V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI
HUZURIDAGI ILMIY DARAJALAR BERUVCHI
DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

MATEMATIKA INSTITUTI

RAHMONOV ASKAR AHMADOVICH

KASR DIFFUZIYA-TO‘LQIN TENGLAMASI UCHUN KOEFFITSIYENTLI TESKARI MASALALARNING KORREKTLIGI

01.01.02 – Differensial tenglamalar va matematik fizika

**FIZIKA-MATEMATIKA FANLARI BO‘YICHA DOKTORLIK (DSc)
DISSERTATSIYASI AVTOREFERATI**

TOSHKENT – 2025

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Оглавление автореферата докторской диссертации (DSc) по физико-математическим наукам

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Dissertatsiya V.I. Romanovskiy nomidagi Matematika institutida bajarilgan.

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Yetakchi tashkilot:	**O‘zbekiston Milliy universiteti**

Dissertatsiya himoyasi V.I. Romanovskiy nomidagi Matematika instituti huzuridagi DSc.02/30.12.2019.FM.86.01 raqamli Ilmiy kengashning 2025 yil “27” may soat 17:00 dagi majlisida bo‘lib o‘tadi. (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871) 207-91-40, e-mail: uzbmath@umail.uz, Website: www.mathinst.uz).

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Dissertatsiya mavzusining dolzarbligi va zarurati. So‘nggi vaqtlarda fanning jadal rivojlanayotgan sohalaridan biri – bu kasr tartibli differensial tenglamalar hisoblanadi. Butun bo‘lmagan tartib hisobi - bu an’anaviy ravishda “Kasr hisob” deb ataladigan matematikaning bir bo‘limi bo‘lib, unda klassik hosilalar va integrallarni interpolyatsiya qilishga harakat qilinadi va ularni har qanday tartib uchun umumlashtiradi, bunda hosila tartibi natural bo‘lishi shart emas. Ushbu soha oxirgi yillarda katta ahamiyat kasb etmoqda, chunki u klassik modellar orqali to‘g‘ri ifodalanmagan murakkab tizimlar va hodisalarni tavsiflash imkonini beradi. Xususan, butun bo‘lmagan hosila tartiblarini o‘z ichiga olgan kasr differensial tenglamalar, xotira, irsiy ta’sirlar va lokal bo‘lmagan o‘zaro ta’sirlar ko‘rsatilgan fizik, biologik va muhandislik jarayonlarini aniqroq modellashtirish imkonini beradi. Kasr diffuziya-to‘lqin tenglamalari uchun teskari koeffitsiyentli masalalarning korrektligi matematik modellashtirish sohasidagi eng muhim masalalardan biridir. Teskari masalalar, kuzatilgan ma’lumotlarga asoslanib, tenglamalardagi boshqaruvchi noma’lum parametrlarni, masalan, koeffitsiyentlarni aniqlash maqsad qilinadi. Ular, tasvirni qayta tiklash, materiallarni xarakterlash va bir jinsli muhitda anomal diffuziya va to‘lqin tarqalishini o‘rganish kabi ko‘plab amaliy masalalarda juda muhimdir. Zamonaviy tizimlarning ortib borayotgan murakkabligi, ayniqsa fraktal yoki nolokallik xususiyatlarga ega bo‘lgan tizimlarning ko‘pligi, ushbu teskari masalalarni to‘g‘ri va samarali hal qilish uchun qat’iy usullarni talab qiladi. Shu sababli, bunday masalalarni korrektligini o‘rganishga bo‘lgan talab, nazariy va amaliy jihatlardan kelib chiqadi, chunki fraktal modellarning fizika, moliya, biotexnologiya va muhandislik kabi sohalarda haqiqiy jarayonlarni tavsiflashda zaruriyati ortib bormoqda. Ushbu teskari masalalarining matematik va hisoblash jihatlarini tushunish, tajriba ma’lumotlarini yanada aniqroq talqin qilish va real vaqt rejimida ilovalar uchun algoritmlarni takomillashtirish imkonini beradi.

Hozirgi kunda jahonda xususiy hosilali kasr tartibli tenglamalarni yechishni o’rganish bilan bir qatorda tenglamaning koeffitsiyentlari, uning o‘ng tomonini va integral ostidagi yadroni aniqlashga oid teskari masalalarni yechishni o‘rganish juda muhim ahamiyatga ega. Teskari masalalarni o‘rganish bizga kasr tartibli tenglama bilan berilgan jarayonlarni o‘rganish, tahlil qilish va boshqarish imkoniyatlarini beradi. Masalan, bir jinsli tor tebranish jarayonlarini olsak, yechimning boshlang‘ich holatlarini xarakteristik uchburchak asosida bilishimiz bizga tor tebranishini chegaradan boshqarish imkoniyatini beradi. Vaqt bo‘yicha kasr tartibli tenglamani qarasak, bunda kasr tartibli hosilaning tartibi 1 va 2 oralig‘ida bo‘lsa, bir vaqtda diffuziya va to‘lqin tarqalish jarayonlarini aks ettiradi, agar 0 va 1 oralig‘ida bo‘lsa, sekin sodir bo‘luvchi diffuziya jarayonlarini ifodalaydi. Masalan, optik tolalar orqali signalning tarqalishi, elektromagnit to‘lqinlarning to‘g‘ri yo‘nalishda o‘tishini va ularning materialga ta’sirini modellashtirishda kasr differensial tenglamalar ishlatiladi va shuning uchun ham kasr tartibli differensial tenglamalarni o‘rganish dolzarb ahamiyatga egadir.

Mamlakatimizda zamonaviy fanga ilmiy va amaliy ahamiyatga ega boʻlgan differensial tenglamalar va matematik fizikaning zamonaviy muammolariga alohida e’tibor qaratilgan. Shu jumladan, kasr hosilali tenglamalar uchun toʻgʻri va teskari masalalarni tadqiq etishga alohida e’tibor qaratilgan. Bu yoʻnalish dunyo ilm-fanida nisbatan yangi boʻlishiga qaramasdan, Oʻzbekiston olimlari bu sohada sezilarli natijalarga erishishdi va ularni faol tadqiq etishda davom etmoqdalar. Matematika fanlarining, xususan, differensial tenglamalar va matematik fizikaning ustuvor yoʻnalishlari boʻyicha dunyo standartlari darajasida ilmiy izlanishlar olib borish V.I. Romanovskiy nomidagi Matematika instituti faoliyatidagi asosiy masaladir[1]. Chegaralangan va chegaralanmagan sohalarda berilgan kasr hosilali differensial tenglamalar uchun toʻgʻri va teskari masalalarni tadqiq etishni rivojlantirish differensial tenglamalar va matematik fizikaning zamonaviy nazariyasidagi muhim yoʻnalish hisoblanadi.

Oʻzbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi «Oʻzbekiston Respublikasini yanada rivojlantirish boʻyicha Harakatlar strategiyasi toʻgʻrisida»gi PF-4947-sonli Farmoni, 2019-yil 9-iyuldagi «Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qoʻllab-quvvatlash, shuningdek Oʻzbekiston Respublikasi Fanlar Akademiyasining V.I. Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari toʻgʻrisida»gi PQ-4387 sonli qarori, 2020-yil 7-maydagi «Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish chora-tadbirlari toʻgʻrisida»gi PQ-4708-sonli qarorlari hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishga ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yoʻnalishlariga mosligi. Dissertatsiya Oʻzbekiston Respublikasi fan va texnologiyalar rivojlanishining IV. «Matematika, mexanika va informatika» ustuvor yoʻnalishi doirasida bajarilgan.

Dissertatsiya mavzusi boʻyicha xorijiy ilmiy tadqiqotlar sharhi. Kasr tartibli hosilali differensial va xususiy hosilali differensial tenglamalar uchun toʻgʻri va teskari masalalar bilan dunyoning deyarli barcha universitetlarida, ilmiy tekshirish institutlarida va ilmiy markazlarida izlanishlar olib borilmoqda. Bunga misol sifatida ba’zilarini sanab oʻtamiz. Dunyoning yetakchi universitetlaridan Jon Xopkins universiteti (AQSh), Tokio universiteti (Yaponiya), Berlin amaliy ilmlar texnika universiteti (Germaniya), New Haven universiteti (AQSh), Berlin universiteti (Germaniya), Belorussiya davlat universiteti (Belorussiya), Xokkaydo universiteti (Yaponiya), La Rochelle universitetida (Fransiya), Amaliy matematika va avtomatlashtirish instituti (Rossiya), Ukraina Milliy Fanlar Akademiyasining Matematika Instituti (Ukraina), Yaqin Sharq universiteti (Turkiya), Xalqaro qozoq-turk universiteti (Qozogʻiston) va boshqalar.

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[1] Oʻzbekiston Respublikasi Vazirlar Mahkamasining 2017-yil 18-maydagi “Oʻzbekiston Respublikasi Fanlar akademiyasining yangidan tashkil etilgan ilmiy-tadqiqot muassasalari faoliyatini tashkil etish chora-tadbirlari toʻgʻrisida”gi 292-sonli qarori.

Kasr tartibli differensial tenglamalarning koeffitsiyentlarini aniqlash bo‘yicha teskari masalalar Tokio universiteti professori M. Yamamoto (Yaponiya), Ho Chi Minh universiteti professori N.H. Tuan (Vetnam), Janubiy-Sharqiy universiteti professori H. Wang (Xitoy), Nankin axborot fanlari va texnologiyalari universiteti professori B. Wu (Xitoy) va boshqalar tomonidan o‘rganilgan. Ular tomonidan ishlab chiqilgan usul teskari masalaning yechimi yagonaligini ta’minlab beradi xolos. Amerika Qo‘shma Shtatlari New Haven universiteti professori S. Umarov va O‘zbekiston Respublikasi FA qoshidagi Matematika instituti professori R. Ashurovlar tomonidan kasr tartibli diffuziya tenglamasining kasr hosila tartibini aniqlash bo‘yicha teskari masala yechimini bir qiymatli aniqlashning yangi shartlari olingan.

Kasr tartibli differensial va xususiy hosilali differensial tenglamalar uchun turli boshlang‘ich, boshlang‘ich-chegaraviy va orqaga qaytish masalalari ko‘pgina olimlar tomonidan o‘rganib kelinmoqda. Kasr diffuziya-to‘lqin tenglamalari uchun Koshi masalalari A. Kochubey (Ukraina), A. Psxu (Rossiya) va boshqalarning ishlarida o‘rganilgan. Diffuziya tenglamalari uchun boshlang‘ich vaqt va tayinlangan vaqtdagi qiymatlarini bog‘lovchi nolokal shart bilan A. Ashyaraliyev (Turkiya) va boshqalarning ishlarida o‘rganilgan. Chegaralangan va chegaralanmagan sohada berilgan kasr tartibli xususiy hosilali differensial tenglamalar uchun koeffitsiyentli teskari masalalar bilan bog‘liq muammolar deyarli o‘rganilmagan.

Mavzuning o‘rganilganlik darajasi. Ma’lumki ko‘plab adabiyotlarda, kasr hosilaning bir nechta ta’riflari taklif qilingan. Masalan, ular orasida ko‘p uchraydigani Riman-Liuvill (D_{0+}^{α}) va Gerasimov-Kaputo (∂_t^{α}) kasr hosilalaridir. Riman-Liuvill kasr hosilasi sof matematikada qo‘llanilishida muhim ahamiyat kasb etsada, Gerasimov-Kaputo kasr hosilasi amaliy masalalarni hal qilishda qo‘llaniladi. Haqiqatdan ham, 1948 yilda birinchilardan bo‘lib Aleksey Gerasimov kasr hosila tushunchasini o‘zining “Chiziqli deformatsiya qonunlarining umumlashmasi va ularning ichki ishqalanish masalalariga qo‘llanilishi” nomli maqolasida keltirib o‘tgan bo‘lsa, undan yigirma yildan so‘ng, aniqrog‘i 1967 yilda Mishel Kaputo “Q chastotaga deyarli bog‘liq bo‘lmagan tarqalishning chiziqli modellari” nomli maqolasida bu tushunchaga to‘xtalib o‘tgan. Gerasimov-Kaputo kasr hosilasi fizik jihatdan talqin qilinadigan boshlang‘ich shartlarni qo‘llashga imkon beradi, biroq Riman-Liuvill hosilasida esa boshlang‘ich shartlarni odatdagidek berib bo‘lmaydi.

Muayyan bir hodisani ifodalaydigan differensial tenglamani ko‘rib chiqishda kasr modellashtirishni qo‘llashning keng tarqalgan usuli, natural tartibdagi hosilalarni natural bo‘lmagan tartibdagi hosilalar bilan almashtirishdir, odatda bu hosilalarning tartibi asl hosilalarning tartibidan kichik yoki unga teng bo‘ladi. Shuning uchun odatdagi yechim maxsus hol sifatida qayta tiklanadi va bu Shnayder, Viss, Maynardi, Luchko, Podlubny, Psxu, Shishkina va Sitnik, Tarasova va Tarasov, Uchaykin kabi olimlarning ishlarida ko‘rish mumkin. Kasr hosilali tenglamalar anomal diffuziya tarqalishida, fizikada, mexanikada, kimyoda, hamda muhandislikdagi turli jarayonlarni o‘rganishda qo‘llaniladi. Diffuziya tenglamasi fraktallardan boshlab, muhim fizik hodisalar uchun matematik modeldir. Issiqlikning normal tarqalishida (issiqlik tenglamasi yoki umumiyroq parabolik tenglamalar bilan tasvirlangan) diffuzion zarrachaning oʻrtacha kvadratik siljishi $t \to \infty$ da $const \cdot t$ kabi harakat qiladi. Matematik nuqtai nazardan oddiy sanalgan anomal diffuziya uchun oʻziga xos harakat $conts \cdot t^{\alpha}$ kabi boʻlib, bu kasr diffuziya tenglamasini kiritilishiga sabab boʻldi. Bundan tashqari, kasr differensial tenglamaning statistik mexanika bilan bogʻlanishi mavjud boʻlib, u M.M. Meerschaert va uning ilmiy hamkorlari tomonidan taqdim etilgan. Bu kabi masalalar tenglamaning fundamental yechimi bilan bevosita bogʻliqdir. Kasr tartibli diffuziya-toʻlqin tenglamasi uchun Koshi masalasining fundamental yechimi 1990 yilda Shnayder-Viss va Kochubeylar $(0 < \alpha < 1)$ tomonidan mustaqil ravishda topilgan. Masalan, Shneider va Wysslar tomonidan

$$\begin{cases} \partial_t^{\alpha} u = \Delta u, \quad x \in \mathbb{R}^N, t > 0, m-1 < \alpha < m, m \in \mathbb{N}, \\ \partial^k u / \partial t^k (x, 0) = f_k(x), 0 \le k \le m-1 \end{cases}$$

koʻrinishidagi Koshi masalasi $0 < \alpha < 2$ uchun qaralgan boʻlib, bu masalaning fundamental yechimi olingan va uning yordamida masala yechimini beruvchi formula qurilgan. Ammo Schneider va Wysslar tomonidan olingan formulaning klassik yechim boʻlishi koʻrsatilmagan, biroq Kochubeyning ishida qurilgan formulaning klassik yechim boʻlishi masalasi toʻliq hal qilingan. Undan tashqari, ixtiyoriy $m \in \mathbb{N}$ uchun 2006 yilda Voroshilov va Kilbaslar tomonidan yuqoridagi Koshi masalasi yechimini beruvchi formula maxsus funksiyalar, ya’ni, Mittag-Leffler $E_{\alpha,\beta}(z)$, Besselning uchinchi tur $K_{\nu}(z)$ va Foksning H-funksiyalari yordamida qurilgan. Biroq, bu ishda olingan formulalar haqiqatdan qaralayotgan Koshi masalasining yechimi boʻlishi ochiq qoldirilgan. Oradan koʻp otmay, ya’ni, 2007 yilda shu mualliflar tomonidan “Doklady Mathematics” jurnalida chop qilingan ishda yuqoridagi ochiq qolgan masala hal qilingan. 2009 yilda Psxu tomonidan umumiy kasr hosila uchun kasr diffuziya-toʻlqin tenglamasining fundamental yechimi topilgan va undan foydalanib, kasr va butun tartibli issiqlik, toʻlqin tarqalish tenglamalari oʻrtasidagi bogʻliqlik keltirib oʻtilgan. Shuni aytish joizki, Psxu tomonidan olingan natija yuqoridagi barcha natijalarni umumlashtirgan va kasr tartibli differensial tenglama uchun Koshi masalasi yechimini olishda foydalaniladigan fundamental yechim va uning xossalari masalasi toʻliq hal etilgan. Tenglamadagi elliptik had umumiyroq boʻlganda, shu kabi masalalar Eydelmen va Kochubeylar tomonidan hal qilingan. Biroq olingan barcha natijalar faqat chiziqli masalalar uchun boʻlib, chiziqli boʻlmagan holatda masalalar yechimini oʻrganish hali ham ochiq qolmoqda.

Shuningdek, ushbu dissertatsiyaning asosiy qismi chegaralangan sohada kasr tartibli differensial tenglamalar uchun boshlangʻich-chegaraviy masalalarni, jumladan, toʻgʻri, orqaga qaytish (teskari vaqtli) va teskari masalalarni oʻrganishga bagʻishlangan. Diffuziya-toʻlqin jarayoni uchun teskari vaqtli masalalar muhandislik sohalarda katta ahamiyatga ega boʻlib, ular jarayonning avvalgi holatini (masalan, boshlangʻich vaqtdagi holatini) uning hozirgi ma’lumotlari asosida aniqlashga qaratilgan. Biroq, bunday masalalarni o‘ziga xosligi shundaki, tenglamada Riman-Liuvill yoki Gerasimov-Kaputo kasr hosilalari qo‘llanilishidan qat’iy nazar, bu masala Adamar ma’nosida nokorrekt hisoblanadi. Shunga qaramay, quyidagi

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) = f(t), \quad 0 < t \leq T, \\ u(T) = \Phi \end{cases}$$

masala $0 < \alpha < 1$ uchun akademik Alimov va professor Ashurovlar tomindan to‘liq tadqiq qilingan, bu yerda A-o‘z-o‘ziga qo‘shma, musbat, chegaralanmagan ixtiyoriy elliptik operator. Yuqoridagi tenglama $u(\xi) = \gamma u(0) + \varphi, \gamma = const \neq 0, 0 < \xi \leq T$ boshlang‘ich shart bilan birgalikda Ashurov va Fayziyevlar tomonidan to‘liq o‘rganilgan. Mualliflar nafaqat orqaga qaytish masalalarni, balki chiziqli teskari masalalarni ham o‘rganishgan, ya’ni manba va boshlang‘ich shartdagi funksiyalarni topish masalalari o‘rganilgan. Undan tashqari, Floridia, Yamamoto va Liu ishlarida kasr diffuziya va diffuziya-to‘lqin tenglamalari uchun teskari vaqtli masalalar va manba funksiyasini aniqlashning teskari masalalari tadqiq qilingan. Biroq, oldingi ishlarda faqat chiziqli to‘g‘ri va teskari masalalar o‘rganilgan bo‘lib, shu bilan birga ushbu dissertatsiya ishi chiziqli bo‘lmagan teskari masalalarga oid ochiq muammolarni yetarlicha o‘rganishga va metodlar taklif etishga bag‘ishlangan. Shuningdek, ushbu dissertatsiya dunyoning yetakchi olimlari, xususan, Al-Salti, Aloroyev, Kirane, Malik, Cheng, Eydelman, Gong, Wei, Wang, Wu, Ismailov, Jin, Rundell, Kochubey, Psxu, Elina, Lopushansky, Lopushanska, Yamamoto, Liao, Li, Liu, Luchko, Rujansky, Settara, Atmania, Sun, Jang, hamda yurtimiz olimlari akademik Alimov, professorlar Ashurov, Durdiyev, Karimov va boshqalarning ilmiy ishlariga tayangan bo‘lib, kasr tartibli xususiy hosilali differensial tenglamalar uchun to‘g‘ri va teskari masalalarni o‘rganishga qaratilgan.

