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**p -ADIC PERIODIC GIBBS MEASURES FOR SOME CLASSICAL
MODELS OF STATISTICAL MECHANICS**

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CONTENTS

	INTRODUCTION (abstract of PhD thesis).....	3
CHAPTER I.	ON P -ADIC ANALYSIS AND P -ADIC MEASURES.....	10
§ 1.1.	Congruences and residues.....	10
§ 1.2.	On p -adic analysis.....	12
§ 1.3.	Non-Archimedean Gibbs measure.....	19
§ 1.4.	On G_2 -periodic p -adic quasi Gibbs measures of the Potts model on a Cayley tree.....	23
	Conclusion of the first chapter.....	40
CHAPTER II.	P -ADIC GENERALIZED GIBBS MEASURES FOR THE ISING MODEL ON THE CAYLEY TREE.....	41
§ 2.1.	Main results about p -adic Gibbs measures for the Ising model on the Cayley tree.....	41
§ 2.2.	Translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three.....	43
§ 2.3.	$G_k^{(2)}$ -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three	50
	Conclusion of the second chapter.....	62
CHAPTER III.	NON-PERIODIC P -ADIC GIBBS MEASURES FOR THE ISING AND POTTS MODELS ON THE CAYLEY TREE.....	63
§ 3.1.	p -adic ART quasi Gibbs measures for the Potts model on the Cayley tree	63
§ 3.2.	ART, (k_0) -translation-invariant, (k_0) -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree.....	69
§ 3.3.	Some constructive p -adic generalized Gibbs measures for the Ising model on the Cayley tree.....	80
	Conclusion of the third chapter.....	96
	CONCLUSION.....	97
	REFERENCES.....	99

INTRODUCTION (abstract of PhD thesis)

Actuality and demand of the theme of dissertation. Many scientific and practical researches devoted to the study of thermodynamic properties of physical and biological systems conducted around the world are often led to the investigation of Archimedean and non-Archimedean models of statistical mechanics. p -adic Gibbs measures are the main object of the theory of phase transitions for non-Archimedean models of statistical mechanics. The study of Gibbs measures is important in solving problems in various fields of science and technology, such as chemistry, physics, service theory, material science, biology. Due to the complexity of fully describing the set of all p -adic Gibbs measures corresponding to the non-Archimedean models of statistical mechanics, the problem of constructing a sufficient number of Gibbs measures corresponding to the given model remains relevant.

Nowadays in the world, one of the actual problems is to determine the existence of p -adic Gibbs measures for the Hamiltonian given on lattice systems, in particular, tree-structured countable graphs, and to describe the set of all p -adic Gibbs measures. One of the important problems of the theory of phase transitions in statistical physics is the problem of determining the existence of at least two p -adic periodic Gibbs measures in the set of all p -adic Gibbs measures. Therefore, describing the set of all periodic and translational-invariant p -adic Gibbs measures for a given Hamiltonian, checking such measures for boundedness, and determining the existence of a phase transition for p -adic generalized Gibbs measures are purposeful scientific researches.

It is known that in our country, in our country, much attention has been paid to develop important directions of scientific and practical application of fundamental sciences, which is one of the main bases of development. In particular, in recent years, significant results have been achieved in the study of translation-invariant, periodic and some non-periodic Gibbs measures generated by constructive methods for classical models of statistical mechanics in the field of

real and p -adic numbers on the Cayley tree. The investigations on the international level in such important areas as the mathematical physics, non-Archimedean analysis, theory of non-Archimedean measures and theory of dynamical systems are considered as the main task of fundamental research¹. In this way, the development of phase transitions theory for the classical models of statistical mechanics plays a crucial role in the implementation this decree.

The subject and object of research of this dissertation are in line with tasks identified in the Decrees and Resolutions of the President of the Republic of Uzbekistan of February 7, 2017, PF-4947 , “On the strategy of action for the further development of the Republic of Uzbekistan”, PQ-4387 dated July 9, 2019 “On state support for the further development of mathematics education and science, as well as measures to radically improve the activities of the Institute of Mathematics named after V.I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan”, PQ-4708 of May 7, 2020 “On measures to improve the quality of education and research in the field of mathematics” as well as in other regulations related to basic sciences.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the priority areas of science and technology of Republic of Uzbekistan IV, “Mathematics, Mechanics and Computer Science”.

The degree of scrutiny of the problem. The concept of Gibbs distribution, which is important for systems with a constant temperature and in thermal equilibrium with the environment, was introduced by the American scientist J.U. Gibbs. The general characteristics of real-valued limit Gibbs measures are given in the works of R.L. Dobrushin, O. Lanford and D. Ruelle. Modern theory of real-valued Gibbs measures is highlighted in works of R. Baxter, H.O. Georgii, V.A. Malishov, R.A. Minlos, K. Preston, D. Ruelle, Ya.G. Sinai, G. Gallavotti, F. Bonetto, J. Gentile, Jin Zin-Justin, N.N. Ganikhodjaev, U.A. Rozikov and F.M.

¹ Decree of Cabinet of Ministers of the Republic of Uzbekistan at the 2017 year 18 May “On measures on the organization of activities of the first created scientific research institutions of the Academy of Sciences of the Republic of Uzbekistan” № 292

Mukhamedov. R.L. Dobrushin proved the theorem about the existence of a real-valued limit Gibbs measure. Gibbs measures for classical models of statistical mechanics are studied by R.L. Dobrushin, G.R. Brightwell, P. Winkler, Yu.M. Suhov, J. Martin, N.N. Ganikhodjaev, U.A. Rozikov, F.M. Mukhamedov, D. Gandolfo, M.M. Rahmatullaev, R.M. Khakimov, G.I. Botirov, O.N. Khakimov, E. Normatov, Sh. Shoyusupov, F. Haydarov, M. Rasulova and others.

In the work of N.N. Ganikhodjaev, F.M. Mukhamedov, and U.A. Rozikov, the uniqueness of the p -adic Gibbs distribution for the Ising model on the lattice \mathbb{Z} was proved. The non-uniqueness of the p -adic Gibbs measure for the Ising-Vannimenus model was shown in the work of N.N. Ganikhodjaev and H.Akin. For the Vannimenus model, translation-invariant and periodic p -adic quasi Gibbs measures on the Cayley tree of order two were studied in the work of O.N. Khakimov. These works were continued for the Ising-Vannimenus model in the work of F.M.Mukhamedov, M.K. Saburov and O.N. Khakimov.

The existence of a unique p -adic Gibbs measure for the Ising model was proved by M. Khamraev, F.M. Mukhamedov. Translation-invariant p -adic generalized Gibbs measures were studied in the works of F.M. Mukhamedov and O.N. Khakimov. The boundedness criterion of the translation-invariant p -adic generalized Gibbs measures was found and it was proved that there does not occur the strong phase transition for these measures. Chaotic problems for p -adic generalized Gibbs measures was studied by F.M. Mukhamedov, O.N. Khakimov, H. Akin, M. Dogan.

In the field of real numbers, using the constructive method for the Ising model the existence of uncountable non-periodic extreme Gibbs measures was proved by P. Bleher and N.N. Ganikhodjaev. H. Akin, U.A. Rozikov and S. Temir studied Gibbs measures which are called ART measures. U.A. Rozikov and M.M. Rahmatullaev studied (k_0) -translation-invariant and (k_0) -periodic Gibbs measures.

Note that, despite there are many scientific works as mentioned above, the full classification of all limiting Gibbs measures for any model on the Cayley tree has not been obtained yet.

Connection of the theme of the dissertation with the research works of scientific research Institute, where the dissertation is carried out. Dissertation research was carried out within the “Fundamental research” branch of the research plan of Namangan state university.

The aim of research work contains describing the limit p -adic translation-invariant, periodic and non-periodic Gibbs measures for Ising and Potts models with a finite set of spin values on the Cayley tree, analyzing the boundedness of these measures.

Research problems are:

to determine sufficient conditions for the existence of translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three;

to find the conditions of the existence of G_2 -periodic p -adic quasi Gibbs measures for the three-state Potts model on the Cayley tree of order two and to check the existence of the phase transitions;

to construct ART p -adic quasi Gibbs measures on the high-order Cayley tree using translation-invariant solutions of the functional equation corresponding to the three-state Potts model on the binary tree and to check these measures for boundedness;

to construct non-periodic ART, (k_0) -translation-invariant and (k_0) -periodic p -adic generalized Gibbs measures on a higher-order Cayley tree tree using translation-invariant and $G_k^{(2)}$ -periodic p -adic Gibbs measures for the Ising model on the Cayley tree of order two and three.

The research object: p -adic Ising and Potts models on the Cayley tree.

The research subject: Theory of groups and graphs, Gibbs measures theory, Algebra and number theory, theory of non-linear discrete dynamical systems, p -adic analysis, non-linear Markov processes.

Research methods: In the research the methods of p -adic analysis, number theory, functional analysis, combinatorics, group theory, measures theory, theory of linear algebra and dynamical systems are used.

The scientific novelty of the research consists of the followings:
The sufficient conditions were found for the existence of the translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three;

the existence conditions of the periodic p -adic quasi Gibbs measures for the three-state Potts model on the Cayley tree of order two were found and the existence of phase transitions was proved;

using translation-invariant solutions of the functional equations for the 3-state Potts model on the lower-order Cayley tree, the construction of p -adic quasi Gibbs measures on a higher-order Cayley tree was given and the boundedness conditions of these measures was found;

using translation-invariant and two periodic solutions of the functional equations for the Ising model on the lower-order Cayley tree, non-periodic various p -adic generalized Gibbs measures on the higher-order Cayley tree were constructed.

Practical results of the research consists of the followings:
The obtained results and the methods used in the dissertation can be taught as a course for graduate students of higher education and after the higher education. Also, the exact expression of parameter values that ensure the existence of the different phase transitions in a set of periodic p -adic generalized Gibbs measures can be used to solve service theory problems.

The reliability of the results of the study. The results have been obtained by using the methods of p -adic analysis, non-Archimedean functional analysis,

non-Archimedean Markov theory of random fields, p -adic probability theory, number theory, discrete dynamical systems, Gibbs and non-Archimedean measure theory. The obtained results are mathematically strongly proved.

Scientific and practical significance of research results. The scientific importance of the results of the research work is explained by the fact that various p -adic models of statistical mechanics can be used in the developing of Gibbs measures theory.

The practical significance of the research determines by the changing of state of physical systems is investigated and it gives opportunity to solve some problems of combinatorics and telecommunications.

Implementation of the research results. The scientific results obtained during the research of the dissertation are implemented in the following research projects:

The obtained results for the p -adic Ising model on the Cayley tree in the foreign grant project number G0003247 “Chaotic and mixed p -adic dynamical systems related to renormalized groups of lattice models” is used to investigation of p -adic generalized Gibbs measures for the Ising model on Cayley trees (United Arab Emirates University Bulletin dated 4 September 2023, UAE). The application of scientific results made it possible to open new insight into the theory of p -adic dynamical systems and practical studies to study the thermodynamic properties of physical systems.

The proof methods of the existence of translation-invariant and periodic p -adic generalized Gibbs measures for the Potts and Ising models are used in the fundamental project YOT-FTEX-2018-154 “Spectra of Hamiltonians and Gibbs measures on lattices Z^d and Cayley trees Γ^k ” to investigate existence of translation-invariant and periodic p -adic generalized Gibbs measures for some classical models of statistical mechanics with continuum spin value (Reference No. 04/11-5431 of the National university of Uzbekistan named after Mirzo Ulugbek dated September 9, 2023). The application of the scientific work made it possible

to expand the set of translation-invariant and periodic Gibbs measures for some models of statistical mechanics with continuum spin values.

Approbation of the research results. The main results of the research have been discussed in 3 international and 4 national scientific conferences.

Publications of the research results. On the topic of the dissertation 6 research papers have been published in the scientific journals, all of them are included in the list of journals proposed by the Higher Attestation Commission of the Republic of Uzbekistan for defending the PhD thesis, including 5 in foreign and 1 in Republic journals.

The structure and volume of the dissertation. The dissertation consists of an introduction, three chapters, conclusion and bibliography. The volume of the thesis is 111 pages.

CHAPTER I. ON p -ADIC ANALYSIS AND p -ADIC MEASURES

This chapter consists of two main parts. The first part contains paragraphs 1.1, 1.2, 1.3 and the second part includes paragraph 1.4. The first part is devoted to present the main definitions and known results. In this part, we present main concepts and results of number theory, p -adic analysis and non-Archimedean measure theory. The second part is devoted to G_2 -periodic p -adic quasi Gibbs measures for Potts model on a Cayley tree.

§ 1.1. Congruences and residues

The concepts and the results in this paragraph are given in [87].

Definition 1.1.1. Let m be a positive integer. If a and b are integers, we say that a is *congruent to b modulo m* , if m divides $(a - b)$.

If m divides c we write $m|c$, if m does not divide c we write $m \nmid c$. If a is congruent to b modulo m , we write $a \equiv b \pmod{m}$. If $m \nmid (a - b)$, we write $a \not\equiv b \pmod{m}$, and say that a and b are *incongruent modulo m* .

The greatest common divisor of a and b denoted by (a, b) .

Proposition 1.1.1. If a, b, c, d, l, k and m are integers such that $m > 0$, $d = (k, m)$, $a \equiv b \pmod{m}$, and $c \equiv l \pmod{m}$ then

- 1) $a + c \equiv b + l \pmod{m}$,
- 2) $ac \equiv bl \pmod{m}$,
- 3) $ak \equiv bk \pmod{m}$,
- 4) if $k \neq 0$ then $ak \equiv bk \pmod{m|k|}$ (here $|k|$ is absolute value of k),
- 5) if $k \neq 0$, $ak \equiv bk \pmod{m}$ then $a \equiv b \left(\pmod{\frac{m}{d}} \right)$.

Proposition 1.1.2. Let a and b are integers with $d = (a, b)$. The equation $ax + by = c$ has no integer solutions if $d \nmid c$. If $d|c$, then there are infinitely many integer solutions. Moreover, if $x = x_0, y = y_0$ is a particular solutions of the equation, then all solutions are given by

$$x = x_0 + \frac{b}{d}n, \quad y = y_0 - \frac{a}{d}n \quad (1.1.1)$$

where n is an integer.

Theorem 1.1.1. Let a , b and m are integers such that $m > 0$, $d = (a, m)$.

If $d \nmid b$ then the linear congruence

$$ax \equiv b \pmod{m} \quad (1.1.2)$$

has no solutions. If $d|b$ then the linear congruence (1.1.2) has exactly d incongruent solutions modulo m .

Corollary 1.1.1. Let a and m are relatively prime integers with $m > 0$ and b is an integer, then the linear congruence (1.1.2) has a unique solution modulo m .

We now present the main theorem concerning the solution of systems of linear congruences in one unknown. This theorem is called the *Chinese remainder theorem*, most likely because of the contributions of Chinese mathematicians such as Ch'in Chiu-Shao to its solution.

Theorem 1.1.2 (The Chinese Remainder Theorem). Let m_1, m_2, \dots, m_r be pairwise relatively prime positive integers. Then the system of congruences

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \vdots \\ x \equiv a_r \pmod{m_r} \end{cases} \quad (1.1.3)$$

has a unique solution modulo $M = m_1 m_2 \dots m_r$. Moreover,

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \dots + a_r M_r y_r, \quad (1.1.4)$$

where $M_k = \frac{M}{m_k}$, y_k is a solution of the linear congruence

$$M_k y \equiv 1 \pmod{m_k}, \quad k = \overline{1, r}.$$

Definition 1.1.2. If m is a positive integer, we say that an integer a is a *quadratic residue* of m if $(a, m) = 1$ and the congruence

$$x^2 \equiv a \pmod{m} \quad (1.1.5)$$

has a solution. If the congruence (1.1.5) has no solution, we say that a is a *quadratic nonresidue* of m .

Theorem 1.1.3. If p is an odd prime, then the following assertions hold:

- 1) The congruence $x^2 \equiv -1 \pmod{p}$ is solvable iff $p \equiv 1 \pmod{4}$;
- 2) The congruence $x^2 \equiv 2 \pmod{p}$ is solvable iff $p \equiv \pm 1 \pmod{8}$;
- 3) The congruence $x^2 \equiv -2 \pmod{p}$ is solvable iff $p \equiv 1$ or $p \equiv 3 \pmod{8}$;
- 4) The congruence $x^2 \equiv 3 \pmod{p}$ is solvable iff $p \equiv \pm 1 \pmod{12}$;
- 5) The congruence $x^2 \equiv -3 \pmod{p}$ is solvable iff $p \equiv 1 \pmod{6}$.

§ 1.2. On p -adic analysis

Metric space and normed field. If X is a nonempty set, a distance, or *metric*, on X is a function d from pairs of elements (x, y) of X to the nonnegative real numbers such that

- 1) $d(x, y) = 0$ if and only if $x = y$,
- 2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- 3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A set X together with a metric d is called a *metric space* and denoted by (X, d) .

The metrics d we shall be dealing with will come from *norms* on the field F , which means a map denoted $\|\cdot\|$ from F to the nonnegative real numbers such that

- 1) $\|x\| = 0$ if and only if $x = 0$,
- 2) $\|xy\| = \|x\| \|y\|$ for all $x, y \in F$,

3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in F$ (*the triangle inequality*).

A field F together with a norm $\|\cdot\|$ is called a *normed field* and denoted by $(F, \|\cdot\|)$.

When we say that a metric d “comes from” (or is “*induced by*”) a norm $\|\cdot\|$, we mean that d is defined by: $d(x, y) = \|x - y\|$.

Definition 1.2.1. A sequence $\{a_n\}$ in the field F is said to be

- *bounded*, if there is a constant $C > 0$ such that

$$\|a_n\| \leq C, \forall n \in \mathbb{N};$$

- a *null sequence*, if

$$\lim_{n \rightarrow \infty} \|a_n\| = 0,$$

i.e. for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > N, n \in \mathbb{N}$ we have

$$\|a_n\| < \varepsilon;$$

- a *Cauchy (or fundamental) sequence*, if

$$\lim_{n, m \rightarrow \infty} \|a_n - a_m\| = 0,$$

i.e. for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n, m > N, n, m \in \mathbb{N}$ we have

$$\|a_n - a_m\| < \varepsilon;$$

- *convergent* to $a \in F$ (we write $\lim_{n \rightarrow \infty} a_n = a$) if

$$\lim_{n \rightarrow \infty} \|a_n - a\| = 0,$$

i.e. for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > N, n \in \mathbb{N}$ we have

$$\|a_n - a\| < \varepsilon.$$

Definition 1.2.2. We say that two metrics d_1 and d_2 on field F are

equivalent (denoted by $d_1 \sim d_2$), if a sequence is Cauchy respect to d_1 if and only if it is Cauchy respect to d_2 . We say two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ ($\|\cdot\|_1 \sim \|\cdot\|_2$) if they induced by equivalent metrics [29, 50].

Proposition 1.2.1. [29, 50] Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on field F . They are equivalent if and only if there exists a positive real number α such that

$$\|x\|_2 = \|x\|_1^\alpha, \forall x \in F. \quad (1.2.1)$$

A basic example of a norm on the rational number field \mathbb{Q} is the absolute value $|x|$. The induced metric $d(x, y) = |x - y|$ is the usual concept of distance on the number line.

Definition 1.2.3. A norm is called *non-Archimedean*, if it satisfies the *strong triangle inequality*, i.e.

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (1.2.2)$$

for all $x, y \in F$. A metric is called *non-Archimedean* or *ultra-metric* if

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad (1.2.3)$$

for all $x, y \in F$.

In particular, a metric is non-Archimedean if it is induced by a non-Archimedean norm, since in that case

$$\begin{aligned} d(x, y) &= \|x - y\| = \|x - z + z - y\| \leq \\ &\leq \max\{\|x - z\|, \|z - y\|\} = \max\{d(x, z), d(z, y)\}. \end{aligned}$$

Remark 1.2.1. It is easy to check that the strong triangle inequality yields the triangle inequality.

p -adic norm and p -adic numbers. One metric on \mathbb{Q} , that induced by the ordinary absolute value. Are there any others? Theorem 1.2.1 answers this question.

Definition 1.2.4. Let p be any prime number. For any nonzero integer a , let $\text{ord}_p a$ be the highest power of p which divides a , i.e., the greatest m such that

$$a \equiv 0 \pmod{p^m}.$$

If $a = 0$, we agree to write $\text{ord}_p a = \infty$.

Now for any rational number $x = \frac{a}{b}$, define $\text{ord}_p x = \text{ord}_p a - \text{ord}_p b$. It is easy to check that $\text{ord}_p x$ is well-defined.

Lemma 1.2.1. [50]. Let x and y are non-zero rational numbers. Then

- 1) $\text{ord}_p (xy) = \text{ord}_p x + \text{ord}_p y$,
- 2) $\text{ord}_p \left(\frac{x}{y} \right) = \text{ord}_p x - \text{ord}_p y$,
- 3) $\text{ord}_p (x \pm y) \geq \min \{ \text{ord}_p x, \text{ord}_p y \}$.

Further define a map $|\cdot|_p$ on \mathbb{Q} as follows:

$$|x|_p = \begin{cases} p^{-\text{ord}_p x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \quad (1.2.4)$$

Proposition 1.2.2. [50] The map defined by (1.2.4.) is a norm. Furthermore, this norm is a non-Archimedean norm, i.e.

$$|x + y|_p \leq \max \{ |x|_p, |y|_p \}$$

for all $x, y \in \mathbb{Q}$.

The norm defined by (1.2.4) is called *p-adic norm*.

The strong triangle inequality property follows that

- 1) if $|x|_p \neq |y|_p$ then $|x \pm y|_p = \max \{ |x|_p, |y|_p \}$;
- 2) if $|x|_p = |y|_p$ then $|x \pm y|_p \leq \max \{ |x|_p, |y|_p \}$.

The completion of \mathbb{Q} with respect to the p -adic norm defines the p -adic field \mathbb{Q}_p [29, 50, 98, 102].

Theorem 1.2.1 [Ostrowski's Theorem] [50]. Every non-trivial norm on \mathbb{Q} is equivalent to either p -adic norm for some p , or absolute value.

Ostrowski's theorem means that the completion of the field of rational numbers \mathbb{Q} with non-trivial norm is either the field of real numbers \mathbb{R} or one of the fields of p -adic numbers \mathbb{Q}_p .

Any p -adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)} (x_0 + x_1 p + x_2 p^2 + \dots), \quad (1.2.5)$$

where $x_j, \gamma(x) \in \mathbb{Z}$, $x_0 \neq 0$, $x_j \in \{0, 1, \dots, p-1\}$, $j = 1, 2, \dots$.

In this case $|x|_p = p^{-\gamma(x)}$ (see e.g. [29, 50]).

Theorem 1.2.2. [102] The equation

$$x^2 = a, \quad a \neq 0, \quad a = p^{\gamma(a)} (a_0 + a_1 p + a_2 p^2 + \dots), \quad a_0 \neq 0, \quad a_j \in \{0, 1, \dots, p-1\}, \quad j \in \mathbb{N},$$

has a solution in $x \in \mathbb{Q}_p$ iff the following statements hold:

- i) $\gamma(a)$ is even;
- ii) a_0 is a quadratic residue of p , if $p \neq 2$; the equality $a_1 = a_2 = 0$ hold if $p = 2$.

Corollary 1.2.1. [102] The equation $x^2 = -1$ has a solution in \mathbb{Q}_p , if and only if $p \equiv 1 \pmod{4}$.

In [66] the authors have introduced new symbols “ O ” and “ o ” which allow us to simplify our calculations. Roughly speaking, these symbols replace the notation $\equiv \pmod{p^k}$ without noticing about power of k . Let us recall such notions. A given p -adic number x by $O[x]$ we mean a p -adic number with the norm $p^{-\gamma(x)}$, i.e. $|x|_p = |O(x)|_p$. By $o[x]$, we mean a p -adic number with a norm strictly less than $p^{-\gamma(x)}$, i.e. $|o(x)|_p < |x|_p$. For instance, if $x = 1 + p - p^2$, we can write

$O[1]=x$, $o[1]=x-1$ or $o[p]=x-1-p$. Therefore, the symbols $O[\cdot]$ and $o[\cdot]$ make our work easier when we need to calculate the p -adic norm of p -adic numbers. It is easy to see that $y = O[x]$ if and only if $x = O[y]$.

We give some basic properties of $O[\cdot]$ and $o[\cdot]$, which will be used later on.

