int_{C_\rho}\left\{y^{\beta_0}q_1(-\alpha_0;M;M_0)A_S\left[u(M)\right]-\right.$$

$$-y^{\beta_0}u(M)\left[A_S\left[R_1(-\alpha_0;M;M_0)\right]\ln r^2+A_S\left[Q_1(-\alpha_0;M;M_0)\right]\right]+$$

$$\left.+u(M)y^{\beta_0}q_1(-\alpha_0;M;M_0)A_{0S}(\alpha_0,0)\right\}ds-$$

$$-\int_{C_\rho}\frac{y^{\beta_0}u(M)}{r^2}R_1(-\alpha_0;M;M_0)\left[y^m 2(x-x_0)dy-\right. \qquad (3.35)$$

$$\left.-\frac{4}{m+2}\left(y^{\frac{m+2}{2}}-y_0^{\frac{m+2}{2}}\right)y^{\frac{m}{2}}dx\right].$$

Endi $J(\rho,M_0)$ da qutb koordinatalar sistemasiga o‘tamiz, ya’ni integral o‘zgaruvchisini quyidagicha almashtiramiz:

$$x=x_0+\rho\cos\varphi,\qquad y=y_0+\rho\sin\varphi,$$
$$dx=-\rho\sin\varphi d\varphi,\quad dy=\rho\cos\varphi d\varphi,$$

$$ds=\sqrt{(dx)^2+(dy)^2}=\rho d\varphi,\ y^{\frac{m+2}{2}}=(y_0+\rho\sin\varphi)^{\frac{m+2}{2}}=$$
$$=y_0^{\frac{m+2}{2}}+\frac{m+2}{2}y_0^{\frac{m}{2}}\rho\sin\varphi+O(\rho^2)=$$
$$r^2=\rho^2\cos^2\varphi+\frac{4\rho^2}{(m+2)^2}\left(\frac{m+2}{2}y_0^{\frac{m}{2}}\sin\varphi+O(\rho)\right)^2.$$

Bu tengliklarga asosan:

$$J(\rho;M_0)=\int_0^{2\pi}\Big\{y^{\beta_0}q_1(-\alpha_0;M;M_0)A_S[u(M)]-$$

$$-y^{\beta_0}u(M)\Big[A_S[R_1(-\alpha_0;M;M_0)]\ln r^2+A_S[Q(-\alpha_0;M;M_0)]\Big]+$$

$$+u(M)y^{\beta_0}q_1(-\alpha_0;M;M_0)A_{0S}(\alpha_0;0)\Big\}\Bigg|_{\substack{x=x_0+\rho\cos\varphi\\ y=y_0+\rho\sin\varphi}}\rho d\varphi-$$

$$-\int_0^{2\pi}\frac{(y_0+\rho\sin\varphi)^{\beta_0}u[x_0+\rho\cos\varphi,y_0+\rho\sin\varphi]}{\rho^2\cos^2\varphi+\dfrac{4\rho^2}{(m+2)^2}\left(\dfrac{m+2}{2}y_0^{\frac{m}{2}}\sin\varphi+O(\rho)\right)^2}\times$$

$$\times R_1(-\alpha_0;x_0+\rho\cos\varphi,y_0+\rho\sin\varphi;x_0,y_0)\times$$

$$\times\left[(y_0+\rho\sin\varphi)^m2\rho^2\cos^2\varphi+\frac{4\rho}{m+2}\left(\frac{m+2}{2}y_0^{\frac{m}{2}}\sin\varphi+O(\rho)\right)\times\right. \qquad (3.36)$$

$$\left.\times(y_0+\rho\cos\varphi)^{\frac{m}{2}}\rho\sin\varphi\right]d\varphi.$$

$F(\lambda,\bar{\lambda}1,0)=1$ bo‘lgani uchun:

$$R_1(-\alpha_0;M;M_0)\big|_{M=M_0}=$$

$$=k_1\left(r_1^2\right)^{-a}_{M=M_0}e^{\varphi(-\alpha_0;M;M_0)}R_0(-\alpha_0;M;M_0)\big|_{M=M_0}=$$

$$=k_1\left(\frac{4}{(m+2)^2}y_0^{m+2}\right)^{-a}e^0\left(-\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})}\right)=$$

$$=-k_1\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})}\left(\frac{4}{m+2}\right)^{-2a}y_0^{-a(m+2)}.$$

Endi (3.36) tenglikda $\rho\to 0$ da limitga o‘tib,

$$J(0;M_0)=2k_1\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})}\left(\frac{4}{m+2}\right)^{-2a}u_0(x_0,y_0)\int_0^{2\pi}\frac{y_0^{m/2}d\varphi}{\cos^2\varphi+y_0^m\sin^2\varphi} \qquad (3.37)$$

tenglikni hosil qilamiz.

(3.37)ning o‘ng qismidagi integralni hisoblaymiz:

$$\int_0^{2\pi} \frac{y_0^{m/2} d\varphi}{\cos^2\varphi + y_0^m \sin^2\varphi} = \int_0^{\pi/2} \frac{d(y_0^{m/2} tg\varphi)}{1+(y_0^{m/2} tg\varphi)^2} +$$

$$+ \int_{\pi/2}^{3\pi/2} \frac{d(y_0^{m/2} tg\varphi)}{1+(y_0^{m/2} tg\varphi)^2} + \int_{3\pi/2}^{2\pi} \frac{d(y_0^{m/2} tg\varphi)}{1+(y_0^{m/2} tg\varphi)^2} =$$

$$= \lim_{\varepsilon_1 \to 0} \left(arctg\,(y_0^{m/2} tg\varphi) \right)\Big|_0^{\frac{\pi}{2}-\varepsilon_1} + \lim_{\substack{\varepsilon_2 \to 0 \\ \varepsilon_3 \to 0}} \left(arctg\,(y_0^{m/2} tg\varphi) \right)\Big|_{\frac{\pi}{2}+\varepsilon_2}^{\frac{3\pi}{2}-\varepsilon_3} +$$

$$+ \lim_{\varepsilon_4 \to 0} \left(arctg\,(y_0^{m/2} tg\varphi) \right)\Big|_{\frac{3\pi}{2}+\varepsilon_4}^{2\pi} = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi \; .$$

Shunday qilib,

$$J(0;M_0) = 2k_1 \frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})} \left(\frac{4}{m+2} \right)^{-2a} 2\pi\, u(x_0 y_0), \quad (3.38)$$

bunda k_1 ni

$$k_1 = \frac{1}{4\pi} \left(\frac{4}{m+2} \right)^{2a} \frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})} \quad (3.39)$$

deb olamiz.

Natijada

$$J(0;M_0) = u(x_0, y_0)$$

tenglikka ega bo‘lamiz.

Shunday qilib, (3.31) tenglikni quyidagi ko‘rinishda yozib olamiz:

$$\begin{aligned} u(x_0, y_0) = & \int_{x_1}^{x_2} \Big\{ y^{\beta_0} q_1(-\alpha_0; M; M_0) A_S[u(M)] - \\ & - y^{\beta_0} u(M) A_S[q_1(-\alpha_0; M; M_0)] + \\ & + u(M) y^{\beta_0} q_1(-\alpha_0; M; M_0) A_{0S}(\alpha_0; 0) \Big\}\Big|_{y=\delta}\, ds + \\ & + \int_{\Gamma_\varepsilon} \Big\{ y^{\beta_0} q_1(-\alpha_0; M; M_0) A_S[u(M)] - \\ & - y^{\beta_0} u(M) A_S[q_1(-\alpha_0; M; M_0)] + \\ & + u(M) y^{\beta_0} q_1(-\alpha_0; M; M_0) A_{0S}(\alpha_0; 0) \Big\} ds. \end{aligned} \quad (3.40)$$

Bunda

$$A_S\left[u(M)\right]|_{y=\delta} = -\frac{dx}{ds}\frac{\partial u}{\partial y}\Big|_{y=\delta},$$

$$A_S\left[q_1(-\alpha_0;M;M_0)\right]|_{y=\delta} = -\frac{dx}{ds}\frac{\partial q_1}{\partial y}|_{y=\delta}\,.$$

U holda (3.40) tenglikni

$$\begin{aligned} u(x_0,y_0) = &\int_{x_1}^{x_2}\left\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) y^{\beta_0}\frac{\partial u}{\partial y} - u(M) y^{\beta_0}\frac{\partial q_1}{\partial y}\right\}\Bigg|_{y=\delta} dx + \\ &+ \int_{\Gamma_\varepsilon}\Big\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) A_S\left[u(M)\right] - u(M) y^{\beta_0} A_S\left[q_1(-\alpha_0;M;M_0)\right] + \\ &+ u(M) y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(\alpha_0;0)\Big\} ds \end{aligned} \tag{3.41}$$

ko‘rinishda yozish mumkin.

Endi (3.13)ni hisobga olgan holda, (3.41) tenglikda $\delta \to 0$, $\varepsilon \to 0$ bo‘lganda limitga o‘tsak, quyidagiga ega bo‘lamiz:

$$\begin{aligned} u(x_0,y_0) = &-\int_{-c}^{c}\nu(x) q_1(-\alpha_0;M;M_0)dx + \\ &+ \int_{\Gamma_\varepsilon}\Big\{ y^{\beta_0} q_1(-\alpha_0;M;M_0) A_S\left[u(M)\right] - \\ &- u(M) y^{\beta_0} A_S\left[q_1(-\alpha_0;M;M_0)\right] + \\ &+ u(M) y^{\beta_0} q_1(-\alpha_0;M;M_0) A_{0S}(\alpha_0;0)\Big\} ds. \end{aligned} \tag{3.42}$$

Shunday qilib, (3.1) tenglamaning yechimi $u(x,y)$ ni D soha ichidagi $M_0(x_0,y_0)$ nuqtadagi qiymatini $u(x,y)$ ning D soha chegarasi ∂D dagi qiymati va hosilasi orqali ifodalovchi munosabatga keldik. Bu munosabat N tipidagi masalalarni yechishda asosiy formula hisoblanadi.

4-§. Yuqori yarim tekislikda $L(\alpha_0,u)=0$ tenglama uchun shakli o'zgargan N masalasi .

Endi (3.42) formuladan foydalanib, $y>0$ yarim tekislikda shakli o'zgargan N masalasi yechimini beruvchi formulani keltirib chiqaramiz**.**

$y\geq 0$ yarim tekislikda $y=0$ o'qining $[-R,R]$ kesmasi va

$$\Gamma = L_R : x^2 + \frac{4}{(m+2)^2} y^{m+2} = R^2$$

normal chiziq bilan chegaralangan bir bog'lamli sohani Ω_R orqali belgilab olaylik.

Ω_R sohada (3.42) formulani (3.11) tenglikni e'tiborga olib, ushbu ko'rinishda yozib olamiz:

$$u(x_0,y_0) = -k_1 \int_{-R}^{R} \nu(x)\left(r_1^2\right)^{-a} e^{\varphi(-\alpha_0,M,M_0)}\Big|_{y=0} dx = J_1 + J_2 + J_3 \quad (3.43)$$

bu yerda

$$J_1 = \int_{L_R} y^{\beta_0} q_1(-\alpha_0;M;M_0) A_s[u(M)] ds\,, \quad (3.44)$$

$$J_2 = -\int_{L_R} y^{\beta_0} u(M) A_s[q_1(-\alpha_0;M;M_0)] ds\,, \quad (3.45)$$

$$J_3 = \int_{L_R} y^{\beta_0} u(M) q_1(-\alpha_0;M;M_0) A_{0s}(\alpha_0;0) ds. \quad (3.46)$$

J_1, J_2, J_3 integrallarni

$$R^2 = x^2 + \frac{4}{(m+2)^2} y^{m+2} \to \infty$$

da baholaymiz, bu yerda $(x,y)\in L_R$.

1. Dastlab J_1 ni baholaymiz. (3.28) ga asosan, $|J_1|$ ni quyidagi ko'rinishda yozib olamiz:

$$|J_1| = \left| \int_{L_R} y^{\beta_0+m} q_1(-\alpha_0; M, M_0) \frac{\partial u}{\partial x} dy - \right.$$

$$\left. - \int_{L_R} y^{\beta_0} q_1(-\alpha_0; M, M_0) \frac{\partial u}{\partial y} dx \right| \le \int_{L_R} \left| y^{\beta_0+m} q_1(-\alpha_0; M, M_0) \frac{\partial u}{\partial x} \right| dy +$$

$$+ \int_{L_R} \left| y^{\beta_0} q_1(-\alpha_0; M, M_0) \frac{\partial u}{\partial y} \right| dx = \int_{L_R} \left| y^{\beta_0+m} \right| \left| q_1(-\alpha_0; M, M_0) \right| \left| \frac{\partial u}{\partial x} \right| dy + \quad (3.47)$$

$$+ \int_{L_R} \left| q_1(-\alpha_0; M, M_0) \right| \left| y^{\beta_0} \frac{\partial u}{\partial y} \right| dx = J_{11} + J_{12} \ .$$

(3.47) tenglikning o‘ng tomonida integral o‘zgaruvchilarni ushbu:

$$x = R\cos\varphi, \quad y = \left[\frac{m+2}{2} R \sin\varphi \right]^{\frac{2}{m+2}}, \quad x^2 + \frac{4}{m+2} y^{m+2} = R^2 \quad (3.48)$$

ko‘rinishda almashtiramiz.Bu yerdan:

$$dx = -R\sin\varphi d\varphi, \; dy = \left(\frac{2}{m+2} \right)^{\frac{m}{m+2}} R^{\frac{2}{m+2}} (\sin\varphi)^{-\frac{m}{m+2}} \cos\varphi d\varphi \ . \quad (3.49)$$

(3.49) ga asosan, yetarli katta R lar uchun:

$$r_1^2 = R^2 \left[1 - \frac{2x_0 \cos\varphi}{R} + \frac{4}{m+2} \frac{\sin\varphi \, y_0^{m+2}}{R} + \right.$$

$$\left. + \frac{1}{R^2} \left(x_0^2 + \frac{4}{(m+2)^2} y_0^{m+2} \right) \right] = O(R^2) \quad (3.50)$$

bahoga kelamiz.

Yuqoridagiga o‘xshash, ushbu

$$r^2=O(R^2),\ \sigma=\frac{r^2}{r_1^2}=O(1),\ 1-\sigma=1-\frac{r^2}{r_1^2}=\frac{16}{(m+2)^2}y^{\frac{m+2}{2}}y_0^{\frac{m+2}{2}}\frac{1}{r_1^2}=$$
$$=\frac{8}{m+2}R\sin\varphi\, y_0^{\frac{m+2}{2}}\frac{1}{r_1^2}=O\left(\frac{1}{R}\right) \tag{3.51}$$

baholarni isbotlash murakkab emas.

Endi (3.50) va (3.51)ga asosan, ushbu

$$\left|q_1(-\alpha_0;M,M_0)\right|=$$
$$=\left|k_1\left(r_1^2\right)^{-a}e^{\varphi(-\alpha_0;M,M_0)}F(\lambda,\overline{\lambda},\lambda+\overline{\lambda};1-\sigma)\right|=O\left(\frac{1}{R^{2a}}\right) \tag{3.52}$$

bahoni yetarli katta R lar uchun hosil qilamiz. Oldindan faraz qilamiz:

$$|u(M)|=\frac{1}{R^2},\quad \left|y^{\beta_0}\frac{\partial u}{\partial x}\right|=O\left(\frac{1}{R^{1-2a+\varepsilon}}\right),\quad \left|\frac{\partial u}{\partial x}\right|=O\left(\frac{1}{R^{1+\varepsilon}}\right), \tag{3.53}$$

bu yerda ε yetarli kichik musbat son. Endi (3.53) munosabatlar va (3.50), (3.52) baholarga asosan, yetarli katta R uchun ushbu

$$\left|y^{\beta_0+m}\right|\left|q(-\alpha_0,M,M_0)\right|\left|\frac{\partial u}{\partial x}\right|dy\le CR^{\frac{2(\beta_0+m)}{m+2}}\frac{1}{R^{2a}}\frac{1}{R^{1+\varepsilon}}R^{\frac{2}{m+2}}d\varphi=$$
$$=CR^{2a}R^{\frac{m}{m+2}}\frac{1}{R^{2a}}R^{-1-\varepsilon}R^{\frac{2}{m+2}}d\varphi=\frac{C}{R^{\varepsilon}}d\varphi$$

tengsizlakka kelamiz, ya'ni

$$\left|y^{\beta_0+m}\right|\left\|q_1(-\alpha_0;M,M_0)\right\|\left|\frac{\partial u}{\partial y}\right|dy=O\left(\frac{1}{R^{\varepsilon}}\right)d\varphi\,. \tag{3.54}$$

Shunday qilib, (3.54) da $R\to+\infty$ da limitga o'tib, ushbu

$$\lim_{R\to+\infty}J_{11}=0\,, \tag{3.55}$$

tenglikni hosil qilamiz.

Yuqoridagidek, (3.53) farazga asosan, $|J_{12}| = O\left(\frac{1}{R^{\varepsilon}}\right)$.Bu yerdan

$$\lim_{R\to+\infty} J_{12} = 0 \tag{3.56}$$

tenglikka ega bo‘lamiz. Shunday qilib, (3.55) va (3.56)ga asosan,

$$\lim_{R\to+\infty} J_1 = 0. \tag{3.57}$$

Endi J_2 ni baholaymiz:

$$|J_2| = \left| \int_\Gamma u(M) y^{\beta_0+m} \frac{\partial q_1}{\partial x} dy - \int_\Gamma u(M) y^{\beta_0} \frac{\partial q_1}{\partial y} dx \right| \le$$

$$\le \int_\Gamma |u(M)| \left| y^{\beta_0+m} \frac{\partial q_1}{\partial x} \right| dx + \int_\Gamma |u(M)| \left| y^{\beta_0} \frac{\partial q_1}{\partial y} \right| dx = J_{21} + J_{22}$$

Dastlab, $\frac{\partial q_1}{\partial x}$ ni hisoblaymiz:

$$\begin{aligned} \frac{\partial q_1}{\partial x} = k_1 \Big[& -a(r_1^2)^{-a-2}(r_1^2)'_x e^{\varphi(-\alpha_0,M,M_0)} F(\lambda,\bar{\lambda};\lambda+\bar{\lambda};1-\sigma) + \\ & + (r_1^2)^{-a} e^{\varphi(-\alpha_0,M,M_0)} \varphi'_x(-\alpha_0,M,M_0) F(\lambda,\bar{\lambda};\lambda+\bar{\lambda};1-\sigma) - \\ & - (r_1^2)^{-a} e^{\varphi(-\alpha_0,M,M_0)} \frac{\partial F(\lambda,\bar{\lambda};\lambda+\bar{\lambda};1-\sigma)}{\partial \sigma} \frac{\partial \sigma}{\partial x} \Big]. \end{aligned} \tag{3.58}$$

Bevosita hisoblashlarni bajarib, ushbu munosabatlarga kelamiz:

a) $(r_1^2)'_x = 2(x - x_0)$

b) $\varphi'_x(-\alpha_0, M, M_0) = -\frac{4b}{r_1^2(m+2)} \left(y^{\frac{m+2}{2}} + y_0^{\frac{m+2}{2}} \right)$,

v) $$(\sigma)'_x = \frac{2(x-x_0)}{(r_1^2)^2} \frac{16}{(m+2)^2} y^{\frac{m+2}{2}} y_0^{\frac{m+2}{2}}. \tag{3.59}$$

Endi ushbu hosilani hisoblaymiz:

$$\frac{\partial F(\lambda,\bar{\lambda};\lambda+\bar{\lambda};1-\sigma)}{\partial\sigma}=-\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})}\left[\frac{\partial F(\lambda,\bar{\lambda};1;\sigma)}{\partial\sigma}\ln\sigma+F(\lambda,\bar{\lambda};1;\sigma)\frac{1}{\sigma}\right]+$$
$$+\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma^2(\lambda)\Gamma^2(\bar{\lambda})}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+k)\Gamma(\bar{\lambda}+k)}{(k!)^2}\left[2\frac{\Gamma'(1+k)}{\Gamma(1+k)}-\frac{\Gamma'(\lambda+k)}{\Gamma(\lambda+k)}-\frac{\Gamma'(\bar{\lambda}+k)}{\Gamma\bar{\lambda}+k)}\right]k\sigma^{k-1}$$

tengliklarga kelamiz. Bu yerda ushbu

$$\frac{d}{dx}F(a,b,c;x)=\frac{ab}{c}F(a+1,b+1,c+1;x)$$

formulaga asosan,

$$\frac{\partial F(\lambda,\bar{\lambda},1,\sigma)}{\partial\sigma}=-\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})}\left[\lambda\bar{\lambda}F(1+\lambda,1+\bar{\lambda},2,\sigma)\ln\sigma+F(\lambda,\bar{\lambda},1;\sigma)\frac{1}{\sigma}\right]+$$
$$+\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma^2(\lambda)\Gamma^2(\bar{\lambda})}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+k)\Gamma(\bar{\lambda}+k)}{(k!)^2}\left[2\frac{\Gamma'(1+k)}{\Gamma(1+k)}-\frac{\Gamma'(\lambda+k)}{\Gamma(\lambda+k)}-\frac{\Gamma'(\bar{\lambda}+k)}{\Gamma(\bar{\lambda}+k)}\right]k\sigma^{k-1}$$

tenglikka kelamiz. Bu yerda

$$F(a,b,c,x)=(1-x)^{c-a-b}F(c-a,c-b,c;x)$$

avtotransformatsiya formulasini qo‘llab, ushbu yakuniy natijani hosil qilamiz:

$$\frac{\partial F(\lambda,\bar{\lambda};\lambda+\bar{\lambda};1-\sigma)}{\partial\sigma}=$$
$$=-\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma(\lambda)\Gamma(\bar{\lambda})}\left[\lambda\bar{\lambda}(1-\sigma)^{-2a}F(1-\lambda,1-\bar{\lambda},2;\sigma)\ln\sigma+F(\lambda,\bar{\lambda},1,\sigma)\frac{1}{\sigma}\right]+ \quad (3.60)$$
$$+\frac{\Gamma(\lambda+\bar{\lambda})}{\Gamma^2(\lambda)\Gamma^2(\bar{\lambda})}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+k)\Gamma(\bar{\lambda}+k)}{(k!)^2}\left[2\frac{\Gamma'(1+k)}{\Gamma(1+k)}-\frac{\Gamma'(\lambda+k)}{\Gamma(\lambda+k)}-\frac{\Gamma'(\bar{\lambda}+k)}{\Gamma(\bar{\lambda}+k)}\right]k\sigma^{k-1}.$$

J_2 integralda (3.48) almashtirishlarni bajarib, (3.59), (3.60) ifodalar uchun yetarli katta R larda ushbu

a) $(r_1^2)'_x = O(R)$,

b) $$\varphi'_x(-\alpha_0, M, M_0) = O\left(\frac{1}{R}\right), \tag{3.61}$$

c) $$\frac{\partial F(\lambda, \overline{\lambda}; \lambda + \overline{\lambda}; 1-\sigma)}{\partial \sigma} = O(1),$$

baholarni hosil qilamiz.

Shunday qilib, (3.61) ga asosan, (3.58) dan ushbu

$$\frac{\partial q_1}{\partial x} = O\left(\frac{1}{R^{1+2a}}\right)$$

bahoni hosil qilamiz. Bu yerdan esa, (3.53) farazga asosan, ushbu

$$|u(M)| y^{\beta_0+m} \left|\frac{\partial q_1}{\partial x}\right| dy = O\left(\frac{1}{R^{\varepsilon}}\right) d\varphi,$$

bahoni hosil qilamiz, bu yerda

$$\lim_{R \to +\infty} J_{21} = 0. \tag{3.62}$$

Endi yetarli katta R lar uchun J_{22} integralni baholaymiz.

$$\begin{aligned} \frac{\partial q_1}{\partial y} = k_1 \Big[&-a(r_1^2)^{-a-1}(r_1^2)'_y e^{\varphi(-\alpha_0, M, M_0)} F(\lambda, \overline{\lambda}; \lambda + \overline{\lambda}; 1-\sigma) + \\ &+ (r_1^2)^{-a} e^{\varphi(-\alpha_0, M, M_0)} \varphi'_y(-\alpha_0, M, M_0) F(\lambda, \overline{\lambda}; \lambda + \overline{\lambda}; 1-\sigma) - \\ &- (r_1^2)^{-a} e^{\varphi(-\alpha_0, M, M_0)} \frac{\partial F(\lambda, \overline{\lambda}; \lambda + \overline{\lambda}; 1-\sigma)}{\partial \sigma} \frac{\partial \sigma}{\partial y} \Big]. \end{aligned} \tag{3.63}$$

Bevosita hisoblashlar yordamida ushbu

a) $$(r_1^2)'_y = \frac{4}{m+2} y^{\frac{m}{2}} \left(y^{\frac{m+2}{2}} + y_0^{\frac{m+2}{2}} \right),$$

b) $$\varphi'_y(-\alpha_0, M, M_0) = \frac{2b(x-x_0)y^{\frac{m}{2}}}{r_1^2}, \quad (3.64)$$

c) $$(\sigma')_y = \frac{4}{m+2}\frac{y^{\frac{m}{2}}}{(r_1^2)^2}\left[\frac{16}{(m+2)^2} y^{\frac{m+2}{2}} y_0^{\frac{m+2}{2}} - y_0^{\frac{m+2}{2}}(r_1^2 + r_2^2)\right]$$

,

tenglikni hosil qilish mumkin. Endi J_{21} integralda (3.48) almashtirishlarni bajarib, yetarli katta R uchun ushbu

a) $$(r_1^2)'_y = O\left(R^{\frac{2m+2}{m+2}}\right),$$

b) $$\varphi'_y(-\alpha_0, M, M_0) = O\left(\frac{1}{R^{\frac{2}{m+2}}}\right), \quad (3.65)$$

c) $$(\sigma')_y = O\left(\frac{1}{R^{\frac{4+m}{2+m}}}\right),$$

baholarga ega bo‘lamiz. Shunday qilib, (3.65) baholarga asosan, (3.63)ni hisobga olib, ushbu

$$\left| y^{\beta_0} \frac{\partial q_1}{\partial y} \right| = O\left(\frac{1}{R}\right),$$

bahoga kelamiz.

Bu yerdan esa (3.53) farazga ko‘ra, ushbu

$$|u(M)| \left| y^{\beta_0} \frac{\partial q_1}{\partial y} \right| dx = O\left(\frac{1}{R^{\varepsilon}}\right) d\varphi,$$

bahoni hosil qilamiz, bundan

$$\lim_{R\to+\infty} J_{22}=0. \qquad (3.66)$$

Shunday qilib, (3.66) va (3.62)ga asosan,

$$\lim_{R\to+\infty} J_{2}=0. \qquad (3.67)$$

Endi J_3 ni baholaymiz:

$$|J_3|=\int_{\Gamma}\left|u(M)y^{\beta_0}q_1(-\alpha_0,M,M_0)\right|\frac{|\alpha_0|}{\left|y^{1-m/2}\right|}dy. \qquad (3.68)$$

(3.68) tenglikning o‘ng tomonidagi integralda (3.48) almashtirishlarni bajaramiz va yetarli katta R lar uchun (3.52)ni hisobga olib, ushbu

$$|u(M)|\left|y^{\beta_0}\right|\left|q_1(-\alpha_0,M,M_0)\right|\frac{dy}{|y|^{1-m/2}}=O\left(\frac{1}{R^{\varepsilon}}\right)d\varphi \qquad (3.69)$$

bahoni hosil qilamiz, bu yerdan

$$\lim_{R\to+\infty} J_{3}=0. \qquad (3.70)$$

Endi (3.43) formuladan, J_1, J_2, J_3 integrallarda $R\to+\infty$ da limitga o‘tib, (3.57), (3.67), (3.70) larga asosan, $y>0$ yarim tekislikda (3.1) tenglama uchun shakli o‘zgargan N masalasining yechimini beruvchi

$$u(x_0,y_0)=-\int_{-\infty}^{+\infty}\nu(x)q_1(-\alpha_0,x,0,x_0,y_0)dx=-k_1\int_{-\infty}^{+\infty}\nu(x)\left[(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]^{-a}\times$$

$$\times\exp\left\{-2b\arcsin(x-x_0)\Big/\sqrt{(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}}\right\}dx \qquad (3.71)$$

formulani hosil qilamiz.

Endi hosil qilingan (3.71) yechim haqiqatdan ham, (3.53) munosabatlarni qanoatlantirishini ko‘rsatamiz.

1. $u(x,y)$ yechimning chegaralangan ekanligini ko‘rsatamiz. Buning uchun (3.71) da quyidagicha almashtirishni bajaramiz:

$$x = x_0 + \frac{2}{m+2} y_0^{(m+2)/2} t, \quad dx = \frac{2}{m+2} y_0^{\frac{m+2}{2}} dt. \tag{3.72}$$

U holda (3.71)dan ushbu tenglikni hosil qilamiz:

$$u(x_0, y_0) = -k_1 \left(\frac{2}{m+2} y_0^{\frac{m+2}{2}} \right)^{1-2a} \int_{-\infty}^{+\infty} \nu \left[x_0 + \frac{2}{m+2} y_0^{\frac{m+2}{2}} t \right] \times$$
$$\times (1+t^2)^{-a} \exp\left(-2b \arcsin \frac{t}{\sqrt{1+t^2}} \right) dt, \tag{3.73}$$

$\nu(x)$ funksiyaning (3.5) xossasiga asosan,

$$u(x_0, y_0) = -k_1 \left(\frac{2}{m+2} y_0^{\frac{m+2}{2}} \right)^{1-2a} \int_{-\infty}^{+\infty} \nu_0 \left[x_0 + \frac{2}{m+2} y_0^{\frac{m+2}{2}} t \right] \left(x_0 + \frac{2}{m+2} y_0^{\frac{m+2}{2}} t \right)^{2a-\varepsilon-1} \times$$
$$\times (1+t^2)^{-a} \exp\left(-2b \arcsin \frac{t}{\sqrt{1+t^2}} \right) dt =$$
$$= -k_1 \left(\frac{2}{m+2} y_0^{\frac{m+2}{2}} \right)^{-\varepsilon} \int_{-\infty}^{+\infty} \nu_0 \left[x_0 + \frac{2}{m+2} y_0^{\frac{m+2}{2}} t \right] \left(\frac{x_0}{(2/(m+2)) y_0^{(m+2)/2}} + t \right)^{2a-\varepsilon-1} \times$$
$$\times (1+t^2)^{-a} \exp\left(-2b \arcsin \frac{t}{\sqrt{1+t^2}} \right) dt, \tag{3.74}$$

ko‘rinishda yozib olamiz. Bu yerda $|\nu_0(x)| \le M$.

Agar (x_0, y_0) nuqta

$$x_0^2 + \frac{4}{(m+2)^2} y_0^{m+2} = R^2 \tag{3.75}$$

konturda joylashgan deb faraz qilsak, u holda (3.74) dan ushbu

$$|u(x_0, y_0)| \le \frac{C_1}{R^{\varepsilon}} \tag{3.76}$$

bahoni hosil qilamiz.

2. $\frac{\partial u}{\partial y_0}$ ni hisoblaymiz.(3.74) tenglikdan:

$$\frac{\partial u}{\partial y_0}=-k_1\int\limits_{-\infty}^{+\infty}\nu(x)\left\{(-a)\left[(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]^{-a-1}\frac{4}{m+2}y_0^{m+1}\times\right.$$

$$\left.\times\exp\left(-2b\arcsin(x-x_0)/\sqrt{(x-x_0{}^2)+\frac{4}{(m+2)^2}y_0^{m+2}}\right)\left(2b(x-x_0)\frac{y_0^{m/2}}{r_1^2|_{y=0}}\right)\right\}dx$$

Bu yerda (3.72) almashtirishni bajarib, ushbu

$$\frac{\partial u}{\partial y_0}=-k_1\int\limits_{-\infty}^{+\infty}\nu\left[x_0+\frac{2}{m+2}y_0^{\frac{m+2}{2}}t\right]\exp\left(-2b\arcsin\frac{t}{\sqrt{1+t^2}}\right)\times$$

$$\times\left[(-a)\left(\frac{2}{m+2}\right)^{-1-2a}y_0^{-(a+1)(m+2)+m+1}(1-t^2)^{-a-1}+\right.$$

$$\left.+\left(\frac{2}{m+2}\right)^{-1-2a}(1+t^2)^{-a-1}bt(y_0)^{-(a+1)(m+2)+m+1}\right]\times$$

$$\times\frac{2}{m+2}y_0^{\frac{m+2}{2}}dt=-2k_1\left(\frac{m+2}{2}\right)^{2a}y_0^{-\beta_0}+$$

$$+\int\limits_{-\infty}^{+\infty}\nu\left[x_0+\frac{2}{m+2}y_0^{\frac{m+2}{2}}t\right](1+t^2)^{-a-1}\exp(-2b\arcsin\frac{t}{\sqrt{1+t^2}})(-a+bt)dt,$$

ifodaga ega bo‘lamiz va (3.75) farazga asosan, ushbu

$$\left|y_0^{\beta_0}\frac{\partial u}{\partial y_0}\right|=O\left(\frac{1}{R^{1-2a+\varepsilon}}\right) \qquad (3.77)$$

bahoni hosil qilamiz.

