V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

MATEMATIKA INSTITUTI

SHAKAROVA MARJONA DILSHOD QIZI

KASR TARTIBLI SHREDINGER VA SUBDIFFUZIYA TENGLAMALARI UCHUN TO‘G‘RI VA TESKARI MASALALAR

01.01.02 – Differensial tenglamalar va matematik fizika

FIZIKA-MATEMATIKA FANLARI BO‘YICHA FALSAFA DOKTORI (PhD) DISSERTATSIYASI AVTOREFERATI

TOSHKENT – 2025 yil

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Fizika – matematika fanlari bo‘yicha falsafa doktori (PhD) dissertatsiyasi avtoreferati mundarijasi

Contents of dissertation abstract of doctor of philosophy (PhD) on physical – mathematical sciences

Оглавление автореферата диссертации доктора философии (PhD) по физико – математическим наукам

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Fizika-matematika fanlari bo‘yicha falsafa doktori (PhD) dissertatsiyasi mavzusi O‘zbekiston Respublikasi Oliy ta’lim, Fan va Innovatsiyalar Vazirligi huzuridagi Oliy attestatsiya komissiyasida № B2024.4.PhD/FM1178 raqam bilan ro‘yxatga olingan.

Dissertatsiya V.I. Romanovskiy nomidagi Matematika institutida bajarilgan.
Dissertatsiya avtoreferati uch tilda (o‘zbek, ingliz, rus (резюме)) Ilmiy kengash veb-sahifasi (http://kengash.mathinst.uz) va «Ziyonet» ta’lim axborot tarmog‘ida (www.ziyonet.uz) joylashtirilgan.

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Dissertatsiya himoyasi V.I.Romanovskiy nomidagi Matematika Instituti huzuridagi DSc.02/30.12.2019.FM.86.01 raqamli Ilmiy kengashning 2025 yil «01» iyul soat 17:00 dagi majlisida bo‘lib o‘tadi. (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871)-207-91-40, e-mail: uzbmath@umail.uz, Website: www.mathinst.uz).

Dissertatsiya bilan V.I.Romanovskiy nomidagi Matematika Institutining Axborot-resurs markazida tanishish mumkin (206-raqami bilan ro‘yxatga olingan). (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871)-207-91-40.

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KIRISH (falsafa doktori (PhD) dissertatsiyasi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zaruriyati. So‘nggi o‘n yilliklarda kasr tartibli differensial tenglamalar dunyo miqyosida turli xil anomal hodisalarni modellashtirish uchun muhim matematik vosita sifatida paydo bo‘ldi. Shu sabali ko‘plab matematik olimlar kasr tartibli differensial tenglamalarni yechishning yangi usullarini faol ravishda o‘rganib, ularning yechimlarining xususiyatlarini tadqiq qilishmoqda. Kasr tartibli differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalarni o‘rganishdan tashqari, ushbu tenglamalarning koeffitsiyentlarini, o‘ng tomondagi manba funksiyasini, chegaraviy shartlarni, hamda kasr tartibli hosilaning tartibini aniqlash bilan bog‘liq teskari masalalarni tadqiq etish ham muhim amaliy ahamiyatga egadir. Ushbu dissertatsiya kasr tartibli differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalarni, shuningdek, diffuziya, subdiffuziya va kasr tartibli Shredinger tipidagi tenglamalar uchun manba funksiyasini aniqlashga doir teskari masalalarni o‘rganishga bag‘ishlangan.

Hozirgi kunda kasr tartibli tenglamalar uchun to‘g‘ri va teskari masalalarni o‘rganish fizik jarayonlarni tahlil qilish va boshqarish imkonini beradi. Manba funksiyasini (tenglamaning o‘ng tomoni $F(x,t)$) aniqlashga oid teskari masalalar mexanika, seysmologiya, tibbiy tomografiya va geofizika kabi sohalardagi amaliy ehtiyojlar bilan bog‘liq. Masalan, $F(x,t)=g(t)$ bo‘lgan holda, atom elektr stansiyalarida yuz bergan avariyalar holatlarida manba joylashuvi ma’lum bo‘lishi mumkin, lekin nurlanish quvvatining vaqt bo‘yicha kamayishi noma’lum bo‘lishi mumkin. Bunday fizik jarayonlarni o‘rganish esa ilmiy va amaliy jihatdan katta ahamiyatga ega. Shuningdek, $F(x,t)=f(x)$ bo‘lgan hol noqonuniy oqova suv tashlamalarini aniqlash kabi masalalarni hal qilishda muhim ahamiyat kasb etadi. Bunday masalalarning fizik mohiyati va amaliy ahamiyati ularni turli sohalarda o‘rganishga sabab bo‘lmoqda. Shuning uchun, kasr tartibli tenglamalar uchun to‘g‘ri va teskari masalalarning yechimlarini qurish maqsadli ilmiy tadqiqotlardan hisoblanadi.

Mamlakatimizda aniq va tabiiy fanlar, jumladan, matematika, fizika, biologiya, geologiya sohalarini rivojlantirishga katta e’tibor qaratilmoqda. Jumladan, kasr tartibli va xususiy hosilali differensial tenglamalar nazariyasini rivojlantirish muhim ustuvor yo‘nalishlardan biri sifatida belgilangan. Chunki bu nazariya mexanika, elektronika, boshqaruv tizimlari, fiziologiya va biologik jarayonlarni tushunishda asosiy rol o‘ynaydi. Yuqorida ta’kidlanganidek, muhim yo‘nalishlar bo‘yicha xalqaro standartlar darajasida ilmiy tadqiqot olib borish, matematika fanining asosiy vazifasi va faoliyat yo‘nalishi etib belgilandi[1]. Mamlakatimiz mutaxassislari tomonidan mazkur sohalarda salmoqli natijalar qo‘lga kiritilib, ular yuqoridagi farmon ijrosini ta’minlashda muhim ahamiyat kasb etmoqda.

[1] O’zbekiston Respublikasi Vazirlar Mahkamasining 2017 yil 18 maydagi «O’zbekiston Respublikasi Fanlar akademiyasining yangidan tashkil etilgan ilmiy-tadqiqot muassasalari faoliyatini tashkil etish to’g’risida » gi № 292-sonli qarori.

