**MIRZO ULUG'BEK NOMIDAGI O'ZBEKISTON MILLIY UNIVERSITETI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.03/30.12.2019.FM.01.01 RAQAMLI ILMIY KENGASH**

**O'ZBEKISTON MILLIY UNIVERSITETI**

**AXADQULOV XABIBULLA ABURUYKULOVICH**

# MAXSUSLIKKA EGA BO'LGAN BIR O'LCHOVLI AKSLANTIRISHLAR RENORMALIZATSIYALARINING ASIMPTOTIK HOLATI VA QO'SHMALARI

**01.01.01– Matematik analiz**

**FIZIKA-MATEMATIKA FANLARI BO'YICHA FAN DOKTORI (DSc) DISSERTATSIYASI AVTOREFERATI**

**Toshkent–2025**

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**Fizika va matematika fanlari bo'yicha fan doktori (DSc) dissertatsiyasi avtoreferati mundarijasi**

**Оглавление автореферата диссертации доктора наука (DSc) по физико-математическим наукам**

**Contents of dissertation abstract of doctor of science (DSc) on physical-mathematical sciences**

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**FIZIKA-MATEMATIKA FANLARI BO'YICHA FAN DOKTORI (DSc) DISSERTATSIYASI AVTOREFERATI**

**Toshkent–2025**

**Fizika-matematika fanlari bo'yicha fan doktori (DSc) dissertatsiyasi mavzusi O'zbekiston Respublikasi Vazirlar Mahkamasi huzuridagi Oliy attestatsiya komissiyasida B2022.3.PhD/FM743 raqam bilan ro'yxatga olingan.**

Dissertatsiya Mirzo Ulug'bek nomidagi O'zbekiston Milliy Universiteti va Utara Malayziya Universitetida bajarilgan.

Dissertatsiya avtoreferati uch tilda (o'zbek, ingliz, rus (rezyume)) Ilmiy kengash vebsahifasi (http://kengash.mathinst.uz) va "ZiyoNet" ta'lim axborot tarmog'ida (http:www.ziyonet.uz) joylashtirilgan.

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Doktorlik dissertatsiyasi bilan Mirzo Ulug'bek nomidagi O'zbekiston Milliy universitetining axborot-resurs markazida tanishish mumkin (73-raqam bilan ro'yxatga olingan). Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet kuchasi, 9-uy. Tel.: (+99871) 207-91-40.

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## KIRISh (fan doktori (DSc) dissertatsiyasi annotatsiyasi)

**Dissertatsiya mavzusining dolzarbligi va zarurati.** Jahonda matematika va fizikaning turli sohalarida olib borilayotgan ilmiy va amaliy tadqiqotlar orasida bir o‘lchovli dinamik sistemalar, xususan, interval akslantirishlari va aylana akslantirishlarini o’rganish va amaliyotda qo’llash yetakchi o‘rinlardan birini egallamoqda. Dunyo miqyosida aylana diffeomorfizmlarining sinish tipidagi maxsuslikka ega aylana gomeomrfizmlari yo‘nalishidagi aksariyat ilmiy ishlar renormalizatsiyaga tegishli bo‘lishini hisobga olsak, taassufki uning *P*-gomeomorfizmlar sinfi deb ataluvchi bo‘limi ko‘plab tadbiqiy masalalarni amaliyotga joriy etishni taqozo etadi. Shu jihatdan matematik analiz, funksional analiz va dinamik sistemalar nazariyasini o’rganish bo‘yicha keng qamrovli chora-tadbirlar amalga oshirilib, fan va texnikada undan foydalanish muhim ahamiyatga ega hisoblanadi.

Jahonda, hozirgi kunda aylana akslatirishlar nazariyasini bitta sinishga aylana diffeomorfizmining renormalizatsiyalari Myobius akslantirishlarining ikki parametrli oilasiga qanday tezlik bilan yaqinlashishni o‘rganishga yo‘naltirilgan ilmiy-tadqiqot ishlari olib borilmoqda. Bu borada aylana akslantirishlari eng ko’p qo’llaniladigan sohalardan biri bo‘lishi bilan bir qatorda, egri chiziqli akslatirishlar nazariyasidagi qo’shma gomeomorfizmlarning silliqligiga ta’luqli bo’lgan natijalaridan keng foydalanib, maydonning uzoq muddatdagi xatti-harakatlari bashoratlar qilinib, pirovardida ko‘plam ilmiy izlanishlarda umumlashgan interval almashinuvchi asklanitishlarining qattiqlik masalalarini yechishga alohida e’tibor berilmoqda.

Respublikamizda fundamental fanlarning ilmiy va amaliy tadbiqiga ega bo‘lgan yo’nalishlariga alohida e’tibor berilibgina qolmay, fanlar akademiyasi tashabbusi bilan “Puankare tomonidan kiritilgan kombinatorika nazariyasi bilan bog‘liq burish sonining muhim xossalari”ni o‘rganishga laborator sharoitlar yaratilmoqda, hamda burish sonlarining uzluksiz kasrga yoyilmasi va dinamik bo‘linishlar haqidagi tushunchalarni umumlashtirish yuzasidan keng qamrovli chora tadbirlar amalga oshirilib, muayyan natijalarga erishilmoqda. “Differensial topologiya, kompleks o‘zgaruvchili funksiyalar nazariyasi” fanlarining ustuvor yo‘nalishlarida, hamda dinamik sistemalar nazariyasi, kompleks analizning zamonaviy muammoalari, algebra va uning tadbiqlari kabi ilmiy sohalarida xalqaro standartlar darajasida ilmiy tadqiqotlar olib borish matematika fanining asosiy vazifalar va faoliyat yo‘nalishlari[1] bo‘yicha muhim vazifalar belgilab berilgan. Ushbu vazifalarni amalga oshirishda, jumladan, Danjua tengsizliklari hamda dinamik bo‘linish intervallari uzunliklari nisbati uchun universal baholar olingan holda, yuqoridagiga o‘xshash dinamik bo‘linish intervallari kasr buzilishi (ratio distortion) hamda aylanib kelgan kasr buzilishlari (cross-ratio distortion) uchun aniq baholarni talab qiladigan integrallashgan yechimlarni yaratish muhim ahamiyat kasb etmoqda.

O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-son “O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi

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[1] O‘zbekiston Respublikasi Prezidentining 2020-yil 7-maydagi “Matematika sohasidagi ta`lim sifatini oshirish va ilmiy tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida”gi PQ-4708-son qarori

to‘g‘risida”gi farmoni, 2017-yil 17-fevraldagi PQ-2789 son “Fanlar akademiyasi faoliyati, ilmiy-tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora-tadbirlari to‘g‘risida”, 2020-yil 7-maydagi PQ-4708 son “Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish choratadbirlari to‘g‘risida”gi qarorlari hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy xujjatlarda belgilangan vazifalarni amalga oshirishga ushbu dissertasiya ishi muayyan darajada xizmat qiladi.

**Tadqiqotning respublika fan va texnologiyalari rivojlanishi ustuvor yo‘nalishlariga bog‘liqligi.** Mazkur tadqiqot respublika fan va texnologiyalarni rivojlantirishining IV. “Matematika, mexanika va informatika” ustivor yo‘nalishiga muvofiq bajarilgan.

**Dissertatsiya mavzusi bo‘yicha halqaro ilmiy tadqiqotlar sharhi[2].** Dinamik sistemalar va ularning tadbiqlariga doir ilmiy tadqiqotlar yetakchi xorijiy davlatlardagi oliy ta’lim muassasalari va nufuzli ilmiy tadqiqot markazlarida faol tarzda olib borilmoqda, jumladan Ilg‘or tadqiqotlar markazi (IAS)- Prinston, (AQSh), Tatbiqiy fizika instituti (FAP)- Gyote universiteti, (Germaniya), Dinamik sistemalar va o‘z-o‘zini tashkil etish uchun Maks Plank instituti- Gyottingen, (Germaniya), Nochiziqli va murakkab tizimlar markazi (CENOLIC)- Turin universiteti, (Italiya), Halqaro nazariy fizika markazi (ICTP)- Triest, (Italiya), Filds nomidagi matematika fanlari tadqiqot instituti- Toronto, (Kanada), Nyuton nomidagi matematika fanlari instituti Kembrij universiteti, (Buyuk Britaniya), Anri Puankare instituti- Parij, (Fransiya), Malayziya milliy universiteti (UKM) fan va texnologiyalar fakulteti, matematika bo‘limi (Malayziya), Shimoliy Malayziya universiteti (UUM) matematika bo‘limi (Malayziya) va boshqalar.

So‘nggi vaqtlarda dinamik sistemalar sohasida, jumladan, bir o‘lchovli dinamik sistemalar renormalizatsiyalarning asimptotik tabiati, qattiqlik muammolari hamda qo‘shma akslantirishning xossalari bo‘yicha dunyo miqyosida olib borilgan ilmiy tadqiqotlarda bir nechta e’tiborga loyiq natijalar olindi. Ulardan ba’zi asosiylari quyidagilar:

Fransiyaning Orsay universitetidan Selim Gazuani tomonidan olingan muhim natija: ikkita nuqtada sinishga ega bo‘lgan bo‘lakli affin aylana akslantirishi Lebeg o‘lchovi bo‘yicha Mors-Smeyl tabiatiga ega bo‘lishini ko‘rsatdi. Shveysariyaning Syurix universitetidan Frank Trujillo tomonidan davriy nuqtalarga ega bo‘lmagan bir nechta kritik nuqtali $C^3$ silliq aylana akslantirishining yagona invariant o‘lchovining Xausdorf o‘lchami uchun aniq chegaralar o‘rnatildi. Fransiyaning Orsay universiteti vakili Korinna Ulchigrai tomonidan jenusi ikki bo‘lgan sirtlardagi akslantirishlar uchun qattiqlik muammosi yechilgan bo‘lib, bu natija aylana diffeomorfizmlari uchun Erman teoremasini yuqori tartibli akslantirishlar uchun umumlashtiradi. Bu natija tordagi oqimlar uchun ham muhim hisoblanadi. AQShning Texas universiteti vakili Sasha Koshich tomonidan qatiqlik mauammosiga muhim yechim ko‘rsatilgan: deyarli

[2] Dissertasiya mavzusiga doir xorijiy ilmiy tadqiqotlar tahlili quyidagi manbalarga asosan ajratilgan: Discrete and Continuous Dynamical Systems https://www.aimsciences.org/DCDS, Ergodic Theory and Dynamical System https://tinyurl.com/4wxkt9h4, Nonlinearity https://iopscience.iop.org/journal/0951-7715, Differential Equations and Dynamical Systems https://link.springer.com/journal/12591 .

barcha irratsional burish sonlari uchun sinish nuqtaga ega bo‘lgan ikkita aylana diffeomorfizmlari $C^1$ silliq qo‘shma gomeomorfizmga ega bo‘lib, birorta ham $\alpha > 0$ uchun $C^{1+\alpha}$ silliq qo‘shma gomeomorfizmga ega bo‘lmasligi isbotlangan. Kanadaning Toronto universiteti professori Konstantin Xanin $(0, 1)$ oraliqdagi deyarli barcha irratsional $\rho$ burish sonlari uchun bitta nuqtada sinishga ega va burish soni $\rho$ bo‘lgan aylana diffeomorfizm-lari invariant o‘lchovining Xausdorf o‘lchami nol bo‘lishi isbotlangan. Toshkent shahridagi Turin Politexnika universiteti professori Axtam Djalilov tomonidan bo‘lakli silliq aylana gomeomorfizmlarining Rouzi-Vich renormalizatsiyalari Myobius funksiyalariga yaqinlashishi isbotlangan. Yuqorida keltirilgan natijalar birgalikda bir o‘lchovli dinamik sistemalarning asimptotik tabiatini va xossalarini tushunishga yordam beradi hamda bu sohada akslantirishning renormalizatsiyasi, qattiqligi va boshqa jihatlarini yoritadi.

**Muammoning o‘rganilganlik darajasi.** Anri Puankare dinamik sistemalar sohasi kashfiyotchilaridan biri hisoblanib, “Osmon mexanikasining yangi usullari” (1892-1899) va “Osmon mexanikasi bo‘yicha ma’ruzalar” (1905-1910) nomli ilmiy risolalar muallifidir. Mazkur risolalarda “uch jism” muammosi yechimini hamda traektoriyaning qaytishi haqidagi Puankare teoremasini keltirilgan. Aleksandr Lyapunovning approksimatsiya usullari (1899 yil) oddiy differensial tenglamalar turg‘unligini boholashlarning asosi bo‘lib xizmat qildi. Jorj Devid Birkgof 1913 yilda Puankarening “Oxirgi geometrik teorema” sini isbotladi hamda 1927 va 1931 yillarda statistik mexanika muammolarini yechish orqali dinamik sistemalar sohasiga va ergodiklik nazariyasi rivojiga katta hissa qo‘shdi. Stiven Smeyl fanga Smeyl taqasi tushunchasini kiritdi, va bu yo‘nalishda tadqiqotlar olib bordi, boshqalarni ham ilhomlantiradigan dasturlar taklif qildi. 1964 yilda Aleksandr Nikolaevich Sharkovskiy, agar diskret dinamik sistema uch davrli davriy traektoriyaga ega bo‘lsa, u holda bu sistema barcha davrli davriy traetoriyaga ega bo‘lishi haqida o‘zining Sharkovskiy teoremasini isbotladi.

Parri (1964) tomonidan interval akslantirishlar bo‘lakli chiziqli akslantirishlarga kombinator ekvivalent (yoki yarim qo‘shma) bo‘lishi isbotlangan. Bundan tashqari, o‘sha yilning o‘zida, Sharkovskiy o‘zining mashhur teoremasini e’lon qildi: davri uch bo‘lgan davriy nuqtaning mavjudligidan ixtiyoriy tartibli davriy nuqtaning mavjud bo‘lishi kelib chiqadi. Keyinchalik, Metropolis va Stayn (1973) lar tomonidan hamda Milnor va Terston (1977) larning mashhur qo‘lyozmalarida ikkita bo‘lakli monoton va uzluksiz interval aslantirishlari kombinator ekvivalent bo‘lishi uchun ularning egilish nuqtalari traektoriyalari bir xil tartibga ega bo‘lishi zarur va yetarli ekanligi isbotlangan. Puankare esa davriy nuqtaga ega bo‘lmagan har qanday aylana akslantirishlari chiziqli burishga yarim qo‘shma ekanligini ko‘rsatdi. Danjua (1932) o‘z teoremasida, agar gomeomorfizm va uning teskarisi silliq bo‘lsa, u holda bu yarim qo‘shma aslida qo‘shma gomeomorfizmligini isbotladi.

Danjua teoremasida ushbu $K_n = \max_{\xi} |\log(f^{q_n}(\xi))'|$ ko‘rinishda aniqlangan Danjua ko‘paytmasini aniq baholash muhim rol o‘ynaydi, bu yerda $f'(\xi)$, - $f$ funksiyaning $\xi$ nuqtadagi hosilasini, $q_n$ esa $f$ ning birinchi qaytish vaqtini bildiradi.

Burish soni $\rho$ irratsional, yo‘nalishini saqlovchi shunday $C^1$ -silliq $f$ diffeomorfizmni qaraymizki, $\log f'$ ning variatsiyasi chegaralangan bo‘lsin. Bunday diffeomorfizmlar uchun Danjua tomonidan ushbu $K_n \le v$ tengsizlik isbotlangan, bu yerda $v = \operatorname{var}_{S^1} \log f'$. Keyinchalik bu natija Kasnelson & Ornsteyn, Sinay & Xanin, Xanin & Teplinskiylar tomonidan kengaytirildi. Ularning ishlarda, agar $f$ aylana diffeomorfizmi $C^{2+\varepsilon}(S^1)$, $\varepsilon > 0$ sinfga tegishli bo‘lib, burish soni $\rho$ irratsional bo‘lsa, u holda $K_n$ ning eksponensail ravishda nolga intilishi isbotlangan. Danjua teoremasining o‘rinli bo‘lishi, aylana gomeomorfizmi va chiziqli burishni orasidagi qo‘shma akslantirishning mavjud bo‘lishini kafolatlaydi. Shu nuqtai nazardan, tabiiy savol tug‘iladi: *qo‘shma akslantirish silliq bo‘lishi uchun qanday shartlar mavjud?*

Dastlabki lokal natija Arnold tomonidan isbotlangan. Bu natija Mozer tomonidan kengaytirilgan. 1970-yillar oxirida Erman tomonidan diffeomorfizmning chiziqli burishga yaqinligini talab qilmaydigan global natija olingan. Keyinchalik, Yokkoz esa Erman teoremasini deyarli barcha irratsional burish sonlariga kengaytirdi. 1980-yillarning oxirida Erman nazariyasiga ikkita o‘ziga xos yondashuv paydo bo‘ldi. Katsnelson & Ornsteyn, Xanin & Sinay ishlarida bu yondashuvlarni rivojlantirdi hamda kengroq sinfdagi aylana diffeomorfizmlari uchun Yokkoz teoremasini kengaytirdi. Ular tomonidan, $C^{2+\varepsilon}(S^1)$, $\varepsilon > 0$ sinfdagi aylana gomeomorfizmi va chiziqli burish orasidagi qo‘shma gomeomorfizm hech bo‘lmaganda $C^1$ silliq bo‘lishi isbotlangan.