3. $\frac{\partial u}{\partial x_0}$ ni hisoblaymiz:

$$\frac{\partial u}{\partial x_0}=-k_1\int_{-\infty}^{+\infty}\nu(x)\left\{(-a)\left[(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]^{-a-1}\times\right.$$

$$\times(-2)(x-x_0)\exp\left(-2b\arcsin(x-x_0)/\sqrt{(x-x_0{}^2)+\frac{4}{(m+2)^2}y_0^{m+2}}\right)+$$

$$+\left[(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]^{-a}\times$$

$$\times\exp\left(-2b\arcsin(x-x_0)/\sqrt{(x-x_0{}^2)+\frac{4}{(m+2)^2}y_0^{m+2}}\right)\times$$

$$\left.\times\frac{4b\,y_0^{\frac{m+2}{2}}}{(m+2)\left[(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]}\right\}dx=$$

$$=-k_1\int_{-\infty}^{+\infty}\nu(x)\left[(x-x_0)^2+\frac{4}{(m+2)^2}y_0^{m+2}\right]^{-a-1}\times$$

$$\times\exp\left(-2b\arcsin(x-x_0)/\sqrt{(x-x_0{}^2)+\frac{4}{(m+2)^2}y_0^{m+2}}\right)\times$$

$$\times\left[2a(x-x_0)+\frac{4by_0^{\frac{m+2}{2}}}{m+2}\right]dx.$$

Bu yerda integral o‘zgaruvchisini (3.72) formula bilan almashtirib, ushbu

$$\frac{\partial u}{\partial x_0}=-2k\left(\frac{m+2}{2}\right)^{2a}y_0^{-(m+2)(a+1)+m+2}\times$$

$$\times\int_{-\infty}^{+\infty}\nu\left[x_0+\frac{2}{m+2}y_0^{\frac{m+2}{2}}t\right](1+t^2)^{-a-1}\times$$

$$\times\exp\left(-2b\arcsin\frac{t}{\sqrt{1+t^2}}\right)(at+b)dt,\tag{3.78}$$

tenglikka kelamiz va (3.75)ga asosan, quyidagi bahoni hosil qilamiz:

$$\left|\frac{\partial u}{\partial x_0}\right| = O\left(\frac{1}{R^{1+\varepsilon}}\right). \tag{3.79}$$

Shunday qilib, (3.76), (3.77) va (3.79)ga asosan, $u(x,y)$ yechim (3.55) munosabatni qanoatlantirishiga ishonch hosil qilamiz.

Endi (3.71) yechim

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y} = \nu(x), \quad x\in(-\infty,+\infty), \tag{3.80}$$

shartni qanoatlantirishini ko‘rsatamiz. Buning uchun (3.71) yechimni ushbu

$$u(x,y) = -k_1\int_{-\infty}^{+\infty}\nu(t)\left(r_0^{\,2}\right)^{-a}\exp\left(-2b\arcsin\frac{t-x}{r_0}\right)dx, \tag{3.81}$$

ko‘rinishda yozib olamiz, bu yerda

$$r_0^2 = (x-t)^2 + \frac{4}{(m+2)^2}y^{m+2}.$$

(3.81) tenglikdan y bo‘yicha hosila olamiz:

$$\frac{\partial u}{\partial y} = \frac{4ak_1}{m+2}y^{m+1}\int_{-\infty}^{+\infty}\nu(t)\left(r_0^{\,2}\right)^{-a-1}\exp\left(-2b\arcsin\frac{t-x}{r_0}\right)dt -$$
$$-2bk_1 y^{\frac{m}{2}}\int_{-\infty}^{+\infty}\nu(t)\left(r_0^{\,2}\right)^{-a-1}\exp\left(-2b\arcsin\frac{t-x}{r_0}\right)(t-x)dt. \tag{3.82}$$

(3.82) tenglikda integral o‘zgaruvchini $t = x + \frac{2}{m+2}y^{\frac{m+2}{2}}\xi$ shaklida almashtiramiz, u holda:

$$\frac{\partial u}{\partial y}=2ak_1\left(\frac{2}{m+2}\right)^{-2a}y^{-\beta_0}\int\limits_{-\infty}^{+\infty}\nu\left[x+\frac{2}{m+2}y^{\frac{m+2}{2}}\xi\right]\times$$

$$\times(1+\xi^2)^{-a-1}\exp\left(-2b\arcsin\frac{\xi}{\sqrt{1+\xi^2}}\right)d\xi-$$

$$-2bk_1\left(\frac{2}{m+2}\right)^{-2a}y^{-\beta_0}\int\limits_{-\infty}^{+\infty}\nu\left[x+\frac{2}{m+2}y^{\frac{m+2}{2}}\xi\right]\times \tag{3.83}$$

$$\times(1+\xi^2)^{-a-1}\exp\left(-2b\arcsin\frac{\xi}{\sqrt{1+\xi^2}}\right)\xi d\xi.$$

(3.83) tenglikni $\arcsin x = arctg\dfrac{x}{\sqrt{1-x^2}},\quad |x|<1$ ayniyatga asosan, ushbu

$$\frac{\partial u}{\partial y}=2ak_1\left(\frac{2}{m+2}\right)^{-2a}y^{-\beta_0}\int\limits_{-\infty}^{+\infty}\nu\left[x+\frac{2}{m+2}y^{\frac{m+2}{2}}\xi\right]\times$$

$$\times(1+\xi^2)^{-a-1}\exp(-2barctg\xi)d\xi-$$

$$-2bk_1\left(\frac{2}{m+2}\right)^{-2a}y^{-\beta_0}\int\limits_{-\infty}^{+\infty}\nu\left[x+\frac{2}{m+2}y^{\frac{m+2}{2}}\xi\right]\times \tag{3.84}$$

$$\times(1+\xi^2)^{-a-1}\exp(-2barctg\xi)\xi d\xi.$$

ko‘rinishda yozib olamiz.

Quyidagi integrallarni hisoblaymiz:

$$J_1=\int\limits_{-\infty}^{+\infty}(1+\xi^2)^{-a-1}\exp(-2barctg\xi)d\xi, \tag{3.85}$$

$$J_2=\int\limits_{-\infty}^{+\infty}(1+\xi^2)^{-a-1}\xi\exp(-2barctg)d\xi. \tag{3.86}$$

Dastlab, J_1 ni hisoblaymiz.Buning uchun (3.85) da integral o‘zgaruvchisini $\xi = tgz$ ko‘rinishda almashtiramiz, u holda

$$J_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2bz} \cos^{2a} z dz = e^{-b\pi} \int_0^{\pi} e^{2bt} \sin^{2a} t dt.$$

Bu yerda

$$\int_0^{\pi} e^{-px} \sin^{\mu} x dx = \frac{\pi e^{\frac{-\pi p}{2}}}{2^{\mu}(\mu+1) B\left(\frac{\mu+ip}{2}+1, \frac{\mu-ip}{2}+1\right)}, \quad \operatorname{Re}\mu > -1$$

formuladan foydalanib [9],

$$J_1 = e^{-b\pi} \frac{\pi e^{b\pi}}{2^a (2a+1) B\left(1+\lambda, 1+\overline{\lambda}\right)} = \frac{2a\pi\Gamma(2a)}{2^{2a} \lambda\overline{\lambda}\Gamma(\lambda)\Gamma(\overline{\lambda})}, \qquad (3.87)$$
$$\lambda = a+bi, \ \overline{\lambda} = a-bi$$

tenglikka kelamiz.

Endi J_2 integralni hisoblaymiz. Bu yerda ham J_1 integralni hisoblashdagi usulni takrorlab,

$$J_2 = -\frac{1}{2a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2bz} d\cos^{2a} z = -\frac{b}{a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2bz} \cos^{2a} z dz =$$
$$= -\frac{b}{a} J_1 = -\frac{2b\pi\Gamma(2a)}{2^{2a} \lambda\overline{\lambda}\Gamma(\lambda)\Gamma(\overline{\lambda})} \qquad (3.88)$$

tenglikni hosil qilamiz.

(3.87) va (3.88) tengliklardan ushbu

$$2k_1 \left(\frac{2}{m+2}\right)^{+2a} \left(aJ_1 - bJ_2\right) = 1 \qquad (3.89)$$

tenglikning to‘g‘riligiga ishonch hosil qilish qiyin emas.

Endi $\left| y^{\beta_0} \frac{\partial u}{\partial y} - \nu(x) \right|$ ifodani baholaymiz. (3.84) (3.89) tengliklarga asosan,

$$\left| y^{\beta_0} \frac{\partial u}{\partial y} - \nu(x) \right| = \left| y^{\beta_0} \frac{\partial u}{\partial y} - 2k_1 \left(\frac{2}{m+2} \right)^{-2a} (aJ_1 - bJ_2)\nu(x) \right| =$$

$$= aA \int_{-\infty}^{\infty} \left| \nu\left(x + \frac{2}{m+2} y^{\frac{m+2}{2}} \xi \right) - \nu(x) \right| E(\xi) d\xi + bA \int_{-\infty}^{\infty} \left| \nu\left(x + \frac{2}{m+2} y^{\frac{m+2}{2}} \xi \right) - \nu(x) \right| E(\xi) \xi d\xi. \tag{3.90}$$

Bu yerda $E(\xi) = (1 + \xi^2)^{-a-1} \exp(-2barctg\beta\xi), \quad A = 2k_1 \left(\frac{2}{m+2} \right)^{-2a}$.

$\nu(x)$ funksiyamiz $(-\infty, +\infty)$ oraliqda uzluksiz va chegaralangan bo‘lgani uchun:

$$\left| \nu\left(x + \frac{2}{m+2} y^{\frac{m+2}{2}} \xi \right) - \nu(x) \right| \le 2M. \tag{3.91}$$

$\varepsilon -$ ixtiyoriy kichik musbat son bo‘lsin. U holda shunday N musbat soni mavjudki, J_1 va J_2 integrallarning yaqinlashuvchiligidan:

$$2aMA \int_{-\infty}^{-N} E(\xi) d\xi < \frac{\varepsilon}{6}; \quad 2aMA \int_{N}^{+\infty} E(\xi) d\xi < \frac{\varepsilon}{6} \tag{3.92}$$

$$2bMA \int_{-\infty}^{-N} E(\xi) \xi d\xi < \frac{\varepsilon}{6}; \quad 2bMA \int_{N}^{+\infty} E(\xi) \xi d\xi < \frac{\varepsilon}{6}, \tag{3.93}$$

tengsizliklar o‘rinli bo‘ladi. (3.91),(3.93) tengsizliklarni hisobga olib, (3.90) tengsizlikni quyidagi ko‘rinishda yozib olamiz:

$$\left| y^{\beta_0} \frac{\partial u}{\partial y} - \nu(x) \right| = \frac{4\varepsilon}{6} + aA \int_{-N}^{N} \left| \nu\left(x + \frac{2}{m+2} y^{\frac{m+2}{2}} \xi \right) - \nu(x) \right| E(\xi) d\xi +$$

$$+ bA \int_{-N}^{N} \left| \nu\left(x + \frac{2}{m+2} y^{\frac{m+2}{2}} \xi \right) - \nu(x) \right| E(\xi) |\xi| d\xi. \tag{3.94}$$

$\nu(x)$ funksiyaning uzluksizligidan yetarli kichik y va $|t| < N$ uchun:

$$\frac{aA}{J_1}\left|\nu\left(x+\frac{2}{m+2}y^{\frac{m+2}{2}}t\right)-\nu(x)\right|<\frac{\varepsilon}{6},$$
$$\frac{bA}{J_2}\left|\nu\left(x+\frac{2}{m+2}y^{\frac{m+2}{2}}t\right)-\nu(x)\right|<\frac{\varepsilon}{6}. \quad (3.95)$$

(3.94) tengsizlikdan, (3.95) tengsizlikni hisobga olib,yetarli kichik y lar uchun ushbu:

$$\left|y^{\beta_0}\frac{\partial u}{\partial y}-\nu(x)\right|<\varepsilon,$$

tengsizlikni hosil qilamiz. Bu yerda ε ning ixtiyoriyligidan foydalanib,

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y}=\nu(x), \quad x\in(-\infty,+\infty), \quad (3.96)$$

tenglikni hosil qilamiz.

Shakli o‘zgargan N masalasini yechishdagi usullarni takrorlab, Dirixle masalasi yechimi uchun ushbu

$$u(x,y)=k_2(1-\beta_0)y^{1-\beta_0}\int_{-\infty}^{+\infty}\tau(t)(r_0^2)^{a-1}\exp\left(-2b\arcsin\frac{t-x}{r_0}\right)dt,$$

formulani hosil qilish qiyin emas.

3.2-lemma. $\tau(x)\in C(-\infty,+\infty)$ *va yetarli katta* $|x|$ *lar uchun*

$$|\tau(x)|\le N_0/|x|^{\delta},$$

tengsizlik o‘rinli bo‘lsin, u holda yetarli katta R *lar uchun*

$$\left|u(x,y)\right| \le \frac{N_1}{R^{\delta}} \ , \ \left|u_x\right| \le \frac{N_2}{R^{1+\delta}} \ , \ \left|y^{\beta_0} \frac{\partial u}{\partial y}\right| \le \frac{N_3}{R^{1-2a+\delta}},$$

tengsizliklar o'rinlidir, bu yerda $N_i \ (i=1,2,3)$, δ *musbat o'zgarmas sonlar.*

3.2-lemmaning isboti 3.1-lemmaning isboti kabi bajariladi.

IV BOB. SINGULYAR KOEFFITSIENTLI GELLERSTEDT TENGLAMASI UCHUN TRIKOMI MASALASI

1-§. Trikomi masalasining qo‘yilishi va yechimning yagonaligi.

1. Umumiy tushunchalar. $z = x + iy$ kompleks tekisligining chekli bir bog‘lamli D sohasida ushbu

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + F(x, y, u, u_x, u_y) = 0, \tag{4.1}$$

ikkinchi tartibli xususiy hosilali kvazichiziqli differensial tenglamani o‘rganamiz, bu yerda $A = A(x, y)$, $B = B(x, y)$, $C = C(x, y)$ koeffitsientlar (x, y)o‘zgaruvchilarning haqiqiy funksiyasi bo‘lib, ular bir vaqtda nolga aylanmaydi:

$$A^2(x, y) + B^2(x, y) + C^2(x, y) \neq 0.$$

(4.1) tenglama D sohada elliptik, giperbolik, parabolik turga tegishli deyiladi, agar mos ravishda $B^2 - AC < 0$, $B^2 - AC > 0$, $B^2 - AC = 0$ bo‘lsa.

Agar D soha nuqtalarida $B^2 - AC$ ifoda o‘z ishorasini o‘zgartirsa, u holda (4.1) tenglama D sohada aralash turdagi tenglama deyiladi. γ chiziq nuqtalar $B^2 - AC = 0$ tenglikni qanoatlantirsa, γ chiziq (4.1) tenglamaning parabolik chizig‘i deyiladi (yoki tenglamaning buzilish chizig‘i deyiladi). Aralash turdagi tenglamalarga

$$yu_{xx} + u_{yy} = 0 \quad \text{(Trikomi tenglamasi)} \tag{4.2}$$

$$y^{2n-1}u_{xx} + u_{yy} = 0, \quad n \in N \text{ (Gellerstedt tenglamasi)} \tag{4.3}$$

$$u_{xx} + sign\, y\, u_{yy} = 0 \quad \text{(Lavrentev-Bitsadze tenglamasi)} \tag{4.4}$$

$$K(y)u_{xx} + u_{yy} = 0 \quad \text{(Chapligin tenglamasi)} \tag{4.5}$$

larni keltirish mumkin.

Bu yerda $K(y)$ monoton o‘suvchi funksiya bo‘lib, $K(0) = 0$.

Aralash turdagi tenglamalar uchun birinchi chegaraviy masalani 1923-yilda italyan matematigi Franchesko Trikomi qo‘ygan va uni korrekt ekanligini isbotlagan.

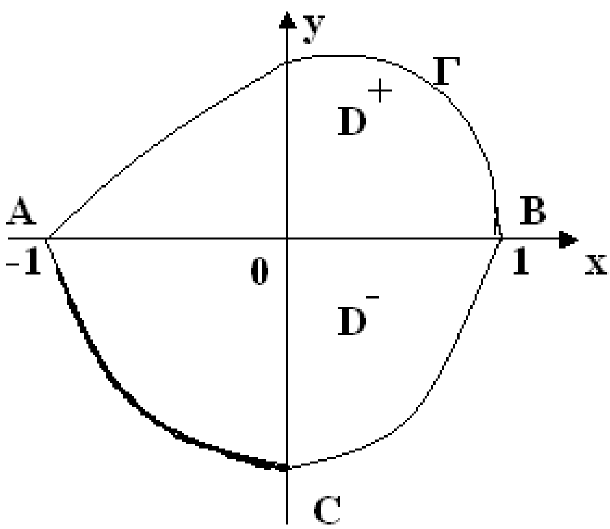
Ushbu bobda singulyar koeffitsientli Gellerstedt tenglamasi uchun Trikomi masalasi o‘rganilgan.

2. Trikomi masalasining qo‘yilishi. Ushbu tenglamani o‘rganamiz:

$$sign\, y|y|^m u_{xx} + u_{yy} + (\beta_0 / y)u_y = 0, \tag{4.6}$$

bu yerda m va β_0 –o‘zgarmas sonlar bo‘lib, ular uchun $m > 0$, $-m/2 < \beta_0 < 1$ tengsizliklar o‘rinli. D soha $z = x + iy$ kompleks tekisligining chekli bir bog‘lamli sohasi bo‘lib, u $y > 0$ yarim tekislikda uchlari $A = A(-1,0)$, $B = B(1,0)$ nuqtalarda yotuvchi, hamda $y > 0$ yarim tekislikda joylashgan Γ Jordan chizig‘i bilan, $y < 0$ yarim tekislikda esa (4.1) tenglamaning AC va BC xarakteristikalari bilan chegaralangan. Ushbu belgilashlarni kiritamiz:

$D^+ = D \cap \{y > 0\}$, $D^- = D \cap \{y < 0\}$.



T masalasi. D sohada (4.1) tenglamaning ushbu

$$u|_\Gamma = \varphi(s), \; 0 \le s \le l, \tag{4.7}$$

$$u|_{AC} = \psi(x), \; x \in [-1,0] \tag{4.8}$$

shartlarni qanoatlantiruvchi $u(x,y) \in C(\overline{D}) \cap C^2(D^+ \cup D^-)$yechimi topilsin. Bu yerda $\varphi(s), \psi(x)$ berilgan funksiyalar, $s - \Gamma$ chiziqning B uchidan $M(x(s), y(s)) \in \Gamma$ nuqtasigacha bo‘lgan BM yoy uzunligi, $l - \Gamma$ chiziq uzunligi. $y = 0$ parabolik buzilish chizig‘ida ushbu

$$\lim_{y\to+0} y^{\beta_0}\frac{\partial u}{\partial y} = \lim_{y\to-0}(-y^{\beta_0})\frac{\partial u}{\partial y}, \quad x\in I \tag{4.9}$$

ulanish sharti bajariladi, $I=(-1,1)$ $y=0$ o‘qining intervali.

3.Regulyar va umumlashgan yechimlar

4.1-ta’rif. D *sohada (4.1) tenglamaning regulyar yechimi deb, ushbu shartlarni qanoatlantiruvchi* $u(x,y)$ *funksiyaga aytiladi:*

1^0) $u(x,y)\in C(\overline{D})$;

2^0) $u(x,y)\in C^2(D^+\cup D^-)$ va (4.1) tenglamani mos ravishda D^+ va D^- sohalarda qanoatlantiradi;

3^0) $v(x)=\lim\limits_{y\to 0}|y|^{\beta_0}\frac{\partial u}{\partial y}$ va $\tau'(x)$ $(\tau(x)=u(x,0))$ funksiyalar $(-1,1)$ intervalda uzluksiz, differensiallanuvchi, shu bilan birga $v(x)$ bu interval chegaralarida $1-2\beta$ dan katta bo‘lmagan tartibda cheksizlikka aylanishi mumkin $\beta=(m+2\beta_0)/2(m+2)$.

4.2-ta’rif. D *sohada (4.1) tenglamaning umumlashgan yechimi deb, ushbu shartlarni qanoatlantiruvchi funksiyaga aytiladi:*

1^0) $u(x,y)\in C(\overline{D})$;

2^0) $u(x,y)\in C^2(D^+)$ va shu sohada (4.1) tenglamani qanoatlantiradi;

3^0) ixtiyoriy $x\in(-1,1)$ uchun

$$\lim_{y\to 0}|y|^{\beta_0}\frac{\partial u}{\partial y}=v(x)$$

mavjud;

4^0) D^- sohada yechim

$$u(x,y)=\gamma_1\int_{-1}^{1}\tau\left[x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{\beta-1}(1+t)^{\beta-1}dt+$$

$$+\gamma_2(-y)^{1-\beta_0}\int_{-1}^{1}\nu\left[x+\frac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{-\beta}(1+t)^{-\beta}dt, \quad (4.10)$$

bu yerda

$$\gamma_1=\frac{\Gamma(2\beta)}{\Gamma^2(\beta)}2^{1-2\beta}\quad,\ \gamma_2=-\frac{\Gamma(2-2\beta)2^{2\beta-1}}{(1-\beta_0)\Gamma^2(1-\beta)}, \quad (4.11)$$

formula bilan aniqlanadi.

Biz quyida ushbu umumlashgan yechimlar sinfini kiritamiz.

4.3-ta'rif. D^- sohada (4.1) tenglamaning yechimi $u(x,y)$ R_1 sinfga tegishli deyiladi, agar $u(x,y)$ yechim (4.10) formula bilan ifodalanib, $\tau(x)$ funksiya $-1\le x<1$ *oraliqda* $\alpha_1>1-\beta$ *ko'rsatkich bilan,* $\nu(x)$ *funksiya esa* $-1\le x<1$ *oraliqda* $\alpha_2>\beta$ *ko'rsatkich bilan Gyolder sinfiga tegishli bo'lsa.*

4.4-ta'rif. D *sohada (4.1) tenglamaning yechimi* R_1 *sinfga tegishli deyiladi, agar* $u(x,y)$ *yechim* D^- *sohada* R_1 *sinfga tegishli bo'lsa.*

4. $\tau(x)$ **va** $\nu(x)$ **noma'lum funksiyalar o'rtasidagi birinchi funksional munosabatni keltirib chiqarish.** (4.10) formuladan, (4.8) shartga asosan, ushbu tenglikka ega bo'lamiz: AC xarakteristikada

$$x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=-1$$

yoki

$$y=-\left(\frac{m+2}{2}(x+1)\right)^{2/(m+2)}.$$

Buni e'tiborga olib, (4.10) formuladan:

$$u(x,y)\big|_{AC} = \gamma_1 \int_{-1}^{1} \tau\big(x+t(x+1)\big)(1-t)^{\beta-1}(1+t)^{\beta-1}dt +$$

$$+\gamma_2\left(\frac{m+2}{2}(x+1)\right)^{\frac{2(1-\beta_0)}{m+2}} \int_{-1}^{1} \nu\big(x+t(x+1)\big)(1-t)^{-\beta}(1+t)^{-\beta}dt = \psi(x), \quad (4.12)$$

$$x \in (-1,0),$$

tenglikka ega bo‘lamiz.

$$1-2\beta = 1-2\frac{m+2\beta_0}{2(m+2)} = \frac{m+2-m-2\beta_0}{(m+2)} = \frac{2(1-\beta_0)}{m+2}$$

tenglikda $X = 1+2x$ almashtirish bajarib, $x \in (-1,0)$ intervalni $X \in (-1,1)$ intervalga akslantiramiz. Eski belgilashni saqlab qolgan holda, (4.12) da x ni $\frac{x-1}{2}$ ga almashtiramiz.

$$\gamma_1 \int_{-1}^{1} \tau\left(\frac{x-1}{2}+\frac{x+1}{2}t\right)(1-t)^{\beta-1}(1+t)^{\beta-1}dt +$$

$$+\gamma_2\left(\frac{m+2}{4}(x+1)\right)^{1-2\beta} \int_{-1}^{1} \nu\left(\frac{x-1}{2}+\frac{x-1}{2}t\right)(1-t)^{-\beta}(1+t)^{-\beta}dt = \quad (4.13)$$

$$= \psi\left(\frac{x-1}{2}\right), \quad x \in (-1,1).$$

(4.13) munosabatda integral o‘zgaruvchisini

$$z = \frac{x-1}{2} + \frac{x+1}{2}t \quad (4.14)$$

ko‘rinishda o‘zgartiramiz:

$$t=-1, \quad z=-1; \quad t=1, \quad z=x$$

$$t = \frac{2z+1-x}{1+x}, \quad 1+t = \frac{2(z+1)}{1+x}, \quad 1-t = \frac{2(x-z)}{1+x}. \quad (4.15)$$

(4.14) va (4.15) ga ko‘ra, (4.13) munosabat ushbu ko‘rinishni oladi:

$$\gamma_1\int_{-1}^{x}\tau(z)\left(\frac{2}{1+x}\right)^{2\beta-1}(1-z)^{\beta-1}(1+z)^{\beta-1}dz+$$

$$+\gamma_2\left(\frac{m+1}{4}(x+1)\right)^{1-2\beta}\int_{-1}^{x}\nu(z)\left(\frac{2}{1+x}\right)^{1-2\beta}(1-z)^{-\beta}(1+z)^{-\beta}dz=\psi\left(\frac{x-1}{2}\right)$$

$$\gamma_1\left(\frac{1+x}{2}\right)^{1-2\beta}\int_{-1}^{x}\tau(z)(1-z)^{\beta-1}(1+z)^{\beta-1}dz+$$

$$+\gamma_2\left(\frac{m+2}{2}\right)^{1-2\beta}\int_{-1}^{x}\nu(z)(1-z)^{-\beta}(1+z)^{-\beta}dz=\psi\left(\frac{x-1}{2}\right),\quad x\in I \tag{4.16}$$

Endi (4.14) ifodani kasr tartibli integro-differensial operatorlar:

$$D_{a,x}^{l}f(x)=\begin{cases}\dfrac{1}{\Gamma(-l)}\displaystyle\int_a^x\frac{f(t)\,dt}{(x-t)^{1+l}}, \text{ агар } l<0\,,\\ \dfrac{d^{n+1}}{dx^{n+1}}D_{a,x}^{l-(n+1)}f(x), \text{агар } l>0\,,\end{cases} \tag{4.17}$$

$$D_{x,b}^{l}f(x)=\begin{cases}\dfrac{1}{\Gamma(-l)}\displaystyle\int_x^b\frac{f(t)\,dt}{(t-x)^{1+l}}, \text{ агар } l<0\,,\\ (-1)^{n+1}\dfrac{d^{n+1}}{dx^{n+1}}f(x), \text{ агар } l>0,\end{cases} \tag{4.18}$$

bu yerda $n=[l]$, yordamida ushbu ko‘rinishda yozib olamiz:

$$\gamma_1\left(\frac{1+x}{2}\right)^{1-2\beta}\Gamma(\beta)\,D_{-1,x}^{-\beta}(1+x)^{\beta-1}\tau(x)+$$

$$+\gamma_2\left(\frac{m+2}{2}\right)^{1-2\beta}\Gamma(1-2\beta)D_{-1,x}^{\beta-1}(1+x)^{-\beta}\nu(x)=\psi\left(\frac{x-1}{2}\right). \tag{4.19}$$

Endi (4.15) tenglikka $D_{-1,x}^{1-\beta}$ operatorni qo‘llaymiz va

$$D_{-1,x}^{1-\beta}\,D_{-1,x}^{\beta-1}(1+x)^{-\beta}\nu(x)=(1+x)^{-\beta}\nu(x), \tag{4.20}$$

$$D_{-1,x}^{1-\beta}(1+x)^{1-2\beta}D_{-1,x}^{-\beta}(1+x)^{\beta-1}\tau(x)=(1+x)^{-\beta}D_{-1,x}^{1-2\beta}\tau(x) \quad (4.21)$$

tengliklarni hisobga olib, (15) tenglikni ushbu ko‘rinishda yozib olamiz:

$$\gamma_1 2^{2\beta-1}\Gamma(\beta)\ (1+x)^{-\beta}D_{-1,x}^{1-2\beta}\tau(x)+$$
$$+\gamma_2\left(\frac{m+2}{2}\right)^{1-2\beta}\Gamma(1-2\beta)(1+x)^{-\beta}\nu(x)=D_{-1,x}^{\beta-1}\psi\left(\frac{x-1}{2}\right),\ x\in I \quad (4.22)$$

yoki

$$\nu(x)=\gamma D_{-1,x}^{1-2\beta}\tau(x)+\psi_1(x). \quad (4.23)$$

Bu yerda

$$\gamma=\frac{2\Gamma(2\beta)\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(1-2\beta)}\left(\frac{m+2}{4}\right)^{2\beta}, \quad (4.24)$$

$$\psi_1(x)=-\gamma\frac{\Gamma(\beta)}{\Gamma(2\beta)}(1+x)^{\beta}D_{-1,x}^{1-\beta}\psi\left(\frac{x-1}{2}\right). \quad (4.25)$$

Shunday qilib, (4.23) tenglik $\tau(x)$ va $\nu(x)$ noma’lum funksiyalar o‘rtasida D^{-} sohadan buzilish chizig‘iga keltirilgan birinchi funksional munosabatni beradi.

5. Trikomi masalasi yechimining yagonaligi.

4.1-teorema. *Agar Trikomi masalasining yechimi $u(x,y)$ AC xarakteristikada nolga teng bo‘lsa, u holda $u(x,y)$ funksiya $\overline{D}^{+}$ sohada o‘zining musbat maksimumini va manfiy minimumini faqat Γ da qabul qiladi.*

Isboti. Haqiqatdan ham, $u(x,y)$ funksiya T masalasining 4.1-teorema shartlarini qanoatlantiruvchi yechimi bo‘lsin. Ravshanki, $u(x,y)$ funksiya o‘zining ekstremumlarini D^{+} soha ichida qabul qilmaydi.

Faraz qilaylik, $u(x,y)$ $\overline{D}^{+}$ sohada o‘zining musbat maksimumini $P(x_0,0)$, $x_0\in I$ nuqtada qabul qilsin, ya’ni $\max\limits_{x,y\in\overline{D}^{+}}u(x,y)=\tau(x_0)$. U holda $\tau(x)$ funksiyaning musbat maksimumi nuqtasi x_0 da, kasr tartibli hosila $D_{-1,x}^{1-2\beta}\tau(x)$ ning qat’iy musbat ekanligidan foydalanib, (4.23) tenglikka mos bir jinsli $(\psi_1(x)\equiv 0)$ tenglikdan $x=x_0$ nuqtada $\nu(x_0)>0$ tengsizlikka kelamiz. Bu tengsizlik ulanish sharti (4.9) ga ziddir.

4.1-teorema isbot bo‘ldi.

2-§. Singulyar koeffitsientli Gellerstedt tenglamasi uchun Trikomi masalasi yechimining mavjudligi.

Trikomi masalasi yechimining mavjudligini isbotlash algoritmini keltiramiz. Dastlab, $\nu(x) = \lim_{y\to 0} |y|^{\beta_0} \frac{\partial u}{\partial y}$ va $\tau(x) = u(x,0)$ funksiyalarni ma'lum deb faraz qilib, D^+ sohada shakli o'zgargan N masalasi, D^- sohada esa shakli o'zgargan Koshi masalasi yechiladi. Keyin bu ikki yechim va uning salmoqli $|y|^{\beta_0} \frac{\partial u}{\partial y}$ hosilalari parabolik buzilish chizig'ining AB kesmasida "ulanadi". Shu usul bilan Trikomi masalasi yechimining mavjudligi ekvivalent ravishda $\nu(x)$ funksiyaga nisbatan singulyar integral tenglamani yechishga olib kelinadi, integral tenglama yagona yechimining mavjudligi Trikomi masalasi yechimining yagonaligi teoremasidan kelib chiqadi. Γ — Jordan chizig'i bo'lib, uning tenglamasi parametrik shaklda berilgan bo'lsin: $x = x(s)$, $y = y(s)$, bu yerda $s - B(1,0)$ nuqtadan hisoblaganda, BM yoy uzunligi. Γ chiziqushbuGellerstedtshartlarniqanoatlantirsin:

1[0]) $x(s)$, $y(s)$ funksiyalar $[0,l]$ kesmada $x'(s)$, $y'(s)$ uzluksizhosilalargaegavabirvaqtdanolgaaylanmaydi; $x''(s)$, $y''(s)$ $[0,l]$ daGyoldershartiniqanoatlantiradi, buyerda l butun Γ chiziquzunligidir;

2[0]) A va B nuqta atrofida Γ chiziq ushbu

$$\left|\frac{dx}{ds}\right| \le C\, y^{m+1}(s), \tag{4.26}$$

shartlarni qanoatlantiradi, bu yerda C o'zgarmas son.