Mamlakatimizda differensial tenglamalar va matematik fizika kabi muhim yo‘nalishlari bo‘yicha xalqaro miqiyosda olib borilayotgan tadqiqotlar, fundamental tadqiqotlarning asosiy yo‘nalishi sifatida qaralayotgani quvonarlidir.

Mazkur dissertatsiya ishining predmeti va tadqiqot ob’yekti O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-sonli "O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi" haqidagi farmonlarida belgilangan vazifalariga mos keladi, 2017 yil 17-fevraldagi PQ-2789 "Fanlar Akademiyasi faoliyati, ilmiy-tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora-tadbirlari to‘g‘risida"gi Prezident qarori, 2018 yil 27 apreldagi "Innovatsion g‘oyalar, texnologiyalar va loyihalarni amaliy joriy qilish tizimini yanada takomillashtirish chora-tadbirlari to‘g‘risida"gi PQ-3682-sonli qarori va 2019 yil 9 iyuldagi "Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab quvvatlash, shuningdek, O‘zbekiston Respublikasi Fanlar Akademiyasining V.I.Romanovskiy nomidagi matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida"gi PQ-4387 sonli Prezident qarori hamda 2020 yil 7 maydagi "Matematika sohasidagi ta’lim sifatini oshirish va ilmiy tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida"gi PQ-4708 sonli Prezident qarori hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga bog‘liqligi. Mazkur dissertatsiya Respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishi doirasida bajarilgan.

Muammoning o‘rganilganlik darajasi. Kasr tartibli differensial tenglamalar va ular bilan bog‘liq bo‘lgan to‘g‘ri va teskari masalalar zamonaviy matematikaning faol tadqiqot sohasiga aylandi. Bu sohada Sh.O. Alimov, R.R.Ashurov, S.R. Umarov, M. Yamamoto, Z. Li, A.V. Psxu, M. Kirane, M. Ruzhansky, D.Q. Durdiyev, Yu.E. Fayziyev, Z.A. Sobirov, E.T. Karimov, B.X. Turmetov, Y. Zhang, X.T. Nguyn, A.S. Malik va boshqa ko‘plab matematik olimlarning salmoqli hissasi bor. Kasr tartibli differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalarning yechimining mavjudligi va yagonaligi yuqoridagi mualliflarning ishlarida batafsil o‘rganilgan. Turli xil diffuziya va subdiffuziya tenglamalarining o‘ng tomonini aniqlashning teskari masalalari amaliy ahamiyatga ega bo‘lib, ko‘pgina olimlarning taqdqiqotlari bunday teskari masalalarni hal qilishga bag‘ishlangan. Ammo, shuni ta’kidlash kerakki, shu paytgacha $F(x,t)$ manba funksiyasini aniqlash bo‘yicha hali umumiy nazariya mavjud emas. Hozirgi kungacha bo‘lgan barcha ishlarda manba funksiyasi $F(x,t)\equiv f(x)g(t)$ bo‘lgan holda, $f(x)$ yoki $g(t)$ funksiyani topish teskari masalalari o‘rganilgan.

Matematik olimlar tomonidan manba funksiyasining vaqtga bog‘liq bo‘lmagan qismi ya’ni $f(x)$ funksiyani topishning teskari masalalari ikkita holda o‘rganilgan: $g(t)\equiv 1$ va $g(t)\not\equiv 1$. Diffuziya va subdiffuziya tenglamalari uchun $g(t)\equiv 1$ va $f(x)$ noma’lum holda R.R. Ashurov, M. Kirane, M. Ruzhansky, N.A.

Asl, K. Furati, S. Liu va boshqalar ishlarida batafsil o‘rganilgan. $g(t) \not\equiv 1$. bo‘lgan holda esa $f(x)$ funksiyani topish teskari masalalari biroz murakkab bo‘lib, bunday masalalarning yechilishi $g(t)$ funksiyaga bog‘liq. S.I. Kabanixin, A.I. Prilepko, D.G. Orlovskii, K.B. Sabitov, V.E. Fedorov, M. Slodicka, K. Van Bokstal kabi olimlarning maqolalarida bunday teskari masalalar o‘rganilgan. Bunday masalalarni o‘rganishda mualliflar, asosan, ikki xil ya’ni oxirgi vaqt $(u(x,T)=\psi(x))$ yoki integral ($\int_0^T u(x,t)dt=\psi(x)$) shartlardan qo‘shimcha shart sifatida foydalangan. Klassik diffuziya tenglamalari uchun birinchi shart bilan bunday teskari masalalar S.I. Kabanikhin, A.I. Prilepko, D.G. Orlovskii, K.B. Sabitov, I.V. Tikhonov va boshqalarning ishlarida o‘rganilgan. Subdiffuziya tenglamasiga kelsak, P. Nyu, M. Slodicka va boshqalarning maqolalarda bunday teskari masalalar tahlil qilingan. Ikkinchi shart bilan bunday teskari masalalar A.I. Prilepko, D.G. Orlovskii, K. Van Bokstal, V.E. Fedorov kabi mualliflar tomonidan diffuziya va subdiffuziya tenglamalari uchun o‘rganilgan.

Manba funksiyasining vaqtga bog‘liq bo‘lgan qismini ya’ni $g(t)$ funksiyani aniqlashning teskari masalalarini o‘rganish biroz murakkabroq bo‘lib, bunday teskari masalalar kam tadqiqotchilar tomonidan o‘rganilgan. Diffuziya tenglamalari uchun A. Ashyralyev, E. Azizbayov, M.J. Damirchi, A. Hazanee, M. Slodicka kabi olimlar bunday teskari masalalarni hal qilishga hissa qo‘shgan. Subdiffuziya tenglamalari uchun bunday teskari masalalar M. Yamamoto, K. Sakamoto, Y. Liu, Z. Li, M. Kirane va T. Wei kabi olimlarning maqolalarida o‘rganilgan.

Dissertatsiya tadqiqotining dissertatsiya bajarilgan Oliy ta’lim muassasasining ilmiy-tadqiqot ishlari rejalari bilan bog‘liqligi.