Erman aylana diffeomorfizmlari va chiziqli burish orasidagi qo‘shia akslantirishning silliqligi uchun ba’zi kriteriyalarni berdi. Bu kriteriyalar qo‘shma gomeomorfizmning $\beta$ -Gyolder uzluksizligi, absolyut uzluksizligi, Lipshis uzluksizligi uchun shartlarni o‘z ichiga oladi. Bundan tashqari, qo‘shma akslantirishning silliqligi uchun zaruriy va yetarli shartlar isbotlangan.

Aylana diffeomorfizmlarining tabiiy umumlashmasi bu sinish tipidagi maxsuslikka ega aylana gomeomrfizmlari bo‘lib, dastlab Erman tomonidan kiritilgan va bu *P*-gomeomorfizmlar sinfi deb ataladi. Dastlab, Erman ikkita sinish nuqtasiga ega bo‘lakli chiziqli aylana gomeomorfizmlari va chiziqli burish orasidagi qo‘shia gomeomorfizmning singulyar bo‘lishini isbotladi. Keyinchalik uning natijasini Djalilov & Xanin, Djalilov, Akin & Temir, Djalilov & Luis, Djalilov, Mayer & Safarovlar ishlarida umumiy *P*-gomeomorfizmlar uchun kengaytirdilar.

Aylana diffeomorfizmlarining yana bir muhim jihati ularning renormalizatsiyasi tabiatidir. Bu yo‘nalishdagi dastlabki natija 90-yillarning boshlarida Xanin & Vul tomonidan olingan. Ular bitta sinishga aylana diffeomorfizmining renormalizatsiyalari Myobius akslantirishlarining ikki parametrli oilasiga eksponensial tezlik bilan yaqinlashishini isbotladilar. Xanin & Xemelev, Xanin & Teplinskiy, Xanin & Yampolskiy, Teplinskiy ishlarida Myobus akslantirishlarni o‘rganish natijasida akslantirishlar fazoning ma’lum bir sohasida renormalizatsiya operatori kritik burish uchun Lendford tomonidan taxmin qilingan xossalarga o‘xshash qat’iy giperbolik xossalarga ega bo‘ladi.

2014-yilda Xanin & Koshich ikkita aylana diffeomorfizmining renormalizatsiyasi bir-biriga yaqinlashishini isbotladi.

Burish soni Diofant shartini fanoatlantiruvchi alana gomeomorfizmlari renormalizatsiyalarining yaqinlashishi, qattiqlik muammosi deb ataladigan ular orasidagi qo‘shma akslantirishning silliqligini ta’minlaydi. Sinish tipidagi maxsuslikka ega hol uchun, qattiqlik masalasiga doir dastlabki natijalar mumkin bo‘lgan irratsional burish sonlari to‘plami Lebeg o‘lchovi nolga teng $C^{2+\alpha}$ silliq aylana diffeomorfizmlari uchun Xanin & Xmelevlar hamda Xanin & Teplinskiylar tomonidan olingan. Bu yo‘nalishdagi eng ajoyib natijalar Xanin & Koshich, Xanin, Koshich & Mazzeolar ishlarida deyarli barcha irratsional burish sonlari uchun isbotlangan.

**Dissertatsiya tadqiqotining dissertatsiya bajarilgan oliy ta’lim muassasasi va ilmiy tadqiqot institutining ilmiy tadqiqot ishlari rejalari bilan bog‘liqligi.**

Dissertatsiya tadqiqoti Mirzo Ulug’bek nomidagi O’zbekiston Milliy Universiteti “Matematik analiz” kafedrasi ilmiy tadqiqot rejasi doirasida hamda Shimoliy Malayziya Universitetining (UUM) FRGS, S/O 13558 “Solutions of nonlinear Volterra-Hammerstein integral equations by combination of multidimensional fixed point and homotopy perturbation methods (MFPM-HPM)” (2016-2018), FRGS S/O 14192 ”Solutions of nonlinear fractional differential equations via a generalized fixed point method and homotopy analysis method” (2019-2020) mavzularidagi ilmiy tadqiqot loyihalari doirasida bajarilgan.

**Tadqiqot ishining maqsadi** sinish tipidagi maxsuslikka ega bo’lgan aylana diffeomorfizmlari o’rtasidagi qo‘shma gomeomorfizmlarning silliqligini, renormalizatsiyasini va qattiqlik xarakteristikalarini tadqiq qilishdan iborat.

**Tadqiqotning vazifalari** quyidagilardan iborat:

Katznelson va Ornsteyn shartlarini qanoatlantiruvchi aylana diffeomorfizmlari uchun kuchaytirilgan Danjua tengsizligini olish;

chegaralanmagan tipli burish soniga ega aylana diffeomorfizmlari va chiziqli burish o’rtasidagi qo‘shma gomeomorfizmning absolyut uzluksizligini o’rganish;

cheskiz ko‘p sondagi sinishga ega bo’lgan ikkita aylana gomeomorfizmlari orasidagi qo‘shma gomeomorfizmining turli darajadagi silliqliklari uchun zarur va yetarli shartlar olish;

ikkitadan sinish nuqtalariga ega bo‘lib xar xil sinish kattaliklariga ega ikkita aylana gomeomorfizmlari orasidagi qo‘shma gomeomorfizmning singulyar funksiyasi bo’lishi ko’rsatish;

bitta sinish nuqtasiga ega bo’lgan aylana gomeomorfizmlari renormalizatsiyalarining Myobius akslantirishlari bilan approksimatsiyasini o‘rganish;

har biri bir xil kattalikdagi bittadan sinishga va bir xil burish soniga ega ikkita aylana gomeomorfizmlari renormalizatsiyalarining yaqinlashishini o’rganish;

har biri bir xil kattalikdagi bittadan sinishga va chegaralangan tipli bir xil burish soniga ega bo’lgan ikkita aylana gomeomorfizmlari orasidagi qo‘shma gomeomorfizmning $C^1$ silliqligini o‘rganish.

**Tadqiqotning ob’ekti.** Aylana gomeomorfizmi va chiziqli burish orasidagi qo‘shma akslantirish, maxsuslikka ega ikkita aylana gomeomrfizmlari orasidagi qo‘shma akslantirish, renormalizatsiya tabiati.

**Tadqiqotning predmeti**. Aylana diffeomorfizmlari, sinish tipidagi maxsusikka ega aylana gomeomorfizmlari.

**Tadqiqotning usullari.** Dissertatsiya ishida kasr buzilishi (ratio distortion), aylanib kelgan kasr buzilishi (cross-ratio distortion), martingal va renormalizatsiyalar yaqinlashishi usullari qo‘llanilgan.

**Tadqiqotning ilmiy yangiligi** quyidagilardan iborat:

Katznelson va Ornsteyn shartlarini qanoatlantiruvchi aylana diffeomorfizmlari uchun kuchaytirilgan Danjua tengsizligi olingan;

chegaralanmagan tipli burish soniga ega aylana diffeomorfizmlari va chiziqli burish orasidagi qo‘shma gomeomorfizm va uning teskarisi absolyut uzluksiz va hosilalari esa $L_2$ fazodan bo’lishi isbotlangan;

cheksiz ko‘p sondagi sinishga ega ikkita aylana gomeomorfizmlari orasidagi qo‘shma gomeomorfizmining Gyolder sinfidan bo’lishi, absolyut uzluksiz bo’lishi va $C^1$ silliq bo’lishi uchun zarur va yetarli shartlar topilgan;

ikkitadan sinish nuqtalariga ega bo‘lib xar xil sinish kattaliklariga ega ikkita aylana gomeomorfizmlari orasidagi qo‘shma gomeomorfizmning singulyar funksiyasi bo’lishi ko’rsatilgan;

bitta sinish nuqtasiga ega bo’lgan aylana gomeomorfizmlari renormalizatsiyalarining Myobius akslantirishlari bilan approksimatsiya qilinishi ko‘rsatilgan;

har biri bir xil kattalikdagi bittadan sinishga va bir xil burish soniga ega ikkita aylana gomeomorfizmlari renormalizatsiyalarining yaqinlashishi Koshi alomatini qo‘llash orqali isbotlangan;

har biri bir xil kattalikdagi bittadan sinishga va chegaralangan tipli bir xil burish soniga ega bo’lgan ikkita aylana gomeomorfizmlari orasidagi qo‘shma gomeomorfizmning $C^1$ silliq funksiya bo’lishi ko’rsatilgan.

**Tadqiqotning amaliy natijalari** quyidagilardan iborat:

danjua tengsizligining boshqa kuchaytirilgan ko‘rinishlari umumlashgan interval almashtirish akslantirishlari uchun kengaytirilgan;

zygmund shartini qanoatlantiruvchi aylana gomemorfizmlari uchun kasr buzilishi (ratio distortion), aylanib kelgan kasr buzilishlarini (cross-ratio distortion) baholash uchun yaratilgan yangi usullar umumlashgan interval almashtirish akslantirishlari tatbiq qilingan;

qo‘shma gomeomorfizmlar silliqligi uchun topilgan yangi kriteriyalar interval akslantirishlari qo‘shmalari uchun ishlatilgan.

**Tadqiqot natijalarining ishonchliligi** matematik analiz, funksional analiz, ehtimollar nazariyasi va garmonik analiz usullaridan, teoremalaridan foydalanilganligi, hamda matematik mulohazalarning va isbotlarning qat’iyligi bilan tasdiqlanadi. Bundan tashqari, dissertatsiyada olingan natijalar yuqori impakt-faktorga ega nufuzli ilmiy jurnallarda chop etilganligi hamda Xalqaro va Respublika konfrensiyalarida ma’ruza qilinganligi bilan asoslanadi.

**Tadqiqot natijalarining ilmiy va amaliy ahamiyati.**

Tadqiqot natijalarining ilmiy ahamiyati, bir o’lchovli dinamik sistemalar nazariyasida quyi silliqlikka ega bo’lgan gomeomorfizmlar qo‘shmalarini o‘rganishda qo‘llanilishi bilan izohlanadi.

Tadqiqot natijalarining amaliy ahamiyati bir va ko’p o‘lchovli dinamik sistemalarning kvazi davriyligini o‘rganish va tadqiq etishda asos sifatida xizmat qilishi bilan belgilanadi.

**Tadqiqot natijalarining joriy qilinishi.** Maxsuslikka ega bo‘lgan bir o‘lchovli akslantirishlar renormalizatsiyalarining asimptotik holati va qo‘shmalari bo‘yicha olingan ilmiy natijalar asosida:

bo’lakli silliq aylana akslantirishlarining qattiqlik muammosini yechish usullaridan Geran Khas S/O 1777 raqamli “Nochiziqli singulyar toq ikkilik tenglamalar uchun hosilasiz kuazi nyuton usuli” mavzusidagi fundamental loyihada singulyar dual fuzzy nochiziqli tenglamalar yechimlarining mavjudligini isbotlashda foydalanilgan (Shimoliy Malayziya Universitetining (UUM) 2024-yil 12-dekabrdagi ma’lumotnomasi). Ilmiy natijaning qo’llanilishi nochiziqli singulyar toq ikkilik tenglamalari yordamida tuzilgan nochiziqli asklantirishlar uchun qo’zg’almas nuqtalarning mavjudligini isbotlash imkonini bergan;

Zygmund shartini qanoatlantiruvchi aylana gomemorfizmlari uchun kasr buzilishi (ratio distortion), aylanib kelgan kasr buzilishlarini (cross-ratio distortion) baholash usullaridan DIP-2017-011 raqamli “Zygmund sinfidagi bo’lakli silliq aylana diffeomorfizmlari uchun qattiqlik muammosi” mavzusidagi fundamental loyihada umumlashgan interval almashinuvchi asklanitishlarining qattiqlik masalalarini yechishda foydalanilgan (Malayziya Milliy Universitetining (UKM) 2024-yil 12-dekabrdagi ma’lumotnomasi). Ilmiy natijaning qo’llanilishi umumlashgan interval almashinuvchi asklanitishlarning Rouzy-Veech renormalizatiyalari yaqinlashishini isbotlash imkonini bergan;

aylana diffeomorfizmlar renormalizatsiyalarini baholash usullaridan DIP-2014-034 raqamli “Konusga ta’luqli metrik fazolarda qozg’almas nuqtalar nazariyasi” mavzusidagi, UKM-MI-OUP-2011(13-00-09-001) raqamli “Matematik tahlil va modellashtirish” mavzusidagi fundamental loyihalarda konusga ta’luqli metrik fazolardagi nochiziqli akslantirishlarning qo’zg’almas nuqtalari turg’un bo’lishini ko’rsatishda foydalanilgan (Malayziya Milliy Universitetining (UKM) 2024-yil 12-dekabrdagi ma’lumotnomasi). Ilmiy natijaning qo’llanilishi Pikard iteratsiyalarining yaqinlashishini isbotlash imkonini bergan;

sinishga ega bo’lgan aylana gomeomorfizmlarining qo’shma gomeomorfizmlarini o’rganish usullari xorijiy ilmiy jurnallardagi maqolalarda (Communications in Mathematical Physics 379(1), 2020; Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire 35(7), 2018; Journal of Statistical Physics 183(2), 2021; Advances in Mathematics 441, 2024; Ergodic Theory and Dynamical Systems 39(9), 2019) foydalanilgan. Ilmiy natijalarning qo’llanilishi qattiqlik muammosini yechish imkoni bergan.

**Tadqiqot natijalarining aprobatsiyasi.** Dissertatsiya natijalari to’rtta xalqaro konferensiyalarda ma’ruzalar qilingan va muhokamadan o‘tkazilgan va beshta ilmiy seminarlarida muhokama qilingan.

**Tadqiqot natijalarining e’lon qilinganligi.** Xorijiy ilmiy jurnallarda jami 20 ta ilmiy maqola chop etilgan bo‘lib, ularning barchasi dissertatsiya mavzusiga tegishli bo‘lib, O‘zbekiston Respublikasi Oliy attestatsiya komissiyasi tomonidan tavsiya etilgan. 1 ta maqola esa chop etilish uchun jurnalga yuborilgan, maqolaning to’liq shakli ArXiv veb sahifasiga joylangan.

**Dissertatsiyaning tuzilishi va hajmi.** Dissertatsiya kirish qismi, oltita bob, xulosa va foydalanilgan adabiyotlar ro‘yxati va 7 ta shakldan tashkil topgan. Dissertatsiya ishi 213 sahifan iborat.

## DISSERTATSIYANING ASOSIY MAZMUNI

Dissertasiya kirish qismida tadqiqotning dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan. Bundan tashqari, mavzu bo‘yicha xorijiy ilmiy tadqiqotlar sharhi, muammoning o‘rganilganlik darajasi keltirilgan, tadqiqot maqsadi, vazifalari, ob’ekti va predmeti tavsiflangan. Dissertatsiya ishida tadqiqotning ilmiy yangiligi va amaliy natijalari batafsil bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan. Shuningdek, kirish qismida tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertatsiya tuzilishi bo‘yicha ma’lumotlar keltirilgan.

**1-bobning asosiy natijalari.** 1-bobning sarlavhasi “Aylana akslantirishlarining kombinatorik va topologik nazariyasi va Danjua tengsizligining anoq baholari” bo‘lib, 1.1-1.3-boblarda aylana aklantirishlari nazarisidagi asosiy tushuchalarning dissertatsiyaga ta’aluqli bo’lgan qismlari to‘liq qayta ko‘rib chiqdilgan. Bunda aylana ta’rifi, aylanada yo‘nalish tushunchasi va aylana akslantirishining asosiy xossalari kabi asosiy elementlarni qayta ko‘rib chiqishni o‘z ichiga oladi. Shunungdek, Puankare tomonidan kiritilgan hamda kombinatorika nazariyasi bilan bog‘liq burish sonining muhim xossalari keltirilgan. Bundan tashqari burish sonlarining uzluksiz kasrga yoyilmasi va dinamik bo‘linishlar haqidagi tushunchalr keltirilgan. Danjua tengsizligi va uning amaliy ahamitiga e’tibor qaratilgan holda, Danjuaning topologik nazariyasi o‘rganilgan.