Proposition 1.2.3. [66] Let $x, y \in \mathbb{Q}_p$. Then the following statements hold:

- 1) $O[x]O[y] = O[xy]$;
- 2) $xO[y] = O[y]x = O[xy]$;
- 3) $O[x]o[y] = o[xy]$;
- 4) $o[x]o[y] = o[xy]$;
- 5) $xo[y] = o[y]x = o[xy]$;
- 6) $\frac{O[x]}{O[y]} = O\left[\frac{x}{y}\right]$ if $y \neq 0$;
- 7) $\frac{o[x]}{O[y]} = o\left[\frac{x}{y}\right]$ if $y \neq 0$.

For each $a \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}. \quad (1.2.6)$$

We recall that

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

and

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}$$

are the set of all p -adic integers and p -adic units, respectively.

The following result is known as the Hensel's lemma.

Lemma 1.2.2 [Hensel's lemma]. [9, 50]. Let $F(x)$ be a polynomial whose coefficients are p -adic integers. Let x^* be a p -adic integer such that for some $i \geq 0$ one has

$$F(x^*) \equiv 0 \pmod{p^{2i+1}}, \quad F'(x^*) \equiv 0 \pmod{p^i}, \quad F'(x^*) \not\equiv 0 \pmod{p^{i+1}}.$$

Then $F(x)$ has a p -adic integer root x_* such that $x_* \equiv x^* \pmod{p^{i+1}}$.

p -adic logarithm is defined by the series

$$\log_p(x) = \log_p(1 + (x-1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad (1.2.7)$$

which converges for $x \in B(1,1)$.

p -adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (1.2.8)$$

which converges for $x \in B\left(0, p^{-\frac{1}{p-1}}\right)$.

We set

$$\mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : |x-1|_p < p^{-\frac{1}{p-1}} \right\}. \quad (1.2.9)$$

This set is the range of the p -adic exponential [17]. In the sequel, the following well known fact will be frequently used without noticing.

Proposition 1.2.4. [17] The set \mathcal{E}_p has the following properties:

- (a) \mathcal{E}_p is a group under multiplication;
- (b) $|a-b|_p < 1$ for all $a, b \in \mathcal{E}_p$;
- (c) if $a, b \in \mathcal{E}_p$ then $|a+b|_p < 1$ if $p=2$ and $|a+b|_p = 1$ if $p > 2$;
- (d) if $a \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $a = \exp_p(h)$.

§ 1.3. Non-Archimedean Gibbs measure

In this section, we give a non-Archimedean analogue of Kolmogorov's extension theorem which was first proved in [24]. We notice that all p -adic measures are considered to be finitely additive, since only discrete measures are countably additive [101].

Let $X \neq \emptyset$ and \mathbb{K} be a non-Archimedean valued field. Assume that \mathfrak{R} is a semiring of subsets of X and $\mu: \mathfrak{R} \rightarrow K$ is a set function. A set function μ is called *non-Archimedean measure* on \mathfrak{R} if

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) \quad (1.3.1)$$

for any $A_1, A_2 \in \mathfrak{R}$ with $A_1 \cap A_2 = \emptyset$.

If $X \in \mathfrak{R}$ and $\mu(X) = 1$ then measure μ on \mathfrak{R} is called *probability measure* (see [47]). Similar to the real case, we first extend this measure from semi-ring \mathfrak{R} to the minimal ring $\tilde{\mathfrak{R}}$ containing \mathfrak{R} . Recall that a non-Archimedean measure $\tilde{\mu}$ on $\tilde{\mathfrak{R}}$ is called *extension* of a non-Archimedean measure μ on \mathfrak{R} if $\mathfrak{R} \subset \tilde{\mathfrak{R}}$ and $\tilde{\mu}(A) = \mu(A)$, $\forall A \in \mathfrak{R}$. We notice that by means of finite additivity, the statements and continuations of measures are standard. For the sake of completeness, we provide such an extension in the p -adic setting.

Proposition 1.3.1. [27] For any non-Archimedean measure μ on semi-ring \mathfrak{R} there exists a unique extension $\tilde{\mu}$ on the minimal ring $\tilde{\mathfrak{R}}$ containing \mathfrak{R} .

Since the proof of Proposition 1.3.1 does not depend on the value of non-Archimedean measure, the proof is similar to the classic (real) case.

Corollary 1.3.1. [24] An extension of any probability measure is a probability measure as well.

Let (X, \mathfrak{B}) be a measurable space, where \mathfrak{B} is an algebra of subsets X . Denote

$$(X^\infty, \mathfrak{B}^\infty) = \prod_{j=1}^{\infty} (X, \mathfrak{B}), \quad (1.3.2)$$

where \mathfrak{B}^∞ is a minimal algebra containing cylindrical subsets of X^∞ , i.e.

$$I_n(B) = \{x \in X^\infty : (x_1, x_2, \dots, x_n) \in \mathfrak{B}\}, \quad B \in \bigotimes_{j=1}^n \mathfrak{B}. \quad (1.3.3)$$

We say a sequence of non-Archimedean probability measures $\{P_n\}_{n=1}^\infty$ on

$\left\{X^n, \bigotimes_{j=1}^n \mathfrak{B}\right\}_{n=1}^\infty$ is a *compatible* if

$$P_{n+1}(B \times X) = P_n(B), \quad \forall n \in \mathbb{N}. \quad (1.3.4)$$

Theorem 1.3.1. [24] (non-Archimedean analogue of Kolmogorov's extension theorem) Let $\{P_n\}_{n=1}^\infty$ be a sequence of non-Archimedean probability

measures on $\left\{X^n, \bigotimes_{j=1}^n \mathfrak{B}\right\}_{n=1}^\infty$. If it is compatible then there exists a unique non-

Archimedean probability measure P defined on $(X^\infty, \mathfrak{B}^\infty)$ such that

$$P(I_n(B)) = P_n(B), \quad \forall B \in \bigotimes_{j=1}^n \mathfrak{B}, \quad \forall n \in \mathbb{N}. \quad (1.3.5)$$

Remark 1.3.1. We notice that in [43] certain results on the existence of probability on a product of non-Archimedean probabilistic spaces have been proved.

One of the important condition is boundedness (see, e.g. the Monna–Springer theory of non-Archimedean integration [56]), namely a p -adic probability measure μ is called bounded if

$$\sup\{|\mu(A)|_p : A \in \mathfrak{B}\} < \infty. \quad (1.3.6)$$

We pay attention to an important special case in which boundedness condition itself provides a fruitful integration theory (see, for example, [40]). Note

that, in general, a p -adic probability measure needs not be bounded [47], [43]. For more detail information about p -adic measures, we refer to [44], [101].

Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles, such that exactly $k + 1$ edges originate from each vertex, where V is the set of vertices, L is the set of edges. Two vertices x and y are called nearest neighbors, if there exist an edge $l \in L$ connecting them. We denote by $l = \langle x, y \rangle$. The distance $d(x, y)$ on the Cayley tree defined by

$$d(x, y) = \min \{d \mid \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V, \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle\}.$$

For fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{j=0}^n W_j, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

direct successors of x .

Let G_k be a free product of $k + 1$ cyclic groups of the second order with generators a_1, a_2, \dots, a_{k+1} , respectively.

The following proposition proved by N.N. Ganikhodjaev.

Proposition 1.3.2. [108] There exists a one-to-one correspondence between the set of vertices V of the Cayley tree $\Gamma^k = (V, L)$ and the group G_k (see Figure 1.1).

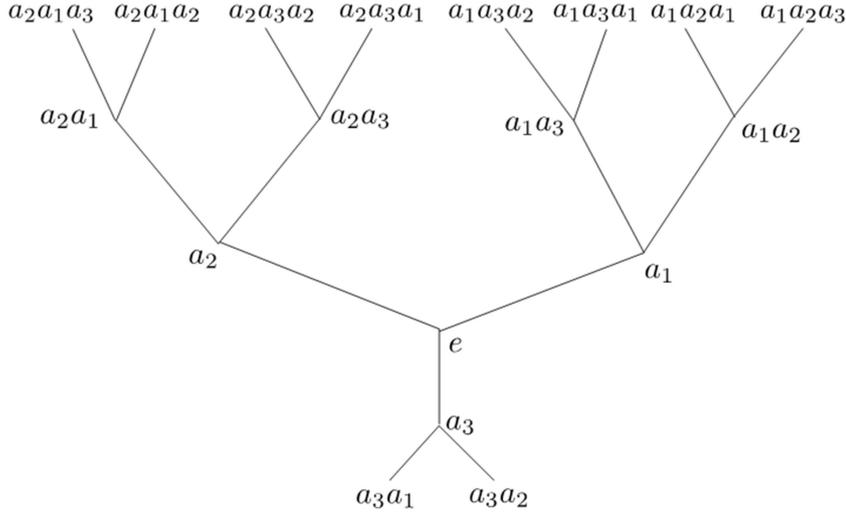


Figure 1.1: The Cayley tree Γ^2 and its group representation.

Let $\Gamma^k = (V, L)$ be Cayley tree and $\Phi \subset \mathbb{Z}$ be a finite set. For $\Lambda \subset V$ a configuration σ_Λ on Λ is defined as a function

$$x \in \Lambda \rightarrow \sigma_\Lambda(x) \in \Phi.$$

The set all configurations on Λ is denoted by $\Omega_\Lambda = \Phi^\Lambda$, in particular, $\Omega = \Omega_V$.

For given configurations $\sigma \in \Omega_\Lambda$ and $\varphi \in \Omega_{V \setminus \Lambda}$ we define a configuration in Ω as follows

$$(\sigma \vee \varphi)(x) = \begin{cases} \sigma(x), & \text{if } x \in \Lambda, \\ \varphi(x), & \text{if } x \in V \setminus \Lambda. \end{cases}$$

The energy of the configuration $\sigma \in \Omega$ is given by the formal Hamiltonian

$$H(\sigma) = \sum_{\substack{\Lambda \subset V: \\ \text{diam}(\Lambda) \leq r}} I(\sigma_\Lambda),$$

where $r \in \mathbb{N}$, $\text{diam}(\Lambda) = \max_{x, y \in \Lambda} d(x, y)$, $I : \Omega_\Lambda \rightarrow \mathbb{Q}_p$ is a given potential.

Let \mathfrak{B} be an algebra which contains cylindrical subsets of Ω .

Definition 1.3.1. [89] Let μ be p -adic probability distributions on (Ω, \mathfrak{B}) . Assume that for all $\Lambda \subset V$ and $\sigma \in \Omega_\Lambda$, $\varphi \in \Omega_{V \setminus \Lambda}$ the following equality holds

$$\mu(\sigma | \varphi) = \frac{\exp_p \{H(\sigma \vee \varphi)\}}{Z_{\Lambda, \varphi}},$$

where $Z_{\Lambda, \varphi}$ is the normalizing constant, i.e.

$$Z_{\Lambda, \varphi} = \sum_{\omega \in \Omega_\Lambda} \exp_p \{H(\omega \vee \varphi)\}.$$

Such a distribution is called p -adic Gibbs distribution.

§ 1.4. On G_2 -periodic p -adic quasi Gibbs measures of the Potts model on a Cayley tree

Semi-infinite Cayley tree. Let $\Gamma_+^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k+1$ edges, except for the root x^0 , which has k edges) (see Fig. 1.2). Here V is the set of vertices and L is the set of edges.

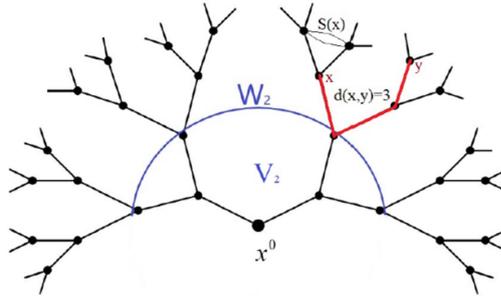


Figure 1.2: A semi-infinite Cayley tree of order two.

We recall a coordinate structure in Γ_+^k : every vertex x (except for x^0) of Γ_+^k has coordinates (i_1, \dots, i_n) , here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0) (see Figure 1.3). Namely, the symbol (0) constitutes level 0, and the sites (i_1, \dots, i_n) form level n (i.e. $d(x^0, x) = n$) of the lattice. Let us define on Γ_+^k binary

operation $\circ : \Gamma_+^k \times \Gamma_+^k \rightarrow \Gamma_+^k$ as follows: for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$ put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m),$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \dots, i_n) \circ (0) = (i_1, \dots, i_n).$$

By means of the defined operation Γ_+^k becomes a non-commutative semigroup with a unit. Let us denote this group (G^k, \circ) . Using this semigroup structure one defines translation $\tau_g : G^k \rightarrow G^k, g \in G_k$ by

$$\tau_g(x) = g \circ x.$$

It is clear that $\tau_{(0)} = id$.

Let $G \subset G^k$ be a sub-semigroup of G^k and $h : G^k \rightarrow Y$ be a Y -valued function defined on G^k . We say that h is G -periodic if $h(\tau_g(x)) = h(x)$ for all $g \in G$ and $x \in G^k$. Any G -periodic function is called *translation-invariant*.

Now for each $m \geq 2$ we put

$$G_m = \{x \in G^k : d(x, x^0) \equiv 0 \pmod{m}\}.$$

One can check that G_m is a sub-semigroup of G^k .

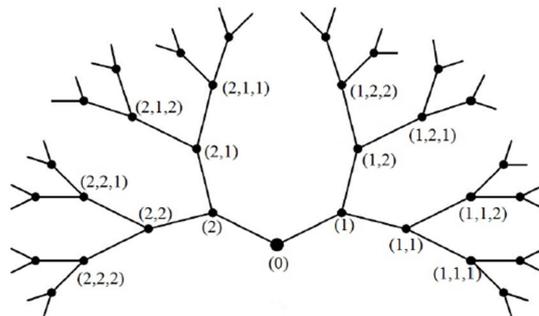


Figure 1.3: A coordinate structure of the semi-infinite Cayley tree of order two.

p -adic quasi Gibbs measure for the Potts model. Let \mathbb{Q}_p be the field of the p -adic numbers and $\Phi = \{1, 2, \dots, q\}$ be a finite set. We consider p -adic Potts model on a Cayley tree.

The formal Hamiltonian of the p -adic Potts model is

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} \delta_{\sigma(x)\sigma(y)} \quad (1.4.1)$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, $\langle x, y \rangle$ stands for nearest neighbor vertices and δ_{ij} is the Kronecker's symbol, i.e.

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (1.4.2)$$

Assume that $\mathbf{h}: V \rightarrow \mathbb{Q}_p^q$ is a mapping, i.e. $\mathbf{h}_x = (h_{1,x}, h_{2,x}, \dots, h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ ($i \in \Phi$) and $x \in V$. Given $n \in \mathbb{N}$, we consider p -adic probability distributions $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x), x}, \quad (1.4.3)$$

here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor

$$Z_n^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x), x}. \quad (1.4.4)$$

We say that p -adic probability distributions (1.4.3) are compatible if all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}). \quad (1.4.5)$$

Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations. We note that a non-Archimedean analogue of the Kolmogorov's extension theorem was proved in

[24]. According to this theorem there exists a unique p -adic measure μ_h on $\Omega = \Phi^V$ such that

$$\mu(\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n) = \mu_h^{(n)}(\sigma_n), \quad (1.4.6)$$

for all $\sigma_n \in \Omega_{V_n}, n \in \mathbb{N}$. Such measure is called a p -adic quasi Gibbs measure (see [58]) corresponding to the Hamiltonian (1.4.1) and vector-valued function $\mathbf{h}_x, x \in V$. By $QG(H)$ we denote the set of all p -adic quasi Gibbs measures associated with function $\mathbf{h} = \{\mathbf{h}_x, x \in V\}$.

We pay attention to an important special case in which boundedness condition (1.3.6) itself provides a phase transition occurs for a given model. If there are at least two distinct bounded p -adic Gibbs measures then one says there exists a *quasi phase transition*. If there are at least two distinct p -adic Gibbs measures μ and ν such that μ is bounded and ν is unbounded, then one says that a *phase transition* occurs. Moreover, if there is a sequence of sets A_n such that $A_n \in \Omega_{V_n}$ with $|\mu(A_n)|_p \rightarrow 0$ and $|\nu(A_n)|_p \rightarrow \infty$ as $n \rightarrow \infty$, then there occurs a *strong phase transition* [58].

The following statement describes conditions \mathbf{h}_x guaranteeing compatibility of $\mu_h^{(n)}$.

Theorem 1.4.1. [58,71] The measure $\mu_h^{(n)}, n = 1, 2, \dots$ (see (1.4.3)) associated with q -state Potts model (1.4.1) satisfy the compatibility condition (1.4.6) if and only if for any $n \in \mathbb{N}$ the following equation holds:

$$\widehat{\mathbf{h}}_x = \prod_{y \in S(x)} F(\widehat{\mathbf{h}}_y, \theta), \quad (1.4.7)$$

here and below a vector $\widehat{h} = (\widehat{h}_1, \widehat{h}_2, \dots, \widehat{h}_{q-1}) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\mathbf{h} = (h_1, h_1, \dots, h_q) \in \mathbb{Q}_p^q$ as follows

$$\hat{h}_i = \frac{h_i}{h_q}, i = 1, 2, \dots, q-1, \theta = \exp_p \{J\}, \quad (1.4.8)$$

and mapping $F: \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \rightarrow \mathbb{Q}_p^{q-1}$ is defined by $F(\mathbf{x}; \theta) = (F_1(\mathbf{x}; \theta), \dots, F_{q-1}(\mathbf{x}; \theta))$ with

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + \theta}, \mathbf{x} = (x_1, x_2, \dots, x_{q-1}) \in \mathbb{Q}_p^{q-1}, i = 1, 2, \dots, q-1. \quad (1.4.9)$$

Remark 1.4.1. Without loss of generality, we may assume that $h_q = 1$. Otherwise, in (1.4.3) we multiply and divide the expression on the right hand side by $\prod_{x \in W_n} h_{q,x}$ and after replacing h_i by h_i / h_q we get the desired equality.

Remark 1.4.2. In [58] the existence of the phase transition for the considered model was established. In [82] it was shown that G_2 -periodic solution of (1.4.7) belonging to \mathcal{E}_p coincides with translation-invariant ones. Therefore, it is natural to find periodic solution of (1.4.7) in a general setting, which allows to find periodic p -adic quasi Gibbs measures.

Remark 1.4.3. In [76] the existence of G_m -periodic p -adic quasi Gibbs measures was found under certain conditions such that $p \geq 3$, $k = 2$, $q = mp^n$ and $0 < |\theta - 1|_p \leq p^{-2n-1}$ for some $m, n \in \mathbb{N}$ and $(m, p) = 1$. However, the explicit form of that measures is not given. We find the explicit form of G_2 -periodic p -adic quasi Gibbs measures. Besides, G_2 -periodic p -adic quasi Gibbs measures we found are not required to satisfy the aforementioned conditions in [76]. Using the explicit form of solutions we can construct some non-periodic p -adic quasi Gibbs measures.

Let us first observe that the set $(\underbrace{1, \dots, 1}_m, h, 1, \dots, 1), (m = 1, \dots, q-1)$ is invariant for the equation (1.4.7). Therefore, in what follows, we restrict ourselves to one of such lines, say, $(h, 1, \dots, 1)$.

Non-translation-invariant G_2 -periodic p -adic quasi Gibbs measure for the Potts model.

In [92] authors studied all translation-invariant p -adic Gibbs measures for the Potts model on the Cayley tree of order two. Recently, in [70, 73] the existence of periodic p -adic Gibbs measures for the p -adic q -state Potts model on the Cayley tree has been carried out by means of chaotic behavior of the function associated with renormalization group. However, explicit construction of such kind of p -adic Gibbs measures is highly a non-trivial task. Therefore, in this section, we are going to construct G_2 -periodic p -adic quasi Gibbs measures for the considered model.

Let G_2 be a sub-semigroup of G_k . Recall that h_x is called G_2 -periodic, if for all $x \in G^k$ and $y \in G_2$, $\sigma(yx) = \sigma(x)$ holds.

We denote

$$h_x = \begin{cases} h_1, & \text{if } x \in G_2, \\ h_2, & \text{if } x \in G^k \setminus G_2. \end{cases} \quad (1.4.10)$$

From the equation (1.4.7) we can get the following system

$$\begin{cases} \hat{h}_1 = (F(\hat{h}_2, \theta))^k, \\ \hat{h}_2 = (F(\hat{h}_1, \theta))^k. \end{cases} \quad (1.4.11)$$

We assume $\hat{h}_i = (h_i^{(1)}, h_i^{(2)}, \dots, h_i^{(q-1)})$. Let $h_i^{(j)} = 1, j = \overline{2, q-1}, i = 1, 2$. Unless otherwise stated, we concentrate on the simplest case where $k = 2$. In this case we can obtain the following system of equations from (1.4.11)

$$\begin{cases} \hat{h}_1^{(1)} = \left(\frac{\theta \hat{h}_2^{(1)} + q - 1}{\hat{h}_2^{(1)} + \theta + q - 2} \right)^2, \\ \hat{h}_2^{(1)} = \left(\frac{\theta \hat{h}_1^{(1)} + q - 1}{\hat{h}_1^{(1)} + \theta + q - 2} \right)^2. \end{cases} \quad (1.4.12)$$

Let

$$f_\theta(h) = \left(\frac{\theta h + q - 1}{h + \theta + q - 2} \right)^2. \quad (1.4.13)$$

Rewriting (1.4.13) we have $f_\theta(h) = \left(\theta - \frac{(\theta-1)(\theta+q-1)}{h+\theta+q-2} \right)^2$. It is clear that if $\theta = 1$ or $\theta = -q + 1$, then $f_\theta(h)$ is a constant function, i.e. $f_\theta(h)$ is a periodic function of any order.

In [73], [75] authors studied dynamics of the function $f_\theta(h)$ and found translation-invariant p -adic quasi Gibbs measures on a Cayley tree of order two. In [69], several methods from Diophantine p -adic equations were used.

There are three fixed points of the function $f_\theta(h)$, i.e. $h_0 = 1$,

$$\begin{aligned} h_1 &= \frac{(\theta-1)^2 - 2(q-1) + (\theta-1)\sqrt{(\theta-1)^2 - 4(q-1)}}{2}, \\ h_2 &= \frac{(\theta-1)^2 - 2(q-1) - (\theta-1)\sqrt{(\theta-1)^2 - 4(q-1)}}{2}. \end{aligned} \quad (1.4.14)$$

Denote $\Delta(\theta, q) = -8 + (\theta-1)^2$.

If $\hat{h}_1^{(1)} = \hat{h}_2^{(1)}$ then we get translation-invariant Gibbs measures. Finding G_2 -periodic (non-translation-invariant) quasi Gibbs measures is equivalent to find $h_1^{(1)} \neq h_2^{(1)}$ solutions of the system (1.4.12). It is equivalent to solve the following equation

$$\frac{f_\theta(f_\theta(h)) - h}{f_\theta(h) - h} = 0. \quad (1.4.15)$$

Simplifying the last equation we obtain

$$Ah^2 + Bh + C = 0, \quad (1.4.16)$$

where

$$A(\theta, q) = (\theta^2 + \theta + q - 2)^2,$$

$$B(\theta, q) = \theta^4 + 4(q-1)\theta^3 + (q^2 + 6q - 12)\theta^2 + (10q^2 - 36q + 32)\theta + 2q^3 - 13q^2 + 26q - 17,$$

$$C(\theta, q) = (\theta^2 + (3q-5)\theta + (q-2)^2)^2.$$

We notice that (1.4.16) has solutions in \mathbb{Q}_p if and only if $\sqrt{D(\theta, q)} \in \mathbb{Q}_p$, where

$$D(\theta, q) = B^2(\theta, q) - 4A(\theta, q)C(\theta, q).$$

For the sake of convenience we denote

$$D_1(\theta, q) = 4q^2(1-q) + (24q - 24q^2)(\theta-1) + (36 - 36q - 3q^2)(\theta-1)^2 - 10q(\theta-1)^3 - 3(\theta-1)^4.$$

After some calculations we can rewrite $D(\theta, q)$ as follows

$$D(\theta, q) = (\theta-1)^2(\theta+q-1)^2 D_1(\theta, q).$$

Thus, we can conclude that $\sqrt{D(\theta, q)}$ exists in \mathbb{Q}_p if and only if $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_p$.

Since (1.4.16) is a quadratic equation we can write formal solutions of (1.4.16) as follows

$$h_{3,4} = \frac{(2q^2 - 2q^3) + (12q - 12q^2)(\theta-1) + (18 - 18q - q^2)(\theta-1)^2 - 4q(\theta-1)^3 - (\theta-1)^4}{2(q + 3(\theta-1) + (\theta-1)^2)} \pm$$

$$\pm \frac{(\theta-1)(\theta+q-1)\sqrt{D_1(\theta, q)}}{2(q + 3(\theta-1) + (\theta-1)^2)}. \quad (1.4.17)$$

Not difficult calculations show that $h_{3,4}$ is not fixed point of $f_\theta(h)$. Now we show that for a given $q \geq 2$ and for each $\theta \in \mathcal{E}_p$ one can find prime integer p such that $h_{3,4} \in \mathbb{Q}_p$.