Dissertatsiya tadqiqoti V.I. Romanovskiy nomidagi Matematika institutida O‘zbekiston Respublikasi Oliy ta’lim, fan va innovatsiyalar vazirligining F-FA-2021-424-sonli ilmiy tadqiqot grantining rejalashtirilgan mavzusiga muvofiq amalga oshirildi.

Tadqiqotning maqsadi subdiffuziya tenglamasi uchun boshlang‘ich-chegaraviy, hamda diffuziya, subdiffuziya va kasr tartibli Shredinger tipidagi tenglamalarining manba funksiyasini topish bo‘yicha teskari masalalarning klassik yechimlarini topishdan iborat.

Tadqiqotning vazifalari:

elliptik qismi ixtiyoriy N o‘lchamli sohada aniqlangan Laplas operatori bo‘lgan Kaputo kasr hosilali subdiffuziya tenglamasi uchun boshlang‘ich-chegaraviy masalaning klassik ma’nosida yagona yechimining mavjud ekanligini isbotlash;

Kaputo va Riman-Liuvill kasr hosilali diffuziya va subdiffuziya tenglamalarining o‘ng tomonining vaqtga bog‘liq bo‘lmagan qismini aniqlash bo‘yicha teskari masalalarning klassik ma’nosida yagona yechimining mavjud ekanligini isbotlash;

Kaputo va Riman-Liuvill kasr hosilali Shredinger va subdiffuziya tenglamalarini o‘ng tomonining vaqtga bog‘liq qismini aniqlash bo‘yicha teskari masalalarning klassik ma’nosida yagona yechimining mavjud ekanligini isbotlash.

Tadqiqotning obyekti klassik diffuziya tenglamasi, Kaputo va Riman-Liuvill kasr hosilali Shredinger va subdiffuziya tenglamalari.

Tadqiqotning predmeti boshlang‘ich-chegaraviy masalasi va tenglamalarning o‘ng tomonini aniqlashning teskari masalalari.

Tadqiqotning usullari. Tadqiqotda funksional analiz usullari, spektral nazariya va Furye usullari qo‘llaniladi.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

elliptik qismi ixtiyoriy N o‘lchamli sohada aniqlangan Laplas operator bo‘lgan Kaputo kasr hosilali subdiffuziya tenglamasi uchun boshlang‘ich-chegaraviy masalaning klassik ma’noda yagona yechimining mavjud ekanligi isbotlangan;

Kaputo va Riman-Liuvill kasr hosilali subdiffuziya va diffuziya tenglamalarining o‘ng tomonining vaqtga bog‘liq bo‘lmagan qismini aniqlash bo‘yicha teskari masalalarning klassik ma’noda yagona yechimining mavjud ekanligini isbotlangan;

Kaputo va Riman-Liuvill kasr hosilali Shredinger va subdiffuziya tenglamalarining o‘ng tomonining vaqtga bog‘liq qismini aniqlash bo‘yicha teskari masalalarning klassik ma’noda yagona yechimining mavjud ekanligini isbotlangan.

Tadqiqotning amaliy natijalari. Ushbu dissertatsiyada olingan natijalar va qo‘llanilgan usullar oliy o‘quv yurtlari magistrantlari va doktorantlari uchun bitiruv kursi sifatida o‘qitilishi mumkin.

Tadqiqot natijalarining ishonchliligi. Natijalar funksional analiz, spektral nazariya va Furye usuli yordamida olingan. Olingan barcha natijalar matematik jihatdan to‘g‘ri.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy ahamiyati shundan iboratki, olingan ilmiy natijalar diffuziya, subdiffuziya va kasr tartibli Shredinger tenglamalari uchun boshlang‘ich-chegaraviy va teskari masalalarni yanada chuqurroq tadqiq qilishda qo‘llanilishi mumkin.

Dissertatsiyaning amaliy ahamiyati shundan iboratki, olingan natijalar texnik, fizik va biologik jarayonlarni matematik modellashtirishda qo‘llanilishi mumkin.

Tadqiqot ishlarning joriy qilinishi. Kasr tartibli Shredinger va subdiffuziya tenglamalari uchun to‘g‘ri va teskari masalalar bo‘yicha dissertatsiya ishida olingan natijalar asosida:

Kaputo va Riman-Liuvill kasr hosilali Shredinger va subdiffuziya tenglamalarining o‘ng tomonining vaqtga bog‘liq qismini aniqlash bo‘yicha teskari masalalarning klassik yechimlaridan 22-11-00064 raqamli “Geosferadagi dinamik jarayonlarni irsiyatni hisobga olgan holda modellashtirish” mavzusidagi xorijiy loyihada subdiffuziya tenglamalari uchun to‘g‘ri va teskari masalalarni tasniflashda foydalanilgan (Kosmofizika tadqiqotlari va radioto‘lqinlar tarqalish instituti 2025 yil 12-maydagi № 211-sonli ma’lumotnoma, Rossiya Federatsiyasi). Ilmiy natijani qo‘llanilishi subdiffuziya tenglamasida radon chiqishining manba funksiyasini aniqlash imkonini;

elliptik qismi ixtiyoriy o‘lchamli sohada aniqlangan Laplas operator bo‘lgan, Kaputo kasr hosilali subdiffuziya tenglamasi uchun to‘g‘ri va teskari masalalarning yechimlaridan 122041800013-4 raqamli “Umumlashgan kasr tartibli differensial operatorli tenglamalar uchun chegaraviy masalalarni o‘rganish, ularni fizik va ijtimoiy-iqtisodiy jarayonlarni modellashtirishda qo‘llash” mavzusidagi xorijiy loyihada turli fizikaviy va biologik jarayonlarni matematik modellashtirishda foydalanilgan (Kabardin-Balkar ilmiy markazi Amaliy matematika va avtomatlashtirish institutining 2025 yil 16-apreldagi № 01-13/48-sonli ma’lumotnomasi, Rossiya Federatsiyasi). Ilmiy natijalarning qo‘llanilishi turli fizikaviy va biologik jarayonlarni matematik modellashtirishda samarali qo‘llanilayotgan kasr tartibli hosila qatnashgan evolyutsion tenglamalar uchun lokal va nolokal chegaraviy masalalarni yechish imkonini bergan.