1.4-bobda, davriy nuqtaga ega bo‘lmagan ixtiyoriy $C^{1+BV}$ ($C^1$ sinfdan bo’lib hosilasining variatsiyasi chegaralgan) diffeomorfizmning chiziqli burishga topologik ekvivalent bo‘lishi haqidagi Danjua (1932) teoremasining umumlashmalari muhokama qilingan. Ma’lumki, $T$ aylana akslantirishi $C^k(S^1)$ sinfga tegishli diffeomorfizm bo‘ladi, agar uning aniqlovchi funksiyasi $C^k(\mathbb{R})$ sinfga tegishli bo‘lsa. Irratsional $\rho := \rho(T)$ burish songa ega $T$ aylana diffeomorfizmini qaraymiz.

Ushbu $K_n = \max_{\xi} |\log(T^{q_n}(\xi))'| = \|\log(T^{q_n})'\|_0$ belgilashni kiritamiz, bu yerda $T'(\xi), T$ akslantirishning $\xi$ nuqtadagi hosilasi. Endi quyi silliqlikka ega aylana diffeomorfizmlari uchun kuchaytirilgan Danjua tengsizligini tadqiq qilishga o‘tamiz. Aylana diffeomorfizmlari qo‘shmasining quyi silliqlik kriteriyalarini keltirish uchun, Kasnelson & Ornshteynlar [73] tomonidan birinchi bo‘lib kiritilgan quyidagi ta’rifni eslatamiz.

**1-Ta’rif.** *$T$ aylana diffeomorfizmi Kasnelson va Ornshteyn sinfiga (qisqalik* *uchun, KO sinfga) tegishli deb aytamiz, agar* $\log T'$ *absolyut uzluksiz, biror* $p > 1$ *uchun* $T''/T' \in L_p$ *bo'lsa.*

Bizning maqsadimiz $K_n$ ketma-ketlik uchun aniq baho olish. Ushbu $d_n = d_n(T) = \| T^{q_n} - Id \|_0$ belgilashni kiritamiz. 1-bobning asosiy natijasi quyidagidan iborat.

**2-Teorema.** *Aytaylik,* $T$ *diffeomorfizm Kasnelson va Ornshteyn shartlarini qanoatlantirsin hamda burish soni* $\rho$ *irratsional bo'lsin. U holda shunday* $C = C(T) > 0$ *o'zgarmas va kvadratlari yig'indisi yaqinlashuvchi bo'lgan shunday* $\tau_n = \tau_n(T)$ *ketma-ketlik mavjud bo'lib, quyidagi munosabat o'rinli bo'ladi:*

$$K_n \le C \sum_{k=1}^{n} \frac{d_n}{d_k} \cdot \tau_k .$$

**2-bobning asosiy natijalari.** 2-bobning sarlavhasi "Quyi silliqlikga ega aylana diffeomorfizmlarining chiziqlilanishi" bo'lib, bunda aylana diffeomorfizmlarini chiziqlilashtirishning murakkab masalasi o'rganildi. Chiziqlilashtirish muammosi aylana diffeomorfizmi va aylanada chiziqli burish o'rtasidagi qo'shma akslantirishning silliqligini anglatadi. 2.2 - 2.5 paragraflarda biz aylana diffeomorfizmlari oilasi $T_t = T + t$ va Danjua tengsizliklari hamda dinamik bo'linish intervallari uzunliklari nisbati uchun universal baholar olindi. Bundan tashqari, dinamik bo'linish intervallari kasr buzilishi (ratio distortion) hamda aylanib kelgan kasr buzilishlari (cross-ratio distortion) uchun aniq baholar olindi. Aytaylik, $\alpha \in (0, 1)$ irratsional son bo'lsin. Berilgan $\alpha$ sonning ushbu

$$\alpha = 1/(a_1 + 1/(a_2 + \cdots)) := [a_1, a_2, ..., a_s, ...)$$

uzluksiz kasrga yoyilmasidan foydalanamiz. Musbat $(a_s,\ s \ge 1)$ sonlar ketma-ketligi *qismiy nisbatlar* deyiladi hamda har bir $\alpha$ uchun yagona ravishda aniqlanadi. Endi berilgan ikkita natural sonlar ketma-ketligi orqali irratsional sonlarning qism to'plamini aniqlaymiz. Aytaylik, $(i_n)$ natural sonlarning qat'iy o'suvchi ketma-ketligi, $(v_n)$ natural sonlarning chegaralanmagan ketma-ketligi, va $M$ esa natural son bo'lsin. Ixtiyoriy $s \in \mathbb{N} \setminus \{i_n,\ n = 1, 2, ...\}$ uchun $a_{i_n} \le v_n$ va $a_s \le M$ bo'ladigan barcha irratsional $\alpha = [a_1, a_2, ..., a_s, ...)$ sonlar to'plamini $I(i_n, v_n, M)$ orqali belgilaymiz va ushbu

$$I(i_n, v_n) = \bigcup_{M=1}^{\infty} I(i_n, v_n, M)$$

to'plamni aniqlaymiz. Quyidagi teorema 2-bobning dastlabki asosiy natijasi hisoblanadi.

**3-Teorema.** *Aytaylik, T Kasnelson va Ornshteyn shartlarini qanoatlantiruvchi aylana diffeomorfizmi bo'lsin. U holda ixtiyoriy chegaralanmagan natural sonlar ketma-ketligi* $(v_n)$ *uchun, shunday o'suvchi* $i_n = i_n(T, v_n)$ *natural sonlar ketma-ketligi mavjudki, barcha* $\hat{\rho} \in I(i_n, v_n)$ *lar uchun* $T_{t_0}$ *va* $T_{\hat{\rho}}$ *orasidagi h qo'shma gomeomorfizm va uning teskarisi* $h^{-1}$ *absolyut uzluksiz va* $h', (h^{-1})' \in L_2$ *bo'ladi. Bu* *yerda* $t_0 = t_0(T, \hat{\rho})$, $t$ *parametrning* $\rho(T_{t_0}) = \hat{\rho}$ *bo'ladigan yagona qiymati.*

Endi $\Delta_\gamma(0) = 0$ va

$$\Delta_\gamma(x) = \frac{x}{(\log \frac{1}{x})^\gamma}, \quad \text{bunda } 0 < x < 1 \text{ va } \gamma > 0$$

shartlarni qanoatlantiruvchi bir-parametrli $\Delta_\gamma : (0,1) \to [0, \infty)$ funksiyalar oilasini qaraylik. $\Delta^2 T'(\xi, \tau)$ orqali $T'$ ning ikkinchi tartibli simmetrik ayirmasini belgilaymiz, ya'ni

$$\Delta^2 T'(\xi, \tau) = T'(\xi + \tau) + T'(\xi - \tau) - 2T'(\xi),$$

bu yerda $\xi \in S^1$ va $\tau \in [0, 1/2]$. Faraz qilaylik, shunday $C > 0$ o'zgarmas mavjud bo'lib, quyidagi tengsizlik o'rinli bo'lsin:

$$\| \Delta^2 T'(\cdot, \tau) \|_{C^\infty(S^1)} \leq C \Delta_\gamma(\tau). \qquad (1)$$

$Z_{\Delta_\gamma}$ orqali $T'$ hosilasi (1) shartni qanoatlantiruvchi $T$ aylana diffeomorfizmlari sinfini belgilaymiz. Quyidagi teorema bu bobdagi keyingi asosiy natijamiz hisoblanadi.

**4-Teorema.** *Aytaylik,* $T \in Z_{\Delta_\gamma}$ *burish soni* $\rho$ *irratsional bo'lgan aylana diffeomorfizmi bo'lib,* $\gamma \in (\frac{1}{2}, 1]$ *bo'lsin. Faraz qilaylik, biror* $\alpha \in (0, \gamma - \frac{1}{2})$ *uchun* $\rho$ *ning qismiy nisbatlari uchun* $a_n \leq Cn^\alpha$, $C > 0$ *bo'lsin. U holda* $T$ *va* $T_\rho$ *orasidagi* $h$ *qo'shma gomeomorfizm va uning teskarisi* $h^{-1}$ *absolyut uzluksiz va* $h', (h^{-1})' \in L_2$ *bo'ladi.*

Endi $C^1$-silliq chiziqlilashtirish holini qaraymiz. Yana $Z_{\Delta_\gamma}$ Zygmund sinfini qaraymiz, biroq $\gamma > 1$ bo'lsin deb faraz qilamiz. Quyidagi teorema bu bobdagi keyingi asosiy natijamiz hisoblanadi.

**5-Teorema.** *Aytaylik,* $T \in Z_{\Delta_\gamma}$ *burish soni* $\rho$ *irratsional bo'lgan aylana diffeomorfizmi bo'lib,* $\gamma > 1$ *bo'lsin. Faraz qilaylik, biror* $\alpha \in (0, \gamma - 1)$ *uchun* $\rho$ *ning qismiy nisbatlari uchun* $a_n \leq Cn^\alpha$, $C > 0$ *bo'lsin. U holda* $T$ *va* $T_\rho$ *orasidagi* $h$ *qo'shma gomeomorfizm va uning* $h^{-1}$ *teskarisi* $C^1$ *diffeomorfizm bo'ladi.*

**3-bobning asosiy natijalari.** 3-bobning sarlavhasi "Maxsuslikka ega aylana diffeomorfizmlari" bo'lib, sinish tipidagi maxsuslikka ega aylana gomeomorfizmlarini tadqiq qilishga bag'ishlanadi.

**6-Ta'rif.** $x_0$ *nuqta* $T$ *aylana akslantirishining sinish nuqtasi deyiladi, agar bir tomonlama hosilalar* $T'(x_0 \pm 0)$ *mavjud, musbat va bir-biriga teng bo'lmasa. Ushbu* $T'(x_0 - 0)/T'(x_0 + 0)$ *nisbatga* $T$ *gomeomorfizmning sinish kattaligi (sakrash nisbati) deyiladi.*

Bu bobda Erman tomonidan kiritilgan sanoqli sondagi sinishga ega aylana gomeomorfizmlari uchun $P$-gomeomorfizm tushunchasi kiritiladi. Ushbu bob $P$-gomeomorfizmlar uchun qat'iy natijalar bayoni bilan boshlanadi hamda bunday akslantirishlarning ergodikligi ko‘rsatiladi. Shundang so‘ng, $P$-gomeomorfizmlar va chiziqli burish orasidagi qo‘shmaning silliqligi uchun turli kriteriyalar keltiriladi.

Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho$ irratsional burishga ega $P$-gomeomorfizmlar bo‘lsin. Erman teoremasiga ko‘ra, $T_f$ va $T_g$ larni bog‘lovchi $T_h : S^1 \to S^1$ qo‘shma akslantirish mavjud bo‘ladi. Ta’kidlab o‘tamizki, agar $T_\varphi$ berilgan aylana gomeomorfizmi bo‘lsa, u holda $\hat{\varphi}_n = \varphi^n - Id$ funksiya barcha $n = 0,1,2,...$ lar uchun $\mathbb{Z}$ davriy funksiya bo‘ladi. Shu sababli bu funksiyalarni aylana akslantirishlari sifatida qarashimiz mumkin. Faraz qilaylie $f, g$ va $h$ lar mos ravishda $T_f, T_g$ va $T_h$ gomeomorfizmlarning aniqlovchi funksiyalari bo‘lsin. Bu aniqlovchi funksiyalardan foydalanib, $H_n : \mathbb{R} \to \mathbb{R}$ funksiyalar ketma-ketligini quyidagicha aniqlaymiz:

$$H_n := H_n(f, g, h) = \frac{1}{n} \sum_{i=0}^{n-1} \left( f^i - \hat{g}_i \circ h \right).$$

Quyidagi teorema 3-bobning dastlabki asosiy natijasi hisoblanadi.

**7-Teorema.** *Aytaylik, $T_f$ va $T_g$ lar bir xil irratsional burish sonli $P$-gomeomorfizmlar va $T_h$ ular orasidagi qo‘shma akslantirish bo‘lsin. U holda yuqoridagi belgilashlar asosida quyidagi tasdiqlar o‘rinli:*

*(a)* $\lim_{n\to\infty} \| H_n - h \|_{C^0} = 0$;

*(b)* $\max_{x\in\mathbb{R}} \left( H_n(x) - x \right) - \min_{x\in\mathbb{R}} \left( H_n(x) - x \right) < 1.$

Endi qo‘shma akslantirishning $\beta$-Gyolder uzluksizligi muhokama qilamiz. $\varphi$ funksiyaning $\beta$-Gyolder yarim normasini qo‘yidagicha aniqlaymiz:

$$[\varphi]_\beta := \sup_{x\neq y} \frac{\varphi(x) - \varphi(y)}{|x - y|^\beta}.$$

**8-Ta’rif.** *$C^\beta$-Gyolder fazosi deb ushbu*

$$\| \varphi \|_{C^\beta} = \| \varphi \|_{C^0} + [\varphi]_\beta$$

*norma chekli bo‘ladigan barcha $\varphi \in C^0$ funksiyalardan tashkil topgan fazoni belgilaymiz.*

Shuni ta’kidlaymizki, $\beta = 1$ bo‘lgan xususiy xolda yuqoridagi fazo ushbu

$$\| \varphi \|_{Lip} = \| \varphi \|_{C^0} + [\varphi]_{Lip}$$

Lipshis normali Lipshis funksiyalari fazosi deyiladi va bu fazoni $\mathrm{Lip}(S^1)$ orqali belgilanadi.

**9-Ta’rif.** *$T_\eta$ aylana gomeomorfizmi $C^\beta(S^1)$ sinfga tegishli deb aytamiz, agar $\eta \in C^\beta$ bo‘lsa.*

Quyidagi teorema bu bobdagi keyingi asosiy natijamiz hisoblanadi.

**10-Teorema.** *Aytaylik, $T_f$ va $T_g$ lar bir xil irratsional burish sonli $P$-gomeomorfizmlar va $T_h$ ular orasidagi qo‘shma akslantirish bo‘lsin. Agar*

$$\sup_{n\in\mathbb{N}}\left[\hat{f}_n-\hat{g}_n\circ h\right]_\beta<+\infty$$

*bo'lsa, u holda* $T_h\in C^\beta(S^1)$.

Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho$ irratsional soniga ega $P$-gomeomorfizmlar bo'lsin. $T_f$ va $T_g$ gomeomorfizmlarga mos $P_n(\xi,T_f)=P_n(T_f)$ va $P_n(T_h(\xi),T_g)=P_n(T_g)$ dinamik bo'linishlarni qaraymiz. $\Delta^n$ orqali $P_n(T_g)$ bo'linishning intervallarini belgilaymiz. $T_h$ funksiya $T_f$ va $T_g$ lar orasidagi qo'shma akslantirish bo'lgani uchun, ixtiyoriy $\Delta^n\in P_n(T_f)$ uchun $T_h(\Delta^n)=\Delta^n$ . Bizning keyingi asosiy natijamiz qo'yidagidan iborat.

**11-Teorema.** *Faraz qilaylik, $T_f$ va $T_g$ lar bir xil irratsional burish sonli $P$-gomeomorfizmlar bo'lib, burish sonining uzluksiz kasrga yoyilmasi $\rho_1=\rho_2=[k_1,k_2,...,k_n,...]$ bo'lsin. Agar $\sum_{n=1}^{\infty}\tau_n^2<\infty$ bo'ladigan shunday $\{\tau_n\}$ ketma-ketlik mavjud bo'lib, barcha $n\ge 1$ lar va har bir o'zaro qo'shni $\Delta_1^n,\Delta_2^n\in P_n(T_f)$ intervallar juftligi uchun*

$$\left|\frac{|\Delta_1|}{|\Delta_2|}-\frac{|\Delta_1|}{|\Delta_2|}\right|\le\tau_n$$

*bo'lsin. U holda $T_f$ va $T_g$ orasidagi $T_h$ qo'shma akslantirish absolyut uzluksiz funksiya bo'ladi.*

Endi qo'shma akslantirishning absolyut uzluksizligi uchun zaruriy shartni keltiramiz. Bizning keyingi asosiy natijamiz qo'yidagidan iborat.