1.Shakli o'zgargan N masalasi (Xolmgren masalasi). $\tau(x)$ va $\nu(x)$ funksiyalar o'rtasidagi ikkinchi funksional munosabatni keltirib chiqarish.

D^+ sohada

$$y^m u_{xx} + u_{yy} + (\beta_0 / y) u_y = 0 \tag{4.27}$$

tenglamaning ushbu

$$u|_{\Gamma} = \varphi(s)\,; \quad \lim_{y\to +0} y^{\beta_0} \frac{\partial u}{\partial y} = \nu(x)\,, \quad x \in I\,, \tag{4.28}$$

shartlarni qanoatlantiruvchi regulyar yechimi topilsin, bu yerda $\varphi(s) \in C[0,l]$, $\nu(x) \in C(I)$ va I intervalning chegaralarida $1-2\beta$ dan kichik tartibda cheksizlikka intilishi mumkin.

Shakli o'zgargan N masalasi yechimi ushbu formula orqali ifodalanadi:

$$u(x,y)=-\int_{-1}^{1}\nu(t)G_1(t,0;x_0,y_0)dt-$$
$$-\int_{0}^{l}\varphi(\xi(s))\eta^{\beta_0}(s)A_s[G_1(\xi(s),\eta(s);x_0,y_0)]ds, \tag{4.29}$$

bu yerda $(\xi(s),\eta(s))\in\Gamma$, $G_1(\xi,\eta;x,y)-$shakli o'zgargan N masalasi uchun (4.27) tenglamaning Grin funksiyasi, $A_s[\,]=\eta^m\frac{d\eta}{ds}\frac{\partial}{\partial\xi}-\frac{d\xi}{ds}\frac{\partial}{\partial\eta}-$konormal hosila. $G_1(x,y;x_0,y_0)$ Grin funksiyasi

$$G_1(x,y;x_0,y_0)=q_1(x,y;x_0,y_0)+\upsilon_1(x,y;x_0,y_0)$$

ko'rinishda ifodalanadi, bu yerda

$$q_1(x,y;x_0,y_0)=k_1r_1^{-2\beta}F(\beta,\beta,2\beta;1-\sigma).$$

(4.27) tenglamaning D^+sohada fundamental yechimi, $\upsilon_1(x,y;x_0,y_0)$ esa regulyar yechimidir,

$$\left.\begin{matrix}r^2\\ r_1^2\end{matrix}\right\}=(x-x_0)^2+\frac{4}{(m+2)^2}\left(y^{\frac{m+2}{2}}\mp y_0^{\frac{m+2}{2}}\right)^2,\quad \sigma=\frac{r^2}{r_1^2}.$$

Agar Γ chiziq (4.27) tenglamaning normal chizig'i

$$\sigma_0: x^2+\frac{4}{(m+2)^2}y^{m+2}=1$$

dan iborat bo'lsa, (4.27) tenglama uchun N masalasining Grin funksiyasi ushbu formula yordamida oshkor ko'rinishda beriladi:

$$G_{01}(x,y;x_0,y_0)=q_1(x,y;x_0,y_0)-R^{-2\beta}q_1(x,y;\bar{x}_0,\bar{y}_0),$$

$$q_1(x,y;x_0,y_0)=k_1\left(r_1^2\right)^{-\beta}F(\beta,\beta,2\beta;1-\sigma),$$

$$\bar{x}_0 = \frac{x_0}{R^2}, \quad \bar{y}_0^{(m+2)/2} = \frac{y_0^{(m+2)/2}}{R^2},$$

$$R^2 = x_0^2 + \frac{4}{(m+2)^2} y_0^{m+2}.$$

Agar Γ Gellerstedt shartlarini qanoatlantiruvchi ixtiyoriy chiziq bo‘lsa, bu holda ixtiyoriy D sohada (4.1) tenglama uchun shakli o‘zgargan N masalasining Grin funksiyasini ushbu ko‘rinishda ifodalash mumkin:

$$G_1(x,y;x_0,y_0) = G_{01}(x,y;x_0,y_0) + H_1(x,y;x_0,y_0), \qquad (4.30)$$

bu yerda

$$H_1(x,y;x_0 y_0) = \int_0^l \rho_1(t;x;y) G_{01}(\xi(t),\eta(t);x_0,y_0)dt. \qquad (4.31)$$

$\rho_1(s;x_0,y_0)$ –zichlik esa

$$\rho_1(s;x_0,y_0) - 2\int_0^l K_1(t,s)\rho_1(t;x_0,y_0)dt = 2A_s[q_1(x(s),y(s);x_0,y_0)]$$

integral tenglamaning yechimi

$$\rho_1(s;x_0,y_0) = 2A(s;x_0,y_0) + 4\int_0^l R_1(t,s)A(t;x_0,y_0)dt, \qquad (4.32)$$

bu yerda

$$K_1(t,s) = y^{\beta_0}(s)A_s[q_1(\xi(t),\eta(t);x(s),y(s))],$$
$$A(s;x_0,y_0) = y^{\beta_0}(s)A_s[q_1(x(s),y(s);x_0,y_0)].$$

Ushbu tenglikningo‘rinli ekanligiga ishonch hosil qilish qiyin emas:

$$y^{\beta_0}(s)A_s[G_1(x(s),y(s);x_0,y_0)] = \rho_1(s;x_0,y_0).$$

Bu tenglikka asosan, (4.29) formulani

$$u(x,y) = -\int_{-1}^{1} \nu(t)G_1(t,0;x,y)dt - \int_0^l \varphi(s)\rho_1(s;x,y)ds, \qquad (4.33)$$

ko‘rinshda yozib olish mumkin. (4.33) formulada $y = 0$ deb va (4.28) hamda (4.30) formulani hisobga olib, ushbu tenglikka kelamiz:

$$\tau(x) = -k_1 \int_{-1}^{1} \left[|x-t|^{-2\beta} - (1-xt)^{-2\beta}\right] \nu(t)dt -$$
$$- \int_{-1}^{1} H(t,x)\nu(t)\,dt - \int_{0}^{l} \varphi(s)\rho_1(s;x,0)ds, \tag{4.34}$$

bu yerda

$$H(x,t) = G_1(t,0;x,0) - G_{01}(t,0;x,0) =$$
$$= \int_{0}^{l} \rho_1(s;t,0)G_{01}(\xi(s),\eta(s);x,0)\,ds \tag{4.35}$$

$\xi(s),\eta(s) \in \Gamma$ (4.34) formula $\tau(x)$ va $\nu(x)$ noma’lum funksiyalar o‘rtasidagi $y=0$ parabolik buzilish chizig‘ining AB kesmasiga D^+ sohadan keltirilgan ikkinchi funksional munosabatni beradi.

4.1-lemma. *Agar* $-1 < x < 1$, $-1 < t < 1$ *bo‘lsa, Grin funksiyasi regulyar qismining* x *bo‘yicha hosilasi uchun ushbu*

$$\left|\frac{\partial H(t,x)}{\partial x}\right| \le C(1-xt)^{-2\beta} \tag{4.36}$$

tengsizlik o‘rinlidir, bu yerda C *faqat* D^+ *sohaga bog‘liqdir.*

Isboti. $R^2 = \xi^2 + \dfrac{4}{(m+2)^2}\eta^{m+2}$ belgilashni kiritib, $G_0(\xi,\eta;x,0)$ uchun ushbu tenglikka kelamiz:

$$G_{01}(\xi,\eta;x,0) = k_1\left[(x-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} -$$
$$- k_1(x^{-2\beta})\left[\left(\xi - \frac{1}{x}\right)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} =$$
$$= k_1\left[R^2 - 2x\xi + x^2\right]^{-\beta} - k_1\left[x^2R^2 - 2x\xi + 1\right]^{-\beta}$$

yoki

$$G_{01}(\xi,\eta\,;x,0)=k_1\left[R_1^{-2\beta}-R_2^{-2\beta}\right]=k_1\left[(R_1^2)^{-\beta}-(R_2^2)^{-\beta}\right],$$

bu yerda

$$R_1^2=R^2-2x\xi+x^2\ \ ,\ \ R_2^2=x^2R^2-2x\xi+1\,.$$

Yetarli kichik η lar uchun:

$$R^2=1+o(\eta^{m+2})\,.$$

Quyidagi hosilani hisoblaymiz:

$$\frac{1}{2\beta k_1}\frac{\partial G_{01}(\xi,\eta\,;x,0)}{\partial x}=$$

$$=\frac{\beta}{2\beta k_1}k_1\left[(R_1^2)^{-\beta-1}(-2\xi+2x)-(R_2^2)^{-\beta-1}(2xR^2-2\xi)\right]=$$

$$=-(x-\xi)\left[R_1^{-2\beta-2}\right]+R_2^{-2\beta-2}\left(x-\xi+x\cdot o(\eta^{m+2})\right)=$$

$$=(x-\xi)\left[R_2^{-2\beta-2}-R_1^{-2\beta-2}\right]+x\,R_2^{-2\beta-2}\cdot o(\eta^{m+2})$$

Ushbu ifodani baholaymiz:

$$\left|R_2^{-2\beta-2}-R_1^{-2\beta-2}\right|=\left|\frac{1}{R_2^{2\beta+2}}-\frac{1}{R_1^{2\beta+2}}\right|=\left|\frac{R_1^{2\beta+2}-R_2^{2\beta+2}}{R_1^{2\beta+2}\cdot R_2^{2\beta+2}}\right|$$

$$f(x)=x^{\beta+1},f'(x)=(\beta+1)x^{\beta}\quad,\quad f''(x)=\beta(\beta+1)x^{\beta-1}>0,$$

demak, $f(x)$ funksiya botiq va botiqlik ta’rifiga ko‘ra, ixtiyoriy x_1, x_2 va ixtiyoriy q_1, q_2 $q_1+q_2=1$ sonlari uchun

$$f(\,q_1x_1+q_2x_2)\le q_1f(x_1)+q_2f(x_2)$$

o‘rinli. Lagranj formulasiga ko‘ra,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(\xi)\,,x_1\le\xi\le x_2\ \ ,\ \ \xi=q_1x_1+q_2x_2\,.$$

Demak,

$$\left|f(x_2)-f(x_1)\right|=\left|x_2-x_1\right|f'(q_1x_1+q_2x_2)\le\left|x_1-x_2\right|\left|q_1f'(x_1)+q_2f'(x_2)\right|$$

yoki

$$\left|x_2^{\beta+1}-x_1^{\beta+1}\right|\le\left|x_1-x_2\right|(\beta+1)\left|q_1x_1^{\beta}+q_2x_2^{\beta}\right|\le\frac{3}{2}\left|x_2-x_1\right|\left|x_1^{\beta}+x_2^{\beta}\right|.$$

$$x_2=R_2^2\,,\;x_1=R_1^2$$

bo‘lsa,

$$\left|R_2^{2\beta+2}-R_1^{2\beta+2}\right|\le\frac{3}{2}\left|R_2^2-R_1^2\right|\left|R_2^{2\beta}+R_1^{2\beta}\right|.$$

Bundan,

$$\left|\frac{1}{R_2^{2\beta+2}}-\frac{1}{R_1^{2\beta+2}}\right|\le\frac{3}{2}\left|R_2^2-R_1^2\right|\left|\frac{R_1^{2\beta}+R_2^{2\beta}}{R_1^{2\beta+2}\cdot R_2^{2\beta+2}}\right|=$$

$$=\frac{3}{2}\left|R_2^2-R_1^2\right|\left|\frac{1}{R_1^{2\beta+2}R_2^2}+\frac{1}{R_2^{2\beta+2}R_1^2}\right|=$$

Agar $-1\le x\le-\frac{1}{2}$ bo‘lsa, $\eta\to0$, $\xi\to\pm1$ bo‘lganda, ushbu tengsizlik o‘rinlidir:

$$\frac{1}{R_1^2}=\frac{1}{R^2-2x\xi+x^2}=$$

$$=\frac{1}{\xi^2+\frac{4}{(m+2)^2}\eta^{m+2}-2x\xi+x^2}=\frac{1}{(\xi-x)^2+\frac{4}{(m+2)^2}\eta^{m+2}}=$$

$$=\frac{(1+x)^2+\frac{4}{(m+2)^2}\eta^{m+2}}{(\xi-x)^2+\frac{4}{(m+2)^2}\eta^{m+2}}\cdot\frac{1}{(1+x)^2+\frac{4}{(m+2)^2}\eta^{m+2}}\le$$

$$\le\frac{c_2}{(1+x)^2+\frac{4}{(m+2)^2}\eta^{m+2}}$$

$$\frac{1}{R_2^2}=\frac{1}{x^2R^2-2x\xi+1}=$$

$$=\frac{1}{x^2\xi^2+x^2\dfrac{4}{(m+2)^2}\eta^{m+2}-2x\xi+1}=\frac{1}{(x\xi-1)^2+x^2\dfrac{4}{(m+2)^2}\eta^{m+2}}=$$

$$=\frac{(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}}{(x\xi-1)^2+x^2\dfrac{4}{(m+2)^2}\eta^{m+2}}\cdot\frac{1}{(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}}\leq$$

$$\leq\frac{C_3}{(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}}.$$

Bu yerda C_2 va C_3 o‘zgarmaslar x,ξ va η larga bog‘liq emas.
Yuqoridagi ikki bahoga asosan, ushbu tengsizlikka kelamiz:

$$\left|\frac{\partial G_{01}}{\partial x}\right|\leq C_4\frac{(x-\xi)(1+x^2)\eta^{m+2}}{\left[(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+2}}+$$

$$+C_5\frac{\eta^{m+2}}{\left[(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}}\leq C_4\frac{(1-x)(1+x^2)\eta^{m+2}}{\left[(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+2}}+$$

$$+C_5\frac{\eta^{m+2}}{\left[(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}}\leq\frac{C_6}{\left[(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta}}.$$

Yuqoridagidek, $\frac{1}{2}\leq x\leq 1$ uchun:

$$\left|\frac{\partial G_{01}}{\partial x}\right|\leq\frac{c_7}{\left[(1+x)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta}}.\qquad(4.38)$$

Shunday qilib, (4.32) va (4.35) ga asosan:

$$\left|\frac{\partial H_1(t,x)}{\partial x}\right| \le \int_0^l |\rho_1(s;t,0)| \left|\frac{\partial G_{01}(\xi(s),\eta(s);x,0)}{\partial x}\right| ds \le$$

$$\le C_6 \int_0^l |\rho_1(s;t,0)| \left[(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} ds \le$$

$$\le C_6 \int_0^l \left[(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} ds \times$$

$$\times \left[2|A(s,t,0)| + 4\int_0^l R_1(t_1,s;2)\, A(t_1,t,0)\, dt_1\right] =$$

$$= 2C_6 \int_0^l |A(s,t,0)| \left[(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} ds +$$

$$+ 4C_6 \int_0^l |A(t_1,t,0)| dt_1 \int_0^l R_1(t_1,s;2) \left[x^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} ds \le$$

$$\le 2c_6 \int_{l-\varepsilon}^{l} |A(s,t,0)| \left[(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} ds + C_8. \qquad (4.39)$$

Bu yerda

$$C_8 = 2C_6 \int_0^{l-\varepsilon} |A(s,t,0)| \left[(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} ds +$$

$$+ 4C_6 \int_0^l | A(t_1,t,0)| dt_1 \int_0^l R_1(t_1,s;2) \left[x^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\beta} ds,$$

$$A(t;x,0) = -\frac{\beta k_1 \eta^{\beta_0}}{r_1^{2\beta}} A_t[\ln r^2] =$$

$$= -\frac{\beta k_1 \eta^{\beta_0}}{r_1^{2\beta}} \left[\eta^m \frac{2(\xi - x)}{r^2} \frac{d\eta}{dt} - \frac{\frac{4}{m+2}\left(\eta^{\frac{m+2}{2}} - y^{\frac{m+2}{2}}\right)\eta^{\frac{m}{2}}}{r^2} \frac{d\xi}{dt} \right]$$

yoki

$$A(s;t,0) = 2\beta k_1 \eta^{\beta_0}(s) \frac{(t-\xi)\eta^m \frac{d\eta}{ds} + \frac{2}{m+2}\eta^{m+1}\frac{d\xi}{ds}}{\left[(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}} . \quad (4.40)$$

$\left|\frac{d\xi}{ds}\right| \le \eta^{m+1}$ tengsizlikka asosan,

$$\left| A(s,t,0) \right| \le C_9' \frac{\left|(t-\xi)\right| \eta^{m+\beta_0} \frac{d\eta}{ds}}{\left[(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}} +$$

$$+ C_{10}' \frac{\eta^{2m+2+\beta_0}}{\left[(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}} = A_1 + A_2 \quad (4.41)$$

(4.41) ifodaningo‘ng tomonidagi ikkinchi qo‘shiluvchisi uchun

$$A_2 = C_{10}' \frac{\eta^{m+2}}{(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}} \times \frac{\eta^{(m+2)\frac{m+2\beta_0}{2(m+2)}} \eta^{\frac{m}{2}}}{\left((t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right)^{\beta}} \le C_{10} \eta^{\frac{m}{2}}$$

tengsizlikni hosil qilamiz. Endi (4.41) dagi birinchi qo‘shiluvchini baholash uchun ushbu tengsizlikdan foydalanamiz:

$$a \cdot b \le \frac{a^p}{p} + \frac{b^q}{q}, \ \frac{1}{p} + \frac{1}{q} = 1,$$

$$p = \frac{2(m+1+\beta_0)}{m+2}, \ q = \frac{2(m+1+\beta_0)}{m+2\beta_0},$$

$$\beta + \frac{1}{2} = \frac{m+2\beta_0}{2(m+2)} + \frac{1}{2} = \frac{2m+2+2\beta_0}{2(m+2)} = \frac{m+1+\beta_0}{m+2},$$

bu yerdan

$$|t-\xi|\eta^{\left(\frac{m}{2}+\beta_0\right)} \le \frac{1}{p}|t-\xi|^{2\frac{m+1+\beta_0}{m+2}} + \frac{1}{q}\eta^{\left(\frac{m}{2}+\beta_0\right)\frac{2(m+1+\beta_0)}{m+2\beta_0}}.$$

Shunday qilib, birinchi qo‘shiluvchi uchun ushbu tengsizlikka ega bo‘lamiz:

$$A_1 = C_9\eta^{\frac{m}{2}+\beta_0} \frac{|t-\xi|\eta^{\frac{m}{2}+\beta_0}\frac{d\eta}{ds}}{\left[(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+\frac{1}{2}+\frac{1}{2}}} \le$$

$$\le C_9\eta^{\frac{m}{2}+\beta_0}\left[\frac{\frac{1}{p}|t-\xi|^{2\frac{m+1+\beta_0}{m+2}} + \frac{1}{q}\eta^{\left(\frac{m}{2}+\beta_0\right)}|t-\xi|^{\frac{2(m+1+\beta_0)}{m+2\beta_0}}}{\left[|t-\xi|^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+\frac{1}{2}+\frac{1}{2}}}\right]\frac{d\eta}{ds}.$$

Faraz qilaylik, $\beta_0 \le 0$ bo‘lsin. U holda:

$$\eta^{\left(\frac{m}{2}+\beta_0\right)\frac{2(m+1+2\beta_0)}{m+2\beta_0}} = \eta^{(m+2\beta_0)m+1+\beta_0\frac{1}{(m+2\beta_0)}} = \eta^{(m+2)\frac{m+1+\beta_0}{m+2}}$$

(4.40) tengsizlikning birinchi qo‘shiluvchisi uchun ushbu tengsizlikni hosil qilamiz

$$A_1 = C'_{10}\eta^{\frac{m}{2}} \frac{1}{\left[\left|t-\xi\right|^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\frac{1}{2}}}$$

tengsizlikka ega bo‘lamiz.

Shunday qilib,

$$\left|A(s;t,0)\right| \le C_9 \frac{\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right|}{\left[(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\frac{1}{2}}} + C_{10}$$

Bu tengsizlikka asosan, (4.39) ni ushbu ko‘rinishda yozib olamiz :

$$\left|\frac{\partial H_1(t,x)}{\partial x}\right| \le 2C_6 \int\limits_{l-\varepsilon}^{l} \left\{ \frac{\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right|}{\left[(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\frac{1}{2}}} + C_{10} \right\} \times$$

$$\times \frac{ds}{\left[(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta}}.$$

Bu yerda $\frac{4\eta^{m+2}}{(m+2)^2} = \tilde{\eta}^2$ belgilash kiritib, ushbu tengsizlikni hosil qilamiz:

$$\left|\frac{\partial H_1(x,t)}{\partial x}\right| \le$$

$$\le C_{11} \int\limits_{l-\varepsilon}^{l} \frac{\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right| ds}{\left[(t-\xi)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\frac{1}{2}} \left((1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right)^{\beta}} + C_{12}\ ,$$

Buyerdaushbualmashtirishnibajaramiz: $\frac{2}{m+2}\eta^{(m+2)/2}=\tilde{\eta}$, $\eta^{\frac{m}{2}}d\eta=d\overline{\eta}$ va ξ ni -1 bilanalmashtirib, ushbutengsizlikkakelamiz:

$$\left|\frac{\partial H_1(x,t)}{\partial x}\right|\le C_{13}\int_0^{\delta}\frac{d\hat{\eta}}{\left[(t+1)^2+\tilde{\eta}^2\right]^{\frac{1}{2}}\left((1+x)^2+\tilde{\eta}^2\right)^{\beta}}+C_{12}\ ,(4.42)$$

ushbu belgilashlarni kiritamiz: $\hat{x}=\max(1+x\ ,1+t)$.

Ushbu integralda

$$B(\hat{x})=\int_0^{\delta}\frac{d\hat{\eta}}{\left[\hat{x}^2+\tilde{\eta}^2\right]^{\frac{1}{2}}\left(\hat{x}^2+\tilde{\eta}^2\right)^{\beta}}\tag{4.43}$$

$\hat{\eta}=\hat{x}\,sh\,z$ almashtirish bajaramiz. $ch^2z-sh^2z=1$ ekanligini hisobga olib, (4.43) ni quyidagicha almashtiramiz:

$$B(\hat{x})=\int_0^{arc\,sh\frac{\delta}{x}}\frac{\hat{x}\,chz\,dz}{\hat{x}\,chz\,\hat{x}^{2\beta}\,ch^{2\beta}z}\le\frac{1}{\hat{x}^{2\beta}}\int_0^{arc\,sh\frac{\delta}{x}}\frac{dz}{ch^{2\beta}z}\le\frac{C'_{14}}{\hat{x}^{14}}$$

Shunday qilib,

$$\left|\frac{\partial H_1(x,t)}{\partial x}\right|\le\frac{C_{14}}{\hat{x}^{2\beta}}\le\frac{C_{14}}{\left(\frac{1+x+1+t}{2}\right)^{2\beta}}\le\frac{C_{14}}{\left(1+\frac{x+t}{2}\right)^{2\beta}}\le\tag{4.44}$$

$$\le\frac{C_{14}}{\left(1+\frac{x+t}{2}-\frac{(1+x)(1+t)}{2}\right)^{2\beta}}=$$

$$=\frac{C_{14}}{\left(\frac{2+x+t-1-t-x-xt}{2}\right)^{2\beta}}=\frac{C_{14}}{\left(\frac{1-xt}{2}\right)^{2\beta}}=\frac{2^{2\beta}C_{14}}{(1-xt)^{2\beta}}.$$

Bu baho $\frac{1}{2}\le x<1$ bo‘lgan holda ham o‘rinlidir.

4.1-lemma isbot bo‘ldi.

3-§. Singulyar integral tenglamani keltirib chiqarish.

Shunday qilib, singulyar koeffitsientli Gellerstedt tenglamasi uchun Trikomi masalasi

$$\nu(x) = \gamma D_{x,1}^{1-2\beta}\tau(x) + \psi_1(x), \tag{4.45}$$

$$\begin{aligned}\tau(x) = &-k_1\int_{-1}^{1}\left[|x-t|^{-2\beta} - (1-xt)^{-2\beta}\right]\nu(t)dt - \\ &-\int_{-1}^{1}H_1(t,x)\nu(t)dt - \int_0^l \varphi(s)\rho_1(s;x,0)ds,\end{aligned} \tag{4.46}$$

ko‘rinishdagi $\tau(x)$ va $\nu(x)$ noma’lum funksiyalarga nisbatan ekvivalent integral tenglamalar sistemasiga olib kelinadi. (4.46) tenglikda quyidagicha shakl almashtirish bajaramiz:

$$\begin{aligned}\tau(x) = &-k_1\int_{-1}^{x}\frac{\nu(t)dt}{(x-t)^{2\beta}} - k_1\int_x^1\frac{\nu(t)dt}{(t-x)^{2\beta}} + k_1\int_{-1}^{1}\frac{\nu(t)dt}{(1-xt)^{2\beta}} - \\ &-\int_{-1}^{1}H_1(t,x)\nu(t)dt - \int_0^l\varphi(s)\rho_1(s;x,o)ds = \\ &= -k_1\Gamma(1-2\beta)D_{-1,x}^{2\beta-1}\nu(x) - k_1\Gamma(1-2\beta)D_{x,1}^{2\beta-1}\nu(x) + \\ &+ k_1\int_{-1}^{1}\frac{\nu(t)dt}{(1-xt)^{2\beta}} - \int_{-1}^{1}H_1(t,x)\nu(t)dt - \int_0^l\varphi(s)\rho_1(s;x,0)ds.\end{aligned} \tag{4.47}$$

Endi (4.47) tenglikdan $\tau(x)$ ning ifodasini (4.45) tenglikka qo‘yib, ushbu

$$\begin{aligned}\nu(x) = \gamma D_{-1,x}^{1-2\beta}\Bigg\{&-k_1\Gamma(1-2\beta)D_{x,1}^{2\beta-1}\nu(x) + k_1\int_{-1}^{1}\frac{\nu(t)dt}{(1-xt)^{2\beta}} - \\ &-\int_{-1}^{1}H_1(t,x)\nu(t)dt - \int_0^1\varphi(s)\rho_1(s;x,y)ds\Bigg\} + \psi_1(x),\end{aligned} \tag{4.48}$$

yoki

$$\nu(x) = -\kappa_1\gamma\, \Gamma(1-2\beta) D_{-1,x}^{1-2\beta} D_{-1,x}^{2\beta-1}\nu(x) -$$

$$-k_1\gamma\, \Gamma(1-2\beta) D_{-1,x}^{1-2\beta} D_{x,1}^{2\beta-1}\nu(x) + \kappa_1\gamma\, D_{-1,x}^{1-2\beta}\left(\int_{-1}^{1}\frac{\nu(t)dt}{(1-xt)^{2\beta}}\right) - \qquad (4.49)$$

$$-\gamma\, D_{-1,x}^{1-2\beta}\left(\int_{-1}^{1} H_1(t,x)\nu(t)dt\right) - \gamma\, D_{-1,x}^{1-2\beta}\left(\int_{0}^{l}\varphi(s)\rho_1(s;x,y)ds\right) + \psi_1(x)$$

tenglikka kelamiz.

4.2-lemma. *Agar* $f(x) \in L(a,b)$ *bo'lsa,* ixtiyoriy $\alpha > 0$ uchun

$$D_{a,x}^{\alpha}\, D_{a,x}^{-\alpha} f(x) = f(x) \qquad (4.50)$$

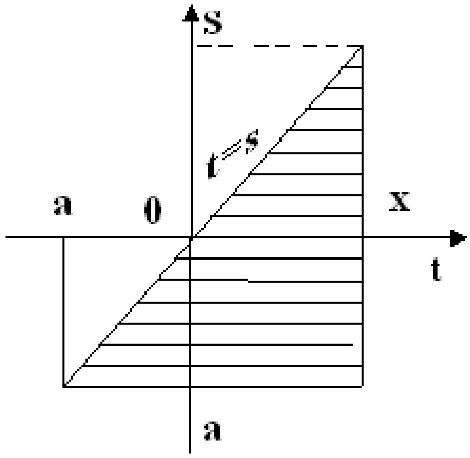
tenglik o'rinlidir.

Isboti. 4.2-lemma isbotini $0<\alpha<1$ holi uchun bajaramiz. Ta'rifga ko'ra,

$$J_1 = D_{a,x}^{\alpha}\, D_{a,x}^{-\alpha} f(x) = \frac{d}{dx} D_{a,x}^{\alpha-1}\, D_{a,x}^{-\alpha} f(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{D_{a,t}^{-\alpha} f(t)dt}{(x-t)^{\alpha}} =$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{dt}{(x-t)^{\alpha}}\int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}}.$$

(4.50) integralda integrallash tartibini o'zgartiramiz: $a \le t \le x$, $a \le s \le t$; $a \le s \le x$, $s \le t \le x$.



$$J_1(x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x f(s)ds \int_s^x \frac{dt}{(x-t)^{\alpha}(t-s)^{1-\alpha}}. \qquad (4.51)$$

(4.51) tenglikdagi ichki integralda $t = s + (x-s)\sigma$ almashtirish bajaramiz:

$$t = s, \sigma = 0; \quad t = x, \quad \sigma = 1,$$
$$x - t = (x - s)(1 - \sigma), \; t - s = (x - s)\sigma,$$
$$J_1(x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx}\int_a^x f(s)ds \int_0^1 \frac{(x-s)d\sigma}{(x-s)(1-\sigma)^{\alpha}\sigma^{1-\alpha}} =$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx}\int_a^x f(s)ds \int_0^1 \sigma^{\alpha-1}(1-\sigma)^{-\alpha} dt.$$

Endi beta funksiyaning aniqlanishiga ko‘ra,

$$\int_0^1 \sigma^{\alpha-1}(1-\sigma)^{\beta-1} dt = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{4.52}$$

bu yerda

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

$J_1(x)$ ni ushbu ko‘rinishda yozib olamiz:

$$J_1(x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx}\int_a^x f(s)ds \cdot B(\alpha, \; 1-\alpha) =$$
$$= \frac{1}{\Gamma(\alpha)\cdot\Gamma(1-\alpha)} \cdot \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} \frac{d}{dx}\int_a^x f(s)ds = f(x),$$

bu yerda $\Gamma(1) = 1$

4.2-lemma isbot bo‘ldi.

4.3-lemma. *Agar* $\varphi(x) \in C^{0,\lambda}(a,b)$ *bo‘lsa, u holda ushbu*

$$D_{a,x}^{\alpha} D_{xb}^{-\alpha} \varphi(x) = \cos\alpha\pi \; \varphi(x) + \frac{\sin\alpha\pi}{\pi}\int_a^b \left(\frac{t-a}{x-a}\right)^{\alpha} \cdot \frac{\varphi(t)dt}{t-x} \tag{4.53}$$

tenglik o‘rinlidir.