Tadqiqot natijalarining aprobatsiyasi. Tadqiqotning asosiy natijalari 12 ta xalqaro va 2 ta respublika ilmiy anjumanlarida muhokama qilindi.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya mavzusi bo‘yicha 20 ta ilmiy ishlar chop etilgan bo‘lib, shundan, 6 ta maqola O‘zbekiston Respublikasi Oliy attestatsiya komissiyasining falsafa doktorlik dissertatsiyalarining asosiy ilmiy natijalarini chop etish uchun tavsiya etilgan ilmiy nashrlarda chop etilgan, shulardan 2 tasi xorijiy va 4 tasi respublika jurnallarida chop etilgan bo‘lib, ulardan 5 tasi SCOPUS ma’lumotlar bazalarida indekslangan va 14 tasi tezisdir.

Dissertatsiyaning tuzilishi va hajmi. Dissertatsiya kirish, to‘rtta bob, xulosa va foydalanilgan adabiyotlar ro‘yxatidan iborat. Dissertatsiya hajmi 105 bet.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismida dissertasiya mavzusining dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan, mavzu bo‘yicha xorijiy ilmiy-tadqiqotlar sharhi, muammoning o‘rganilganlik darajasi keltirilgan, tadqiqot maqsadi, vazifalari, ob’ekti va predmeti tavsiflangan, tadqiqotning ilmiy yangiligi va amaliy natijalari bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertasiya tuzilishi bo‘yicha ma’lumotlar keltirilgan.

Dissertatsiyaning **“Dastlabki ma’lumotlar”** deb nomlangan birinchi bobi yordamchi xususiyatga ega bo‘lib, dissertatsiyani o‘qish qulay bo‘lishi uchun yaratilgan. Bu yerda yangi natijalar yo‘q, faqat kerakli ta’riflar va tasdiqlar to‘plangan.

Keling, birinchi bobdagi ba’zi kerakli ma’lumotlarni keltiraylik.

Hilbert fazosida abstrakt operator. $(\cdot,\cdot)$ skalyar ko‘paytma va norma $\|\cdot\|$ aniqlangan H separabel Hilbert fazosi bo‘lsin. Aytaylik, $A: H \to H$ operator H fazoda aniqlangan bo‘lib, o‘z-o‘ziga qo‘shma, quyidan chegaralangan, musbat aniqlangan bo‘lsin. Faraz qilaylik, A operator kompakt A^{-1} teskari operatorga ega. U holda u to‘la ortonormal $\{v_k\}$ xos vektorlar sistemasiga va ularga mos $\{\lambda_k\}$ musbat xos sonlar to‘plamiga ega bo‘ladi, ya’ni v_k vektorlar va λ_k sonlar

$Av_k = \lambda_k v_k$ tenglikni qanoatlantiradi. Xos sonlarni qayta nomerlash yordamida ularni kamaymaydigan qilib nomerlab olamiz, ya'ni $0 < \lambda_1 \leq \lambda_2 \cdots \to +\infty$.

Keling, H Hilbert fazosida A operatornig darajasi tushunchasini eslatib o'taylik. τ ixtiyoriy haqiqiy son bo'lsin. Biz H da A operatorning darajasini quyidagicha kiritamiz:

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

bu yerda $h_k = (h, v_k)$ lar $h \in H$ elementning Furye koeffitsiyentlari. Shubhasiz, ushbu operatorning aniqlanish sohasi quyidagi ko'rinishga ega bo'ladi:

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

Laplas operatorining o'z-o'ziga qo'shma kengaytmasi. Biz yuqorida Hilbert fazosida abstrakt o'z-o'ziga qo'shma operatorni ko'rib chiqdik. Biz qo'llagan usulda, A operator faqat to'la ortonormal xos vektorlar sistemasiga ega bo'lishi talab qilinganligi sababli, M. Ruzhansky va boshqalarning ishida berilgan har qanday elliptik operatorni A operator deb hisoblash mumkin. Masalan, $L_2(\Omega)$, $\Omega \subset \mathbb{R}^N$, ni H Hilbert fazosi. Aytaylik, A operator $L_2(\Omega)$ da $Ag(x) = -\Delta g(x)$ kabi aniqlangan, $D(A) = \{g \in C^2(\overline{\Omega}) : g(x) = 0, x \in \partial\Omega\}$ aniqlanish sohaga ega bo'lsin. A operatorning $L_2(\Omega)$ dagi o'z-o'ziga qo'shma kengaytmasini $\hat{A}$ bilan belgilaymiz.

Aytaylik, σ ixtiyoriy haqiqiy son bo'lsin. $L_2(\Omega)$ fazoda $\hat{A}$ operatorning darajasini quyidagicha aniqlaymiz:

$$\hat{A}^\sigma g(x) = \sum_{k=1}^{\infty} \lambda_k^\sigma g_k v_k(x), \quad g_k = (g, v_k),$$

va aniqlanish sohasi quyidagi shaklga ega

$$D(\hat{A}^\sigma) = \{g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 < \infty\}.$$

V.A. Il'inning teoremasi. Tenglamaning elliptik qismi Laplas operatori bo'lganda, Furye usulida boshlang'ich-chegaraviy masalalarning yechimlarining mavjudligini isbotlash uchun

$$\sum_{k=1}^{\infty} \lambda_k^\tau |h_k|^2, \quad \tau > \frac{N}{2}, \tag{1}$$

ko'rinishdagi qatorlarning yaqinlashuvchi ekanligini o'rganish kerak bo'ladi, bunda h_k lar $h(x)$ funksiyaning Furye koeffitsiyentlari. τ sonining butun qiymatlarida (1) qatorning yaqinlashishi uchun $h(x)$ funksiyaning qaysi $W_2^k(\Omega)$ klassik Sobolev fazolariga tegishligi bo'lishlik shartlari V.A. Il'inning fundamental ishida ko'rsatilgan. Ushbu shartlarni keltirish uchun biz $\dot{W}_2^1(\Omega)$ sinfni kiritamiz. Ω sohada uzluksiz differensiallanuvchi va $\partial\Omega$ chegaraning atrofida nolga teng barcha funksiyalar to'plamini $W_2^1(\Omega)$ ning normasi bo'yicha yopilmasini belgilaymiz.