Dissertatsiya ishining ilmiy-izlanish rejalari bilan bog‘liqligi. Dissertatsiya ishi O‘zbekiston Respublikasi Fanlar Akademiyasi V.I. Romanovskiy nomidagi Matematika instituti Buxoro bo‘linmasining «Matematik fizikaning teskari masalalari» mavzusidagi kalendar rejasi doirasida bajarilgan.

Tadqiqotning maqsadi kasr differensial va integro-differensial diffuziya-to‘lqin tenglamalari uchun to‘g‘ri, orqaga qaytish va teskari masalalarning yagona yechilishishi o‘rganish.

Tadqiqotning vazifalari quyidagilardan iborat:

vaqt bo‘yicha birinchi va kasr tartibli differensial tenglama uchun Koshi masalasi yechimini qurish, olingan formulaning yechim ekanligini va uning yagonaligini isbotlash;

chegaralanmagan sohada berilgan kasr diffuziya tengmalasidan koeffitsiyentni aniqlashning nochiziqli teskari masalani yechish;

kasr tartibli diffuziya tenglamasi uchun manba funksiyasini aniqlash teskari masalasini tadqiq qilish;

kasr tartibli diffuziya va to‘lqin tenglamalari uchun vaqt bo‘yicha nolokal boshlang‘ich shartli to‘g‘ri va nochiziqli teskari masalalarni yechish;

abstrak kasr diffuziya tenglamasi uchun Koshi masalasidan vaqt o‘zgaruvchisiga bog‘liq koeffitsiyentni topish bo‘yicha to‘g‘ri va nochiziqli teskari masalalarni yechish;

to‘g‘ri masala yechimiga ikkita nuqtada shartlar berish orqali tenglamadagi koeffitsiyent va xotira funksiyasini topishga oid nochiziqli teskari masalani yechish;

nolokal boshlang‘ich shartli kasr tartibli diffuziya tenglamasidan xotira funksiyasini aniqlash;

teskari vaqtli kasr tartibli abstrak Koshi masalasidan vaqt o‘zgaruvchisiga bog‘liq xotira funksiyasini topish bo‘yicha to‘g‘ri va teskari masalalarni yechish.

Tadqiqotning ob’ekti. Riman-Liuvill, Gerasimov-Kaputo kasr tartibli hosilalar, Mittag-Leffler funksiyalari, Foks funksiyasi, vaqt bo‘yicha kasr tartibli tenglamalar.

Tadqiqotning predmeti. Dissertatsiyaning tadqiqot predmeti xususiy hosilalali va kasr tartibli chekli o‘lchamli va ixtiyoriy elliptik operatorli differensial tenglamalar uchun to‘g‘ri, orqaga qaytish va teskari masalalardan iborat.

Tadqiqotning usullari. Dissertatsiya ishida matematik analiz, funksional analiz, differensial tenglamalar va matematik fizika usullaridan foydalanilgan. Matematik fizika usullaridan o‘zgaruvchilarni ajratish usuli va integral energiya usuli qo‘llaniladi hamda Hilbert fazosida xos funksiyalar sistemasining to‘la ekanligi tadbiq qilinadi.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

bir o‘lchamli kasr differensial tenglama uchun qo‘yilgan Koshi masalasining yechimi qurilgan va berilgan funksiyaga yetarlilik shartini qo‘yib, masalaning yagona klassik yechim mavjudligi isbotlangan;

chegaralanmagan sohada kasr diffuziya tenglamasi bilan bog‘liq to‘g‘ri va nochiziqli teskari masalalarning yagona yechilishi uchun berilganlarga yetarlilik shartlar topilgan va tenglamaning o‘ng tomonini topish bo‘yicha to‘g‘ri va chiziqli teskari masalalarning yechimi mavjud va yagonaligi isbotlangan;

kasr tartibli diffuziya va to‘lqin tenglamalari uchun qo‘yilgan vaqt bo‘yicha nolokal boshlang‘ich shartli to‘g‘ri va nochiziqli teskari masalalar yechiming mavjudlik va yagonalik haqidagi teoremalar isbotlangan;

abstrak kasr diffuziya tenglamasi uchun qo‘yilgan Koshi masalasi korrektligi va vaqt o‘zgaruvchisiga bog‘liq koeffitsiyentli teskari masalaning shartli korrektligi ko‘rsatilgan;

kasr diffuziya-to‘lqin tenglamasidan koeffitsiyent va xotira funksiyasini topish haqidagi nochiziqli teskari masalaning yagona yechimi mavjudligi haqidagi teorema isbotlangan;

Gerasimov-Kaputo kasr hosila ishtirok etgan ko‘p o‘lchamli kasr diffuziya tenglama uchun nolokal boshlang‘ich shartli to‘g‘ri va nochiziqli teskari masalalarning yagona yechiluvchanligi haqidagi teorema isbotlangan.

Tadqiqotning amaliy natijalari. Dissertatsiya ishida olingan natijalar nazariy xarakter kasb etib, kasrt tartibli xususiy hosilali differensial tenglamalarni tadqiq qilish usullari taklif etiladi. Shu bilan birga tenglamaning o‘ng tomonini, vaqt o‘zgaruvchisiga bog‘liq koeffitsiyentni hamda xotira funksiyasini aniqlash bo‘yicha teskari masalalarni yechish usullari bayon qilinadi.

Tadqiqot natijalarining ishonchliligi. Tadqiqot natijalarining va xulosalarining ishonchliligi differensial tenglamalar, matematik analiz, funksional analiz va operatorlar nazariyasi, matematik fizika metodlariga asoslangan qat’iy matematik isbotlar bilan asoslangan.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy ahamiyati kasr tartibli differensial va integro-differensial tenglamalar uchun to‘g‘ri va teskari masalalar nazariyasini rivojlantiradi.

Tadqiqot natijalarning amaliy ahamiyati ularning kasr tartibli tenglamalar bilan berilgan jarayonlarni o‘rganishda, nazorat qilishda va boshqarishda qo‘llanilishi bilan belgilanadi.

Tadqiqot natijalarining joriy qilinishi. Kasr duffuziya-to‘lqin tenglamasi uchun to‘g‘ri va teskari masalalar bo‘yicha olingan natijalar asosida:

kasr tartibli diffuziya-to’liq tenglamasi uchun qo‘yilgan nolokal boshlang‘ich va bir jinsli Dirixle shartli masalalarning yechimi mavjud va yagona bo‘lishlik shartlaridan IL-21071166 raqamli «Shamolning past tezligi uchun mo‘ljallangan vertikal o‘qli shamol turbinasini yaratish» mavzusidagi innovatsion loyihada shamol trubinalarining samarali parametrlarini aniqlash maqsadida tuzilgan matematik model tenglamalarini taqribiy yechishda foydalanilgan (Mexanika va inshootlar seysmik mustahkamligi inisitutining 2025 yil 25 martdagi №281-3-sonli ma’lumotnomasi). Ilmiy natijaning qo‘llanilishi Koltunov yadroli giperbolik tipdagi differensial tenglamalar uchun masalalarning sonli modellarini qurish imkonini bergan;

Chegaralangan sohalarda berilgan diffuziya va superdiffuziya differensial tenglamalari uchun koeffitsiyent va xotira funksiyalarini bir qiymatli aniqlash natijalaridan №-122041100096-4 raqamli «Sotsiologiyada, geofizikada va muhandislik fanlarida matematik modellashtirish» mavzusidagi xorijiy loyihasida yer osti qavariq muhitlarni tekshirishda foydalanilgan (Rossiya Fanlar akademiaysining Vladikavkaz ilmiy markazi Federal davlat ilmiy muassasasi Federal ilmiy markazi filali Janubiy matematika institutining, 2025 yil 24 martdagi 18-sonli ma’lumotnomasi, Rossiya Federatsiyasi). Ilmiy natijalarning qo‘llanilishi yer osti qovushqoq muhitlarda to‘lqin hodisasini kuzatish imkoni bergan;

subdiffuziya tenglamalari uchun orqaga qaytish, nolokal boshlang‘ich va birinchi chegaraviy masalalardan vaqt o’zgaruvchisiga bog‘liq koeffitsiyentni bir qiymatli aniqlash natijalaridan AP09258836 raqamli “Anomal diffuziya uchun differensial matematik modellarga sonli algoritmlarni ishlab chiqish” mavzusidagi xorijiy fundamental loyihasida haroratning o‘zgarishini tekshirishda foydalanilgan (Hoja Ahmad Yassaviy nomidagi xalqaro Qozoq-Turk universitetining, 2025 yil 2 apreldagi 04/857-sonli ma’lumotnomasi, Qozog‘iston). Ilmiy natijalarning qo‘llanilishi suv havzalardagi issiqlik almashinish jarayonini kuzatish imkoni bergan;

nolokal boshlang‘ich shartli kasr tartib subdiffuziya va superdiffuziya tenglamalari uchun boshlang‘ich-chegaraviy masalalardan koeffisient va xotira funksiyasini topishdan xorijiy ilmiy jurnallardaGI maqolalarda nolokal bigarmonik operatorli kasr parabolik tenglama uchun teskari masalalarni echishda foydalanilgan (Fractal and Fractional, 2023, 7(5) 404, Physica Scripta, 2024, 99(10) 105242, AIMS Mathematics, 2023, 9(3), 6832-6849). Ilmiy natijalarni qo‘llanishi ikki o‘lchamli nolokal shartli kasr parabolik tenglamadan manba funsiyasini topish imkonini bergan.

Tadqiqot natijalarining aprobatsiyasi. Dissertatsiyaning asosiy natijalari quyidagi xalqaro va respublika ilmiy konferensiyalarda muhokama qilingan: «6-Xalqaro matematika fanlari konferensiyasi ICMS» (Istanbul 2022), «Ilmiy amaliy matematika va axborot texnologiyalarining zamonaviy muammolari al-Xorazmiy» (Farg‘ona 2021), «Amaliy matematika va axborot texnologiyalarining dolzarb muammolari-al-Xorazmiy» (Samarqand 2023), «Operator algebralar, no-assotsiativ strukturalar va ularga oid muammolar» (Toshkent 2022), «Fizika, matematika va mexanikaning dolzarb muammolari» (Buxoro 2023), «Tartib tahlili va matematik modellashtirishga oid muammolar, XVII. Operator nazariyasi va differensial tenglamalar» (Vladikavkaz 2021), «Matematik analiz va zamonaviy matematik fizika qo‘llanmalari» (Samarqand 2022), «Tartib tahlili va matematik modellashtirishga oid muammolar, XVII. Operator nazariyasi va differensial tenglamalar» (Vladikavkaz 2023), «Amaliy matematika va axborot texnologiyalarining zamonaviy muammolari al-Xorazmiy» (Toshkent 2024), «Qozog‘iston Respublikasining Fan kuni munosabati bilan an’anaviy Xalqaro aprel matematika konferensiyasi» (Olmaota 2023).

Dissertatsiya ishi O‘zbekiston Milliy Universitetining “Differensial tenglamalar va matematik fizikaning zamonaviy muammolari”, Matematika institutining “differensial tenglamalar va matematik fizika” hamda Matematika instituti Buxoro bo‘linmasining “Matematik fizikaning zamonaviy masalalari” nomli seminlarida muhokama qilindi.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya mavzusi bo‘yicha jami 27 ta ilmiy ishlar chop etilgan bo‘lib, shundan 15 tasi O‘zbekiston Respublikasi Oliy attestatsiya komissiyasi doktorlik (DSc) dissertatsiyalarining asosiy ilmiy natijalarini chop etish uchun tavsiya etilgan ilmiy nashrlarda chop etilgan, 9 tasi xorijiy jurnallarda chop etilgan, 6 tasi respublika jurnallarida chop etilgan ilmiy maqolalar va 12 ta tezis nashr etilgan.

Dissertatsiyaning tuzilishi va hajmi. Dissertatsiya kirish qismi, 4 ta bob, xulosa va foydalanilgan adabiyotlar ro‘yxatidan tashkil topgan. Dissertatsiyaning hajmi 212 betni tashkil etadi.

DISSERTATSIYANING ASOSIY MAZMUNI

Dissertatsiyaning kirish qismida dissertatsiya mavzusining dolzarbligi va zaruratini asoslash, tadqiqotning respublika fan va texnikasini rivojlantirishning ustuvor yo‘nalishlariga muvofiqligi, dissertatsiya mavzusi bo‘yicha xorijiy mutaxassislar tomonidan olib borilgan ilmiy ishlarga taqriz berilgan va mahalliy olimlar tomonidan masalaning o‘rganilganlik darajasi, maqsad, vazifalar, tadqiqot ob’ekti va predmeti, ilmiy yangilik bayoni, olingan natijalarning nazariy va amaliy ahamiyati, asosiy natijalarni amalga oshirish to‘g‘risidagi ma’lumotlar aks ettirilgan va nashr etilgan ishlar va dissertatsiya tuzilishi haqida ma’lumot berilgan.

Dissertatsiya ishining birinchi bobi **«Boshlang‘ich tushunchalar va vaqt-kasr differensial tenglama uchun Koshi masalasi»** deb nomlangan. Bu bobning birinchi ikki bo‘limida, dissertatsiya ishining tahlili uchun zarur bo‘lgan asosiy dastlabki tushunchalar, ta’riflar va teoremalar kiritilgan.

Bu bobning uchinchi paragrafi birinchi va α-tartibli bir o‘lchamli differensial tenglama uchun Koshi masalasini yechishga qaratilgan.

Ushbu

$$\rho\frac{\partial u(t,x)}{\partial t}+\mathbb{D}_t^{(\alpha)}u(t,x)-u_{xx}(t,x)=0,\quad t>0, x\in\mathbb{R},\tag{1}$$

$$u(0,x)=\varphi(x),\quad x\in\mathbb{R}\tag{2}$$

Koshi masalasi berilgan bo‘lsin, bu yerda ρ berilgan musbat haqiqiy son, $\mathbb{D}_t^{(\alpha)}$ esa $\alpha\in(0,1)$-tartibli Djrbashian-Kaputo kasr hosilasi, ya’ni,

$$\left(\mathbb{D}_t^{(\alpha)}u\right)(t,x)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_0^t(t-s)^{-\alpha}u(s,x)ds-t^{-\alpha}u(0,x)\right]$$

$$=\frac{d}{dt}(I_{0+}^{1-\alpha}u)(t,x)-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}u(0,x)\tag{3}$$

bo‘lib, bunda α-tartibli Riman-Liuvill chap kasr integrali

$$(I_{0+}^{\alpha}u)(t,x)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s,x)ds\tag{4}$$

ko‘rinishida aniqlangan va φ berilgan funksiya.

(1), (2) masala yechimining mavjudlik shartlarini aniqlashdan oldin, avvalo, klassik yechimining ta’rifini beraylik.

1-ta’rif. *Agar (i) har bir* $x\in\mathbb{R}$ *da* $u_t(t,x)\in C(t>0)$*;*

(ii) har bir $t>0$ *da* $u(t,x)$ *funksiya* x *o‘zgaruvchi bo‘yicha ikki marta uzluksiz differensiallanuvchi;*

(iii) ixtiyoriy $x\in\mathbb{R}$ *da* $u(t,x)\in C(t\geq 0)$ *va uning kasr* $(I_{0+}^{1-\alpha}u)(t,x)$ *integrali* $t>0$ *lar uchun* t *bo‘yicha uzluksiz differnsiallanuvchi;*

(iv) (1) tenglama va (2) boshlang‘ich shartni odatdagi ma’noda qanoqlantirsa, u holda $u(t,x)$ $(t\geq 0, x\in\mathbb{R})$ *funksiyaga (1), (2) masalaning klassik yechimi deyiladi.*

Bu paragrafning asosiy natijasi quyidagi teoremalarda o‘z aksini topgan.

1-teorema. *Faraz qilaylik,* $0<\alpha<1$ *va* $\rho>0$ *bo‘lsin. Agar* $\varphi(x)\in C(\mathbb{R})\cap L^{\infty}(\mathbb{R})$ *bo‘lsa, u holda (1), (2) Koshi masalasining yagona klassik yechimi mavjud va bu yechim*

$$u(t,x)=\int_{\mathbb{R}}\mathcal{G}^{\alpha}(t,x-\xi)\varphi(\xi)d\xi\tag{5}$$

formula orqali beriladi, bu yerda

$$\mathcal{G}^{\alpha}(t,x):=\mathcal{G}_1^{\alpha}(t,x)+\mathcal{G}_2^{\alpha}(t,x),$$

$$\mathcal{G}_1^\alpha(t,x) = \frac{1}{2\pi^{1/2}|x|}\sum_{k=0}^{\infty}\frac{(-1)^k}{\rho^k k!}t^{(1-\alpha)k}H_{1,2}^{2,0}\left[\frac{\sqrt{\rho}}{2\sqrt{t}}|x|\,\middle|\begin{matrix}(1+(1-\alpha)k,1/2)\\(1/2,1/2),(1+k,1/2)\end{matrix}\right],$$

$$\mathcal{G}_2^\alpha(t,x) = \frac{1}{2\pi^{1/2}|x|}\sum_{k=0}^{\infty}\frac{(-1)^k}{\rho^{k+1} k!}t^{(1-\alpha)(k+1)}H_{1,2}^{2,0}\left[\frac{\sqrt{\rho}}{2\sqrt{t}}|x|\,\middle|\begin{matrix}(1+(1-\alpha)(k+1),1/2)\\(1/2,1/2),(1+k,1/2)\end{matrix}\right],$$

va $H_{p,q}^{m,n}(z) -$ *Foksning H-funksiyasi. Undan tashqari, barcha tayin* $t > 0$ *lar uchun bu yechim* $|x| \to \infty$ *da nolga intiladi.*

2-teorema. *Faraz qilaylik,* $u(t,x)$ *funksiya (1), (2) masalaning klassik yechimi bo‘lib,* $\varphi(x) \equiv 0$ *bo‘lsin. Agar*

$$|u(t,x)| \le M_1 \exp\{M_2|x|^\mu\}, \quad M_1, M_2 > 0, \ \mu < 2$$

bo‘lsa, u holda $u(t,x) \equiv 0.$

Dissertatsiyaning ikkinchi bobi **«Cheksiz va chegaralangan sohalarda vaqt-kasr hosilali diffuziya tenglamasidan koeffitsiyent va manba funksiyalarini aniqlash»** deb nomlangan bo‘lib, unda vaqt va fazoviy o‘zgaruvchilarga bog‘liq koeffitsiyent va manba funksiyalarni qo‘shimcha shartlar orqali aniqlash masalasi tadqiq qilinadi.