Isboti. Integro-differensial operatorning ta’rifiga ko‘ra,

$$J_2(x) = D_{a,x}^{\alpha} D_{a,b}^{-\alpha}\varphi(x) = \frac{d}{dx} D_{a,x}^{\alpha-1} D_{a,a}^{-\alpha}\varphi(x) =$$

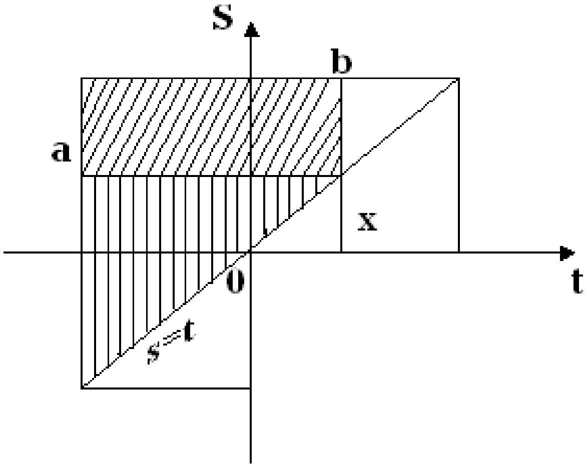
$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \frac{d}{dx} \int_a^x \frac{dt}{(x-t)^{\alpha}} \int_t^b \frac{\varphi(s)ds}{(s-t)^{1-\alpha}},$$

bu yerda

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}$$

tenglikdan foydalanamiz va integrallash tartibini o‘zgartiramiz:

$$D:\begin{cases} a \le t \le x \\ t \le s \le b \end{cases} \quad D_1:\begin{cases} a \le s \le x \\ a \le t \le s \end{cases} \quad D_2:\begin{cases} x \le s \le b \\ a \le t \le x. \end{cases}$$



Natijada ushbu tenglikka ega bo‘lamiz:

$$J_2(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\times$$

$$\times\left[\int_a^x \varphi(s)ds\int_a^s (x-t)^{-\alpha}(s-t)^{\alpha-1}dt + \cdot\int_x^b \varphi(s)ds\int_a^x (x-t)^{-\alpha}(s-t)^{\alpha-1}dt\right].$$

(4.53) tenglikning ichki integrallarida ushbu almashtirishni bajaramiz:

$$z = \frac{s-t}{x-t}$$

ya’ni

$$t=\frac{s-xz}{1-z},\; x-t=\frac{x-s}{1-z},\; s-t=\frac{z(x-s)}{1-z}. \qquad (4.54)$$

(4.54) ni hisobga olib, (4.53) tenglikni ushbu ko‘rinishda yozib olamiz:

$$I_2(x)=\frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\times$$

$$\times\left[\int_a^x \varphi(s)ds\int_a^{\frac{(s-a)}{(x-a)}}\frac{z^{\alpha-1}dz}{1-z}-\int_x^b\varphi(s)ds\int_{\frac{(s-a)}{(x-a)}}^{\infty}\frac{z^{\alpha-1}dz}{1-z}\right] \qquad (4.55)$$

Ushbu integralni kiritamiz:

$$f_\varepsilon(x)=\int_a^{x-\varepsilon}\varphi(s)ds\int_0^{\frac{(s-a)}{(x-a)}}\frac{z^{\alpha-1}dz}{1-z}-\int_{x+\varepsilon}^{b}\varphi(s)ds\int_{\frac{(s-a)}{(x-a)}}^{\infty}\frac{z^{\alpha-1}dz}{1-z}. \qquad (4.56)$$

Bu integralda x bo‘yicha differensiallash operatsiyasini bajarib,

$$\frac{df_\varepsilon(x)}{d(x)}\left[\varphi(x-\varepsilon)\int_a^{\frac{(x-\varepsilon-a)}{(x-a)}}\frac{z^{\alpha-1}dz}{1-z}+\varphi(x+\varepsilon)\int_{\frac{(x+\varepsilon-a)}{(x-a)}}^{\infty}\frac{z^{\alpha-1}dz}{1-z}+\right.$$

$$\left.+\int_a^{x-a}\left(\frac{s-a}{x-a}\right)^{\alpha}\frac{\varphi(s)ds}{s-x}+\int_{x+\varepsilon}^{b}\left(\frac{s-a}{x-a}\right)^{\alpha}\frac{\varphi(s)ds}{s-x}\right] \qquad (4.57)$$

tenglikka kelamiz. (4.57) tenglikda $\varepsilon\to 0$da limitga o‘tib, ushbu

$$J_2(x)=\frac{\sin\alpha\pi}{\pi}\lim_{\varepsilon\to 0}\frac{df_\varepsilon(x)}{d(x)}=\frac{\sin\alpha\pi}{\pi}\left[\varphi(x)\int_0^{\infty}\frac{z^{\alpha-1}dz}{1-z}+\int_a^b\left(\frac{s-a}{x-a}\right)^{\alpha}\frac{\varphi(s)ds}{s-x}\right]$$

tenglikni hosil qilamiz.

Ushbu integralni hisoblaymiz:

$$\int_0^{\infty}\frac{z^{\alpha-1}dz}{1-z}=\lim_{\varepsilon\to 0}\left[\int_0^1\frac{z^{\alpha-1}dz}{(1-z)^{1-\varepsilon}}-\int_1^{\infty}\frac{z^{\alpha-1}dz}{(1-z)^{1-\varepsilon}}\right]=$$
$$=\lim_{\varepsilon\to 0}\left[B(\alpha,\varepsilon)-B(1-\alpha-\varepsilon,\varepsilon\right]=$$
$$=\lim_{\varepsilon\to 0}\Gamma(\varepsilon)\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}-\frac{\Gamma(1-\alpha-\varepsilon)}{\Gamma(1-\alpha)}\right]=\pi\, ctg\alpha\pi\,.$$

Shunday qilib,

$$J_2(x)=\frac{\sin\alpha\pi\alpha}{\pi}\left[\pi\, ctg\alpha\,\pi\,\varphi(x)+\int_a^b\left(\frac{s-a}{x-a}\right)^{\alpha}\frac{\varphi(s)ds}{s-x}\right]=$$
$$=\cos\alpha\pi\,\varphi(x)+\frac{\sin\alpha\pi}{\pi}\int_a^b\left(\frac{s-a}{x-a}\right)^{\alpha}\frac{\varphi(s)ds}{s-x}\,.$$

4.3-lemma isbot bo‘ldi.

4.4-lemma. *Agar* $v(x)\in L[-1,1]$ *bo‘lsa,* $0<\alpha<1$ *uchun ushbu*

$$D_{-1,x}^{1-\alpha}\int_{-1}^{1}\frac{v(t)dt}{(1-xt)^{\alpha}}=\frac{1}{\Gamma(\alpha)}\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-\alpha}\frac{v(t)dt}{1-xt}\quad(4.58)$$

tenglik o‘rinlidir.

Isboti. Integro-differensial operatorlar ta’rifiga ko‘ra,

$$J_3(x)=D_{-1,x}^{1-\alpha}\int_{-1}^{1}\frac{v(t)dt}{(1-xt)^{\alpha}}=\frac{d}{dx}D_{-1,x}^{-\alpha}\int_{-1}^{1}\frac{v(t)dt}{(1-xt)^{\alpha}}=$$
$$=\frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_{-1}^{x}\frac{dt}{(x-t)^{1-\alpha}}\int_{-1}^{1}\frac{v(s)ds}{(1-st)^{\alpha}}.\quad(4.58)$$

Bu tenglikda integrallash tartibini o‘zgartiramiz:

$$J_3(x)=\frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_{-1}^{1}v(s)ds\int_{-1}^{x}\frac{dt}{(x-t)^{1-\alpha}(1-st)^{\alpha}}.\quad(4.59)$$

Endi (4.59) tenglikning ichki integralida $t=-1+(1+x)\sigma$ almashtirish bajaramiz:

$$x-t=(x+1)-(1+x)\sigma=(1+x)(1-\sigma)\,,$$
$$1-st=1+s-s(1+x)\sigma=(1+s)\left(1-\frac{s(1+x)}{1+s}\right)\sigma.$$

Shunday qilib,

$$J_3(x)=\frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_{-1}^{1}v(s)ds\int_{a}^{1}\frac{(1+x)d\sigma}{(1+x)^{1-\alpha}(1-\sigma)^{1-\alpha}}\cdot\frac{1}{(1+s)^{\alpha}\left(1-\frac{s(1+x)}{1+s}\sigma\right)^{\alpha}}=$$
$$=\frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_{-1}^{1}v(s)\left(\frac{1+x}{1+s}\right)^{\alpha}\int_{-1}^{1}(1-\sigma)^{\alpha-1}\left(1-\frac{s(1+x)}{1+s}\sigma\right)^{-\alpha}d\sigma.$$

Bu yerda gipergeometrik funksiyaning integral ifodasidan foydalanamiz:

$$\int_{0}^{1}t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}=\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F(a,b,c;z). \quad (4.60)$$

(4.60) tenglikka ko‘ra,

$$a-1=0\,,\ a=1\,,\ c-a-1=\alpha-1\,,\ c=1+\alpha\,,\ b=\alpha\,.$$

Shunday qilib,

$$J_3(x)=\frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_{-1}^{1}v(s)\left(\frac{1+t}{1+x}\right)^{\alpha}\frac{\Gamma(1)\Gamma(\alpha)}{\Gamma(1+\alpha)}F\left(1,\alpha,1+\alpha;\frac{s(1+x)}{1+s}\right)ds=$$
$$=\frac{1}{\alpha\Gamma(\alpha)}\int_{-1}^{1}v(s)\frac{d}{dx}\left[\left(\frac{1+x}{1+s}\right)^{\alpha}F\left(\alpha,1,1+\alpha\,;\,\frac{s(1+x)}{1+s}\right)\right]ds. \quad (4.61)$$

Endi

$$\frac{d}{dx}z^{a}F(a,b,c;z)=a\,z^{a-1}F(a+1,b,c;z)$$

formulani e’tiborga olib, (4.61) tenglikda differensiallash operatsiyasini bajaramiz:

$$J_3(x)=\frac{1}{\Gamma(\alpha)}\int_{-1}^{1}v(s)\left(\frac{1+x}{1+s}\right)^{\alpha-1}\frac{1}{1+s}F\left(\alpha+1,1,1+\alpha;\frac{s(1+x)}{1+s}\right)ds. \quad (4.62)$$

Bu yerda

$$F(a,b,в;z)=(1-x)^{-a}$$

tenglikdan foydalanib, (4.62) tenglikni quyidagi ko'rinishda yozib olamiz:

$$J_3(x)=\frac{1}{\Gamma(\alpha)}\int_{-1}^{1}v(s)\left(\frac{1+x}{1+s}\right)^{\alpha-1}\frac{1}{1+s}\left(1-\frac{s+sx)}{1+s}\right)^{-1}ds=$$

$$=\frac{1}{\Gamma(\alpha)}\int_{-1}^{1}v(s)\left(\frac{1+x}{1+s}\right)^{1-\alpha}\frac{v(t)dt}{1-xt}$$

4.4-lemma isbot bo'ldi.

Endi (4.50), (4.53) va (4.58) formulalarga ko'ra, (4.49) munosabatni ushbu ko'rinishda yozib olamiz:

$$v(x)=-k_1\gamma\Gamma(1-2\beta)\Big[v(x)+\cos(1-2\beta)\pi\, v(x)+$$

$$+\frac{\sin(1-2\beta)\pi}{\pi}\int_{-1}^{1}\left(\frac{1+x}{1+s}\right)^{1-2\beta}\frac{v(t)dt}{t-x}-\frac{1}{\Gamma(1-2\beta)\Gamma(2\beta)}\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\frac{v(t)dt}{1-xt}\Big]-$$

$$-\gamma\frac{d}{dx}D_{-1,x}^{-2\beta}\left(\int_{-1}^{1}H_1(t,x)v(t)dt\right)-\gamma\frac{d}{dx}D_{-1,x}^{-2\beta}\left(\int_{0}^{l}\varphi(s)\rho_1(s;x,0)ds\right)+\psi_1(x)$$

yoki

$$\begin{aligned}v(x)=-k_1\gamma\Gamma(1-2\beta)\Big[&(1-\cos 2\beta)\pi\, v(x)+\frac{\sin 2\beta\pi}{\pi}\\ &\int_{-1}^{1}\left(\frac{1+x}{1+s}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)v(t)dt\Big]-\\ &-\gamma\frac{d}{dx}\frac{1}{\Gamma(2\beta)}\int_{-1}^{x}\frac{ds}{(x-s)^{1-2\beta}}\int_{-1}^{1}H_1(t,s)v(t)dt-\\ &-\gamma\frac{d}{dx}\frac{1}{\Gamma(2\beta)}\int_{-1}^{x}\frac{dt}{(x-t)^{1-2\beta}}\int_{0}^{l}\varphi(s)\rho_1(s;t,0)ds+\psi_1(x),\end{aligned} \quad (4.63)$$

bu yerda $1-\cos 2\beta\pi = 2\sin^2\beta\pi$

$$k_1\gamma\,\Gamma(1-2\beta)=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2\beta}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)}\frac{2\Gamma(2\beta)\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(1-2\beta)}\left(\frac{4}{m+2}\right)^{-2\beta}$$

$$\Gamma(1-2\beta)=\frac{\Gamma(\beta)\Gamma(1-\beta)}{2\pi}=\frac{\pi}{2\pi\sin\beta\pi}=\frac{1}{2\sin\beta\pi}.$$

Bu tengliklarga asosan, (4.63) tenglikni ushbu ko‘rinishda yozib olamiz:

$$v(x)=-\frac{1}{2\sin\beta\pi}\times$$

$$\times\left[2\sin^2\beta\pi\, v(x)+\frac{2\sin\beta\pi\cos\beta\pi}{\pi}\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\alpha}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)v(t)dt\right]+$$

$$+\frac{\gamma}{\Gamma(2\beta)}\frac{d}{dx}\int_{-1}^{1}v(t)dt\int_{-1}^{x}\frac{H_1(t,s)}{(x-s)^{1-2\beta}}ds+\frac{\gamma}{\Gamma(2\beta)}\frac{d}{dx}\times$$

$$\times\int_{-1}^{x}\frac{dt}{(x-s)^{1-2\beta}}\int_{0}^{l}\varphi(s)\rho(s,t,0)ds=\psi_1(x)$$

yoki

$$v(x)+\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\alpha}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)v(t)dt+$$

$$+\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\frac{d}{dx}\int_{-1}^{1}v(t)dt\times$$

$$\times\int_{-1}^{x}\frac{H_1(t,s)}{(x-s)^{1-2\beta}}ds+\frac{\gamma}{\Gamma(2\beta)(1+\sin\beta\pi)}\frac{d}{dx}\int_{-1}^{x}\frac{dt}{(x-s)^{1-2\beta}}\int_{0}^{l}\varphi(s)\rho_1(s,t,0)ds= \tag{4.64}$$

$$=\frac{\psi_1(x)}{(1+\sin\beta\pi)}.$$

Bu yerdan

$$v(x)+\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\alpha}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)v(t)dt+$$

$$+\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\frac{d}{dx}\int_{-1}^{1}v(t)dt\int_{-1}^{1}\frac{H_1(t,s)}{(x-s)^{1-2\beta}}ds=F(x), \tag{4.65}$$

bu yerda

$$F(x) = -\frac{\gamma}{\Gamma(2\beta)(1+\sin\beta\pi)}\frac{d}{dx}\int_{-1}^{x}\frac{dt}{(x-s)^{1-2\beta}}\int_{0}^{l}\varphi(s)\rho_1(s,t,0)ds +$$
$$+\frac{\psi_1(x)}{(1+\sin\beta\pi)}.$$

Ushbu integralni o‘rganamiz:

$$H_2(x,t) = \int_{-1}^{x}(x-s)^{2\beta-1}H_1(t,s)ds = -\frac{1}{2\beta}\int_{2\beta-1}^{x}H_1(t,s)ds(x-s)^{2\beta} =$$
$$= -\frac{1}{2\beta}H_1(t,s)(x-s)^{2\beta}\Big|_{-1}^{x} + \frac{1}{2\beta}\int_{-1}^{x}(x-s)^{2\beta}\frac{\partial H_1(t,s)}{\partial s}ds =$$
$$= \frac{1}{2\beta}H_1(t,-1)(1+x)^{2\beta} + \frac{1}{2\beta}\int_{-1}^{x}(x-s)^{2\beta}\frac{\partial H_1(t,s)}{\partial s}ds$$

$\frac{\partial H_1(t,s)}{\partial s}$ ning mavjudligi 4.3-lemmadan kelib chiqadi, $H_1(t_1,-1)=0$ ekanligidan,

$$H_2(x,t) = \frac{1}{2\beta}\int_{-1}^{x}(x-s)^{2\beta}\frac{\partial H_1(t,s)}{\partial s}ds.$$

Oxirgi tenglikni x bo‘yicha differensiallab, ushbu

$$\frac{\partial H_2(x,t)}{\partial x} = \int_{-1}^{x}(x-s)^{2\beta-1}\frac{\partial H_1(t,s)}{\partial s}ds$$

tenglikka kelamiz. Shunday qilib,

$$\frac{d}{dx}\int_{-1}^{1}v(t)dt\int_{-1}^{x}\frac{H_1(t,x)}{(x-s)^{1-2\beta}}ds = \Gamma(2\beta)$$
$$\frac{d}{dx}\int_{-1}^{1}H_2(x,t)v(t)dt = \int_{-1}^{1}v(t)dt\int_{-1}^{x}(x-s)^{2\beta-1}\frac{\partial H_1(t,s)}{\partial s}ds \tag{4.66}$$

(4.66) tenglikka asosan, (4.65) tenglamani ushbu ko‘rinishda yozib olamiz:

$$v(x)+\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)v(t)dt+$$
$$+\int_{-1}^{1}M(x,t)v(t)dt=F(x), \tag{4.67}$$

bu yerda

$$M(x,t)=\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\int_{-1}^{x}(x-s)^{2\beta-1}\frac{\partial H_1(t,s)}{\partial s}ds. \tag{4.68}$$

4-§. Singulyar integral tenglamaning ozod hadi $F(x)$ ni o‘rganish.

Trikomi masalasi integral tenglamasi (4.65) ning ozod hadi

$$F(x)=-\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\frac{d}{dx}\int_{-1}^{x}\frac{dt}{(x-t)^{1-2\beta}}\cdot\int_{0}^{l}\varphi(s)\rho_1(s;t;0)ds+$$
$$+\frac{\psi_1(x)}{1+\sin\beta\pi} \tag{4.69}$$

ni o‘rganamiz.

Dastlab ushbu

$$f_1(x)=\int_{0}^{l}\varphi(s)\rho_1(s;t;0)ds \tag{4.70}$$

integralni o‘rganamiz.

4.5-lemma. *Agar $\varphi(s)$ α ko‘rsatkich bilan Gyolder shartini qanoatlantirsa, shu bilan birga A nuqtaning (B nuqtaning) yetarli kichik atrofida*

$$|\varphi(s)|\le c\eta^{1+2\beta}\quad\left(|\varphi(s)|\le c\eta^{1-\beta_0}\right) \tag{4.71}$$

tengsizliklar o‘rinli bo‘lsa, u holda

$$|f'(x)|\le C_1\left((1+x)^{\frac{4\beta_0-m^2}{(m+2)^2}}+(1-x)^{-2\beta}\right) \tag{4.72}$$

tengsizlik o‘rinlidir.

Isboti. (4.32) tenglikni hisobga olib, (4.70) ifodani quyidagicha almashtiramiz:

$$f_1(x)=\int_0^l \varphi(s)\left[2A(s;x,0)+4\int_0^l R_1(t,s;2)A(t;x,0)dt\right]ds=$$
$$=2\int_0^l \varphi(s)A(s;x,0)ds+4\int_0^l A(t;x,0)dt\int_0^l R_1(t,s;2)\varphi(s)ds=$$
$$=2\int_0^l \varphi(s)A(s;x,0)ds+4\int_0^l A(s;x,0)ds\int_0^l R_1(s,t;2)\varphi(t)dt,$$

bu yerda $R_1(s,t;2)$, $K_1(s,t)$ yadroning rezolventasi.

Bu yerdan

$$f'_1(x)=2\int_0^l \varphi(s)A'_x(s;t;0)ds+$$
$$+4\int_0^l A'_x(s;x,0)ds\int_0^l R_1(s;t;2)\varphi(t)dt. \tag{4.73}$$

(4.40) formulaga asosan, $A(s,x;0)$ dan x bo‘yicha hosila olamiz:

$$A'_x(s,x;0)=2\beta k_1\eta^{\beta_0}(s)\frac{\eta^m\dfrac{d\eta}{ds}}{\left[(x-\xi)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}}-$$
$$-2\beta(\beta+1)k_1\eta^{\beta_0}\frac{(x-\xi)\eta^m\dfrac{d\eta}{ds}+\dfrac{2}{m+2}\eta^{m+1}\dfrac{d\xi}{ds}}{\left[(x-\xi)^2+\dfrac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+2}}2(x-\xi). \tag{4.75}$$

(4.74) tenglikdan (4.75) ga asosan, yetarli kichik $l-s$ uchun, ya’ni $A(-1,0)$ ning yetarli kichik atrofida

$$\left|f_1'(x)\right| < C_2 \int\limits_{l-\delta}^{l} \frac{\eta^{1+2\beta}\,\eta^{\frac{m}{2}+\beta_0}\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right|ds}{\left[(\xi - x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}} + 0(1),.$$

tengsizlikka kelamiz. $x = -1$ nuqtaatrofidayetarlikichik δ laruchun, ya'ni

$$\lim_{\xi \to -1} \frac{(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}}{(\xi - x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}} = 1,$$

limitdan foydalanib, ushbu

$$\frac{1}{(\xi - x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}} < \frac{C_3}{(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}},$$

tengsizlikka ega bo'lamiz. Bu tengsizlikka asosan, ushbu

$$\left|f_1'(x)\right| < C_4 \int\limits_{l-\delta}^{l} \frac{\eta^{1+2\beta}\,\eta^{\frac{m}{2}+\beta_0}\cdot\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right|d\eta}{\left[(1+x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta+1}} + 0(1) \qquad (4.76)$$

munosabatni hosil qilamiz.

(4.76) tenglikning o'ng tomonidagi integralda $\frac{2}{m+2}\eta^{\frac{m+2}{2}} = (1+x)\widetilde{\eta}$ almashtirish bajaramiz: $\eta^{\frac{m}{2}}d\eta = (1+x)d\widetilde{\eta}$.

Bunga asosan, (4.76) ni ushbu

$$\left|f_1'(x)\right| \le C_4 \times \int_{\frac{2}{m+2}\cdot\frac{(l-\delta)^{\frac{m+2}{2}}}{1+x}}^{\frac{2}{m+2}\cdot\frac{l^{\frac{m+2}{2}}}{1+x}} \frac{\left[(1+x)\tilde{\eta}\right]^{\frac{2}{m+2}\left(1+2\beta+\frac{m+2\beta_0}{2}\right)}(1+x)d\tilde{\eta}}{(1+x)^{2(\beta+1)}\left[1+\tilde{\eta}^2\right]^{\beta+1}} + 0(1),$$

(4.77)

ko‘rinishda yozib olamiz. Bu integrallar yetarli kichik $1+x$ lar uchun xosmas integrallardir va bu integralning yaqinlashuvchi bo‘lishi uchun

$$2\beta + 2 - \frac{2}{m+2}\left(1 + 2\beta + \frac{m+2\beta_0}{2}\right) = 2 - \frac{2(1+2\beta)}{m+2} > 1$$

tengsizlik o‘rinli bo‘lishi kerak, ya’ni

$$\beta_0 < \frac{m^2}{4} \quad (0 < m < 2)$$

tengsizlik bajarilganda, (4.77) xosmas integral yaqinlashuvchi bo‘ladi. Bundan, (4.77) tengsizlikka asosan,

$$\left|f_1'(x)\right| \le C_5(1+x)^{\frac{4\beta_0 - m^2}{(m+2)^2}}. \tag{4.79}$$

Yuqoridagi hisoblashlarni takrorlab, (4.71) ning ikkinchi shartiga asosan, $1-x$ miqdor kichik bo‘lganda, ushbu

$$\left|f_1'(x)\right| \le C_0(1-x)^{-2\beta} \tag{4.80}$$

tengsizlikni hosil qilish mumkin.

4.6-lemma. *Agar $\varphi(s)$ 4.5-lemmaning shartlarini qanoatlantirsa, $\psi'(x)$ esa $-1<x<1$ oraliqda δ ko‘rsatkich bilan Gyolder shartini qanoatlantiruvchi birinchi tartibli hosilaga ega bo‘lsa, u holda $F(x)$ funksiya $-1<x<1$ oraliqda $\beta+\varepsilon$, $\varepsilon>0$ ko‘rsatkich bilan Gyolder shartini qanoatlantiradi.*

Isboti. (4.69) va (4.70) formulalarga ko‘ra,

$$F(x) = -\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\frac{d}{dx}\int_{-1}^{x}\frac{f_1(t)dt}{(x-t)^{1-2\beta}} + \frac{\psi_1(x)}{1+\sin\beta\pi}. \tag{4.81}$$

(4.81) tenglikdagi qo‘shiluvchilarni o‘rganamiz.

a) Dastlab $\psi_1(x)$ ni o‘rganamiz. (4.25) formulaga ko‘ra:

$$\psi_1(x) = -\gamma\left(\frac{\Gamma(\beta)}{\Gamma(2\beta)}\right)(1+x)^{\beta}\, D_{-1,x}^{1-\beta}\,\psi\left(\frac{(x-1)}{2}\right) =$$

$$= -\frac{\gamma(1+x)^{\beta}}{\Gamma(2\beta)}\frac{d}{dx}\int_{-1}^{x}\frac{\psi\left(\frac{(t-1)}{2}\right)dt}{(x-t)^{1-\beta}}. \tag{4.82}$$

Bu yerda bo‘laklab integrallash operatsiyasini bajaramiz:

$$\psi_1(x) = \frac{\gamma\,(1+x)^{\beta}}{\beta\,\Gamma(2\beta)}\frac{d}{dx}\int_{-1}^{x}\psi\left(\frac{(t-1)}{2}\right)d(x-t)^{\beta} =$$

$$= \frac{\gamma(1+x)^{\beta}}{\Gamma(2\beta)}\frac{d}{dx}\left[\psi\left(\frac{(t-1)}{2}\right)(x-t)^{\beta}\Bigg|_{-1}^{x} - \frac{1}{2}\int_{-1}^{x}\psi'\left(\frac{(t-1)}{2}\right)\cdot(x-t)^{\beta}\,dt\right] = \tag{4.83}$$

$$= -\frac{\gamma(1+x)^{\beta}}{2\,\Gamma(2\beta)}\int_{-1}^{x}\frac{\psi'\left(\frac{(t-1)}{2}\right)dt}{(x-t)^{1-\beta}}.$$

Bu yerda 1.3-teoremadan foydalanamiz.

Bu teoremaga asosan, (4.83) munosabatdan $\psi_1(x)$ funksiyaning $(-1,1)$ oraliqda $\beta+\delta$ ko‘rsatkich bilan Gyolder shartini qanoatlantirishi kelib chiqadi.

v) Endi (4.81) dagi birinchi qo‘shiluvchini o‘rganamiz. (4.72) tenglikka ko‘ra,

$$f_2(x) = \frac{d}{dx}\int_{-1}^{x}\frac{f_1(t)dt}{(x-t)^{1-2\beta}} = -\frac{1}{2\beta}\frac{d}{dx}\int_{-1}^{x}f_1(t)d(x-t)^{2\beta} =$$

$$= -\frac{1}{2\beta}\frac{d}{dx}\left[f_1(t)d(x-t)^{2\beta}\Big|_{-1}^{x} - \int_{-1}^{x}(x-t)f'_1(t)dt\right] = \tag{4.84}$$

$$= -\frac{1}{2\beta}\frac{d}{dx}\left[-f_1(-1)d(1+x)^{2\beta} - \int_{-1}^{x}(x-t)^{2\beta}f'_1(t)dt\right] = \int_{-1}^{x}\frac{f_1(t)dt}{(x-t)^{1-2\beta}}.$$

(4.84) formuladan ko‘rinib turibdiki, $f_2(x)$ funksiya 2β ko‘rsatkich bilan $(-1{,}1)$ kesmada Gyolder shartini qanoatlantiradi va (4.79), (4.80) baholarga ko‘ra, $f_2(x)$ funksiya $x=1$ nuqtada logarifmik maxsuslikka ega bo‘ladi.

Shunday qilib, $F(x)$ funksiya $-1<x<1$ intervalda $\gamma>\beta$ ko‘rsatkich bilan Gyolder shartini qanoatlantiradi, hamda $x=-1$ nuqtada

$$2\beta+\frac{4\beta-m^2}{m+2}>0\ \left(\beta_0>-\frac{m}{m+2}\right)$$

tartibda nolga aylanadi, $x=1$ nuqtada esa logarifimik maxsuslikka ega bo‘ladi.

5-§. Singulyar integral tenglamani regulyarizatsiyalash

(4.67) integral tenglamani ushbu ko‘rinishda yozib olamiz:

$$\nu(x)+\lambda\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\nu(t)dt=f(x), \qquad (4.86)$$

bu yerda

$$\lambda=\frac{\cos\beta\pi}{\pi(1+\sin\beta)},\quad f(x)=-\int_{-1}^{1}M(x,t)\nu(t)dt+F(x). \qquad (4.87)$$

(4.86)-tenglama Trikomi singulyar integral tenglamasi deyiladi. (4.86) integral tenglama N.N. Musxelishvili [28], F.D. Gaxovlar [7] tomonidan o‘rganilgan tenglamalar sinfiga kirmaydi, chunki (4.86) integral tenglama yadrosining «singulyar bo‘lmagan» qismi $\frac{1}{1-xt}$, $-1\le x,t\le 1$ kvadratda integrallanuvchi emas. Bu integral tenglama S.G. Mixlin tomonidan o‘rganilgan [26]. (4.86) tenglamani Karleman usulini qo‘llab yechish mumkin va yechim oshkor formula orqali ifodalanadi.

4.7-lemma. *Agar* $f(x)$ *funksiya* $-1<x<1,$ *oraliqda Gyolder shartini qanoatlantirsa va* $f(x)\in L_p(-1{,}1), p>1$ *bo‘lsa, u holda (4.86) tenglamaning* $(1+x)^{1-2\beta}\nu(x)$ *ifoda* $x=$-1 *nuqtada chegaralangan va* $x=1$ *nuqtada chegaralanmagan yechimi ushbu*

$$\nu(x)=\frac{1+\sin\beta\pi}{2}f(x)-$$

$$-\frac{\cos\beta\pi}{2\pi}\int_{-1}^{1}\left[\frac{(1-t)}{(1+x)}\frac{(1+t)}{(1-x)}\right]^{\frac{1}{2}-\beta}\cdot\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)f(t)dt \qquad (4.88)$$

formula bilan ifodalanadi va bu yechim Gyolder sinfiga tegishli bo'ladi.

Isboti. $\rho(x)=(1+x)^{1-2\beta}\nu(x)$ belgilashni kiritamiz va (4.86) tenglamani ushbu ko'rinishda yozib olamiz:

$$\rho(x)+\lambda\int_{-1}^{1}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\rho(t)dt=g(x), \quad (4.89)$$

bu yerda $g(x)=(1+x)^{1-2\beta}f(x)$.

z-kompleks tekislikning ixtiyoriy nuqtasi bo'lsin. (4.89) tenglamani yechish uchun Karleman g'oyasiga ko'ra,

$$\Phi(z)=\frac{1}{2\pi i}\int_{-1}^{1}\left(\frac{1}{t-z}-\frac{1}{1-zt}\right)\rho(t)dt, \qquad (4.90)$$

funksiyanikiritamiz; $\Phi(-1)=0$, $\Phi(\infty)=0$. $\Phi(z)$ funksiyayuqorivaquyiyarimtekisliklardagolomorfdir. $\Phi^{+}(x)$ va $\Phi^{-}(x)$ orqalimosravishda $\Phi(z)$ funksiyamizning $z-$argument x haqiqiyo'qqayuqorivaquyitekislikdanintilgandagilimitqiymatinitushunamiz.

Ushbu

$$\Phi\left(\frac{1}{z}\right)=z\Phi(z), \qquad (4.91)$$

munosabatni tekshirib ko'rish qiyin emas. $\frac{1}{z}$ almashtirish yuqori yarim tekislikni quyi yarim tekislikka va aksincha quyi yarim tekislikni yuqori yarim tekislikka akslantiradi, shu bilan birga $(-1,1)$ oraliq $\Delta=(-\infty,-1)\cup(1,+\infty)$ to'plamga akslanadi.

(4.91) tenglikdan $y\to+0$, $y\to-0$ da limitga o'tib,

$$\Phi^{+}\left(\frac{1}{x}\right)=x\Phi^{-}(x), \quad \Phi^{-}\left(\frac{1}{x}\right)=x\Phi^{+}(x), \qquad (4.92)$$

tengliklarni hosil qilamiz.

Soxotskiy-Plemel formulalariga ko‘ra,

$$\Phi^{+}(x)-\Phi^{-}(x)=\rho(x), \qquad (4.93)$$

$$\Phi^{+}(x)+\Phi^{-}(x)=\frac{1}{\pi i}\int_{-1}^{1}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\rho(t)dt, \qquad (4.94)$$

va bu munosabatga ko‘ra, (4.89) tenglama

$$\Phi^{+}(x)(1+\lambda\pi i)-\Phi^{-}(x)(1-\lambda\pi i)=g(x), \quad x\in(-1,1), \quad (4.95)$$

ko‘rinishni oladi.

Endi (4.65) tenglikda x ni $\frac{1}{x}$ bilan almashtirib va (4.92) tengliklarni hisobga olib, ushbu tenglikka kelamiz:

$$x\Phi^{-}(x)(1+\lambda\pi i)-x\Phi^{+}(x)(1-\lambda\pi i)=g\left(\frac{1}{x}\right), \quad x\in\Delta$$

yoki

$$\Phi^{+}(x)(1-\lambda\pi i)-\Phi^{-}(x)(1+\lambda\pi i)=-\frac{1}{x}g\left(\frac{1}{x}\right), \quad x\in\Delta, \qquad (4.96)$$

(4.95) va (4.96) tenglamalarni birlashtirib, ularni bitta tenglama ko‘rinishida yozib olamiz:

$$\Phi^{+}(x)-G(x)\Phi^{-}(x)=h(x), \quad x\in J\cup\Delta \qquad (4.97)$$

bu yerda

$$G(x)=\begin{cases}\dfrac{1-\lambda\pi i}{1+\lambda\pi i}, & x\in I,\\[2mm] \dfrac{1+\lambda\pi i}{1-\lambda\pi i}, & x\in\Delta,\end{cases} \qquad (4.98)$$

$$h(x)=\begin{cases} \dfrac{1}{1+\lambda\,\pi i}\; g(x),\; x\in I\;, \\ -\dfrac{1}{(1-\lambda\;\pi i)x}\, g\left(\dfrac{1}{x}\right),\;\; x\in\Delta, \end{cases} \qquad (4.99)$$

$1-\lambda\pi i=\sqrt{1+\lambda^2\pi^2}\; e^{-iarctg\lambda\pi}$, $1+\lambda\pi i=\sqrt{1+\lambda^2\pi^2}\; e^{iarctg\,\lambda\pi}$, $arctg\lambda\pi=\theta\,\pi$, belgilash kiritib,

$$1-\lambda\;\pi i=\sqrt{1+\lambda^2\pi^2}\; e^{-\theta\pi i}\; 1+\lambda\;\pi i=\sqrt{1+\lambda^2\pi^2}\; e^{\theta\pi i}$$

tengliklarga ega bo‘lamiz.