Theorem 1.4.2. For every $q \geq 2$ one can find a prime $p(q)$ such that the $h_{3,4}$ exist in \mathbb{Q}_p .

Proof. Since, $h_{3,4} \in \mathbb{Q}_p$ is equivalent to $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_p$. We rewrite $D_1(\theta, q)$ as follows

$$\begin{aligned} D_1(\theta, q) = & 12^2 - 288(\theta - 1) + 117(\theta - 1)^2 + 30(\theta - 1)^3 - 3(\theta - 1)^4 \\ & - (132 - 168(\theta - 1) + 18(\theta - 1)^2 + 10(\theta - 1)^3)(q + 3) \\ & + (40 - 24(\theta - 1) - 3(\theta - 1)^2)(q + 3)^2 - 4(q + 3)^3 \end{aligned} \quad (1.4.18)$$

or

$$\begin{aligned} D_1(\theta, q) = & 48^2 - 1728(\theta - 1) + 132(\theta - 1)^2 + 80(\theta - 1)^3 - 3(\theta - 1)^4 \\ & - (832 - 408(\theta - 1) - 12(\theta - 1)^2 + 10(\theta - 1)^3)(q + 8) \\ & + (100 - 24(\theta - 1) - 3(\theta - 1)^2)(q + 8)^2 - 4(q + 8)^3. \end{aligned} \quad (1.4.19)$$

In order to establish the existence of $\sqrt{D_1(\theta, q)}$, we consider several cases.

Case 1. Let p be a prime divisor of $q + 3$ such that $p \notin \{2, 3\}$.

We are going to show that $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_p$. By (1.4.18), we get

$$D_1(\theta, q) = 12^2(1 + o[1]).$$

Using the last equality and due to Theorem 1.2.2 we have $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_p$.

Case 2. Let $q + 3 = 2^a 3^b$, where a and b are some non-negative integers.

First, we assume that $b \geq 2$ and $p = 3$. Then we get $q + 3 = o[3]$. Due to

$$\theta - 1 = o[1], \text{ from (1.4.18), we obtain } D_1(\theta, q) = 12^2(1 + o[1]).$$

Hence, thanks to Theorem 1.2.2 we have $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_3$.

Let us suppose that $b = 1$. Then we have $q + 8 = 3 \cdot 2^a + 5$. One can see that $q + 8$ has a prime divisor $p \notin \{2, 3\}$.

Then due to (1.4.18), in \mathbb{Q}_p we have the following representation

$$D_1(\theta, q) = 48^2(1 + o[1]).$$

Again according to Theorem 1.2.2 we conclude that $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_p$.

Now we assume that $b = 0$. Then keeping in mind $q \geq 2$ one has $a \geq 3$.

It is easy to see that $2 \nmid (q + 8)$. So, we have to consider two cases:

either $q + 8 = 3^c$ or $q + 8$ has a prime divisor $p > 3$.

One can see that if $q + 8 = 3^c$ holds then $c \geq 3$. In this case we consider $D_1(\theta, q)$ in \mathbb{Q}_3 . We have

$$1728(\theta - 1) = o[3^2], \quad 132(\theta - 1)^2 = o[3^2],$$

$$80(\theta - 1)^3 = o[3^2], \quad 3(\theta - 1)^4 = o[3^4],$$

$$(832 - 408(\theta - 1) - 12(\theta - 1)^2 + 10(\theta - 1)^3)(q + 8) = o[3^2],$$

$$(100 - 24(\theta - 1) - 3(\theta - 1)^2)(q + 8)^2 = o[3^5],$$

$$4(q + 8)^3 = o[3^8].$$

Hence, for $p = 3$ from (1.4.19) we obtain $D_1(\theta, q) = 48^2 + o[9]$, which yields that $D_1(\theta, q) = 48^2(1 + o[1])$. Then due to Theorem 1.2.2 we conclude that $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_3$.

Suppose that $q + 8$ has a prime divisor $p > 3$. Then for this prime number from (1.4.19) we immediately get $D_1(\theta, q) = 48^2 + o[1]$ in \mathbb{Q}_p . The last equality together with $48^2 = O[1]$ implies that $D_1(\theta, q) = 48^2(1 + o[1])$. Again by Theorem

1.2.2 we conclude that $\sqrt{D_1(\theta, q)} \in \mathbb{Q}_p$ where $p > 3$ is a prime divisor of $q + 8$. The theorem is proved.

Boundedness of translation-invariant and G_2 -periodic p -adic quasi Gibbs measures and phase transitions

Lemma 1.4.1. [78] Let \mathbf{h} be a solution of (1.4.7), and $\mu_{\mathbf{h}}$ be an associated with p -adic quasi Gibbs measure. Then for the corresponding partition function $Z_n^{(\mathbf{h})}$ the following equality holds:

$$Z_n^{(\mathbf{h})} = A_{\mathbf{h}, n-1} Z_{n-1}^{(\mathbf{h})}, \quad (1.4.20)$$

where

$$A_{\mathbf{h}, n} = \prod_{x \in W_n} a_{\mathbf{h}}(x), \quad \prod_{y \in S(x)} \sum_{j=1}^q \exp_p \{J \delta_{ij}\} h_{j,y} = a_{\mathbf{h}}(x) h_{i,x}, \quad a_{\mathbf{h}}(x) \in \mathbb{Q}_p, \quad i = 1, 2, \dots, q.$$

Using Lemma 1.4.1 we get the following statement.

Lemma 1.4.2. Let $k = 2$. If \mathbf{h} be a translation-invariant solution of (1.4.7) then for the corresponding partition function $Z_n^{(\mathbf{h})}$ the following equality holds:

$$Z_n^{(\mathbf{h})} = (h + \theta + q - 2)^{2^{n+1}-2} (h + q - 1). \quad (1.4.21)$$

Proof. It is easy to check that $\mathbf{h} = (h, 1, \dots, 1)$ is a translation-invariant solution of (1.4.12), where h is a fixed point of (1.4.13). Since $\theta = \exp_p \{J\}$, using (1.4.4) we get $Z_1^{(\mathbf{h})} = (h + \theta + q - 2)^2 (h + q - 1)$. Then by Lemma 1.4.1 we obtain the following equalities:

$$a_{\mathbf{h}}(x) = \frac{(\theta h_{1,y} + q - 1)^2}{h_{1,x}} = \frac{(\theta h + q - 1)^2}{h} = (h + \theta + q - 2)^2,$$

$$A_{\mathbf{h}, n} = (h + \theta + q - 2)^{2^{n+1}}, \quad Z_n^{(\mathbf{h})} = (h + \theta + q - 2)^{2^{n+1}-2} (h + q - 1),$$

where $h = h_0, h = h_1, h = h_2$. The lemma is proved.

Lemma 1.4.3. Let $k = 2$. If $h_{3,4}$ be G_2 -periodic (non-translation-invariant) solutions of (1.4.7) then for the corresponding partition function $Z_n^{(h)}$ the following equalities hold:

If n is odd, then

$$Z_n^{(h)} = (h_3 + \theta + q - 2)^{\frac{2^{n+2}-2}{3}} (h_4 + \theta + q - 2)^{\frac{2^{n+1}-4}{3}} (h_4 + q - 1); \quad (1.4.22)$$

If n is even, then

$$Z_n^{(h)} = (h_3 + \theta + q - 2)^{\frac{2^{n+1}-2}{3}} (h_4 + \theta + q - 2)^{\frac{2^{n+2}-4}{3}} (h_4 + q - 1). \quad (1.4.23)$$

Proof. Let

$$h_{\sigma(x),x} = \begin{cases} h_{1,x}, & \text{if } \sigma(x) = 1, \\ 1, & \text{if } \sigma(x) \neq 1, \end{cases} \quad h_{1,x} = \begin{cases} h_3, & \text{if } x \in W_n, n \text{ is odd,} \\ h_4, & \text{if } x \in W_n, n \text{ is even.} \end{cases}$$

Due to (1.4.12), we have

$$\begin{cases} h_3 = \left(\frac{\theta h_4 + q - 1}{h_4 + \theta + q - 2} \right)^2, \\ h_4 = \left(\frac{\theta h_3 + q - 1}{h_3 + \theta + q - 2} \right)^2. \end{cases} \quad (1.4.24)$$

Using (1.4.4) and (1.4.24), we get $Z_1^{(h)} = (h_3 + \theta + q - 2)^2 (h_4 + q - 1)$.

We consider following cases.

Case 1. Let n be odd. By lemma (1.4.1) and (1.4.24) we get

$$a_h(x) = \frac{(\theta h_{1,y} + q - 1)^2}{h_{1,x}} = \frac{(\theta h_4 + q - 1)^2}{h_3} = (h_4 + \theta + q - 2)^2,$$

$$A_{h,n-1} = (h_3 + \theta + q - 2)^{2^n}, A_{h,n} = (h_4 + \theta + q - 2)^{2^{n+1}},$$

$$Z_n^{(h)} = (h_3 + \theta + q - 2)^{\frac{2^{n+2}-2}{3}} (h_4 + \theta + q - 2)^{\frac{2^{n+1}-4}{3}} (h_4 + q - 1).$$

Case 2. Let n be even. By Lemma 1.4.1 and (1.4.24) we get

$$a_h(x) = \frac{(\theta h_{1,y} + q - 1)^2}{h_{1,x}} = \frac{(\theta h_3 + q - 1)^2}{h_4} = (h_3 + \theta + q - 2)^2,$$

$$A_{h,n-1} = (h_4 + \theta + q - 2)^{2^n}, A_{h,n} = (h_3 + \theta + q - 2)^{2^{n+1}},$$

$$Z_n^{(h)} = (h_3 + \theta + q - 2)^{\frac{2^{n+1}-2}{3}} (h_4 + \theta + q - 2)^{\frac{2^{n+2}-4}{3}} (h_4 + q - 1).$$

The lemma is proved.

Case $q = 3$.

We consider the case $q = 3$, i.e. spin values are 1,2,3. In this case $D_1(\theta) = -72 - 144(\theta - 1) - 99(\theta - 1)^2 - 30(\theta - 1)^3 - 3(\theta - 1)^4$. Using (1.4.12) we can obtain following solutions:

$$h_{3,4} = \frac{-36 - 72(\theta - 1) - 45(\theta - 1)^2 - 12(\theta - 1)^3 - (\theta - 1)^4}{2(3 + 3(\theta - 1) + (\theta - 1)^2)^2} \pm \frac{(\theta - 1)(\theta + 2)\sqrt{-72 - 144(\theta - 1) - 99(\theta - 1)^2 - 30(\theta - 1)^3 - 3(\theta - 1)^4}}{2(3 + 3(\theta - 1) + (\theta - 1)^2)^2}. \quad (1.4.25)$$

Lemma 1.4.4. Let $|Per_2(f_\theta(h))|$ be number of 2-periodic points of $f_\theta(h)$.

Then the following statements are true:

- 1) if $p \equiv 1(\text{mod } 8)$ or $p \equiv 3(\text{mod } 8)$, then $|Per_2(f_\theta(h))| = 5$;
- 2) if $p = 2$ or $p \equiv 5(\text{mod } 8)$ or $p \equiv 7(\text{mod } 8)$, then $|Per_2(f_\theta(h))| = 1$;

Proof. a) Let $p = 2$. Due to $\theta \in \mathcal{E}_2$ we have $|(\theta - 1)^2|_2 \leq \frac{1}{16}$. Hence, using the

strong triangle inequality one gets $|-8 + (\theta - 1)^2|_2 = \frac{1}{8}$. This means that $|D(\theta)|_2 = \frac{1}{8}$.

Then thanks to Theorem 1.2.2 there does not exist $\sqrt{\Delta(\theta)}$ in \mathbb{Q}_2 . Consequently, $h_1, h_2 \notin \mathbb{Q}_2$.

Now we will check the existence of solutions h_3 and h_4 in \mathbb{Q}_2 . In order to check it we calculate the norm of $D_1(\theta)$. Again using $|\theta - 1|_2 \leq \frac{1}{4}$ we obtain the followings:

$$|-144(\theta - 1)|_2 \leq \frac{1}{2^6}, \quad |-99(\theta - 1)^2|_2 \leq \frac{1}{2^4},$$

$$|-30(\theta - 1)^3|_2 \leq \frac{1}{2^7}, \quad |-3(\theta - 1)^4|_2 \leq \frac{1}{2^8}.$$

Keeping in mind $|-72|_2 = \frac{1}{2^3}$, due to the strong triangle inequality the last

inequalities follow that $|D_1(\theta)|_2 = \frac{1}{8}$. Then according to Theorem 1.2.2, we

conclude that $\sqrt{D_1(\theta)} \notin \mathbb{Q}_2$. This means that $h_3, h_4 \notin \mathbb{Q}_2$. Thus in this case we have shown that $|Per_2(f_\theta(h))| = 1$.

b) $p \geq 3$. Since $\theta \in \mathcal{E}_p$ we get $|\theta - 1|_p < 1$. Then thanks to Theorem 1.2.2 the existence of $\sqrt{\Delta(\theta)}$ in \mathbb{Q}_p is equivalent to the existence of $\sqrt{-2}$. We notice that $\sqrt{-2} \in \mathbb{Q}_p$ if -2 is a quadratic residue by modulo p . -2 is a quadratic residue by modulo p if and only if $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$ (see [87]).

So, we infer that

$$h_1, h_2 \in \mathbb{Q}_p \quad p \equiv 1 \pmod{8} \text{ or } p \equiv 3 \pmod{8}.$$

Now we will check the existence of solutions h_3 and h_4 in \mathbb{Q}_p . Due to $\theta \in \mathcal{E}_p$ we have

$$|(24q - 24q^2)(\theta - 1) + (36 - 36q - 3q^2)(\theta - 1)^2 - 10q(\theta - 1)^3 - 3(\theta - 1)^4|_p \leq p^{-1}.$$

By Lemma 1.4.1, the existence $\sqrt{D_1(\theta)}$ is equivalent to the existence of $\sqrt{-72} = 6\sqrt{-2}$ or existence of $\sqrt{-2}$. We notice that $\sqrt{-2} \in \mathbb{Q}_p$ if -2 is a quadratic residue by modulo p . By Theorem 1.1.3, we have $\sqrt{-2}$ exists in \mathbb{Q}_p if and only if $p \equiv 1(\text{mod } 8)$ or $p \equiv 3(\text{mod } 8)$. So, we infer that

$$h_3, h_4 \in \mathbb{Q}_p \Leftrightarrow p \equiv 1(\text{mod } 8) \text{ or } p \equiv 3(\text{mod } 8). \quad (1.4.26)$$

In conclusion, we get

$$h_1, h_2, h_3, h_4 \in \mathbb{Q}_p \quad p \equiv 1(\text{mod } 8) \text{ or } p \equiv 3(\text{mod } 8)$$

Thus in this case we have shown that if $p \equiv 1(\text{mod } 8)$ or $p \equiv 3(\text{mod } 8)$ then $|Per_2(f_\theta(h))| = 5$, if $p = 2$, $p \equiv 5(\text{mod } 8)$ and $p \equiv 7(\text{mod } 8)$, then $|Per_2(f_\theta(h))| = 1$.

Finally, we have finished to prove Lemma 1.4.4.

Remark 1.4.4. Note that all prime numbers are in the forms of $p = 2$, $p \equiv 1(\text{mod } 8)$, $p \equiv 3(\text{mod } 8)$, $p \equiv 5(\text{mod } 8)$, $p \equiv 7(\text{mod } 8)$. If $p \not\equiv 1(\text{mod } 8)$, $p \not\equiv 3(\text{mod } 8)$ then $p = 2$ or $p \equiv 5(\text{mod } 8)$ or $p \equiv 7(\text{mod } 8)$.

The following theorem is immediate from Lemma 1.4.4.

Theorem 1.4.3. Let $q = 3$. Following statements are true for the p -adic Potts model on the Cayley tree of order two

- 1) if $p \equiv 1(\text{mod } 8)$ or $p \equiv 3(\text{mod } 8)$, then there are three translation-invariant p -adic quasi Gibbs measures and two G_2 -periodic, non-translation-invariant p -adic quasi Gibbs measures;
- 2) if $p = 2$ or $p \equiv 5(\text{mod } 8)$ or $p \equiv 7(\text{mod } 8)$, then there is one translation-invariant p -adic quasi Gibbs measure;

Lemma 1.4.5. The norms of all h_i , $i = 0, 1, \dots, 4$ are equal to one, here h_i defined by (1.4.14) and (1.4.17).

Proof. From $\theta \in \mathcal{E}_p$ and properties of p -adic norm we can get easily the result of Lemma 1.4.5.

Theorem 1.4.4. Let $q = 3$. Following statements are true for the p -adic Potts model on the Cayley tree of order two

1) if $p = 2$ or $p \equiv 5 \pmod{8}$ or $p \equiv 7 \pmod{8}$, then μ_{h_0} measure is unbounded;

2) if $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}, p \neq 3$ then only μ_{h_0} measure is bounded,

$\mu_{h_1}, \mu_{h_2}, \mu_{h_3}, \mu_{h_4}$ measures are unbounded;

3) if $p = 3$, then all $\mu_{h_0}, \mu_{h_1}, \mu_{h_2}, \mu_{h_3}, \mu_{h_4}$ measures are unbounded.

Proof.

Case 1. If $p = 2$ or $p \equiv 5 \pmod{8}$ or $p \equiv 7 \pmod{8}$, then exists only μ_{h_0} translation-invariant measure. Note that $|h_0 + \theta + q - 2|_p = |\theta + 2|_p = 1$. By Lemma 1.4.2 we get

$$\lim_{n \rightarrow \infty} |\mu_{h_0}^{(n)}|_p = \lim_{n \rightarrow \infty} \left| \frac{1}{3(\theta + 2)^{2^{n+1}-2}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x), x} \right|_p = 1.$$

Case 2. If $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}, p \neq 3$ then there exist $\mu_{h_i}, i = 0, \dots, 5$ measures. For the translation-invariant solutions we have $|h_0 + \theta + 1|_p = 1$. Since

$$|\theta - 1|_p < 1, \quad |h_{1,2} + 2|_p = \left| \frac{(\theta - 1)^2 \pm (\theta - 1)\sqrt{(\theta - 1)^2 - 4(q - 1)}}{2} \right|_p < 1,$$

then we get $|h_{1,2} + \theta + 1|_p = |h_{1,2} + 2 + \theta - 1|_p \leq \max\{|h_{1,2} + 2|_p, |\theta - 1|_p\} < 1$.

By Lemma 1.4.2, we have $\lim_{n \rightarrow \infty} |\mu_{h_0}^{(n)}|_p = 1$,

$$\lim_{n \rightarrow \infty} |\mu_{h_{1,2}}^{(n)}|_p = \lim_{n \rightarrow \infty} \left| \frac{1}{(h_{1,2} + 2)(h_{1,2} + \theta + 1)^{2^{n+1}-2}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x), x} \right|_p = \infty.$$

For two periodic non-translation-invariant solutions we have $|\theta - 1|_p < 1$,

$$\begin{aligned} |h_{3,4} + 2|_p &= \\ &= \left| \frac{-72(\theta - 1) - 45(\theta - 1)^2 - 12(\theta - 1)^3 - (\theta - 1)^4 \pm (\theta - 1)(\theta + 2)\sqrt{D_1(\theta)}}{2(3 + 3(\theta - 1) + (\theta - 1)^2)^2} \right|_p < 1. \end{aligned}$$

Consequently, we get

$$|h_{3,4} + \theta + 1|_p = |h_{3,4} + 2 + \theta - 1|_p \leq \max\{|h_{3,4} + 2|_p, |\theta - 1|_p\} < 1.$$

By Lemma 1.4.3 we have

$$\lim_{n \rightarrow \infty} |\mu_{h_{3,4}}^{(n)}|_p = \infty.$$

Case 3. If $p = 3$, then there exist μ_{h_i} , $i = 0, \dots, 5$ measures. Note that

$$|h_0 + \theta + q - 2|_3 = |\theta + 2|_3 < 1.$$

By Lemma 1.4.2 we get

$$\lim_{n \rightarrow \infty} |\mu_{h_0}^{(n)}|_3 = \lim_{n \rightarrow \infty} \left| \frac{1}{3(\theta + 2)^{2^{n+1}-2}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x), x} \right|_3 = \infty.$$

We can prove that μ_{h_1} , μ_{h_2} , μ_{h_3} , μ_{h_4} measures are unbounded as in the case 2.

Using this results we get the following theorem:

Theorem 1.4.5. Let $p \equiv 1(\text{mod } 8)$ or $p \equiv 3(\text{mod } 8)$, $p \neq 3$. Then for p -adic 3-state Potts model on a Cayley tree of order two there exists a phase transition.

CONCLUSION OF THE FIRST CHAPTER

The first chapter is devoted to present the important concepts and main results related to topic of the dissertation, and obtained results, G_2 -periodic p -adic quasi Gibbs measures for the Potts model on the Cayley tree of order two.

Section 1.1 contains important concepts and main results related to the topics congruences and residues of number theory. The linear congruences, system of linear congruences, quadratic residues and some main theorems related to solving congruences are given in the section 1.1.

Section 1.2 contains basic concepts of metric and normed spaces, p -adic norm, p -adic numbers and their main properties, Ostrowski's theorem, Hensel's lemma, important functions of p -adic measure theory and their essential properties.

Section 1.3 is devoted to present mainly, non-Archimedean analogue of Kolmogorov's extension theorem.

Section 1.4 is devoted to present G_2 -periodic p -adic quasi Gibbs measures for the Potts model on the Cayley tree of order two. If $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$, $p \neq 3$, then the existence of a phase transition for 3-state p -adic Potts model on the Cayley tree of order two was proved.

CHAPTER II. p -ADIC GENERALIZED GIBBS MEASURES FOR THE ISING MODEL ON THE CAYLEY TREE

This chapter is devoted to the p -adic Ising model on the Cayley tree. This chapter consists of three sections. The first section contains main known results related to p -adic Ising model on the Cayley tree. The second and third sections include translation-invariant and $G_k^{(2)}$ -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three, respectively.

§ 2.1. Main results about p -adic Gibbs measures for the Ising model on the Cayley tree

Let \mathbb{Q}_p be a field of p -adic numbers and $\Phi = \{-1, 1\}$. A formal Hamiltonian $H : \Omega \rightarrow \mathbb{Q}_p$ of the p -adic Ising model is defined by

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y), \quad (2.1.1)$$

where $J \in B\left(0, p^{\frac{1}{1-p}}\right)$.

Let $h : x \in V \setminus \{x^0\} \rightarrow h_x \in \mathbb{Q}_p$ be a function. We define p -adic probability generalized Gibbs distribution $\mu_h^{(n)}$ on Ω_{V_n} by

$$\mu_h^{(n)}(\sigma) = \frac{1}{Z_n^{(h)}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_x^{\sigma(x)}, \quad n = 1, 2, \dots, \quad (2.1.2)$$

where, $\sigma \in \Omega_{V_n}$, $Z_n^{(h)}$ is the corresponding normalizing constant:

$$Z_n^{(h)} = \sum_{\sigma \in \Omega_{V_n}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_x^{\sigma(x)}. \quad (2.1.3)$$

The compatibility conditions for $\mu_h^{(n)}$, $n \geq 1$ are given by the equality

$$\sum_{\varphi^{(n)} \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu_h^{(n-1)}(\sigma_{n-1}). \quad (2.1.4)$$

In this case, by the p -adic analogue of Kolmogorov theorem ([24]) there exists a unique measure μ_h on the set Ω such that $\mu_h(\{\sigma|_{V_n} \equiv \sigma_n\}) = \mu_h^{(n)}(\sigma_n)$, for all $n \in \mathbb{N}$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$.

A limiting p -adic distribution generated by (2.1.2) is called p -adic *generalized Gibbs measure* [30].

Theorem 2.1.1. [30] The sequence of p -adic distributions $\{\mu_h^{(n)}\}_{n \geq 1}$ determined by formula (2.1.2) is consistent if and only if for any $x \in V \setminus \{x^0\}$, the following equation holds

$$h_x^2 = \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta}, \quad (2.1.5)$$

where $\theta = \exp_p \{2J\}$.

It is known [18] that the ferromagnetic Ising model with nearest-neighbor interaction on a Cayley tree has a trivial partition function unless an external magnetic field is presented.