Dastlab, ushbu

$$\left(\mathbb{D}_t^{(\alpha)}u\right)(t,\bar{x}) - \Delta u(t,\bar{x}) + q(t,x)u(t,\bar{x}) = f(t,\bar{x}), \quad (t,\bar{x}) \in \mathbb{R}_T^2 \qquad (6)$$

vaqt-kasr diffuziya tenglamasini

$$u(0,\bar{x}) = \varphi(\bar{x}), \quad \bar{x} \in \mathbb{R}^2 \qquad (7)$$

boshlang‘ich shart bilan birgalikda qaraymiz, bu yerda $\bar{x} = (x,y) \in \mathbb{R}^2, \mathbb{R}_T^2 = \{(t,\bar{x}) : \bar{x} \in \mathbb{R}^2, 0 < t \le T\}$, $\Delta := \partial_x^2 + \partial_y^2$ va $f(t,\bar{x}), \varphi(\bar{x})$ berilgan yetarlicha silliq funksiyalar, $\mathbb{D}_t^\alpha, 0 < \alpha < 1$ Djrbashiyan-Kaputo kasr hosilasi ((3) formulaga qarang).

1-teskari masala. *(6), (7) to‘g‘ri masalaning yechimi haqida*

$$u(t,x,0) = g(t,x), \quad x \in \mathbb{R}, t \ge 0 \qquad (8)$$

qo‘shimcha shartdan foydalanib $q(t,x), x \in \mathbb{R}, t \ge 0$ *funksiyani aniqlashdan iborat, bu yerda* $g(t,x)$ *berilgan yetarlicha silliq funksiya.*

2-ta’rif. $u(t,\bar{x})$ *funksiya (6), (7) masalaning klassik yechimi deyiladi, agar*

(i) har bir $t > 0$ *uchun* $\bar{x}$ *bo‘yicha ikki marta uzluksiz differensiallanuvchi;*

(ii) ixtiyoriy $\bar{x} \in \mathbb{R}^2$ *da* t *o‘zgaruvchining funksiyasi sifatida* $[0,T]$ *da uzluksiz, va uning* $I_{0+}^{1-\alpha}u(t,\bar{x})$ *kasr integrali* t *bo‘yicha* $t > 0$ *da uzluksiz differensiallanuvchi;*

(iii) (6) tenglama va (7) boshlang‘ich shartni odatdagi ma’noda qanoatlantirsa.

1-lemma. *Agar berilgan funksiyalar* $q(t,x) \in C\left([0,T]; C_b^l(\mathbb{R})\right)$, $f(t,\bar{x}) \in C\left([0,T]; C_b^{l,l}(\mathbb{R}^2)\right)$, $\varphi(\bar{x}) \in C_b^{l,l}(\mathbb{R}^2)$ *sinflardan bo‘lsa, u holda (6), (7) to‘g‘ri* *masalaning* $u(t,\bar{x}) \in C^{1-\alpha,2}(\mathbb{R}_T^2) \cap C(\overline{\mathbb{R}}_T^2)$ *sinfga tegishli yagona yechimi mavjud, bu yerda* $l \in (0,1)$.

3-teorema. *Agar* $f(t,\bar{x}) \in C\left([0,T]; C_b^{l,l+1}(\mathbb{R}^2)\right)$, $\varphi(\bar{x}) \in C_b^{l,l+2}(\mathbb{R}^2)$, $g(t,x) \in C^1([0,T]; C_b^{l+2}(\mathbb{R}))$, $|g(t,x)| \geq g_0 > 0$ *bo'lsa, u holda shunday yetarlicha kichik* $T^* \in (0,T)$ *soni topiladiki (6)-(8) teskari masalaning* $q(t,x) \in C([0,T^*]; C_b^l(\mathbb{R}))$ *sinfga tegishli yagona yechimi mavjud, bu yerda* $l \in (0,1)$.

Aytaylik T tayin musbat haqiqiy son bo'lsin. $\Omega(\gamma_0)$ ($\gamma_0 > 0$ tayinlangan haqiqiy son) orqali $\max\{\| f \|^{l,l+1}, |\varphi|^{l,l+2}, \| g \|_1^l\} \leq \gamma_0$ tengsizlikni va 3-teorema shartlarini qanoatlantiruvchi berilgan (f,φ,g) funksiyalar to'plamini belgilaymiz, $Q(\gamma_1)$ orqali esa $\| q \|^l \leq \gamma_1$ tengsizlikni qanoatlantiruvchi $q(t,x) \in C\left([0,T^*]; C_b^l(\mathbb{R})\right)$ funksiyalar sinfini belgilaylik, bu yerda γ_1 musbat haqiqiy son va $l \in (0,1)$.

4-teorema. *Faraz qilaylik* $(f,\varphi,g) \in \Omega(\gamma_0), (\tilde{f},\tilde{\varphi},\tilde{g}) \in \Omega(\gamma_0)$ *va* $(q,\tilde{q}) \in Q(\gamma_1)$ *bo'lsin. U holda (6)-(8) teskari masalaning yechimi uchun quyidagi turg'unlik bahosi o'rinli*

$$\| q - \tilde{q} \|^l \leq c\left(\| f - \tilde{f} \|^{l,l+1} + |\varphi - \tilde{\varphi}|^{l,l+2} + \|g - \tilde{g}\|_1^l\right),$$

bu yerda c *o'zgarmas* T, l, γ_0, γ_1 *larga bog'liq.*

Bu bobning keyingi paragrafida, biz ikkita teskari masalani, ya'ni vaqt va fazoviy o'zgaruvchilarga bog'liq manba funksiyalarini topish masalalarini tadqiq qilamiz.

Ushbu

$$Q := D \times (0,T], \quad D := \{(x,y): 0 < x < l, 0 < y < q\}$$

parallelepiped sohada quyidagi to'g'ri masalani qarab chiqamiz: Q sohada aniqlangan va

$$u(t,x,y) \in X_T, \tag{9}$$

$$\partial_t^\alpha u - a^2\left(u_{xx} + u_{yy}\right) = F(t,x,y), \ (t,x,y) \in Q, \tag{10}$$

$$u(t,0,y) = u(t,l,y) = 0, \quad 0 \leq y \leq q, \ 0 \leq t \leq T, \tag{11}$$

$$u(t,x,0) = u(t,x,q) = 0, \ 0 \leq x \leq l, \ 0 \leq t \leq T, \tag{12}$$

$$u(0,x,y) = \varphi(x,y), \quad 0 \leq x \leq l, \ 0 \leq y \leq q \tag{13}$$

shartlarni qanoatlantiruvchi $u(t,x,y)$ funksiyani topish talab etilsin, bu yerda $X_T := \{u: u \in C(\bar{Q});\ u(t,\cdot,\cdot) \in C^2(D),\ t \in (0,T];\ u_t(\cdot,x,y) \in C(0,T] \cap L^1(0,T),\ (x,y) \in D\}$ va φ berilgan funksiya.

(10) tenglama uchun qo'yilgan boshlang'ich-chegaraviy shartlar asosida quyidagi teskari masalalarni ko'rib chiqamiz.

2-teskari masala. *Faraz qilaylik,* $F(t,x,y) = f(x,y)g(t)$ *bo'lsin. Quyidagi*

$$g(t) \in AC[0,T], \tag{14}$$

$$u(t,x_0,y_0) = h(t), \quad 0 \leq t \leq T \tag{15}$$

qo'shimcha va (9)-(13) shartlarni qanoatlantiruvchi $u(t,x,y)$ *va* $g(t)$ *funksiyalar juftligini toping, bu yerda* $(x_0,y_0) \in D$ *berilgan tayin nuqta,* $\varphi(x,y)$, $h(t)$,

$f(x,y)$ *−berilgan yetarlicha silliq funksiyalar bo'lib,* $\varphi(x_0,y_0)=h(0)$ *muvofiqlik sharti o'rinli.*

3-teskari masala. *Faraz qilaylik,* $F(t,x,y)=f(x,y)g(t)$ *bo'lsin. Ushbu*

$$f(x,y)\in C(\overline{D}), \tag{16}$$

$$u(t_0,x,y)=\psi(x,y),\quad (x,y)\in\overline{D} \tag{17}$$

qo'shimcha shart va (9)-(13) tengliklarni qanoatlantiruvchi $u(t,x,y)$ *va* $f(x,y)$ *funksiyalar juftligini toping, bu yerda* $t_0\in(0,T]$ *berilgan son,* $\varphi(x,y)$*,* $\psi(x,y)$*,* $g(t)$ *−ma'lum yetarlicha silliq funksiyalar.*

Dastlab to'g'ri masala yechimi uchun quyidagi energiya integral tengsizligini keltiramiz.

2-lemma. *Faraz qilaylik,* u *funksiya (9)-(13) masalaning klassik yechimi bo'lsin, u holda*

$$\frac{1}{2\Gamma(1-\alpha)}\iint_D\int_0^T\frac{u^2}{(T-t)^\alpha}dtdxdy+a^2\iint_D\int_0^T\left(u_x^2+u_y^2\right)dtdxdy$$
$$\le\frac{T^{1-\alpha}}{2\Gamma(2-\alpha)}\iint_D\varphi^2dxdy+\iint_D\int_0^T uFdtdxdy$$

tengsizlik o'rinli.

(9)-(13) masala yechimining yagonaligi 2-lemmadan bevosita kelib chiqadi. Endi (9)-(13) masalaning klassik yechimi mavjudligiga to'xtalamiz.

5-teorema. *Aytaylik, berilgan* $\varphi(x,y)\in C^3(\overline{D})$ *funksiya ushbu*

$$\varphi(0,y)=\varphi_{xx}(0,y)=\varphi(l,y)=\varphi_{xx}(l,y),\quad 0\le y\le q,$$
$$\varphi(x,0)=\varphi_{yy}(x,0)=\varphi(x,q)=\varphi_{yy}(x,q),\quad 0\le x\le l$$

va $F(t,x,y)\in C\left([0,T];C^2(\overline{D})\right)\cap AC\left([0,T];C(\overline{D})\right)$ *funksiya esa ushbu*

$$F(t,0,y)=F(t,l,y)=0,\quad 0\le y\le q,\quad 0\le t\le T,$$
$$F(t,x,0)=F(t,x,q)=0,\quad 0\le x\le l,\quad 0\le t\le T$$

shartlarni qanoatlantirsin. U holda (9)-(13) to'g'ri masala yagona $u\in X_T$ *klassik yechimga ega.*

Yuqorida olingan natijadan foydalangan holda, 2- va 3-teskari masalalarni o'rganamiz.

6-teorema. *Faraz qilaylik* $\varphi(x,y)\in C^4(\overline{D})$*,*

$$\varphi(0,y)=\varphi_{xx}(0,y)=\varphi(l,y)=\varphi_{xx}(l,y),\quad 0\le y\le q,$$
$$\varphi(x,0)=\varphi_{yy}(x,0)=\varphi(x,q)=\varphi_{yy}(x,q),\quad 0\le x\le l$$

bo'lib, $f(x,y)\in C^4(\overline{D})$ *funksiya esa*

$$f(0,y)=f_{xx}(0,y)=f(l,y)=f_{xx}(l,y),\quad 0\le y\le q,$$
$$f(0,x)=f_{yy}(0,x)=f(x,q)=f_{yy}(x,q),\quad 0\le x\le l$$

shartlarni qanoatlantirsin. Undan tashqari, ushbu

$$f(x_0,y_0)\ne0,\quad h''(t)\in L^1(0,T), h(0)=\varphi(x_0,y_0)$$

shartlar bajarilsin. U holda 2-teskari masalaning $g(t)\in AC[0,T]$ *sinfga tegishli yagona yechimi mavjud.*

3-lemma. *Agar* $g(t)\in AC[0,T]$ *va* $|g(t)|\ge g_0=const>0$ *bo'lsa, u holda shunday* $C_0>0$ *topiladiki, barcha* $m,n\in\mathbb{N}$ *lar uchun*

$$|g_{mn}(t_0)| \geq \frac{C_0}{m^2+n^2}$$

tengsizlik o‘rinli bo‘ladi, bu yerda

$$g_{mn}(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-a^2\lambda_{mn}^2(t-\tau)^\alpha) g(\tau) d\tau$$

va C_0 o‘zgarmas m, n larga bog‘liq emas.

7-teorema. *Faraz qilaylik, berilgan $\varphi(x,y)$ funksiya 6-teorema shartlarini, shuningdek $\psi(x,y) \in C^4(\bar{D})$ funksiya*

$$\psi(0,y) = \psi_{xx}(0,y) = \psi(l,y) = \psi_{xx}(l,y), \quad 0 \leq y \leq q,$$
$$\psi(0,x) = \psi_{yy}(0,x) = \psi(x,q) = \psi_{yy}(x,q), \quad 0 \leq x \leq l$$

shartlarni qanoatlantirsin. Undan tashqari, $g(t)$ funksiya 3-lemma shartlarini bajarsin, u holda 3-teskari masala yagona yechimga ega.

Dissertatsiyaning uchinchi bobi **«Kasr tartibli diffuziya-to‘lqin tenglamalaridan koeffitsiyentni aniqlash teskari masalalari»** deb nomlangan bo‘lib, vaqt o‘zgaruvchisiga bo‘g‘liq bo‘lgan koeffitsiyentni aniqlash masalasiga bag‘ishlangan.

Dastlab, kasr diffuziya tenglamasi uchun koeffitsiyentli teskari masalani ko‘rib chiqamiz. Faraz qilaylik, Ω-chegarasi $\partial\Omega$ yetalicha silliq, $\mathbb{R}^N$ da chegaralangan soha bo‘lsin. $Q_T := \{(t,x) : x \in \Omega \subset \mathbb{R}^N, 0 < t \leq T\}$ sohada quyidagi kasr-diffuziya

$$\mathbb{D}_t^{(\alpha)} u(t,x) = Au(t,x) + q(t)u(t,x) + f(t,x), \quad (t,x) \in Q_T \tag{18}$$

tenglamasini qarab chiqamiz, bu yerda $0 < \alpha < 1$ va

$$-Au(t,x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a_{ij}(x) \frac{\partial}{\partial x_j} u(t,x)\right), \quad (t,x) \in Q_T$$

simmetrik va tekis elliptik operator bo‘lib, aniqlanish sohasi $D(-A) = H^2(\Omega) \cap H_0^1(\Omega)$ ga teng. Undan tashqari uning koeffitsiyentlari uchun ushbu

$$a_{ij} = a_{ji}, \quad 1 \leq i,j \leq N, \quad a_{ij} \in C^1(\bar{\Omega}),$$
$$\nu_1 \sum_{i=1}^N |\xi_i|^2 \leq \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \leq \nu_2 \sum_{i=1}^N |\xi_i|^2, x \in \bar{\Omega}, \xi \in \mathbb{R}^N, \nu_1, \nu_2 > 0$$

shartlar o‘rinli.

(18) tenglamani

$$u(0,x) + \beta u(T,x) = \varphi(x), \quad x \in \Omega \tag{19}$$

nolokal boshlang‘ich shart va

$$u(t,x) = 0, \quad x \in \partial\Omega, \quad t \in (0,T) \tag{20}$$

bir jinsli chegaraviy shart hamda birinchi tur

$$\int_\Omega \omega(x)u(t,x)dx = h(t), \quad 0 \leq t \leq T \tag{21}$$

integral shart bilan birga qaraymiz, bu yerda $\beta \geq 0$ va $f(t,x), \varphi(x), \omega(x), h(t)$ –ma’lum yetarlicha silliq funksiyalar.

4-teskari masala. *(18) tenglama va (19)-(21) shartlarni qanoatlantiruvchi* $(u,q)\in Y_T$ *funksiyalar juftligini toping, bu yerda*

$$Y_T = C\big([0,T];H^2(\Omega)\big)\times C[0,T].$$

Berilganlar uchun quyidagilarni talab qilamiz:

(A1) $\varphi\in H^2(\Omega)\cap H_0^1(\Omega),\, f\in C([0,T];D(-A)^{\varepsilon})$ bu yerda $0<\varepsilon<1$;

(A2) $h(0)+\beta h(T)=(\omega,\varphi)$;

(A3) $\mathbb{D}_t^{(\alpha)}h\in C[0,T]$ bo‘lib, barcha $t\in[0,T]$ uchun ushbu

$$|h(t)|\geq\frac{1}{h_0}>0,\;\; t\in[0,T]$$

tengsizlikni qanoatlantirsin, bu yerda h_0 musbat o‘zgarmas son;

(A4) $\omega(x)\in L^2(\Omega)$.

8-teorema. *Faraz qilaylik (A1)-(A4) shartlar bajarilsin. U holda (18)-(21) teskari masalaning yetarlicha kichik* $T>0$ *larda yagona* $(u,q)\in Y_T$ *yechimi mavjud.*

9-teorema. *Aytaylik berilgan funksiyalar uchun (A1)-(A4)shartlar bajarilsin. Agar* $(u_i,q_i)\in Y_T$ $(i=1{,}2)$ *funksiyalar (18)-(21) teskari masalaning ikkita teng bo‘lmagan yechimlari bo‘lsa, u holda barcha* $0\leq t\leq T$ *lar uchun* $(u_1,q_1)=(u_2,q_2)$.

Bobning ikkinchi paragrafida bir o‘lchamli vaqt-kasr diffuziya-to‘lqin tenglamasida vaqtga bog‘liq koeffitsiyentni aniqlash teskari masalasini tadqiq qilamiz.

$D_T:=\{(t,x):0<t\leq T, 0<x<1\}$ sohada ushbu

$$\mathbb{D}_t^{(\alpha)}u(t,x)-\mathcal{L}u(t,x)=q(t)u(t,x)+f(t,x),\quad (t,x)\in D_T \tag{22}$$

kasr to‘lqin tenglamasini qaraylik, bu yerda $\mathbb{D}_t^{(\alpha)}$ esa $\alpha\in(1{,}2)$ -tartibli Djrbashian-Kaputo kasr hosilasi, ya’ni,

$$\left(\mathbb{D}_t^{(\alpha)}u\right)(t,x)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial}{\partial t}\int_0^t(t-s)^{-\alpha+1}u_s(s,x)ds-t^{-\alpha+1}\frac{u_t(0,x)}{\Gamma(2-\alpha)},$$

va $\mathcal{L}:=\partial_{xx}$.

Yuqoridagi kasr to‘lqin tenglamasini quyidagi nolokal

$$u(0,x)+\delta_1u(T,x)=\varphi(x),\quad u_t(0,x)+\delta_2u_t(T,x)=\psi(x),\;\; x\in(0{,}1) \tag{23}$$

boshlang‘ich va

$$u(t,0)=u(t,1)=0,\quad 0<t<T \tag{24}$$

chegaraviy hamda

$$\int_0^1\omega(x)u(t,x)dx=h(t),\quad 0\leq t\leq T \tag{25}$$

integral shart bilan birgalikda qaraymiz, bu yerda $\delta_1,\delta_2\geq0$ va $f(t,x),\varphi(x),\psi(x),\omega(x),h(t)$ –berilgan funksiyalar.