(4.98) va (4.99) belgilashlarga asosan (4.95) va (4.96) tengliklar ushbu ko‘rinishni oladi:

$$\Phi^+(x)-G(x)\Phi^-(x)=h(x)\;,\quad x\in I\cup\Delta. \qquad (4.100)$$

Shunday qilib, (4.89) integral tenglamani yechish masalasi kompleks o‘zgaruvchili funksiyalar nazariyasining ushbu chegaraviy masalasiga olib kelindi: yuqori va quyi yarim tekisliklarda golomorf, $\Phi(\infty)=0$ hamda (4.100) chegaraviy shartni qanoatlantiruvchi $\Phi(z)$ funksiya topilsin.

Bu masala yechimini oshkor ko‘rinishda hosil qilamiz. Dastlab qo‘yilgan masalaga mos bir jinsli masalani yechamiz,ya’ni yuqori va quyi yarim tekislikda golomorf hamda $I\bigcup\Delta$ oraliqda

$$X^+(x)=G(x)X^-(x) \qquad (4.101)$$

chegaraviy shartni qanoatlantiruvchi yechimni topamiz, ya’ni bizning asosiy maqsadimiz $G(x)$ funksiyani golomorf $X(z)$ funksiyaning $X^+(x)$ va $X^-(x)$ chegaraviy qiymatlarining $X^+(x)/X^-(x)$ nisbati orqali ifodalash, ya’ni $G(x)$ funksiyani faktorizatsiyalash, (4.101) tenglamaning xususiy yechimlaridan biri

$$X(z)=\exp\left\{\frac{1}{2\pi i}\int_{-1}^{1}\left(\frac{1}{t-z}-\frac{z}{1-zt}\right)\ln G(t)dt\right\}=\exp\left\{-\theta\int_{-1}^{1}\left(\frac{1}{t-z}-\frac{z}{1-zt}\right)dt\right\}, \qquad (4.102)$$

ko‘rinishda bo‘ladi, hamda bu funksiya uchun

$$X\left(\frac{1}{z}\right)=X(z), \qquad (4.103)$$

munosabat o‘rinli. (4.102) dan

$$X(z)=\exp\{-\theta\pi[\ln(t-z)+\ln(1-zt)]|_{-1}^{1}\}=$$
$$=\exp\{-\theta\pi[2\ln(1-z)-\ln(-1-z)-\ln(1+z)]\},$$

tenglikka ega bo‘lamiz. Oxirgi tenglikdan:

$$X^{+}(x)=\exp\{-\theta\pi[2\ln(1-x)+i\cdot 0-\ln|1+x|+i\pi-\ln|1+x|-i\cdot 0]\}=$$
$$=\exp\left\{-\theta\pi\left[\ln\frac{1-x}{1+x}+i\pi\right]\right\}=\left(\frac{1+x}{1-x}\right)^{\theta\pi}e^{-\theta\pi i}\ ,\quad x\in I, \tag{4.104}$$

$$X^{-}(x)=\exp\{-\theta\pi[2\ln(1-x)+i\cdot 0-\ln|1+x|-i\pi-\ln|1+x|-i\cdot 0]\}=$$
$$=\exp\left\{-\theta\pi\left[\ln\frac{1-x}{1+x}-i\pi\right]\right\}=\left(\frac{1+x}{1-x}\right)^{\theta\pi}e^{\theta\pi i},\quad x\in I. \tag{4.105}$$

(4.104) va (4.105) formulalardan, (4.103) ga ko‘ra,

$$X^{+}(x)=\left|\frac{1+x}{1-x}\right|^{\theta\pi}e^{\theta\pi i}\ ,\ \ x\in\Delta, \tag{4.106}$$

$$X^{-}(x)=\left|\frac{1+x}{1-x}\right|^{\theta\pi}e^{-\theta\pi i}\ ,x\in\Delta. \tag{4.107}$$

Shunday qilib, (4.101) ga ko‘ra, (4.100) tenglama

$$\frac{\Phi^{+}(x)}{X^{+}(x)}-\frac{\Phi^{-}(x)}{X^{-}(x)}=\frac{h(x)}{X^{+}(x)}\ \ ,\ x\in J\cup\Delta \tag{4.108}$$

ko‘rinishni oladi.

Bu tenglamaning xususiy yechimlaridan biri

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\int_{-1}^{1}\frac{h(t)dt}{X^+(t)(t-z)} + \frac{1}{2\pi i}\int_{\Delta}\frac{h(t)dt}{X^+(t)(t-z)} \qquad (4.109)$$

ko‘rinishga ega. (4.99) ga ko‘ra, (4.109) yechimni ushbu ko‘rinishda ifodalaymiz:

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\int_{-1}^{1}\frac{g(t)}{(1+\lambda\pi i)(t-z)}\frac{dt}{X^+(t)} - \frac{1}{2\pi i}\int_{\Delta}\frac{g\left(\frac{1}{t}\right)}{(1-\lambda\pi i)t}\cdot\frac{dt}{X^+(t)(t-z)}. \qquad (4.110)$$

Oxirgi integralda $t = \frac{1}{\xi}$ almashtirishni bajaramiz. Bu almashtirishda $\Delta = (-\infty,-1)\cup(1,\infty)$ oraliq $(-1,0)\cup(0,1)$ oraliqqa akslanadi. Natijada (4.110) ushbu ko‘rinishni oladi ($X^+\left(\frac{1}{t}\right) = X^-(t)$ tenglikni hisobga olganda $t \in I$):

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\int_{-1}^{1}\frac{g(t)dt}{(1+\lambda\pi i)X^+(t)(t-z)} - \frac{1}{2\pi i}\int_{-1}^{1}\frac{g(t)dt}{(1-\lambda\pi i)X^-(t)(1-zt)}. \qquad (4.111)$$

Endi (4.101) formulaga ko‘ra,

$$(1+\lambda\pi i)X^+(t) = (1-\lambda\pi i)X^-(t),$$

tengliklarni hisobga olib, (4.111) tenglikni ushbu

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\int_{-1}^{1}\frac{g(t)dt}{(1+\lambda\pi i)X^t(t)}\left(\frac{1}{t-z} - \frac{1}{1-zt}\right)dt, \qquad (4.112)$$

ko‘rinishda yozib olamiz.

Endi umumiy yechimni topish uchun ushbu

$$\frac{\Phi^+(x)}{X^+(x)} - \frac{\Phi^-(x)}{X^-(x)} = 0, \qquad (4.113)$$

bir jinsli tenglamaning umumiy yechimini topamiz. (4.113) tenglamadan ko‘rinib turibdiki, $\chi(z)=\dfrac{\Phi(z)}{X(z)}$ funksiya oddiy qutb bo‘lishi mumkin bo‘lgan $z=-1$ va $z=1$ maxsus nuqtalardan tashqari barcha nuqtalarda golomorf funksiya va Liuvillning umumlashgan teoremasiga ko‘ra, bu funksiya ushbu

$$\chi(z)=\frac{c_1}{(1+z)}+\frac{c_2}{(1-z)},$$

ko‘rinishda bo‘ladi.

Teorema (Liuvillning umumlashgan va analitik davom ettirish haqida). Agar $F_1(z)$ va $F_2(z)$ kompleks o‘zgaruvchili funksiyalar chekli sondagi $z_0=\infty$; $z_k(k=1,2,...,n)$ nuqtalardan tashqari mos ravishda yuqori va quyi yarim kompleks tekisliklarda analitik va ta’kidlangan nuqtalarda bosh qismi

$$G_0(z)=c_1^0 z+...+c_{m_0}^0 z^{m_0},$$

$$G_k\left(\frac{1}{z-z_k}\right)=\frac{c_1^k}{z-z_k}+...+\frac{c_{m_k}^k}{\left(z-z_k\right)^{m_k}}$$

ko‘rinishda bo‘lgan qutblarga ega bo‘lsa, va haqiqiy o‘qda ustma-ust tushsa $F_1^+(x)=F_2^-(x)$, u holda bu funksiyalar butun kompleks tekislikda yagona, yaxlit

$$F(z)=c+G_0(z)+\sum_{k=1}^{n} G_k\left(\frac{1}{z-z_k}\right),$$

ratsional funksiyani ifoda etadi, bu yerda $c-$ixtiyoriy o‘zgarmas son. z_k qutblar yarim tekisliklarda joylashishi bilan birgalikda ular haqiqiy o‘qda ham yotishi mumkin.

Shunday qilib, (4.108) tenglamaning yechimi ushbu

$$\frac{\Phi(z)}{X(z)}=\frac{1}{2\pi i}\int_{-1}^{1}\frac{g(t)}{(1+\lambda\pi i)X^+(t)}\left(\frac{1}{t-z}-\frac{1}{1-zt}\right)dt+\frac{c_1}{1+z}+\frac{c_2}{1-z},$$

yoki

$$\Phi(x)=\frac{X(z)}{2\pi i}\int_{-1}^{1}\frac{g(t)}{(1+\lambda\pi i)X^{t}(t)}\left(\frac{1}{t-z}-\frac{1}{1-zt}\right)dt+\left(\frac{c_1}{1+z}+\frac{c_2}{1-z}\right)X(z),\tag{4.114}$$

ko‘rinishda bo‘ladi.

Bu yerdan (4.93) formulaga ko‘ra,

$$\rho(x)=\Phi^{+}(x)-\Phi^{-}(x)=\frac{X^{+}(x)}{2}\cdot\frac{g(x)}{(1+\lambda\pi i)X^{+}(x)}+$$
$$+\frac{X^{+}(x)}{2\pi i}\int_{-1}^{1}\frac{g(t)}{(1+\lambda\pi i)X^{+}(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)dt+$$
$$+\left(\frac{c_1}{1+x}+\frac{c_2}{1-x}\right)-X^{+}(x)-\left[-\frac{X^{-}(x)}{2}\cdot\frac{g(x)}{(1+\lambda\pi i)X^{+}(x)}+\right.$$
$$\left.+\frac{X^{-}(x)}{2\pi i}\int_{-1}^{1}\frac{g(t)}{(1+\lambda\pi i)X^{+}(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)dt+\left(\frac{c_1}{(1+x)}+\frac{c_2}{1-x}\right)X^{-}(x)\right]$$

yoki

$$\rho(x)=\frac{g(x)}{2(1+\lambda\pi i)}\left(1+\frac{X^{-}(x)}{X^{+}(x)}\right)+$$
$$+\frac{X^{+}(x)-X^{-}(x)}{2\pi i(1+\lambda\pi i)}\int_{-1}^{1}\frac{g(t)}{X^{+}(t)}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)dt+\tag{4.115}$$
$$+\left(X^{+}(x)-X^{-}(x)\right)\left(\frac{c_1}{1+x}+\frac{c_2}{1-x}\right).$$

Ushbuifodalarnihisoblaymiz:

$$1+\frac{X^{-}(x)}{X^{+}(x)}=1+\frac{1+\lambda\pi i}{1-\lambda\pi i}=\frac{2}{1-\lambda\pi i},\tag{4.116}$$

$$X^{+}(x)-X^{-}(x)=\left(\frac{1+x}{1-x}\right)^{\theta\pi}\left(e^{-\theta\pi i}-e^{\theta\pi i}\right)=-i2\sin(\theta\pi)\left(\frac{1+x}{1-x}\right)^{\theta\pi},\tag{4.117}$$

$$\frac{X^+(x)-X^-(x)}{(1+\lambda\pi i)X^+(t)}=\frac{\left(\frac{1+x}{1-x}\right)^{\theta\pi}}{\sqrt{1+\lambda^2\pi^2}}\frac{\left(e^{-\theta\pi i}-e^{\theta\pi i}\right)}{e^{\theta\pi i}\cdot\left(\frac{1+t}{1-t}\right)^{\theta\pi}e^{-\theta\pi i}}=$$

$$=\frac{-i2\cdot\sin(\theta\pi)}{\sqrt{1-\lambda^2\pi^2}}\left(\frac{1-t}{1+t}\right)^{\theta\pi}\left(\frac{1+x}{1-x}\right)^{\theta\pi},\qquad(4.118)$$

$$\sin\theta\pi=\frac{tg\,\theta\pi}{\sqrt{1+tg^2\theta\pi}}=\frac{\lambda\pi}{\sqrt{1+\lambda^2\pi^2}}.\qquad(4.119)$$

(4.116), (4.117), (4.118) tengliklarnihisobgaolib, (4.115) yechimniushbu

$$\rho(x)=\frac{g(x)}{1+\lambda^2\pi^2}-\frac{\lambda}{1+\lambda^2\pi^2}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{\theta\pi}\left(\frac{1+x}{1-x}\right)^{\theta\pi}\cdot\left(\frac{1}{t-x}-\frac{1}{1-xt)}\right)g(t)dt-$$

$$-i2\sin(\theta\pi)\left(\frac{1+x}{1-x}\right)^{\theta\pi}\left(\frac{c_1}{1+x}+\frac{c_2}{1-x}\right)-i2\sin(\theta\pi)\left(\frac{1+x}{1-x}\right)^{\theta\pi}\left(\frac{c_1}{1+x}+\frac{c_2}{1-x}\right)\qquad(4.120)$$

yoki $\rho(x)=(1+x)^{1-2\beta}\nu(x)$ ekanligini hisobga olib, ushbu ko‘rinishda yozib olamiz:

$$\nu(x)=\frac{1}{1+\lambda^2\pi^2}\times$$

$$\times\left[f(x)-\lambda\int_{-1}^{1}\left[\frac{(1-t)}{(1+t)}\frac{(1+x)}{(1-x)}\right]^{2\theta}\cdot\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)f(t)dt\right]-\qquad(4.121)$$

$$-i\cdot2\cdot\sin\theta\pi\left[c_1(1+x)^{2\beta-2+\theta\pi}+c_2(1-x)^{-\theta\pi-1}\right].$$

(4.86) singulyar integral tenglama yechimini $x=-1$ da chegaralangan va $x=1$ da $1-2\beta$ dan kichik maxsuslikka ega bo‘lgan funksiyalar sinfidan izlaganimiz uchun (4.120) formulada $c_1=0, c_2=0$ deb olishimiz kerak. Shunday qilib, (4.86) tenglamaning $(-1,1)$ intervalda Gyolder sinfiga tegishli va bu interval chegaralari

$x=-1$ da chegaralangan, $x=1$ da $1-2\beta$ tartibdan kichik maxsuslikka ega bo'lgan funksiyalar sinfidagi yechimi

$$\nu(x)=\frac{1}{1+\lambda^2\pi^2}\left[f(x)-\lambda\int_{-1}^{1}\left[\frac{(1-t)}{(1+t)}\frac{(1+x)}{(1-x)}\right]^{2\theta}\times\right.$$
$$\left.\times\left(\frac{(1+t)}{(1+x)}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\right]f(t)dt \qquad (4.122)$$

ko'rinishda bo'ladi.

Ushbu ifodani hisoblaymiz:

$$\theta=\frac{1}{\pi}arctg\lambda\pi\text{, bu yerda }\lambda=\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)},$$

$$\theta=\frac{1}{\pi}arctg\frac{\cos\beta\pi}{(1+\sin\beta}=\frac{1}{\pi}arctg\frac{\cos^2\frac{\beta\pi}{2}-\sin^2\frac{\beta\pi}{2}}{\left(\cos\frac{\beta\pi}{2}+\sin\frac{\beta\pi}{2}\right)^2}=\frac{1}{\pi}arctg\left(\frac{1-tg\frac{\beta\pi}{2}}{1+tg\frac{\beta\pi}{2}}\right)=$$
$$=\frac{1}{\pi}arctgtg\left(\frac{\pi}{4}-\frac{\beta\pi}{2}\right)=\frac{1}{\pi}\left(\frac{\pi}{4}-\frac{\beta\pi}{2}\right)=\frac{1}{4}-\frac{\beta}{2}. \qquad (4.123)$$

(4.123) tenglikni e'tiborga olib, (4.112) yechimni ushbu ko'rinishda yozib olamiz:

$$\nu(x)=\frac{1}{1+\lambda^2\pi^2}\left[f(x)-\lambda\int_{-1}^{1}\left[\frac{(1-t)}{(1+t)}\frac{(1+x)}{(1-x)}\right]^{\frac{1}{2}-\beta}\times\right.$$
$$\left.\times\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\right]f(t)dt \qquad (4.124)$$

yoki (1.123) tenglikni hisobga olib, (124) ni ushbu

$$\nu(x)=\frac{1+\sin\beta\pi}{2}f(x)-$$

$$-\frac{\cos\beta\pi}{2\pi}\int_{-1}^{1}\left[\frac{(1-t)}{(1+x)}\frac{(1+t)}{(1-x)}\right]^{\frac{1}{2}-\beta}\cdot\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)f(t)dt \qquad (4.125)$$

ko‘rinishda yozib olamiz.

4.6-lemma isbot bo‘ldi.

6-§. Integral operator xossasi.

4.8-lemma. *Agar* $k(x)\in C[-1,1]$ *funksiya* μ $(0<\mu<1)$ *ko‘rsatkich bilan* $-1<x<1$ *oraliqda Gyolder shartini qanoatlantirsa, u holda*

$$M(x)=\int_{-1}^{1}\left(\frac{(1-t)(1+t)}{(1-x)(1+x)}\right)^{\frac{1}{2}-\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)k(t)dt \qquad (4.126)$$

integral $-1<x<1$ *oraliqda* μ *ko‘rsatkich bilan Gyolder shartini qanoatlantiradi va* $M(x)$ *integral* $x\to -1$ *da cheklidir.*

Isboti. (4.126) integralni ushbu ko‘rinishda yozib olamiz:

$$M(x)=\int_{-1}^{1}\left[\frac{(1+t)(1-t)}{(1+x)(1-x)}\right]^{\frac{1}{2}-\beta}\frac{(1+x)(1-t)}{1-xt}\cdot\frac{k(t)-k(x)}{t-x}dt+$$

$$+k(x)\int_{-1}^{1}\left[\frac{(1+t)(1-t)}{(1+x)(1-x)}\right]^{\frac{1}{2}-\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)dt=M_1(x)+k(x)M_2(x). \qquad (4.127)$$

Bu yerda

$$M_1(x)=\int_{-1}^{1}\left[\frac{(1+t)(1-t)}{(1+x)(1-x)}\right]^{\frac{1}{2}-\beta}\frac{(1+x)(1-t)}{1-xt}\cdot\frac{k(t)-k(x)}{t-x}dt\ ,(4.128)$$

$$M_2(x)=\int_{-1}^{1}\left[\frac{(1+t)(1-t)}{(1+x)(1-x)}\right]^{\frac{1}{2}-\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)dt\ . \qquad (4.129)$$

$M_2(x)$ integralni hisoblaymiz, buning uchun integral o‘zgaruvchisini quyidagicha almashtiramiz:

$$\xi=\frac{(1+x)(1-t)}{(1+t)(1-x)},\quad \mathrm{t}=\frac{1+x\text{-}(1\text{-}x)\xi}{1+x+(1-x)\xi},$$

$$dt=-\frac{2(1-x)(1+x)d\xi}{(1+x+(1+x)\xi)^2},\ 1+t=\frac{2(1+x)}{1+x+(1-x)\xi},$$

$$1-t=\frac{2(1-x)\xi}{1+x+(1-x)\xi},\quad t\text{-}x=\frac{(1+x)(1-x)(1-\xi)}{1+x+(1-x)\xi},$$

$$1-xt=\frac{(1+x)(1-x)(1+\xi)}{1+x+(1-x)\xi}, \qquad (4.130)$$

va bu almashtirishlarni hisobga olib, (4.129) integralni quyidagicha yozib olamiz:

$$M_2(x)=\int_0^{\infty}\frac{2^{4\alpha}\xi^{2\alpha}}{\left(1+x+\left(1-x\right)\xi\right)^{4\alpha}}\times$$

$$\times\left(\frac{1+x+\left(1-x\right)\xi}{1-\xi}-\frac{1+x+\left(1-x\right)\xi}{1+\xi}\right)\frac{2d\,\xi}{(1+x+(1-x)\xi)^2}=$$

$$=\int_0^{\infty}\frac{2^{2+4\alpha}\xi^{1+2\alpha}}{\left(1+x+\left(1-x\right)\xi\right)^{1+4\alpha}}\cdot\frac{d\,\xi}{1-\xi^2}=\int_0^{\infty}\frac{2^{1+4\alpha}\xi^{1+2\alpha}d\,\xi}{\left(1+x+\left(1-x\right)\xi\right)^{1+4\alpha}}\left(\frac{1}{1-\xi}+\frac{1}{1+\xi}\right)$$

$$=\int_0^{\infty}\frac{2^{1+4\alpha}\xi^{1+2\alpha}d\,\xi}{\left(1+x+\left(1-x\right)\xi\right)^{1+4\alpha}(1-\xi)}+\int_0^{\infty}\frac{2^{1+4\alpha}\xi^{1+2\alpha}d\xi}{\left(1+x+\left(1-x\right)\xi\right)^{1+4\alpha}(1+\xi)},\tag{4.131}$$

bu yerda $2\alpha=\frac{1}{2}-\beta$.Endi (4.131) dagi ushbu

$$T_1\left(\alpha,\beta;x\right)=\int_0^{\infty}\frac{t^{\alpha-1}dt}{\left[1+x+\left(1-x\right)t\right]^{\beta}\left(1-t\right)},\tag{4.132}$$

$$T_2\left(\alpha,\beta;x\right)=\int_0^{\infty}\frac{t^{\alpha-1}dt}{\left[1+x+\left(1-x\right)t\right]^{\beta}\left(1+t\right)}\tag{4.133}$$

ikki integralni hisoblaymiz. Bu yerda α va β o‘zgarmas sonlar ushbu shartlarni qanoatlantiradi: $0<\alpha<1+\beta,\ \ \alpha>\beta.$

1. Dastlab, $\mathrm{T}_1(\alpha,\beta;x)$ ni hisoblaymiz:

$$T_1(\alpha,\beta;x)=\lim_{\varepsilon\to0}\left[\int_0^{1}\frac{t^{\alpha-1}dt}{\left[1+x+(1-x)t\right]^{\beta}(1-t)^{1-\varepsilon}}-\int_1^{\infty}\frac{t^{\alpha-1}dt}{\left[1+x+(1-x)t\right]^{\beta}(t-1)^{1-\varepsilon}}\right]=$$

$$=\lim_{\varepsilon\to 0}\left[(1+x)^{-\beta}\int_0^1 t^{\alpha-1}(1-t)^{\varepsilon-1}\left(1-\frac{x-1}{1+x}t\right)^{-\beta}dt-(1-x)^{-\beta}\int_1^{\infty}t^{\alpha-1}(t-1)^{\varepsilon-1}\left(t-\frac{1+x}{x-1}\right)^{-\beta}dt\right].$$

Ikkinchi integralda $t=\frac{1}{\xi}$ almashtirish bajarib, $T_1(\alpha,\beta;x)$ ni ushbu ko‘rinishda yozib olamiz:

$$T_1(\alpha,\beta;x)=\lim_{\varepsilon\to 0}\left[(1+x)^{-\beta}\int_0^1 t^{\alpha-1}(1-t)^{\varepsilon-1}\left(1-\frac{x-1}{1+x}t\right)^{-\beta}dt-\right.$$
$$\left.-(1-x)^{-\beta}\int_0^1 \xi^{\beta-\alpha-\varepsilon}(1-\xi)^{\varepsilon-1}\left(1-\frac{1+x}{x-1}\xi\right)^{-\beta}d\xi\right]. \qquad (4.134)$$

(4.134) da gipergeometrik funksiyaning integral ifodasidan foydalanib, ushbu tenglikka ega bo‘lamiz:

$$T_1(\alpha,\beta;x)=\lim_{\varepsilon\to 0}\left[\frac{\Gamma(\alpha)\Gamma(\varepsilon)}{\Gamma(\alpha+\varepsilon)}(1+x)^{-\beta}F\left(\alpha,\beta,\alpha+\varepsilon;\frac{x-1}{x+1}\right)-\right.$$
$$\left.-\frac{\Gamma(\beta-\alpha-\varepsilon+1)\Gamma(\varepsilon)}{\Gamma(\beta-\alpha+1)}(1+x)^{-\beta}F\left(\beta-\alpha-\varepsilon+1,\ \beta;\beta-\alpha+1;\frac{1+x}{x-1}\right)\right]. \qquad (4.135)$$

(4.135) tenglikning o‘ng tomonining birinchi va ikkinchi qo‘shiluvchilaridagi gipergeometrik funksiyalarni mos ravishda

$$F(a,b,c,z)=(1-z)^{-в}F\left(c-a,b,c;\frac{z}{z-1}\right) \qquad (4.136)$$

va

$$F(a,b,c;1-z)=z^{-в}F\left(c-a,b,c;\frac{z-1}{z}\right) \qquad (4.137)$$

formulalar bilan almashtiramiz, ya’ni

$$F\left(\alpha,\beta,\alpha+\varepsilon;\frac{x-1}{x+1}\right)=\left(1-\frac{x-1}{x+1}\right)^{\beta}F\left(\varepsilon,\beta,\alpha+\varepsilon;\frac{1-x}{2}\right) \quad (4.138)$$

va

$$F\left(\beta-\alpha-\varepsilon+1,\beta,\beta-\alpha+1;1-\frac{2}{1-x}\right)=$$
$$=\left(\frac{2}{1-x}\right)^{-\beta}F\left(\varepsilon,\beta,\beta-\alpha+1;\frac{1+x}{2}\right) \quad (4.139)$$
,

tengliklarni hisobga olib, (4.135) tenglikni quyidagicha yozib olamiz:

$$T_1(\alpha,\beta;x)=\lim_{\varepsilon\to 0}\left[\frac{\Gamma(\alpha)\Gamma(\varepsilon)}{\Gamma(\alpha+\varepsilon)}(1+x)^{-\beta}\cdot\left(\frac{2}{1+x}\right)^{-\beta}\cdot F\left(\varepsilon,\beta,\alpha+\varepsilon;\frac{1-x}{2}\right)-\right.$$
$$\left.-\frac{\Gamma(\beta-\alpha-\varepsilon+1)\Gamma(\varepsilon)}{\Gamma(\beta-\alpha+1)}(1-x)^{-\beta}\cdot\left(\frac{2}{1-x}\right)^{-\beta}F\left(\varepsilon,\beta,\beta-\alpha+1;\frac{1+x}{2}\right)\right]. \quad (4.140)$$

Endi (4.140) ning birinchi qo‘shiluvchisidagi gipergeometrik funksiyaga Bols formulasini

$$F(a,b,c,z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z)+$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b,c-a-b+1;1-z)$$

qo‘llab, ushbu tenglikka ega bo‘lamiz:

$$T_1(\alpha,\beta;x)=\lim_{\varepsilon\to 0}\left\{\frac{\Gamma(\alpha)\Gamma(\varepsilon)}{\Gamma(\alpha+\varepsilon)}2^{-\beta}\left[\frac{\Gamma(\alpha+\varepsilon)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\alpha-\beta+\varepsilon)}\cdot F\left(\varepsilon,\beta,\beta-\alpha+1;\frac{1+x}{2}\right)+\right.\right.$$

$$\left.+\frac{\Gamma(\alpha+\varepsilon)\Gamma(\beta-\alpha)}{\Gamma(\varepsilon)\Gamma(\beta)}\cdot\left(\frac{1+x}{2}\right)^{\alpha-\beta}F\left(\alpha,\alpha+\varepsilon-\beta;\alpha-\beta+1;\frac{1+x}{2}\right)\right]-$$

$$\left.-\frac{\Gamma(\beta-\alpha-\varepsilon+1)\Gamma(\varepsilon)}{\Gamma(\beta-\alpha+1)}2^{-\beta}F\left(\varepsilon,\beta,\beta-\alpha+1;\frac{1+x}{2}\right)\right\}$$

yoki

$$T_1(\alpha,\beta;x)=$$

$$=\lim_{\varepsilon\to 0}\left\{2^{-\beta}\Gamma(\varepsilon)\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta+\varepsilon)}-\frac{\Gamma(\beta-\alpha+1-\varepsilon)}{\Gamma(\beta-\alpha+1)}\right]\cdot F\left(\varepsilon,\beta,\beta-\alpha+1;\frac{1+x}{2}\right)+\right.$$
$$\left.+2^{-\beta}\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\cdot\left(\frac{1+x}{2}\right)^{\alpha-\beta}F\left(\alpha,\alpha+\varepsilon-\beta;\alpha-\beta+1;\frac{1+x}{2}\right)\right\}. \quad (4.141)$$

Ushbu limitni hisoblaymiz:

$$\lim_{\varepsilon\to 0}\Gamma(\varepsilon)\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha+\varepsilon)}-\frac{\Gamma(1-\alpha-\varepsilon)}{\Gamma(1-\alpha)}\right]=$$

$$=\lim_{\varepsilon\to 0}\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left[\frac{\Gamma(\alpha)\Gamma(1-\alpha)-\Gamma(\alpha+\varepsilon)\Gamma(1-(\alpha+\varepsilon))}{\Gamma(\alpha+\varepsilon)\Gamma(1-\alpha)}\right]=$$

$$=\lim_{\varepsilon\to 0}\frac{\Gamma(1+\varepsilon)}{\varepsilon}\cdot\frac{\dfrac{\pi}{\sin\alpha\pi}-\dfrac{\pi}{\sin(\alpha+\varepsilon)\pi}}{\Gamma(\alpha+\varepsilon)\Gamma(1-\alpha)}= \quad (4.142)$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\cdot\frac{\dfrac{\pi(\sin(\alpha+\varepsilon)\pi-\sin\alpha\pi)}{\sin\alpha\pi\cdot\sin\alpha\pi}}{\dfrac{\pi}{\sin\alpha\pi}}=\sin\alpha\pi\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\cdot\frac{2\cdot\sin\dfrac{\varepsilon\pi}{2}\cdot\cos\dfrac{(2\alpha+\varepsilon)\pi}{2}}{\sin^2\alpha\pi}=\pi ctg\alpha\pi.$$

(4.141) formulada $\varepsilon\to 0$ da limitga o‘tib va (4.142) tenglikni hisobga olib, ushbu yakuniy tenglikka kelamiz:

$$T_1(\alpha,\beta;x)=2^{-\beta}\pi ctg(\alpha-\beta)\pi+2^{-\beta}\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\cdot\left(\frac{1+x}{2}\right)^{\alpha-\beta}\cdot$$
$$\cdot F\left(\alpha,\alpha-\beta,\alpha-\beta+1;\frac{1+x}{2}\right) \qquad (4.143)$$

Endi (4.133) ifodani, ya’ni $T_2(\alpha,\beta;x)$ ni hisoblaymiz:

$$T_2(\alpha,\beta;x)=\int_0^\infty\frac{t^{\alpha-1}dt}{[1+x+(1-x)t]^\beta(1+t)}. \qquad (4.144)$$

Bu integralda $z=\frac{1}{1+t}, t=\frac{1}{z}-1=\frac{1-z}{z}, dt=-\frac{1}{z^2}dz$,

$$1+x+(1-x)\frac{1-z}{z}=\frac{1-x}{z}\left(1-\frac{2x}{x-1}z\right)$$

almashtirish bajaramiz va natijada ushbu tenglikni hosil qilamiz:

$$T_2(\alpha,\beta;x)=(1-x)^{-\beta}\int_0^1 z^{\beta-\alpha}(1-z)^{\alpha-1}\left(1-\frac{2x}{x-1}z\right)^{-\beta}dz.$$

Bu yerda gipergeometrik funksiyaning integral ifodasidan foydalanib, ushbu tenglikni hosil qilamiz:

$$T_2(\alpha,\beta;x)=(1-x)^{-\beta}\frac{\Gamma(\beta-\alpha+1)\Gamma(\alpha)}{\Gamma(\beta+1)}F\left(\beta-\alpha+1,\beta,\beta+1;\frac{2x}{x-1}\right)$$

Oxirgi tenglikda (4.136) formulani qo‘llab, ushbu tenglikka kelamiz:

$$T_2(\alpha,\beta;x)=\frac{\Gamma(\alpha)\cdot\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}(1+x)^{-\beta}F\left(\alpha,\beta,\beta+1;\frac{2x}{x+1}\right). \qquad (4.145)$$