**12-Teorema.** *Aytaylik, $T_f$ va $T_g$ lar bir xil irratsional burish sonli $P$-gomeomorfizmlar bo'lsin. Agar $T_h$ qo'shma akslantirish absolyut uzluksiz funksiya bo'lsa, u holda barcha $\delta>0$ lar uchun*

$$\lim_{n\to\infty}\ell\left(\tilde{x}:\ \tilde{x}\in S^1, |\log DT_g^{q_n}(T_h(\tilde{x}))-\log DT_f^{q_n}(\tilde{x})|\ge\delta\right)=0.$$

Qo'shma akslantirishning Lipshis uzluksizligi. Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho=\rho(T_f)=\rho(T_g)$ irratsional burish sonli $P$-gomeomorfizmlar va $T_h$ ular orasidagi qo'shma akslantirish bo'lsin. Bu bo'limda $T_h$ qo'shma akslantirishning Lipshis uzluksizligi uchun yetarli va zaruriy shartlarni keltiramiz. Quyidagi tasdiqdan boshlaymiz. Aytaylik, $T_\Gamma$ aylana gomeomorfizmi bo'lsin. Agar $T_\Gamma$ —Lipshis uzluksiz bo'lsa, u holda

$$[\Gamma]_{Lip}=\sup_{x\ne y}\frac{|\Gamma(x)-\Gamma(y)|}{|x-y|}<+\infty.$$

Bu tasdiqdan va Lebeg teoremasidan deyarli hamma joyda $DT_\Gamma(x)$ mavjud va $|DT_\Gamma|\le[\Gamma]_{Lip}$ bo'lishi kelib chiqadi. Oxirgi munosabatlar $DT_\Gamma\in L^\infty(S^1)$ va $\|DT_\Gamma\|_{L^\infty}=[\Gamma]_{Lip}$ bo'lishini bildiradi, bu yerda $\|\cdot\|_{L^\infty}$ - $L^\infty(S^1)$ fazodagi norma.

Qo‘shma akslantirishning Lipshis uzluksizligi uchun zaruriy shartni keltiramiz. Bizning keyingi asosiy natijamiz qo‘yidagidan iborat.

**13-Teorema.** *Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho$ irratsional burish sonli $P$-gomeomorfizmlar bo‘lsin. Agar $T_h \in Lip(S^1)$ bo‘lsa, u holda*

$$\sup_{n\in\mathbb{N}} \| \log Df^n - \log Dg^n(h) \|_{L^\infty} < +\infty .$$

Endi qo‘shma akslantirishning Lipshis uzluksizligi uchun yetarli shartni keltiramiz. Bizning keyingi asosiy natijamiz qo‘yidagidan iborat.

**14-Teorema.** *Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho$ irratsional burish sonli $P$-gomeomorfizmlar bo‘lsin. Faraz qilaylik, $T_h$ absolyut uzluksiz va*

$$\sup_{n\in\mathbb{N}} \| D f_n - D(g_n \circ h) \|_{L^\infty} < +\infty$$

*bo‘lsin. U holda $T_h \in Lip(S^1)$.*

Bu bo‘limda ikkita sinish ekvivalent $P$-gomeomorfizmlar orasidagi qo‘shma gomeomorfizmning $C^1$-silliqligi uchun zarur va yetarli shartni keltiramiz. Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho = \rho(T_f) = \rho(T_g)$ irratsional burish sonli $P$-gomeomorfizmlar va $T_h$ ular orasidagi qo‘shma akslantirish bo‘lsin. Quyidagi ta’rifni keltiramiz.

**15-Ta’rif.** *$T_f$ va $T_g$ $P$-gomeomorfizmlar sinish ekvivalent deyiladi, agar*

(1) $T_h(BP(T_f)) = BP(T_g)$;

(2) *barcha $b \in BP(T_f)$ lar uchun $\sigma_{T_g}(T_h(b)) = \sigma_{T_f}(b)$, bu yerda $BP(T_f)$ va $BP(T_g)$ lar mos ravishda $T_f$ va $T_g$ larning sinish nuqtalari to‘plamidir.*

Endi ikkita sinish ekvivalent aylana gomeomorfizmlari orasidagi qo‘shma gomeomorfizmning $C^1$-silliqligi uchun kriteriyani keltiramiz.

**16-Teorema.** *Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho$ irratsional burish sonli sinish ekvivalent $P$-gomeomorfizmlar bo‘lsin. $T_h$ qo‘shma akslantirish $C^1$-diffeomorfizm bo‘lishi uchun*

$$\sup_{n\in\mathbb{N}} \| \log Df^n - \log Dg^n(h) \|_{L^\infty} < +\infty$$

*bo‘lishi zarur va yetarli.*

Quyida biz qo‘shma akslantirishning silliqligi uchun ko‘proq mos keladigan shartlarni taqdim etamiz. Bizning keyingi asosiy natijamiz quyidagicha

**17-Teorema.** *Aytaylik, $T_f$ va $T_g$ lar bir xil $\rho$ irratsional burish soniga ega $P$-gomeomorfizmlar bo‘lsin. Agar $\sum_{n=1}^{\infty} \tau_n^2 < \infty$ bo‘ladigan shunday $\{\tau_n\}$ ketma-ketlik mavjud bo‘lib, barcha $n \geq 1$ lar va har bir o‘zaro qo‘shni $\Delta_1, \Delta_2 \in P_n(T_f)$ yoki $\Delta_1, \Delta_2 \subset \Delta \in P_{n-1}(T_f)$ intervallar juftligi uchun*

$$\left| \log \frac{|\Delta_1|}{|\Delta_2|} - \log \frac{|\Delta_1|}{|\Delta_2|} \right| \leq \tau_n$$

*bo'lsa, u holda* $T_f$ *va* $T_g$ *lar orasidagi* $T_h$ *qo'shma gomeomorfizm* $C^1$ *- silliq bo'ladi, bu yerda* $\Delta_i = T_h(\Delta_i)$, $i = 1,2$.

**4-bobning asosiy natijalari.** 4-bobning sarlavhasi "Ikkita sinish nuqtalariga ega bo'lakli silliq aylana akslantirishlari orasidagi qo'shmalar" bo'lib, har biri ikkita sinishga ega ikkita bo'lakli silliq aylana akslantirishlari orasidagi qo'shma gomeomorfizmning singulyarligini tadqiq qilishga bag'ishlanadi. Ushbu tadqiqot ikkita turli hollarni o'z ichiga oladi: 1-hol, sinish kattaliklari ko'paytmasi birga teng, 2-hol, sinish kattaliklari ko'paytmasi birdan farqli. 1-hol uchun bizning asosiy natijamiz qo'yidagidan iborat.

**18-Teorema.** *Aytaylik,* $T_i \in C^{2+\alpha}(S^1 \backslash \{a_i, b_i\}), i = 1,2,$ *larning har biri ikkita* $a_i$ *va* $b_i$ *nuqtalarda sinishga ega* $P$*-gomeomorfizmlar bo'lsin. Faraz qilaylik,*

1) *burish sonlari* $\rho(T_i)$, $i = 1,2$ *irratsional va teng, ya'ni* $\rho(T_1) = \rho(T_2) = \rho, \rho \in \mathbb{R}\backslash\mathbb{Q}$;
2) *ularning sinish kattaliklari ko'paytmalari* $\sigma_{T_i}(a_i) \cdot \sigma_{T_i}(b_i)$ *har xil, ya'ni* $\sigma_{T_1}(a_1) \cdot \sigma_{T_1}(b_1) \neq \sigma_{T_2}(a_2) \cdot \sigma_{T_2}(b_2)$;
3) $\mu_1([a_1,b_1]) = \mu_2([a_2,b_2])$ , *bu yerda* $T_i, i = 1,2$ *ning invariant ehtimollik o'lchovi.*

*U holda* $T_1$*, va* $T_2$ *lar orasidagi* $T_\psi$ *qo'shma gomeomorfizm singulyar bo'ladi.*

2-holda biz Katsnelson-Ornshteyn (K.O) ning umumlashtirilgan shartlarini qanoatlantiradigan ikkita uzilish nuqtasiga ega bo'lgan $T$ aylana gomeomorfizmlari sinfidan foydalandik, ya'ni $DT$ ning har bir uzluksizlik intervalida $\log DT$ funksiya absolyut uzluksiz va biror $p > 1$ uchun $D\log DT \in L^p$.

2-hol uchun bizning asosiy natijamiz qo'yidagidan iborat.

**19-Teorema.** *Aytaylik,* $T_i, i = 1,2$ *lar ikkita* $a_i$ *va* $b_i$ *nuqtalarda sinishga ega bo'lakli* $C^1$ *silliq aylana gomeomorfizmlari bo'lsin. Faraz qilaylik,*

1) $T_i, i = 1,2$ *larning* $\rho(T_i)$ *burish sonlari chegaralangan tipli irratsional (ya'ni* $\rho(T_i)$ *lar uzluksiz kasrlaridagi koeffitsientlar chegaralangan) bo'lib, ular bir xil bo'lsin;*
2) $\sigma_{T_1}(a_1)\sigma_{T_1}(b_1) = \sigma_{T_2}(a_2)\sigma_{T_2}(b_2)$;
3) *barcha* $b \in BP(T_2)$ *lar uchun* $\sigma_{T_1}(a_1) \neq \sigma_{T_2}(b)$;
4) $T_i, i = 1,2$ *larning sinish nuqtalari bitta traektoriyada yotmasin;*
5) $T_i, i = 1,2$ *lar biror* $p > 1$ *uchun (K.O) shartlarni qanoatlantirsin.*

*U holda* $T_1$ *va* $T_2$ *lar orasidagi* $h$ *qo'shma gomeomorfizm singulyar bo'ladi.*

**5-bobning asosiy natijalari.** 5-bobning sarlavhasi "Sinish tipidagi maxsuslikka ega aylana diffeomorfizmlarining renormalizatsiyalari" bo'lib, bu sinish tipidagi maxsuslikka ega aylana diffeomorfizmlari renormalizatsiyalarini o'rganishga bag'ishlanadi. Aylanada $\xi_0 \in S^1$ sinish nuqtani olamiz hamda $n$ juft bo'lganda aylananing $[\xi_0, T^{q_n}(\xi_0)]$ yoyini, $n$ toq bo'lganda $[T^{q_n}(\xi_0), \xi_0]$ yoyini $n$*-fundamental interval* deb ataymiz va uni $I_0^n := I_0^n(\xi_0)$ orqali aniqlaymiz. Ikkita yonma-yon $I_0^{n-1}$ va

$I_0^n$ fundamental intervallar birlashmasi $\xi_0$ nuqtaning *$n$ -renormalizatsion atrofi* deyiladi va uni $V_n$ orqali aniqlaymiz. Ushbu $V_n$ renormalizatsion atrofda $\pi_n = (T^{q_n}, T^{q_{n-1}}): V_n \to V_n$ Puankare akslan-tirishini quyidagicha aniqlaymiz:

$$\pi_n(\xi) = \begin{cases} T^{q_n}(\xi), & \xi \in I_0^{n-1}, \\ T^{q_{n-1}}(\xi), & \xi \in I_0^n. \end{cases}$$

Renormalizatsiya usulining asosiy g‘oyasi $\pi_n$ ning $n \to \infty$ dagi asimptotik xossasini bilishdan iborat. Buning uchun koordinatalarni almashtirishdan foydalaniladi. Aytaylik, $A_n : \mathbb{R} \to S^1$ — shunday affin akslantirishi bo‘lsinki, $A_n([-1,0]) = I_0^n$ hamda $A_n(0) = \xi_0$ , $A_n(-1) = T^{q_n}(\xi_0)$ . Shunday $a_n$ musbat sonni tanlaymizki, $A_{n-1}(a_n) = T^{q_n}(\xi_0)$ bo‘lsin. Tushunarliki,

$$A_{n-1} : [0, a_n] \to I_0^n \text{ va } A_{n-1} : [-1, 0] \to I_0^{n-1}.$$

Ushbu $(f_n, g_n) = A_{n-1}^{-1} \circ \pi_n \circ A_{n-1}$ ko‘rinishda aniqlangan $(f_n, g_n) : [-1, a_n] \to$ $[-1, a_n]$ funksiyalar juftligi $T$ ning $\xi_0$ ga nisbatan *$n$ -renormalizatsiyasi* deyiladi, bu yerda $A_{n-1}^{-1}$ — teskari akslantirishning qismi bo‘lib, $V_n$ intervalni $[-1, a_n]$ ga akslantiradi. Bu bobda bitta $\xi_0$ nuqtada sinishga ega, $T'$ hosilasining o‘zgarishi chegaralangan va yuqoridagi (1) tengsizlikni qanoatlantiruvchi $T$ aylana diffeomorfizmlari sinfini o‘rganamiz. Bu sinfni $D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ orqali belgilaymiz. Aytaylik, $T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ va uning burish soni irratsional bo‘lsin. Ikkita Myobius funksiyalarini quyidagicha aniqlaymiz:

$$F_n(z) = \frac{a_n + (a_n + b_n \tilde{m}_n)z}{1 + (1 - \tilde{m}_n)z}, \qquad G_n(z) = \frac{-a_n \hat{m}_n + (\hat{m}_n - b_n)z}{a_n \hat{m}_n + (1 - \tilde{m}_n)z}$$

bu yerda

$$\tilde{m}_n = \exp\left( \sum_{i=0}^{q_n - 1} \frac{T'(\xi_i) - T'(\xi_{i+q_{n-1}})}{2T'(\xi_i)} \right), \quad \hat{m}_n = \exp\left( \sum_{j=0}^{q_{n-1}-1} \frac{T'(\xi_{j+q_n}) - T'(\xi_j)}{2T'(\xi_{j+q_n})} \right)$$

hamda $\xi_i$, $\xi_{i+q_{n-1}}$ va $\xi_j$, $\xi_{j+q_n}$ nuqtalar mos ravishda $I_i^{n-1}$ va $I_j^n$ intervallarning chetki nuqtalari. Mazkur bobdagi birinchi asosiy natijasi quyidagi teoremadir.

**20-Teorema.** *Aytaylik, $T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ va $\gamma \in (0,1]$ bo‘lsin. Faraz qilaylik, $T$ ning burish soni irratsional bo‘lsin. U holda shunday $C = C(T) > 0$ o‘zgarmas va $N_0 = N_0(T)$ natural sonlar mavjudki, barcha $n \ge N_0$ lar uchun quyidagi tengsizliklar o’rinli*

$$\| f_n - F_n \|_{C^1([-1,0])} \le \frac{C}{n^\gamma}, \qquad \| g_n - G_n \|_{C^1([0,a_n])} \le \frac{C}{n^\gamma}.$$

Shuni ta’kidlaymizki, $D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ sinf $\gamma$ o‘sib borgan sari “yaxshilanib” boradi. Bu esa $(f_n, g_n)$ renormalizatsiyaning xossalarini yaxshiroq tushunish imkonini beradi. Kelgusida $\gamma > 1$ holni qaraymiz. Keyingi asosiy toeremani bayon qilishdan oldin qo‘yidagi Myobius funksiyalarini aniqlab olamiz

$$F_n(z)=\frac{a_n+(a_n+b_nm_n)z}{1+(1-m_n)z}, \quad G_n(z)=\frac{-a_nc_n+(c_n-b_nm_n)z}{a_nc_n+(m_n-c_n)z}$$

bu yerda

$$c_n=c^{(-1)^n}, \ b_n=\frac{|I_0^{n-1}|-|I_{q_{n-1}}^n|}{|I_0^{n-1}|}, \quad m_n=\exp\left((-1)^n\sum_{i=0}^{q_n-1}\int_{I_i^{n-1}}\frac{T''(x)}{2T'(x)}dx\right),$$

$c$ sinish kattaligi va $I_i^{n-1}$ esa $I_0^{n-1}$ intervalning *i*- iteratsiyasidir. Ushbu bobdagi ikkinchi asosiy natija quyidagi teoremadir.

**21-Teorema.** *Aytaylik,* $T\in D^{1+Z_\gamma}(S^1\setminus\{\xi_0\})$ *va* $\gamma>1$ *bo'lsin. Faraz qilaylik,* $T$ *ning burish soni irratsional bo'lsin. U holda shunday* $C=C(T)>0$ *o'zgarmas va* $N_0=N_0(T)$ *natural sonlar mavjudki, barcha* $n\geq N_0$ *lar uchun quyidagi tengsizlar o'rinli*

$$\|f_n-F_n\|_{C^1([-1,0])}\leq\frac{C}{n^\gamma}, \qquad \|g_n-G_n\|_{C^1([0,a_n])}\leq\frac{C}{n^\gamma}.$$

$$\|f_n''-F_n''\|_{C^0([-1,0])}\leq\frac{C}{n^{\gamma-1}}, \qquad \|g_n''-G_n''\|_{C^0([0,a_n])}\leq\frac{C}{n^{\gamma-1}}.$$

$$|a_n+b_nm_n-c_n|\leq\frac{Ca_n}{n^\gamma}.$$

Endi ikkita $T, T\in D^{1+Z_\gamma}(S^1\setminus\{\xi_0\})$, $\gamma>1$ aylana diffeomorfizmlarini qaraymiz. Ushbu $M_e$ va $M_o$ lar orqali uzluksiz kasrga yoyilmasining mos ravishda juft-indeksli va toq-indeksli hadlari chegaralangan irratsional burish sonlari to'plamini belgilab olamiz. Mazkur bobdagi uchinchi asosiy natija quyidagi teoremadir.