5-teskari masala. *(22) tenglama va (23)-(25) shartlarni qanoatlantiruvchi* $(u,q)\in C([0,T];H^2(0{,}1))\cap C^1([0,T];L^2(0{,}1))\times C[0,T]$ *funksiyalar juftligini toping.*

Quyidagi

$$\rho(\eta) = 1 + (\delta_1 + \delta_2)E_{\alpha,1}(-\eta) +$$
$$+\delta_1\delta_2\left[\left(E_{\alpha,1}(-\eta)\right)^2 + \eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta)\right], \quad \eta > 0.$$

funksiyani kiritib, mos ravishda $\eta_1 < \eta_2 < \cdots < \eta_N$ lar orqali $\rho(\eta) = 0$ tenglamaning ildizlarini belgilaymiz. Λ orqali esa

$$\Lambda = \bigcup_{k=1}^{\infty}\left\{\left(\frac{\eta_1}{(\pi k)^2}\right)^{1/\alpha}, \dots, \left(\frac{\eta_N}{(\pi k)^2}\right)^{1/\alpha}\right\}$$

to‘plamni belgilaylik.

Berilganlar uchun quyidagilarni talab qilamiz:

(B1) $f \in C([0,T]; D(-\mathcal{L})^{\frac{1}{\alpha}})$), $\varphi \in H^2(0,1) \cap H_0^1(0,1)$, $\psi \in H_0^1(0,1)$;

(B2) $h(0) + \delta_1 h(T) = (\omega, \varphi)$, $h'(0) + \delta_2 h'(T) = (\omega, \psi)$;

(B3) $h \in C^2[0,T]$ bo‘lib, barcha $t \in [0,T]$ uchun ushbu

$$|h(t)| \geq \frac{1}{h_0} > 0$$

tengsizlikni qanoatlantirsin, bu yerda h_0 musbat o‘zgarmas son;

(B4) $T \notin \Lambda$;

(B5) $\omega(x) \in H_0^2(0,1)$.

10-teorema. *Faraz qilaylik (B1)-(B5) shartlar bajarilsin. U holda yetarlicha kichik* $T > 0$ *lar uchun (22)-(25) teskari masalaning* $(u,q) \in C\left([0,T]; H^2(0,1)\right) \cap C^1\left([0,T]; L^2(0,1)\right) \times C[0,T]$ *sinfga tegishli yagona yechimi mavjud.*

Aytaylik, H separabel Gilbert fazosi bo‘lsin. Unda skalyar ko‘paytma va norma aniqlangan bo‘lib, ularni mos ravishda $(\cdot,\cdot)$ va $\|\cdot\|$ kabi belgilaylik. $A: H \to H$ operator H Gilbert fazosida aniqlangan, o‘z-o‘ziga qo‘shma, musbat, chegaralanmagan ixtiyoriy operator bo‘lsin. Faraz qilaylik, A operator H Gilbert fazosida to‘la ortonormal $\{e_n\}$ xos funksiyalar sistemasiga va $\{\lambda_n\}$ musbat xos qiymatlar to‘plamiga ega bo‘lsin. Xos qiymatlarni qayta nomerlash yordamida ularni kamaymaydigan qilib nomerlab olamiz, ya’ni $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$ kabi yozib olamiz.

Bobning oxirgi uchinchi paragrafida kasr diffuziya tenglamasidan

$$\partial_t^\alpha u(t) + Au(t) + q(t)u(t) = f(t), \quad 0 < t \leq T \tag{26}$$

vaqt o‘zgaruvchisiga bog‘liq koeffitsiyentni aniqlashning to‘g‘ri va teskari masalalarini tadqiq qilamiz, bu yerda A operator yuqorida aniqlangan.

Berilgan $\alpha \in (0,1)$, $q(t) \in C[0,T]$ va $f(t) \in C([0,T]; H)$ larga ko‘ra (26) tenglama hamda

$$u(0) = \varphi \tag{27}$$

boshlang‘ich shartni qanoatlantiruvchi $u(t) \in C\left([0,T]; D(A)\right)$ funksiyani topishga *to‘g‘ri masala* deyiladi, bu yeda $\varphi \in H$. To‘g‘ri masala uchun yagona yechimning mavjudligini isbotlaymiz va bu yechim uchun silliqlik shartlarini keltirib o‘tamiz. Undan tashqari, (26) tenglamadagi $q(t)$ koeffitsiyentni topishning quyidagi teskari masalasini ko‘rib chiqamiz.

6-teskari masala. *Berilgan* α, $f(t)$ *va* φ *lar bo'yicha (26) tenglama va (27) boshlang'ich shart hamda*

$$\Phi[u(t)] = h(t), \quad 0 \leq t \leq T \tag{28}$$

qo'shimcha shartni qanoatlantiruvchi (u, q) *funksiyalar juftligini toping, bu yerda* $h: [0, T] \to \mathbb{R}$ *ma'lum funksiya,* $\Phi: D(\Phi) \subset H \to \mathbb{R}$ *ma'lum chiziqli chegaralangan funksional.*

Faraz qilaylik $0 < \varepsilon < 1$ bo'lib, berilganlar uchun quyidagilar bajarilsin:

(C1) $\varphi \in D(A^{\varepsilon+1}), f \in C\big([0,T]; D(A^{\varepsilon})\big)$;

(C2) $h \in AC[0,T]$ va $0 < \frac{1}{h_0} \leq |h(t)|$ bo'lsin, bu yerda $h_0 > 0$ ma'lum haqiqiy son;

(C3) $h(0) = \Phi[\varphi]$;

(C4) $\Phi: \{\Phi[e_k]\} \in l^2(\mathbb{N})$, bu yerda $l^2(\mathbb{N})$ kvadrati bilan jamlanuvchi sonli ketma-ketliklar fazosi.

To'g'ri masala yechimining mavjudlik, yagonalik va turg'unligi haqidagi quyidagi teorem o'rinli.

11-teorema. *Faraz qilaylik, ba'zi* $\varepsilon \in (0,1)$ *lar uchun* $\varphi \in D(A^{\varepsilon+1})$ *va* $f \in C\big([0,T]; D(A^{\varepsilon})\big)$ *bo'lib,* $q(t) \in C[0,T]$ *bo'lsin. U holda (26), (27) masalaning yagona* $u \in C\big([0,T]; D(A)\big)$ *yechimi mavjud bo'lib,* $\partial_t^{\alpha} u \in C([0,T]; H)$ *bo'ladi. Undan tashqari shunday o'zgarmas* $c > 0$ *son topiladiki, ushbu*

$$\| u \|_{C([0,T];D(A))}$$
$$\leq c E_{\alpha\varepsilon,1}\big(\Gamma(\alpha\varepsilon) T \| q \|_{C[0,T]}\big) \Big[\| \varphi \|_{D(A^{\varepsilon+1})} + \| f \|_{C([0,T];D(A^{\varepsilon}))}\Big]$$

tengsizlik bajariladi va yechim

$$u(t) = Z(t)\varphi + \mathcal{H}(f)(t) - \big(\mathcal{H}(q)\big)(u)(t)$$

ko'rinishida tasvirlanadi, bu yerda $\mathcal{H}: C([0,T]; D(A^{\varepsilon})) \to C([0,T]; D(A^{\varepsilon}))$ *operator quidagi ko'rinishda aniqlangan*

$$\mathcal{H}(F)(t) = \int_0^t Y(t-s) F(s) ds,$$

$$Y(t)\phi = \sum_{k=1}^{\infty} (\phi, e_k) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^{\alpha}) e_k, \quad \phi \in H, t > 0,$$

$$Z(t)\varphi = \sum_{k=1}^{\infty} (\varphi, e_k) E_{\alpha,1}(-\lambda_k t^{\alpha}) e_k.$$

(26), (27) masala yechimining berilganlarga uzluksiz bog'liqligi to'g'risida quyidagi teorema o'rinli.

12-teorema. *Faraz qilaylik, 11-teoremadagi shartlar bajarilsin. U holda (26), (27) masalaning yechimi berilganlarga uzluksiz bog'liq bo'ladi, ya'ni*

$$\| u - \hat{u} \|_{C([0,T];D(A))} \leq C[\| \varphi - \hat{\varphi} \|_{D(A^{\varepsilon+1})} + \| q - \hat{q} \|_{C[0,T]}$$
$$+ \| f - \hat{f} \|_{C([0,T];D(A^{\varepsilon}))}]$$

tengsizlik o‘rinli, bu yerda $c > 0$ *o‘zgarmas son* α, T, ε $\| f \|_{C([0,T];D(A^{\varepsilon}))}$ *va* $\| q \|_{C[0,T]}$ *larga bog‘liq.*

Paragrafning asosiy natijasi quyida keltirilgan.

13-teorema. *Faraz qilaylik, (C1)-(C4) shartlar bajarilsin. U holda yetarlicha kichik* $T > 0$ *larda teskari masalaning* $q(t) \in C[0,T]$ *sinfga tegishli yagona yechimi mavjud.*

14-teorema. *Faraz qilaylik, (C1)-(C4) shartlar bajarilsin va* u_i $(i = 1,2)$ *lar* $\| q_i \|_{C[0,T]} \leq R$ $(i = 1,2)$ *shartlarni qanoatlantiruvchi* $q = q_i \in C[0,T]$ *larga mos (26)-(28) masalaning yechimi bo‘lsin. Undan tashqari, ushbu*

$$|\Phi[u_2](t)| \geq \nu_1^{-1} > 0, \qquad (|\Phi[u_2](t)| \geq \nu_2^{-1} > 0),\ \ t \in [0,T]$$

tengsizlikni qanoatlantiruvchi $\nu_1 > 0$ $(\nu_2 > 0)$ *son mavjud bo‘lsin. U holda* $R, T, \alpha, \varepsilon$ *va* h_0 *larga bog‘liq shunday* $\tilde{C} > 0$ *o‘zgarmas topiladiki ushbu*

$$\tilde{C}^{-1} \| \partial_t^{\alpha}(\Phi[u_1] - \Phi[u_2]) \|_{C[0,T]} \leq \| q_1 - q_2 \|_{C[0,T]}$$
$$\leq \tilde{C} \| \partial_t^{\alpha}(\Phi[u_1] - \Phi[u_2]) \|_{C[0,T]}$$

va

$$\| u_1 - u_2 \|_{C([0,T];D(A))} \leq \tilde{C} \| q_1 - q_2 \|_{C[0,T]}$$

tengsizliklar bajariladi.

Dissetatsiya ishining so‘ngi bobi «**Kasr tartibli integro-differensial tenglamalar uchun teskari masalalar**» deb nomlangan bo‘lib, bu yerda kasr tartibli integro-differensial tenglamalardan vaqt o‘zgaruvchisiga bog‘liq xotira funksiyasi hamda koeffitsiyentni topish masalalari o‘rganilgan.

Bobning birinchi paragrafida kasr tartibli integro-differensial diffuziya tenglamasidan yadroni aniqlash masalasini tadqiq qilamiz.

Quyidagi

$$\partial_t^{\alpha} u(t,x) + Au(t,x) = \int_0^t k(t-s)u(s,x)ds + f(t,x),\ \ (t,x) \in Q_T \qquad (29)$$

vaqt-kasr tartibli integro-differensial diffuziya tenglamasini ushbu

$$u(T,x) - \beta u(0,x) = \varphi(x), \quad x \in \Omega \qquad (30)$$

nolokal boshlang‘ich shart hamda bir jinsli

$$u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega \qquad (31)$$

chegaraviy shart bilan birgalikda qaraymiz. Bu yerda $0 < \alpha < 1$ va aniqlanish sohasi $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ bo‘lgan simmetrik tekis elliptik A operator

$$Au(t,x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{ij}(x) \frac{\partial}{\partial x_j} u(t,x) \right) + c(x)u(t,x), \qquad (t,x) \in Q_T$$

ko‘rinishida berilgan bo‘lib, uning koeffitsiyentlari uchun

$$a_{ij} = a_{ji}, \quad 1 \leq i,j \leq d, \quad a_{ij} \in C^1(\overline{\Omega}), \quad c(x) \in C(\overline{\Omega}),\ c(x) \geq 0, \quad x \in \overline{\Omega}$$

shartlar o‘rinli va shunday $\mu > 0$ o‘zgarmas soni mavjudki

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\bar{\xi}_j \geq \mu \sum_{i=1}^{N} |\xi_i|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^N$$

tengsizlik bajariladi.

Teskari masala

$$\mathcal{L}[u(t,\cdot)] := \int_\Omega \phi(x)u(t,x)dx = h(t), \quad t \in [0,T] \tag{32}$$

qo‘shimcha integral shart yordamida $k(t)$ funksiyani topishdan iborat, bu yerda $\phi(x), h(t)$ berilgan funksiyalar.

X_T orqali

$$X_T := \{u : u \in C([0,T]; D(A^\gamma)) \text{ va } \partial_t u \in L^1(0,T; L^2(\Omega))\}$$

to‘plamni belgilaymiz, bu yerda $\partial_t u$ hosila umumlashgan ma’noda tushuniladi, bundan tashqari X_T va $L^1(0,T)$ larning topologik ko‘paytmasini

$$Y_T = X_T \times L^1(0,T)$$

ko‘rinishida belgilaymiz va undagi normani esa

$$\| (u,k) \|_{Y_T} := \| u \|_{C([0,T];D(A^\gamma))} + \| u_t \|_{L^1(0,T;L^2(\Omega))} + \| k \|_{L^1(0,T)}$$

kabi aniqlanadi.

7-teskari masala. *(29)-(31) masala va (32) shartni qanoatlantiruvchi* $(u,k) \in X_0^T \times L^1(0,T)$ *funksiyalar juftligini toping.*

Paragraf davomida biz $0 < \varepsilon < 1, \gamma > 0$ va

$$\gamma_0 \geq \gamma > \max\left\{\varepsilon, \frac{N}{4} - 1\right\}, N = 1,2,\dots .$$

tengsizliklar bajarilsin deb qaraymiz. Shu bilan birga, faraz qilaylik berilgan funksiyalar uchun quyidagi shartlar bajarilsin:

(D1) $\varphi \in D(A^{\gamma_0}), f \in X_T$;

(D2) $h(T) - \beta h(0) = \mathcal{L}[\varphi], \ \mathcal{L}[Au](0) = \mathcal{L}[f](0)$;

(D3) $\partial_t^\alpha h \in C^1[0,T]$ va $\partial_t^\alpha h(0) = 0$ hamda $h(0) \neq 0$;

(D4) $\phi \in H_0^2(\Omega)$;

(D5) $\beta \notin [0,1]$.

15-teorema. *Faraz qilaylik, (D1)-(D5) shartlar bajarilsin, u holda (29)-(32) teskari masalaning yetarlicha kichik* $T > 0$ *larda yagona* $(u,k) \in Y_T$ *yechimi mavjud*

Bobning ikkinchi paragrafi ko‘p o‘lchamli vaqt-kasr integro-differensial tenglama bilan bog‘liq teskari masalani yechishga bag‘ishlangan bo‘lib, bunda tenglamada qatnashayotgan koeffitsiyent hamda integral ostidagi yadro funksiyasi aniqlanadi.

Bizga quyidagi vaqt-kasr tartibli inetgro-differensial

$$\mathbb{D}_t^{(\alpha)} u(t,x) + Au(t,x) = q(t)u_t(t,x) + k * u(t,x) + f(t,x), \ (t,x) \in Q_T \tag{33}$$

tenglama berilgan bo‘lsin, bu yerda $1 < \alpha < 2$, $\mathbb{D}_t^{(\alpha)}$ esa (22) tenglamadagidek aniqlangan hamda aniqlanish sohasi $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ bo‘lgan simmetrik, tekis elliptik A operator

$$Av(t,x) \equiv -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i} v(t,x)\right) + c(x)v(t,x), \quad (t,x) \in Q_T$$

ko‘rinishida berilgan bo‘lib, uning koeffitsiyentlari uchun

$$a_{ij} = a_{ji} \in C^1(\Omega), \quad c \in C(\overline{\Omega}), \quad c(x) \geq 0, \quad x \in \overline{\Omega}$$

shartlar o‘rinli va shunday $\mu > 0$ o‘zgarmas soni mavjudki

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\bar{\xi}_j \geq \mu \sum_{i=1}^{N} |\xi_i|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^N$$

tengsizlik bajariladi.

(33) tenglamani ushbu

$$u(0,x) = a(x), \quad u_t(0,x) = b(x), \quad x \in \Omega \tag{34}$$

boshlang‘ich shartlar va bir jinsli

$$u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega \tag{35}$$

chegaraviy shart bilan birgalikda qaraymiz.

Agar $q(t)$, $k(t)$, $f(t,x)$, $a(x)$ va $b(x)$ funksiyalar ma’lum bo‘lsa, u holda (33)-(35) masaladan u funksiyani topishga *to‘g‘ri masala* deyiladi. Teskari masalada to‘g‘ri masala yechimiga nisbatan ushbu

$$u(t,x_i) = h_i(t), \quad t \in (0,T) \tag{36}$$

qo‘shimcha shartlar yordamida $q(t)$ va $k(t)$ funksiyalarni topish talab qilinadi, bu yerda $h_i(t), i = 1,2$ berilgan funksiayalar va $x_i \in \Omega$ $(i = 1,2)$ berilgan nuqtalar.

Quyidagi masalani tadqiq qilamiz.

8-teskari masala. *(33) tenglama va (34)-(36) shartlarni qanoatlantiruvchi* $u \in C\left([0,T]; D\left(A^{\gamma+\frac{1}{\alpha}}\right)\right) \cap C^1\big([0,T]; D(A^\gamma)\big)$, $q \in C^1[0,T]$ *va* $k \in C[0,T]$ *funksiyalarni toping, bu yerda* $\gamma > 0$ *o‘zgarmas son.*

X_T orqali

$$X_T := C\left([0,T]; D\left(A^{\gamma+\frac{1}{\alpha}}\right)\right) \cap C^1\big([0,T]; D(A^\gamma)\big)$$

kesishmani va

$$Y_T := X_T \times C^1[0,T] \times C[0,T],$$

hamda Y_T da normani

$$\| (u,q,k) \|_{Y_T} := \| u \|_{X_T} + \| q \|_{C^1[0,T]} + \| k \|_{C[0,T]}$$

kabi aniqlaymiz.

Faraz qilaylik, $\gamma_0 > \frac{N}{2} + 1$ va

$$\frac{N}{4} + 1 < \gamma \leq \gamma_0$$

tengsizlik bajarilsin. Undan tashqari, berilgan funksiyalar quyidagi shartlarni qanoatlantirsin:

(E1) $h_i \in C^3[0,T]$ va $h_i''(0) = 0, i = 1,2$, $a \in D\left(A^{\gamma_0+\frac{1}{\alpha}}\right), b \in D(A^{\gamma_0})$, $f \in C^1\big([0,T]; D(A^\gamma)\big)$;

(E2) $h_i'(0)q(0) = \partial_t^\alpha h_i(0) + Aa(x_i) - f_i(0,x_i), \ i = 1,2$;

(E3) $a(x_i) = h_i(0), \ b(x_i) = h_i'(0), \ i = 1,2$;

(E4) $p(t) = h_1'(t)h_2(0) - h_2'(t)h_1(0) \neq 0$ va $p(t) \in C^1[0,T]$ bo‘lib, ushbu

$$|p(t)| \geq \frac{1}{p_0} > 0, \qquad t \in [0,T]$$

tengsizlik bajarilsin, bu yerda p_0 ma'lum musbat o'zgarmas son.

16-teorema. *Faraz qilaylik, (E1)-(E4) shartlar bajarilsin. U holda yetarlicha kichik* $T > 0$ *larda (33)-(36) masalaning* $(u,q,k) \in Y_T$ *yagona yechimi mavjud.*

Ushbu bobning so'ngi paragrafida vaqt-kasr integro-differensial diffuziya tenglamasi uchun orqaga qaytish va teskari masalalarni tadqiq qilamiz. Teskari masala tenglamada qatnashayotgan xotira funksiyasini orqaga qaytish masalasining yechimiga nisbatan qo'shimcha shart yordamida aniqlashdan iborat. Bunda Banax fazosida teskari masala yechimining lokal mavjudligi va yagonaligiga to'xtalamiz.