(4.143) va (4.145) formulalarda α ni $2+2\alpha$ ga, β ni $1+4\alpha$ ga almashtirib, ushbu tengliklarga ega bo‘lamiz:

$$\int_0^{\infty}\frac{t^{1+2\alpha}dt}{[1+x+(1-x)t]^{1+4\alpha}(1-t)}=$$

$$=2^{-1-4\alpha}\pi ctg(1-2\alpha)\pi+2^{-1-4\alpha}\frac{\Gamma(2+2\alpha)\Gamma(2\alpha-1)}{\Gamma(1+4\beta)}\left(\frac{1+x}{2}\right)^{-1-2\alpha}\times \quad (4.146)$$
$$\times F\left(2+2\alpha,1-2\alpha,2-2\alpha;\frac{1+x}{2}\right),$$

$$\int_0^{\infty}\frac{t^{1+2\alpha}dt}{[1+x+(1-x)t]^{1+4\alpha}(1+t)}=$$

$$=\frac{\Gamma(2+2\alpha)\Gamma(2\alpha)}{\Gamma(2+4\beta)}(1+x)^{-1-4\beta}\cdot F\left(2+2\alpha,1+4\alpha,2+4\alpha;\frac{2x}{1+x}\right) \quad (4.147)$$

(4.146) va (4.147) tengliklarga ko‘ra, (4.122) dan $M_2(x)$ uchun ushbu tenglikni hosil qilamiz:

$$M_2(x)=-\pi ctg2\alpha\pi+\frac{(1+2\alpha)\Gamma(2\alpha)\Gamma(2\alpha-1)}{2\Gamma(4\alpha)}\times$$
$$\times\left(\frac{1+x}{2}\right)^{1-2\alpha}F\left(2+2\alpha,1-2\alpha;2-2\alpha,\frac{1+x}{2}\right)+ \quad (4.148)$$
$$+\frac{(1+2\alpha)\Gamma^2(2\alpha)}{2(1+4\alpha)\Gamma(4\alpha)}(1+x)^{-1-4\alpha}F\left(2+2\alpha,1+4\alpha,2+4\alpha;\frac{2x}{1+x}\right).$$

Shunday qilib, (4.148) ni (4.117) ga qo‘yib, ushbu tenglikka ega bo‘lamiz:

$$M(x)=\int_{-1}^{1}\left[\frac{(1+t)(1-t)}{(1+x)(1-x)}\right]^{\frac{1}{2}-\beta}\frac{(1+x)(1-t)}{1-xt}\cdot\frac{k(t)-k(x)}{t-x}dt+$$
$$+k(x)\Bigg[-\pi tg\beta\pi+\frac{\left(\frac{3}{2}-\beta\right)\Gamma\left(\frac{1}{2}-\beta\right)\Gamma\left(-\frac{1}{2}-\beta\right)}{2\Gamma(1-2\beta)}\times$$

$$\times\left(\frac{1+x}{2}\right)^{\frac{1}{2}+\beta}F\left(\frac{5}{2}-\beta,\frac{1}{2}+\beta,\frac{3}{2}+\beta;\frac{1+x}{2}\right)$$

$$\left.+\frac{\left(\frac{3}{2}-\beta\right)\Gamma^2\left(\frac{1}{2}-\beta\right)}{2(2-2\beta)\Gamma(1-2\beta)}(1+x)^{-2+2\beta}F\left(\frac{5}{2}-\beta,2-2\beta;3-2\beta;\frac{2x}{1+x}\right)\right]. \quad (4.149)$$

(4.136) formulaga ko‘ra,

$$F\left(\frac{5}{2}-\beta,2-2\beta;3-2\beta;\frac{2x}{1+x}\right)=$$

$$=\left(1-\frac{2x}{1+x}\right)^{2\beta-2}F\left(\frac{1}{2}-\beta,2-2\beta;3-2\beta;\left(\frac{2x}{1+x}-1\right)\right)= \quad (4.150)$$

$$=\left(\frac{1-x}{1+x}\right)^{2\beta-2}F\left(\frac{1}{2}-\beta,2-2\beta;3-2\beta;\frac{2x}{x-1}\right).$$

(4.150) tenglikka asosan, (4.149) tenglikni ushbu ko‘rinishda yozib olamiz:

$$M(x)=\int_{-1}^{1}\left[\frac{(1+t)(1-t)}{(1+x)(1-x)}\right]^{\frac{1}{2}-\beta}\frac{(1+x)(1-t)}{1-xt}\cdot\frac{\kappa(t)-\kappa(x)}{t-x}dt+$$

$$+k(x)\left[-\pi tg\beta\pi+\frac{(3-2\beta)\Gamma\left(\frac{1}{2}-\beta\right)\Gamma\left(-\frac{1}{2}-\beta\right)}{4\Gamma(1-2\beta)}\times\right.$$

$$\times\left(\frac{1+x}{2}\right)^{\frac{1}{2}+\beta}F\left(\frac{5}{2}-\beta,\frac{1}{2}+\beta,\frac{3}{2}+\beta;\frac{1+x}{2}\right)+ \quad (4.151)$$

$$\left.+\frac{(3-2\beta)\Gamma^2\left(\frac{1}{2}-\beta\right)}{8(1-\beta)\Gamma(1-2\beta)}(1-x)^{2\beta-2}F\left(\frac{1}{2}-\beta,2-2\beta;3-2\beta;\frac{2x}{x-1}\right)\right].$$

(4.151) tenglikdan ko‘rinib turibdiki, $M(x)$ funksiyamiz $-1<x<1$ oraliqda μ ko‘rsatkich bilan Gyolder shartini qanoatlantiradi. Endi $M(x)$ funksiyani $x=-1$ da chekli ekanligini ko‘rsatamiz. Buning uchun (4.116) tenglikda integral o‘zgaruvchisi

t ning o‘rniga yangi $\xi=\dfrac{(1+x)(1-t)}{(1-x)(1+t)}$ o‘zgaruvchi kiritib va (4.121) munosabatlarni hisobga olib, ushbu munosabatga ega bo‘lamiz:

$$M(x)=\int_0^\infty k\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]\frac{2^{1-2\beta}\xi^{\frac{1}{2}-\beta}}{(1+x+(1-x)\xi)^{1-2\beta}}\times$$

$$\times\left(\frac{1+x+(1-x)\xi}{1-\xi}-\frac{1+x+(1-x)\xi}{1+\xi}\right)\frac{2d\xi}{(1+x+(1-x)\xi)^2}=$$

$$=\int_0^\infty \kappa\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]\frac{2^{2-2\beta}\xi^{\frac{3}{2}-\beta}}{(1+x+(1-x)\xi)^{2-2\beta}}\left(\frac{1}{1-\xi}+\frac{1}{1+\xi}\right)d\xi=$$

$$=\int_0^\infty k\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]\frac{2^{2-2\beta}\xi^{\frac{1}{2}-\beta}}{(1+x+(1-x)\xi)^{2-2\beta}}\left(\frac{2\xi}{1-\xi^2}\right)d\xi=$$

$$=2^{2-2\beta}\int_0^\infty k\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]\frac{\xi^{\frac{3}{2}-\beta}d\xi}{(1+x+(1-x)\xi)^{2-2\beta}(1-\xi)}+$$

$$+2^{2-2\beta}\int_0^\infty k\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]\frac{\xi^{\frac{3}{2}-\beta}d\xi}{[1+x+(1-x)\xi]^{2-2\beta}(1+\xi)}=$$

$$=2^{2-2\beta}\left(I_1(x)+I_2(x)\right). \tag{4.152}$$

Dastlab, ikkinchi $I_2(x)$ integralni o‘rganamiz. O‘rta qiymat haqidagi teoremani qo‘llab, ushbu tenglikka ega bo‘lamiz:

$$I_2(x)=K(\theta)\int_0^\infty\frac{\xi^{\frac{3}{2}-\beta}d\xi}{[1+x+(1-x)\xi]^{2-2\beta}(1+\xi)}, \tag{4.153}$$

bu yerda $0<\theta<1$. Endi (4.143), (4.145) formulalarni qo‘llab, ya’ni bu formulada $\alpha\Rightarrow\frac{5}{2}-\beta,\quad \beta\Rightarrow 2\text{-}2\beta$ almashtirishlar bajarib, ushbu tenglikni hosil qilamiz:

$$I_2(x)=k(\theta)\frac{\Gamma\left(\frac{5}{2}-\beta\right)\Gamma\left(\frac{1}{2}-\beta\right)}{\Gamma(3-2\beta)}(1+x)^{2\beta-2}\cdot F\left(\frac{5}{2}-\beta,2-2\beta,3-2\beta;\frac{2x}{1+x}\right)=$$

$$=k(\theta)\frac{(3-2\beta)\Gamma^2\left(\frac{1}{2}-\beta\right)}{8\cdot(1-\beta)\Gamma(1-2\beta)}\cdot F\left(\frac{5}{2}-\beta,2-2\beta,3-2\beta;\frac{2x}{1+x}\right).$$

Bu yerda

$$F(a,b,c;z)=(1-z)^{-b}F\left(a,b,c;\frac{z}{z-1}\right)$$

formulani qo‘llab, ushbu tenglikka kelamiz:

$$I_2(x)=K(\theta)\frac{(3-2\beta)\Gamma^2\left(\frac{1}{2}-\beta\right)}{8\cdot(1-\beta)\Gamma(1-2\beta)}(1-x)^{2\beta-2}\cdot F\left(\frac{1}{2}-\beta,2-2\beta,3-2\beta;\frac{2x}{x-1}\right).$$

Bu formuladan ko‘rinib turibdiki, $x\to-1$ da $I_2(x)$-cheklidir.

Endi $I_1(x)$ integralni o‘rganamiz. ξ ning ikkita ξ_1 va ξ_2 qiymatlarini $0<\xi_1<1,\quad \xi_2>1$ oraliqlarda fiksirlaymiz va $I_1(x)$ integralni $(0,\xi_1),\ (\xi_1,\xi_2)(\xi_2,\infty)$ oraliqlar bo‘yicha olingan uchta $I_{11}(x),I_{12}(x),I_{13}(x)$ integrallarga ajratamiz:

$$I_1(x)=\int_0^\infty k\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]\frac{\xi^{\frac{3}{2}-\beta}d\xi}{[1+x+(1-x)\xi]^{2-2\beta}(1-\xi)}=$$

$$=I_{11}(x)+I_{12}(x)+I_{13}(x).$$

1. $I_{11}(x)\quad x=-1$ bo‘lganda integral ostidagi ifoda $\xi=0$nuqtada $\frac{1}{\xi^{\frac{1}{2}-\beta}}$ tartibda cheksizlikka aylanadi, ya’ni $\frac{1}{2}-\beta<1$. Demak, $I_{11}(x)\quad x=-1$ da chegaralangan.

2. $I_{13}(x)$ ni $x=-1$ bo‘lganda tekshiramiz. Bu holda integral ostidagi ifoda $\xi\to\infty$ da $\dfrac{1}{\xi^{\frac{3}{2}-\beta}}$ tartibda nolga aylanadi va $\dfrac{3}{2}-\beta>1$. Demak bu integral $x=-1$ da chekli.

Endi $I_{12}(x)$ni o‘rganamiz va uni ushbu ko‘rinishda yozib olamiz :

$$I_{12}(x)\ =k(x)\int_{\xi_1}^{\xi_2}\frac{\xi^{\frac{3}{2}-\beta}d\xi}{\left[1+x+(1-x)\xi\right]^{2-2\beta}(1-\xi)}+$$

$$+\int_{\xi_1}^{\xi_2}\left\{k\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]-k(x)\right\}\frac{\xi^{\frac{3}{2}-\beta}d\xi}{\left[1+x+(1-x)\xi\right]^{2-2\beta}(1-\xi)}$$

Bu yerda birinchi integral Koshi ma’nosida integrallanuvchi, $k(x)$ esa μ ko‘rsatkich bilan Gyolder shartini qanoatlantirgani uchun:

$$\left|k\left[\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}\right]-k(x)\right|\le C\left|\frac{1+x-(1-x)\xi}{1+x+(1-x)\xi}-x\right|^{\mu}=$$

$$=C\left|\frac{(1-x)(1+x)}{(1+x)+(1-x)\xi}\right|^{\mu}|1-\xi|^{\mu}\ .$$

Bu tengsizlikdan ko‘rinib turibdiki, $I_{12}(x)$ $x=-1$ da chekli. Shunday qilib, $I_1(x)$ $x=-1$ nuqtada chekli ekan.

4.7-lemma isbot bo‘ldi.

7-§. Fredgolm integral tenglamasini o‘rganish.

Endi (4.115) yechimni (4.87) tenglikka asosan, ushbu ko‘rinishda yozib olamiz:

$$\nu(x)=\frac{1+\sin\beta\pi}{2}\left[F(x)-\int_{-1}^{1}M(x,t)\nu(t)dt\right]-$$

$$-\frac{\cos\beta\pi}{2\pi}\int_{-1}^{1}\left[\frac{(1-t)(1+t)}{(1-x)(1+x)}\right]^{\frac{1}{2}-\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\times$$

$$\times\left[F(t)-\int_{-1}^{1}M(t,s)\nu(s)ds\right]dt$$

yoki

$$\nu(x)+\int_{0}^{1}K(x,t)\nu(t)dt=g_1(x). \qquad (4.154)$$

Bu yerda

$$K(x,t)=\frac{1+\sin\beta\pi}{2}\left[M(x,t)-\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\times\right.$$

$$\left.\times\int_{-1}^{1}\left[\frac{(1-s)(1+s)}{(1-x)(1+x)}\right]^{\frac{1}{2}-\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)M(s,t)ds\right]\ g_1(x)=$$

$$=\frac{1+\sin\beta\pi}{2}\left[F(x)-\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\times\right. \qquad (4.155)$$

$$\left.\times\int_{-1}^{1}\left[\frac{(1-s)(1+s)}{(1-x)(1+x)}\right]^{\frac{1}{2}-\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)F(s)ds\right].$$

$K(x,t)$ yadroni o‘rganamiz.

(4.68) tenglikka ko‘ra:

$$M(x,t)=\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\int_{-1}^{x}(x-s)^{2\beta-1}\frac{\partial H_1(t,s)}{\partial s}ds. \qquad (4.156)$$

Bu yerdan (4.36) tenglikka ko‘ra:

$$\left|M(x,t)\right| \le C\int_{-1}^{x}(x-s)^{2\beta-1}(1-ts)^{-2\beta}\,ds. \qquad (4.157)$$

(4.157) integralda $s=-1+(1+x)\sigma$ almashtirish bajarib,

$$x-s=(1+x)(1-\sigma),\ 1-ts=1+t-(1+x)t\ ,$$
$$\sigma=(1+t)\left(1-\frac{(1+x)t}{1+t}\sigma\right).$$

(4.157) tengsizlikni quyidagi shakldа yozib olamiz:

$$\left|M(x,t)\right| \le C\int_{0}^{1}(1+x)^{2\beta-1}\cdot(1-\sigma)^{2\beta-1}\times$$
$$\times(1+t)^{-2\beta}\left(1-\frac{(1+x)t}{1+t}\sigma\right)^{-2\beta}(1+x)d\sigma.$$

Bu yerda gipergeometrik funksiyalarning integral ifodasidan foydalanib,

$$\left|M(x,t)\right| \le \frac{C}{2\beta}\left(\frac{1+x}{1+t}\right)^{2\beta}F\left(1,2\beta;1+2\beta;\frac{(1+x)t}{1+t}\right), \qquad (4.158)$$

tengsizlikka ega bo‘lamiz. Bu ifodadan ko‘rinib turibdiki, $M(x,t)$ funksiya $(1,1)$ nuqtada $\ln(1-xt)$ logarifmik maxsuslikka ega.

Endi $M(x,t)$ funksiyani $(-1,-1)$ nuqtada o‘rganish uchun (4.158) ni (4.127) formulaga asosan,

$$\left|M(x,t)\right| \le \frac{C}{2\beta}\left(\frac{1+x}{1-xt}\right)^{2\beta}F\left(2\beta;2\beta,1+2\beta;\frac{(1+x)t}{xt-1}\right),$$

shakldа yozib olamiz, bu yerdan esa $(-1,-1)$ nuqta atrofida

$$M(x,t)=0(1). \quad (1+x=\rho\cos\varphi, 1+t=\rho\sin\varphi).$$

Shunday qilib, (4.155) formuladan

$$\left|K(x,t)\right| \le C_1(1-x)^{\beta-\frac{1}{2}}, \tag{4.159}$$

tengsizlikka kelamiz.

Endi $g_1(x)$ ni o‘rganamiz. 4.6-lemmaga ko‘ra, $F(x)$ $-1<x<1$ oraliqda $\beta+\varepsilon$ ko‘rsatkich bilan Gyolder shartini qanoatlantiradi. $x=-1$ da 2β dan katta bo‘lmagantartibli nolga ega, $x=1$ da esa logarifmik maxsuslikka ega, bundan $g_1(x)$ ni $-1\le x<1$ da uzluksiz va $g_1(x)=0(1)(1-x)^{\beta-\frac{1}{2}}$ ekanligi kelib chiqadi.

Shunday qilib, biz

$$\nu(x)+\int_{-1}^{1}K(x,t)\nu(t)dt=g_1(x), \tag{4.160}$$

tenglamaga keldik va bu tenglamaga Fredgolm nazariyasini qo‘llash mumkin.

Endi (4.160) tenglamaga mos

$$\nu(x)+\int_{-1}^{1}K(x,t)\nu(t)dt=0, \tag{4.161}$$

bir jinsli tenglama $\nu(x)\in L_p[-1,1]$, $p>\dfrac{m+2}{2(1-\beta_0)}$ sinfda faqat $\nu(x)\equiv 0$ yechimga ega ekanligini ko‘rsatamiz. Teskarisini faraz qilamiz, (4.159) bahoga asosan, $\nu(x)\in L_2[-1,1]$. (4.155) va (4.68) tengliklarga asosan, (4.161) tenglamani ushbu ko‘rinishda yozib olamiz:

$$\nu(x)+\int_{-1}^{1}\left\{\frac{1+\sin\beta\pi}{2}\left[M(x,t)-\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\cdot\right.\right.$$

$$\left.\left.\cdot\int_{-1}^{1}\left[\frac{(1-s)(1+s)}{(1-x)(1+x)}\right]^{\frac{1}{2}-\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)M(s,t)ds\right\}\nu(t)dt=0$$

yoki

$$\nu(x)+\frac{1+\sin\beta\pi}{2}\int_{-1}^{1}M(x,t)\nu(t)dt=$$

$$=\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\int_{-1}^{1}\left[\frac{(1-s)(1+s)}{(1-x)(1+x)}\right]^{\frac{1}{2}-\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)\times \tag{4.162}$$

$$\times\int_{-1}^{1}M(s,t)\nu(t)dt=0$$

,

bu yerda

$$k(x) = -\frac{1+\sin\beta\pi}{2}\int_{-1}^{1} M(x,t)\nu(t)dt\,, \qquad (4.163)$$

belgilash kiritib, (4.162) tenglamani ushbu ko‘rinishda yozib olamiz:

$$\nu(x) = k(x) - \frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\int_{-1}^{1}\left[\frac{(1-t)(1+t)}{(1-x)(1+x)}\right]^{\frac{1}{2}-\beta} \times \\ \times\left(\frac{1}{t-x} - \frac{1}{1-xt}\right)k(t)dt \qquad (4.164)$$

Endi (4.163) ni (4.68) tenglikka asosan, quyidagicha almashtiramiz:

$$k(x) = -\frac{1+\sin\beta\pi}{2}\cdot\int_{-1}^{1} M(x,t)\nu(t)dt = \\ = -\frac{1+\sin\beta\pi}{2}\cdot\int_{-1}^{1}\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\left(\int_{-1}^{x}(x-s)^{2\beta-1}\frac{\partial \mathrm{H}_1(t,s)}{\partial s}ds\right)\nu(t)dt = \\ = -\frac{\gamma}{2\Gamma(2\beta)}\int_{-1}^{x}(x-s)^{2\beta-1}ds\int_{-1}^{1}\frac{\partial \mathrm{H}_1(t,s)}{\partial s}\nu(t)dt = \\ = -\frac{\gamma}{2\Gamma(2\beta)}\int_{-1}^{x}(x-s)^{2\beta-1}k_1(s)ds, \qquad (4.165)$$

bu yerda

$$k_1(s) = \int_{-1}^{1}\frac{\partial \mathrm{H}_1(t,s)}{\partial s}\nu(t)dt\,.$$

Ushbu

$$\left(\int_{-1}^{1} x(t)y(t)dt\right)^2 \le \int_{-1}^{1} x^2(t)dt\int_{-1}^{1} y^2(t)dt$$

Koshi-Bunyakovskiy tengsizligi va (4.36):

$$\left|\frac{\partial \mathrm{H}_1(t,x)}{\partial x}\right| \le c(1-xt)^{-2\beta},$$

tengsizlikka ko‘ra,

$$|k_1(s)| \le \sqrt{\int_{-1}^{1}\left(\frac{\partial \mathrm{H}_1(t,s)}{\partial s}\right)^2 dt} \cdot \sqrt{\int_{-1}^{1}\nu^2(t)dt},$$

Bundan esa $4\beta<1,\ m<2-4\beta_0$ -shart bajarilganda $|k_1(s)|<c$ bahoga kelamiz.

Bundan esa kasr tartibli integral operator xossasiga ko‘ra, $k(x)\ -1<x<1$ oraliqda 2β tartibda Gyolder shartini qanoatlantirishi kelib chiqadi. Endi $\nu(x)$ ni (4.23):

$$\tau(x)=\gamma\int_{-1}^{x}(x-t)^{-2\beta}\nu(t)dt, \qquad (4.166)$$

tenglikka qo‘yib, bu yerda $\psi_1(x)\equiv 0, \tau(x)$ ni topamiz. (4.166) tenglikda $\nu(x)\in H_{2\beta}(I)$ bo‘lgani uchun, $\tau(x)\ -1<x<1$ oraliqda $1-\varepsilon$ (ε yetarli kichik son) ko‘rsatkich bilan Gyolder shartini qanoatlantiradi. Shunday qilib, $\tau(x)$ va $\nu(x)$ funksiyalar (4.1) tenglama yechimi $u(x,y)$ ni R_1 sinfga tegishli yechimini aniqlaydi va bu yechim Γ chiziqda hamda AC xarakteristikada 0 ga aylanadi. Bu yerdan esa R_1 sinfda Trikomi masalasi yechimining yagonaligiga ko‘ra, $u(x,y)\equiv 0$ bo‘lgani uchun $\nu(x)\equiv 0$ ekanligi kelib chiqadi. Shunday qilib, (4.161) tenglama ixtiyoriy o‘ng tomon uchun yechimga ega.

(4.155) formuladan, 4.6-lemmaga asosan, $g_1(x)$ funksiya $\beta+\varepsilon$ ko‘rsatkich bilan $-1<x<1$ oraliqda Gyolder shartini qanoatlantiradi.

Endi 5-lemmani

$$\int_{-1}^{1}K(x,t)\nu(t)dt,$$

integralga qo‘llab, $\nu(x)$ funksiya $-1<x<1$ oraliqda $\beta+\varepsilon$ ko‘rsatkich bilan Gyolder shartini qanoatlantirishiga ishonch hosil qilamiz. (4.166) formulaga ko‘ra, $\tau(x)\in H_\alpha,\ \alpha>1\text{-}\beta.$

Shunday qilib, ushbu natija o‘rinlidir:

Agar $\varphi(s)$ 4.7-lemmaning shartlarini, $\psi(\eta)$ esa 4.6-lemmaning shartlarini qanoatlantirsa, Γ-esa 59-betdagi shartlarni qanoatlantirsa, u holda D sohada (4.1) tenglamaning R_1 sinfda $u\big|_{\Gamma}=\varphi(s),\ u\big|_{AC}=\psi(x),$ shartlarni qanoatlantiruvchi yagona yechimi mavjud.

8-§. Elliptik soha chegarasi normal chiziq bilan ustma-ust tushgan holda Trikomi masalasi

Agar elliptik soha chegarasi normal chiziq σ_0 :

$$x^2 + \frac{4}{(m+2)^2} y^{m+2} = 1$$

bilan ustma-ust tushsa, Trikomi masalasi yechimini oshkor ko‘rinishda berish mumkin. Haqiqatdan ham, D_0^+ sohada $H_1(x,t)$ funksiya nolga teng bo‘ladi, ya’ni $H_1(x,t) \equiv 0$ va (4.67) singulyar integral tenglama ushbu

$$\nu(x) + \frac{cos\beta\pi}{\pi(1+\sin\beta\pi)} \int_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{1-2\beta} \left(\frac{1}{t-x} - \frac{1}{1-xt}\right) \nu(t) dt = F(x) \quad (4.167)$$

sodda ko‘rinishni oladi, bu yerda (4.66) (4.83) formulalarga ko‘ra:

$$F(x) = \frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)} \frac{d}{dx} \int_{-1}^{x} \frac{\Phi(t)dt}{(x-t)^{1-2\beta}} + \frac{\psi_1(x)}{1+\sin\beta\pi} \quad (4.168)$$

$$\Phi(x)=-\int_0^l \varphi(s)\rho_1(s,x,0)ds=-\int_0^l \varphi(s)\eta^{\beta_0}(s)A_s[G_{01}(\xi,\eta;x,0)ds]=$$

$$=-k_1\beta(m+2)(1-x^2)\int_0^l \frac{\eta^{\beta_0-1}(s)\varphi(\xi(s))d\xi(s)}{\left[(x-\xi(s))^2+\frac{4}{(m+2)^2}\eta^{m+2}(s)\right]^{\beta+1}}= \tag{4.169}$$

$$=-k_1\beta(m+2)(1-x^2)\int_0^l \frac{\eta^{\beta_0-1}(s)\varphi(\xi(s))d\xi(s)}{\left[x^2-2x\xi(s)+1\right]^{\beta+1}}$$

$$\psi(x)=-\frac{\gamma(1+x)^\beta d\xi(s)}{2\Gamma(2\beta)}\int_{-1}^x \frac{\psi'((t-1)/2)}{(x-t)^{1-\beta}}dt \tag{4.170}$$

$F(x)$funksiyani o‘rganamiz. Trikomi masalasi yechimi $u(x,y)$ ni normal chiziq σ_0 dagi qiymat $\varphi(x)$ ni ushbu

$$\varphi(x)=y^{2(1-\beta_0)}\varphi_1(x), \tag{4.171}$$

ko‘rinishda ifodalash mumkin deb faraz qilamiz, bu yerda $\varphi_1(x),\in C[-1,1]$.

Normal chiziq tenglamasidan

$$y=\left(\frac{m+2}{2}\right)^{\frac{2}{m+2}}(1-x^2)^{\frac{1}{m+2}},$$

tenglikni e’tiborga olib va (4.169) tenglikda $t=\xi(s)$ almashtirish bajarib, $\Phi(x)$ funksiyani ushbu ko‘rinishda yozib olamiz:

$$\Phi(x)=\bar{k}_1(1-x^2)\int_{-1}^1 \varphi_1(t)\cdot\frac{(1-t^2)^{\frac{1}{2}-\beta}}{\left[x^2-2xt+1\right]^{\beta+1}}dt, \tag{4.172}$$

bu yerda $\bar{k}_1=2k_1\beta\left(\frac{m+2}{2}\right)^{2\beta_0}$. (4.172) tenglikdan ko‘rinib turibdiki, $\Phi(x)$ funksiya $(-1,1)$ intervalda istalgan tartibli hosilaga ega. $\Phi(x)$ va uning hosilasi $\Phi'(x)$

ning $x \to -1$ va $x \to 1$ da o‘zini tutishini o‘rganamiz. Agar (4.172) formulaning o‘ng tomonida x ni $-x$ ga almashtirsak, butun ifoda o‘zgarmaydi, faqat $\varphi_1(t)$ ning argumenti t ning o‘rniga $-t$ ga almashadi. Demak, $\Phi(x)$ funksiyaning $x=-1$ va $x=1$ nuqta atrofida o‘zini tugatishi bir xildir. Shuning uchun, $\Phi(x)$ va uning hosilasi $\Phi'(x)$ ni $x=-1$ nuqtada o‘rganish yetarlidir. (4.172) ifodani x bo‘yicha differensiallab, ushbu

$$\Phi'(x) = -2\bar{k}_1 x \int_{-1}^{1} \varphi_1(t) \frac{\left(1-t^2\right)^{\frac{1}{2}-\beta}}{\left[x^2-2xt+1\right]^{\beta+1}} dt -$$
$$- \bar{k}_1(\beta+1)\left(1-x^2\right)\int_{-1}^{1} \varphi_1(t) \frac{\left(1-t^2\right)^{\frac{1}{2}-\beta} \cdot 2(x-t)}{\left[x^2-2xt+1\right]^{\beta+2}} dt \qquad (4.173)$$
,

tenglikka kelamiz.

(4.173) tenglikka o‘rta qiymat haqidagi teoremani qo‘llab va $x - t = x + 1 - (t + 1)$ ayniyatni hisobga olib, uni ushbu ko‘rinishda yozib olamiz:

$$\Phi'(x) = 2\bar{k}_1 \frac{x\varphi_1(t_1)}{(1+x)^{2+2\beta}} \int_{-1}^{1} (1+t)^{\frac{1}{2}-\beta}(1-t)^{\frac{1}{2}-\beta} \cdot \left(1 \frac{2x}{(1-x)^2}(1+t)\right)^{-1-\beta} dt -$$
$$- \bar{k}_1(1+\beta)\varphi_1(t_2) - \frac{(1-x)}{(1+x)^{2+2\beta}} \int_{-1}^{1} (1+t)^{\frac{1}{2}-\beta}(1-t)^{\frac{1}{2}-\beta} \cdot \left(1 - \frac{2x}{(1+x)^2} 1+t\right)^{-2-\beta} dt -$$
$$- \bar{k}_1(1+\beta)\varphi_1(t_3) \frac{(1-x)}{(1+x)^{3+2\beta}} \int_{-1}^{1} (1+t)^{\frac{3}{2}-\beta}(1-t)^{\frac{1}{2}-\beta} \cdot \left(1 - \frac{2x}{(1+x)^2} 1+t\right)^{-2-\beta} dt. \qquad (4.174)$$

Bu tenglikning o‘ng tomonidagi integrallarda integral o‘zgaruvchisini $t = -1 + 2\sigma$ shaklda almashtiramiz, natijada (4.174) ifoda ushbu ko‘rinishni oladi:

$$\Phi'(x)=-2^{3-2\beta}\,\overline{k}_1\,\frac{x\varphi_1(t_1)x}{(1+x)^{2+2\beta}}\int_0^1\sigma^{\frac{1}{2}-\beta}(1-\sigma)^{\frac{1}{2}-\beta}\cdot\left(1-\frac{4x}{(1+x)^2}\sigma\right)^{-1-\beta}d\sigma-$$

$$-2^{2-2\beta}(1+\beta)\varphi_1(t_2)\,\frac{(1-x)}{(1+x)^{2+2\beta}}\int_0^1\sigma^{\frac{1}{2}-\beta}(1-\sigma)^{\frac{1}{2}-\beta}\cdot\left(1-\frac{4x}{(1+x)^2}\sigma\right)^{-2-\beta}d\sigma-$$

$$-2^{3-2\beta}(1+\beta)\varphi_1(t_3)\frac{(1-x)}{(1+x)^{3+2\beta}}\int_0^1\sigma^{\frac{3}{2}-\beta}(1-\sigma)^{\frac{1}{2}-\beta}\cdot\left(1-\frac{4x}{(1+x)^2}\sigma\right)^{-2-\beta}d\sigma.$$

(4.174)

(4.174) tenglikning o‘ng tomoni Gaussning gipergeometrik funksiyasi orqali ifodalanadi. (1.17) formulani hisobga olib, (4.174) ni quyidagi ko‘rinishda yozib olamiz:

$$\Phi'(x)=-2^{3-2\beta}\,\overline{k}_1\,\frac{\Gamma^2\left(\frac{3}{2}-\beta\right)}{\Gamma(3-2\beta)}\cdot\frac{\varphi\ \ (t_1)\cdot x}{(1-x)^{2+2\beta}}F\left(\frac{3}{2}-\beta,1+\beta;3-2\beta;\frac{4x}{(1+x)^2}\right)-$$

$$-2^{2-2\beta}(1+\beta)\frac{\Gamma^2\left(\frac{3}{2}-\beta\right)}{\Gamma(3-2\beta)}\cdot\frac{\varphi_1(t_2)(1-x)}{(1+x)^{2+2\beta}}F\left(\frac{3}{2}-\beta,2+\beta;3-2\beta;\frac{4x}{(1+x)^2}\right)-$$

$$-2^{3-2\beta}(1+\beta)\frac{\Gamma\left(\frac{5}{2}-\beta\right)}{\Gamma(4-2\beta)}\Gamma\left(\frac{3}{2}-\beta\right)\frac{\varphi_1(t_3)(1-x)}{(1+x)^{3+2\beta}}F\left(\frac{5}{2}-\beta,2+\beta;4-2\beta;\frac{4x}{(1+x)^2}\right).$$

(4.175)

Bu yerdan (1.24)

$$F\left(a,b,c:\frac{4x}{(1+x)^2}\right)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-b)\Gamma(b)}\left(\frac{1-x}{1+x}\right)^{-2a}F\left(a,c-b,a-b+1;\left(\frac{1+x}{1-x}\right)^2\right)+$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}\left(\frac{1-x}{1+x}\right)^{-2b}F\left(c-a,b,b-a+1;\left(\frac{1+x}{1-x}\right)\right)^2$$

formulaga asosan, $\varphi'(x)$ funksiya $4\beta<1$ $(m<2-4\beta_0)$ bo‘lganda $x=-1$ nuqta atrofida chegaralangan degan xulosaga kelamiz. Yuqoridagi hisoblashlarni takrorlab,

$x=-1$ nuqta atrofida ikkinchi tartibli hosila uchun $\Phi''(x)=0\left((1+x)^{-4\beta}\right)$ baho o‘rinli ekanligini ko‘rsatish qiyin emas.