In [30] the uniqueness of p -adic Gibbs measure for Ising model is proved and p -adic generalized Gibbs measures were introduced. Furthermore, in [30] the translation-invariant and $G_k^{(2)}$ -periodic p -adic generalized Gibbs measures for the same model on the Cayley trees have been studied. In [70] the authors proved the existence of infinitely many periodic p -adic generalized Gibbs measures for the p -adic Ising model on the semi-infinite Cayley tree of order k . Recently, in [67] the authors studied translation-invariant generalized p -adic Gibbs measures for the Ising model on Cayley trees of order k . Moreover, the set of fixed points of the Ising-Potts mapping was described, which allows to detect the existence of the phase transition.

According to [59], [70], [85], the chaos of the RG (renormalization group) implies vastness of the periodic p -adic Gibbs measures of the Ising model on semi-infinite Cayley tree.

§ 2.2. Translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three

Let G_k^* be a normal subgroup of the group G_k (see Section 1.3). A function h_x (a configuration $\sigma(x), x \in V$) is called G_k^* -periodic if $h_{yx} = h_x$ (resp., $\sigma(yx) = \sigma(x)$) for any $x \in G_k$ and $y \in G_k^*$. A G_k -periodic function is called *translation-invariant*. The Gibbs measure is called G_k^* -periodic (translation-invariant) if it corresponds to G_k^* -periodic (translation-invariant) function h .

We consider translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree Γ^3 . Thanks to Theorem 2.1.1, in order to find all translation-invariant measures it is enough to consider the following equation

$$h^2 = \left(\frac{\theta h^2 + 1}{h^2 + \theta} \right)^3. \quad (2.2.1)$$

Denoting $z = h^2$, from the last one we get

$$z = \left(\frac{\theta z + 1}{z + \theta} \right)^3, \quad (2.2.2)$$

which is equivalent to

$$(z^2 - 1)(z^2 + (3\theta - \theta^3)z + 1) = 0. \quad (2.2.3)$$

The equation (2.2.3) has at least two solutions $z_{1,2} = \pm 1$. If there exists $\sqrt{\theta^2 - 4}$ in \mathbb{Q}_p then (2.2.6) has exactly four solutions, which are $z_{1,2}$ and

$$z_{3,4} = \frac{\theta^3 - 3\theta \pm (\theta^2 - 1)\sqrt{\theta^2 - 4}}{2}.$$

Denote $\Delta(\theta) = \theta^2 - 4$.

Lemma 2.2.1. A number $\sqrt{\Delta(\theta)}$ exists in \mathbb{Q}_p if and only if $p \equiv 1 \pmod{6}$.

Proof. Since $\theta \in \mathcal{E}_p$, due to Lemma 1.2.4 we have $|\theta^2 - 1|_p < 1$. It yields that $|\Delta(\theta) + 3|_p < 1$. Then according to Theorem 1.2.2, a number $\sqrt{\Delta(\theta)}$ exists in \mathbb{Q}_p if and only if $\sqrt{-3} \in \mathbb{Q}_p$. Again thanks to Theorem 1.2.2, the existence of $\sqrt{-3}$ is equivalent to the solvability of $x^2 + 3 \equiv 0 \pmod{p}$. By Theorem 1.1.3, it is known that the congruence $x^2 + 3 \equiv 0 \pmod{p}$ is solvable if and only if $p \equiv 1 \pmod{6}$. Since $z_3 z_4 = 1$, one can conclude that the existence of $\sqrt{z_3}$ implies the existence of $\sqrt{z_4}$. For this reason it is enough to check the existence of $\sqrt{z_3}$. We give the set of all translation-invariant p -adic generalized Gibbs measures for the Ising model.

Lemma 2.2.2. The number $\sqrt{z_3}$ exists in \mathbb{Q}_p if and only if $p \equiv 1 \pmod{12}$.

Proof. Assume that z_3 is a solution of (2.2.1). Then due to Lemma 2.2.1 we have $p \equiv 1 \pmod{6}$. In order to find $|z_3 + 1|_p$, we consider

$$z_3 + 1 = \frac{\theta^3 - 3\theta + 2 + (\theta^2 - 1)\sqrt{\Delta(\theta)}}{2} = \frac{(\theta - 1)(\theta^2 + \theta - 2 + (\theta + 1)\sqrt{\Delta(\theta)})}{2}.$$

Since $p \neq 2$ and $|\theta - 1|_p < 1$ one has $|z_3 + 1|_p < 1$. Again keeping in mind $p \neq 2$ and thanks to Theorem 1.2.2 the existence of $\sqrt{z_3}$ is equivalent to the existence of the number $\sqrt{-1}$. Then using Corollary 1.2.1 we conclude that $\sqrt{z_3} \in \mathbb{Q}_p$ if and only if $p \equiv 1 \pmod{4}$. Using The Chinese Remainder Theorem, combining $p \equiv 1 \pmod{6}$ with $p \equiv 1 \pmod{4}$ we get $p \equiv 1 \pmod{12}$.

Proposition 2.2.1 Let \mathcal{N}_p be a number of solutions of (2.2.1). Then

$$\mathcal{N}_p = \begin{cases} 2, & \text{if } p \not\equiv 1 \pmod{4}, \\ 4, & \text{if } p \equiv 5 \pmod{12}, \\ 8, & \text{if } p \equiv 1 \pmod{12}. \end{cases}$$

Proof. Assume that $p \not\equiv 1 \pmod{4}$. Then due to Corollary 1.2.1 we have $\sqrt{z_2} \notin \mathbb{Q}_p$. Moreover, according to Lemma 2.2.2 one has $\sqrt{z_3}, \sqrt{z_4} \notin \mathbb{Q}_p$. Hence, in this case (2.2.1) has exactly two solutions: $h_1 = \sqrt{z_1}$ and $-h_1$. Now, we suppose that $p \equiv 1 \pmod{4}$ and $p \not\equiv 1 \pmod{3}$, i.e. $p \equiv 5 \pmod{12}$. Then again due to Corollary 1.2.1 there exists $\sqrt{z_2}$ in \mathbb{Q}_p and thanks to Theorem 1.2.2 $z_{3,4} \notin \mathbb{Q}_p$. Consequently, (2.2.1) has exactly four solutions: $\pm h_1, h_2 = \sqrt{z_2}$ and $-h_2$. Let us assume that $p \equiv 1 \pmod{12}$. In this case according to Corollary 1.2.1 and by Theorem 1.2.2 there exist $\sqrt{z_2}$ and $\sqrt{z_{3,4}}$ in \mathbb{Q}_p , which imply that the equation (2.2.1) has exactly eight solutions: $\pm h_1, \pm h_2, h_3 = \sqrt{z_3}$ and $-h_3$.

We denote by $TIpGGM$ the set of all translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order two. Notation $|A|$ means the cardinality of the set A .

Theorem 2.2.1. The following assertions hold for the number of translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order two

$$|TIpGGM| = \begin{cases} 1, & \text{if } p \not\equiv 1 \pmod{4}, \\ 2, & \text{if } p \equiv 5 \pmod{12}, \\ 4, & \text{if } p \equiv 1 \pmod{12}. \end{cases}$$

Proof. The proof follows from Theorem 2.1.1 and Proposition 2.2.1.

Now we study the boundedness of p -adic translation-invariant generalized Gibbs measures $\mu_h, i = 1, 2, 3, 4$. We need some auxiliary lemmas.

Lemma 2.2.3. [91] Let h be a translation-invariant solution to equation (2.1.5) and μ_h be a corresponding translation-invariant p -adic generalized Gibbs measure. Then for normalizing constant $Z_{n,h}$ we have

$$Z_{n+1,h} = A_{n,h} Z_{n,h}, \quad (2.2.4)$$

where

$$A_{n,h} = \prod_{x \in W_n} a_h(x),$$

$$a_h(x) h_x^t = \prod_{y \in S(x)} \sum_{u \in \{-1,1\}} \exp\{Jtu\} h_y^u, \quad a_h(x) \in \mathbb{Q}_p, \quad t \in \{-1,1\}.$$

Lemma 2.2.4. The norms of the solutions h_1, h_2, h_3, h_4 are equal to one.

Proof. Since $|z_i|_p = 1, i = 1, 2, 3, 4$ one can immediately obtain $|\sqrt{z_i}|_p = 1, i = 1, 2, 3, 4.$

Lemma 2.2.5. Let h be a translation-invariant solution of (2.1.5) then for the corresponding partition function $Z_n^{(h)}$ the following equalities hold:

$$Z_1^{(h)} = \frac{(\theta h^2 + 1)^{k+1} + (h^2 + \theta)^{k+1}}{\theta^{\frac{k+1}{2}} h^{k+1}},$$

$$Z_n^{(h)} = \left(\frac{(\theta h^2 + 1)(h^2 + \theta)}{\theta h^2} \right)^{\frac{(k+1)(k^n - k)}{2(k-1)}} Z_1^{(h)}.$$

Proof. We put $t=1$ and $t=-1$ into the formula Lemma 2.2.3 and multiply them one gets

$$a_h^2(x) = \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y} \prod_{y \in S(x)} \frac{h_y^2 + \theta}{\theta h_y},$$

where $\theta = \exp_p\{2J\}$. Since h_x is a translation-invariant solutions of (2.1.5). Then we get

$$a_h(x) = \left(\frac{(\theta h^2 + 1)(h^2 + \theta)}{\theta h^2} \right)^{\frac{k}{2}}.$$

Using the last equality and according to Lemma 2.2.3 we obtain

$$A_{h,n} = \left(\frac{(\theta h^2 + 1)(h^2 + \theta)}{\theta h^2} \right)^{\frac{k|W_n|}{2}},$$

or

$$A_{h,n} = \left(\frac{(\theta h^2 + 1)(h^2 + \theta)}{\theta h^2} \right)^{\frac{(k+1)k^n}{2}}.$$

Substituting the expression of $A_{h,n}$ into (2.1.4) we have

$$Z_n^{(h)} = \left(\frac{(\theta h^2 + 1)(h^2 + \theta)}{\theta h^2} \right)^{\frac{(k+1)(k^n - k)}{2(k-1)}} Z_1^{(h)}.$$

Let $h_{\sigma(x),x} = h^{\sigma(x)}$, then by (2.1.3) we get

$$Z_1^{(h)} = \frac{(\theta h^2 + 1)^{k+1} + (h^2 + \theta)^{k+1}}{\theta^{\frac{k+1}{2}} h^{k+1}}.$$

Lemma 2.2.6. Let $k = 3$. For the normalizing constants Z_{n+1,h_i} , $i = \overline{1,4}$ we have

- $|Z_{n,h_1}|_p = \begin{cases} 1, & \text{if } p \neq 2, \\ 2^{-2(3^n+1)}, & \text{if } p = 2. \end{cases}$
- $|Z_{n+1,h_i}|_p \leq |\theta - 1|_p^{2(3^n-1)}$, $i = 2, 3, 4$.

Proof. 1) By Lemma 2.2.5, for the normalizing constant Z_{n,h_1} we obtain

$$|Z_{n,h_1}|_p = \left| \frac{\theta + 1}{\theta} \right|_p^{2(3^n-3)} \cdot \left| \frac{2(\theta + 1)}{\theta^2} \right|_p^4 = \begin{cases} 1, & \text{if } p \neq 2, \\ 2^{-2(3^n+1)}, & \text{if } p = 2. \end{cases}$$

2) Let h_2 be a solution to (2.2.1). Then due to Lemma 2.2.5 we have

$$Z_{n,h_2} = \left(\frac{\theta-1}{\theta} \right)^{2(3^n-3)} \cdot \frac{2(\theta-1)^4}{\theta^2}.$$

Thus we obtain

$$\left| Z_{n+1,h_2} \right|_p = |\theta-1|_p^{2(3^n-1)}.$$

3) Let h_3 be a solution of (2.2.1).

$$\begin{aligned} |h_3^2 + 1|_p = |z_3 + 1|_p &= \left| \frac{\theta^3 - 3\theta + (\theta^2 - 1)\sqrt{\Delta(\theta)}}{2} + 1 \right|_p = \\ &= \left| \frac{(\theta-1)((\theta-1)(\theta+2) + (\theta+1)\sqrt{\Delta(\theta)})}{2} \right|_p = |\theta-1|_p. \end{aligned}$$

Since $\theta \in \mathcal{E}_p$ using the strong triangle inequality we have

$$|\theta h_3^2 + 1|_p = |h_3^2 + \theta|_p = |h_3^2 + 1|_p = |\theta - 1|_p.$$

Using $|h_4|_p = 1$ and $z_3 z_4 = 1$ we obtain the following

$$|h_3^2 + 1|_p = \left| \frac{1}{h_4^2} + 1 \right|_p = |h_4^2 + 1|_p.$$

It follows that

$$Z_{n,h_3} = \left(\frac{(\theta z_3 + 1)(z_3 + \theta)}{\theta z_3} \right)^{3^n-3} \cdot \frac{(\theta z_3 + 1)^4 + (z_3 + \theta)^4}{\theta^2 z_3^2}$$

or

$$\left| Z_{n,h_3} \right|_p = \left| Z_{n,h_4} \right|_p \leq |\theta-1|_p^{2(3^n-1)}.$$

Theorem 2.2.2. Let H be the Hamiltonian of the Ising model on a Cayley tree order three. Then the following statements are hold:

- i) if $p = 2$ then the unique p -adic translation-invariant generalized Gibbs measure μ_{h_1} is not bounded,
- ii) if $p \neq 2$ then among the p -adic translation-invariant generalized Gibbs measures only μ_{h_1} is bounded.

Proof. (i) Let $p = 2$. In this case due to Theorem 2.2.1 there exists a unique translation-invariant p -adic generalized Gibbs measure μ_{h_1} . Then by Lemma 2.2.6 we have

$$|\mu_{h_1}^{(n)}(\sigma)|_2 = 2^{2(3^n+1)}, \quad \forall n \in \mathbb{N}, \forall \sigma \in \Omega_{V_n}.$$

Hence, $|\mu_{h_1}^{(n)}(\sigma)|_2 \rightarrow \infty$ as $n \rightarrow \infty$. It means that the measure μ_{h_1} is not bounded.

- (ii) Let $p \neq 2$. According to Lemma 2.2.6 one has

$$|\mu_{h_1}^{(n)}(\sigma)|_2 = 1, \quad \forall n \in \mathbb{N}, \forall \sigma \in \Omega_{V_n},$$

which implies boundedness of limiting measure μ_{h_1} . Moreover, if there exists at least one of the measures μ_{h_2} , μ_{h_3} and μ_{h_4} then again using Lemma 2.2.6 we can verify that they are not bounded.

Thanks to Theorem 2.2.1 and Theorem 2.2.2 we get the following

Theorem 2.2.3. If $p \equiv 1 \pmod{12}$ then for the Ising model on a Cayley tree of order three a phase transition occurs.

§ 2.3. $G_3^{(2)}$ -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three

Let $G_k^{(2)}$ be the subgroup in G_k consisting of all words of even length, i.e. $G_k^{(2)} = \{x \in G_k : |x| \text{ is even}\}$, where $|x|$ is the length of x . In fact, $G_k^{(2)}$ is a normal subgroup of G_k of index two, i.e.

$$G_k / G_k^{(2)} = \{G_k^{(2)}, G_k \setminus G_k^{(2)}\}.$$

In [30] the translation-invariant and $G_k^{(2)}$ -periodic generalized Gibbs measures for Ising model on the Cayley tree of order two were studied. In this section, we are aiming to construct $G_k^{(2)}$ -periodic generalized Gibbs measures of the Ising model on the Cayley tree of order three.

According to Theorem 2.1.1, in order to find all $G_k^{(2)}$ -periodic measures it is sufficient to consider the following system of equations

$$\begin{cases} \tilde{h}_1 = \left(\frac{\theta \tilde{h}_2 + 1}{\tilde{h}_2 + \theta} \right)^k, \\ \tilde{h}_2 = \left(\frac{\theta \tilde{h}_1 + 1}{\tilde{h}_1 + \theta} \right)^k. \end{cases} \quad (2.3.1)$$

Denote

$$f(h) = \left(\frac{\theta h + 1}{h + \theta} \right)^k. \quad (2.3.2)$$

We consider the case $k = 3$. Solving the system of equations (2.3.1) is equivalent to find two-periodic fixed points of $f(h)$. We may find two-periodic non-translation-invariant generalized Gibbs measures by solving the following equation:

$$\frac{f(f(h)) - h}{f(h) - h} = 0. \quad (2.3.3)$$

Simplifying the last one, we have

$$\begin{aligned} & (\theta^3 h^2 + (3\theta^2 - 1)h + \theta^3) \cdot ((\theta^6 + 3\theta^4 + 3\theta^2 + 1)(h^4 + 1) + \\ & + (6\theta^5 + 20\theta^3 + 6\theta)(h^3 + h) + (24\theta^4 + 24\theta^2)h^2) = 0. \end{aligned} \quad (2.3.4)$$

First, we consider the following equation

$$\theta^3 h^2 + (3\theta^2 - 1)h + \theta^3 = 0. \quad (2.3.5)$$

Its solutions are

$$h_{1,2} = \frac{1 - 3\theta^2 \pm (\theta^2 - 1)\sqrt{1 - 4\theta^2}}{2\theta^3}. \quad (2.3.6)$$

Lemma 2.3.1. The solutions (2.3.6) belong to \mathbb{Q}_p iff $p \equiv 1 \pmod{6}$.

Proof. Let us denote $D(\theta) = 1 - 4\theta^2$. We rewrite $D(\theta) = -3 - 8(\theta - 1) - 4(\theta - 1)^2$. It is clear that the equation (2.3.5) has a solution in \mathbb{Q}_p if and only if $\sqrt{D(\theta)}$ exists in \mathbb{Q}_p . Thanks to Theorem 1.2.2, $\sqrt{D(\theta)}$ exists in \mathbb{Q}_p if and only if the congruence $x^2 + 3 \equiv 0 \pmod{p}$ has a solution. By Theorem 1.1.3, the congruence $x^2 + 3 \equiv 0 \pmod{p}$ has a solution if and only if $p \equiv 1 \pmod{6}$.

Now, we are going to investigate the second factor of (2.2.4), i.e.

$$(\theta^6 + 3\theta^4 + 3\theta^2 + 1)(h^4 + 1) + (6\theta^5 + 20\theta^3 + 6\theta)(h^3 + h) + (24\theta^4 + 24\theta^2)h^2 = 0. \quad (2.3.7)$$

Dividing both sides of (2.3.7) by h^2 and introducing a new variable $z = h + \frac{1}{h}$ one finds

$$(\theta^6 + 3\theta^4 + 3\theta^2 + 1)z^2 + (6\theta^5 + 20\theta^3 + 6\theta)z - 2\theta^6 + 18\theta^4 + 18\theta^2 - 2 = 0. \quad (2.3.8)$$

Clearly, its solutions are

$$z_{1,2} = \frac{-(3\theta^5 + 10\theta^3 + 3\theta) \pm (\theta^2 - 1)^2 \sqrt{(2\theta^2 + 1)(\theta^2 + 2)}}{(\theta^2 + 1)^3}. \quad (2.3.9)$$

Assume that $\theta = 1 + \theta_s p^s + \theta_{s+1} p^{s+1} + \dots$, where $\theta_s \neq 0$.

Lemma 2.3.2. For the solutions of the equation (2.3.8) the following statements hold:

- 1) if $p \neq 3$ then $z_{1,2} \in \mathbb{Q}_p$;
- 2) if $p = 3$ and $|\theta - 1|_3 \leq \frac{1}{9}$ then $z_{1,2} \in \mathbb{Q}_3$.

Proof. With the notation $\Delta(\theta) = (2\theta^2 + 1)(\theta^2 + 2)$, we have

$$\Delta(\theta) = 9 + 18(\theta - 1) + 17(\theta - 1)^2 + 8(\theta - 1)^3 + 2(\theta - 1)^4.$$

The equation (2.3.8) has a solution in \mathbb{Q}_p if and only if $\sqrt{\Delta(\theta)}$ exists in \mathbb{Q}_p .

Let us consider several cases with respect to p .

Case 1. Let $p \geq 5$. Since $\theta \in \mathcal{E}_p$, by rewriting $\Delta(\theta) = 9 + o[1]$, and according to Theorem 1.2.2, we get $z_{1,2} \in \mathbb{Q}_p$.

Case 2. Let $p = 2$. Then $\Delta(\theta) = 1 + o[4]$. Again by Theorem 1.2.2, one finds $z_{1,2} \in \mathbb{Q}_2$.

Case 3. Let $p = 3$.

Case 3.1. Let $s \geq 2$. Then $\Delta(\theta) = 3^2(1 + o[1])$. By Theorem 1.2.2, we infer $z_{1,2} \in \mathbb{Q}_3$.

Case 3.2. Let $s = 1$. Then it is not difficult to check that the existence of $\sqrt{\Delta(\theta)}$ in \mathbb{Q}_3 depends on the coefficients of θ .

Keeping in mind $z = h + \frac{1}{h}$, we obtain $h = \frac{z \pm \sqrt{z^2 - 4}}{2}$.

Therefore, let

$$h_{3,4} = \frac{z_1 \pm \sqrt{z_1^2 - 4}}{2}, \quad h_{5,6} = \frac{z_2 \pm \sqrt{z_2^2 - 4}}{2}.$$

Using (2.3.9) we find

$$\sqrt{z^2 - 4} = \frac{(\theta^2 - 1)\sqrt{u(\theta) \pm v(\theta)\sqrt{\Delta(\theta)}}}{(\theta^2 + 1)^3},$$

where

$$u(\theta) = -2(\theta^8 + 11\theta^6 + 24\theta^4 + 11\theta^2 + 1),$$

$$v(\theta) = 2(3\theta^5 + 10\theta^3 + 3\theta).$$

Denote $\Delta_1(\theta) = u(\theta) \pm v(\theta)\sqrt{\Delta(\theta)}$. The existence of h in \mathbb{Q}_p is equivalent to the existence of $\sqrt{\Delta_1(\theta)}$ in \mathbb{Q}_p .

Lemma 2.3.3. $\sqrt{\Delta_1(\theta)}$ exists in \mathbb{Q}_p if and only if one of the following conditions hold:

- 1) $p \equiv 1 \pmod{6}$;
- 2) $p = 2, |\theta - 1|_2 = \frac{1}{4}$;
- 3) $p = 3, \text{ord}_3 |\theta - 1|_3$ is odd, $|\theta - 1|_3 \leq \frac{1}{27}$ and $|\theta - 1|_3 (\theta - 1) \in \mathcal{E}_3$.

Proof. Case 1. Let $p \geq 5$. Then $z_{1,2}$ exists in \mathbb{Q}_p . Rewrite $\sqrt{\Delta(\theta)}, u(\theta), v(\theta)$ as follows

$$\begin{aligned} \sqrt{\Delta(\theta)} &= \sqrt{\left(3 + 3(\theta - 1) + \frac{4}{3}(\theta - 1)^2\right)^2 + \frac{2}{9}(\theta - 1)^4} = \\ &= \left(3 + 3(\theta - 1) + \frac{4}{3}(\theta - 1)^2\right) \sqrt{1 + o[(\theta - 1)^3]} = \end{aligned}$$

$$= 3 + 3(\theta - 1) + \frac{4}{3}(\theta - 1)^2 + o[(\theta - 1)^3],$$

$$u(\theta) = -96 - 384(\theta - 1) - 696(\theta - 1)^2 + o[(\theta - 1)^2],$$

$$v(\theta) = 32 + 96(\theta - 1) + 120(\theta - 1)^2 + o[(\theta - 1)^2].$$

It follows that

$$\Delta_1(\theta) = -96 - 384(\theta - 1) - 696(\theta - 1)^2 + o[(\theta - 1)^2] \pm \left(96 + 384(\theta - 1) + \frac{2072}{3}(\theta - 1)^2 + o[(\theta - 1)^2] \right).$$

Case 1.1. Let $\Delta_1(\theta) = u(\theta) + v(\theta)\sqrt{\Delta(\theta)}$. Then

$$\Delta_1(\theta) = (\theta - 1)^2 \left(-\frac{16}{3} + o[1] \right).$$

According to Theorem 1.2.2, $\sqrt{\Delta_1(\theta)}$ exists in \mathbb{Q}_p iff $-\frac{1}{3}$ is a quadratic residue modulo p , i.e. the congruence $3x^2 + 1 \equiv 0 \pmod{p}$ is solvable. Using the substitution $y = \frac{1}{x}$ we get $y^2 \equiv -3 \pmod{p}$. The congruence $y^2 \equiv -3 \pmod{p}$ is solvable iff $p \equiv 1 \pmod{6}$ (see Theorem 1.1.3).