Bizga

$$\partial_t^\alpha u(t) + Au(t) = \int_0^t k(t-s)u(s)ds + f(t), \quad t \in (0,T) \tag{37}$$

ko'rinishidagi kasr tartibli integro-differensial tenglama berilgan bo'lsin, bu yerda $T > 0$ tayin son, va $\alpha \in (0,1)$ bo'lib, A operator esa uchinchi bobning oxirgi paragrafida keltirilgan.

Berilgan $\alpha, k(t)$ va $f(t)$ lar bo'yicha (40) tenglamani va

$$u(T) = \varphi \tag{38}$$

shartni qanoatlantiruvchi $u(t): [0,T] \to H$ funksiyani topishga *orqaga qaytish* masalasi deyiladi, bu yerda $\varphi \in H$ va $f: [0,T] \to H$ ma'lum funksiya.

9-teskari masala. *Berilgan* $\alpha, f(t)$ *va* φ *lar bo'yicha (37) tenglama va (38) shart hamda*

$$\Phi[u(t)] = h(t), \quad t \in [0,T] \tag{39}$$

qo'shimcha shartni qanoatlantiruvchi $u: [0,T] \to H$ *va* $k: (0,T) \to \mathbb{R}^+$ *funksiyalar juftligini toping, bu yerda* $h: [0,T] \to \mathbb{R}$ *ma'lum funksiya,* $\Phi: D(\Phi) \subset H \to \mathbb{R}$ *ma'lum chiziqli chegaralangan funksional, va* $D(\Phi) = \{u \in H: Au \in H\}$.

X_T orqali

$$X_T := W_1^1(0,T;H) \cap L^1\big(0,T;D(A)\big)$$

funksiyalar fazosini va undagi normani esa

$$\| u \|_{X_T} := \| u \|_{W_1^1(0,T;H)} + \| u \|_{L^1(0,T;D(A))}$$

kabi aniqlaymiz. Xuddi shunday

$$Y_T := X_T \times L^1(0,T)$$

va undagi normani

$$\| (u,k) \|_{Y_T} := \| u \|_{X_T} + \| k \|_{L^1(0,T)}$$

ko'rinishida aniqlaymiz.

3-ta'rif. *(37) tenglama va (38), (39) shartlarni qanoatlantiruvchi* $(u,k) \in Y_T$ *funksiyalar juftligini topish masalasiga teskari masala deyiladi.*

Faraz qilaylik, $0 < \varepsilon < 1$ bo'lib, berilgan funksiyalar quyidagi shartlarni qanoatlantirsin:

(F1) $\varphi \in D(A^{\varepsilon+1}), f \in C\big([0,T]; D(A^\varepsilon)\big) \cap C^1([0,T];H)$;

(F2) $h(T) = \Phi[\varphi],\ \Phi[Au](0) = \Phi[f](0)$;

(F3) $\partial_t^\alpha h \in C^1[0,T]$ va $\partial_t^\alpha h(0) = 0$, hamda $h(0) \neq 0$ bo'lsin;

(F4) $\Phi\colon \{\lambda_n \Phi[e_n]\} \in l^2(\mathbb{N})$.

Paragrafning asosiy natijasi quyidagi teoremalardan iborat.

17-teorema. *Faraz qilaylik, ba'zi* $\varepsilon \in (0,1)$ *uchun* $\varphi \in D(A^{\varepsilon+1}), f \in C\big([0,T]; D(A^{\varepsilon})\big) \cap C^1([0,T]; H)$ *va* $k \in L^1(0,T)$ *bo'lsin. U holda (37), (38) maslaning yagona* $u \in X_T$ *yechimi mavjud va* $\partial_t^{\alpha} u \in L^1(0,T;H)$ *bo'ladi.*

18-teorema. *Faraz qilaylik, (F1)-(F4) shartlar bajarilsin. U holda yetarlicha kichik* $T > 0$ *larda (37)-(39) teskari masalaning* $(u,k) \in Y_T$ *yagona yechimi mavjud.*

XULOSA

Ushbu dissertatsiya kasr tartibli integro-differensial operatorlarni o‘z ichiga olgan xususiy hosilali differensial tenglamalar uchun to‘g‘ri, orqaga qaytish va teskari masalalarni o‘rganishga bag‘ishlangan.

Dissertatsiya ishning asosiy natijalari quyidagilardan iborat:

1. Vaqt bo‘yicha birinchi va kasr tartibli xususiy hosilali differensial tenglama uchun qo‘yilgan Koshi masalasi yechimini beruvchi formula olindi va uning klassik yechim bo‘lishi isbotlandi.
2. Ikki o‘lchamli kasr tartibli diffuziya tenglamasi uchun to‘g‘ri va teskari masalalarning korrektligi ko‘rsatildi.
3. Parallelepiped sohada qo‘yilgan kasr tartibli diffuziya tenglamasi uchun manba funksiyalarni aniqlashga doir chiziqli teskari masalalar yechimining mavjudligi va yagonaligi kabi natijalari olindi.
4. Kasr tartibli diffuziya differensial tenglamasi uchun qo‘yilgan nolokal boshlang‘ich va bir jinsli Dirixle chegaraviy shartli masalaning yagona yechimga ega bo‘lishi ko‘rsatildi. Bundan tashqari, Gauss-Zeydel iteratsion usuli qo‘llash orqali vaqtga bog‘liq koeffitsiyentni aniqlash teskari masalasining yagona yechimga egaligi ko‘rsatildi.
5. Bir o‘lchamli kasr tartibli to‘lqin tenglamasidan vaqtga bog‘liq koeffitsiyentni aniqlashga doir teskari masala lokal tarzda yechildi. Bu holatda ikkita boshlang‘ich shart noan’anaviy shaklda berilgan bo‘lib, muammoni sezilarli darajada murakkablashtirdi.
6. Abstrakt kasr tartibli diffuziya tenglamasi uchun Koshi masalasi, hamda vaqtga bog‘liq koeffitsiyentli teskari masalaning yagona yechimga ega ekanligi isbotlandi.
7. Integral shart bilan berilgan nolokal integro-differensial tenglamadan vaqtga bog‘liq yadro funksiyasini aniqlashga doir teskari masala yagona yechimga ega ekanligi isbotlandi.
8. Ko‘p o‘lchamli kasr tartibli integro-differensial tenglamadan ikkita vaqtga bog‘liq koeffitsiyentni aniqlash muammosi to‘liq yechildi. Bu holda, to‘g‘ri masala yechimiga nisbatan chegaraviy shartlar ikkita turli xil nuqtalarda berilgan.
9. Kasr tartibli integro-differensial tenglamadan vaqt o‘zgaruvchisiga bog‘liq yadro funksiyasini aniqlashga doir teskari masalaning yagona yechimga ega ekanligi isbotlandi.

SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES DSc.02/30.12.2019.FM.86.01 INSTITUTE OF MATHEMATICS NAMED AFTER V.I. ROMANOVSKIY

INSTITUTE OF MATHEMATICS

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CORRECTNESS OF INVERSE COEFFICIENT PROBLEMS FOR FRACTIONAL DIFFUSION-WAVE EQUATIONS

01.01.02 – Differential equations and mathematical physics

ABSTRACT
OF THE DOCTORAL (DSc) DISSERTATION
ON PHYSICAL AND MATHEMATICAL SCIENCES

TASHKENT – 2025

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INTRODUCTION (abstract of doctoral dissertation)

Actuality and demand of the theme of the dissertation. The Non-Integer Order Calculus, traditionally known as Fractional Calculus**,** is the branch of mathematics that tries to interpolate the classical derivatives and integrals and generalizes them for any orders, not necessarily integer order. This field has gained increasing importance in recent years due to its ability to describe complex systems and phenomena that cannot be adequately captured by classical models. In particular, fractional differential equations, which involve derivatives of non-integer order, provide more accurate models for various physical, biological, and engineering processes, especially those exhibiting memory, hereditary effects, and non-local interactions. The correctness of inverse coefficient problems for fractional diffusion-wave equations addresses one of the most critical challenges in the field of mathematical modeling. These inverse problems aim to determine unknown parameters, such as coefficients in the governing equations, based on observed data. They are of immense importance in applications such as image reconstruction, material characterization, and the study of anomalous diffusion and wave propagation in heterogeneous media. The increasing complexity of modern systems, especially those with fractal-like structures or non-local properties, demands robust methods to solve these inverse problems accurately and efficiently. Thus, the demand for studying the correctness of such problems is driven by both theoretical and practical considerations, as fractional models are becoming essential for describing real-world phenomena in areas like physics, finance, biology, and engineering. Understanding the mathematical and computational aspects of these inverse problems will enable more precise interpretations of experimental data and improve the design of algorithms for real-time applications.

Nowadays it is very important in the world to learn how to solve partial differential equations, as well as to learn how to solve inverse problems related to determining the coefficients of an equation, its right-hand side and the kernel under the integral. Investigating inverse problems allows us to study, analyze, and control processes given by fractional differential equations. For example, if we consider the vibration processes of a homogeneous string, the known initial conditions of the solution based on the characteristic triangle provide the possibility to control the string vibration from the boundaries. If we consider the fractional differential equation with respect to time, and if order of the fractional derivative is between 1 and 2, then it simultaneously describes diffusion and wave propagation processes. If the order is between 0 and 1, it describes slow diffusion processes. For instance, fractional differential equations are used to model the propagation of signals through optical fibers, the transmission of electromagnetic waves in a straight line, and their interaction with materials. That is why it is important to study fractional equations.

Special interest is given to modern areas of differential equations and mathematical physics in Uzbekistan, as they are of scientific and practical importance in contemporary science. This includes special attention to the study of equations in branched structures and the investigation of direct and inverse problems for equations with fractional derivatives. Although these areas are relatively new in global science, scientists in Uzbekistan have achieved significant results in them and continue to actively conduct research. Conducting scientific research at international standards in priority areas of mathematical sciences, particularly differential equations and mathematical physics, is the main goal and focus of the Romanovskiy Institute of Mathematics[1]. The advancement of studies on direct and inverse problems for integro-differential equations with fractional derivatives is one of the most important areas of modern differential equation theory and mathematical physics. It undoubtedly plays a key role in realizing the aforementioned statement by the Cabinet of Ministers of the Republic of Uzbekistan.

The subject and object of research of this dissertation are in line with tasks identified in the Decrees and Resolutions of the President of the Republic of Uzbekistan of February 8, 2017, PQ-4947, “On the strategy of action development of the Republic of Uzbekistan”, PQ-4387 date July 9, 2019 “On state support for the further development of mathematics education and science, as well as measures to improve the activities of the Institute of Mathematics named after V.I.Romanovsky of the Academy of Sciences of the Republic of Uzbekistan”, PQ-4708 of May 7, 2020 “On measures to improve the quality of education and research in field of mathematics” as well as in other regulations related to basic sciences.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed by the priority areas of science and technology of the Republic of Uzbekistan IV, «Mathematics, Mechanics and Computer Science».

Review of foreign scientific researches on the topic the dissertation. Almost all universities, research institutes and scientific centers of the world are conducting researches on direct and inverse problems for fractional differential and partial differential equations. We will list some of them as an example. As we said above, they are Johns Hopkins University (USA), Tokyo University (Japan), Berlin University of Applied Sciences (Germany), New Haven University (USA), Berlin University (Germany), Belorussian State University (Belarus), Hokkaido University (Japan), at La Rochelle University (France), Institute of Applied Mathematics and Automation (Russia), Institute of Mathematics NAS of Ukraine (Ukraine), Middle East University (Turkey), International Kazakh-Turkish University (Kazakhstan) and others.

Inverse problems on determining the source terms, the coefficients and the memory functions of the fractional differential equations were studied by Professor M. Yamamoto of Tokyo University (Japan), N.H. Tuan of Ho Chi Minh University

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[1] Decree of Cabinet of Ministers of the Republic of Uzbekistan at the 2017 year 18 May "On measures for the organization of activities of newly established scientific organizations of the Academy of Sciences of the Republic of Uzbekistan" No. 292.

of Banking, H. Wang of Southeast University (China), B. Wu of College of Mathematics and Physics, Nanjing University of Information Science and Technology (China), and others. Their method ensured the uniqueness of the solution of the inverse problem only. Professor S. Umarov of the New Haven University of the United States of America and Professor R. Ashurov of the Mathematical Institute of the FA of the Republic of Uzbekistan obtained new conditions for single-valued determination of the solution of the inverse problem for determining the order of the fractional derivative.

The various initial, initial-boundary and retrospective problems for fractional order differential equations and partial differential equations have been studied by many scientists. Cauchy problems for fractional diffusion-wave equations have been studied in the works of A. Kochubei (Ukraine), A. Pskhu (Russia) and others. The nonlocal condition connecting initial and time-fixed values for diffusion equations was studied in the works of A. Ashyaraliev (Turkey) and others. The questions related to coefficient inverse problems for fractional partial derivative equations in an unbounded domain are practically not studied.

The degree of scrutiny of the problem. In the literature, several definitions of the fractional derivatives have been proposed. For instance, the Riemann-Liouville derivative (D_{0+}^{α}) and the Gerasimov-Caputo derivative (∂_t^{α}). The Riemann-Liouville derivative played an important role for its application in pure mathematics while Caputo derivative has been introduced to respond to applied problems. Indeed, M. Caputo was the first to give applications of fractional calculus to mechanics, especially to linear models of viscoelasticity. Caputo derivatives allow the use of physically interpretable initial conditions, which is not permitted by the Riemann-Liouville derivative.

Considering a differential equation that describes a specific phenomenon, a common way to use fractional modeling is to replace the integer order derivatives by non-integer derivatives, usually with order lower than or equal to the order of the original derivatives, so that the usual solution may be recovered as a particular case and this can be observed in the works of Mainardi, Podlubny, Shishkina and Sitnik, Tarasova and Tarasov, Uchaikin. Equations with fractional derivatives are applied in studying of anomalous diffusion and various processes in physics, mechanics, chemistry and engineering. The diffusion equation is a mathematical model of important physical phenomena ranging from fractals. In normal diffusion (described by the heat equation or more general parabolic equations) the mean square displacement of a diffusive particle behaves like $const \cdot t$ for $t \to \infty$. A typical behavior for anomalous diffusion is $const \cdot t^{\alpha}$, and this was the reason to invoke fractional diffusion equation, where this anomalous behavior is an easy mathematical fact. For connections to statistical mechanics is given by Meerschaert. Such issues are directly related to the fundamental solution of the equation. An expression for the fundamental solution of the Cauchy problem was found independently by Schneider and Wyss and Kochubei. For example, by Schneider and Wyss discussed the Cauchy problem in the form

$$\begin{cases}\partial_t^\alpha u = \Delta u, \quad x \in \mathbb{R}^N, t > 0, m-1 < \alpha < m, m \in \mathbb{N},\\ \partial^k u/\partial t^k(x,0) = f_k(x), 0 \le k \le m-1.\end{cases}$$

Moreover, in 2006 Voroshilov and Kilbas derived formulae that provide solutions when $0 < \alpha < 2$. The authors present a formula that offers a solution to the problem. However, there is no proof that this is indeed the solution. But, Kochubei gives a proof that the obtained formula is indeed a solution (for $m = 1$). After few years later, the fundamental solution of the fractional diffusion-wave equation was found by Pskhu for general fractional derivative, and using it, the connection between fractional order equations and integer order heat and wave propagation equations was demonstrated. Later, in 2014, a formula was obtained by Eidelman and Kochubei which gives the solution to the Cauchy problem for the elliptic part in the general case. However, all the obtained results are only for linear Cauchy problems, and the study of the problem in the nonlinear case remains open.

Moreover, the main part of this dissertation consists of studying initial-boundary value problems for fractional order equations in a bounded domain, such as direct, backward and inverse problems. The backward problems for the diffusion-wave process are of great importance in engineering fields and are aimed at determining the previous state of a physical field (for example, at initial time) based on its current information. However, the peculiarity of such problems is that, regardless of whether the Riemann-Liouville or Caputo derivative is used in the equation, this problem is ill-posed in the sense of Hadamard. Nevertheless, the problem in the form

$$\begin{cases}\partial_t^\alpha u(t) + Au(t) = f(t), \quad 0 < t \le T,\\ u(T) = \Phi\end{cases}$$

has been studied by academician Alimov and professor Ashurov (for $0 < \alpha < 1$), where A is a given self-adjoint, positive, unbounded arbitrary operator. The problem has been investigated by Ashurov and Fayziev under the condition $u(\xi) = \gamma u(0) + \varphi, \gamma = const \neq 0, 0 < \xi \le T$. The authors have not only studied the backward problem, but also the linear inverse problem, i.e. the problems of finding the right side and the initial conditions. In the works of Floridia, Yamamoto, and Liu, fractional diffusion and fractional diffusion-wave equations have also been studied. In the above mentioned works, not only backward problems but also inverse problems related to the identification of source functions (inverse problems) have been studied. However, in the previous works, only linear problems were studied, while this dissertation adequately addresses the open problems related to the nonlinear case. Moreover, this dissertation builds upon the scientific work of leading scholars in the world, such as Al-Salti, Aloroev, Kirane, Malik, Cheng, Eidelman, Gong, Wei, Ismailov, Jin, Rundell, Kochubei, Lopushansky, Lopushanska, Yamammoto, Liao, Li, Liu, Luchko, Ruzhansky, Settara, Atmania, Sun, Zhang as well as researchers in our country academician Alimov, professors Ashurov, Durdiev, Karimov and others, who focus on direct and inverse problems.

The connection of the theme of the thesis with the research plans of the higher education institute, where the research on the thesis is carried out. The dissertation work was carried out within the framework of the calendar plan on the topic "Inverse Problems of Mathematical Physics" of the Bukhara branch of the V.I. Romanovskiy Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan.

The aim of research work: To study the unique solvability of direct, backward and inverse problems for fractional differential and integrodifferential diffusion-wave equations.

Research problems:

Construction and investigation of the solution of the Cauchy problem for the diffusion equation with time derivatives of first and fractional orders;

Investigation of the inverse problem of determination of the coefficient from the fractional diffusion equation;

Solvability of the inverse problem on determination of the source function from the fractional diffusion equation;

Study of direct and inverse coefficient problems with nonlocal initial conditions for fractional diffusion and wave equations;

Solving direct and inverse problems of finding time-dependent coefficient in fractional differential equations;

Investigation of direct and inverse problems of finding the time-dependent coefficient from the abstract Cauchy problem for the fractional diffusion equation;

Determination of coefficient and memory function by conditions given at two points concerning the solution of a direct problem;

Investigating the retrospective and inverse problems for determining the memory function in the integro-differential fractional diffusion equation.

The research object. Riemann-Liuville, Gerasimov-Kaputo fractional derivatives, Mittag-Leffler functions, Fox's H-function, fractional time equations.

The research subject. The research subject of the dissertation consists of direct and inverse problems for differential equations with finite dimension and arbitrary elliptic operators with partial derivatives and fractional derivatives.

Research methods. Mathematical analysis, functional analysis, differential equations and mathematical physics methods were used in the thesis work. The methods of separation of variables and integral energy from the methods of mathematical physics is used, and the completeness of the system of eigenfunctions in the Hilbert space is applied.