Shunday qilib, agar $x\to-1$ ga yoki $x\to 1$ ga intilganda, $\Phi''(x)$ esa $4\beta<1$ tartibda cheksizlikka intiladi. (4.172) formuladan $\varphi(-1)=0,\ \varphi(1)=0$ ekanligi bevosita kelib chiqadi.

Trikomi masalasi yechimi $u(x,y)$ning $\xi=-1$ (AC) xarakteristikadagi chegaraviy qiymatini $\psi(x)=(1+x)^2\psi_1(x)$ ko‘rinishda ifodalash mumkin bo‘lsin, bu yerda $\psi_1(x)\in C^2[-1,0]$ va $\psi_1''(x), -1\le x<0$ oraliqda $\delta>0$ ko‘rsatkich bilan Gyolder shartini qanoatlantirsin.

$\Phi(x)$ va $\psi(x)$ funksiyalarning yuqorida keltirilgan xossalariga asosan, (4.168) formuladan $F(x)$ ni quyidagi ko‘rinishda yozib olamiz:

$$F(x)=\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\left[\int_{-1}^{x}\frac{\Phi'(t)dt}{(x-t)^{1-2\beta}}-\frac{(1+x)^{\beta}}{2}\int_{-1}^{x}(x-t)^{\beta-1}\psi'\left(\frac{(t-1)}{2}\right)dt\right].$$

Bu tenglikni x bo‘yicha bir marta differensiallab, ushbu tenglikni hosil qilamiz:

$$F'(x)=\frac{\gamma}{(1+\sin\beta\pi)\Gamma(2\beta)}\left[\int_{-1}^{x}\frac{\Phi''(t)dt}{(x-t)^{1-2\beta}}-\beta\frac{(1+x)^{\beta-1}}{2}\int_{-1}^{x}(x-t)^{\beta-1}\psi''\left(\frac{(t-1)}{2}\right)dt-\right.$$
$$\left.-\frac{(1-x)^{\beta}}{4}\int_{-1}^{x}(x-t)^{\beta-1}\cdot\psi''\left(\frac{(t-1)}{2}\right)dt\right].\qquad(4.176)$$

Bu tenglikdan ko‘rinib turibdiki, $F(x)\in C[-1,1]$, $F'(x)$ hosila esa $-1<x<1$ oraliqda $\mu>\beta$ ko‘rsatkich bilan Gyolder shartini qanoatlantiradi va $x=\pm1$ nuqta atrofida 2β tartibda cheksizlikka aylanadi.

Endi (4.167) singulyar integral tenglamaning yechimini topamiz, uning yechimi (4.115) formulaga ko‘ra, ushbu ko‘rinishda bo‘ladi:

$$\nu(x)=\frac{1+\sin\beta\pi}{2}\left[F(x)-\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}\int_{-1}^{1}\left[\frac{(1-t)(1+t)}{(1-x)(1+x)}\right]^{\frac{1}{2}-\beta}\times\right.$$
$$\left.\times\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)F(t)dt\right].\qquad(4.177)$$

Bu formuladagi integral 4.7-lemmaga ko‘ra, $-1 < x < 1$ oraliqda Gyolder shartini qanoatlantiradi va $x = -1$ da cheklidir.

$\nu(x)$ funksiyamiz $x = 1$ nuqtada $1 - 2\beta$ dan kichik bo‘lgan tartibda cheksizlikka aylanishi mumkin.

(4.177) ni (4.23) formulaga qo‘yib, $\tau(x)$ ni topamiz.

(4.177) va (4.23) formulalardan, $\nu(x)$ va $\tau(x)$ ni (4.10) Darbu formulasiga qo‘yib, Trikomi masalasining D_0^- - sohadagi yechimini topamiz. D_0^+ sohada esa yechim

$$u(x, y) = -\int_{-1}^{1} \nu(t) G_{01}(t, 0; x, y) dt - \int_{0}^{l} \varphi(s) \eta^{\beta_0}(s) A_s \left[G_{01}(\xi, \eta; x, y)\right] ds$$

formula bilan aniqlanadi. Shunday qilib, biz quyidagi teoremani isbotladik:

Teorema. *Agar* $\varphi(x)$ *ni (4.171) shaklda ifodalash mumkin bo‘lsa va* $\psi(x) \in C^2(-1,0) \cap C^{2,\delta}$ *(-1,0) sinfga tegishli bo‘lsa, G chiziq esa* σ_0 *normal chiziq bilan ustma-ust tushsa, u holda* D_0 *sohada (4.6) tenglamaning* $u\big|_{\sigma_0} = \varphi(s),\ u\big|_{AC} = \psi(x)$ *shartlarni qanoatlantiruvchi yagona yechimi mavjud.*

V BOB. SINGULYAR KOEFFITSIYENTLI GELLERSTEDT TENGLAMASI UCHUN XARAKTERISTIKADA VA BUZILISH CHIZIG‘IDA FRANKL SHARTIGA EGA BO‘LGAN NOLOKAL MASALA.

5.1-§. FH masalasining qo‘yilishi.

Ushbu tenglamani o‘rganamiz:

$$signy|y|^m u_{xx} + u_{yy} + (\beta_0 / y)u_y = 0\,, \tag{5.1}$$

bu yerda m va β_0– o‘zgarmas sonlar bo‘lib, ular uchun $m > 0, -m/2 < \beta_0 < 1$, tengsizliklar o‘rinli. Ω soha $z = x + iy$ kompleks tekisligining chekli bir bog‘lamli sohasi bo‘lib, u $y > 0$ tekisligida joylashgan va uchlari $A = A(-1,0), B = B(1,0)$ nuqtalarda yotuvchi (5.1) tenglamaning normal chizig‘i $\sigma_0 : x + \dfrac{4}{(m+2)^2} y^{m+2} = 1$ bilan, $y < 0$ tekislikda (5.1) tenglamaning AC va BC xarakteristikalari bilan chegaralangan bo‘lsin. Ushbu belgilashlarni kiritamiz: $\Omega^+ = \Omega \cap \{y > 0\}$, $\Omega^- = \Omega \cap \{y < 0\}$ C_0 va C_1 orqali esa $O(0,0)$ nuqtadan chiquvchi xarakteristikaning AC va BC xarakteristikalar bilan kesishish nuqtasini belgilaymiz. $J = (-1,1)$ $y = 0$ o‘qining intervali.

***FH*masalasi.** Ω sohada quyidagi shartlarni qanoatlantiruvchi $u(x,y) \in C(\overline{\Omega})$ funksiya topilsin:

1. $u(x,y) \in C^2(\Omega^+)$ va bu sohada (5.1) tenglamani qanoatlantiradi;

2. $u(x,y)$ funksiya $\Omega^- \setminus (OC_0 \cup OC_1)$ sohada (5.1) tenglamaning R_1 sinfga tegishli bo‘lgan umumlashgan yechimi [38];

3. $J = AB$ -intervalda ushbu ulanish shartlari bajariladi:

$$\lim_{y \to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} = \lim_{y \to +0} y^{\beta_0} \frac{\partial u}{\partial y}\,, \quad x \in J \setminus \{0\} \tag{5.2}$$

va bu limitlar $x = -1$, $x = 0$, $x = 1$ nuqtalarda $1 - 2\beta$ dan kichik tartibda maxsuslikka ega bo‘lishi mumkin, bu yerda $\beta = (m + 2\beta_0)/2(m+2)$;

4. $u(x,y)\big|_{\sigma_0} = \varphi(x)$, $x \in J$ (5.3)

$$a(x) D_{-1,x}^{1-\beta} u[\theta(x)] - b(x) D_{x,1}^{1-\beta} u[\theta(-x)] = \psi(x), x \in J \tag{5.4}$$

$$u(x,0)-u(-x,0)=f(x), \quad x\in \overline{J}, \tag{5.5}$$

bu yerda $\theta(x_0)=\dfrac{x_0-1}{2}-i\left[\dfrac{m+2}{4}(1+x_0)\right]^{2/(m+2)}$ $(x_0,0)$, $x_0\in J$ nuqtadan chiquvchi xarakteristikaning AC xarakteristika bilan kesishish nuqtasining affiksi. $a(x), b(x), \psi(x), f(x), \varphi(x)\in C(\overline{J})\cap C^{1,\alpha}(J)$ berilgan funksiyalar bo'lib, ular uchun $a(-x)=b(x)$, $a(x)\cdot b(x)\neq 0, \psi(-x)=-\psi(x)$, $f(-x)=-f(x)$, $\varphi(\pm 1)=0$, $f(\pm 1)=0$ munosabatlar o'rinli.

(5.4) va (5.1) shartlar Frankl shartlari bo'lib, ular mos ravishda AC xarakteristikada va AB buzilish chizig'ida berilgan.

5.2-§. FH masalasi yechimining yagonaligi.

(5.5) tenglamaning (1.116) va (1.117) shartlarni qanoatlantiruvchi shakli o'zgargan Koshi masalasi yechimini beruvchi (1.135) Darbu formulasidan foydalanib, ushbu qiymatlarni hosil qilamiz:

$$\begin{aligned} u[\theta(x)] &= \gamma_1\Gamma(\beta)((1+x)/2)^{1-2\beta} D_{-1,x}^{-\beta}(1+x)^{\beta-1}\tau(x)+ \\ &+\gamma_2\Gamma(1-\beta)((m+2)/2)^{1-2\beta} D_{-1,x}^{\beta-1}(1+x)^{-\beta}\nu(x), \end{aligned} \tag{5.6}$$

$$\begin{aligned} u[\theta(-x] &= \gamma_1\Gamma(\beta)((1-x)/2)^{1-2\beta} D_{x,1}^{-\beta}(1-x)^{\beta-1}\tau(-x)+ \\ &+\gamma_2((m+2)/2)^{1-2\beta}\Gamma(1-\beta) D_{x,1}^{\beta-1}(1-x)^{-\beta}\nu(-x), \end{aligned} \tag{5.7}$$

bu yerdan

$$\begin{aligned} D_{-1,x}^{1-\beta}u[\theta(x] &= \gamma_1\Gamma(\beta)2^{2\beta-1}(1+x)^{-\beta} D_{-1,x}^{1-2\beta}\tau(x)+ \\ &+\gamma_2\Gamma(1-\beta)((m+2)/2)^{1-2\beta}(1+x)^{-\beta}\nu(x), \end{aligned} \tag{5.8}$$

$$\begin{aligned} D_{x,1}^{1-\beta}u[\theta(-x] &= \gamma_1\Gamma(\beta)2^{\beta-1}(1-x)^{-\beta} D_{x,1}^{1-2\beta}\tau(-x)+ \\ &+\gamma_2\Gamma(1-\beta)((m+2)/2)^{1-2\beta}(1-x)^{-\beta}\nu(-x). \end{aligned} \tag{5.9}$$

(5.8) va (5.9) tengliklardan (5.4) va (5.5) nolokal shartlarga asosan,

$$
\begin{gathered}
(1-x)^{\beta} a(x)\nu(x)-(1+x)^{\beta} b(x)\nu(-x)= \\
=\gamma[(1-x)^{\beta} a(x)D_{-1,x}^{1-2\beta}\tau(x)- \\
-(1+x)^{\beta} b(x)D_{x,1}^{1-2\beta}\tau(x)]+\psi_1(x)
\end{gathered} \tag{5.10}
$$

munosabatga kelamiz, bu yerda

$$
\begin{gathered}
\psi_1(x)=\frac{(1-x)^{\beta}(1+x)^{\beta}\psi(x)}{\gamma_2\Gamma(1-\beta)((m+2)/2)^{1-2\beta}}+\gamma(1+x)^{\beta} b(x)D_{x,1}^{1-2\beta} f(x) \\
\gamma=-\frac{\gamma_1\Gamma(\beta)2^{2\beta-1}}{\gamma_2\Gamma(1-\beta)((m+2)/2)^{1-2\beta}}.
\end{gathered} \tag{5.11}
$$

Teorema 5.1. Agar $a(x)$ va $b(x)$ funksiyalar uchun ushbu

$$
a(x)\cdot b(x)<0, \quad x\in\overline{J}, \tag{5.12}
$$

tengsizlik o‘rinli bo‘lib, $\varphi(x)\equiv 0$, $\psi(x)\equiv 0$, $f(x)\equiv 0$ bo‘lsa, u holda FH masalasi faqat trivial yechimga ega.

Isbot. Teskari tasdiqni faraz qilamiz, $\overline{\Omega}^{+}$ sohada $u(x,y)\neq 0$ bo‘lsin. Xopf prinsipiga ko‘ra, $u(x,y)$ funksiya o‘zining eng katta musbat qiymatini va eng kichik manfiy qiymatini Ω^{+} sohaning ichki nuqtalarida qabul qilmaydi. $u(x,y)\big|_{\sigma_0}=0$ bo‘lgani uchun bu qiymatlarga σ_0 nuqtalarida ham erishilmaydi. Faraz qilaylik, $u(x,y)$ funksiya o‘zining eng katta musbat va eng kichik manfiy qiymatlarini $AB\setminus\{O\}$ nuqtalarda qabul qilsin, u holda (5.5) shartga mos bir jinsli shartga asosan, bu ekstremum qiymatlar $(x_0,0)$ va $(-x_0,0)$ nuqtalarda qabul qilinadi. $\tau(x)$ funksiyaning musbat maksimumini (manfiy minimumini) qabul qiluvchi x_0 nuqtada kasr tartibli $D_{-1,x_0}^{1-2\beta}\tau(x)$ va $D_{x_0,1}^{1-2\beta}\tau(x)$ hosilalar qiymati qat’iy musbat (qat’iy manfiy) bo‘lgani uchun va bu nuqtalarda lemma 2.1 ga asosan,

$$
\nu(x_0)=\lim_{y\to+0} y^{\beta_0}\frac{\partial u(x_0,y)}{\partial y}<0 \ \ (>0)
$$

bo‘lgani uchun, (5.10) tenglikka mos bir jinsli ushbu munosabat

$$(1-x)^{\beta} a(x)\nu(x)-(1+x)^{\beta} b(x)\nu(-x)=$$
$$=\gamma[(1-x)^{\beta} a(x)D_{-1,x}^{1-2\beta}\tau(x)-(1+x)^{\beta} b(x)D_{x,1}^{1-2\beta}\tau(x)] \quad (5.13)$$

$(x_0,0)$ va $(-x_0,0)$ nuqtalarda o‘rinli emas, chunki bu nuqtalarda (5.13)tenglikning o‘ng va chap tomonlari turli ishorali.

Shunday qilib, $u(x,y)$ yechim o‘zining eng katta musbat va eng kichik manfiy qiymatlarini $O(0,0)$ nuqtada erishadi. Bundan esa Ω^+ sohada $u(x,y)=const$, ya’ni o‘zgarmas ekanligi kelib chiqadi, lekin $u|_{\sigma_0}=0$ bo‘lgani uchun $u(x,y)\equiv 0$ $(x,y)\in\overline{\Omega}^+$. Bundan $u(x,y)\equiv 0$ $(x,y)\in\Omega$ ekanligi kelib chiqadi.

Teorema 5.1 isbot bo‘ldi.

5.3-§. FH masalasi yechimining mavjudligi.

Ω^+ sohada (5.1) tenglama uchun Dirixle va shakli o‘zgargan N masalalarining (2.183), (2.184) yechimlaridan foydalanib, $y=0$ o‘qida mos ravishda:

$$\nu(x)=-k_2(1-\beta_0)\frac{m+2}{2}\left[\int_{-1}^{1}\frac{x-t}{|x-t|^{2-2\beta}}\tau'(t)dt+\right.$$
$$\left.+(1-2\beta)\int_{-1}^{1}\frac{\tau(t)dt}{(1-xt)^{2-2\beta}}\right]+\Phi_2(x), \quad (5.14)$$

$$\tau(x)=-k_1\int_{-1}^{1}\nu(t)[|x-t|^{-2\beta}-(1-xt)^{-2\beta}]dt+\Phi_1(x)\,, \quad (5.15)$$

$$\Phi_2(x)=k_2(1-\beta)(1-\beta_0)(m+2)(1-x^2)\int_{-1}^{1}\varphi(t)(1-2xt+x^2)^{\beta-2}dt, \quad (5.16)$$

$$\Phi_1(x)=k_1\beta(m+2)(1-x^2)\int_{-1}^{1}\varphi(t)\eta^{\beta_0-1}(t)(1-2xt+x^2)^{-\beta-1}dt, \quad (5.17)$$

bu yerda

$$k_1 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2\beta}\frac{\Gamma^2(\beta)}{\Gamma(2\beta)},\ k_2 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)}$$

$$\eta(t) = \left(\left(\frac{m+2}{2}\right)^2(1-t^2)\right)^{1/(m+2)}.$$

(5.14) munosabatda x ni $-x$ ga almashtirib va ushbu

$$\tau(x) = \tau(-x) + f(x), \quad \tau'(x) = -\tau'(-x) + f'(x). \tag{5.18}$$

munosabatlarni hisobga olib, quyidagi tenglikni hosil qilamiz:

$$\nu(-x) = \nu(x) + F_2(x), \tag{5.19}$$

bu yerda

$$F_2(x) = k_2(1-\beta_0)\frac{m+2}{2}\left[\int_{-1}^{1}\frac{(x-t)f'(t)dt}{|x-t|^{2-2\beta}} + (1-2\beta)\int_{-1}^{1}\frac{f(t)dt}{(1-xt)^{2-2\beta}}\right] + \\ + \Phi_2(-x) - \Phi_2(x) \tag{5.20}$$

(5.19) ga asosan, (5.10) tenglikdan ushbu

$$\left[(1-x)^{\beta}a(x) - (1+x)^{\beta}b(x)\right]\nu(x) = \\ \gamma\left[(1-x)^{\beta}a(x)D_{-1,x}^{1-2\beta}\tau(x) - (1+x)^{\beta}b(x)D_{x,1}^{1-2\beta}\tau(x)\right] + \psi_2(x) \tag{5.21}$$

munosabatga kelamiz, bu yerda

$$\psi_2(x) = \psi_1(x) + (1+x)^{\beta}b(x)F_2(x). \tag{5.22}$$

Dastlab, faraz qilaylik,

$$(1-x)^{\beta}a(x) - (1+x)^{\beta}b(x) \neq 0 \tag{5.23}$$

bo‘lsin. (5.15) tenglikka $D_{-1,x}^{1-2\beta}$ va $D_{x,1}^{1-2\beta}$ operatorlarini qo‘llab, ushbu munosabatlarni hosil qilamiz:

$$D_{-1,x}^{1-2\beta}\tau(x) = -k_1\Gamma(1-2\beta)\ \times$$
$$\times\left[(1-\cos 2\beta\pi)\nu(x) + \frac{\sin 2\beta\pi}{\pi}\int_{-\pi}^{\pi}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x} - \frac{t}{1-xt}\right)\nu(t)\,dt\right] + \tag{5.24}$$
$$+ D_{-1,x}^{1-2\beta}\Phi_1(x) \quad x \in J\ ,$$

$$D_{x,1}^{1-2\beta}\tau(x) = -k_1\Gamma(1\text{-}2\beta)\left[(1-\cos 2\beta\pi)\nu(x) - \frac{\sin 2\beta\pi}{\pi}\times\right.$$
$$\left.\times\int_{-\pi}^{\pi}\left(\frac{1-t}{1-x}\right)^{1-2\beta}\left(\frac{1}{t-x} + \frac{t}{1-xt}\right)\nu(t)\,dt\right] + D_{x,1}^{1-2\beta}\Phi_1(x), \qquad x \in J. \tag{5.25}$$

Endi (5.24) va (5.25) formulalarga asosan, (5.21) ifodani ushbu ko‘rinishda yozib olamiz:

$$[(1-x)^{\beta} a(x) - (1+x)^{\beta} b(x)]\nu(x) + \lambda(1-x)^{\beta} a(x)\times$$
$$\times\int_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x} - \frac{1}{1-xt}\right)\nu(t)dt + \tag{5.26}$$
$$+\lambda(1+x)^{\beta} b(x)\int_{-1}^{1}\left(\frac{1-t}{1-x}\right)^{1-2\beta}\left(\frac{1}{t-x} + \frac{1}{1-xt}\right)\nu(t)dt = \psi_2(t)$$

bu yerda $\lambda = \cos\beta\pi / \pi(1+\sin\beta\pi)$

$$\psi_3(x) = \frac{1}{1+\sin\beta\pi}\left[\psi_2(x) + \gamma(1-x)^{\beta} a(x) D_{-1,x}^{1-2\beta}\Phi_1(x) - \right.$$
$$\left. -\gamma(1+x)^{\beta} b(x) D_{-1,x}^{1-2\beta}\Phi_1(x)\right] \tag{5.27}$$

Ushbu tengliklarning to‘g‘riligini bevosita tekshirib ko‘rish mumkin:

$$\left(\frac{1+t}{1+x}\right)\left(\frac{1}{t-x} - \frac{1}{1-xt}\right) = \frac{1}{t-x} - \frac{t}{1-xt}\ ,$$

$$\left(\frac{1-t}{1-x}\right)\left(\frac{1}{t-x} + \frac{1}{1-xt}\right) = \frac{1}{t-x} - \frac{t}{1-xt}\ .$$

Bu tengliklarga asosan, (5.26) tenglamani quyidagi ko‘rinishda yozib olamiz:

$$
\begin{aligned}
&[(1-x)^{\beta}a(x)-(1+x)^{\beta}b(x)]\nu(x)+ \\
&+\lambda\int_{-1}^{1}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)K(x,t)\nu(t)dt+\psi_3(x),
\end{aligned}
\tag{5.28}
$$

bu yerda

$$
K(x,t)=\left(\frac{1+x}{1+t}\right)^{2\beta}(1-x)^{\beta}a(x)+\left(\frac{1-x}{1-t}\right)^{2\beta}(1+x)^{\beta}b(x). \tag{5.29}
$$

Ushbu

$$
\frac{1}{t-x}-\frac{t}{1-xt}=\frac{2(1-t^2)}{(1+x^2)(1+t^2)\left(\frac{2t}{1+t^2}-\frac{2x}{1+x^2}\right)}
$$

tenglikka asosan, (5.28) tenglamada quyidagi

$$
s=\frac{2t}{1+t^2},\ \ t=\frac{s}{1+\sqrt{1-s^2}};\ y=\frac{2x}{1+x^2},\ x=\frac{y}{1+\sqrt{1-y^2}}
$$

almashtirishlarni bajarib,

$$
\begin{aligned}
&A(y)+\rho(y)\frac{B(y)}{\pi}\int_{-1}^{1}\frac{\rho(s)ds}{s-y}= \\
&=\lambda\int_{-1}^{1}\frac{K(x,x)-K(x,t)}{s-y}\rho(s)ds+\psi_2(x)(1+x^2)
\end{aligned}
\tag{5.30}
$$

,

singulyar integral tenglamani hosil qilamiz. Bu yerda

$$
\rho(y)=(1+x^2)\nu(x),\ \ A(y)=(1-x)^{\beta}a(x)-(1+x)^{\beta}b(x)
$$
$$
B(y)=\lambda\pi k(x,x)=\lambda\pi\, K\left((1-x)^{\beta}a(x)+(1+x)^{\beta}b(x)\right)
$$

$$A^2(y)+B^2(y)=\left[(1-x)^{\beta}a(x)-(1+x)^{\beta}b(x)\right]^2+$$
$$+\lambda^2\pi^2[(1-x)^{\beta}a(x)+(1+x)^{\beta}b(x)]^2\neq 0,$$

bo‘lgani uchun, (5.30) tenglama normal tipdagi singulyar integral tenglamadir.

(5.30) tenglama yechimi $\rho(x)$ funksiyani $H(-1,1)$ Gyolder sinfida izlaymiz va bu funksiya $x=-1,\ x=1$ nuqtalarda chegaralangan bo‘lsin.(5.30) singulyar integral tenglama yechimi indeksini hisoblaymiz.

Ushbu funksiyani tuzib olamiz:

$$G(y)=\frac{A(y)-iB(y)}{A(y)+iB(y)}=\frac{[(1-x)^{\beta}a(x)-(1+x)^{\beta}b(x)]-i\lambda\pi[(1-x)^{\beta}a(x)+(1+x)^{\beta}b(x)]}{[(1-x)^{\beta}a(x)-(1+x)^{\beta}b(x)+i\lambda\pi[(1-x)^{\beta}a(x)+(1+x)^{\beta}b(x)]}$$

I.N. Musxelishvili formulasiga ko‘ra,

$$\alpha_k+i\beta_k=\frac{G(c_k)}{2\pi i}$$

bu yerda "-" ishorasi $c_0=-1$ qiymatga, "$+$" ishorasi $c_1=1$ qiymatga mos keladi.

$$\alpha_0+i\beta_0=-\frac{\ln G(c_0)}{2\pi i}=-\frac{\ln G(-1)}{2\pi i}=-\frac{1}{2\pi i}\ln\frac{1-i\lambda\pi}{1+i\lambda\pi}=$$
$$=-\frac{1}{2\pi i}\left[\ln\left|\frac{1-i\lambda\pi}{1+i\lambda\pi}\right|+i(\arg\frac{1-i\lambda\pi}{1+i\lambda\pi}+2k\pi)\right]=$$
$$=-\frac{1}{2\pi}\arg\frac{1+\sin\beta\pi-i\cos\beta\pi}{1+\sin\beta\pi+i\cos\beta\pi}-k=\frac{1}{\pi}arctg\,\frac{\cos\beta\pi}{1+\sin\beta\pi}-k=$$
$$=\frac{1}{\pi}arctg\left(\frac{\pi}{4}-\frac{\beta\pi}{2}\right)=\frac{1}{2}\left(\frac{1}{2}-\beta\right)-k,$$

bu yerdan $\alpha_0=\frac{1}{2}\left(\frac{1}{2}-\beta\right)+k$, λ_0 butun sonni shunday tanlaymizki, $0<\alpha+\lambda_0<1$ tengsizlik bajarilsin, ya’ni $\lambda_0=k$.

Yuqoridagiga o‘xshash

$$\alpha_1+i\beta_1=\frac{\ln G(c_1)}{2\pi i}=\frac{\ln G(1)}{2\pi i}=\frac{1}{2\pi i}\ln\frac{1+i\lambda\pi}{1-i\lambda\pi}=\frac{1}{2}\left(\frac{1}{2}-\beta\right)+k,$$

ya'ni $\alpha_1 = \frac{1}{2}\left(\frac{1}{2} - \beta\right) + k.\ \lambda_1$ butun sonni shunday tanlaymizki, $0 < \alpha_1 + \lambda_1 < 1$ tengsizlik o'rinli bo'lsin, ya'ni $\lambda_1 = -k$. Endi χ indeksni hisoblaymiz: $\chi = -(\lambda_0 + \lambda_1) = 0$. $X(z)$ kanonik funksiya ushbu ko'rinishda bo'ladi:

$$X(z) = (1 - z^2)^{\frac{1}{2}\left(\frac{1}{2} - \beta\right)} \omega(z),$$

bu yerda $\omega(z) \in H$ va $\omega(z)$ nolga aylanmaydi.

(5.30) tenglama Karleman usuli yordamida Fredgolmning ikkinchi tur integral tenglamasiga ekvivalent ravishda olib kelinadi va bu tenglama FH masalasi yechimining yagonalik teoremasiga ko'ra birdan-bir yechimga ega.

Shunday qilib, FH masalada (5.23) shart bajarilganda yagona yechimga ega.

5.4-§. FH masalasini $(1-x)^{\beta} a(x) - (1+x)^{\beta} b(x) = 0$ bo'lgan holda o'rganish.

FH masalasini (5.23) shart bajarilmagan holda o'rganamiz, ya'ni:

$$(1-x)^{\beta} a(x) - (1+x)^{\beta} b(x) = 0,\ x \in J. \qquad (5.31)$$

Bu holda (5.21) munosabat ushbu ko'rinishda bo'ladi:

$$D_{-1,x_0}^{1-2\beta} \tau(x) - D_{x,1}^{1-2\beta} \tau(x) = \psi_2(x). \qquad (5.32)$$

Dastlab, bu holda ham FH masalasi yechimi yagona ekanligini isbotlaymiz. (5.32) tenglikda $\psi_2(x) \equiv 0$ bo'lsin. (5.32) ni ushbu ko'rinishda yozib olamiz:

$$\frac{d}{dx}\left[D_{-1,x}^{-2\beta} \tau(x) + D_{-1,x}^{-2\beta} \tau(x)\right] = 0,$$

yoki

$$\left[D_{-1,x}^{-2\beta} \tau(x) + D_{-1,x}^{-2\beta} \tau(x)\right] = c = const. \qquad (5.33)$$

(5.33) tenglikkaoperatorini qo'llab va

$$D_{-1,x}^{2\beta} D_{x,1}^{-2\beta} \tau(x) = \cos 2\beta\pi \tau(x) + \frac{\sin 2\beta\pi}{\pi} \int_{-1}^{1} \left(\frac{1+t}{1+x} \right)^{2\beta} \frac{\tau(t)dt}{t-x},$$

$$D_{-1,x}^{2\beta} D_{-1,x}^{-2\beta} = \tau(x), \quad D_{-1,x}^{2\beta} c = \frac{(1+x)^{-2\beta}}{\Gamma(1-2\beta)} c,$$

formulalarni hisobga olib, ushbu singulyar integral tenglamani hosil qilamiz:

$$\varphi(x) + \frac{tg\beta\pi}{\pi} \int_{-1}^{1} \frac{\varphi(t)dt}{t-x} = f_0(x), \quad (5.34)$$

bu yerda

$$\varphi(x) = (1+x)^{2\beta} \tau(x), \; f_0(x) = \frac{c}{2\Gamma(1-2\beta)\cos^2 \beta\pi}.$$

(5.34) integral tenglama yechimini (-1,1) intervalda Gyolder sinfiga tegishli, $x=-1, x=1$ nuqtalarda esa uzluksiz bo‘lgan funksiyalar sinfida izlaymiz, ya’ni $h(-1,1)$ sinfda. Bu sinfda (5.34) tenglama indeksini hisoblaymiz. (5.34) singulyar integral tenglamada $A=1, B=tg\beta\pi$.

Ushbu fuksiyani tuzamiz:

$$G(t) = \frac{A(t) - iB(t)}{A(t) + iB(t)},$$

Bevosita hisoblash yordamida ko‘rsatish mumkinki, $\alpha_0 + i\beta_0 = \beta - k$, ya’ni $\alpha_0 = \beta - k_1, \; \beta_0 = 0; \; \alpha_1 + i\beta_1 = -\beta + k$, ya’ni $\alpha_1 = -\beta + k, \; \beta_1 = 0$. Endi λ_0 va λ_1 butun sonlarni shunday tanlaymizki, $0 < \alpha_k + \lambda_k < 1, \; k = 0,1$ tengsizliklar o‘rinli bo‘lsin, bunda $\lambda_0 = k, \; \lambda_1 = 1 - k$.

Demak, (5.34) tenglama indeksi:

$$\chi = -(\lambda_0 + \lambda_1) = -1.$$

Shunday qilib, (5.34) tenglama yechimining $h(-1,1)$ sinfdagi indeksi -1 ga teng, kanonik funksiya esa ushbu ko‘rinishda bo‘ladi:

$$X(z) = (1+z)^{\alpha_0 + \lambda_0} (1-z)^{\alpha_1 + \lambda_1} = (1+z)^{\beta} (1-z)^{1-\beta}.$$

Indeks $\chi < 0$, demak, singulyar integral tenglamalar uchun Nyoter nazariyasi Fredgolm integral tenglamalar nazariyasi bilan ustma-ust tushmaydi (bu nazariyalar faqat $\chi = 0$ bo‘lgandagina ustma-ust tushadi).

$\chi < 0$ bo‘lganda (5.34) tenglamasining yagona yechimi ushbu

$$\int_{-1}^{1} \frac{t^{\mu} f_0(t)dt}{[a(t)+ib(t)]X^{+}(t)} = 0, \ (\mu = 0,1 \ldots -\chi - 1) \quad (5.35)$$

zaruriy va yetarli shartlar bajarilgandagina o‘rinli bo‘ladi.

Bizning tenglamamiz uchun $\chi = -1$.

Demak, $\mu = 0, \ f_0(t) = \dfrac{c}{2\Gamma(1-2\beta)\cos^2 \beta\pi}$ va (5.35) shart ushbu ko‘rinishni oladi:

$$\int_{-1}^{1} \frac{f(t)dt}{(a(t)+ib(t)X^{+}(t))} = \frac{c}{2\cos^2 \beta\pi\Gamma(1-2\beta)(1+itg\beta\pi)} \int_{-1}^{1} \frac{dt}{X^{+}(t)} = 0$$

,

bu yerdan $c = 0$ ekanligi kelib chiqadi.