Case 1.2. Let $\Delta_1(\theta) = u(\theta) - v(\theta)\sqrt{\Delta(\theta)}$. Then $\Delta_1(\theta) = -192 + o[1]$. Again thanks to Theorem 1.2.2, $\sqrt{\Delta_1(\theta)}$ exists in \mathbb{Q}_p iff -3 is a quadratic residue modulo p , i.e. $p \equiv 1 \pmod{6}$.

Case 2. Let $p = 3$. Then

$$\Delta(\theta) = 3(1 + o[1]), \quad \Delta_1(\theta) = 3(-32 - 128\theta_s \cdot 3^s \pm 32 + o[3^s]).$$

Case 2.1. Let

$$\Delta_1(\theta) = u(\theta) + v(\theta)\sqrt{\Delta(\theta)}.$$

Then

$$\Delta_1(\theta) = 3^{s+1}(\theta_s + o[1]).$$

Due to Theorem 1.2.2 and Lemma 2.3.2, $\sqrt{\Delta_1(\theta)} \in \mathbb{Q}_3$ iff $s \geq 3$, s is odd and $\theta_s = 1$, i.e. $\text{ord}_3 |\theta - 1|_3$ is odd, $|\theta - 1|_3 \leq \frac{1}{27}$ and $|\theta - 1|_3 (\theta - 1) \in \mathcal{E}_3$.

Case 2.2. Let

$$\Delta_1(\theta) = u(\theta) + v(\theta)\sqrt{\Delta(\theta)}.$$

Then $\Delta_1(\theta) = 3(2 + o[1])$. Due to Theorem 1.2.2, $\sqrt{\Delta_1(\theta)} \notin \mathbb{Q}_3$.

Case 3. Let $p = 2$. Then

$$\Delta(\theta) = 3 + 3(\theta - 1) + \frac{4}{3}(\theta - 1)^2 + o[2^{4s}],$$

$$\begin{aligned} \Delta_1(\theta) = & -96 - 384(\theta - 1) - 696(\theta - 1)^2 - 744(\theta - 1)^3 - 518(\theta - 1)^4 \pm \\ & \pm \left(96 + 384(\theta - 1) + \frac{2072}{3}(\theta - 1)^2 + 728(\theta - 1)^3 + 520(\theta - 1)^4 \right) + o[2^{4s}]. \end{aligned}$$

Case 3.1. Let

$$\Delta_1(\theta) = u(\theta) + v(\theta)\sqrt{\Delta(\theta)}.$$

Then

$$\Delta_1(\theta) = 16(\theta - 1)^2(1 + 2^2 - 2^s + o[2^2]).$$

According to Theorem 1.2.2, $\sqrt{\Delta_1(\theta)} \in \mathbb{Q}_2$ iff $s = 2$, i.e. $|\theta - 1|_2 = \frac{1}{4}$.

Case 3.2. Let

$$\Delta_1(\theta) = u(\theta) - v(\theta)\sqrt{\Delta(\theta)}.$$

Then

$$\Delta_1(\theta) = 2^6(1 + 2^2 + o[2^2]).$$

According to Theorem 1.2.2, $\sqrt{\Delta_1(\theta)} \notin \mathbb{Q}_2$.

Theorem 2.3.1. Let M be the number of p -adic non-translation-invariant $G_k^{(2)}$ -periodic generalized Gibbs measures for the Ising model on the Cayley tree of order three. Then the following assertions hold:

$$M = \begin{cases} 6, & p \equiv 1 \pmod{6}; \\ 2, & p = 2, |\theta - 1|_2 = \frac{1}{4}; \\ 2, & p = 3, \text{ord}_3 |\theta - 1|_3 \text{ is odd, } |\theta - 1|_3 \leq \frac{1}{27} \text{ and } |\theta - 1|_3 (\theta - 1) \in \mathcal{E}_3; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $h_i, i = \overline{1,6}$ be the solutions of the equation (2.3.4).

If $p \equiv 1 \pmod{6}$ then thanks to Lemmas 2.3.1 and 2.3.2, $h_i \in \mathbb{Q}_p, i = \overline{1,6}$, i.e. $N = 6$

. Let $p = 3, \text{ord}_3 |\theta - 1|_3$ be odd, $|\theta - 1|_3 \leq \frac{1}{27}$ and $|\theta - 1|_3 (\theta - 1) \in \mathcal{E}_3$ then due to

Lemma 2.3.2, $h_{5,6} \in \mathbb{Q}_3$ and $h_i \notin \mathbb{Q}_3, i = \overline{1,4}$, i.e., $M = 2$ (see the cases 2.1 and 2.2 of the proof of Lemma 2.3.2).

Let $p = 2, |\theta - 1|_2 = \frac{1}{4}$ then due to Lemma 2.3.2, $h_{5,6} \in \mathbb{Q}_2$ and $h_i \notin \mathbb{Q}_2$

$i = \overline{1,4}$, i.e., $M = 2$ (see the cases 3.1 and 3.2 of the proof of Lemma 2.3.2.)

The rest of the proof is straightforward by Lemmas 2.3.1 and 2.3.2.

Remark 2.3.1. In the field of real numbers the maximal number of solutions of (2.3.4) is two, however, this number for the field of p -adic numbers equals six.

Boundedness of $G_k^{(2)}$ -periodic p -adic generalized Gibbs measures and phase transitions

In this section we are aiming to study the boundedness of the measures found in Theorem 2.3.2.

Lemma 2.3.4. Let h_i ($i = \overline{1,6}$) be the solutions of (2.3.4). Then the following statements hold true:

$$1) h_i h_{i+1} = 1, f(h_i) = h_{i+1}, f(h_{i+1}) = h_i, i \in \{1, 3, 5\};$$

$$2) |h_i|_p = 1, i = \overline{1,6};$$

$$3) |h_i + \theta|_p < 1, |h_i + 1|_p < 1, |\theta h_i + 1|_p < 1, i = \overline{1,6}.$$

Proof. 1) Let $i=1$. Since h_1, h_2 are solutions of the equation $\theta^3 h^2 + (3\theta^2 - 1)h + \theta^3 = 0$, then we have $h_1 h_2 = 1$. It is easy to see that $h_2^2 - 1 \neq 0$, and

$$(h_2^2 - 1)(\theta^3 h_2^2 + (3\theta^2 - 1)h_2 + \theta^3) = 0.$$

The last equality gives us

$$h_2(\theta^3 h_2^3 + 3\theta^2 h_2^2 + 3\theta h_2 + 1) - h_2^3 - 3\theta h_2^2 - 3\theta^2 h_2 - \theta^3 = 0,$$

which implies

$$\frac{1}{h_2} = \left(\frac{\theta h_2 + 1}{h_2 + \theta} \right)^3, h_1 = \left(\frac{\theta h_2 + 1}{h_2 + \theta} \right)^3, h_1 = f(h_2).$$

By the same argument we can show

$$h_3 h_4 = 1, h_5 h_6 = 1,$$

and

$$h_2 = f(h_1), h_3 = f(h_4), h_4 = f(h_3), h_5 = f(h_6), h_6 = f(h_5).$$

2) Let $p \equiv 1 \pmod{6}$. Since $\theta \in \mathcal{E}_p$, one has

$$|h_1|_p = \left| \frac{1 - 3\theta^2 + (\theta^2 - 1)\sqrt{1 - 4\theta^2}}{2\theta^3} \right|_p = \left| \frac{(\theta - 1)(\theta + 1)(-3 + \sqrt{1 - 4\theta^2}) - 2}{2\theta^3} \right|_p = 1.$$

Let $i \in \{3, 4, 5, 6\}$, $p \equiv 1 \pmod{6}$ or $p = 3$. Then one gets

$$|z|_p = \left| \frac{-16 + o[1]}{2} \right|_p = 1, |h_i|_p = \left| \frac{z \pm \sqrt{z^2 - 4}}{2} \right|_p = \left| \frac{z \pm o[1]}{2} \right|_p = 1.$$

Let $p = 2$. According to Lemma 2.3.3, there exists $h_{5,6}$ iff $|\theta - 1|_2 = \frac{1}{4}$, which implies

$$|z_{1,2}|_2 = \left| \frac{-16 + o[16]}{2} \right|_2 = \frac{1}{8}, |h_{5,6}|_2 = \left| \frac{z_2 \pm \sqrt{z_2^2 - 4}}{2} \right|_2 = \frac{2^{-1}}{2^{-1}} = 1.$$

3) Let $i \in \{1, 2\}$, $p \equiv 1 \pmod{6}$. Then one gets

$$|h_i + \theta|_p = \left| \theta + \frac{1 - 3\theta^2 \pm (\theta^2 - 1)\sqrt{1 - 4\theta^2}}{2\theta^3} \right|_p = |\theta - 1|_p \left| -1 + 2\theta^2 \pm \sqrt{1 - 4\theta^2} \right|_p < 1.$$

Let $i \in \{3, 4, 5, 6\}$, $p \equiv 1 \pmod{6}$ or $p = 3$. Then it yields

$$\begin{aligned} |h_i + \theta|_p &= \left| \theta + \frac{z \pm \sqrt{z^2 - 4}}{2} \right|_p = \left| z + 2\theta \pm \sqrt{z^2 - 4} \right|_p = \\ &= \left| (\theta - 1)(16 + o[1]) \pm (\theta^2 - 1)\sqrt{u(\theta) \pm v(\theta)\sqrt{\Delta(\theta)}} \right|_p = |\theta - 1|_p \left| 16 + o[1] \pm \sqrt{\Delta_1(\theta)} \right|_p < 1. \end{aligned}$$

Let $p = 2$. Then it follows that

$$|h_i + \theta|_2 = 2 |\theta - 1|_p \left| 16 + o[1] \pm \sqrt{\Delta_1(\theta)} \right|_p < 1.$$

Using the strong triangle inequality we obtain

$$|h_i + 1|_p = |h_i + \theta + (1 - \theta)|_p \leq \max\{|h_i + \theta|_p, |\theta - 1|_p\} < 1,$$

$$|\theta h_i + 1|_p = |\theta(h_i + \theta) + (1 - \theta^2)|_p \leq \max\{|h_i + \theta|_p, |\theta^2 - 1|_p\} < 1.$$

Lemma 2.3.5. Let $k = 3$, $i \in \{1, 3, 5\}$. If h_i and h_{i+1} are $G_3^{(2)}$ -periodic (non-translation-invariant) solutions of (2.1.5) then for the corresponding partition function $Z_m^{(h)}$ the following equality holds:

$$Z_m^{(h)} = \begin{cases} \frac{1}{\theta^2} \left((h_i + \theta)^4 + (\theta h_i + 1)^4 \right), & \text{if } m = 1, \\ \theta^{3^m - 3} \cdot (h_{i+1} + \theta)^{\frac{3^m - 3}{2}} \cdot (h_i + \theta)^{\frac{3^{m-1} - 1}{2}} Z_1^{(h)}, & \text{if } m \geq 3 \text{ and } m \text{ is odd,} \\ \theta^{3^m - 3} \cdot (h_{i+1} + \theta)^{\frac{3^m - 1}{2}} \cdot (h_i + \theta)^{\frac{3^{m-1} - 3}{2}} Z_1^{(h)}, & \text{if } m \text{ is even.} \end{cases} \quad (2.3.10)$$

Proof. Let

$$h_{\sigma(x), x} = \begin{cases} h_{1,x}, & \text{if } \sigma(x) = 1, \\ 1, & \text{if } \sigma(x) = -1, \end{cases}$$

and

$$h_{1,x} = \begin{cases} h_i, & \text{if } |x| \text{ is odd,} \\ h_{i+1}, & \text{if } |x| \text{ is even,} \end{cases}$$

Let $|x| = n$, here $|x|$ is the length of the word x , then one gets

$$h_{1,x} = \begin{cases} h_i, & \text{if } n \text{ is odd,} \\ h_{i+1}, & \text{if } n \text{ is even,} \end{cases}$$

Due to Lemma 2.3.4,

$$\begin{cases} h_i = \left(\frac{\theta h_{i+1} + 1}{h_{i+1} + \theta} \right)^3, \\ h_{i+1} = \left(\frac{\theta h_i + 1}{h_i + \theta} \right)^3. \end{cases} \quad (2.3.11)$$

Using (2.1.3) and (2.3.11), we get

$$Z_1^{(h)} = \frac{1}{\theta^2} \left((h_i + \theta)^4 + (\theta h_i + 1)^4 \right).$$

We consider the following cases:

Case 1. Let n be odd. By Lemmas 2.3.4 and 2.2.3,

$$a_h(x) = \frac{(\exp_p(-J) + \exp_p(J)h_{i+1})^3}{h_i} = \exp_p(-3J) \frac{(1 + \theta h_{i+1})^3}{h_i} = \frac{1}{\sqrt{\theta^3}} (h_{i+1} + \theta)^3,$$

$$A_{h,n} = \theta^{2 \cdot 3^n} (h_i + \theta)^{4 \cdot 3^{n-1}},$$

$$Z_{n+1}^{(h)} = \prod_{k \text{ is odd}}^n (A_{h,k}) \prod_{k \text{ is even}}^n (A_{h,k}) Z_1^{(h)} = \theta^{3^{n+1}-3} \cdot (h_{i+1} + \theta)^{\frac{3^{n+1}-1}{2}} \cdot (h_i + \theta)^{\frac{3^n-3}{2}} Z_1^{(h)}.$$

Case 2. Let n be even. Again thanks to by Lemmas 2.3.4 and 2.2.3,

$$a_h(x) = \frac{(\exp_p(-J) + \exp_p(J)h_i)^3}{h_{i+1}} = \exp_p(-3J) \frac{(1 + \theta h_i)^3}{h_{i+1}} = \frac{1}{\sqrt{\theta^3}} (h_i + \theta)^3,$$

$$A_{h,n} = \theta^{2 \cdot 3^n} (h_{i+1} + \theta)^{4 \cdot 3^{n-1}},$$

$$Z_{n+1}^{(h)} = \prod_{k \text{ is odd}}^n (A_{h,k}) \prod_{k \text{ is even}}^n (A_{h,k}) Z_1^{(h)} = \theta^{3^{n+1}-3} \cdot (h_{i+1} + \theta)^{\frac{3^{n+1}-3}{2}} \cdot (h_i + \theta)^{\frac{3^n-1}{2}} Z_1^{(h)}.$$

With the notation $m = n + 1$ we get the assertion of the lemma.

Theorem 2.3.2. Let $k = 3$. If one of the following conditions holds true

- 1) $p = 2$, $|\theta - 1|_2 = \frac{1}{4}$;
- 2) $p = 3$, $\text{ord}_3 |\theta - 1|_3$ is odd, $|\theta - 1|_3 \leq \frac{1}{27}$ and $|\theta - 1|_3 (\theta - 1) \in \mathcal{E}_3$;
- 3) $p \equiv 1 \pmod{6}$,

then all non-translation invariant $G_k^{(2)}$ -periodic p -adic generalized Gibbs measures are unbounded.

Proof. Using by (2.3.2), Proposition 2.1.3. and Lemma 2.3.4 one has

$$|\mu_h^{(n)}(\sigma)|_p = \left| Z_{n,h}^{-1} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x} \right|_p = |Z_n^{(h)}|_p^{-1}.$$

First of all, we find $|Z_1^{(h)}|_p$. Thanks to (2.3.3) and Lemma 2.3.4, one gets

$$|Z_1^{(h)}|_p = \left| \frac{1}{\theta^2} \left((h_i + \theta)^4 + (\theta h_i + 1)^4 \right) \right|_p < 1.$$

Let n be odd. Then it follows that

$$\lim_{n \rightarrow \infty} \left| \mu_{h_{i,i+1}}^{(n)} \right|_p = \lim_{n \rightarrow \infty} \left| \theta^{3^n-3} \cdot (h_{i+1} + \theta)^{\frac{3^n-3}{2}} \cdot (h_i + \theta)^{\frac{3^{n-1}-1}{2}} Z_1^{(h)} \right|_p^{-1} = \infty.$$

Let n be even. Then it gives

$$\lim_{n \rightarrow \infty} \left| \mu_{h_{i,i+1}}^{(n)} \right|_p = \lim_{n \rightarrow \infty} \left| \theta^{3^n-3} \cdot (h_{i+1} + \theta)^{\frac{3^n-1}{2}} \cdot (h_i + \theta)^{\frac{3^{n-1}-3}{2}} Z_1^{(h)} \right|_p^{-1} = \infty.$$

Remark 2.3.2. Let $k=3, p \geq 3$. It is easy to see that $h_0=1$ is an arbitrary periodic fixed point of $f(h)$. In previous paragraph we proved that the measure μ_{h_0} corresponding to this fixed point h_0 is bounded. If there exists unbounded measure μ_{h_i} ($\neq \mu_{h_0}$) then according to Theorem 2.3.2 there occurs a phase transition.

Theorem 2.3.3. Let $k=3$. If either $p=3$, $\text{ord}_3 |\theta-1|_3$ is odd, $|\theta-1|_3 \leq \frac{1}{27}$, $|\theta-1|_3$ ($\theta-1 \in \mathcal{E}_3$ or $p \equiv 1 \pmod{6}$) then there exists a phase transition.

Proof. The proof is straightforward by Theorem 2.3.2 and Remark 2.3.2.

CONCLUSION OF THE SECOND CHAPTER

The second chapter contains three sections. Section 2.1 is devoted to present main known results related to p -adic Ising model on the Cayley tree. The uniqueness of p -adic Gibbs measure for the Ising model is proved by F.M.Mukhamedov, M. Khamraev, and O.N.Khakimov. The concept of p -adic generalized Gibbs measure was introduced by F.M.Mukhamedov and O.N.Khakimov. Until now, translation-invariant, periodic and non-periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree have been studied.

Section 2.2 is devoted to translation-invariant p -adic generalized Gibbs measures for the Ising model. In this section all translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three were found. The number of such measures is at most four. In particular, if $p \equiv 1 \pmod{12}$, it was proved that there are four translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three and there is a phase transition.

Section 2.3 is devoted to $G_k^{(2)}$ -periodic p -adic generalized Gibbs measures for the Ising model. In this section all non-translation-invariant $G_k^{(2)}$ -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three were found. The boundedness of these measures was checked.

CHAPTER III. NON-PERIODIC p -ADIC GIBBS MEASURES FOR THE ISING AND POTTS MODELS ON THE CAYLEY TREE

This chapter is devoted to some non-periodic p -adic Gibbs measures for the Potts and Ising models on the Cayley tree. Using constructive methods, we construct non-periodic p -adic Gibbs measures for the Potts and Ising models on the Cayley tree. In this chapter we consider ART, (k_0) -translation-invariant, (k_0) -periodic, p -adic analogue of the Bleher-Ganikhodjaev constructions. Using these constructions we construct p -adic Gibbs measures for the Potts and Ising models on the Cayley tree.

§ 3.1. p -adic ART quasi Gibbs measures for the Potts model on the Cayley tree

It is always interesting to study non-periodic Gibbs measures. In [2] some non-periodic Gibbs measures that are called ART measures were investigated. In this paragraph we are going to study p -adic ART quasi Gibbs measures for 3-state Potts model on the Cayley tree of order k , using by translation-invariant and G_2 -periodic solutions of the functional equation (1.4.7) for $k=2$. Recall that each solution of (1.4.7) defines a p -adic quasi Gibbs measure for Potts model. One can see that $h_x=1$ for all $x \in V$ is a solution of (1.4.7) for any $k \geq 2$. Now we construct new solutions of (1.4.7) for $k \geq 3$. In the case $k=2$ and $q=3$ all translation-invariant solutions of (1.4.7) had been found in [75]. For $k \geq 3$ we set up some solutions of (1.4.7) using (1.4.14) and (1.4.17).

(a_1) Let V^k be the set of all vertices of the Cayley tree Γ_+^k . Since $k > 2$ one can consider V^2 as a subset of V^k . Define the following function

$$\tilde{h}_x^i = \begin{cases} h_i^{(2)}, & \text{if } x \in V^2, \\ 1, & \text{if } x \in V^k \setminus V^2, \end{cases} \quad (3.1.1)$$

where $i = \overline{0,2}$, $h_0^{(2)}, h_1^{(2)}, h_2^{(2)}$ are translation-invariant solutions of the functional equation (1.4.7) for $k = 2$. This function on the Cayley tree of order $k = 3$ is shown in Figure 3.1.

Now we shall check that (3.1.1) satisfies (1.4.7) on Γ_+^k . Let $x \in V^2 \subset V^k$. For $i = \overline{0,2}$ we have

$$\begin{aligned} \widetilde{h}_x^{(i)} &= \prod_{y \in S(x)} F(\widetilde{h}_y^{(i)}, \theta) = \prod_{y \in S(x)} \left(\frac{\theta h_y^{(i)} + 2}{h_y^{(i)} + \theta + 1} \right) = \\ &= \prod_{y \in S(x) \cap V^2} \left(\frac{\theta h_y^{(i)} + 2}{h_y^{(i)} + \theta + 1} \right) \times \prod_{y \in S(x) \cap (V^k \setminus V^2)} \left(\frac{\theta h_y^{(i)} + 2}{h_y^{(i)} + \theta + 1} \right) = \\ &= \prod_{y \in S(x) \cap V^2} \left(\frac{\theta h_y^{(i)} + 2}{h_y^{(i)} + \theta + 1} \right) = \left(\frac{\theta h_i^{(2)} + 2}{h_i^{(2)} + \theta + 1} \right)^2 = h_i^{(2)} \end{aligned}$$

Here we used

$$\prod_{y \in S(x) \cap (V^k \setminus V^2)} \left(\frac{\theta h_y^{(i)} + 2}{h_y^{(i)} + \theta + 1} \right)^2 = 1 \quad \prod_{y \in S(x) \cap (V^k \setminus V^2)} \left(\frac{\theta h_y^{(i)} + 2}{h_y^{(i)} + \theta + 1} \right)^2 = 1.$$

Thus $\widetilde{h}_x^{(i)}, i = \overline{0,2}$ satisfies the functional equation (1.4.7). We denote by $\mu_{\widetilde{h}_x^{(i)}}, i = \overline{0,2}$ the Gibbs measures corresponding to $\widetilde{h}_x^{(i)}, i = \overline{0,2}$ and these measures are called p -adic ART quasi Gibbs measures.

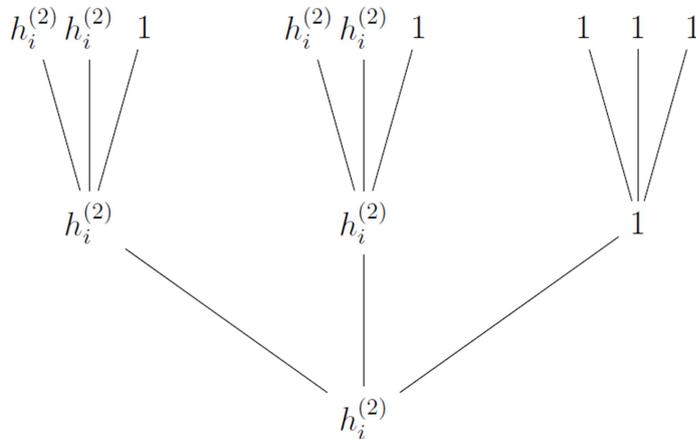


Figure 3.1: The function $\widetilde{h}_x^{(i)}, i = \overline{0,2}$ on the Cayley tree of order three.

(a₂). Let $k \geq 3$. We shall construct new p -adic (non-periodic) Gibbs measures using by $h_0 = 1, h_3, h_4$ and $h_i^{(1)} = h_i, i = 0, 3, 4$. Define the following function

$$\widetilde{h}_x^{(i)} = \begin{cases} h_i^{(1)}, & \text{if } x \in V^2 \cap G_2, \\ F^2(h_i^{(1)}, \theta), & \text{if } x \in V^2 \cap (G^k \setminus G_2), \\ 1, & \text{if } x \in V^k \setminus V^2. \end{cases} \quad (3.1.2)$$

This function on the Cayley tree of order $k = 3$ is shown in Figure 3.2.

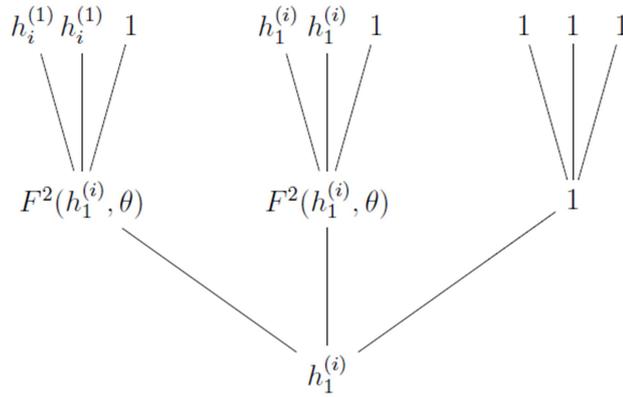


Figure 3.2: The function $\widetilde{h}_x^{(i)}, i = 0, 3, 4$ on the Cayley tree of order three.