Demak, agar $h(x)$ funksiya quyidagi

$$h(x)\in W_2^{\left[\frac{N}{2}\right]+1}(\Omega) \text{ va } h(x),\ \Delta h(x),\ \cdots,\ \Delta^{\left[\frac{N}{4}\right]}h(x)\in \dot{W}_2^1(\Omega), \tag{2}$$

shartlarni qanoatlantirsa, u holda (1) qator $\tau=\left[\frac{N}{2}\right]+1$ bo‘lganda yaqinlashuvchi bo‘ladi.

Xuddi shunday, agar (1) da τ ni $\left[\frac{N}{2}\right]+3$ ga almashtirsak, yaqinlashish shartlari quyidagi ko‘rinishga ega bo‘ladi:

$$h(x)\in W_2^{\left[\frac{N}{2}\right]+3}(\Omega), \quad \text{va} \quad h(x),\Delta h(x),\cdots,\Delta^{\left[\frac{N}{4}\right]+1}h(x)\in \dot{W}_2^1(\Omega). \tag{3}$$

Hölder sinflari. Laplas operatori bir o‘lchamli holatda Hölder sinflarini quyidagicha kiritish biz uchun qulaydir**.** $\omega_g(\delta)$ qiymat $g(x)\in C[0,\pi]$ funksiyaning uzluksizlik moduli bo‘lsin ya’ni:

$$\omega_g(\delta)=\sup_{|x_1-x_2|\le\delta}|g(x_1)-g(x_2)|, \quad x_1,x_2\in[0,\pi].$$

Agar ba’zi $a>0$ uchun quyidagi tengsizlik

$$\omega_g(\delta)\le C\delta^a \tag{4}$$

o‘rinli bo‘lsa, (bu yerda C o‘zgarmas δ ga bog‘liq emas va $g(0)=g(\pi)=0$), u holda $g(x)$ funksiya $C^a[0,\pi]$ Hölder sinfiga tegishli deyiladi. Keling, biz bunday C o‘zgarmaslarning eng kichigini $\|g\|_{C^a[0,\pi]}$ orqali belgilaymiz.

$g''(x)$ funksiya (4) shartni qanoatlantirsin, ya’ni $\omega_{g''}(\delta)\le C\delta^a$, va $g(0)=g(\pi)=0$, $g''(0)=g''(\pi)=0$ shartlar bajarilsin. Bu shartlarni bajaruvchi $g(x)$ funksiyalar sinfini $C_2^a[0,\pi]$ orqali belgilaymiz.

Dissertatsiyaning asosiy natijalari ikkinchi bobdan boshlanadi. Ushbu bob **“Elliptik qismi Laplas operator bo‘lgan diffuziya va subdiffuziya tenglamalari uchun to‘g‘ri va teskari masalalar”** deb nomlanadi.

Ushbu bobning birinchi bo‘limida N o‘lchamli Ω sohada elliptik qismi Laplas operatori bo‘lgan Kaputo kasr hosilali subdiffuziya tenglamasi uchun boshlang‘ich-chegaraviy masala o‘rganildi.

Aytaylik, B Banax fazosi va $AC[0,T]$ to‘plam $[0,T]$ da aniqlangan absolyut uzluksiz funksiyalar to‘plami bo‘lsin. Biz $AC([0,T];B)$ orqali $[0,T]$ da absolyut uzluksiz va qiymatlari B dan bo‘lgan funksiyalar fazosini belgilaymiz. $C([0,T];B)$ fazo ham xuddi shunday aniqlanadi.

1-masala. Aytaylik, $\rho\in(0,1)$ bo‘lsin. Quyidagi boshlang‘ich-chegaraviy masalani

$$\begin{cases} D_t^\rho u(x,t)-\Delta u(x,t)=F(x,t)\equiv f(x)g(t), \quad x\in\Omega, \quad t\in(0,T], \\ u(x,t)|_{\partial\Omega}=0, \\ u(x,0)=\varphi(x), \quad x\in\overline{\Omega}, \end{cases} \tag{5}$$

qanoatlantiruvchi, hamda $D_t^\rho u(x,t)\in C(\overline{\Omega}\times(0,T])$, $\Delta u(x,t)\in C(\overline{\Omega}\times(0,T])$ xossalarga ega bo‘lgan $u(x,t)\in AC([0,T];C(\overline{\Omega}))$ funksiyani toping. Biz bu qidirilayotgan $u(x,t)$ funksiyani masalaning klassik yechimi deb ataymiz.

Bu yerda $g(t)\in C[0,T]$, $f(x)$ va $\varphi(x)$ esa $\Omega\subset R^N$ sohada berilgan uzluksiz funksiyalar.

1-teorema. *Aytaylik, $g(t)\in C[0,T]$ bo‘lsin. Bundan tashqari, $\varphi(x)$ va $f(x)$ funksiyalar (2) shartlarni qanoatlantirsin. U holda (5) boshlang‘ich-chegaraviy masalaning yagona yechimi mavjud va u quyidagi qator ko‘rinishga ega bo‘ladi:*

$$u(x,t)=\sum_{k=1}^{\infty}\left[\varphi_k E_\rho(-\lambda_k t^\rho)+f_k\int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)g(t-\eta)d\eta\right]v_k(x), \qquad (6)$$

va bu qator $x\in\overline{\Omega}$ da va barcha $t\in[0,T]$ lar uchun absolyut va tekis yaqinlashadi.

Bu yerda φ_k, f_k lar mos ravishda, $\varphi(x)$, $f(x)$ funksiyalarining Furye koeffitsientlaridir, masalan $\varphi_k=(\varphi,v_k)$.

Ushbu bobning ikkinchi bo‘limida (5) boshlang‘ich-chegaraviy masaladagi subdiffuziya tenglamasining o‘ng tomonidagi vaqtga bog‘liq bo‘lmagan qismini topish teskari masalasi o‘rganildi.

2-masala. Aytaylik, $\rho\in(0,1]$ bo‘lsin. (5) boshlang‘ich-chegaraviy masalani va quyidagi qo‘shimcha shartni

$$u(x,t_0)=\psi(x),\quad x\in\overline{\Omega}, \qquad (7)$$

qanoatlantiruvchi, hamda $D_t^\rho u(x,t),\Delta u(x,t)\in C(\overline{\Omega}\times(0,T])$ xossalarga ega bo‘lgan $u(x,t)\in AC([0,T];C(\overline{\Omega}))$ va $f(x)\in C(\overline{\Omega})$ funksiyalar juftligini toping.