**22-Teorema.** *Aytaylik,* $T, T\in D^{1+Z_\gamma}(S^1\setminus\{\xi_0\})$ *va* $\gamma>1$ *bo'lsin. Faraz qilaylik,* $T$ *va* $T$ *lar bir xil* $c$ *sinish kattaligiga va bir xil* $\rho\in M_e$, $c>1$ *yoki* $\rho\in M_o$, $0<c<1$ *burish soniga ega bo'lsin. U holda shunday* $C=C(T,\tilde{T})>0$ *o'zgarmas va* $N_0=N_0(T,\tilde{T})$ *natural sonlar mavjud bo'lib, barcha* $n\geq N_0$ *lar uchun quyidagi tengsizliklar o'rinli*

$$||f_n-\tilde{f}_n||_{C^1([-1,0])}\leq\frac{C}{n^\gamma}, \quad ||f_n''-\tilde{f}_n''||_{C^0([-1,0])}\leq\frac{C}{n^{\gamma-1}}$$

*bunda* $\tilde{f}_n$, $\tilde{T}$ *ning n-renormalizatsiyasi* .

**6-bobning asosiy natijalari.** Oltinchi bobda sinish tipidagi maxsuslikka ega aylana diffeomorfizmlari uchun qattiqlik muammosini batafsil tadqiq qilamiz. Bu bobda bitta $\xi_0$ nuqtada sinishga ega, $T'$ hosilasining o'zgarishi chegaralangan va yuqoridagi (1) tengsizlikni qanoatlantiruvchi $T$ aylana diffeomorfizmlari sinfini qaraymiz. Bu sinfni $D^{1+Z_\gamma}(S^1\setminus\{\xi_0\})$ orqali belgilaymiz. Berilgan $m\in\mathbb{N}$ uchun quyidagilarni aniqlaymiz:

$$D_m^{(1)}=\{c\in\mathbb{R}_+\setminus\{1\}:\ c^{4m}-c^2<1\};\ D_m^{(2)}=\{c\in\mathbb{R}_+\setminus\{1\}:\ c^{4m+2}+c^{4m}>1\}.$$

Quyidagi teorema mazkur bobdagi asosiy natija hisoblanadi.

**23-Teorema.** *Aytaylik,* $\gamma>2$ *va* $m\in\mathbb{N}$ *bo'lsin. Faraz qilaylik,* $T$ *va* $T$ *lar* *bitta $\xi_0$ nuqtada sinishga ega va quyidagi shartlarni qanoatlan-tiruvchi aylana diffeomorfizmlari bo'lsin:*

*a)* $T, T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$;

*b)* $T$ *va* $T$ *lar chegaralangan tipli bir xil* $\rho$ *irratsional burish songa ega bo'lib,* $s(\rho) = m$ *bo'lsin;*

*c)* $T$ *va* $T$ *lar* $\xi_0$ *nuqtada bir xil* $c \in \mathbb{R}_+ \setminus \{1\}$ *sinish kattaligiga ega;*

*d)* $c > 1$ *bo'lganda* $c \in D_m^{(1)}$ *va* $0 < c < 1$ *bo'lganda* $c \in D_m^{(2)}$.

*U holda* $C^1$*-silliq shunday* $h$ *aylana diffeomorfizmi va shunday* $A > 0$ *o'zgarmas son mavjud bo'lib,* $h \circ T = T \circ h$ *va barcha* $x, y \in S^1$, $x \neq y$ *lar uchun quyidagi baho o'rinli*

$$| Dh(x) - Dh(y) | \leq A\omega_\gamma(| x - y |).$$

## XULOSA

Xulosa qilib aytganda, ushbu dissertatsiya ishida kuchaytirilgan Danjua tengsizligi, aylana diffeomorfizmlari uchun chiziqlilashtirish muammosi, sanoqli sondagi sinish nuqtalariga ega bo'lgan ikki aylana gomeomorfizmlari orasidagi qo'shma akslantirishning regulyarligi uchun har xil turdagi kriteriyalar, ikkita sinish nuqtalariga ega aylana gomeomorfizmlari orasidagi qo'shma akslantirishning singulyarligi, bitta sinish nuqtasiga ega aylana gomeomorfizmlari renormalizatsiyasi yaqinlashishi, bitta sinish nuqtasiga ega aylana gomeomorfizmlari uchun qattiqlik masalasi o'rganildi. Aniqroq aytganda,

1-bobda aylana gomeomorfizmlari nazariyasiga oid fundamental tushunchalarni kiritdik, ular aylana akslantirishlarining kombinatorik, topologik va silliqlik nazariyalarini qamrab oldi. Biz, shuningdek, uzluksiz kasrlar, dinamik bo'linishlar va martingal tushunchalarini qayta ko'rib chiqdik. Shuningdek kuchaytirilgan Danjua tengsizligini isbotladik.

2-bobda biz silliq aylana diffeomorfizmlari oilasini $T_t$ ni o'rganib chiqdik. Danjua tengsizligi va kasr buzilishi (ratio distortion) uchun aniq baholar olindi. Irratsional burishga mos barcha $t$ parametrlar uchun $K_n(t)$ Danjua ko'paytmasining $\ell_2$ normada yaqinlashish teoremasini isbotladik. Bundan tashqari, biz Zigmund shartlarini qanoatlantiruvchi, chegaralanmagan tipli irratsional burish soniga ega aylana diffeomorfizmlari uchun uning chiziqli burish orasidagi qo'shma akslantirishning silliq bo'lishini ko'rsatdik. Bobning oxirida biz asosiy teoremalarning bir nechta kengaytmalarini muhokama qildik.

3-bobda biz Erman tomonidan kiritilgan va sanoqli sondagi sinish nuqtalariga ega aylana akslantirishlari uchun $P$-gomeomorfizm tushunchasini ko'rib chiqdik. Dastlab $P$-gomeomorfizmlarning ergodikligini isbotladik.

Keyinchalik biz ikkita gomeomorfizm orasidagi qo'shma akslantirishning silliqligini baholash kriterielarini taqdim etdik, jumladan:

- Qo'shma akslantirishning $\beta$-Gyolder uzluksizligi uchun yetarli shart;
- Qo'shma akslantirishning absolyut uzluksizligi sharti (ham yetarli, ham zarur);
- Lipshis uzluksizligi uchun shart (ham yetarli, ham zarur);

- $C^1$ -silliqlik uchun yetarli va zarur shart.

Va nihoyat, qo‘shma akslantirishning silliqligini ta’minlash uchun yanada takomil shartni taqdim etdik.

4-bobda har biri ikkita sinish nuqtasiga ega ikkita bo‘lakli-silliq aylana akslantirishlari orasidagi qo‘shma akslantirishning singulyarligini o‘rganib chiqdik. Bunda ikkita hol qaraldi: 1-holda, turli xil sinish kattaliklari ko‘paytmasiga ega va 2-holda, bir xil sinish kattaliklari ko‘paytmasiga ega. Biz har ikkala holat uchun alohida usullardan foydalangan holda qo‘shma akslantirishning singulyarligini o‘rgandik.

• 1-holda natijalarni isbotlash uchun aylanib kelgan kasr (cross-ratio) usulidan foydalanildi. Dastlab bir nechta kerakli lemmalar isbotlandi, so‘ngra bu lemmalardan foydalanib qo‘shma akslantirishning singulyarligi isbotlandi.

• 2-holda baritsentrik koeffitsientlar uchun universal baholar olindi, shu bilan birga, $f^{q_n}$ ning ketma-ket sinish nuqtalari uchun va $\log Df^{q_n}$ ning ayirmalari uchun universal baholar olindi. Natijada, qo‘shma akslantirishning singulyar bo‘lishi isbotlandi.

Keyinchalik ushbu bobdagi asosiy teoremalarning ba’zi kengaytmalarini muhokama qildik.

Beshinchi bob sinish tipidagi maxsuslikka ega aylana diffeomorfizmlari renormalizatsiyalarini o‘rganishga bag‘ishlandi. Bu bobda tizimli ravishda quyidagi natijalar olindi:

- sinishga ega va Zigmund shartini qanoatlantiruvchi aylana diffeomorfizmlari kasr buzilishlari (ratio distortions) uchun aniq baholar olindi;
- nisbiy koordinatalar nuqtai nazaridan intervallarning kasr buzilishlari chuqur tadqiq qilindi;
- kasr buzilishlar uchun aniq baholar topdildi, bu esa ular qanday tabiatini tushunishga yordam berdi;
- nisbiy koordinatalar qanday ishlashi haqida tushunchalar berildi;
- 5.6-bo‘limda aylana gomeomorfizmlari renormalizatsiyasining Möbius akslantirishi bilan approksimatsiyasi isbotlandi;
- 5.6-bo‘lim renormalizatsiyalarning yaqinlashishi haqidagi teoremalarni taqdim etish bilan yakunlandi;
- 5.8-bo‘limda bir nechta sinishga ega aylana diffeomorfizmlarini kengaytirishdan iborat umumlashgan interval akslantirishlari uchun Ruzi-Vech renormalizatsiyalarini muhokama qildik.

Oltinchi bobda sinish tipidagi maxsuslikka ega aylana diffeomorfizmlari uchun qattiqlik muammosi o‘rganildi.

- dastlab, qattiqlikni masalasini tushunishda muhim hisoblangan sinishi-yekvivalent diffeomorfizmlar uchun kogomologik tenglamaning yechimlari sharti isbotlandi;
- keyin biz aylananing dinamik bo‘linishi segmentlari uchun umumiy baholar topildi;
- qaytanormallangan nuqtalarning farqi ularning renormalizatsiyalari yaqinlashishidan foydalanib baholandi;
- qattiqlik teoremasi isbotlandi.

**SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES DSc.03/30.12.2019.FM.01.01 AT THE NATIONAL UNIVERSITY OF UZBEKISTAN**

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**NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER MIRZO ULUG'BEK**

**AHADKULOV HABIBULLA ABURUYKULOVICH**

# THE ASYMPTOTIC BEHAVIOUR OF THE RENORMALIZATION AND CONJUGATIONS OF ONE DIMENSIONAL MAPS WITH SINGULARITIES

**01.01.01– Mathematical Analysis**

**ABSTRACT OF DOCTORAL DISSERTATION (DSc)**
**ON PHYSICAL AND MATHEMATICAL SCIENCES**

**Tashkent–2025**

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## INTRODUCTION (abstract of DSc dissertation)

**Actuality and demand of the theme of dissertation.** In the world, considering that most scientific studies on circle homeomorphisms with break-type singularities of circle diffeomorphisms are related to renormalization, it should be noted that its subclass known as $P-$homeomorphisms requires the practical implementation of numerous applied problems. In this regard, comprehensive measures have been taken to study mathematical analysis, functional analysis, and the theory of dynamical systems, and their application in science and technology is considered to be of great importance.

Worldwide, current scientific research in the theory of circle maps is directed toward studying how rapidly the renormalizations of a circle diffeomorphism with a single break converge to a two-parameter family of Möbius transformations. In this regard, circle maps, being one of the most widely used areas, have also been extensively applied in the results related to the smoothness of conjugate homeomorphisms in the theory of curvature transformations. These have been used to predict long-term behavior in fields, with a particular focus on solving rigidity problems related to generalized interval exchange maps in various scientific studies.

In our country, special attention is not only given to the scientific and practical application of fundamental sciences, but also, under the initiative of the academy of sciences, specialized laboratory environments are being established to study "The important properties of rotation numbers related to the combinatorial theory introduced by Poincaré". Moreover, substantial steps are being taken toward the generalization of concepts of the continuous fraction expansion of rotation numbers and dynamic partitions, leading to significant results. In the priority areas of "Differential topology and theory of functions of a complex variable", as well as in scientific fields such as dynamical systems, modern problems of complex analysis, algebra and its applications, conducting scientific research at the level of international standards has been identified[1] as one of the main objectives and directions of activity in the field of mathematics. In the process of implementing these tasks, upon obtaining universal estimates for Denjoy's inequalities and the ratio of lengths of intervals of dynamical partition, it is becoming increasingly important to develop integrated solutions similar to above, that require precise estimates for ratio distortion and cross-ratio distortion of the intervals of dynamical partition.

This dissertation contribute to the implementation of the tasks outlined in the Presidential Decree No. PF-4947, dated February 7, 2017, "On the Strategy for the Further Development of the Republic of Uzbekistan," the Presidential Resolution No. PQ-2789, dated February 17, 2017, "On measures to further improve the activities of the Academy of Sciences, organization, management, and financing of scientific research," and the Presidential Resolution No. PQ-4708, dated May 7, 2020, "On measures to improve the quality of education in mathematics and develop

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[1] Presidential Decree No. PQ-4708 of the Republic of Uzbekistan, dated May 7, 2020, "On measures to improve the quality of education in mathematics and to develop scientific research".

scientific research", as well as other relevant normative-legal documents related to this field.

**Connection of the research to priority areas development of science and technology of the Republic.** This study was carried out in accordance with the priority direction of the development of science and technology in the Republic of Uzbekistan IV-"Mathematics, Mechanics and Computer Science".

**Review of foreign research on the topic of the dissertation[2].** Scientific research on dynamic systems and their applications is actively conducted at leading higher education institutions and prestigious research centers in foreign countries. These include the Institute for Advanced Study (IAS) in Princeton (USA), the Institute for Applied Physics (FAP) at Goethe University (Germany), the Max Planck Institute for Dynamics and Self-Organization in Göttingen (Germany), the Center for Nonlinear and Complex Systems (CENOLIC) at the University of Turin (Italy), the International Center for Theoretical Physics (ICTP) in Trieste (Italy), the Fields Institute for Research in Mathematical Sciences in Toronto (Canada), the Isaac Newton Institute for Mathematical Sciences at Cambridge University (United Kingdom), the Henri Poincaré Institute in Paris (France), the Faculty of Science and Technology, Department of Mathematics, National University of Malaysia (UKM) (Malaysia), the Department of Mathematics at Northern University of Malaysia (UUM) (Malaysia), and others.

Recently, several notable results have been achieved in the field of dynamic systems, particularly in the global scientific community. These include studies on the asymptotic nature of renormalizations in one-dimensional dynamic systems, rigidity problems, and the properties of composite mappings. Some of the key findings are as follows:

A significant result by Selim Ghazouani from the University of Orsay, France, demonstrates that a generic piecewise affine circle homeomorphism with two breakpoints, concerning the Lebesgue measure, exhibits Morse-Smale behavior. Frank Trujillo, from the Universität Zürich, Switzerland, has successfully established explicit bounds for the Hausdorff dimension of the unique invariant measure of $C^3$ multicritical circle maps that lack periodic points. Corinna Ulcigrai, representing the University of Orsay, France, has proven a rigidity result for foliations on surfaces of genus two, extending Herman's theorem on circle diffeomorphisms to higher-genus cases. This has implications for flows on the torus as well. Sasa Kocić, from the University of Texas at Austin, USA, has addressed a generic rigidity problem. For almost all irrational rotation numbers, it has been shown that two circle diffeomorphisms with a break are $C^1$-rigid but not $C^{1+\alpha}$-rigid, for any $\alpha > 0$. Konstantin Khanin, at the University of Toronto, Canada, has demonstrated that for almost all irrational numbers $\rho$ in the interval $(0,1)$, the Hausdorff dimension of the invariant measure of circle diffeomorphisms with breaks

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[2] The analysis of foreign scientific studies related to the dissertation topic is based on the following sources: Discrete and Continuous Dynamical Systems (https://www.aimsciences.org/DCDS), Ergodic Theory and Dynamical Systems (https://tinyurl.com/4wxkt9h4), Nonlinearity (https://iopscience.iop.org/journal/0951-7715), Differential Equations and Dynamical Systems (https://link.springer.com/journal/12591).

and rotation number $\rho$ is zero. Akhtam Dzhalilov, affiliated with Turin Polytechnic University in Uzbekistan, has established that the study of Rauzy-Veech renormalizations of piecewise smooth circle homeomorphisms approximates the Möbius function. These findings collectively contribute to our understanding of the behavior and properties of one-dimensional dynamical systems, shedding light on various aspects of renormalization, rigidity, and mapping in this field.