Now we shall check that (3.1.2) satisfies (1.4.7) on Γ_+^k . Let $x \in V^2 \cap G_2$. For $i = 0, 3, 4$, we have

$$h_i^{(1)} = \prod_{y \in S(x)} F(h_y, \theta) = \prod_{y \in S(x)} F(F^2(h_y^{(1)}, \theta)) = F^2(F^2(h_i^{(1)}, \theta)).$$

Let $x \in V^2 \cap (G^k \setminus G_2)$.

$$F^2(h_i^{(1)}, \theta) = \prod_{y \in S(x)} F(h_y, \theta) = F^2(h_1, \theta).$$

If $x \in V^k \setminus V^2$ then it is easy to see that $h_x^{(i)}$ satisfies (1.4.7). Let $q = 3, k > 3$.

Consequently, we have the following theorem:

Theorem 3.1.1. If $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$ then there exist at least four p -adic ART (non-periodic) quasi Gibbs measures for the Potts model on a Cayley tree of order $k \geq 3$.

Lemma 3.1.1. Let $k \geq 3$. If μ_h is the p -adic ART quasi Gibbs measure constructed by rule (a_1) then for the corresponding partition function $Z_n^{(h)}$ the following equality holds:

$$Z_n^{(h)} = (\theta + 2)^{\frac{(k-2)(k^n-k)}{k-1}} (h + \theta + 1)^{\frac{2(k^n-k)}{k-1}} [(\theta h + 2)^k + 2(h + \theta + 1)^k] \quad (3.1.3)$$

Proof. Since $\theta = \exp_p \{J\}$, using (1.4.4) we get $Z_1^{(h)} = (\theta h + 2)^k + 2(h + \theta + 1)^k$. Then by lemma 4.1.1 we obtain the following equalities:

$$a_h(x) = \frac{\prod_{y \in S(x)} (\theta h_{1,y} + 2)}{h_{1,x}} = \frac{\prod_{y \in S(x)} (\theta h + 2)}{h} = (h + \theta + 1)^2 (\theta + 2)^{k-2},$$

$$A_{h,n} = (h + \theta + 1)^{2k^n} (\theta + 2)^{(k-2)k^n},$$

$$Z_n^{(h)} = (\theta + 2)^{\frac{(k-2)(k^n-k)}{k-1}} (h + \theta + 1)^{\frac{2(k^n-k)}{k-1}} [(\theta h + 2)^k + 2(h + \theta + 1)^k],$$

where $h = h_i, i = 0, 1, 2$. The lemma is proved.

Lemma 3.1.2. Let $k \geq 3$. If μ_h is the p -adic ART quasi Gibbs measure constructed by rule (a_2) then for the corresponding partition function $Z_n^{(h)}$ the following assertions hold:

if n is odd, then

$$Z_n^{(h)} = (\theta + 2)^{\frac{(k-2)(k^n-k)}{k-1}} (h_3 + \theta + 1)^{\frac{2(k^{n+1}-k^2)}{k^2-1}} (h_4 + \theta + 1)^{\frac{2(k^n-k)}{k^2-1}} Z_1^{(h)}; \quad (3.1.4)$$

If n is even, then

$$Z_n^{(h)} = (\theta + 2)^{\frac{(k-2)(k^n-k)}{k-1}} (h_3 + \theta + 1)^{\frac{2(k^n-k^2)}{k^2-1}} (h_4 + \theta + 1)^{\frac{2(k^{n+1}-k)}{k^2-1}} Z_1^{(h)}. \quad (3.1.5)$$

where $Z_1^{(h)} = (\theta h_3 + 2)^k + 2(h_3 + \theta + 1)^k$.

Proof. Let $|x| = n$ and

$$h_{1,x} = \begin{cases} h_3, & \text{if } n \text{ is odd;} \\ h_4, & \text{if } n \text{ is even.} \end{cases}$$

Due to (1.4.12), we have

$$\begin{cases} h_3 = \left(\frac{\theta h_4 + 2}{h_4 + \theta + 1} \right)^2; \\ h_4 = \left(\frac{\theta h_3 + 2}{h_3 + \theta + 1} \right)^2. \end{cases} \quad (3.1.6)$$

Using (1.4.4) and (3.1.6), we get $Z_1^{(h)} = (h_3 + \theta + q - 2)^2 (h_4 + q - 1)$.

Consider following cases

Case 1. Let n be odd. By Lemma 1.4.1 and (3.1.6) we get the following equalities:

$$a_h(x) = \frac{\prod_{y \in S(x)} (\theta h_{1,y} + 2)}{h_{1,x}} = \frac{(\theta h_4 + 2)^2 (\theta + 2)^{k-2}}{h_3} = (h_4 + \theta + 1)^2 (\theta + 2)^{k-2},$$

$$A_{h,n-1} = (h_3 + \theta + 1)^{2k^{n-1}} (\theta + 2)^{(k-2)k^{n-1}}, A_{h,n} = (h_4 + \theta + 1)^{2k^n} (\theta + 2)^{(k-2)k^n},$$

$$Z_n^{(h)} = (\theta + 2)^{\frac{(k-2)(k^n-k)}{k-1}} (h_3 + \theta + 1)^{\frac{2(k^{n+1}-k^2)}{k^2-1}} (h_4 + \theta + 1)^{\frac{2(k^n-k)}{k^2-1}} Z_1^{(h)}.$$

Case 2. Let n be even. By Lemma 1.4.1 and (3.1.6) we get the following equalities:

$$a_h(x) = \frac{\prod_{y \in S(x)} (\theta h_{1,y} + 2)}{h_{1,x}} = \frac{(\theta h_3 + 2)^2 (\theta + 2)^{k-2}}{h_4} = (h_3 + \theta + 1)^2 (\theta + 2)^{k-2},$$

$$A_{h,n-1} = (h_4 + \theta + 1)^{2k^{n-1}} (\theta + 2)^{(k-2)k^{n-1}}, A_{h,n} = (h_3 + \theta + 1)^{2k^n} (\theta + 2)^{(k-2)k^n},$$

$$Z_n^{(h)} = (\theta + 2)^{\frac{(k-2)(k^n-k)}{k-1}} (h_3 + \theta + 1)^{\frac{2(k^n-k^2)}{k^2-1}} (h_4 + \theta + 1)^{\frac{2(k^{n+1}-k)}{k^2-1}} Z_1^{(h)}.$$

Finally, the lemma 3.1.2 is proved.

Remark 3.1.1. If $k=2$ in Lemma 3.1.1 and Lemma 3.1.2 then we get Lemma 1.4.2 and Lemma 1.4.3, respectively.

Theorem 3.1.2. Let $q=3$ and $k \geq 3$. Following statements are true for p -adic Potts model on a Cayley tree of order k

- 1) if $p=2$ or $p \equiv 5 \pmod{8}$ or $p \equiv 7 \pmod{8}$, then ART quasi Gibbs measure $\mu_{h_0}^{\sim}$ is unbounded;
- 2) if $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$, $p \neq 3$ then only ART quasi Gibbs measure $\mu_{h_0}^{\sim}$ is bounded;
- 3) if $p=3$, then all ART quasi Gibbs measures constructed by rules (a_1) and (a_2) are unbounded.

Proof.

Case 1. If $p=2$ or $p \equiv 5 \pmod{8}$ or $p \equiv 7 \pmod{8}$, then exists only ART quasi Gibbs measure $\mu_{h_0}^{\sim}$. Note that $|h_0 + \theta + q - 2|_p = |\theta + 2|_p = 1$. By Lemma 3.1.1 we get

$$\lim_{n \rightarrow \infty} |\mu_{h_0}^{(n)}|_p = \lim_{n \rightarrow \infty} \left| \frac{1}{3(\theta + 2)^{\frac{k(k^n-1)}{k-1}}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x} \right|_p = 1.$$

Case 2. If $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$, $p \neq 3$ then there exist ART quasi Gibbs measures constructed by rules (a_1) and (a_2) . Since $\theta \in \mathcal{E}_p$, for h_0, h_1, h_2 we have

$$|\theta + 2|_p = 1, |h_0 + \theta + 1|_p = 1, |h_{1,2} + \theta + 1|_p < 1, 0 < |Z_1^{(h)}|_p < 1.$$

By Lemma 3.1.1 we have $\lim_{n \rightarrow \infty} |\mu_{h_0}^{(n)}|_p = 1$,

$$\lim_{n \rightarrow \infty} |\mu_{h_{1,2}}^{(n)}|_p = \lim_{n \rightarrow \infty} \left| \frac{1}{(\theta + 2)^{\frac{(k-2)(k^n-k)}{k-1}} (h + \theta + 1)^{\frac{2(k^n-k)}{k-1}} Z_1^{(h)}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x} \right|_p = \infty.$$

For h_3, h_4 we have

$$|\theta + 2|_p = 1, |h_{3,4} + \theta + 1|_p < 1, 0 < |Z_1^{(h)}|_p = |(\theta h_3 + 2)^k + 2(h_3 + \theta + 1)^k|_p < 1.$$

By Lemma 3.1.2 we have

$$\lim_{n \rightarrow \infty} |\mu_{h_{3,4}}^{(n)}|_p = \infty.$$

Case 3. If $p = 3$, then there exist μ_{h_i} , $i = 0, 1, 2, 3, 4$ measures. Note that $|h_0 + \theta + 1|_3 = |\theta + 2|_3 < 1$. By Lemma 3.1.2 we obtain

$$\lim_{n \rightarrow \infty} |\mu_{h_0}^{(n)}|_p = \lim_{n \rightarrow \infty} \left| \frac{1}{3(\theta + 2)^{\frac{k(k^n - 1)}{k-1}}} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x), x} \right|_p = \infty.$$

We can prove that $\mu_{h_{1,2}}, \mu_{h_{3,4}}$ measures are unbounded as in the case 2.

The theorem is proved.

§ 3.2. ART, (k_0) -translation-invariant, (k_0) -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree.

In this paragraph we are going to describe new p -adic generalized Gibbs measures of the Ising model on the Cayley tree of order $k \geq 3$ by method ART (see [2]). We recall that each solution of (2.1.5) define a p -adic generalized Gibbs measures for Ising model on the Cayley tree of order $k \geq 1$. One can see that $h_x = 1$ for all $x \in V$ is a solution of (2.1.5) for any $k \geq 1$. Now we construct new solutions of (2.1.5) for $k \geq 3$. If $k = 2$ then all translation-invariant solutions of (2.1.5) can be found from the following equation

$$h^2 = \left(\frac{\theta h^2 + 1}{h^2 + \theta} \right)^2. \quad (3.2.1)$$

In [30] it has been proved that the equation (3.2.1) has a unique solution if $p \not\equiv 1 \pmod{4}$ and it has exactly three solutions if $p \equiv 1 \pmod{4}$. In what follows we assume that $p \equiv 1 \pmod{4}$. In this case due to results of [30] the followings

$$h_0^{(2)} = 1, h_{1,2}^{(2)} = \frac{\theta - 1 \pm \sqrt{(\theta - 3)(\theta + 1)}}{2} \quad (3.2.2)$$

are solutions of (3.2.1). For $k \geq 3$ we construct some solutions of (2.1.5) using (3.2.2). Let V^k be the set of all vertices of the Cayley tree Γ^k . Since $k > 2$ one can consider V^2 as a subset of V^k . Define the following function

$$\widetilde{h}_x^{(i)} = \begin{cases} h_i^{(2)}, & \text{if } x \in V^2, \\ 1, & \text{if } x \in V^k \setminus V^2, \end{cases} \quad (3.2.3)$$

where $i = 1, 2$. This function on the Cayley tree of order $k = 3$ is shown in Figure 3.3.

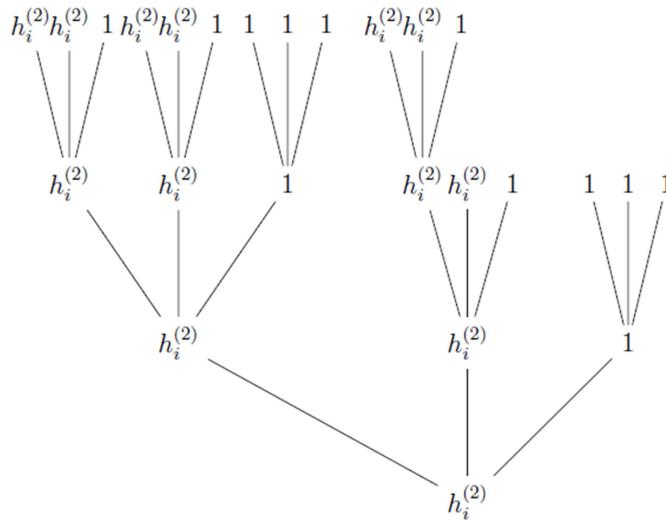


Figure 3.3: The function $\widetilde{h}_x^{(i)}, i = 1, 2$ on the Cayley tree of order three.

Now we shall check that (3.2.3) satisfies (2.1.5) on Γ^k . Let $x \in V^2, k \geq 3$. For $i = 1, 2$ we have

$$\begin{aligned}
(\widetilde{h}_x^{(i)})^2 &= \prod_{y \in S(x)} \frac{\theta(\widetilde{h}_y^{(i)})^2 + 1}{(\widetilde{h}_y^{(i)})^2 + \theta} = \prod_{y \in S(x) \cap V^2} \frac{\theta(\widetilde{h}_y^{(i)})^2 + 1}{(\widetilde{h}_y^{(i)})^2 + \theta} \times \prod_{y \in S(x) \cap (V^k \setminus V^2)} \frac{\theta(\widetilde{h}_y^{(i)})^2 + 1}{(\widetilde{h}_y^{(i)})^2 + \theta} = \\
&= \prod_{y \in S(x) \cap V^2} \frac{\theta(\widetilde{h}_y^{(i)})^2 + 1}{(\widetilde{h}_y^{(i)})^2 + \theta} = \left(\frac{\theta(h_i^{(2)})^2 + 1}{(h_i^{(2)})^2 + \theta} \right)^2 = (h_i^{(2)})^2,
\end{aligned}$$

here we used

$$\prod_{y \in S(x) \cap (V^k \setminus V^2)} \frac{\theta(\widetilde{h}_y^{(i)})^2 + 1}{(\widetilde{h}_y^{(i)})^2 + \theta} = 1.$$

If $x \in V^k \setminus V^2$ then it is easy to see that $S_k(x) \subset V^k \setminus V^2$. Therefore we have

$$(\widetilde{h}_x^{(i)})^2 = \prod_{y \in S(x)} \frac{\theta(\widetilde{h}_y^{(i)})^2 + 1}{(\widetilde{h}_y^{(i)})^2 + \theta} = 1.$$

Thus, $\widetilde{h}_x^{(i)}, i=1,2$ satisfies the functional equation (2.1.5) and we denote by $\mu_{\widetilde{h}_x^{(i)}}, i=1,2$ the Gibbs measures corresponding to $\widetilde{h}_x^{(i)}, i=1,2$ and these measures are called p -adic ART generalized Gibbs measures. Thus, we have the following result:

Theorem. 3.2.1. Let $p \equiv 1 \pmod{4}$. Then there exists at least three non-periodic p -adic generalized Gibbs measures for the Ising model on a Cayley tree of order $k \geq 3$.

(k_0) -translation-invariant p -adic generalized Gibbs measures. In [86], [93] for the Ising model on Cayley tree of order k , the new sets of non-translation-invariant Gibbs measures are constructed.

Now we shall construct a new solution of functional equation (2.1.5) on the Cayley tree of order $k \geq 3$ by using well known solutions (3.2.2).

Having some p -adic generalized Gibbs measures on a Cayley tree of order (k_0) , we will construct new type of p -adic generalized Gibbs measures on a Cayley tree of order $k > k_0$. We call these measures p -adic (k_0) -translation-

invariant generalized Gibbs measures. We consider two cases $k_0 = 2$ and $k_0 = 3$. Having some p -adic generalized Gibbs measures on a Cayley tree of order (k_0) , we will construct new type of p -adic generalized Gibbs measures on a Cayley tree of order $k > k_0$. We call these measures (k_0) -translation-invariant p -adic generalized Gibbs measures. We consider two cases $k_0 = 2$ and $k_0 = 3$.

Case $k_0 = 2$. Let $k > 2$, such that $k - k_0$ is even. For $x \in V$, by $S_{k_0}(x)$ we denote an arbitrary set of k_0 vertices of the set $S(x)$, and the remaining $k - k_0$ vertices are denoted by $S_{k-k_0}(x)$.

We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in \{h_1^{(2)}, h_2^{(2)}\}$) as follows:

(a₁) If on the vertex of the x , we have $h_x = h_1^{(2)}$, then at each vertex from $S_{k_0}(x)$ we put the value $h_1^{(2)}$, and on each vertex of $S_{k-k_0}(x)$ we put one of the values of $h_1^{(2)}$ and $h_2^{(2)}$, so that it satisfies the following equality

$$\prod_{y \in S_{k-k_0}(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = 1. \quad (3.2.4)$$

Note that relation (3.2.4) holds, if we put on half vertex of $S_{k-k_0}(x)$ value $h_1^{(2)}$ and another half vertex of $S_{k-k_0}(x)$ value $h_2^{(2)}$. Indeed, it is easy to check that

$$h_1^{(2)} \cdot h_2^{(2)} = 1, \text{ and}$$

$$\frac{\theta(h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \times \frac{\theta(h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} = 1.$$

Consequently,

$$\begin{aligned} \prod_{y \in S_{k-k_0}(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} &= \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} = \\ &= \left(\frac{\theta(h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \times \frac{\theta(h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} \right)^{\frac{k-k_0}{2}} = 1. \end{aligned}$$

(a₂) If on the vertex of the x , we have $h_x = h_x^{(2)}$, then at each vertex of $S_{k_0}(x)$ we put the value $h_2^{(2)}$, and on each vertex of $S_{k-k_0}(x)$ we put one of the values of $h_1^{(2)}$ and $h_2^{(2)}$, so that it satisfies the equality (3.2.4) (see Figure. 3.4).

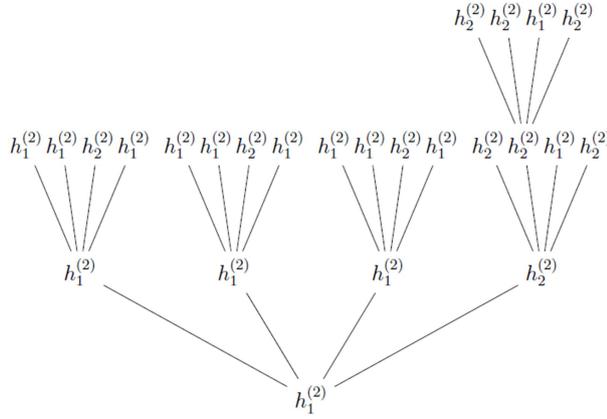


Figure 3.4: This is an example of (k_0) -translation-invariant function h_x on the vertices of the Cayley tree of order 4 in the case $k_0 = 2$.

Proposition 3.2.1. Let $p \equiv 1 \pmod{4}$. Then any set of quantities according to the rules (a₁), (a₂) on the Cayley tree Γ^k satisfies the functional equation (2.1.5).

Proof. Let $h_x = h_1^{(2)}$, then we have

$$\begin{aligned} (h_1^{(2)})^2 &= \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \prod_{y \in S_{k_0}(x)} \frac{\theta(h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \times \\ &\times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} = \\ &= \left(\frac{\theta(h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \right)^{k_0} \times 1 = \left(\frac{\theta(h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \right)^2. \end{aligned}$$

Let $h_x = h_2^{(2)}$, then

$$\begin{aligned} (h_2^{(2)})^2 &= \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \prod_{y \in S_{k_0}(x)} \frac{\theta (h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta (h_1^{(2)})^2 + 1}{(h_1^{(2)})^2 + \theta} \times \\ &\times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta (h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} = \left(\frac{\theta (h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} \right)^{k_0} \times 1 = \left(\frac{\theta (h_2^{(2)})^2 + 1}{(h_2^{(2)})^2 + \theta} \right)^2. \end{aligned}$$

Consequently, the set of quantities $h = \{h_x, x \in V\}$ constructed by rules (a_1) , (a_2) satisfies the functional equation (2.1.5). The proposition is proved.

A measure, corresponding to the set of quantities constructed by rules (a_1) , (a_2) is called a (2)-translation-invariant p -adic generalized Gibbs measure. Then thanks to Proposition 3.2.1 we can prove the following result:

Theorem 3.2.2. Let $p \equiv 1 \pmod{4}$, $k_0 = 2$ and $k \geq 4$ be even. Then there exist exactly two (2)-translation-invariant p -adic generalized Gibbs measures.

Case $k_0 = 3$. In this case we get the equation (2.2.8) and its solutions are:

$$z_{1,2} = \pm 1, \quad z_{3,4} = \frac{\theta^3 - 3\theta \pm (\theta^2 - 1)\sqrt{(\theta^2 - 4)}}{2},$$

where $h^2 = z$ and $p \equiv 1 \pmod{12}$. Now we construct a new solution of functional equation (2.1.5) on the Cayley tree of order $k \geq 3$ using the solutions z_1, z_2, z_3, z_4 .

Let $k \geq 4$ and $k - k_0$ be even. For $x \in V$, by $S_{k_0}(x)$ we denote an arbitrary set of k_0 vertices of the set $S(x)$, and the remaining $k - k_0$ vertices are denoted by $S_{k-k_0}(x)$.

We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in \{h_1, h_2, h_3, h_4\}$, h_i is a solution corresponding z_i , $i \in \{1, 2, 3, 4\}$) as follows:

(a_3) Let us denote $h_1^* = h_2$, $h_2^* = h_1$, $h_3^* = h_4$, $h_4^* = h_3$. If on the vertex of the x , we have $h_x = h_i$, then at each vertex from $S_{k_0}(x)$ we put the value h_i , and on each vertex of $S_{k-k_0}(x)$ we put one of the values of h_i and h_i^* , so that its satisfy (2.1.5) equality.

Note that the equation (2.1.5) holds, if we put on half vertex of $S_{k-k_0}(x)$ value h_i and another half vertex of $S_{k-k_0}(x)$ value h_i^* . Because it is easy to check that $h_i \cdot h_i^* = 1$, and

$$\frac{\theta(h_i)^2 + 1}{(h_i)^2 + \theta} \times \frac{\theta(h_i^*)^2 + 1}{(h_i^*)^2 + \theta} = 1.$$

Consequently,

$$\prod_{y \in S_{k-k_0}(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_i)^2 + 1}{(h_i)^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_i^*)^2 + 1}{(h_i^*)^2 + \theta} = 1.$$

(a₄) If on the vertex of the x , we have $h_x = h_i^*$, then at each vertex from $S_{k_0}(x)$ we put the value h_i^* , and on each vertex of $S_{k-k_0}(x)$ we put one of the values of h_i and h_i^* , so that it satisfies the equation (2.1.5) (see Figure 3.5).

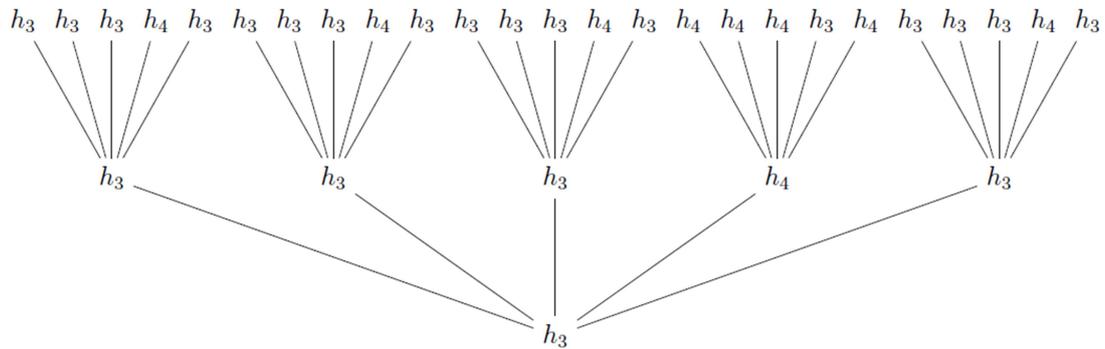


Figure 3.5 : This is an example of (k_0) -translation-invariant function h_x on the vertices of the Cayley tree of order 5 in the case $k_0 = 3$.