Shunday qilib, biz

$$\varphi(x) + \frac{tg\beta\pi}{\pi} \int_{-1}^{1} \frac{\varphi(t)dt}{t-x} = 0 \quad (5.36)$$

tenglamaga kelamiz. Bu tenglama uzluksiz funksiyalar sinfida faqat $\varphi(x) \equiv 0$ trivial yechimga ega bo‘ladi, demak $\tau(x) \equiv 0$. Shunday qilib, biz FH masalasi yechimining yagonaligini (5.23) shart buzilgan holda isbotladik.

Endi (5.23) shart buzilgan holda FH masalasi yechimining mavjudligini ko‘rsatamiz.

(5.31) shartga asosan,

$$a(x) = (1+x)^{\beta} \mu(x), \quad b(x) = (1-x)^{\beta} \mu(x),$$

bu yerda $\mu(x) \in C^{(0,\alpha)}[-1,1]$ va $\mu(x) \neq 0 \ \forall x \in [-1,1]$. Bu holda (5.21) tenglamani ushbu ko‘rinishda yozib olish mumkin:

$$D_{-1,x}^{1-2\beta}\tau(x) + D_{x,1}^{1-2\beta}\tau(x) = -\frac{\psi_2(x)}{\gamma(1-x^2)^{\beta}\mu(x)}$$

yoki

$$D_{-1,x}^{-2\beta}\tau(x) - D_{x,1}^{-2\beta}\tau(x) = \int_{-1}^{x}\psi_4(t)dt + c\ , \qquad (5.37)$$

bu yerda

$$\psi_4(x) = -\psi_2(x)/\gamma(1-x^2)^{\beta}\,\mu(x)\,. \qquad (5.38)$$

(5.37) tenglikka $D_{-1,x}^{2\beta}$ operatorni qo‘llab, ushbu integral tenglamaga kelamiz:

$$\varphi(x) + \frac{tg\beta\pi}{\pi}\int_{-1}^{1}\frac{\varphi(t)dt}{t-x} = F(x)\ , \qquad (5.39)$$

bu yerda

$$\varphi(x) = (1+x)^{2\beta}\tau(x)\ ,$$

$$F(x) = \frac{(1+x)^{2\beta}}{2\Gamma(1-2\beta)\cos^2\beta\pi}\int_{-1}^{x}\frac{\psi_4(t)dt}{(x-t)^{2\beta}} + \frac{c}{2\Gamma(1-2\beta)\cos^2\beta\pi}. \qquad (5.40)$$

Yuqorida (5.39) tenglamaning $h(-1,1)$ sinfdagi indeksi $\chi = -1$ ekanligini ko‘rsatgan edik. Demak, (5.39) tenglamaning yagona yechimi:

$$\int_{-1}^{1}\frac{t^{\mu}F(t)dt}{[a(t)+ib(t)]X^{+}(t)} = 0,\ (\mu = 0,1\ldots - \chi - 1), \qquad (5.41)$$

zaruriy va yetarli shart bajarilgandagina mavjud bo‘ladi.

Bu yerda $a(t) = 1, b(t) = tg\beta\pi,\ X^{+}(t) = (1-t^2)^{\beta}$.

$\chi = -1$ bo‘lgani uchun $\mu = 0$. (5.40) ga asosan, (5.41) tenglikni ushbu ko‘rinishda yozib olamiz:

$$\int_{-1}^{1}\frac{cdt}{(1+t)^{\beta}(1-t)^{\beta}} + \int_{-1}^{1}\frac{(1+t)^{2\beta}dt}{(1+t)^{\beta}(1-t)^{\beta}}\int_{-1}^{t}\frac{\psi_4(s)ds}{(t-s)^{2\beta}} = 0, \qquad (5.42)$$

bu yerda integrallash tartibini o‘zgartirib va xosmas intergrallarni hisoblab, c uchun quyidagi qiymatni hosil qilamiz:

$$c=-\frac{2^{\beta}\Gamma(\beta)\Gamma(1-2\beta)\sin\beta\pi}{\Gamma(1-\beta)}\int_{-1}^{1}\frac{\psi_4(s)F\left(\beta,-\beta;1-\beta;\frac{1-s}{2}\right)}{(1-s)^{\beta}}ds.$$

Shunday qilib, (5.42) shart bajarilganda (5.39) tenglamaning yagona yechimi mavjud.

(5.39) tenglamani Karleman usulida yechamiz. Shu maqsadda,

$$\Phi(z)=\frac{1}{2\pi i}\int_{-1}^{1}\frac{\varphi(t)dt}{t-z},\qquad \Phi(\infty)=0\,, \tag{5.43}$$

funksiyani kiritamiz.

Soxotskiy-Plemel formulalariga asosan,

$$\Phi^{+}(x)+\Phi^{-}(x)=\frac{1}{\pi i}\int_{-1}^{1}\frac{\varphi(t)dt}{t-x}, \tag{5.44}$$

$$\Phi^{+}(x)-\Phi^{-}(x)=\varphi(x)\,. \tag{5.45}$$

Bu yerda

$$\Phi^{+}(x)=\lim_{y\to+0}\Phi(x+iy),$$
$$\Phi^{-}(x)=\lim_{y\to-0}\Phi(x+iy),$$

(5.44) va (5.45) formulalarga asosan, (5.39) tenglama ushbu ko‘rinishni oladi:

$$\Phi^{+}(x)=G(x)\Phi^{-}(x)+f(x)\,,\qquad (-\infty<x<+\infty) \tag{5.46}$$

bu yerda

$$G(x)=\begin{cases}\dfrac{1-itg\beta\pi}{1+itg\beta\pi}, & -1\le x\le 1,\\ 1\quad, & x\in(-\infty,-1)\cup(1,+\infty);\end{cases} \tag{5.47}$$

$$f(x)=\begin{cases}\dfrac{F(x)}{1+itg\beta\pi}, & -1\le x\le 1,\\ 0\quad, & x\in(-\infty,-1)\cup(1,+\infty).\end{cases} \tag{5.48}$$

Shunday qilib, biz golomorf funksiyalar uchun Riman masalasiga keldik: yuqori va quyi yarim tekisliklarda golomorf, cheksiz uzoqlashgan nuqtada $\Phi(\infty)=0$, haqiqiy o‘qda esa (5.46) shartni qanoatlantiruvchi $\Phi(z)$ funksiya topilsin.

Ushbu funksiyani o‘rganamiz:

$$\Psi(z)=\exp\left\{\frac{1}{2\pi i}\int_{-1}^{1}\frac{\ln G(t)dt}{t-z}\right\}=\exp\{-\beta[\ln(1-z)-\ln(-1-z)]\}=$$
$$=\left(\frac{1-z}{1+z}\right)^{\beta}e^{i\beta\arg(-1-z)} \qquad (5.49)$$

(5.49) tenglikdan ushbu chegaraviy qiymatlarni hosil qilamiz:

$$\Psi^{+}(x)=\left(\frac{1-x}{1+x}\right)^{\beta}e^{-i\beta\pi},\ \Psi^{-}(x)=\left(\frac{1-x}{1+x}\right)^{\beta}e^{i\beta\pi}. \qquad (5.50)$$

Endi kanonik funksiyani tuzamiz:

$$\Pi(z)=(1+z)^{\lambda_0}(1-z)^{\lambda_1}$$
$$X(z)=e^{\psi(z)}\Pi(z)=\left(\frac{1-z}{1+z}\right)^{\beta}e^{i\beta\arg(-1-z)}(1+z)^{0}(1-z)^{1}.$$

Shunday qilib,

$$X^{+}(x)=(1+x)^{\beta}(1-x)^{1-\beta}e^{-i\beta\pi}$$
$$X^{-}(x)=(1+x)^{\beta}(1-x)^{1-\beta}e^{i\beta\pi} \qquad (5.51)$$

(5.51) chegaraviy qiymatlarga asosan, (5.46) tenglamani ushbu ko‘rinishda yozib olamiz:

$$\frac{\Phi^{+}(x)}{X^{+}(x)}=\frac{\Phi^{-}(x)}{X^{-}(x)}+\frac{F(x)}{X^{+}(x)}. \qquad (5.52)$$

(5.52) tenglamaning xususiy yechimlaridan biri ushbu ko‘rinishda bo‘ladi:

$$\Phi(z)=\frac{X(z)}{2\pi i}\int_{-1}^{1}\frac{F(t)dt}{t-z}. \qquad (5.53)$$

(5.53) ga asosan, $\varphi(x)$ ni topamiz:

$$\varphi(x)=\Phi^{+}(t)-\Phi^{-}(t)=\cos^{2}\beta\pi F(x)-$$
$$-\frac{\sin 2\beta\pi}{2\pi}\int_{-1}^{1}\left(\frac{1-x}{1-t}\right)^{1-\beta}\left(\frac{1+x}{1+t}\right)^{\beta}\frac{F(t)dt}{t-x}. \qquad (5.54)$$

(5.54) formula (5.34) singulyar integral tenglama yechimini beradi.

Endi $F(x)$ ning ifodasini (5.40) formuladan (5.54) yechimga qo‘yib, ushbu ifodaga ega bo‘lamiz:

$$\varphi(x)=\frac{c}{2\Gamma(1-2\beta)}+\frac{(1+x)^{2\beta}}{2\Gamma(1-2\beta)}\int_{-1}^{x}\frac{\psi_{4}(t)dt}{(x-t)^{2\beta}}-\frac{c\cdot tg\beta\pi}{2\pi\Gamma(1-2\beta)}(1+x)^{\beta}(1-x)^{1-\beta}\times$$

$$\times\int_{-1}^{1}\frac{(1+t)^{-\beta}}{(1-t)^{1-\beta}}\frac{dt}{t-x}-\frac{tg\beta\pi}{2\pi\Gamma(1-2\beta)}(1+x)^{\beta}(1-x)^{1-\beta}\times\int_{-1}^{1}\frac{(1+t)^{\beta}}{(1-t)^{1-\beta}}\frac{dt}{t-x}\int_{-1}^{1}\frac{\psi_{4}(s)ds}{(t-s)^{2\beta}}. \qquad (5.55)$$

Ushbu tenglikka asosan,

$$\int_{-1}^{1}\frac{(1+t)^{-\beta}}{(1-t)^{1-\beta}}\frac{dt}{t-x}=\pi ctg\beta\pi\frac{(1+x)^{-\beta}}{(1-x)^{1-\beta}} \qquad (5.55)$$

ifoda ushbu ko‘rinishda bo‘ladi:

$$\varphi(x)=\frac{(1+x)^{2\beta}}{2\Gamma(1-2\beta)}\int_{-1}^{x}\frac{\psi_{4}(t)dt}{(x-t)^{2\beta}}-\frac{tg\beta\pi}{2\pi\Gamma(1-2\beta)}(1+x)^{\beta}(1-x)^{1-\beta}\times$$
$$\times\int_{-1}^{1}\frac{(1+t)^{\beta}dt}{(1-t)^{2\beta}(t-x)}\int_{-1}^{1}\frac{\psi_{4}(s)ds}{(t-s)^{2\beta}}. \qquad (5.56)$$

(5.56) formulada $\varphi(x)=(1+x)^{2\beta}\tau(x)$ tenglikni hisobga olib, uni ushbu ko‘rinishda yozamiz:

$$\tau(x)=\frac{1}{2\Gamma(1-2\beta)}\int_{-1}^{x}\frac{\varphi_4(t)dt}{(x-t)^{2\beta}}-\frac{tg\beta\pi}{2\pi\Gamma(1-2\beta)}\frac{(1-x)^{1-\beta}}{(1+x)^{\beta}}\times$$

$$\times\int_{-1}^{1}\frac{(1+t)^{\beta}}{(1-t)^{1-\beta}}\frac{dt}{t-x}\int_{-1}^{t}\frac{\psi_4(s)ds}{(t-s)^{2\beta}} \qquad (5.57)$$

bu yerda ushbu

$$\left(\frac{1+t}{1+x}\right)^{\beta}=\left(\frac{1+t}{1+x}\right)^{\beta}\frac{t-x}{1+t}-\left(\frac{1+t}{1+x}\right)^{\beta-1}$$

ayniyatni e'tiborga olib, (5.57) yechimni quyidagi ko'rinishda yozib olamiz:

$$\tau(x)=\frac{1}{2\Gamma(1-2\beta)}\int_{-1}^{x}\frac{\psi_4(t)dt}{(x-t)^{2\beta}}-\frac{tg\beta\pi}{2\pi\Gamma(1-2\beta)}(1+x)^{1-\beta}(1-x)^{1-\beta}\times$$

$$\times\int_{-1}^{1}\frac{(1+t)^{\beta-1}}{(1-t)^{1-\beta}}\frac{dt}{t-x}\int_{-1}^{t}\frac{\psi_4(s)ds}{(t-s)^{2\beta}}-\frac{tg\beta\pi}{2\pi\Gamma(1-2\beta)}\frac{(1-x)^{1-\beta}}{(1+x)^{\beta}}\times$$

$$\times\int_{-1}^{1}\frac{(1+t)^{\beta-1}}{(1-t)^{1-\beta}}dt\int_{-1}^{t}\frac{\psi_4(s)ds}{(t-s)^{2\beta}}. \qquad (5.58)$$

(5.58) tenglikning oxirgi integralida integrallash tartibini o'zgartirib, ushbu tenglikni hosil qilamiz:

$$I=\int_{-1}^{1}\frac{(1+t)^{\beta-1}}{(1-t)^{1-\beta}}dt\int_{-1}^{t}\frac{\psi_4(s)ds}{(t-s)^{2\beta}}=\int_{-1}^{1}\psi_4(s)\int_{s}^{1}\frac{(1+t)^{\beta-1}}{(1-t)^{1-\beta}}\frac{ds}{(t-s)^{2\beta}}.$$

Endi ichki integralda $t=1+(s-1)\sigma$ almashtirish bajarib va gipergeometrik funksiyaning integral ifodasi hamda uning $F(a,b,c;x)=(1-x)^{-a}$ xossasidan foydalanib, ushbu qiymatni hosil qilamiz:

$$I=2^{2\beta-1}\frac{\Gamma(\beta)\Gamma(1-2\beta)}{\Gamma(1-\beta)}\int_{-1}^{1}\frac{\varphi(s)ds}{(1-s^2)^{\beta}}, \qquad (5.59)$$

(5.59) tenglikka asosan, (5.58) yechimni ushbu ko'rinishda yozib olamiz:

$$\tau(x)=\frac{1}{2\Gamma(1-2\beta)}\int_{-1}^{x}\frac{\psi_4(t)dt}{(x-t)^{2\beta}}-\frac{tg\beta\pi}{2\pi\Gamma(1-2\beta)}(1+x)^{1-\beta}\times$$

$$\times\int_{-1}^{1}\frac{(1+t^2)^{\beta-1}}{t-x}dt\int_{-1}^{t}\frac{\psi_4(s)ds}{(t-s)^{2\beta}}-\frac{2^{2\beta-2}tg\beta\pi\Gamma(\beta)}{\pi\Gamma(1-\beta)}\times \tag{5.60}$$

$$\times\frac{(1-x)^{1-\beta}}{(1+x)^{\beta}}\int_{-1}^{1}\frac{\psi_4(s)ds}{(1-s^2)^{2\beta}}.$$

(5.60) yechim uzluksiz bo‘lishi uchun

$$\int_{-1}^{1}\frac{\psi_4(s)ds}{(1-s^2)^2}=0 \tag{5.61}$$

tenglikning bajarilishi zarur. Bu holda (5.34) singulyar integral tenglamaning uzluksiz yechimi ushbu ko‘rinishda bo‘ladi:

$$\tau(x)=\frac{1}{2\Gamma(1-2\beta)}\int_{-1}^{x}\frac{\psi_4(t)dt}{(x-t)^{2\beta}}-\frac{tg\beta\pi}{2\pi\Gamma(1-2\beta)}\int_{-1}^{1}\frac{(1-t^2)^{\beta-1}}{t-x}ds\int_{-1}^{t}\frac{\psi_4(s)ds}{(t-s)^{2\beta}}. \tag{5.62}$$

Shunday qilib, FH masalasi ushbu Dirixle masalasiga olib kelindi: Ω^+ sohada (5.1) tenglamaning ushbu

$$u|_{\sigma_0}=\varphi(x),\ u(x.0)=\tau(x)\,,\ x\in\bar{J} \tag{5.63}$$

shartlarni qanoatlantiruvchi regulyar yechimi $u(x,y)\in C(\overline{\Omega})$ funksiya topilsin. Bu yerda $\tau(x)$ (5.62)formula bilan ifodalanadi va $\varphi(0)=\tau(1),\varphi(l)=\tau(-1)$ shartlari bajariladi. Dirixle masalasining yechimi (2.183) formula bilan beriladi.

Faraz qilaylik, $\psi_4(x)$ ushbu shartlarni qanoatlantirsin:

1. $\psi_4(x)\in C^1[0,1]$;

$$2.\ \int_0^t (1-t)^{1-2\beta}\psi'_2(t)dt = O(1-t)^{1-\beta+\varepsilon} \tag{5.64}$$

ε -ixtiyoriy yetarli kichik son.

(5.62) formulada bo‘laklab integrallash amalini bajarib,

$$\tau(x) = \frac{\psi_4(-1)}{2\Gamma(2-2\beta)}(1+x)^{1-2\beta} + \frac{1}{2\Gamma(2-2\beta)}\times$$
$$\times \int_{-1}^{x}\psi'_4(t)(x-t)^{1-2\beta}dt - \frac{tg\beta\pi\psi_4(-1)(1-x^2)^{1-\beta}}{2\pi\Gamma(2-2\beta)}\times$$
$$\times \int_{-1}^{1}\frac{(1+t)^{-\beta}(1-t)^{\beta-1}}{t-x}dt - \frac{tg\beta\pi(1-x^2)^{1-\beta}}{2\pi\Gamma(2-2\beta)}\times \tag{5.65}$$
$$\times \int_{-1}^{1}\frac{(1-t^2)^{\beta-1}}{t-x}dt\int_{-1}^{1}\psi'_4(s)(t-s)^{1-2\beta}ds$$

tenglikni hosil qilamiz.

Ushbu

$$\int_{-1}^{1}\frac{(1+t)^{-\beta}(1-t)^{\beta-1}}{t-x}dt = \frac{\pi ctg\beta\pi}{(1+x)^{\beta}(1-x)^{1-\beta}}$$

tenglikni hisobga olib, (5.65) munosabatni ushbu ko‘rinishda yozib olamiz:

$$\tau(x) = \frac{1}{2\Gamma(2-2\beta)}\int_{-1}^{x}\psi'_4(t)(x-t)^{1-2\beta}dt - \frac{tg\beta\pi(1-x^2)^{1-\beta}}{2\pi\Gamma(2-2\beta)}\int_{-1}^{1}\frac{(1-t^2)^{\beta-1}}{t-x}dt\int_{-1}^{1}\psi'_4(s)(t-s)^{1-2\beta}ds. \tag{5.66}$$

Bu yerdan (5.64) ni hisobga olib, $\tau(x)\in C[-1,1]\cap C^1(-1,1)$ va $\tau'(x)$ hosila (-1,1) interval chegaralarida β dan katta bo‘lmagan tartibda cheksizlikka aylanishini ko‘rish qiyin emas, shu bilan birga $\tau(-1)=\tau(1)=0$

Endi $\nu(x)$ ni topamiz. Buning uchun, (2.183) formula yordamida Dirixle masalasini yechamiz. Bu yechimdan y bo‘yicha hosila olib, ushbu tenglikka kelamiz:

$$\frac{\partial u}{\partial y}=k_2(1-\beta_0)\int_{-1}^{1}\tau(t)\frac{\partial}{\partial y}y^{\beta_0}\left\{\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}-\right.$$

$$\left.-\left[(1-xt)^2+\frac{4t^2}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right\}dt+k_2(1-\beta_0)(m+2)\int_{-1}^{1}\varphi(t)\times$$

$$\times\frac{\partial}{\partial y}\left\{(1-R^2)y^{1-\beta_0}\left(r_1^2\right)^{\beta-2}F(1-\beta,2-\beta,2-2\beta;1-\sigma)\right\}dt. \quad (5.67)$$

Ushbu tenglikning to‘g‘riligini bevosita tekshirib ko‘rish mumkin:

$$\frac{\partial}{\partial y}\left\{y^{1-\beta_0}\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right\}=$$

$$=\frac{m+2}{2}y^{-\beta_0}\frac{\partial}{\partial t}\left\{(x-t)\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right\}. \quad (5.68)$$

Endi (5.67) tenglikning o‘ng tomonidagi birinchi integralda (5.68) tenglikni e’tiborga olib, bo‘laklab integrallash operatsiyasini bajaramiz va hosil bo‘lgan yangi tenglikni y^{β_0} ga ko‘paytirib, u nolga intilganda limitga o‘tib, $\nu(x)$ ni topamiz:

$$\nu(x)=-k_2(1-\beta_0)\frac{m+2}{2}\left\{\frac{\tau(1)}{(1-x)^{1-2\beta_0}}+\frac{\tau(-1)}{(1+x)^{1-2\beta_0}}+\int_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\beta_0}}-\right.$$

$$\left.-(2\beta-1)\int_{-1}^{1}\frac{\tau(t)dt}{(1-xt)^{2-2\beta}}\right\}+\Phi(x), \quad (5.69)$$

bu yerda

$$\Phi(x)=k_2(1-\beta)(1-\beta_0)(m+2)\,(1-x^2)\int_{-1}^{1}(x^2-2xt+1)^{\beta-1}\varphi(t)dt.$$

FH masalasida $u(x,y)$ funksiyaning σ_0 normal chiziqdagi qiymati $\varphi(x)$ ni ushbu ko‘rinishda ifodalash mumkin deb faraz qilamiz:

$$\varphi(x)=y\varphi_1(x),\quad (x,y)\in\sigma_0$$

bu yerda $\varphi_1(x) \in C^{(0,\alpha)}[-1,1]$. U holda normal chiziqning $\sigma_0 : x^2 + \frac{4}{(m+2)^2} y^{m+2} = 1$ tenglamasidan foydalanib, $\Phi(x)$ ni quyidagicha tasvirlaymiz:

$$\Phi(x) = 2k_2(1-\beta)(1-\beta_0)\left(\frac{m+2}{2}\right)^{(m+4)(m+2)} \int_{-1}^{1} \frac{(1-t^2)^{2/m+2}\varphi_1(t)dt}{(x^2-2xt+1)^{2-\beta}}$$

$\Phi(x)$ funksiya ifodasidan ko‘rinib turibdiki, $\Phi(x) \in C[-1,1]$ va $(-1,1)$ intervalda $\Phi(x)$ ixtiyoriy tartibli hosilasiga ega. (5.69) formuladan ko‘rinib turibdiki, 2.2 teoremaga asosan, $\nu(x)$ funksiyamiz (-1,1) intervalda $\beta-\varepsilon$ ko‘rsatkich bilan Gyolder shartini qanoatlantiradi.

Endi $\tau(x)$ va $\nu(x)$ ma’lum bo‘lgandan keyin Ω^- sohada yechimni shakli o‘zgargan Koshi masalasi yechimi sifatida tiklaymiz. Bu yechim R_1 sinfga tegishli.

A D A B I YO T L A R

1. Алимов Ш.А. Об одной спектральной задаче типа задачи Бицадзе-Самарского // ДАН СССР.1986, т. 287, т. No 6.с. 1289-1290.

2. Бейтмен Г.,ЭрдейиА. Высшие трансцендентные функции.Гипергеометрическая функция. Функция Лежандра. М.: Наука. 1965. Т.1.-296 с.

3. Бердышев А.С.,Тойжонова Г.Д. Краевая задача с наклонной производной для параболо-гиперболического уравнения в области с отходом от характеристики / / Известия НАН Республики Казахстан. Серия физ.-мат, наук, 1995, N 5, с. 13-20.

4. Бицадзе А.В. Уравнения смешанного типа. М.: Изд-во АН СССР. 1959,-164 с.

5. Бицадзе А.В. К теории одного класса уравнений смешанного типа. В сб. «Некоторые проблемы математики и механики». Л.: Наука, 1970, с. 112-119.

6. Бицадзе А.В. К теории уравнений смешанного типа, порядок которых вырождается вдоль линии изменения типа. В сб. «Механика сплошной среды и родственные проблемы анализа». М.: Наука, 1972, с.47-52.

7. Гахов Ф.Д. Краевые задачи. М.: Наука, 1977, -640 с.

8. Гахов Ф.Д., Черский Ю.Н. Уравнения типа свертки. М.: Наука,1978,-269 с.

9. Градштейн И.С., Рыжик И.М. Таблицы интегралов, сумм, рядов и произведений. М.: Наука 1971,-1108 с.

10. Девингталь Ю.В. К вопросу о существовании и единственностирешения задачи Франкля. // УМН XIV, 1(85), 1959, 177-182 с.

11. Джураев Т.Д. Краевые задачи для уравнений смешанного и смешанно-составного типов.Ташкент. Фан, 1979, -238 с.

12. Елеев В.А. О некоторых краевых задачах для одного вырождающегося гиперболического уравнения второго порядка // Дифференциальные уравнения. 1976, т. 12, No 1. с. 46-58.

13. Кальменов Т.Ш. Критерий единственности решения задачи Дарбу для одного вырождающегося гиперболического уравнения // Дифференциальные уравнения. 1971, т. 7, No 1. с.178-181.

14. Кальменов Т.Ш., Садыбеков М.А. О задаче Дирихле и нелокальных краевых задачах для волнового уравнения // Дифференциальные уравнения. 1990, т.26, No 1. с. 60-65.

15. Кожанов А.И. Краевые задачи для уравнений математической физики нечётного порядка. Новосибирск. Изд-во Института математика им.С.Л.Соболева СО РАН. 1990,-132 с.

16. Линь Цзянь-бин. О некоторых задачах Франкля // Вестник ЛГУ, сер. Матем, мех., и астр. 3, 13, 1961, 18-39 с.

17. Мирсабуров М. Задача типа задачи Бицадзе-Самарского для одного класса уравнений смешанного типа. / / Дифференциальные уравнения. 1995, т.35, No5. с.829-834.

18. Мирсабуров М. Об одном простом обобщении задачи Трикоми // Узбекский математический журнал. 1996, No 4.с. 46-54.

19. Мирсабуров М. Задача Бицадзе-Самарского для вырождающегося эллиптического уравнения. / / Докл. АН РУз. 1998 , No2. с. 6-9.

20. Мирсабуров М., Салахитдинов М.С. Об аналоге задачи Бицадзе-Самарского для одного класса уравнений гиперболического типа // Докл. РАН 1999, No 1. с.10-12.

21. Мирсабуров М. Нелокальная краевая задача для уравнения Геллерстедта / / Математические заметки.2000. т.67, вып. 5, с. 721-729.

22. Мирсабуров М. Краевая задача для одного класса уравнений смешанного типа с условием Бицадзе-Самарского на параллельных характеристиках / / Дифференциальные уравнения. 2001, т.37, No 9. с. 1281-1284.

23. Мирсабуров М. Нелокальная краевая задача для вырождающегося эллиптического уравнения // Дифференциальные уравнения. 2002, т.38, No1. с. 129-131.

24. Михлин С.Г. Об интегральном уравнении F.TRICOMI / / ДАН СССР. 1948, Т.59,No 6. с. 1053-1056.

25. Михлин С.Г. Лекции по линейным интегральным уравнениям. М.:Физ.мат.гиз, 1959.-232 с.

26. Моисеев Е.И. Уравнения смешанного типа со спектральным параметром. М.: Изд-во МГУ 1988.-149 с.

27. Нахушев А.М. Уравнения математической биологии. 1995. М.: Высшая школа.-301 с.

28. Пономарев С.М. К задаче на собственные значения для уравнения Лаврентьева-Бицадзе / / ДАН СССР. 1977, Т.233, No 1, с. 39-40 .

29. Салахитдинов М.С. Уравнения смешанно-составного типа.Ташкент: Фан. 1974-156с.

30. Салоҳиддинов М. Математик физика тенгламалари.Тошкент-«Ўзбекистон» нашриёти, 2002й.-448 б.

31. Салахитдинов М.С., Мирсабуров М. Нелокальние задачи для уравнений смешанного типа с сингулярными коэффициентами. 2005. “Universitet”. “Yangiyo‘lpoligrafservis”-224 с.

32. Салахитдинов М.С., Мирсабуров М. О некоторых краевых задачах длягиперболического уравнения вырождающегося внутри области / / Дифференциальные уравнения. 1981, т 17, No 1. с. 129-136.

33. Салахитдинов М.С., Мирсабуров М. О двух нелокальных краевых задачах для вырождающегося гиперболического уравнения //Дифференциальные уравнения. 1982, т 18, No 1. с. 116-127.

34. Салахитдинов М.С., Бердышев А. Краевые задачи для параболо-гиперболического уравнения в области с отходом от характериситики //ДАН СССР. 1992, т. 327.No 3. с. 303-305.

35. Салахитдинов М.С.,Уринов А.К. Краевые задачи для уравнений смешанного типа со спектральным параметром. Ташкент. Фан, 1997, –165 с.

36. Салахитдинов М.С., Мирсабуров М. Об одном аналоге задачи БицадзеСамарского для вырождающегося эллиптического уравнения./ / Сибирский математический журнал, 1999, т. 40 No 1. с. 177-182.

37. Салахитдинов М.С., Мирсабуров М. Задачи Бицадзе-Самарского для одного класса вырождающегося гиперболических уравнений //Дифференциальные уравнения.2002, т. 38, No 2, с. 271-276.

38. Смирнов М.М. Уравнения смешанного типа. М.: Наука. 1970,-296 с.

39. Солдатов А.П. Одномерные сингулярные оператора и краевые задачи теории функции М.: Наука, 1991.

40. Трикоми Ф. О линейных уравнениях в частных производных второго порядка смешанного типа М.-Л. Гос.тех.издат 1947,-192 с.

41. Франкль Ф.И. Обтекание профилей газом с местной сверхзвуковой зоной, оканчивающейся прямим скачком уплотнения //Прикладная математика и механика 1996 т. 20, №2 с.196-202 .

42. Gellerstedt S. Quelgues problemes mixted pour l’eguation $y^m z_{xx} + z_{yy} = 0$ / / Arkiv for matematik, astronomi och fysik.1938, vand 26 A.No 3, p. 1-32.

43. Holmgren E. Sur un porleme aux limites pour l’eqution $y^m z_{xx}+z_{yy}=0$// A Arkiv for matematik, astronomi och fysik.1927, vand 19B.No 14.

M U N D A R I J A

O‘QUV ADABIYOTINING
NASHR RUXSATNOMASI

Termiz davlat universiteti rektorining 20 22 yil "21" noyabr dagi "233-U" sonli buyrug'iga asosan

M.Mirsobirov
(muallifning familiyasi, ismi-sharifi)

60540100-Matematika, 70540101-Matematika
(ta'lim yo'nalishi (mutaxassisligi))
(Differensial tenglamalar va matematik fizika) ning talabalari va magistratura mutaxassisligi uchun tavsiya etilgan

Singulyar koeffitsiyentli Gellerstedt tenglamasi uchun
(o'quv adabiyotining nomi va turi: darslik, o'quv qo'llanma)
trikomi masalasi nomli o'quv qo'llanma

O'zbekiston Respublikasi Vazirlar Mahkamasi tomonidan litsenziya berilgan nashriyotlarda nashr etishga ruxsat berildi.

Rektor ____________ A. Maraximov

Ro'yxatga olish raqami
223-U-003

Qaydlar uchun

Qaydlar uchun

Qaydlar uchun

Qaydlar uchun

M. Mirsaburov

SINGULYAR KOEFFITSIYENTLI GELLERSTEDT TENGLAMASI UCHUN TRIKOMI MASALASI

(O'quv qo'llanma)

Ushbu o'quv qo'llanma Termiz davlat universiteti ilmiy kengashining 3.11.2022-yilda bo'lib o'tgan yig'ilishi tomonidan 3-bayonnoma bilan nashrga tavsiya etilgan.

Muharrir: Alisher MUQIMOV
Mas'ul muharrir: Qo'shon ABDULLAYEV
Texnik muharrir: Juma SHAYMATOV
Musahhih: Gulchehra RAHMATOVA
Sahifalovchi: Zulfiya TURSUNOVA

Nashriyot litsenziyasi AI № 013. 20.07.2018.
Bosishga 26.12.2022-yilda ruxsat etildi. Bichimi 60x84 $^1/_{16}$. Ofset qog'ozi. TimesUZ garniturasi. Shartli bosma tabog'i 10. Nashr bosma tabog'i 9,3. 289-sonli shartnoma.
16-sonli buyurtma. Adadi 300 nusxada. Erkin narxda.

NASHRIYOT VA BOSMAXONA MANZILI:
"Surxon-nashr" nashriyoti, Termiz shahri,
"Ayritom" ko'chasi, 25/15-uy.

Tel: (+998 55) 451-24-74