**The degree of scrutiny of the problem.** Henri Poincaré, a French mathematician, is considered the founder of dynamical systems. His influential works, "New Methods of Celestial Mechanics" (1892-1899) and "Lectures on Celestial Mechanics" (1905-1910) explored the complex three-body problem and led to the Poincaré recurrence theorem, which states that certain systems will return close to their starting state after a long time. Aleksandr Lyapunov contributed to the field in 1899 by developing methods for studying the stability of dynamical systems, laying the foundation for modern stability theory. In 1913, George David Birkhoff proved Poincaré's "Last Geometric Theorem" and later published Dynamical Systems (1927). His ergodic theorem helped resolve a key issue in statistical mechanics and had a significant impact on dynamics. Stephen Smale introduced the Smale horseshoe and a research program that spurred further exploration in dynamical systems. In 1964, Oleksandr Sharkovsky developed Sharkovsky's theorem, showing that if a discrete dynamical system has a periodic point of period three, it must also have periodic points of all other periods.

Parry (1964) demonstrated that interval maps are combinatorially equivalent (or semi-conjugate) to piecewise linear maps. Subsequently, Metropolis, and Stein (1973) and, in their well-known handwritten notes, Milnor and Thurston (1977) proved that two piecewise monotone continuous interval maps are combinatorially equivalent if and only if the orbits of their turning points follow the same order. In the circle map case, Poincaré showed that any homeomorphism of the circle without periodic points is semi-conjugated to a rotation. Denjoy's (1932) theorem strengthens this result by asserting that this semi-conjugacy is in fact a conjugacy provided both the homeomorphism and its inverse are smooth.

In Denjoy's theorem, a key role is played by the precise estimation of Denjoy's product defined as $K_n = \max_{\xi} |\log(f^{q_n}(\xi))'|$ where, $f'(\xi)$ signifies the derivative of the function $f$ at the point $\xi$, and $q_n$ represents the first return time of $f$. He proved that if an orientation-preserving $C^1$-diffeomorphism of the circle with an irrational rotation number $\rho$, such that $\log f'$ possesses bounded variation then $K_n \leq v$ where $v$ is the total variation of $\log f'$ over the unit circle $S^1$. This result was extended by Katznelson & Ornstein, Sinai & Khanin and Khanin & Teplinsky. Summarizing their results, they demonstrated that for $C^{2+\varepsilon}(S^1)$ smooth circle diffeomorphisms, Denjoy's product tends to zero with an exponential rate.

Denjoy's theorem implies the existence of the conjugation map between the circle homeomorphism and linear rotation. Within this context, a natural question arises: *what are the conditions that the conjugation map is smooth?* Arnold obtained the first "local" result, that is, requiring closeness of the diffeomorphism to the linear rotation, in analytic setting. Moser extended this finding for $C^{\infty}$. In the late 1970s, Herman established the first global result, one that did not necessitate the closeness of the diffeomorphism to the linear rotation. Subsequently, Yoccoz, extended Herman's theorem to almost all irrational rotation numbers. In the late 1980s, two distinctive approaches to Herman's theory occurred. The first one is by Katznelson and Ornstein and the second one is by Khanin and Sinai whose extend Yoccoz's result for a wider class of circle diffeomorphisms. They proved that the conjugation map between $C^{2+\varepsilon}(S^1)$ circle diffeomorphism and linear rotation is at least $C^1$ smooth.

Herman gave some criteria for the smoothness of the conjugation map between circle diffeomorphisms and linear rotation. These criteria include conditions for: $\beta$-Hölder continuity, absolute continuity of the conjugation map, Lipschitz continuity of the conjugation map. Additionally, proved both sufficient and necessary conditions for $C^1$ smoothness of the conjugation map.

A natural generalization of circle diffeomorphisms is circle homeomorphisms with break-type singularities, a topic originally introduced by Herman. This class is known as $P$-homeomorphisms. Herman proved the singularity of conjugation between piecewise linear circle homeomorphisms having two break points and the linear rotation. His result was extended in the works of Dzhalilov & Khanin, Dzhalilov, Akin, & Temir, Dzhalilov & Liousse, as well as Dzhalilov, Mayer & Safarov to general $P$-homeomorphisms in different settings.

Another key aspect of circle diffeomorphisms is their renormalization behavior. The first result in this direction was obtained Khanin & Vul in at the beginning of the 90's. They proved that the renormalizations of a such circle diffeomorphism converge exponentially fast to a two-parameter family of Möbius transformations. The investigations of those Möbius transformations in the works of Khanin & Khemelev, Khanin & Teplinsky, Khanin & Yamplosky showed that the renormalization operator possesses strong hyperbolic properties in a certain domain of that space, which are analogous to those predicted by Lanford in the case of critical rotations.

In 2014, Khanin & Kocić proved that the renormalization of two circle diffeomorphisms converges with each other.

The convergence of the renormalizations, along with certain Diophantine conditions on the rotation numbers, implies the smoothness of the conjugation map between two circle diffeomorphisms, which is known as the rigidity problem. In the case of a break type singularity, the first rigidity results for $C^{2+\alpha}$ circle diffeomorphisms were obtained by Khanin & Khmelev, and Khanin & Teplinsky for a set of irrational rotation numbers which has zero Lebesgue measure. The most remarkable results in this direction were obtained by Khanin & Kocić and Khanin, Kocić, & Mazzeo for almost all irrational rotation numbers.

**Connection of the theme of the dissertation with the research works of higher education and research institute, where the dissertation is carried out.**

The research presented in this dissertation was carried out within the framework of the scientific research plan of the “Mathematical analysis” department at the National University of Uzbekistan named after Mirzo Ulugbek, as well as within the scientific research projects of Universiti Utara Malaysia (UUM): FRGS, S/O 13558 “Solutions of nonlinear Volterra-Hammerstein integral equations by combination of multidimensional fixed point and homotopy perturbation methods (MFPM-HPM)” (2016–2018), and FRGS S/O 14192 “Solutions of nonlinear fractional differential equations via a generalized fixed point method and homotopy analysis method” (2019–2020).

**The aim of the research work** is to investigate the smoothness, renormalization, and rigidity characteristics of conjugate homeomorphisms between circle diffeomorphisms with break-type singularities.

**Research problems are as follows:**

to obtain enhanced Denjoy’s inequality for circle diffeomorphisms satisfying the conditions of Katznelson and Ornstein;

to study the absolute continuity of the conjugate homeomorphism between circle diffeomorphisms with rotation number of unbounded type and linear rotation;

to establish necessary and sufficient conditions for various degrees of smoothness of the conjugate homeomorphism between two circle homeomorphisms with infinitely many breaks;

to demonstrate that the conjugate homeomorphism between two circle homeomorphisms, each having two break points with different sizes, is a singular function;

to study the approximation of the renormalizations of circle homeomorphisms with a single break point by Möbius transformations;

to investigate the convergence of the renormalizations of two circle homeomorphisms, each having one break of equal size and the same rotation number;

to study the $C^1$ smoothness of the conjugate homeomorphism between two circle homeomorphisms each having one break of equal size and the same rotation number of bounded type.

**Research object.** Conjugation maps between a circle homeomorphism and linear rotation, between two circle homeomorphisms with singularities, and renormalization behaviour.

**Research subject.** Circle diffeomorphisms, and circle homeomorphisms with singularities of break type.

**Research methods**. In the dissertation, ratio distortion, cross-ratio distortion, martingales, and renormalization convergent methods have been used.

**The scientific novelty of the research.** The scientific novelty of the research is as follows:

obtained enhanced version of the Denjoy inequality for circle diffeomorphisms satisfying the Katznelson and Ornstein conditions;

it was proved that the conjugate homeomorphism and its inverse between a circle diffeomorphism with unbounded type of rotation number and the linear rotation are absolutely continuous, and their derivatives belong to $L_2$ space;

necessary and sufficient conditions were established for the conjugate homeomorphism between two circle homeomorphisms with infinitely many break points to belong to the Hölder class, to be absolutely continuous, and to be $C^1$ smooth;

it was shown that the conjugate homeomorphism between two circle homeomorphisms, each having two break points of different sizes, is a singular function;

it was demonstrated that the renormalizations of circle homeomorphisms with a single break point can be approximated by Möbius transformations;

the convergence of the renormalizations of two circle diffeomorphisms, each having a single break of equal size and the same rotation number, was proved;

it was shown that the conjugate homeomorphism between two circle homeomorphisms with a break point of the same size, with the same irrational rotation number of bounded type is a $C^1$ smooth function.

**Practical results of the research are the following:**

the analogues of a more robust version of Denjoy's inequality can be extended to generalized interval exchange maps;

a newly developed method for estimating ratio and cross-ratio distortions in homeomorphisms that adhere to Zygmund conditions can be effectively applied to generalized interval exchange maps;

the newly developed criteria can be effectively employed for the conjugations of interval maps.

**The reliability of the research results.** The research results' reliability is confirmed by the use of methods and theorems from mathematical analysis, functional analysis, probability theory, and harmonic analysis, as well as by the rigor of mathematical reasoning and proofs. In addition, the results obtained in the dissertation are supported by their publication in prestigious scientific journals with high impact factors and by their presentation at international and national conferences.

**Scientific and practical significance of the research results.**

The scientific significance of the research results lies in their application to the study of conjugations of homeomorphisms with low smoothness in one-dimensional dynamical systems.

The practical significance of the research results lies in their potential use as a basis for studying and investigating the quasi-periodicity of one and multi-dimensional dynamical systems.

**Implementation of the research results.** The scientific results obtained in the dissertation have been practically implemented in the following:

the methods developed for solving the rigidity problem of piecewise smooth circle maps were applied in the fundamental research project Geran Khas S/O 1777, titled "Derivative-free quasi-newton-like method for solving singular dual fuzzy nonlinear equations," to prove the existence of solutions to singular dual fuzzy nonlinear equations (according to the letter from Universiti Utara Malaysia (UUM), dated December 12, 2024). The application of the scientific results enabled the proof of the existence of fixed points of nonlinear mappings constructed using nonlinear singular odd binary equations;

the techniques for estimating ratio distortion and cross-ratio distortion for circle homeomorphisms satisfying the Zygmund condition were used in the fundamental research project DIP-2017-011, titled "Rigidity for piecewise smooth circle diffeomorphisms of Zygmund class," in solving the rigidity problems for generalized interval exchange maps (according to the letter from Universiti Kebangsaan Malaysia (UKM), dated December 12, 2024). The application of the scientific results enabled the proof of convergence of Rauzy–Veech renormalizations for generalized interval exchange maps;

the methods for estimating of the convergence of the renormalizations of circle diffeomorphisms were applied in the fundamental research projects DIP-2014-034, "Fixed point theory in cone metric spaces," and UKM-MI-OUP-2011(13-00-09-001), "Mathematical analysis and modeling," in proving the stability of fixed points for nonlinear mappings in cone-related metric spaces (according to the letter from Universiti Kebangsaan Malaysia (UKM), dated December 12, 2024). The application of the scientific results enabled the proof of convergence of Picard iterations;

the methods used in the investigations of the conjugations between circle homeomorphisms with break points were utilized in international scientific journal articles (Communications in Mathematical Physics 379(1), 2020; Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire 35(7), 2018; Journal of Statistical Physics 183(2), 2021; Advances in Mathematics 441, 2024; Ergodic Theory and Dynamical Systems 39(9), 2019). The application of the scientific results made it possible to solve the rigidity problem.

**Approbation of the research results.** The results of the dissertation have been presented and discussed at four international conferences and have also been reviewed during five scientific seminars.

**Publication of Research Results.** A total of 20 scientific articles related to the dissertation topic have been published in foreign scientific journals recommended by the Higher Attestation Commission of the Republic of Uzbekistan. Additionally, one article has been submitted to a journal for publication, and its full version has been uploaded to the ArXiv web site.

**Structure and volume of the dissertation.** The dissertation consists of an introduction, six chapters, a conclusion, a list of references, and seven figures. The total pages of the dissertation is 213.

## MAIN RESULTS OF THE DISSERTATION

In the introduction, we address the relevance and demand of the research theme, emphasizing its alignment with the current scientific and technological priorities. We provide a review of international scientific research related to the topic, highlighting the degree of scrutiny the problem has received. The object and subject of study are clearly defined, and the aim of the research is outlined. We describe the research methods used and present the scientific novelty and practical results of the work. The scientific and practical significance of the research outcomes is discussed, along with their potential for implementation. Finally, we outline the structure of the dissertation.

**Main Results of Chapter 1**. The title of Chapter 1 is “Combinatorial and topological theory of the circle maps and precise bounds for Denjoy’s inequality”. In Chapters 1.1 to 1.3, we revisited key ideas from the Theory of Circle Maps. This included reviewing basic concepts like the definition of a circle, an orientation on a circle, and the main definitions of circle maps. We revised the definition of rotation numbers, focusing on the combinatorial theory introduced by Poincaré. We also recalled the concept of continued fraction expansions of rotation numbers and dynamical partitions. Along the way, we explained Denjoy’s topological theory, playing an important role in the theory of circle homeomorphisms. We emphasized Denjoy’s inequality and its practical applications.

In Chapter 1.4, we discussed about an extension of a theorem proved by Denjoy (1932), stating that any $C^{1+BV}$ (a class of $C^1$ diffeomorphism whose derivative has bounded variation) circle diffeomorphism without periodic points is topologically equivalent to a rotation. Recall that the map $T$ is a diffeomorphism of class $C^k(S^1)$, if its lift functions $f \in C^k(\mathbb{R})$. Consider a circle diffeomorphism $T$ with irrational rotation number $\rho := \rho(T)$. Denote $K_n = \max_{\xi} |\log(T^{q_n}(\xi))'|$ $=\| \log(T^{q_n})' \|_0$, where $T'(\xi)$ is the derivative of $T$ at the point $\xi$. Our focus shifts to the exploration of a more robust version of Denjoy’s inequality for circle diffeomorphisms with lower smoothness of regularities. To precisely lower the smoothness criteria for circle diffeomorphisms, we revisit the following definition, which was initially introduced by Katznelson & Ornstein.

**Definition 1.** *We say that a circle diffeomorphism $T$ belongs to Katznelson and Ornstein class (KO class, in short) if $\log T'$ is absolutely continuous and $T''/T' \in L_p(S^1, d\ell)$ for some $p > 1$.*

Our aim is to get sharp estimate for the sequence $K_n$. Denote by $d_n = d_n(T)$ $=\| T^{q_n} - Id \|_0$. The main result of Chapter 1 is the following.

**Theorem 2.** *Let diffeomorphism $T$ satisfy KO conditions and the rotation number $\rho$ is irrational. Then there exists a constant $C = C(T) > 0$ and a sequence $\tau_n = \tau_n(T)$ whose sum of squares converges, such that*

$$K_n \leq C \sum_{k=1}^{n} \frac{d_n}{d_k} \cdot \tau_k .$$

**Main Results of Chapter 2.** In Chapter 2, is titled as “Linearization of circle diffeomorphisms with low smoothness of regularities” where we studied a linearization problem for circle diffeomorphisms. The linearization problem refers for the smoothness of the conjugation map between a circle diffeomorphism and linear rotation on the circle. In Chapters 2.2 to 2.5 we investigated the family of circle diffeomorphisms $T_t = T + t$ and universal estimates for Denjoy’s inequality, and ratio of lengths of intervals of the dynamical partitions. Furthermore, we obtained precise bounds for these ratio and cross-ratio distortions.

Let $\alpha \in (0,1)$ be an irrational number. We use the continued fraction representation

$$\alpha = 1/(a_1 + 1/(a_2 + \cdots)) := [a_1, a_2, ..., a_s, ...].$$

of a given number $\alpha$. The sequence of positive integers $(a_s)$ with $s \geq 1$, called *partial quotients*, and is uniquely determined for each $\alpha$. Now we define a subset of irrational numbers by using two given sequences of natural numbers. Let $(i_n)$ be a strictly increasing sequence of natural numbers, $(v_n)$ be an unbounded sequence of natural numbers and $M$ be a natural number. Denoting the set of all irrational numbers $\alpha = [a_1, a_2, ..., a_s, ...)$ such that $a_{i_n} \leq v_n$ and $a_s \leq M$ for any $s \in \mathbb{N} \setminus \{i_n, n = 1, 2, ...\}$ by $I(i_n, v_n, M)$, we set

$$I(i_n, v_n) = \bigcup_{M=1}^{\infty} I(i_n, v_n, M).$$

Our first main result in Chapter 2 is given by the following theorem.