Proposition 3.2.2. Let $k \geq 5$ be odd number and $p \equiv 1 \pmod{12}$. Then any set of quantities constructed by rules $(a_3), (a_4)$ on a Cayley tree of order k , satisfies the functional equation (2.1.5) .

Proof. Let $h_x = h_3$, then we get

$$\begin{aligned}
(h_3)^2 &= \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \prod_{y \in S_{k_0}(x)} \frac{\theta(h_3)^2 + 1}{(h_3)^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_3)^2 + 1}{(h_3)^2 + \theta} \times \\
&\times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_4)^2 + 1}{(h_4)^2 + \theta} = \left(\frac{\theta(h_3)^2 + 1}{(h_3)^2 + \theta} \right)^{k_0} \times 1 = \left(\frac{\theta(h_3)^2 + 1}{(h_3)^2 + \theta} \right)^3.
\end{aligned}$$

Let $h_x = h_4$, then

$$\begin{aligned}
(h_4)^2 &= \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \prod_{y \in S_{k_0}(x)} \frac{\theta(h_4)^2 + 1}{(h_4)^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_4)^2 + 1}{(h_4)^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_3)^2 + 1}{(h_3)^2 + \theta} = \\
&= \left(\frac{\theta(h_4)^2 + 1}{(h_4)^2 + \theta} \right)^{k_0} \times 1 = \left(\frac{\theta(h_4)^2 + 1}{(h_4)^2 + \theta} \right)^3.
\end{aligned}$$

Consequently, the set of quantities $h = \{h_x, x \in V\}$ constructed by (a_3) , (a_4) satisfies the functional equation (2.1.5). Proposition 3.2.2 is proved.

A measure that corresponds to the set of quantities constructed by (a_3) , (a_4) is called a (3)-translation-invariant p -adic generalized Gibbs measure.

Then due to Proposition 3.2.2 we get the following theorem.

Theorem 3.2.3. Let $k \geq 5$ be odd number and $k_0 = 3$, $p \equiv 1 \pmod{12}$.

Then, there exist exactly four (3)-translation-invariant p -adic generalized Gibbs measures for the Ising model on a Cayley tree of order k .

(k_0) -periodic p -adic generalized Gibbs measures. We notice that for each $G_k^{(2)}$ -periodic generalized Gibbs measures could be constructed by solutions of the following equation:

$$\theta h^2 + (\theta - 1)h + \theta = 0 \quad (3.2.5)$$

The solutions of (3.2.5) are

$$h_{3,4}^{(2)} = \frac{1 - \theta \pm \sqrt{(\theta - 1)^2 - 4\theta^2}}{2\theta} \quad (3.2.6)$$

We are going to construct the new set of measures for the p -adic Ising model on the Cayley tree. Now we shall construct a new solution of functional equation (2.1.5) on the Cayley tree of order $k \geq 2$ with the solutions (3.2.6).

It is easy to see that:

$$h_3^{(2)} h_4^{(2)} = 1, \frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \times \frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} = 1.$$

We are going to prove the following equalities:

$$(h_3^{(2)})^2 = \left(\frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} \right)^2, (h_4^{(2)})^2 = \left(\frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \right)^2.$$

Keeping in mind that $h_4^{(2)}$ is a solution of the equation (3.2.6), we can obtain following equality:

$$\theta(h_4^{(2)})^2 + (\theta - 1)h_4^{(2)} + \theta = 0$$

or

$$-\theta(h_4^{(2)})^2 + h_4^{(2)} - \theta h_4^{(2)} - \theta = 0.$$

From last equalities one has

$$-\theta(h_4^{(2)})^2 + h_4^{(2)} - \theta h_4^{(2)} - \theta + h_4^{(2)}(\theta(h_4^{(2)})^2 + (\theta - 1)h_4^{(2)} + \theta) = 0.$$

The last one implies that

$$\theta(h_4^{(2)})^3 - (h_4^{(2)})^2 + h_4^{(2)} - \theta = 0, \frac{(h_4^{(2)})^2 + \theta}{\theta(h_4^{(2)})^2 + 1} = h_4^{(2)}.$$

Since $h_4^{(2)} = \frac{1}{h_3^{(2)}}$, one gets

$$\frac{(h_4^{(2)})^2 + \theta}{\theta(h_4^{(2)})^2 + 1} = \frac{1}{h_3^{(2)}},$$

or

$$\left(\frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} \right)^2 = (h_3^{(2)})^2. \quad (3.2.7)$$

(3.2.7) and $h_4^{(2)} = \frac{1}{h_3^{(2)}}$ yields that

$$\left(\frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \right)^2 = (h_4^{(2)})^2. \quad (3.2.8)$$

We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in \{h_3^{(2)}, h_4^{(2)}\}$), as follows:

(a_5) If on the vertex of the x , we have $h_x = h_i, i = 3, 4$, then at each vertex from $S_{k_0}(x)$ we put the value h_i^* , and on each vertex of $S_{k-k_0}(x)$ we put one of the values of h_i and h_i^* , so the equality (2.1.5) holds.

(a_6) If on the vertex of the x , we have $h_x = h_i^*$, then at each vertex from $S_{k_0}(x)$ we put the value h_i , and on each vertex of $S_{k-k_0}(x)$ we put one of the values of h_i and h_i^* , so that they satisfy the equation (2.1.5) (see Figure 3.6).

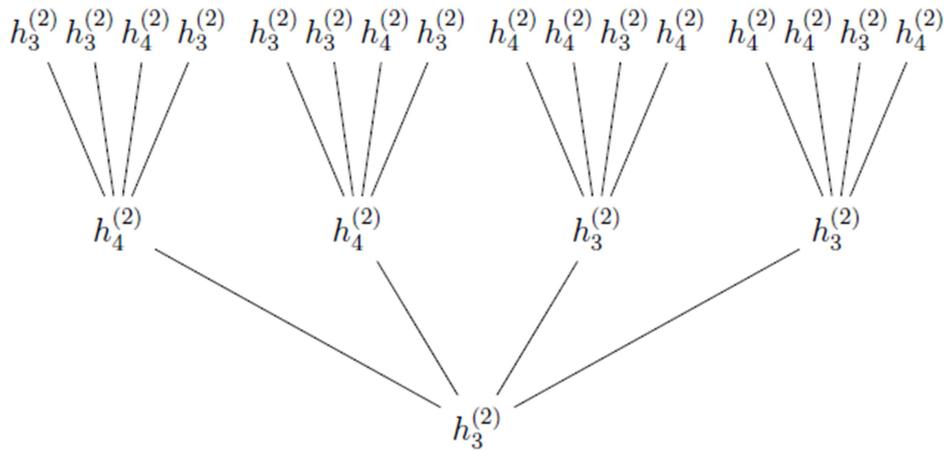


Figure 3.6: This is an example of (k_0) -periodic function h_x on the vertices of the Cayley tree of order 5 in the case $k_0 = 2$.

Proposition 3.2.3. Let $p \equiv 1 \pmod{4}$. Then any set of quantities constructed by rules (a_5) and (a_6) on a Cayley tree of order k , satisfies the functional equation (2.1.5).

Proof. Let $h_x = h_3^{(2)}$, then

$$\begin{aligned}
(h_3^{(2)})^2 &= \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \prod_{y \in S_{k_0}(x)} \frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \times \\
&\times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} = \left(\frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} \right)^{k_0} \times 1 = \left(\frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} \right)^2.
\end{aligned}$$

Let $h_x = h_4^{(2)}$, then

$$\begin{aligned}
(h_4^{(2)})^2 &= \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \prod_{y \in S_{k_0}(x)} \frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \times \\
&\times \prod_{y \in S_{\frac{k-k_0}{2}}(x)} \frac{\theta(h_4^{(2)})^2 + 1}{(h_4^{(2)})^2 + \theta} = \left(\frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \right)^{k_0} \times 1 = \left(\frac{\theta(h_3^{(2)})^2 + 1}{(h_3^{(2)})^2 + \theta} \right)^2.
\end{aligned}$$

Consequently, (k_0) -periodic set of quantities $h = \{h_x, x \in V\}$ satisfies the functional equation (2.1.5). The proposition is proved.

A measure that corresponds to a (k_0) -periodic set of quantities is called a p -adic (k_0) -periodic generalized Gibbs measure. According to Proposition 3.2.3, we have the following theorem.

Theorem 3.2.4. Let $k \geq 4$ be even number and $p \equiv 1 \pmod{4}$. Then there exists (2) -periodic p -adic generalized Gibbs measures for Ising model on a Cayley tree of order k .

§ 3.3 Some constructive p -adic generalized Gibbs measures for the Ising model on the Cayley tree

On the semi-infinite Cayley tree of order two, we denote by $h_i^{(i)}$ ($i=0,1,2$) and $h_j^{(p)}$ ($j=1,2$) the translation-invariant and G_2 -periodic solutions of the equation (2.1.5), respectively, i.e. let $p \equiv 1 \pmod{4}$ then (see [30])

$$h_0^{(i)} = 1, h_{1,2}^{(i)} = \frac{\theta - 1 \pm \sqrt{(\theta - 3)(\theta + 1)}}{2}, h_{1,2}^{(p)} = \frac{1 - \theta \pm \sqrt{(\theta - 1)^2 - 4\theta^2}}{2\theta}. \quad (3.3.1)$$

Denote

$$f(h) = \frac{\theta h^2 + 1}{h^2 + \theta}. \quad (3.3.2)$$

It is easy to see that

$$\begin{aligned} h_1^{(i)} h_2^{(i)} &= 1; \\ h_1^{(p)} h_2^{(p)} &= 1; \\ f(h_1^{(i)}) f(h_2^{(i)}) &= 1; \\ f(h_1^{(p)}) f(h_2^{(p)}) &= 1; \\ (h_i^{(i)})^2 &= f^2(h_i^{(i)}), i=1,2,3; \\ (h_j^{(p)})^2 &= f^2(h_{3-j}^{(p)}), j=1,2. \end{aligned} \quad (3.3.3)$$

Let $k \geq 3, k_0 = 2$. For $x \in V$, by $S_{k_0}(x)$ we denote an arbitrary set of k_0 vertices of the set $S(x)$, and remaining $k - k_0$ vertices is denoted by $S_{k-k_0}(x)$. Let $k - k_0 = a + b + c$, where a and b are even, c is even or odd.

We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in \{1, h_1^{(i)}, h_2^{(i)}, h_1^{(p)}, h_2^{(p)}\}$) as follows:

(A₁) if at vertex x we have $h_x = h_i^{(i)}$ ($i=1,2$), then the function h_y , which gives p -adic values to each vertex $y \in S(x)$ by the following rule

$$h_y = \begin{cases} h_i^{(t)} & \text{on } \frac{a}{2} + 2 \text{ vertices of } S(x), \\ h_{3-i}^{(t)} & \text{on } \frac{a}{2} \text{ vertices of } S(x), \\ h_i^{(p)} & \text{on } \frac{b}{2} \text{ vertices of } S(x), \\ h_{3-i}^{(p)} & \text{on } \frac{b}{2} \text{ vertices of } S(x), \\ 1 & \text{on } c \text{ vertices of } S(x). \end{cases} \quad (3.3.4)$$

(A_2) if at vertex x we have $h_x = h_i^{(p)}$ ($i=1,2$), then the function $h_y, y \in S(x)$ is defined as follows

$$h_y = \begin{cases} h_i^{(t)}, & \text{on } \frac{a}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(t)}, & \text{on } \frac{a}{2} \text{ vertices of } S(x); \\ h_i^{(p)}, & \text{on } \frac{b}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(p)}, & \text{on } \frac{b}{2} + 2 \text{ vertices of } S(x); \\ 1, & \text{on } c \text{ vertices of } S(x). \end{cases} \quad (3.3.5)$$

(A_3) if at vertex x we have $h_x = 1$, then the function h_y is defined by the following rule for all vertices $y \in S(x)$.

$$h_y = \begin{cases} h_i^{(t)}, & \text{on } \frac{a}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(t)}, & \text{on } \frac{a}{2} \text{ vertices of } S(x); \\ h_i^{(p)}, & \text{on } \frac{b}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(p)}, & \text{on } \frac{b}{2} \text{ vertices of } S(x); \\ 1, & \text{on } c + 2 \text{ vertices of } S(x). \end{cases} \quad (3.3.6)$$

(See the Figure 3.7).

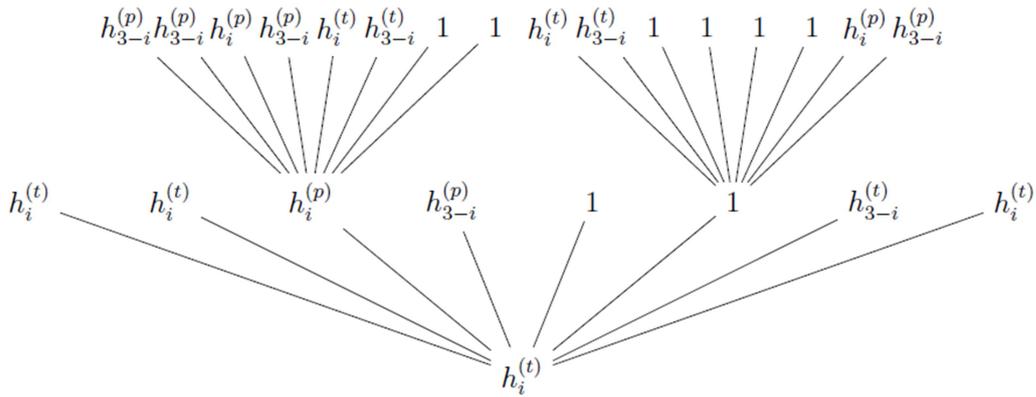


Figure 3.7: This is an example of the function h_x on the vertices of the semi-infinite Cayley tree of order 8 in the case $a = 2, b = 2, c = 2, i = 1, 2$.

Lemma 3.3.1. Let $p \equiv 1 \pmod{4}$. Then any set of quantities according to the rules (A_1) , (A_2) and (A_3) on the Cayley tree Γ_+^k satisfy the functional equation (2.1.5).

Proof. Let $h_x = h_i^{(t)}$, then using (3.3.4) we get

$$\prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \left(\frac{\theta (h_i^{(t)})^2 + 1}{(h_i^{(t)})^2 + \theta} \right)^{a+2} \cdot \left(\frac{\theta (h_{3-i}^{(t)})^2 + 1}{(h_{3-i}^{(t)})^2 + \theta} \right)^{\frac{a}{2}} \cdot \left(\frac{\theta (h_i^{(p)})^2 + 1}{(h_i^{(p)})^2 + \theta} \right)^{\frac{b}{2}} \cdot \left(\frac{\theta (h_{3-i}^{(p)})^2 + 1}{(h_{3-i}^{(p)})^2 + \theta} \right)^{\frac{b}{2}} = \left(\frac{\theta (h_i^{(t)})^2 + 1}{(h_i^{(t)})^2 + \theta} \right)^2 = (h_i^{(t)})^2.$$

Let $h_x = h_i^{(p)}$, then according to (3.3.4) we have

$$\prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} = \left(\frac{\theta (h_i^{(t)})^2 + 1}{(h_i^{(t)})^2 + \theta} \right)^{\frac{a}{2}} \cdot \left(\frac{\theta (h_{3-i}^{(t)})^2 + 1}{(h_{3-i}^{(t)})^2 + \theta} \right)^{\frac{a}{2}} \cdot \left(\frac{\theta (h_i^{(p)})^2 + 1}{(h_i^{(p)})^2 + \theta} \right)^{\frac{b}{2}} \cdot \left(\frac{\theta (h_{3-i}^{(p)})^2 + 1}{(h_{3-i}^{(p)})^2 + \theta} \right)^{\frac{b}{2}+2} = \left(\frac{\theta (h_{3-i}^{(p)})^2 + 1}{(h_{3-i}^{(p)})^2 + \theta} \right)^2 = (h_i^{(p)})^2.$$

The rest of the proof runs as before.

Remark 3.3.1.

- If $a = b = c = 0$ in case (A_1) and (A_3) then p -adic generalized Gibbs measures corresponding to set of quantities h_x are translation-invariant, the figure for case (A_2) , we get p -adic generalized $G_2^{(2)}$ -periodic Gibbs measures;
- If $a = b = 0, c \neq 0$ in case (A_3) and in cases $(A_1), (A_2)$ then p -adic generalized Gibbs measures corresponding to set of quantities h_x are translation-invariant and ART Gibbs measures, respectively (see [2]).
- If $b = c = 0, a \neq 0$ in case (A_1) then p -adic generalized Gibbs measures corresponding to set of quantities h_x are k_0 -translation-invariant (see [86]);
- If $a = c = 0, b \neq 0$ in case (A_2) then p -adic generalized Gibbs measures corresponding to set of quantities h_x are k_0 -periodic (see [86]);
- In other cases we get new measures except for previous known ones.

Let $k \geq 3, k_0 = 2$. Let $k - k_0 = a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3$, where a_s and b_s are even, c_s is even or odd and $s = 1, 2, 3$. We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in \{1, h_1^{(t)}, h_2^{(t)}, h_1^{(p)}, h_2^{(p)}\}$) as follows:

(B_1) if at vertex x we have $h_x = h_i^{(t)}$ ($i = 1, 2$), then the function h_y , which gives p -adic values to each vertex $y \in S(x)$ by the following rule

$$h_y = \begin{cases} h_i^{(t)}, & \text{on } \frac{a_1}{2} + 2 \text{ vertices of } S(x); \\ h_{3-i}^{(t)}, & \text{on } \frac{a_1}{2} \text{ vertices of } S(x); \\ h_i^{(p)}, & \text{on } \frac{b_1}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(p)}, & \text{on } \frac{b_1}{2} \text{ vertices of } S(x); \\ 1, & \text{on } c_1 \text{ vertices of } S(x). \end{cases} \quad (3.3.7)$$

(B_2) if at vertex x we have $h_x = h_i^{(p)}$ ($i = 1, 2$), then the function $h_y, y \in S(x)$ is defined as follows

$$h_y = \begin{cases} h_i^{(t)}, & \text{on } \frac{a_2}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(t)}, & \text{on } \frac{a_2}{2} \text{ vertices of } S(x); \\ h_i^{(p)}, & \text{on } \frac{b_2}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(p)}, & \text{on } \frac{b_2}{2} + 2 \text{ vertices of } S(x); \\ 1, & \text{on } c_2 \text{ vertices of } S(x). \end{cases} \quad (3.3.8)$$

(B_3) if at vertex x we have $h_x = 1$, then the function h_y is defined by the following rule for all vertices $y \in S(x)$.

$$h_y = \begin{cases} h_i^{(t)}, & \text{on } \frac{a_3}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(t)}, & \text{on } \frac{a_3}{2} \text{ vertices of } S(x); \\ h_i^{(p)}, & \text{on } \frac{b_3}{2} \text{ vertices of } S(x); \\ h_{3-i}^{(p)}, & \text{on } \frac{b_3}{2} \text{ vertices of } S(x); \\ 1, & \text{on } c_3 + 2 \text{ vertices of } S(x). \end{cases} \quad (3.3.9)$$

(See the Figure 3.8)

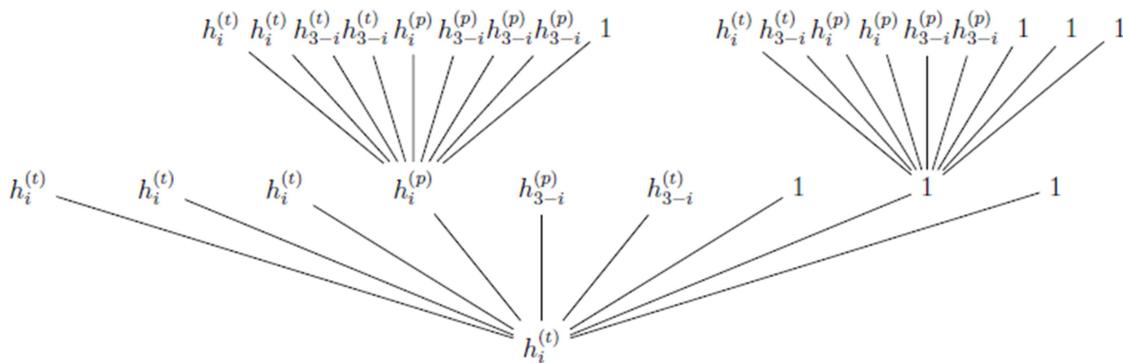


Figure 3.8: This is an example of the function h_x on the vertices of the semi-infinite Cayley tree of order 9 in the case $a_1 = 2, b_1 = 2, c_1 = 3, a_2 = 4, b_2 = 2, c_2 = 1,$

$$a_3 = 2, b_3 = 4, c_3 = 1, i = 1, 2.$$

Lemma 3.3.2. Let $p \equiv 1 \pmod{4}$. Then any set of quantities according to the rules $(B_1), (B_2)$ and (B_3) on the Cayley tree Γ_+^k satisfy the functional equation (2.1.5).

Proof. The proof is similar to the proof of Lemma 3.3.1.

Remark 3.3.2. Let $k_0 = 3$ and $p \equiv 1 \pmod{12}$. As above, using the translation-invariant and G_2 -periodic solutions of the equation (2.1.5) for $k = 3$ (see Chapter II), we can construct set of quantities h_x which satisfies (2.1.5).

Let us consider the following equation that we shall use it later

$$x_1 + x_2 + \dots + x_m = n, \quad (3.3.10)$$

here $m, n \in \mathbb{N}, m \leq n, x_i \in \{0, 1, 2, \dots, n\}, i = \overline{1, m}$.

Lemma 3.3.3. [27] Let N be the number of non-negative integer solutions of the equation (3.3.10). Then

$$N = C_{n+m-1}^{m-1},$$

here $C_n^m = \frac{n!}{m!(n-m)!}, n! = 1 \cdot 2 \cdot \dots \cdot n, n \geq m$.

Lemma 3.3.4. Let $p \equiv 1 \pmod{4}$ and T be the number of choices of the quantities h_x for each level on the Cayley tree Γ_+^k in Lemma 3.3.2. Then the following statements hold:

$$T = \begin{cases} \frac{k(k+2)}{8}, & \text{if } k \text{ is even;} \\ \frac{(k-1)(k+1)}{8}, & \text{if } k \text{ is odd.} \end{cases} \quad (3.3.11)$$

Proof. Case 1. Let k be even. Then the equality $k - k_0 = a_i + b_i + c_i$ yields that c_i is even ($i=1,2,3$). Thus, we can rewrite a_i, b_i, c_i as $a_i = 2a_i^*, b_i = 2b_i^*, c_i = 2c_i^*$ where $a_i^*, b_i^*, c_i^* \in \mathbb{N}$. Using the last equalities we have

$$a_i^* + b_i^* + c_i^* = \frac{k-2}{2}.$$

Thanks to Lemma 3.3.3, the number of solutions of the last equation equals $C_{\frac{k+2}{2}}^2 = \frac{k(k+2)}{8}$. It follows that $T = \frac{k(k+2)}{8}$.

Case 2. Let k be odd. Then the equality $k - k_0 = a_i + b_i + c_i$ yields that c_i is odd ($i=1,2,3$). Thus, we can rewrite a_i, b_i, c_i as $a_i = 2a_i^*, b_i = 2b_i^*, c_i = 2c_i^* + 1$ where $a_i^*, b_i^*, c_i^* \in \mathbb{N}$. Using the last equalities we have

$$a_i^* + b_i^* + c_i^* = \frac{k-3}{2}.$$

Thanks to Lemma 3.3.3, the number of solutions of the last equation equals

$$C_{\frac{k+1}{2}}^2 = \frac{(k-1)(k+1)}{8}.$$

It follows that

$$T = \frac{(k-1)(k+1)}{8}.$$

Corollary 3.3.1. Let $p \equiv 1 \pmod{4}$. There are at least countable non-periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree Γ_+^k .

Proof. According to Lemma 3.3.4, there are finite quantities h_x for each level on the semi-infinite Cayley tree Γ_+^k . There are countable levels on the semi-infinite Cayley tree Γ_+^k . It follows the assertion of the lemma.