Bu yerda $\psi(x)$ funksiya $\Omega\in R^N$ sohada berilgan ma’lum funksiya, t_0 esa $(0,T]$ segmentda berilgan ixtiyoriy fiksirlangan nuqta.

Bu teskari masalani yechish uchun, biz (5) boshlang‘ich-chegaraviy masalaning (6) yechimiga (7) qo‘shimcha shartni qo‘llaymiz va ψ_k orqali $\psi(x):\psi_k=(\psi,v_k)$ funksiyaning Furye koeffitsientlarini belgilaymiz. U holda quyidagi tenglikka ega bo‘lamiz:

$$\sum_{k=1}^{\infty} f_k b_{k,\rho}(t_0)v_k(x)=\sum_{k=1}^{\infty}\psi_k v_k(x)-\sum_{k=1}^{\infty}\varphi_k E_\rho(-\lambda_k t_0)v_k(x),$$

bu yerda

$$b_{k,\rho}(t)=\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(-\lambda_k(t-s)^\rho)g(s)ds.$$

Biz bu tenglikdan f_k koeffitsientlarni topish uchun, Furye koeffitsientlarini tenglashtiramiz va quyidagi tenglikka ega bo‘lamiz:

$$f_k b_{k,\rho}(t_0)=\psi_k-\varphi_k E_\rho(-\lambda_k t_0).$$

Bu tenglikka ko‘ra, f_k koeffitsientlarni topish uchun $b_{k,\rho}(t_0)$ ifodaning noldan farqli bo‘lishi muhim. Agar $g(t)$ funksiya ishorasi almashmaydigan funksiya bo‘lsa, $b_{k,\rho}(t_0)$ ifoda barcha k larda noldan farqli. Aks holda ya’ni $g(t)$ funksiya ishorasi almashinuvchi funksiya bo‘lsa, u holda $b_{k,\rho}(t_0)=0$ bo‘lishi mumkin. Bu holda yechimning yagonaligi yo‘qolishi mumkin. Keling, bu hol uchun quyidagi misolni keltiramiz.

1-misol *($g(t)$ funksiya ishorasi almashinuvchi bo‘lganda). Quyidagi $u(x,t)$ va $f(x)$ funksiyalar juftligini topish bo‘yicha teskari masala berilgan bo‘lsin:*

$$\begin{cases} D_t^\rho u(x,t)-\Delta u(x,t)=f(x)g(t), \quad (x,t)\in\Omega\times(0,T], \\ u(x,t)|_{\partial\Omega}=0, \\ u(x,0)=0, \quad x\in\overline{\Omega}, \\ u(x,t_0)=0, \quad x\in\overline{\Omega}. \end{cases} \tag{8}$$

Laplas operatorining bir jinsli Dirixle chegaraviy shartlarni ya’ni

$$\begin{cases} -\Delta v=\lambda v \\ v(x)|_{\partial\Omega}=0 \end{cases}$$

qanoatlantiruvchi λ ga mos *v xos funksiyasini olamiz. Bundan tashqari, $t_0=1$ va $T(t)=t^\rho(1-t^b)$, $b>0$ bo‘lsin. U holda $g(t)=D_t^\rho T(t)+\lambda T(t)$ uchun, $u(x,t)=T(t)v(x)$ va $f(x)=v(x)$ yechimlar (8) masalani qanoatlantiradi.*

U holda, biz (8) masalaning $(u,f)=(0,0)$ trivial yechimidan tashqari, $u(x,t)=T(t)v(x),\ f(x)=v(x)$. notrivial yechimiga ham ega bo‘lamiz.

Bu misolda, $b=0.1$ va $\rho=0.5$ deb parametrlarni tanlash orqali, $g(t)$ funksiyaning ishorasi almashinuvchi ekanini ko‘rsatish qiyin emas:

$$g(t)=\frac{\rho B(\rho,1-\rho)}{\Gamma(1-\rho)}-\frac{(b+\rho)t^\rho B(b+\rho,1-\rho)}{\Gamma(1-\rho)}+\lambda t^\rho(1-t^b),$$

va

$$g(0)=0.5\Gamma(0.5)=\frac{\sqrt{\pi}}{2}>0, \qquad g(1)=\frac{\sqrt{\pi}}{2}-\frac{0.6\Gamma(0.6)}{\Gamma(1.1)}<0.$$

Keling, N natural sonlar to‘plamini ikkita guruhga ajrataylik $N=K_\rho\cup K_{0,\rho}$:

1) Agar $b_{k,\rho}(t_0)=0$ bo‘lsa, u holda $k\in K_{0,\rho}$;

2) Aks holda esa ya’ni agar $b_{k,\rho}(t_0)\neq 0$ bo‘lsa, u holda $k\in K_\rho$.

Endi (5), (7) teskari masalani yechish uchun $g(t)$ funksiyaning ishorasi almashmaydigan va almashinuvchi hollarini alohida qaraymiz.

Birinchi, $g(t)$ funksiya ishorasi almashmaydigan hol uchun quyidagi asosiy lemmani keltiramiz.

1-lemma. *Aytaylik, $g(t)\in C[0,T]$, $g(t)\geq 0$ (yoki $g(t)\leq 0$), $t\in[0,T]$ va $\exists\tau\in[0,t_0]$, $g(\tau)\neq 0$ bo‘lsin. U holda barcha $k\in N$ lar uchun quyidagi baho o‘rinli:*

$$|b_{k,\rho}(t_0)|\geq\frac{C_0}{\lambda_k}.$$

Bu yerda $C_0>0$ o‘zgarmas son t_0 ga bog‘liq.

Endi 1-lemmaga ko‘ra, $g(t)$ funksiya ishorasi almashmaydigan holda quyidagi natijani taqdim etamiz.