**Theorem 3.** *Let $T$ be a KO diffeomorphism of the circle with irrational rotation number. Then for any unbounded sequence of natural numbers $(v_n)$, there exists a strictly increasing sequence $i_n = i_n(T, v_n)$ of natural numbers, such that for any $\hat{\rho} \in I(i_n, v_n)$, the conjugating map $h$ between $T_{t_0}$ and $T_{\hat{\rho}}$ and its inverse $h^{-1}$ are absolutely continuous and $h', (h^{-1})' \in L_2$. Here $t_0 = t_0(T, \hat{\rho})$ is the unique value of a parameter $t$ such that $\rho(T_{t_0}) = \hat{\rho}$.*

Let us consider the following one-parameter family of functions: $\Delta_\gamma : (0,1) \to [0, \infty)$, $\Delta_\gamma(0) = 0$ and

$$\Delta_\gamma(x) = \frac{x}{(\log \frac{1}{x})^\gamma}, \quad \text{where } 0 < x < 1 \text{ and } \gamma > 0.$$

Denote by $\Delta^2 T'(\xi, \tau)$ the *second symmetric difference* of $T'$ i.e.,

$$\Delta^2 T'(\xi, \tau) = T'(\xi + \tau) + T'(\xi - \tau) - 2T'(\xi),$$

where $\xi \in S^1$ and $\tau \in [0, 1/2]$. Suppose that there exists a constant $C > 0$ such that the following inequality holds:

$$\| \Delta^2 T'(\cdot, \tau) \|_{C^\infty(S^1)} \leq C \Delta_\gamma(\tau). \qquad (1)$$

Denote by $Z_{\Delta_\gamma}$ the class of circle diffeomorphisms $T$, whose derivatives $T'$ satisfy (1). Our next main result in this chapter is the following theorem.

**Theorem 4.** *Let $T \in Z_{\Delta_\gamma}$ be a circle diffeomorphism with irrational rotation number $\rho$ and $\gamma \in (\frac{1}{2}, 1]$. Suppose that for some $\alpha \in (0, \gamma - \frac{1}{2})$ the partial quotients of $\rho$ satisfies $a_n \leq C n^\alpha$, $C > 0$. Then the conjugating map $h$ between $T$ and $T_\rho$ and its inverse $h^{-1}$ are absolutely continuous and $h', (h^{-1})' \in L_2$.*

We next consider the case of $C^1$-smooth linearization. We again consider the Zygmund class $Z_{\Delta_\gamma}$ but now assume that $\gamma > 1$. Our next main result in this chapter is as follows.

**Theorem 5.** *Let $T \in Z_{\Delta_\gamma}$ be a circle diffeomorphism with irrational rotation number $\rho$ and $\gamma > 1$. Suppose that for some $\alpha \in (0, \gamma - 1)$ the partial quotients of $\rho$ satisfies $a_n \le Cn^\alpha$, $C > 0$. Then the conjugating map $h$ between $T$ and $T_\rho$ and its inverse $h^{-1}$ are $C^1$ diffeomorphisms.*

At the end of the chapter, we discussed several extensions of the main theorems

**Main Results of Chapter 3.** Chapter 3 with title "Circle diffeomorphisms with singularities" is dedicated to the investigation of circle homeomorphisms with break type of the singularities.

**Definition 6.** *The point $x_0$ is said to be a break point of the circle map $T$ is one sided derivatives $T'(x_0 \pm 0)$ exist, positive and do not equal to each other. The ration $T'(x_0-0)/T'(x_0+0)$ is called the jump ratio of $T$ at $x_0$.*

In this chapter we revise the concept of $P$-homeomorphisms for circle homeomorphisms with countably many break points which was originally proposed by Herman. The chapter begins with a presentation of rigorous results for $P$-homeomorphisms and subsequently demonstrates the ergodicity of these homeomorphisms. The discussion then proceeds to provide various criteria for assessing the smoothness of the conjugation map between two $P$-homeomorphisms. Let $T_f$ and $T_g$ be $P$-homeomorphisms with the same irrational rotation number $\rho$. According to Herman's theorem there exists a conjugation $T_h : S^1 \to S^1$ between $T_f$ and $T_g$. Note that if $T_\varphi$ is a given circle homeomorphism then the function $\varphi^n = \varphi - Id_{\mathbb{R}}$ is a $\mathbb{Z}$-periodic function for each $n = 0,1,2,\ldots$. Thus and so we can also consider these functions as a circle maps. Let $f, g$ and $h$ be the lift homeomorphisms of $T_f$, $T_g$ and $T_h$ respectively. Define a sequence of functions $H_n : \mathbb{R} \to \mathbb{R}$ as follows:

$$H_n := H_n(f,g,h) = \frac{1}{n}\sum_{i=0}^{n-1}\left(f^i - \hat{g}_i \circ h\right).$$

The following is our first main result.

**Theorem 7.** *Let $T_f$ and $T_g$ be $P$-homeomorphisms with the same irrational rotation number and $T_h$ be the conjugating map between $T_f$ and $T_g$. Then under the notations above, the following statements hold:*

(a) $\lim_{n\to\infty} \| H_n - h \|_{C^0} = 0$;

(b) $\max_{x\in\mathbb{R}}\left(H_n(x) - x\right) - \min_{x\in\mathbb{R}}\left(H_n(x) - x\right) < 1$.

Our next discussion is about $\beta$-Hölder continuity of the conjugation map.
Define $\beta$-Hölder seminorm of $\varphi$ as follows

$$[\varphi]_\beta := \sup_{x\neq y}\frac{\varphi(x)-\varphi(y)}{|x-y|^\beta}.$$

**Definition 8.** *The* $C^\beta(S^1)$-Hölder *space, consists of all functions* $\varphi\in C^0(S^1)$ *for which the norm*

$$\|\varphi\|_{C^\beta}=\|\varphi\|_{C^0}+[\varphi]_\beta$$

*is finite.*

Note that in the special case $\beta=1$ the above space is called the space of Lipschitz functions with the (Lipschitz) norm

$$\|\varphi\|_{Lip}=\|\varphi\|_{C^0}+[\varphi]_{Lip}$$

and we denote it by $\mathrm{Lip(S^1)}$.

**Definition 9.** *We say that a circle homeomorphism* $T_\eta$ *belongs to* $C^\beta(S^1)$ *if* $\eta\in C^\beta(S^1)$.

The following is our next main result.

**Theorem 10.** *Let* $T_f$ *and* $T_g$ *be* $P$*-homeomorphisms with the same irrational rotation number and* $T_h$ *be the conjugating map between* $T_f$ *and* $T_g$. *If*

$$\sup_{n\in\mathbb{N}}\left[\hat{f}_n-\hat{g}_n\circ h\right]_\beta<+\infty$$

*then* $T_h\in C^\beta(S^1)$.

Let $T_f$ and $T_g$ be $P$-homeomorphisms with identical irrational rotation number $\rho$. Now, we consider dynamical partitions $P_n(\xi,T_f)=P_n(T_f)$ and $P_n(T_h(\xi),T_g)=P_n(T_g)$ appropriate to the homeomorphisms $T_f$ and $T_g$. Denote by $\Delta^n$ intervals of partition of $P_n(T_g)$. Since the function $T_h$ is a conjugation function between $T_f$ and $T_g$, so we have $T_h(\Delta^n)=\Delta^n$ for any $\Delta^n\in P_n(T_f)$.

Our next main result is the following.

**Theorem 11.** *Let* $T_f$ *and* $T_g$ *be* $P$*-homeomorphisms with the identical irrational rotation numbers with continued fraction expansion* $\rho_1=\rho_2=[k_1,k_2,...,k_n,...]$. If *there exists a sequence* $\{\tau_n\}$ *such that* $\sum_{n=1}^{\infty}(k_n\tau_n)^2<\infty$ *with*

$$\left|\frac{|\Delta_1|}{|\Delta_2|}-\frac{|\Delta_1|}{|\Delta_2|}\right|\le\tau_n$$

*for each pair of adjacent intervals* $\Delta_1,\Delta_2\in P_n(T_f)$ *for all* $n\ge1$. *Then the conjugation* $T_h$ *between* $T_f$ *and* $T_g$ *is an absolutely continuous function.*

Next we present a necessary condition for the absolute continuity of the conjugation map. Our next main result is the following.

**Theorem 12.** *Let $T_f$ and $T_g$ be $P$-homeomorphisms with identical irrational rotation number $\rho$. If the conjugating map $T_h$ is an absolutely continuous function, then for all $\delta > 0$ we have*

$$\lim_{n\to\infty} \ell\left(\tilde{x}:\ \tilde{x}\in S^1, |\log DT_g^{q_n}(T_h(\tilde{x})) - \log DT_f^{q_n}(\tilde{x})| \geq \delta\right) = 0.$$

Further we investigated Lipschitz continuity of the conjugating map. Let $T_f$ and $T_g$ be $P$-homeomorphisms with identical irrational rotation number $\rho = \rho(T_f) = \rho(T_g)$ and let $T_h$ be the conjugating map between $T_f$ and $T_g$. We begin with the following fact. Let $T_\Gamma$ be a circle homeomorphism. If $T_\Gamma$ is a Lipschitz homeomorphism, then

$$[\Gamma]_{Lip} = \sup_{x\neq y}\frac{|\Gamma(x)-\Gamma(y)|}{|x-y|} < +\infty.$$

This and according to Lebesgue's theorem it follows that $DT_\Gamma(x)$ exists and $|DT_\Gamma| \leq [\Gamma]_{Lip}$ almost everywhere, which means that $DT_\Gamma \in L^\infty(S^1)$ and $\|DT_\Gamma\|_{L^\infty} = [\Gamma]_{Lip}$, where $\|\cdot\|_{L^\infty}$ is the $L^\infty(S^1)$ norm. Now we formulate a necessary condition for the Lipschitz continuity of the conjugating map. The following is our next main result.

**Theorem 13.** *Let $T_f$ and $T_g$ be $P$-homeomorphisms with identical irrational rotation number $\rho$. If $T_h \in Lip(S^1)$ then*

$$\sup_{n\in\mathbb{N}} \|\log Df^n - \log Dg^n(h)\|_{L^\infty} < +\infty.$$

Now we formulate a sufficient condition for the Lipschitz continuity of the conjugating map. Our next main result is the following.

**Theorem 14.** *Let $T_f$ and $T_g$ be $P$-homeomorphisms with identical irrational rotation number $\rho$. Suppose that the conjugation $T_h$ is absolutely continuous and*

$$\sup_{n\in\mathbb{N}} \|D f_n - D(g_n \circ h)\|_{L^\infty} < +\infty,$$

*then $T_h \in Lip(S^1)$.*

We provided a necessary and sufficient condition for the $C^1$-smoothness of the conjugating map between two break equivalent $P$-homeomorphisms. Let $T_f$ and $T_g$ be $P$-homeomorphisms, with identical irrational rotation number $\rho = \rho(T_f) = \rho(T_g)$ and let $T_h$ be the conjugacy. We need the following definition.

**Definition 15.** *The $P$-homeomorphisms $T_f$ and $T_g$ are said to be break equivalent if*

1) $T_h(BP(T_f)) = BP(T_g)$;

2) $\sigma_{T_g}(T_h(b)) = \sigma_{T_f}(b)$ *for all* $b \in BP(T_f)$

*where $BP(T_f)$ and $BP(T_g)$ are the set of break points of $T_f$ and $T_g$ respectively.*

Now we will formulate a criteria for the $C^1$-smoothness of the conjugating map between two break equivalent circle homeomorphisms. The following is our next main result.

**Theorem 16.** *Let $T_f$ and $T_g$ be break equivalent $P$-homeomorphisms with irrational rotation number $\rho$. Then the conjugating map $T_h$ is a $C^1$-diffeomorphism if and only if*

$$\sup_{n\in\mathbb{N}} \| \log Df^n - \log Dg^n(h) \|_{L^\infty} < +\infty .$$

We provided more applicable sufficient condition for $C^1$ smoothness of the conjugation map.

The following is our next main result.

**Theorem 17.** *Let $T_f$ and $T_g$ be $P$-homeomorphisms with the identical irrational rotation numbers.* If *there exists a sequence $\{\tau_n\}$ such that $\sum_{n=1}^{\infty} \tau_n < \infty$ and*

$$\left| \log \frac{|\Delta_1|}{|\Delta_2|} - \log \frac{|\bar{\Delta}_1|}{|\bar{\Delta}_2|} \right| \le \tau_n$$

*for each pair of adjacent intervals $\Delta_1, \Delta_2 \in P_n(T_f)$ or $\Delta_1, \Delta_2 \subset \Delta \in P_{n-1}(T_f)$ for all $n \ge 1$. Then the conjugation $T_h$ between $T_f$ and $T_g$ is $C^1$ smooth, where $\bar{\Delta}_i = T_h(\Delta_i),\ i = 1,2$.*

**Main Results of Chapter 4.** The title of Chapter 4 is "Conjugations between piecewise-smooth circle maps with two break points". In this chapter we investigated the singularity of the conjugation map between two piecewise-smooth circle maps, each having two breakpoints. Consider two cases: Case 1, characterized by a different product of jumps and Case 2 characterized by the identical product of jumps.

For the Case 1, our main result is the following.

**Theorem 18.** *Let $T_i \in C^{2+\alpha}(S^1 \backslash \{a_i, b_i\}), i = 1,2,$ be $P$- homeomorphisms each with two break points $a_i, b_i$. Assume*

1) *their rotation numbers $\rho(T_i),\ i = 1,2,$ are irrational and coincide i.e. $\rho(T_1) = \rho(T_2) = \rho, \rho \in \mathbb{R}\backslash\mathbb{Q}$;*
2) *the products of their jump ratios $\sigma_{T_i}(a_i) \cdot \sigma_{T_i}(b_i)$ do not coincide i.e. $\sigma_{T_1}(a_1) \cdot \sigma_{T_1}(b_1) \neq \sigma_{T_2}(a_2) \cdot \sigma_{T_2}(b_2)$;*
3) *$\mu_1([a_1, b_1]) = \mu_2([a_2, b_2])$, where $\mu_i$ is the invariant probability measure of $T_i, i = 1,2$.*

*Then the map $T_\psi$ conjugating $T_1$ and $T_2$, is singular.*

In Case 2, we employed with the class of circle homeomorphisms $T$ with two break points satisfying the generalized conditions of Katznelson-Ornstein (K.O), that is, $\log DT$ is absolutely continuous on each continuity intervals of $DT$ and $D\log DT \in L^p$ for some $p > 1$.

For the Case 2, our main result is the following.