Scientific novelty of the research work is presented as follows:

an explicit formula has been constructed providing a solution to the Cauchy problem posed for a one-dimensional fractional differential equation and the sufficient conditions for the given function to possess a classical solution were established;

the conditions for the uniqueness of solutions to direct and nonlinear inverse problems related to the fractional diffusion equation in an unbounded domain are established. Additionally, sufficient conditions for the unique identification of the source terms from the fractional diffusion equation have been provided;

the uniqueness and existence results are proved for a non-local initial and boundary value problem which is formulated for fractional diffusion-wave equations;

the existence and well-posedness of the Cauchy problem for the abstract fractional diffusion equation are proven, and the conditional well-posedness of the inverse problem with time-dependent coefficients is established;

theorems are proven that establish the existence and uniqueness of the solution to the nonlinear inverse problem of determining the coefficient and memory function in the fractional diffusion-wave equation;

a theorem is proved concerning the uniqueness of solvability for nonlocal initial direct and nonlinear inverse problems associated with the multidimensional fractional diffusion equation, incorporating the fractional Gerasimov-Kaputo derivative.

Practical results of the research. The present results are theoretical in character, and methods for studying nonlinear fractional-order differential equations are proposed. At the same time, methods for solving inverse problems related to finding the right-hand side of the equation, the time-dependent coefficient and the memory function are presented.

The reliability of the results of the study. The results have been obtained by using the methods of modern methods of mathematical physics, mathematical analysis, differential equations, the theory of special functions for the construction of fundamental solutions, finding exact solutions of boundary value problems and solving the theoretical problems of elliptic equations.

Scientific and practical significance of the research results. The scientific significance of the research results is that the scientific results obtained in the work can be used in the theory of fractional differential and integro-differential equations.

The practical importance of the results obtained in the dissertation work is determined by their application in the study, control and management of processes given by fractional equations.

Implementation of the research results. Based on the results obtained for the correctness of inverse coefficient problems for fractional diffusion-wave equations:

the conditions for the existence and uniqueness of solutions to nonlocal initial and homogeneous Dirichlet boundary value problems for the fractional-order diffusion-complete equation were used in the innovative project No. IL-21071166 titled “Development of a Vertical Axis Wind Turbine Designed for Low Wind Speeds” for the approximate solution of mathematical model equations developed to determine effective parameters of wind turbines (based on Reference No. 281-3 dated March 25, 2025, from the Institute of Mechanics and Seismic Stability of Structures). The application of this scientific result made it possible to construct numerical models for problems related to differential equations of hyperbolic type with Koltunov kernels;

the results on the unique determination of coefficients and memory functions for diffusion and superdiffusion differential equations given in bounded domains were used in the foreign project No. 122041100096-4 titled "Mathematical Modeling in Sociology, Geophysics, and Engineering Sciences" for the investigation of subsurface convex media (based on Reference No. 18 dated March 24, 2025, from the Southern Mathematical Institute, a branch of the Federal Scientific Center of the Vladikavkaz Scientific Center of the Russian Academy of Sciences, Russian Federation). The application of these scientific results enabled the observation of wave phenomena in subsurface viscous media;

the results on the unique determination of time-dependent coefficients from backward, nonlocal initial, and first boundary value problems for subdiffusion equations were used in the foreign fundamental project No. AP09258836 titled "Development of Numerical Algorithms for Differential Mathematical Models of Anomalous Diffusion" for the investigation of temperature variations (based on Reference No. 04/857 dated April 2, 2025, from Khoja Akhmet Yassawi International Kazakh-Turkish University, Kazakhstan). The application of these scientific results enabled the observation of heat exchange processes in water reservoirs;

the results on determining coefficients and memory functions from initial-boundary value problems for fractional-order subdiffusion and superdiffusion equations with nonlocal initial conditions were used in international scientific journal articles to solve inverse problems for fractional parabolic equations with nonlocal biharmonic operators (Fractal and Fractional, 2023, 7(5): 404; Physica Scripta, 2024, 99(10): 105242; AIMS Mathematics, 2023, 9(3): 6832–6849). The application of these scientific results made it possible to identify the source function from a two-dimensional fractional parabolic equation with nonlocal conditions.

Approbation of the research results. The main result of the dissertation was discussed at the following international and republic scientific conferences: "6^{th} International conference of mathematical sciences ICMS" (Istanbul 2022), "Modern problems of applied mathematics and information technologies al-Khwarizmi" (Fergana 2021), "Actual problems of applied Mathematics and information technologies-al-Khwarizmi" (Samarkand 2023), "Operator algebras, non-associative structures and related problems" (Tashkent 2022), "Actual problems of physics, mathematics and mechanics" (Bukhara 2023), "Order analysis and related questions of mathematical modelling, XVII. Operator theory and differential equations" (Vladikavkaz 2021), "Mathematical analysis and its applications in modern mathematical physics" (Samarkand 2022), "Order analysis and related questions of mathematical modelling, XVII. Operator theory and differential equations" (Vladikavkaz 2023), "Modern problems of applied mathematics and information technologies Al-Khwarizmi" (Tashkent 2024), “Traditional International April Mathematical Conference in honour of the Day of Science of the Republic of Kazakhstan” (Almaty 2023).

This dissertation was discussed at the seminars “Differential equations and modern problems of mathematical physics” at the National University of Uzbekistan, “Differential equations and mathematical physics” at the Institute of Mathematics of the Academy of Sciences of Uzbekistan, and “Modern problems of mathematical physics” at the Bukhara branch of the Institute of Mathematics.

Publications of the research results. On the topic of the dissertation, 27 scientific papers were published, 15 of which are included in the list of scientific publications proposed by the Higher Attestation Commission of the Republic of Uzbekistan for the defence of theses of the Doctor of Philosophy, including 9 of them published in foreign journals and 6 in national scientific journals and 12 abstracts.

The structure and volume of the dissertation. The dissertation consists of an introduction, four chapters, a conclusion, and a bibliography. The volume of the thesis is 212 pages.

THE MAIN CONTENT OF THE DISSERTATION

In the introduction besides the motivation of research theme and correspondence to the priority research areas of science and technology of the Republic, we present a review of international research on the theme of the dissertation and the degree of scrutiny of the problem, formulate our goals and objectives, identify the object and subject of study, and state scientific novelty and practical results of the research. Moreover, we reduce the theoretical and practical importance of the obtained results, and give information on the implementation of the research results, the published works and the structure of dissertation.

The opening chapter of this dissertation is titled **«Preliminaries and Cauchy Problem for a Time-Fractional Differential Equation»**. In the first two sections of this chapter, essential preliminary concepts, definitions, and propositions are introduced to establish the foundational framework required for the subsequent development and analysis in the dissertation.

The third section of this chapter focuses on solving the Cauchy problem for a one-dimensional differential equation of first and α-order.

Consider the Cauchy problem

$$\rho\frac{\partial u(t,x)}{\partial t}+\mathbb{D}_t^{(\alpha)}u(t,x)-u_{xx}(t,x)=0,\quad t>0, x\in\mathbb{R},\tag{1}$$

$$u(0,x)=\varphi(x),\quad x\in\mathbb{R},\tag{2}$$

where ρ is a positive constant, $\mathbb{D}_t^{(\alpha)}$ is the $\alpha\in(0,1)$-order regularized fractional derivative, i.e.,

$$\left(\mathbb{D}_t^{(\alpha)}u\right)(t,x)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_0^t(t-s)^{-\alpha}u(s,x)ds-t^{-\alpha}u(0,x)\right]$$

$$= \frac{d}{dt}(I_{0+}^{1-\alpha}u)(t,x) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}u(0,x), \qquad (3)$$

here the *Riemann-Liuville* left fractional integral of order α is defined as

$$(I_{0+}^{\alpha}u)(t,x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s,x)ds, \qquad (4)$$

φ is a given function.

Before establishing the existence conditions for the solution of problem (1), (2), let us first give a definition of a classical solution.

Definition 1. *A function* $u(t,x)$ $(t > 0, x \in \mathbb{R})$ *is a classical solution to Cauchy problem (1), (2) if*

(i) for each $x \in \mathbb{R}$, $u_t(t,x) \in C(t > 0)$*;*

(ii) $u(t,x)$ *is twice continuously differentiable with respect to* x *for every* $t > 0$*;*

(iii) for every $x \in \mathbb{R}$, $u(t,x) \in C(t \geq 0)$ *and its fractional integral* $(I_{0+}^{1-\alpha}u)(t,x)$ *is continuously differentiable in* t *for* $t > 0$;

(iv) $u(t,x)$ *satisfies Eqs. (1) and (2).*

The main results of this section is the following statements.

Theorem 1. *Let* $0 < \alpha < 1$. *If* $\varphi(x) \in C(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, *then the unique solution of Cauchy problem (1), (2) exists and it is described by formula*

$$u(t,x) = \int_{\mathbb{R}} \mathcal{G}^{\alpha}(t, x-\xi)\varphi(\xi)d\xi, \qquad (5)$$

where

$$\mathcal{G}^{\alpha}(t,x) := \mathcal{G}_1^{\alpha}(t,x) + \mathcal{G}_2^{\alpha}(t,x),$$

$$\mathcal{G}_1^{\alpha}(t,x) = \frac{1}{2\pi^{1/2}|x|}\sum_{k=0}^{\infty}\frac{(-1)^k}{\rho^k k!}t^{(1-\alpha)k}H_{1,2}^{2,0}\left[\frac{\sqrt{\rho}}{2\sqrt{t}}|x| \,\middle|\, \begin{matrix}(1+(1-\alpha)k, 1/2)\\ (1/2,1/2),(1+k,1/2)\end{matrix}\right],$$

$$\mathcal{G}_2^{\alpha}(t,x) = \frac{1}{2\pi^{1/2}|x|}\sum_{k=0}^{\infty}\frac{(-1)^k}{\rho^{k+1}k!}t^{(1-\alpha)(k+1)}H_{1,2}^{2,0}\left[\frac{\sqrt{\rho}}{2\sqrt{t}}|x| \,\middle|\, \begin{matrix}(1+(1-\alpha)(k+1), 1/2)\\ (1/2,1/2),(1+k,1/2)\end{matrix}\right]$$

and $H_{p,q}^{m,n}(z)$ *is the Fox's H-function. Moreover, this solution tends to zero as* $|x| \to \infty$ *for any fixed* $t > 0$.

Theorem 2. *Let* $u(t,x)$ *be a classical solution of the problem (1), (2) with* $\varphi(x) \equiv 0$. *Suppose that*

$$|u(t,x)| \leq M_1 \exp\{M_2|x|^{\mu}\}, \quad M_1, M_2 > 0, \ \mu < 2.$$

Then $u(t,x) \equiv 0$.

The second chapter of the dissertation is titled «**Determining a coefficient and source terms from the time-fractional diffusion equation in both unbounded and bounded domains**», in which the problem of determining the time- and spatial variable-dependent coefficients through additional conditions are explored. This is done within the context of the two-dimensional fractional diffusion equation, both in an unbounded domain and for functions dependent on both time and space variables in a bounded domain.

Consider the following time-fractional diffusion equation:

$$\left(\mathbb{D}_t^{(\alpha)}u\right)(t,\bar{x}) - \Delta u(t,\bar{x}) + q(t,x)u(t,\bar{x}) = f(t,\bar{x}), \quad (t,\bar{x}) \in \mathbb{R}_T^2 \qquad (6)$$

at condition

$$u(0,\bar{x}) = \varphi(\bar{x}), \quad \bar{x} \in \mathbb{R}^2, \qquad (7)$$

where $\bar{x} = (x,y) \in \mathbb{R}^2, \mathbb{R}_T^2 = \{(t,\bar{x}): \bar{x} \in \mathbb{R}^2, 0 < t \leq T\}$, $\Delta := \partial_x^2 + \partial_y^2$, and $f(t,\bar{x}), \varphi(\bar{x})$ are given smooth functions, $\mathbb{D}_t^{\alpha}, 0 < \alpha < 1$, is a regularized fractional derivative (see (3)).

Inverse problem 1. *Find the function* $q(t,x), x \in \mathbb{R}, t \geq 0$ *in (6), if the solution to Cauchy problem (6), (7) satisfies*

$$u(t,x,0) = g(t,x), \quad x \in \mathbb{R}, t \geq 0, \qquad (8)$$

where $g(t,x)$ *is given function.*

Definition 2. *We call a function* $u(t,\bar{x})$ *a classical solution to Cauchy problem (6) and (7), if:*

(i) $u(t,\bar{x})$ *is twice continuously differentiable in* $\bar{x}$ *for each* $t > 0$*;*

(ii) for each $\bar{x} \in \mathbb{R}^2$ $u(t,\bar{x})$ *is continuous in* t *on* $[0,T]$*, and its fractional integral* $I_{0+}^{1-\alpha}u(t,\bar{x})$ *is continuously differentiable in* t *for* $t > 0$*;*

(iii) $u(t,\bar{x})$ *satisfies (6) and (7).*

Lemma 1. *If* $q(t,x) \in C\left([0,T]; C_b^l(\mathbb{R})\right)$, $f(t,\bar{x}) \in C\left([0,T]; C_b^{l,l}(\mathbb{R}^2)\right)$, $\varphi(\bar{x}) \in C_b^{l,l}(\mathbb{R}^2)$*, then there exists a unique solution of the direct problem (6), (7)* $u(t,\bar{x}) \in C^{1-\alpha,2}(\mathbb{R}_T^2) \cap C(\overline{\mathbb{R}}_T^2)$*, where* $l \in (0,1)$.

Theorem 3. *If* $f(t,\bar{x}) \in C\left([0,T]; C_b^{l,l+1}(\mathbb{R}^2)\right)$, $\varphi(\bar{x}) \in C_b^{l,l+2}(\mathbb{R}^2)$, $g(t,x) \in C^1\left([0,T]; C_b^{l+2}(\mathbb{R})\right)$, $|g(t,x)| \geq g_0 > 0$*, then there exists a number* $T^* \in (0,T)$*, such that there exists a unique solution* $q(t,x) \in C([0,T^*]; C_b^l(\mathbb{R}))$ *of the inverse problem (6)-(8).*

Let T be a positive fixed number. Consider the set $\Omega(\gamma_0)$ ($\gamma_0 > 0$ is some fixed number) of the given functions (f,φ,g) for which all conditions from Theorem 3 are fulfilled and so that $\max\left\{\| f \|^{l,l+1}, |\varphi|^{l,l+2}, \| g \|_1^l\right\} \leq \gamma_0$, and by $Q(\gamma_1)$ we denote the class of functions $q(t,x) \in C\left([0,T^*]; C_b^l(\mathbb{R})\right)$, satisfying the inequality $\| q \|^l \leq \gamma_1$ with some fixed positive number γ_0.

Theorem 4. *Let* $(f,\varphi,g) \in \Omega(\gamma_0), \left(\tilde{f},\tilde{\varphi},\tilde{g}\right) \in \Omega(\gamma_0)$ *and* $(q,\tilde{q}) \in Q(\gamma_1)$*. Then for the solution of the inverse problem (9)-(11) the following stability estimate is valid:*

$$\| q - \tilde{q} \|^l \leq c\left(\| f - \tilde{f} \|^{l,l+1} + \left|\varphi - \tilde{\varphi}\right|^{l,l+2} + \|g - \tilde{g}\|_1^l\right),$$

where the constant c *depends only on* T, l, γ_0, γ_1.

In the next section of this chapter, we investigate two inverse problems, namely finding source functions that depend on both time and space variables.

Now in the parallelepiped

$$Q := D \times (0,T], \quad D := \{(x,y): 0 < x < l, 0 < y < q\},$$

we consider the following direct problem: find in the domain Q a function $u(t,x,y)$ such that

$$u(t,x,y) \in X_T, \tag{9}$$

$$\partial_t^\alpha u - a^2\left(u_{xx} + u_{yy}\right) = F(t,x,y),\ (t,x,y) \in Q, \tag{10}$$

$$u(t,0,y) = u(t,l,y) = 0, \quad 0 \le y \le q,\ 0 \le t \le T, \tag{11}$$

$$u(t,x,0) = u(t,x,q) = 0,\ 0 \le x \le l,\ 0 \le t \le T, \tag{12}$$

$$u(0,x,y) = \varphi(x,y), \quad 0 \le x \le l,\ 0 \le y \le q, \tag{13}$$

where $X_T := \{u: u \in C(\bar{Q});\ u(t,\cdot,\cdot) \in C^2(D),\ t \in (0,T];\ u_t(\cdot,x,y) \in C(0,T] \cap L^1(0,T),\ (x,y) \in D\}$, and φ is a given function.

On the base of this initial-boundary value problem for Eq. (10) we consider the following inverse problems.

Inverse problem 2. *Suppose $F(t,x,y) = f(x,y)g(t)$. It is required to find functions $u(t,x,y)$ and $g(t)$ which satisfy conditions (9)-(13) and, in addition, the following ones:*

$$g(t) \in AC[0,T], \tag{14}$$

$$u(t,x_0,y_0) = h(t), \quad 0 \le t \le T, \tag{15}$$

where (x_0,y_0) is a given fixed point in domain D, $\varphi(x,y)$, $h(t)$, $f(x,y)$ are given sufficiently smooth functions, wherein $\varphi(x_0,y_0) = h(0)$.

Inverse problem 3. *Suppose $F(t,x,y) = f(x,y)g(t)$. Find functions $u(t,x,y)$ and $f(x,y)$ which satisfy equalities (9)-(13) and, in addition, the following ones:*

$$f(x,y) \in C(\bar{D}), \tag{16}$$

$$u(t_0,x,y) = \psi(x,y), \quad (x,y) \in \bar{D}, \tag{17}$$

where t_0 is a given fixed point in the half-interval $(0,T]$, wherein $\varphi(x,y)$, $\psi(x,y)$, $g(t)$ are given functions.

We first derive the following inequality for direct problem.

Lemma 2. *Let u be a classical solution of the problem (9)-(13), then we have*

$$\frac{1}{2\Gamma(1-\alpha)}\iint_D \int_0^T \frac{u^2}{(T-t)^\alpha}dtdxdy + a^2 \iint_D \int_0^T \left(u_x^2 + u_y^2\right)dtdxdy$$
$$\le \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)}\iint_D \varphi^2 dxdy + \iint_D \int_0^T uFdtdxdy.$$

The uniqueness of the classical solution immediately folows from Lemma 2.

Theorem 5. *Let $\varphi(x,y) \in C^3(\bar{D})$ such that*

$$\varphi(0,y) = \varphi_{xx}(0,y) = \varphi(l,y) = \varphi_{xx}(l,y), \quad 0 \le y \le q,$$
$$\varphi(x,0) = \varphi_{yy}(x,0) = \varphi(x,q) = \varphi_{yy}(x,q), \quad 0 \le x \le l$$

and $F(t,x,y) \in C\left([0,T];C^2(\bar{D})\right) \cap AC\left([0,T];C(\bar{D})\right)$ satisfies conditions of

$$F(t,0,y) = F(t,l,y) = 0, \quad 0 \le y \le q, \quad 0 \le t \le T,$$
$$F(t,x,0) = F(t,x,q) = 0, \quad 0 \le x \le l, \quad 0 \le t \le T.$$

Then the direct problem (9)-(13) has the unique solution $u \in X_T$.

Using the results obtained above, we now focus on the study of inverse problems.