2-teorema. *Aytaylik,* $g(t)\in C[0,T]$, $g(t)\ge 0$ (*yoki* $g(t)\le 0$), $t\in[0,T]$ *va* $\exists\,\tau\in[0,t_0]$, $g(\tau)\ne 0$ *bo‘lsin. Bundan tashqari,* $\varphi(x)$ *funksiya (2) shartlarni va* $\psi(x)$ *funksiya (3) shartlarni qanoatlantirsin. U holda (5), (7) teskari masalaning yagona yechimi mavjud va u quyidagi qator ko‘rinishiga ega:*

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{b_{k,\rho}(t_0)}\left[\psi_k-\varphi_k E_\rho(-\lambda_k t_0^{\rho})\right]v_k(x), \tag{9}$$

$$u(x,t)=\sum_{k=1}^{\infty}\varphi_k E_\rho(-\lambda_k t^{\rho})v_k(x)+\sum_{k=1}^{\infty}\frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)}\left[\psi_k-\varphi_k E_\rho(-\lambda_k t_0^{\rho})\right]v_k(x), \tag{10}$$

va bu qator $x\in\overline{\Omega}$ *da va barcha* $t\in[0,T]$ *lar uchun absolyut va tekis yaqinlashadi.*

Bu yerda φ_k, ψ_k *lar mos ravishda,* $\varphi(x)$, $\psi(x)$ *funksiyalarining Furye koeffitsientlaridir.*

Endi $g(t)$ funksiya ishorasi almashinuvchi bo‘lgan holda quyidagi lemmani keltiramiz.

2-lemma. *Aytaylik,* $\rho\in(0,1]$, $g(t)\in C^1[0,T]$ *va* $g(t_0)\ne 0$ bo‘lsin. *U holda shunday* k_0 *soni mavjudki barcha* $k\ge k_0$ *lar uchun quyidagi baho o‘rinli:*

$$|b_{k,\rho}(t_0)|\ge\frac{C_0}{\lambda_k}, \tag{11}$$

bu yerda $C_0>0$ *o‘zgarmas son* ρ, k_0 *va* t_0 *ga bog‘liq.*

2-lemmaga ko‘ra, $g(t)$ funksiya ishorasi almashmaydigan holda quyidagi tasdiqlarni olishimiz mumkin.

1-tasdiq. *Agar 2-lemmaning barcha shartlari bajarilsa, u holda (11) quyi baho barcha* $k\in K_\rho$ *lar uchun o‘rinli.*

2-tasdiq. *Agar 2-lemmaning barcha shartlari bajarilsa va* $K_{0,\rho}$ *to‘plam bo‘sh bo‘lmasa, u holda* $K_{0,\rho}$ *to‘plam chekli elementlarga ega bo‘ladi.*

Endi $g(t)$ funksiya ishorasi almashinuvchi bo‘lgan hol uchun quyidagi natijani taqdim etamiz.

3-teorema. *Aytaylik,* $\rho\in(0,1]$, $g(t)\in C^1[0,T]$ *va* $g(t_0)\ne 0$ *bo‘lsin. Bundan tashqari,* $\varphi(x)$ *funksiya (2) shartlarni va* $\psi(x)$ *funksiya (3) shartlarni qanoatlantirsin.*

1. Agar $K_{0,\rho}$ *to‘plam bo‘sh bo‘lsa, u holda (5), (7) teskari masalaning yagona yechimi mavjud va u (9)-(10) ko‘rinishda aniqlanadi.*

2. Agar $K_{0,\rho}$ *to‘plam bo‘sh bo‘lmasa, u holda (5), (7) teskari masalaning yechimining mavjudligi uchun quyidagi shartlarning bajarilishi zarur va yetarli:*

$$\psi_k=\varphi_k E_\rho(-\lambda_k t_0^{\rho}),\quad k\in K_{0,\rho}. \tag{12}$$

Bu holda, (1), (3) teskari masalaning yechimi mavjud, ammo yagona emas:

$$f(x)=\sum_{k\in K_\rho}\frac{1}{b_{k,\rho}(t_0)}\left[\psi_k-\varphi_k E_\rho(-\lambda_k t_0{}^\rho)\right]v_k(x)+\sum_{k\in K_{0,\rho}} b_k v_k(x),$$

$$u(x,t)=\sum_{k=1}^{\infty}[\varphi_k E_\rho(-\lambda_k t^\rho)+f_k b_{k,\rho}(t)]v_k(x),$$

bu yerda b_k, $k\in K_{0,\rho}$, *ixtiyoriy haqiqiy sonlar.*

1-izoh. *Shuni ta'kidlashimiz kerakki, (12) shartlar ham zarur, ham yetarli. Bu shartlarni tekshirish biroz qiyin. Shuning uchun biz (12) shartlarni tekshirish oson bo'lgan quyidagi yetarli shartlar bilan almashtirishimiz mumkin:*

$$\varphi_k=(\varphi,v_k)=0,\quad \psi_k=(\psi,v_k)=0,\quad k\in K_{0,\rho}.$$

Shuni ta'kidlash kerakki, ushbu bo'limda teskari masala uchun keltirilgan natijalar diffuziya tenglamalari uchun ham mutlaqo yangidir.

Uchinchi bob **"Diffuziya va subdiffuziya tenglamalari uchun teskari masalalarning kuchli yechimlari"** deb nomlanadi.

Ushbu bobning birinchi bo'limida H separabel Hilbert fazosida Riman-Liuvill kasr hosilali subdiffuziya tenglamasining elliptik qismi abstrakt operator bo'lgan holda, tenglamaning o'ng tomonidagi vaqtga bog'liq bo'lmagan qismini topish teskari masalasi o'rganildi.

3-masala. Aytaylik, $\rho\in(0,1)$ bo'lsin. Quyidagi Koshi masalasini

$$\begin{cases}\partial_t^\rho u(t)+Au(t)=g(t)f+p(t), & t\in(0,T],\\ \lim\limits_{t\to0} J_t^{\rho-1}u(t)=\varphi,\end{cases}\tag{13}$$

va

$$u(t_0)=\psi,\tag{14}$$

qo'shimcha shartni qanoatlantiruvchi, hamda $\partial_t^\rho u(t),\ Au(t)\in C((0,T];H)$ xossalarga ega bo'lgan $t^{1-\rho}u(t)\in C([0.T];H)$ funksiya va $f\in H$ elementni toping. Bu yerda $g(t)\in C[0,T]$, $p(t)\in C((0,T];H)$, φ, ψ esa H Hilbert fazosida berilgan ma'lum elementlar va t_0 esa $(0,T]$ da ixtiyoriy fiksirlangan nuqta.