Lemma 3.3.5. [81] If $|A_i - 1|_p \leq M$ and $|A_i|_p = 1, i = \overline{1, n}$ then

$$\left| \prod_{i=1}^n A_i - 1 \right|_p \leq M.$$

Corollary 3.3.2. If $|a_i|_p = |b_i|_p = 1, i = \overline{1, n}$ then

$$\left| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right|_p \leq \max_{1 \leq i \leq n} |a_i - b_i|_p.$$

Proof. If $A_i = \frac{a_i}{b_i}, |a_i|_p = 1, i = \overline{1, n}$ and $M = \max_{1 \leq i \leq n} |A_i - 1|_p$ in Lemma 3.3.5,

then we get the assertion of Corollary 3.3.2.

p -adic analogue of the Bleher-Ganikhodjaev construction. In real case this construction was studied in [8]. We are aiming to investigate this construction in p -adic case. Consider an infinite path $\pi = x^0 = x_0 < x_1 < \dots$ on the semi-Cayley tree of order k Γ_+^k (the notation $x < y$ meaning that paths from the root to y go through x). We assign the set of p -adic numbers $h^\pi = \{h_x^\pi, x \in V \subset \Gamma_+^k\}$ satisfying the equation (2.1.5) to the path π . For $x \in W_n$, the set h^π is unambiguously defined by the conditions

$$h_x^\pi = \begin{cases} \frac{1}{h_*}, & \text{if } x \prec x_n, x \in W_n; \\ h_*, & \text{if } x_n \prec x, x \in W_n. \end{cases} \quad (3.3.12)$$

where $n = 1, 2, 3, \dots$, $x \prec x_n$ (resp. $x_n \prec x$) means that x is on the left (resp. right) from the path π and h_* is translation-invariant solution of the equation (2.1.5) (see Figure 3.9).

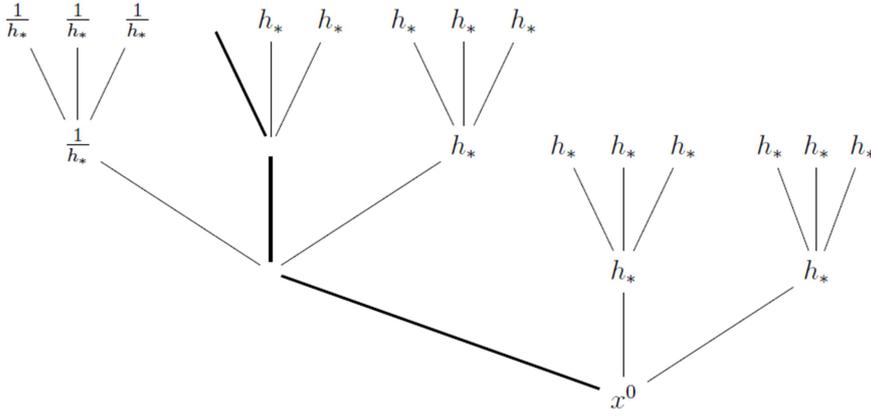


Figure 3.9: This figure illustrates the function h_x^π on the semi-Cayley tree of order three.

Lemma 3.3.6. For any infinite path π , there exists a unique set of numbers $h^\pi = \{h_x^\pi, x \in V \subset \Gamma_+^k\}$ satisfying (3.3.12) and (2.1.5).

Proof. On W_n we define the set

$$h_x^\pi = \begin{cases} \frac{1}{h_*}, & \text{if } x \prec x_n, x \in W_n; \\ h_*, & \text{if } x_n \prec x, x \in W_n; \\ h_x^{(n)}, & \text{if } x = x_n. \end{cases} \quad (3.3.13)$$

where $n=1,2,3,\dots$, and $h_{x_n}^{(n)}$ is an arbitrary p -adic numbers such that $(h_{x_n}^{(n)})^2 \in \mathbb{Z}_p^* \setminus B(-1,1)$. We extend the definition of $h_x^{(n)}$ for all $x \in V_n = \bigcup_{m=0}^n W_m$ using recursion equations (2.1.5). We now prove that the limit

$$h_x^2 = \lim_{n \rightarrow \infty} (h_x^{(n)})^2 \quad (3.3.14)$$

exists for every fixed $x \in V \subset \Gamma_+^k$ and h_x is independent of the choice of $h_x^{(n)}$ for $x = x_n$. If $x \in W_{n-1}$ and $x \prec x_{n-1}$ then $(h_x^{(n)})^2 = \prod_{y \in W_n, y > x} f(h_y^{(n)}, \theta) = f^k(\frac{1}{h_*}, \theta) = \left(\frac{1}{h_*}\right)^2$. Similarly, if $x \in W_{n-1}$ and $x_{n-1} \prec x$ then $(h_x^{(n)})^2 = \prod_{y \in W_n, y > x} f(h_y^{(n)}, \theta) = f^k(h_*, \theta) = h_*^2$.

Consequently, for any $x \in W_m$, $m \leq n$ we have

$$h_x^{(n)} = \begin{cases} \frac{1}{h_*}, & \text{if } x \prec x_m, x \in W_m, \\ h_*, & \text{if } x_m \prec x, x \in W_m. \end{cases} \quad (3.3.15)$$

This implies that the limit (3.3.14) exists for $x \in W_m$ and $x \neq x_m$ and

$$h_x = \begin{cases} \frac{1}{h_*}, & \text{if } x \prec x_m, x \in W_m, \\ h_*, & \text{if } x_m \prec x, x \in W_m. \end{cases}$$

Therefore, we only need to establish that the limit (3.3.14) exists for $x = x_m$.

Let $1 \leq l \leq n$. Then

$$\left(h_{x_{l-1}}^{(n)}\right)^2 = \prod_{y \in W_n, y > x_{l-1}} f(h_y^{(n)}, \theta). \quad (3.3.16)$$

Consider two sets $\{\bar{h}_x^{(n)}, x \in V_n\}$ and $\{\widetilde{h}_x^{(n)}, x \in V_n\}$ which correspond to two values $\bar{h}_x^{(n)}$ and $\widetilde{h}_x^{(n)}$ for $x = x_n$, in (3.3.13) then from (3.3.16) we get

$$\left| \left(\bar{h}_{x_{l-1}}^{(n)}\right)^2 - \left(\widetilde{h}_{x_{l-1}}^{(n)}\right)^2 \right|_p = \left| \prod_{y \in W_n, y > x_{l-1}} f(\bar{h}_y^{(n)}, \theta) - \prod_{y \in W_n, y > x_{l-1}} f(\widetilde{h}_y^{(n)}, \theta) \right|_p. \quad (3.3.17)$$

Applying Corollary 3.3.2 we can rewrite (3.3.17) as

$$\left| \left(\bar{h}_{x_{l-1}}^{(n)}\right)^2 - \left(\widetilde{h}_{x_{l-1}}^{(n)}\right)^2 \right|_p \leq \max_l \left| f(\bar{h}_{x_l}^{(n)}, \theta) - f(\widetilde{h}_{x_l}^{(n)}, \theta) \right|_p. \quad (3.3.18)$$

Since $\left(h_x^{(n)}\right)^2 \in \mathbb{Z}_p^* \setminus B(-1, 1)$, using (3.3.18) we obtain

$$\left| \left(\bar{h}_{x_{l-1}}^{(n)}\right)^2 - \left(\widetilde{h}_{x_{l-1}}^{(n)}\right)^2 \right|_p \leq |\theta^2 - 1|_p \max_l \left| \left(\bar{h}_{x_l}^{(n)}\right)^2 - \left(\widetilde{h}_{x_l}^{(n)}\right)^2 \right|_p. \quad (3.3.19)$$

Iterating the last inequality we get

$$\left| \left(\bar{h}_{x_m}^{(n)}\right)^2 - \left(\widetilde{h}_{x_m}^{(n)}\right)^2 \right|_p \leq |\theta^2 - 1|_p^{n-m} \max_l \left| \left(\bar{h}_n^{(n)}\right)^2 - \left(\widetilde{h}_n^{(n)}\right)^2 \right|_p. \quad (3.3.20)$$

For arbitrary $N, M > n$, we now consider the set $\{h_x^{(N)}, x \in V_N\}$ and $\{h_x^{(M)}, x \in V_M\}$ determined by initial conditions of form (3.3.13) for $x \in W_N$ and $x \in W_M$

respectively and by the recursion equation (2.1.5). We set $\bar{h}_{x_n}^{(n)} = h_{x_n}^{(N)}$, $\widetilde{h}_{x_n}^{(n)} = h_{x_n}^{(M)}$. Then inequality (3.3.20) implies

$$\left| \left(h_{x_m}^{(N)} \right)^2 - \left(h_{x_m}^{(M)} \right)^2 \right|_p \leq |\theta^2 - 1|_p^{n-m} \max_l \left| \left(h_{x_n}^{(N)} \right)^2 - \left(h_{x_n}^{(M)} \right)^2 \right|_p \leq |\theta^2 - 1|_p^{n-m}. \quad (3.3.21)$$

Since $|\theta^2 - 1|_p < 1$, the last estimate implies that the sequence $h_{x_m}^{(n)}$ satisfies the Cauchy criterion as $n \rightarrow \infty$ for a fixed m ; therefore, the limit (3.3.14) exists and it is independent of the choice $h_{x_n}^{(n)}$ in (3.3.13). Because, by construction, the set $\{h_x^{(n)}\}$ satisfies the equation (2.1.5) before taking the limit, so does $\{h_x\}$. The uniqueness of $\{h_x\}$ obviously follows from estimate (3.3.20).

The boundedness of the constructive measures.

Lemma 3.3.7. Let $p \equiv 1 \pmod{4}$ and $h_0^{(t)}, h_{1,2}^{(t)}, h_{1,2}^{(p)}$ be the solutions of (2.1.5) that is given in (3.3.1). Then the following statements hold true:

- $|h_i^{(t)}|_p = 1$, $|(h_i^{(t)})^2 + \theta|_p \leq |\theta - 1|_p$, $i = \overline{0, 2}$;
- $|h_i^{(p)}|_p = 1$, $|(h_i^{(p)})^2 + \theta|_p \leq |\theta - 1|_p$, $i = \overline{1, 2}$.

Proof. The proof is straightforward from (3.3.3) and $\theta \in \mathcal{E}_p$.

Theorem 3.3.1. Let $p \equiv 1 \pmod{4}$ and $a^2 + b^2 \neq 0$. Then the measures corresponding to the set of quantities according to the rules (A_1) , (A_2) and (A_3) are unbounded.

Proof. Let the values h_x defined by rules $(A_1), (A_2), (A_3)$.

Case A_1 . Let at vertex x we have $h_x = h_i^{(t)}$ ($i = 1, 2$). Then according to Lemmas 3.3.7 and 2.2.3 we have

if $s = 1$ then

$$\begin{aligned} |a_h(x)|_p &= \left| \prod_{y \in S(x)} \left(\exp_p \{-J\} h_y + \exp_p \{J\} h_y^{-1} \right) \right|_p = \left| \prod_{y \in S(x)} (h_y^2 + \theta) \right|_p \\ &= \left| \left((h_i^{(t)})^2 + \theta \right)^{\frac{a}{2}+2} \right|_p \left| \left((h_{3-i}^{(t)})^2 + \theta \right)^{\frac{a}{2}} \right|_p \left| \left((h_i^{(p)})^2 + \theta \right)^{\frac{b}{2}} \right|_p \left| \left((h_{3-i}^{(p)})^2 + \theta \right)^{\frac{b}{2}} \right|_p \left| (\theta + 1)^c \right|_p \\ &\leq |\theta - 1|_p^{k-c}, \end{aligned}$$

if $s = -1$ then

$$\begin{aligned}
|a_h(x)|_p &= \left| \prod_{y \in S(x)} (\exp_p \{J\} h_y + \exp_p \{-J\} h_y^{-1}) \right|_p = \left| \prod_{y \in S(x)} (\theta h_y^2 + 1) \right|_p \\
&= \left| \left((\theta h_i^t)^2 + 1 \right)^{\frac{a}{2}+2} \right|_p \left| \left((\theta h_{3-i}^t)^2 + 1 \right)^{\frac{a}{2}} \right|_p \left| \left((\theta h_i^p)^2 + 1 \right)^{\frac{b}{2}} \right|_p \left| \left((\theta h_{3-i}^p)^2 + 1 \right)^{\frac{b}{2}} \right|_p \left| (\theta + 1)^c \right|_p \\
&= \left| \left((h_i^t)^2 + \theta \right)^{\frac{a}{2}+2} \right|_p \left| \left((h_{3-i}^t)^2 + \theta \right)^{\frac{a}{2}} \right|_p \left| \left((h_i^p)^2 + \theta \right)^{\frac{b}{2}} \right|_p \left| \left((h_{3-i}^p)^2 + \theta \right)^{\frac{b}{2}} \right|_p \\
&\leq |\theta - 1|_p^{k-c}.
\end{aligned}$$

Using the equality Lemma 2.2.3 we obtain

$$|A_{h,n}|_p = \left| \prod_{x \in W_n} a_h(x) \right|_p \leq |\theta - 1|_p^{(k-c)|W_n|} = |\theta - 1|_p^{(k-c)k^n}.$$

The last inequality and the equality (2.1.3) follow that

$$|Z_n^{(h)}|_p = \left| \prod_{i=1}^{n-1} A_{h,i} Z_1^{(h)} \right|_p \leq |\theta - 1|_p^{\frac{(k-c)(k^n-k)}{k-1}} \cdot |Z_1^{(h)}|_p \leq |\theta - 1|_p^{\frac{(k-c)(k^n-k)}{k-1}}. \quad (3.3.22)$$

(2.1.2), (3.3.22) and Lemma 3.3.7 give

$$|\mu_h^{(n)}(\sigma)|_p = \left| (Z_n^{(h)})^{-1} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_x^{(\sigma)} \right|_p = |Z_n^{(h)}|_p^{-1} \geq |\theta - 1|_p^{-\frac{(k-c)(k^n-k)}{k-1}}. \quad (3.3.23)$$

The inequality (3.3.23) yields

$$\lim_{n \rightarrow \infty} |\mu_h^{(n)}|_p = +\infty.$$

Case A_1, A_2 . The rest of the proof runs as before.

Remark 3.3.2. Let $p \equiv 1 \pmod{4}$. It is easy to see that $h_0^{(t)} = 1$ is an arbitrary periodic fixed point of $f(h)$. As in the proof of Theorem 3.3.1, we can prove that the measure μ_{h_0} is bounded.

Theorem 3.3.2. Let $p \equiv 1 \pmod{4}$ and $a_i^2 + b_i^2 \neq 0$, $i=1,2,3$. Then the measures corresponding to the set of quantities according to the rules (B_1) , (B_2) and (B_3) are unbounded.

Proof. Analysis is similar to that in the proof of Theorem 3.3.1.

Theorem 3.3.3. Let $p \geq 3$, h_x^π be the set of quantities defined by (3.3.13) and $h_* \neq 1$. Then the measures corresponding to the set of quantities h_x^π are unbounded.

Proof. Case 1. Let at vertex x we have $h_x = h_*$, here $h_*(\neq 1)$ be translation-invariant solution of (2.1.5). It is not difficult to check that $|h_*|_p = 1$

and

$$|h_*^2 + \theta|_p = \left| \frac{1}{h_*^2} + \theta \right|_p \leq |\theta - 1|_p.$$

Then according to Lemma 2.2.3 we have

if $s = 1$ then

$$\begin{aligned} |a_h(x)|_p &= \left| \prod_{y \in S(x)} (\exp_p \{-J\} h_y + \exp_p \{J\} h_y^{-1}) \right|_p = \\ &= \left| \prod_{y \in S(x)} (h_y^2 + \theta) \right|_p = \left| (h_*^2 + \theta)^k \right|_p \leq |\theta - 1|_p^k, \end{aligned}$$

if $s = -1$ then

$$\begin{aligned} |a_h(x)|_p &= \left| \prod_{y \in S(x)} (\exp_p \{J\} h_y + \exp_p \{-J\} h_y^{-1}) \right|_p = \\ &= \left| \prod_{y \in S(x)} (\theta h_y^2 + 1) \right|_p = \left| \prod_{y \in S(x)} (\theta h_y^2 + 1) \right|_p = \\ &= \left| (\theta h_*^2 + 1)^k \right|_p = \left| h_*^2 (h_*^2 + \theta)^k \right|_p \leq |\theta - 1|_p^k. \end{aligned}$$

By Lemma 2.2.3 we obtain

$$|A_{h,n}|_p = \left| \prod_{x \in W_n} a_h(x) \right|_p \leq |\theta - 1|_p^{|W_n|} = |\theta - 1|_p^{k^{n+1}}.$$

From the last inequality and the equality (2.1.3), it follows that

$$|Z_n^{(h)}|_p = \left| \prod_{i=1}^{n-1} A_{h,i} Z_1^{(h)} \right|_p \leq |\theta - 1|_p^{\frac{k^{n+1}-k^2}{k-1}} \cdot |Z_1^{(h)}|_p \leq |\theta - 1|_p^{\frac{k^{n+1}-k^2}{k-1}}. \quad (3.3.24)$$

(2.1.3), (3.3.24) and Lemma 2.2.3 give

$$|\mu_h^{(n)}(\sigma)|_p = \left| (Z_n^{(h)})^{-1} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_x^{(\sigma)} \right|_p = |Z_n^{(h)}|_p^{-1} \geq |\theta - 1|_p^{\frac{k^{n+1}-k^2}{k-1}}. \quad (3.3.25)$$

The inequality (3.3.25) yields

$$\lim_{n \rightarrow \infty} |\mu_h^{(n)}|_p = +\infty.$$

Case 2. Let at vertex x we have $h_x = \frac{1}{h_*}$, here $h_*(\neq 1)$ be translation-invariant solution of (2.1.5). As in Case 1, we can see that the measure is unbounded.

Case 3. Let at vertex x we have $h_x = h_{x_n}^{(n)}$, here $h_{x_n}^{(n)}$ be an arbitrary p -adic numbers such that $(h_{x_n}^{(n)})^2 \in \mathbb{Z}_p^* \setminus B(-1,1)$. Then according to Lemma 3.3.8 we have

if $s = 1$ then

$$\begin{aligned} |a_h(x)|_p &= \left| \prod_{y \in S(x)} (\exp_p \{J\} h_y + \exp_p \{-J\} h_y^{-1}) \right|_p = \left| \prod_{y \in S(x)} (h_y^2 + \theta) \right|_p \\ &= \left| (h_*^2 + \theta) \right|_p \left| \left(\frac{1}{h_*^2} + \theta \right) \right|_p^r \left| \left((h_{x_{n+1}}^{(n+1)})^2 + \theta \right) \right|_p \leq |\theta - 1|_p^{k-1}, \end{aligned}$$

here $q + r + 1 = k$.

if $s = -1$ then

$$\begin{aligned} |a_h(x)|_p &= \left| \prod_{y \in S(x)} \left(\exp_p \{J\} h_y + \exp_p \{-J\} h_y^{-1} \right) \right|_p = \left| \prod_{y \in S(x)} (\theta h_y^2 + 1) \right|_p \\ &= \left| (\theta h_*^2 + 1) \right|_p^l \left| \left(\theta \frac{1}{h_*^2} + 1 \right)^m \right|_p \left| \theta (h_{x_{n+1}}^{(n+1)})^2 + 1 \right|_p \leq \theta - 1 \Big|_p^{k-1}, \end{aligned}$$

here $l + m + 1 = k$.

Using Lemma 2.2.3 we obtain

$$|A_{h,n}|_p = \left| \prod_{x \in W_n} a_h(x) \right|_p \leq \theta - 1 \Big|_p^{(k-1)|W_n|} = \theta - 1 \Big|_p^{(k-1)k^n}.$$

The last inequality and the equality (2.1.3) follow that

$$|Z_n^{(h)}|_p = \left| \prod_{i=1}^{n-1} A_{h,i} Z_1^{(h)} \right|_p \leq \theta - 1 \Big|_p^{k^n - k} \cdot |Z_1^{(h)}|_p \leq \theta - 1 \Big|_p^{k^n - k} \quad (3.3.26)$$

(2.1.3), (3.3.26) and Lemma 2.2.3 give

$$|\mu_h^{(n)}(\sigma)|_p = \left| (Z_n^{(h)})^{-1} \exp_p \{H_n(\sigma)\} \prod_{x \in W_n} h_x^{(\sigma)} \right|_p = |Z_n^{(h)}|_p^{-1} \geq \theta - 1 \Big|_p^{-(k^n - k)} \quad (3.3.27)$$

The inequality (3.3.27) yields

$$\lim_{n \rightarrow \infty} |\mu_h^{(n)}|_p = +\infty.$$

The theorem is completely proved.

Theorem 3.3.4. Let $p \geq 3$, h_x^π be the set of quantities defined by (3.3.13)

and $h_* = 1$. Then the measures corresponding to the set of quantities h_x^π are bounded.

Proof. The proof is similar to proof of Theorem 3.3.3.

Corollary 3.3.3. Let $p \geq 3$, h_x^π be the set of quantities defined by (3.3.13)

Then the measures correspond to the set of quantities h_x^π are bounded if and only if

$h_* = 1$.

Proof. The proof is straightforward from Theorems 3.3.3 and 3.3.4.

Theorem 3.3.5. Let $p \geq 3$, $h_* \notin \{-1, 1\}$ be the translation-invariant solution of the functional equation (2.1.5). Then there exists a phase transition.

Proof. The proof is straightforward from Corollary 3.3.3.

Theorem 3.3.6. The measure which is defined in Lemma 3.3.6 depends on the path π . Moreover, there exists a set of uncountable non-periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree.

Proof. In the real case, this theorem was proved in [8]. The same conclusion can be drawn for the p -adic case.

CONCLUSION OF THE THIRD CHAPTER

This chapter consists of three sections. Section 3.1 is devoted to the ART p -adic quasi Gibbs measures for 3-state Potts model on the Cayley tree of order two. Using the translation-invariant and G_2 -periodic solutions of the functional equation for the 3-state Potts model on the Cayley tree of order two, ART p -adic quasi Gibbs measures on a higher-order Cayley tree were constructed, the conditions of existence of these measures were determined and their boundedness was checked.

Section 3.2 is devoted to (k_0) -translation-invariant, (k_0) -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree. Using translation-invariant and $G_k^{(2)}$ -periodic solutions of the functional equations for the Ising model on the Cayley tree of order two and three, ART, (k_0) -translation-invariant, (k_0) -periodic p -adic generalized Gibbs measures on a higher-order Cayley tree were constructed.

Section 3.3 is devoted to some constructive p -adic generalized Gibbs measures for the Ising model on the Cayley tree. It was proved that this construction generalizes some known constructions, i.e. at some specific values of the parameter, the construction coincides with one of the constructions, ART, (k_0) -translation-invariant, and (k_0) -periodic. Using this construction, some non-periodic p -adic generalized Gibbs measures were constructed. The conditions of existence of these measures were determined and the boundedness of these measures was checked.

In Section 3.3 a p -adic analogue of the Bleher-Ganikhodjaev construction, providing the existence of uncountable non-periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree, was constructed. The conditions of existence of found measures were determined and boundedness of these measures were checked. New condition of existence a phase transition was found.

CONCLUSION

The dissertation is devoted to investigate the translation-invariant, periodic and non-periodic p -adic Gibbs measures for finite state p -adic Potts and Ising models, furthermore, to check the boundedness of found measures and to study phase transitions for these models.

The main results of the research are as follows:

1. It was proved that if $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$, then there exist two non-translation-invariant G_2 -periodic p -adic quasi Gibbs measures for the 3-state Potts model on the Cayley tree of order two;
2. If $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$, $p \neq 3$, then the existence of a phase transition for 3-state p -adic Potts model on the Cayley tree of order two was proved;
3. All translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three were found. In particular, if $p \equiv 1 \pmod{12}$, it was proved that there are four translation-invariant p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three and a phase transition occurs.
4. All non-translation-invariant $G_k^{(2)}$ -periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree of order three were found. The boundedness of these measures was checked;
5. Using translation-invariant and G_2 -periodic solutions of the functional equations for the 3-state Potts model on the Cayley tree of order two, ART p -adic quasi Gibbs measures on a higher-order Cayley tree were constructed, the conditions of existence of these measures were determined and their boundedness was checked;
6. Using translation-invariant and $G_k^{(2)}$ -periodic solutions of the functional equations for the Ising model on the Cayley tree of order two and three, ART, (k_0) -translation-invariant, (k_0) -periodic p -adic generalized Gibbs measures

on a higher-order Cayley tree were constructed, It was proved that this construction generalizes some known constructions, i.e. at some specific values of the parameter, the construction coincide with one of the constructions, ART, (k_0) -translation-invariant, and (k_0) -periodic. Using this construction, some non-periodic p -adic generalized Gibbs measures were constructed. The conditions of existence of these measures were determined and boundedness of these measures was checked;

7. A p -adic analogue of the Bleher-Ganikhodjaev construction, providing the existence of uncountable non-periodic p -adic generalized Gibbs measures for the Ising model on the Cayley tree, was constructed. The conditions of existence of found measures were determined and boundedness of these measures was checked. The new condition of the existence a phase transition was found.

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