Theorem 6. *Let* $\varphi(x,y) \in C^4(\bar{D})$,
$$\varphi(0,y) = \varphi_{xx}(0,y) = \varphi(l,y) = \varphi_{xx}(l,y), \quad 0 \le y \le q,$$
$$\varphi(x,0) = \varphi_{yy}(x,0) = \varphi(x,q) = \varphi_{yy}(x,q), \quad 0 \le x \le l,$$
while $f(x,y) \in C^4(\bar{D})$,
$$f(0,y) = f_{xx}(0,y) = f(l,y) = f_{xx}(l,y), \quad 0 \le y \le q,$$
$$f(0,x) = f_{yy}(0,x) = f(x,q) = f_{yy}(x,q), \quad 0 \le x \le l.$$
Besides
$$f(x_0,y_0) \neq 0, \quad h''(t) \in L^1(0,T), h(0) = \varphi(x_0,y_0),$$
then the inverse problem 2 has the unique solution $g(t)$ *in the class* $AC[0,T]$.

Lemma 3. *If* $g(t) \in AC[0,T]$ *and* $|g(t)| \ge g_0 = const > 0$, *then there exists a constant* $C_0 > 0$ *such that for all* $m,n \in \mathbb{N}$ *the estimate*
$$|g_{mn}(t_0)| \ge \frac{C_0}{m^2+n^2}$$
is fulfilled, where
$$g_{mn}(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-a^2\lambda_{mn}^2(t-\tau)^\alpha) g(\tau)d\tau$$
and $E_{\alpha,\beta}(\cdot)$ *is the two parametric Mittag-Leffler function and* C_0 *does not depend on* m,n.

Theorem 7. *Let functions* $\varphi(x,y)$ *satisfies conditions of Theorem 6, while* $\psi(x,y) \in C^4(\bar{D})$,
$$\psi(0,y) = \psi_{xx}(0,y) = \psi(l,y) = \psi_{xx}(l,y), \quad 0 \le y \le q,$$
$$\psi(0,x) = \psi_{yy}(0,x) = \psi(x,q) = \psi_{yy}(x,q), \quad 0 \le x \le l.$$
In addition, the function $g(t)$ *is subject to the conditions of Lemma 3, then the inverse problem 3 has a unique solution.*

The third chapter of the dissertation is titled «**Inverse problems of coefficient determination for fractional-order diffusion-wave equations**» and is devoted to the problem of determining the coefficient that depends solely on the time variable.

First, we consider an inverse coefficient problem for a fractional diffusion equation. Let Ω be a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. Consider the following fractional-diffusion equation in $Q_T := \{(t,x): x \in \Omega \subset \mathbb{R}^N, 0 < t \le T\}$:
$$\mathbb{D}_t^{(\alpha)} u(t,x) = Au(t,x) + q(t)u(t,x) + f(t,x), \quad (t,x) \in Q_T, \tag{18}$$
where $0 < \alpha < 1$ and the operator $-A$ is a symmetric uniformly elliptic operator defined on $D(-A) = H^2(\Omega) \cap H_0^1(\Omega)$ given by
$$Au(t,x) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a_{ij}(x)\frac{\partial}{\partial x_j}u(t,x)\right), \quad (t,x) \in Q_T,$$
in which the coefficients satisfy
$$a_{ij} = a_{ji}, \quad 1 \le i,j \le N, \quad a_{ij} \in C^1(\bar{\Omega}),$$

$$\nu_1 \sum_{i=1}^{N} |\xi_i|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le \nu_2 \sum_{i=1}^{N} |\xi_i|^2, \qquad x \in \bar{\Omega}, \xi \in \mathbb{R}^N, \nu_1, \nu_2 > 0.$$

We supplement the equation (18) with the nonlocal initial condition

$$u(0,x) + \beta u(T,x) = \varphi(x), \quad x \in \Omega, \tag{19}$$

the boundary condition

$$u(t,x) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \tag{20}$$

and integral over-determination condition

$$\int_{\Omega} \omega(x) u(t,x) dx = h(t), \quad 0 \le t \le T, \tag{21}$$

where $\beta \ge 0$ and $f(t,x), \varphi(x), \omega(x), h(t)$ are known functions.

Inverse problem 4. *Find* $(u,q) \in Y_T$ *to satisfy (18)-(20) and the additional measurement (21), where*

$$Y_T = C\big([0,T]; H^2(\Omega)\big) \times C[0,T].$$

We make the following assumptions:

(A1) $\varphi \in H^2(\Omega) \cap H_0^1(\Omega), f \in C([0,T]; D(-A)^{\varepsilon})$ where $0 < \varepsilon < 1$;

(A2) $h(0) + \beta h(T) = (\omega, \varphi)$;

(A3) $\mathbb{D}_t^{(\alpha)} h \in C[0,T]$ satisfies the following inequality:

$$|h(t)| \ge \frac{1}{h_0} > 0, \quad \text{for all} \quad t \in [0,T],$$

where h_0 is a positive constant;

(A4) $\omega(x) \in L^2(\Omega)$.

Theorem 8. *Under hypotheses (A1)-(A4), there exists a solution* $(u,q) \in Y_T$ *of the inverse problem (18)-(21) for small positive* T.

Theorem 9. *Let* $T > 0$. *Under hypotheses (A1)-(A4), if the inverse problem (18)-(21) has two solutions* $(u_i, q_i) \in Y_T$ $(i = 1,2)$, *then* $(u_1, q_1) = (u_2, q_2)$ *for* $0 \le t \le T$.

In the second paragraph of the chapter we discuss an inverse problem of determining a coefficient, depending only on time in a one-dimensional time fractional-wave equation.

Let $D_T := \{(t,x): 0 < t \le T, 0 < x < 1\}$. We consider the following fractional-wave equation:

$$\mathbb{D}_t^{(\alpha)} u(t,x) - \mathcal{L}u(t,x) = q(t)u(t,x) + f(t,x), \quad (t,x) \in D_T, \tag{22}$$

where $1 < \alpha < 2$, $\mathbb{D}_t^{(\alpha)}$ is the Dzhrbashyan-Caputo fractional derivative, that is,

$$\mathbb{D}_t^{(\alpha)} u(t,x) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha+1} u_s(s,x) ds - t^{-\alpha+1} \frac{u_t(0,x)}{\Gamma(2-\alpha)},$$

and $\mathcal{L} := \partial_{xx}$.

We supplement the above fractional wave equation with the nonlocal initial conditions

$$u(0,x) + \delta_1 u(T,x) = \varphi(x), \quad u_t(0,x) + \delta_2 u_t(T,x) = \psi(x), \ x \in (0,1), \tag{23}$$

the boundary conditions

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{24}$$

and integral over-determination condition of the first kind

$$\int_0^1 \omega(x)u(t,x)dx = h(t), \quad 0 \le t \le T, \tag{25}$$

where $\delta_1, \delta_2 \ge 0$ and $f(t,x), \varphi(x), \psi(x), \omega(x), h(t)$ are known functions.

Inverse problem 5. *Find* $(u,q) \in C([0,T]; H^2(0,1)) \cap C^1([0,T]; L^2(0,1)) \times C[0,T]$ *to satisfy (22)-(24) and the additional measurement (25).*

We set

$$\rho(\eta) := 1 + (\delta_1 + \delta_2)E_{\alpha,1}(-\eta) +$$

$$+\delta_1\delta_2\left[\left(E_{\alpha,1}(-\eta)\right)^2 + \eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta)\right], \quad \eta > 0$$

and $\{\eta_1, \dots, \eta_N\} = \{\eta > 0 : \rho(\eta) = 0\}$ with $\eta_1 < \dots < \eta_N$. Moreover, by Λ we set

$$\Lambda = \bigcup_{k=1}^{\infty}\left\{\left(\frac{\eta_1}{(\pi k)^2}\right)^{1/\alpha}, \dots, \left(\frac{\eta_N}{(\pi k)^2}\right)^{1/\alpha}\right\}.$$

We make the following assumptions:

(B1) $f \in C([0,T]; D((-\mathcal{L})^{\frac{1}{\alpha}}))$, $\varphi \in H^2(0,1) \cap H_0^1(0,1)$, $\psi \in H_0^1(0,1)$;

(B2) $h(0) + \delta_1 h(T) = (\omega, \varphi)$, $h'(0) + \delta_2 h'(T) = (\omega, \psi)$;

(B3) $h \in C^2[0,T]$ satisfies the following inequality:

$$|h(t)| \ge \frac{1}{h_0} > 0 \quad \text{for all} \quad t \in [0,T],$$

where h_0 is a positive constant;

(B4) $T \notin \Lambda$;

(B5) $\omega(x) \in H_0^2(0,1)$.

Theorem 10. *Under hypotheses (B1)-(B5), there exists a solution* $(u,q) \in C([0,T]; H^2(0,1)) \cap C^1([0,T]; L^2(0,1)) \times C[0,T]$ *of the inverse problem (22)-(25) for small* $T > 0$.

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and $A: H \to H$ is an arbitrary unbounded positive selfadjoint operator in H. Suppose that A has a complete in H system of orthonormal eigenfunctions e_n and a countable set of positive eigenvalues λ_n. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e., $0 < \lambda_1 \le \lambda_2 \le \dots \to +\infty$.

The third last paragraph of the chapter, we discuss a direct and an inverse problems of determining a coefficient, depending only on time in a time fractional-diffusion equation

$$\partial_t^\alpha u(t) + Au(t) + q(t)u(t) = f(t), \quad 0 < t \le T, \tag{26}$$

where an operator A is defined above.

First we consider the following *direct problem:* given $\alpha \in (0,1)$, $q(t) \in C[0,T]$ and $f(t) \in C([0,T]; H)$, find a function $u(t) \in C\big([0,T]; D(A)\big)$ satisfies the Eq. (26) and the initial condition

$$u(0) = \varphi. \tag{27}$$

Here φ is a given element of H. For this direct problem, we prove the uniquely existence of the solution and derive some regularity results. In the main part of this section, based direct problem, we consider the following inverse problem of finding the coefficient $q(t)$ in the Eq. (26).

Inverse problem 6. *Given* α, $f(t)$ *and* φ, *find a pair of functions* (u,q) *satisfying the problem* (26), (27*) and the additional condition*

$$\Phi[u(t)] = h(t), \quad 0 \le t \le T, \tag{28}$$

where $h\colon [0,T] \to \mathbb{R}$ *is a given function,* $\Phi\colon D(\Phi) \subset H \to \mathbb{R}$ *is a known linear bounded functional.*

We set $0 < \varepsilon < 1$ and make the following assumptions.

(C1) $\varphi \in D(A^{\varepsilon+1})$, $f \in C\big([0,T]; D(A^{\varepsilon})\big)$;

(C2) $h(t) \in AC[0,T]$ and satisfy the conditions $0 < \frac{1}{h_0} \le |h(t)|$, where h_0 is given number;

(C3) $h(0) = \Phi[\varphi]$;

(C4) Φ: $\{\Phi[e_k]\} \in l^2(\mathbb{N})$, where $\mathbb{N}$ is a natural number set.

We have the existence, uniqueness, and regularity of the solution for the direct problem.

Theorem 11. *Let* $\varphi \in D(A^{\varepsilon+1})$ *and* $f \in C\big([0,T]; D(A^{\varepsilon})\big)$ *for some* $\varepsilon \in (0,1)$, *and* $q \in C[0,T]$. *Then there exists a unique solution* $u \in C\big([0,T]; D(A)\big)$ *to (26), (27) such that* $\partial_t^{\alpha} u \in C([0,T]; H)$. *Moreover there exists a constant* $c > 0$ *such that*

$$\| u \|_{C([0,T];D(A))} \le c E_{\alpha\varepsilon,1}\big(\Gamma(\alpha\varepsilon) T \| q \|_{C[0,T]}\big) \Big[\| \varphi \|_{D(A^{\varepsilon+1})} + \| f \|_{C([0,T];D(A^{\varepsilon}))}\Big],$$

and we have

$$u(t) = Z(t)\varphi + \mathcal{H}(f)(t) - \big(\mathcal{H}(q)\big)(u)(t),$$

where $\mathcal{H}\colon C\big([0,T]; D(A^{\varepsilon})\big) \to C\big([0,T]; D(A^{\varepsilon})\big)$ *by*

$$\mathcal{H}(F)(t) = \int_0^t Y(t-s)F(s)ds,$$

$$Y(t)\phi = \sum_{k=1}^{\infty} (\phi, e_k) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^{\alpha}) e_k, \quad \phi \in H, t > 0,$$

$$Z(t)\varphi = \sum_{k=1}^{\infty} (\varphi, e_k) E_{\alpha,1}(-\lambda_k t^{\alpha}) e_k.$$

The continuous dependence of the solution to problem (26), (27) on the data is given by the following theorem.

Theorem 12. *Under the same conditions as Theorem 11, the solution of the direct problem (26), (27) depends continuously on the given data, that is*

$$\| u - \hat{u} \|_{C([0,T];D(A))} \le C[\| \varphi - \hat{\varphi} \|_{D(A^{\varepsilon+1})} + \| q - \hat{q} \|_{C[0,T]} + \| f - \hat{f} \|_{C([0,T];D(A^{\varepsilon}))}],$$

where $c > 0$ *depending on* $\alpha, T, \varepsilon \parallel f \parallel_{C([0,T];D(A^{\varepsilon}))}$ *and* $\parallel q \parallel_{C[0,T]}$.

The main results of this section are presented below.

Theorem 13. *Under hypotheses (C1)-(C4), there exists a sufficiently small* $T > 0$ *such that inverse problem 6 has a unique solution* $q(t) \in C[0,T]$.

Theorem 14. *Let conditions (C1)-(C4) be fulfilled and* u_i *be the solution of (26)-(28) for* $q = q_i \in C[0,T]$ *with* $\parallel q_i \parallel_{C[0,T]} \leq R$ $(i = 1,2)$. *Assume that there exists* $\nu > 0$ *such that*

$$|\Phi[u_2](t)| \geq \nu^{-1} > 0, \qquad for\ \ all \qquad t \in [0,T].$$

Then there exists a constant $\tilde{C} > 0$ *depending on* $R, T, \alpha, \varepsilon$ *and* h_0 *such that*

$$\tilde{C}^{-1} \parallel \partial_t^{\alpha}(\Phi[u_1] - \Phi[u_2]) \parallel_{C[0,T]} \leq \parallel q_1 - q_2 \parallel_{C[0,T]}$$
$$\leq \tilde{C} \parallel \partial_t^{\alpha}(\Phi[u_1] - \Phi[u_2]) \parallel_{C[0,T]},$$

and

$$\parallel u_1 - u_2 \parallel_{C([0,T];D(A))} \leq \tilde{C} \parallel q_1 - q_2 \parallel_{C[0,T]}.$$

The final chapter of the dissertation is called «**Inverse problems for a time-fractional integro-differential equation**», and investigates the problem of finding a memory function and coefficient dependent on the time variable from fractional-order integro-differential equations.

In the first paragraph of chapter four we consider an inverse problem of determining the kernel of a fractional diffusion integrodifferential equation.

We study the following time-fractional integro-differential diffusion equation

$$\partial_t^{\alpha} u(t,x) + Au(t,x) = \int_0^t k(t-s)u(s,x)ds + f(t,x), \ \ (t,x) \in Q_T \qquad (29)$$

with the non-local initial condition

$$u(T,x) - \beta u(0,x) = \varphi(x), \quad x \in \Omega, \qquad (30)$$

the boundary condition

$$u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega. \qquad (31)$$

Here $0 < \alpha < 1$ and the operator A is a symmetric uniformly elliptic operator defined on $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ given by

$$Au(t,x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{ij}(x)\frac{\partial}{\partial x_j}u(t,x)\right) + c(x)u(t,x), (t,x) \in Q_T,$$

in which the coefficients satisfy

$$a_{ij} = a_{ji}, \quad 1 \leq i,j \leq d, \quad a_{ij} \in C^1(\overline{\Omega}), \quad c(x) \in C(\overline{\Omega}), \ c(x) \geq 0, \quad x \in \overline{\Omega}$$

and there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\bar{\xi}_j \geq \mu \sum_{i=1}^{N} |\xi_i|^2, \quad for\ \ all \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^N.$$

The inverse problem is to reconstruct $k(t)$ according to the additional data

$$\mathcal{L}[u(t,\cdot)] := \int_{\Omega} \phi(x)u(t,x)dx = h(t), \quad t \in [0,T], \qquad (32)$$

where $\phi(x), h(t)$ are given functions.

We define Banach space X_T by

$$X_T := \{u : u \in C([0,T]; D(A^\gamma)) \ \ and \ \ \partial_t u \in L^1(0,T; L^2(\Omega))\},$$

where $\partial_t u$ means a distributional sense. Furthermore, we set the topological product

$$Y_T = X_T \times L^1(0,T)$$

endowed with the norm

$$\| (u,k) \|_{Y_T} := \| u \|_{C([0,T];D(A^\gamma))} + \| u_t \|_{L^1(0,T;L^2(\Omega))} + \| k \|_{L^1(0,T)}.$$

Inverse problem 7. *Find* $(u,k) \in X_0^T \times L^1(0,T)$ *to satisfy (29)-(31) and the additional measurement (32).*

Throughout this section, we set $0 < \varepsilon < 1$ and $\gamma > 0$ such that

$$\gamma_0 \geq \gamma > \max\left\{\frac{N}{4} - 1, \varepsilon\right\}, \quad N = 1,2, \dots .$$

We make the following assumptions:

(D1) $\varphi \in D(A^{\gamma_0})$, $f \in X_T$;

(D2) $h(T) - \beta h(0) = \mathcal{L}[\varphi]$, $\mathcal{L}[Au](0) = \mathcal{L}[f](0)$;

(D3) $\partial_t^\alpha h \in C^1[0,T]$ and $\partial_t^\alpha h(0) = 0$ and satisfy the condition $h(0) \neq 0$;

(D4) $\phi \in H_0^2(\Omega)$.

(D5) $\beta \notin [0,1]$.

Theorem 15. *Under hypotheses (D1)-(D5), there exists a unique solution* $(u,k) \in Y_T$ *of the inverse problem (29)-(32) for small* $T > 0$.

In the second paragraph of this chapter is dedicated to deriving a unique solution to the inverse problem associated with a multidimensional time-fractional integro-differential equation.

We consider a time-fractional integro-differential equation with a fractional derivative in time t:

$$\mathbb{D}_t^{(\alpha)} u(t,x) + Au(t,x) = q(t)u_t(t,x) + \int_0^t k(\tau)u(t-\tau,x)d\tau + f(t,x),$$
$$(t,x) \in Q_T, \qquad (33)$$

where $1 < \alpha < 2, \mathbb{D}_t^{(\alpha)}$ is defined in Eq.(22), and the operator A is a symmetric uniformly elliptic operator defined on $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ given by

$$Av(t,x) \equiv - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial}{\partial x_i} v(t,x) \right) + c(x)v(t,x), \quad (t,x) \in Q_T,$$

in which the coefficient satisfy

$$a_{ij} = a_{ji} \in C^1(\Omega), \quad c \in C(\overline{\Omega}), \quad c(x) \geq 0, \quad x \in \overline{\Omega}$$

and there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\bar{\xi}_j \geq \mu \sum_{i=1}^{N} |\xi_i|^2, \quad for \ \ all \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^N.$$

We supplement the above fractional wave equation with the following initial conditions:

$$u(0,x) = a(x), \quad u_t(0,x) = b(x), \quad x \in \Omega \qquad (34)$$

and the zero boundary condition:

$$u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega. \tag{35}$$

If $q(t)$, $k(t)$, $f(t,x)$, $a(x)$ and $b(x)$ are known, then problem (33)-(35) is called a *direct problem*. The inverse problem is to reconstruct $q(t)$ and $k(t)$ according to the additional data

$$u(t,x_i) = h_i(t), \quad t \in (0,T), \tag{36}$$

where $h_i(t), i = 1,2$ are given functions and $x_i \in \Omega$ $(i = 1,2)$ are given points.