**Theorem 19.** *Let $T_i$, $i = 1,2$ be piecewise-smooth $C^1$ circle homeomorphisms with two break points $a_i, b_i$. Assume that*

1) *the rotation numbers $\rho(T_i)$ of $T_i$, $i = 1,2$ are irrational of bounded type (i.e. the coefficients in the continued fraction expansion of $\rho(T_i)$ are bounded) and coincide;*
2) *$\sigma_{T_1}(a_1)\sigma_{T_1}(b_1) = \sigma_{T_2}(a_2)\sigma_{T_2}(b_2)$;*
3) *$\sigma_{T_1}(a_1) \neq \sigma_{T_2}(b)$ for all $b \in BP(T_2)$;*
4) *the break points of $T_i$, $i = 1,2$ do not lie on the same orbit;*
5) *$T_i$, $i = 1,2$ satisfy KO conditions for the same $p > 1$.*

*Then the map $h$ conjugating $T_1$ and $T_2$ is a singular function.*

**Main Results of Chapter 5.** The title of Chapter 5 is “Renormalizations of circle diffeomorphisms with break type of singularities”. In this chapter we studied the renormalizations of circle diffeomorphisms with a single break point. Taking the break point $\xi_0 \in S^1$, we define the *n-th fundamental segment* $I_0^n := I_0^n(\xi_0)$ as the circle arc $[\xi_0, T^{q_n}(\xi_0)]$ if $n$ is even and $[T^{q_n}(\xi_0), \xi_0]$ if $n$ is odd. The union of two consequent fundamental segments $I_0^{n-1}$, $I_0^n$ is called the *$n$-th renormalization neighborhood* of $\xi_0$ and we denote it by $V_n$. On $V_n$ we define the Poincaré map $\pi_n = (T^{q_n}, T^{q_{n-1}}) : V_n \to V_n$ as follows

$$\pi_n(\xi) = \begin{cases} T^{q_n}(\xi), & \xi \in I_0^{n-1}, \\ T^{q_{n-1}}(\xi), & \xi \in I_0^n. \end{cases}$$

The main idea of renormalization method is to study the behaviour of $\pi_n$ as $n \to \infty$. For this, rescaling the coordinates are usually used. Let $A_n : \mathbb{R} \to S^1$ be an affine covering map such that $A_n([-1,0]) = I_0^n$, with $A_n(0) = \xi_0$ and $A_n(-1) = T^{q_n}(\xi_0)$. Define $a_n$ to be a positive number such that $A_{n-1}(a_n) = T^{q_n}(\xi_0)$. Obviously, $A_{n-1} : [0, a_n] \to I_0^n$ and $A_{n-1} : [-1, 0] \to I_0^{n-1}$. A pair of functions $(f_n, g_n) : [-1, a_n] \to [-1, a_n]$ defined by $(f_n, g_n) = A_{n-1}^{-1} \circ \pi_n \circ A_{n-1}$, is called the $n$ th *renormalization* of $f$ with respect to $\xi_0$, where $A_{n-1}^{-1}$ is the inverse branch that maps $V_n$ onto $[-1, a_n]$. In this chapter we studied the class of circle diffeomorphisms $T$ with break point $\xi_0$, whose derivatives $T'$ have bounded variation and satisfying the inequality (1). We denote this class $D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$. Let $T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ and its rotation number is irrational. Define two Möbius transformations as follow

$$F_n(z) = \frac{a_n + (a_n + b_n\tilde{m}_n)z}{1 + (1 - \tilde{m}_n)z}, \qquad G_n(z) = \frac{-a_n\hat{m}_n + (\hat{m}_n - b_n)z}{a_n\hat{m}_n + (1 - \tilde{m}_n)z}$$

where

$$\tilde{m}_n = \exp\left(\sum_{i=0}^{q_n - 1} \frac{T'(\xi_i) - T'(\xi_{i+q_{n-1}})}{2T'(\xi_i)}\right), \qquad \hat{m}_n = \exp\left(\sum_{j=0}^{q_{n-1} - 1} \frac{T'(\xi_{j+q_n}) - T'(\xi_j)}{2T'(\xi_{j+q_n})}\right)$$

and the points $\xi_i$, $\xi_{i+q_{n-1}}$ and $\xi_j$, $\xi_{j+q_n}$ are endpoints of the intervals $I_i^{n-1}$, $I_j^n$ respectively.

Our first main result in this chapter is the following.

**Theorem 20.** *Let $T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ and $\gamma \in (0,1]$. Suppose the rotation number of $T$ is irrational. There exists a constant $C = C(T) > 0$ and a natural number $N_0 = N_0(T)$ such that*

$$\| f_n - F_n \|_{C^1([-1,0])} \le \frac{C}{n^\gamma}, \qquad \| g_n - G_n \|_{C^1([0,a_n])} \le \frac{C}{n^\gamma}.$$

*for all $n \ge N_0$.*

Note that the class $D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ will be "better" when $\gamma$ increases. This gives more opportunities to better understand the behavior of $(f_n, g_n)$. Further, we consider the case $\gamma > 1$. To formulate our next main result we define the following Möbius transformations

$$F_n(z) = \frac{a_n + (a_n + b_n m_n)z}{1 + (1 - m_n)z}, \quad G_n(z) = \frac{-a_n c_n + (c_n - b_n m_n)z}{a_n c_n + (m_n - c_n)z}$$

where

$$c_n = c^{(-1)^n}, \; b_n = \frac{|I_0^{n-1}| - |I_{q_{n-1}}^n|}{|I_0^{n-1}|}, \quad m_n = \exp\left( (-1)^n \sum_{i=0}^{q_n - 1} \int_{I_i^{n-1}} \frac{T''(x)}{2T'(x)} dx \right),$$

$c$ is the break size, and $I_i^{n-1}$ is *i*-th iteration of $I_0^{n-1}$.

Our second result in this chapter is the following.

**Theorem 21.** *Let $T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ and $\gamma > 1$. Suppose the rotation number of $T$ is irrational. There exists a constant $C = C(T) > 0$ and a natural number $N_0 = N_0(T)$ such that*

$$\| f_n - F_n \|_{C^1([-1,0])} \le \frac{C}{n^\gamma}, \qquad \| g_n - G_n \|_{C^1([0,a_n])} \le \frac{C}{n^\gamma}.$$

$$\| f_n'' - F_n'' \|_{C^0([-1,0])} \le \frac{C}{n^{\gamma-1}}, \qquad \| g_n'' - G_n'' \|_{C^0([0,a_n])} \le \frac{C}{n^{\gamma-1}}.$$

*and*

$$| a_n + b_n m_n - c_n | \le \frac{C a_n}{n^\gamma}.$$

*for all $n \ge N_0$.*

Further, consider two circle diffeomorphisms $T, T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ for $\gamma > 1$. Let $M_e$ and $M_o$ denote the sets of all irrational rotations whose continued fraction expansions have bounded even-indexed and odd-indexed terms, respectively.

Our third result in this chapter is the following.

**Theorem 22.** *Let $T, T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ and $\gamma > 1$. Assume that $T$ and $T$ have the same break size $c$ and the same rotation number $\rho \in M_e$ in the case of*

*$c>1$, or $\rho \in M_o$ in the case of $0<c<1$. There exists a constant $C=C(T,\tilde{T})>0$ and a natural number $N_0 = N_0(T,\tilde{T})$ such that*

$$||f_n - \tilde{f}_n||_{C^1([-1,0])} \le \frac{C}{n^\gamma}, \quad || f_n'' - \tilde{f}_n''||_{C^0([-1,0])} \le \frac{C}{n^{\gamma-1}}$$

*for all $n \ge N_0$ where $\tilde{f}_n$ is n-th renormalization of $\tilde{T}$.*

At the end of Chapter 5 we discussed the Rauzy-Veech renormalizations of the generalized interval exchange maps which extend circle diffeomorphisms with multiple breaks.

**Main Results of Chapter 6.** In Chapter 6, titled "Rigidity problem for circle diffeomorphisms with break type of singularities" we investigated the rigidity problem for circle diffeomorphisms characterized by singularities of the break type. In this chapter we studied the class of circle diffeomorphisms $T$ with break point $\xi_0$, whose derivatives $T'$ have bounded variation and satisfying the inequality (1). We denote this class $D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$. Let $m \in \mathbb{N}$. Define

$$D_m^{(1)} = \{c \in \mathbb{R}_+ \setminus \{1\}:\ c^{4m} - c^2 < 1\}; \quad D_m^{(2)} = \{c \in \mathbb{R}_+ \setminus \{1\}:\ c^{4m+2} + c^{4m} > 1\}.$$

The following is our main theorem in chapter.

**Theorem 23.** *Let $\gamma > 2$ and $m \in \mathbb{N}$. Let $T$ and $T$ be two circle diffeomorphisms with the break point $\xi_0$ satisfying the following conditions:*

*(a) $T, T \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$;*

*(b) $T$ and $T$ have the same irrational rotation number $\rho$ of bounded type such that $s(\rho) = m$;*

*(c) $T$ and $T$ have the same size $c \in \mathbb{R}_+ \setminus \{1\}$ of the break point $\xi_0$;*

*(d) $c \in D_m^{(1)}$ in case of $c>1$ or $c \in D_m^{(2)}$ in case of $0<c<1$.*

*Then there exists a $C^1$-smooth circle diffeomorphism $h$ and a constant $A>0$ such that $h \circ T = T \circ h$ and*

$$| Dh(x) - Dh(y) | \le A\omega_\gamma(| x - y |)$$

*for any $x, y \in S^1$ such that $x \ne y$, where $Dh$ is the derivative of $h$.*

## CONCLUSION

In conclusion, in this dissertation we studied enhanced version of Denjoy's inequality, linearization problem for circle diffeomorphisms, various type of criteria for the regularity of conjugation maps between two circles having countable many break points, the singularities of the conjugation maps between two circle homeomorphisms with two break point, the renormalization convergence of circle homeomorphisms with a break point, and rigidity problems for circle homeomorphisms with a single break point. More precisely, In Chapter 1, we introduced fundamental concepts related to the theory of circle homeomorphisms, covering combinatorial, topological, and smooth theories of circle maps. We also revisited the concepts of continued fractions, dynamical partitions, and the concepts of martingales. Finally, we proved an enhanced version of Denjoy's inequality.

In Chapter 2, we studied a family of smooth circle diffeomorphisms $T_t$. We established uniform estimates for Denjoy's inequality and ratio distortions. We proved $\ell_2$ convergence theorem for Denjoy's product $K_n(t)$ for all $t$ corresponding to irrational rotation numbers. Moreover, we showed the smoothness of the conjugation map between circle diffeomorphisms satisfying Zygmund conditions and linear rotation for unbounded irrational rotation numbers. At the end of the chapter, we discussed several extensions of the main theorems.

In Chapter 3 we reviewed the concept of $P$-homeomorphisms for circle maps with countably many break points, as introduced by Herman. First, we proved the ergodicity of $P$-homeomorphisms.

Then, we provided criteria for assessing the smoothness of the conjugation map between two $P$-homeomorphisms, including:

- A sufficient condition for $\beta$-Hölder continuity of the conjugation map;
- A condition for absolute continuity of the conjugation map (both sufficient and necessary);
- A condition for Lipschitz continuity (both sufficient and necessary);
- A condition sufficient and necessary condition for $C^1$ smoothness.

Finally, we presented a more practical condition for ensuring $C^1$ smoothness of the conjugation map.

In Chapter 4, we investigated the singularities of the conjugation map between two piecewise-smooth circle maps, each with two break points. The analysis covered two cases: Case 1, with different jump products, and Case 2, with identical jump products. We established the singularities of the conjugation map for both cases, using distinct methods for each.

- In Case 1, we used powerful tools of the Cross-ratio distortion to substantiate our findings. Initially, we proved several necessary lemmas, and then, using these lemmas, we proved the singularities of the conjugation map.
- In Case 2, we obtained universal bounds for the barycentric coefficients, along with universal bounds for the consecutive break points of $T^{q_n}$, and universal estimates for the differences of $\log DT^{q_n}$. Subsequently, we proved the singularities of the conjugation map.

Finally, we discussed some extensions of the main theorems from this chapter.

In Chapter 5 explored the complex details of renormalizations for circle transformations that have break points. This chapter followed a structured approach, covering various important aspects:

- First, we estimated the ratio distortions, focusing on circle diffeomorphisms with breaks that satisfy the Zygmund condition.

•Next, we looked closely at the distortions of the intervals in terms of relative coordinates.

•We then found precise estimates for the ratios of distortions, helping us understand how they behave.

•We also provided insights into how relative coordinates behave.

•In Section 5.6 we provided the proofs of approximations of the renormalization with Möbius transformations.

•Section 5.6 concluded by presenting theorems about the convergence of renormalizations.

•Finally, in Section 5.8 we discussed the Rauzy-Veech renormalizations of the generalized interval exchange maps which extend circle diffeomorphisms with multiple breaks.

In Chapter 6, we studied the rigidity problem in circle diffeomorphisms with break-type singularities.

•First, we proved a condition for solutions to the cohomological equation for break-equivalent diffeomorphisms, which was key to understanding rigidity.

•Then we found general estimates for the segments of the dynamical partition of the circle.

•We estimated the difference of rescaled points by using the convergence of their renormalizations.

•Finally, we proved the rigidity theorem.

**НАУЧНЫЙ СОВЕТ ПО ПРИСУЖДЕНИЮ УЧЕНЫХ СТЕПЕНЕЙ DSc.03/30.12.2019.FM.01.01 В НАЦИОНАЛЬНОМ УНИВЕРСИТЕТЕ УЗБЕКИСТАНА**

**НАЦИОНАЛЬНЫЙ УНИВЕРСИТЕТ УЗБЕКИСТАНА ИМЕНИ МИРЗО УЛУГБЕК**

**АХАДКУЛОВ ХАБИБУЛЛА АБУРУЙКУЛОВИЧ**

# АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ РЕНОРМАЛИЗАЦИИ И СОПРЯЖЕНИЯ ОДНОМЕРНЫХ ОТОБРАЖЕНИЙ С ОСОБЕННОСТЯМИ

**01.01.01–Математический анализ**

**АННОТАЦИЯ ДОКТОРСКОЙ ДИССЕРТАЦИИ (DSc) ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ**

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## ВВЕДЕНИЕ (аннотация докторской диссертации)

**Объект исследования:** Отображения сопряжения между гомеоморфизмом окружности и линейным вращением, между двумя гомеоморфизмами окружности с особенностями и поведением перенормировки.

**Научная новизна исследования состоит в следующем:**

расширено неравенство Данжуа для диффеоморфизмов окружности, удовлетворяющих условиям Кацнельсона и Орнштейна;

доказано, что сопряжённая гомеоморфизм и его обратная функция между диффеоморфизмом окружности с числом вращения неограниченного типа и линейным вращением являются абсолютно непрерывными, а их производные принадлежат функциональным пространствам $L_2$;

установлены необходимые и достаточные условия, при которых сопрягающее гомеоморфизм между двумя гомеоморфизмами окружности с бесконечным числом изломов принадлежит классу Гёльдера, является абсолютно непрерывным и $C^1$ гладким;

показана сингулярность отображения сопряжения между двумя гомеоморфизмами окружности, каждый из которых имеет две точки излома разной величины, является сингулярной функцией;

демонстрировано, что ренормализации гомеоморфизмов окружности с одной точкой излома были аппроксимированы преобразованиями Мёбиуса;

доказана сходимость ренормализаций двух диффеоморфизмов окружности, каждый из которых имеет по одной точке излома одинакового размера и одинаковое число вращений;

показано, что сопрягающий гомеоморфизм между двумя гомеоморфизмами окружности с точкой излома одинакового размера и одинаковым иррациональным числом вращения ограниченного типа является $C^1$ гладкой функцией.

**Внедрение результатов исследования.** Научные результаты, полученные в диссертации, были практически реализованы в следующих направлениях:

методы, разработанные для решения задачи жёсткости кусочно-гладких отображений окружности, были применены в фундаментальном исследовательском проекте Geran Khas S/O 1777 под названием “Derivative-free quasi-newton-like method for solving singular dual fuzzy nonlinear equations” для доказательства существования решений сингулярных двойственных нечетких нелинейных уравнений (согласно письму Universiti Utara Malaysia (UUM) от 12 декабря 2024 года). Применение научных результатов позволило доказать существование неподвижных точек нелинейных отображений, построенных с использованием нелинейных сингулярных нечетных бинарных уравнений;

методы при оценивание ratio distortion и cross-ratio distortion для гомеоморфизмов окружности, удовлетворяющих условиям гладкости

Зигмунда, были использованы в фундаментальном исследовательском проекте DIP-2017-011 под названием “Rigidity for piecewise smooth circle diffeomorphisms of Zygmund class” при решении задач жёсткости для обобщенных отображений перекладывания интервалов (согласно письму Universiti Kebangsaan Malaysia (UKM) от 12 декабря 2024 года). Применение научных результатов позволило доказать сходимость ренормализаций Rauzy–Veech для обобщенных отображений перекладывания интервалов;

методы при оценивание ренормализаций диффеоморфизмов окружности были применены в фундаментальных исследовательских проектах DIP-2014-034 “Fixed point theory in cone metric spaces” и UKM-MI-OUP-2011(13-00-09-001) “Mathematical analysis and modeling” для доказательства устойчивости неподвижных точек нелинейных отображений в метрических пространствах, связанных с конусами (согласно письму Universiti Kebangsaan Malaysia (UKM) от 12 декабря 2024 года). Применение научных результатов позволило доказать сходимость итераций Пикара;

методы, использованные при изучении сопряжений между гомеоморфизмами окружности с точками разрыва, были использованы в статьях международных научных журналов (Communications in Mathematical Physics 379(1), 2020; Annales de l’Institut Henri Poincaré (C) Analyse Non Linéaire 35(7), 2018; Journal of Statistical Physics 183(2), 2021; Advances in Mathematics 441, 2024; Ergodic Theory and Dynamical Systems 39(9), 2019). Применение научных результатов позволило решить задачу жёсткости.

**Структура и объем диссертации.** Диссертация состоит из введения, шести глав, заключения, списка литературы и семи рисунков. Общий объем диссертации составляет 213 страниц.

**ЭЪЛОН ҚИЛИНГАН ИШЛАР РЎЙХАТИ**
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