We investigate the following inverse problem.

Inverse problem 8. *Find* $u \in C\left([0,T]; D(A^{\gamma+\frac{1}{\alpha}})\right) \cap C^1([0,T]; D(A^{\gamma}))$, $q \in C^1[0,T]$ *and* $k \in C[0,T]$ *to satisfy* (33)-(35) *and the additional measurement* (36), *with some positive constant* γ.

We define Banach space X_0^T by

$$X_T := C([0,T]; D(A^{\gamma+\frac{1}{\alpha}})) \cap C^1([0,T]; D(A^{\gamma})).$$

Furthermore, we set

$$Y_T = X_T \times C^1[0,T] \times C[0,T]$$

endowed with the norm

$$\| (u,q,k) \|_{Y_T} := \| u \|_{X_T} + \| q \|_{C^1[0,T]} + \| k \|_{C[0,T]}.$$

We set $\gamma_0 > \frac{N}{2} + 1, \gamma > 0$ and

$$\frac{N}{4} + 1 < \gamma \leq \gamma_0.$$

We make the following assumptions:

(E1) $h_i \in C^3[0,T]$ with $h_i''(0) = 0, i = 1,2$, $a \in D\left(A^{\gamma_0+\frac{1}{\alpha}}\right), b \in D(A^{\gamma_0})$, $f \in C^1([0,T]; D(A^{\gamma}))$;

(E2) $h_i'(0)q(0) = \partial_t^{\alpha} h_i(0) + Aa(x_i) - f(0,x_i), i = 1,2$;

(E3) $a(x_i) = h_i(0),\ b(x_i) = h_i'(0),\ i = 1,2$;

(E4) $p(t) = h_1'(t)h_2(0) - h_2'(t)h_1(0) \neq 0$ and $p(t) \in C^1[0,T]$ satisfies the following inequality:

$$|p(t)| \geq \frac{1}{p_0} > 0, \quad t \in [0,T],$$

where p_0 is a given positive constant.

Theorem 16. *Let (E1)-(E4) hold. Then, there exists a unique solution* $(u,q,k) \in Y_T$ *of the inverse problem (33)-(36) for small* $T > 0$.

In the last section of this chapter, we investigate backward and inverse problems for the time-fractional integrodifferential diffusion equation. We prove a local in-time existence and uniqueness theorems for the inverse problem of memory reconstruction for abstract fractional integrodifferential equation in Banach space

Consider the time-fractional integrodifferential diffusion equation:

$$\partial_t^{\alpha} u(t) + Au(t) = \int_0^t k(t-s)u(s)ds + f(t), \quad t \in (0,T), \tag{37}$$

where $T > 0$ is a fixed final time, and $\alpha \in (0,1)$, where the operator A is defined in the last paragraph of the third chapter.

The main subject of this section is the following two: backward and inverse problems.

Backward problem given $k(t)$ and $f(t)$, find a function $u(t)$ such that $u\colon [0,T] \to H$ satisfies the equation (37) and the final time condition

$$u(T) = \varphi, \tag{38}$$

where φ is a given element of H, $f\colon [0,T] \to H$ is a known function.

Inverse problem 9. *Given* $\alpha, f(t)$ *and* φ, *determine a pair of functions* $u\colon [0,T] \to H$ *and* $k\colon (0,T) \to \mathbb{R}^+$ *satisfying* (37), (38) *and the additional condition*

$$\Phi[u(t)] = h(t), \quad t \in [0,T], \tag{39}$$

where $h\colon [0,T] \to \mathbb{R}$ *is a given function,* $\Phi\colon D(\Phi) \subset H \to \mathbb{R}$ *is a known linear bounded functional, where* $D(\Phi) = \{u \in H\colon Au \in H\}$.

We define Banach space X_T by

$$X_T := W_1^1(0,T;H) \cap L^1\big(0,T;D(A)\big).$$

Furthermore, we set the topological product

$$Y_T := X_T \times L^1(0,T)$$

endowed with the norm

$$\| (u,k) \|_{Y_T} := \| u \|_{X_T} + \| k \|_{L^1(0,T)}.$$

Definition 3. *A pair of functions* $(u,k) \in Y_T$ *satisfying conditions (37)-(39) is called the solution of inverse problem 9.*

Throughout this section, we set $0 < \varepsilon < 1$ and make the following assumptions.

(F1) $\varphi \in D(A^{\varepsilon+1}), f \in C\big([0,T]; D(A^{\varepsilon})\big) \cap C^1([0,T];H)$;

(F2) $h(T) = \Phi[\varphi]$, $\Phi[Au](0) = \Phi[f](0)$;

(F3) $\partial_t^{\alpha} h \in C^1[0,T]$ and $\partial_t^{\alpha} h(0) = 0$ and satisfy the condition $h(0) \neq 0$;

(F4) $\Phi\colon \{\lambda_n \Phi[e_n]\} \in l^2(\mathbb{N})$, where $l^2(\mathbb{N})$ is the space of square summable sequences.

Main results are as follow:

Theorem 17. *Let* $\varphi \in D(A^{\varepsilon+1}), f \in C\big([0,T]; D(A^{\varepsilon})\big) \cap C^1([0,T];H)$ *and* $k \in L^1(0,T)$ *for some* $\varepsilon \in (0,1)$. *Then there exists a unique solution* $u \in X_T$ *to (37)-(38) with* $\partial_t^{\alpha} u \in L^1(0,T;H)$.

Theorem 18. *Under hypotheses (F1)-(F4), there exists a unique solution* $(u,k) \in Y_T$ *of the inverse problem (37)-(39) for small* $T > 0$.

CONCLUSION

This dissertation is dedicated to the investigation of direct, backward, and inverse problems for partial differential equations involving fractional-order integral-differential operators.

The main conclusions of this scientific work are as follows:

1. The formula for the solution of the Cauchy problem for the partial derivative equation of first and fractional order in time is obtained and it is proved that it is a classical solution.
2. The correctness of the direct and inverse problems for the two-dimensional fractional diffusion equation was established.
3. Existence and uniqueness results were obtained for solutions to linear inverse problems, specifically for determining source functions, posed for the fractional-order diffusion equation in a parallelepiped domain.
4. The uniqueness of the solution was established for the problem involving a fractional-order diffusion differential equation with a nonlocal initial condition and homogeneous Dirichlet boundary conditions. In addition, the inverse problem of determining a time-dependent coefficient was solved using the Gauss-Seidel iterative method.
5. The inverse problem of determining the time-dependent coefficient from the one-dimensional fractional wave equation was solved locally. In this case, both initial conditions were given in nonlocal form, significantly complicating the problem.
6. The correctness of the solution to the Cauchy problem and the time-dependent coefficient problem posed for the abstract fractional diffusion equation was proven.
7. The inverse problem of determining the time-dependent kernel from a non-local integro-differential equation with an integral condition was successfully solved.
8. The problem of determining two time-dependent coefficients from a multidimensional fractional integro-differential equation, with boundary conditions specified at two distinct points with respect to the solution of the direct problem, was fully solved.
9. The unique solvability of the inverse problem of determining the kernel for a time-fractional integro-differential equation was established.

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ИНСТИТУТ МАТЕМАТИКИ

РАХМОНОВ АСКАР АХМАДОВИЧ

КОРРЕКТНОСТЬ ОБРАТНЫХ КОЭФФИЦИЕНТНЫХ ЗАДАЧ ДЛЯ ДРОБНЫХ ДИФФУЗИОННО-ВОЛНОВЫХ УРАВНЕНИЙ

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ВВЕДЕНИЕ (аннотация докторской диссертации)

Целью исследования является изучение однозначной разрешимости прямых, обратных и ретроспективных задач для дробно-дифференциальных и интегро-дифференциальных диффузионно-волновых уравнений.

Объект исследования: начально-краевые задачи и обратные задачи для диффузионно-волнового уравнения дробного порядка в ограниченных и неограниченных областях.

Научная новизна исследования состоит в следующем:

получена явная формула для решения задачи Коши для одномерного дробно-дифференциального уравнения и установлены достаточные условия для того, чтобы полученная функция имела классическое решение;

получены условия единственности решений прямых и нелинейных обратных задач, связанных с уравнением дробной диффузии, в неограниченной области. Кроме того, получены достаточные условия для однозначного определения исходных членов из уравнения дробной диффузии;

доказано существование и единственность решения для нелокальной начальной и краевой задачи для дробных диффузионно-волновых уравнений;

доказано существование и корректность постановки задачи Коши для абстрактного дробного уравнения диффузии, а также установлена условная корректность постановки обратной задачи с зависящими от времени коэффициентами;

доказана теорема о существовании и единственности решения нелинейной обратной задачи определения коэффициента и функции памяти в дробном диффузионно-волновом уравнении;

доказана теорема о единственности разрешимости нелокальных начальных прямых и нелинейных обратных задач, связанных с многомерным дробным уравнением диффузии с дробной производной Герасимова-Капуто.

Внедрение результатов исследования. Научные результаты, полученные в ходе исследования диссертации, реализованы в следующих научно-исследовательских проектах:

условия единственности и существования для приближенных решений уравнений в построенной математической модели используются для определения эффективных параметров ветроэнергетических установок в инновационном проекте No. IL-21071166 на тему «Разработка вертикальной осевой ветряной турбины для низких скоростей ветра», направлена на определение эффективных параметров ветряных турбин (Справка № 281-3 от 25 марта 2025 года, выданная Институтом сейсмостойкости сооружений и механики). Научные результаты обеспечили разработку вычислительных моделей для решения гиперболических дифференциальных уравнений с ядром Колтунова;

полученные результаты однозначного определения коэффициентов и функций памяти для заданных дифференциальных уравнений диффузии и супердиффузии в ограниченных областях были использованы в зарубежном фундаментальном проекте «Математическое моделирование в социологии, геофизике и технике» под номером №-122041100096-4. Они были применены при исследовании подземных сжимающихся сред (согласно свидетельству № 18 от 24 марта 2025 года Владикавказского научного центра Российской академии наук, Федерального государственного научного учреждения, Федерального научного центра-филиала Южного института математики, Российская Федерация). Полученные научные результаты позволили наблюдать волновые процессы в вязких подземных средах;

полученные результаты определения зависящего от времени коэффициента для уравнений субдиффузии в обратной, нелокальной начальной и первой краевой задачах были использованы для проверки изменения температуры в зарубежном фундаментальном проекте № AP09258836 «Разработка численных алгоритмов для дифференциальных математических моделей аномальной диффузии». Это приложение было использовано для изучения изменения температуры (Справка № 04/857 от 2 апреля 2025 г. Международного казахско-турецкого университет им. Ходжи Ахмета Ясави, Казахстан). Полученные научные результаты обеспечили возможность эффективного наблюдения за процессами теплообмена в водных объектах;

результаты по определению коэффициента и функции памяти из начально-краевых задач для дробных субдиффузионных и супердиффузионных уравнений с нелокальными начальными условиями были использованы в статьях, опубликованных в зарубежных научных журналах, для решения обратных задач для дробных параболических уравнений с нелокальным бигармоническим оператором (Fractal and Fractional, 2023, 7(5): 404; Physica Scripta, 2024, 99(10): 105242; AIMS Mathematics, 2023, 9(3): 6832-6849). Применение научных результатов позволило определить источник в двумерном дробном параболическом уравнении с нелокальными условиями.

Структура и объем диссертации. Диссертация состоит из введения, четырех глав, заключения и списка использованной литературы. Объем диссертации составляет 212 страниц.

E’LON QILINGAN ISHLAR RO‘YXATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo‘lim (part 1; 1 часть)

1. Durdiev D., Rahmonov A. *Global solvability of inverse coefficient problem for one fractional diffusion equation with initial non-local and integral overdetermination conditions.*// Fractional Calculus and Applied Analysis. -2025. -Vol. 27. -P. 117-145. (Scopus, IF=0,84).
2. Rahmonov A.A. *Inverse problem for the time-fractional diffrential equation.*// Samarkand University Scientifi Bulletin. -2024. -Vol. 5(147/2). -P. 113-119. (01.00.00; №2).
3. Rahmonov A.A. *Well-posedness of the inverse problem for a time-fractional integro-differential equation.*// Bulletin of the Institute of Mathematics. -2024. -Vol. 7. no. 5, -P. 48-65. (01.00.00; №17).
4. Rahmonov A.A. *Inverse problem for the diffusion equation with fractional time derivative.*// Scientific Reports of Bukhara State University. -2024. -Vol. 8. -P. 20-24. (01.00.00; №3).
5. Rahmonov A.A. *Determination of a coefficient and kernel in a* d*-dimensional fractional integro-differential equation.*// Vladikavkaz Mathematical Journal. -2024. -Vol. 26(3). -P. 72-85. (Scopus, IF=0,21).
6. Rahmonov A.A. *Recovering the time-dependent coefficient in fractional wave equation.*// Uzbek Mathematical Journal. -2024. -Vol. 68(2). -P.125-140. (01.00.00; №6, Scopus, IF=0,15).
7. Rahmonov A., Akramova D., Elmuradova H., Togaev F. *Determination of a coefficient and kernel in a two-dimensional fractional integrodifferential equation.*// Lobachevskii Journal of Mathematics. -2024. -Vol. 45(2). -P. 800-818. (Scopus, IF=0,45).
8. Rahmonov A.A. *Determination of a kernel in a nonlocal problem for the time-fractional integro-differential equation.*// Eurasian Journal of Mathematical and Computer applications. -2024. -Vol. 12(2). -P. 107-133. (Scopus, IF=0,33).
9. Durdiev D.K., Sultanov M.A., Rahmonov A.A., Nurlanuly Y. *Inverse problems for a time-fractional diffusion equation with unknown right-hand side.*// Progress in Fractional Differentiation and Applications. -2023. -Vol. 9(4). -P. 639-653. (Scopus, IF=0,33).
10. Durdiev D.K., Rahmonov A.A. *Inverse coefficient problem for a fractional wave equation with time-nonlocal and integral overdetermination conditions.*// Boletín de la Sociedad Matemática Mexicana. - 2023. -Vol. 29(50). -P. 1-33. (Scopus, IF=0,4).
11. Durdiev D.K., Shishkina E.L., Rahmonov A.A. *The explicit formula for a solution of wave differential equation with fractional derivatives in the multi-dimensional space.*// Bulletin of the Institute of Mathematics. -2022. -Vol. 5(2). -P. 1-12. (01.00.00; №17).

12. Durdiev D.K., Rahmonov A.A. *A multidimensional diffusion coefficient determination problem for the time-fractional equation.*// Turkish Journal of Mathematics. -2022. -Vol. 46(6). -P. 2250-2263. (Scopus, IF=0,41).
13. Rahmonov A.A., Bozorov Z.R. *Recovering time dependent function for the fractional diffusion equation in a finite domain.*// Uzbek Mathematical Journal. -2022. -Vol. 66(2). -P. 135-149. (01.00.00; №6, Scopus, IF=0,15).
14. Sultanov M.A., Durdiev D.K., Rahmonov A.A. *Construction of an explicit solution of a time-fractional multidimensional differential equation.*// Mathematics. -2021. -Vol. 9(2052). (Scopus, IF=0,48).
15. Durdiev D.K., Rahmonov A.A., Bozorov Z.R. *A two-dimensional diffusion coefficient determination problem for the time-fractional equation.*// Mathematical Methods in the Applied Sciences. -2021. -Vol. 44. -P. 10753-10761. (Scopus, IF=0,61).

II bo‘lim (part 2; 2 часть)

16. Rahmonov A.A., Durdiev D.K., Akramova D.I. *A 2D inverse problem for a fractional-wave equation.*// AIP Conf. Proc. -2024, 3004. 040012. (Scopus).
17. Durdiev D.K., Sultanov M.A., Rahmonov A.A., Turebekov R.Z., Nurlanuly Y. *Source identification problems for time-fractional diffusion equation.*// 6^{th} International conference of mathematical sciences ICMS 2022. 20-24 July. Istanbul. -P. 66.
18. Durdiev D.K., Rahmonov A.A. *The explicit formula for solution of wave differential equation with fractional derivatives in the multi-dimensional space.*// Modern problems of applied mathematics and information technologies al-Khwarizmi 2021. 15-17 November. Fergana. -P. 119.
19. Rahmonov A.A. *Determination of a coefficient and kernel in a* $d-$*dimensional fractional integrodifferential equation.*// Actual problems of applied Mathematics and information technologies-al-Khwarizmi 2023. 25-26 September. Samarkand. -P. 201.
20. Rahmonov A.A. *A time-nonlocal inverse problem for a fractional diffusion-wave equation with an integral overdetermination condition.*// Operator algebras, non-associative structures and related problems 2023. 14-15 September. Tashkent. -P. 173-174.
21. Durdiev D.K., Rahmonov A.A. *Global solvability of inverse coefficient problem for one fractional diffusion equation with initial non-local and integral overdetermination conditions.*// Actual problems of physics, mathematics and mechanics 2023. May 14-15. Bukhara. -P. 131.
22. Durdiev D.K., Rahmonov A.A. *A multi-dimensional diffusion coefficient determination problem for the time-fractional equation.*// Order analysis and related questions of mathematical modelling, XVII. Operator theory and differential equations 2021. 20-25 September. Vladikavkaz. -P. 86-88.
23. Durdiev D.K., Rahmonov A.A., Z.R. Bozorov. *A 2D diffusion coefficient determination problem for the time-fractional equation.*// Order analysis and related questions of mathematical modelling, XVII. operator theory and

differential equations 2021. 20-25 September. Vladikavkaz. -P. 89-90.

24. Rahmonov A.A. *Well-posedness of the inverse problem for a time-fractional integro-differential equation.*// Traditional International April Mathematical Conference in honour of the Day of Science of the Republic of Kazakhstan, 2023. Almaty. -P. 214.
25. Rahmonov A., Safarov J., Boboev S. *Determination of a coefficient and kernel in a two-dimensional fractional integro-differential equation.*// Mathematical analysis and its applications in modern mathematical physics 2022. 23-24 September, Samarkand. -P. 215-216.
26. Rahmonov A. *Determination of a coefficient and kernel in a two-dimesional fractional integro-differential equation.*// Order analysis and related questions of mathematical modelling, XVII. Operator theory and differential equations 2023.June 29- July 5. Vladikavkaz. -P. 217-218.
27. Rahmonov A. *An inverse source problem for a fractional diffusion-wave equation.*// IX International scientific conference «Modern problems of applied mathematics and information technologies Al-Khwarizmi 2024». 22-23 October. Tashkent. -P. 183-184.

Avtoreferat “O‘zbekiston matematika jurnali” tahririyatida 2025 yil 28 aprelda tahrirdan o‘tkazilib, o‘zbek, ingliz va rus tillaridagi matnlar o‘zaro muvofiqlashtirildi.

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