Bu teskari masala uchun olingan natijalarni $g(t)$ funksiya ishorasi almashmaydigan va almashinuvchi hollar uchun alohida taqdim etamiz.

4-teorema. *Aytaylik,* $\varphi\in H$, $\psi\in D(A)$ *va ba'zi* $\epsilon\in(0,1)$ *uchun* $t^{1-\rho}p(t)\in C([0,T];D(A^\epsilon))$ *bo'lsin. Bundan tashqari,* $g(t)$ *funksiya 1-lemmaning barcha shartlarini qanoatlantirsin. U holda (13)-(14) teskari masalaning yagona yechimi mavjud va u quyidagi qator ko'rinishiga ega:*

$$f=\sum_{k=1}^{\infty}\frac{1}{b_{k,\rho}(t_0)}\left[\psi_k-t_0^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t_0^\rho)-P_k(t_0)\right]v_k,\tag{15}$$

$$u(t)=\sum_{k=1}^{\infty}\left[t^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t^\rho)+P_k(t)\right]v_k\tag{16}$$

$$+\sum_{k=1}^{\infty}\frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)}\left[\psi_k-t_0^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t_0^\rho)-P_k(t_0)\right]v_k,$$

bu yerda

$$P_k(t)=\int_0^t (t-\eta)^{\rho-1}E_{\rho,\rho}(-\lambda_k(t-\eta)^\rho)p_k(\eta)d\eta$$

va φ_k, ψ_k *lar mos ravishda,* φ, ψ *elementlarining Furye koeffitsientlaridir.*

5-teorema. *Aytaylik,* $\varphi\in H$, $\psi\in D(A)$ *va ba'zi* $\epsilon\in(0,1)$ *uchun* $t^{1-\rho}p(t)\in C([0,T];D(A^\epsilon))$ *bo'lsin. Bundan tashqari,* $g(t)$ *funksiya 2-lemmaning barcha shartlarini qanoatlantirsin.*

1) Agar $K_{0,\rho}$ *to'plam bo'sh bo'lsa, u holda (13)-(14) teskari masalaning yagona yechimi mavjud va u (15)-(16) qator ko'rinishda aniqlanadi.*

2) Agar $K_{0,\rho}$ *to'plam bo'sh bo'lmasa, u holda (13)-(14) teskari masalaning yechimining mavjudligi uchun quyidagi shartlarning bajarilishi zarur va yetarli:*

$$\psi_k=t_0^{\rho-1}\varphi_kE_{\rho,\rho}(-\lambda_kt_0^\rho)+P_k(t_0),\quad k\in K_{0,\rho}. \tag{17}$$

Bu holda, (13)-(14) teskari masalaning yechimi mavjud, ammo yagona emas:

$$f=\sum_{k\in K_\rho}\frac{1}{b_{k,\rho}(t_0)}\left[\psi_k-t_0^{\rho-1}\varphi_kE_{\rho,\rho}(-\lambda_kt_0^\rho)-P_k(t_0)\right]v_k+\sum_{k\in K_{0,\rho}}b_kv_k,$$

$$u(t)=\sum_{k=1}^{\infty}[t^{\rho-1}\varphi_kE_{\rho,\rho}(-\lambda_kt^\rho)+f_k+P_k(t)]v_k,$$

bu yerda b_k, $k\in K_{0,\rho}$, *ixtiyoriy haqiqiy sonlar.*

2-izoh. (17) *zarur va yetarli shartlarni ham tekshirish oson bo'lgan ortogonallik shartlariga almashtirishimiz mumkin:*

$$\varphi_k=(\varphi,v_k)=0,\quad \psi_k=(\psi,v_k)=0,\quad p_k(t)=(p(t),v_k)=0,\quad k\in K_{0,\rho}.$$

Ushbu bobning ikkinchi bo'limida nolokal chegaraviy shartli subdiffuziya tenglamasining o'ng tomonidagi vaqtga bog'liq bo'lmagan qismini topish teskari masalasi H separabel Hilbert fazosida o'rganildi.

4-masala. Aytaylik, $\rho\in(0,1]$ bo'lsin. Quyidagi nolokal chegaraviy masalani

$$\begin{cases} D_t^\rho u(t)+Au(t)=g(t)f, & t\in(0,T],\\ u(T)=u(0), \end{cases} \tag{18}$$

va

$$\int_0^T u(t)dt=\psi, \tag{19}$$

qo'shimcha shartni qanoatlantiruvchi, hamda $D_t^\rho u(t), Au(t)\in C((0,T];H)$ xossalarga ega bo'lgan $u(t)\in AC([0,T];H)$ funksiya va $f\in H$ elementni toping. Bu yerda $g(t)\in C[0,T]$, φ, ψ esa H Hilbert fazosida berilgan ma'lum elementlar.

Biz (18)-(19) teskari masalani yechish uchun, (18) nolokal chegaraviy masalaning yechimidan foydalanamiz:

Agar $f\in H$ *va* $g(t)\in C[0,T]$ *bo'lsa, u holda (18)* nolokal chegaraviy *masalaning yagona yechimi mavjud:*

$$u(t)=\sum_{k=1}^{\infty}\frac{f_k}{1-E_\rho(-\lambda_kT^\rho)}\left[(1-E_\rho(-\lambda_kT^\rho))b_{k,\rho}(t)+E_\rho(-\lambda_kt^\rho)b_{k,\rho}(T)\right]v_k. \tag{20}$$

Bu yerda $f_k, k \geq 1$, *lar* f *elementning Furye koeffitsientlari.*

Bu teskari masalani yechish uchun, (18) nolokal-chegaraviy masalaning (20) yechimiga (19) qo‘shimcha shartni qo‘llaymiz va ψ_k orqali $\psi(x): \psi_k = (\psi, v_k)$ ning Furye koeffitsientlarini belgilaymiz. U holda quyidagi tenglikka ega bo‘lamiz:

$$\sum_{k=1}^{\infty} f_k \left[(1 - E_\rho(-\lambda_k T^\rho)) G_{k,\rho}(T) + T E_{\rho,2}(-\lambda_k T^\rho) b_{k,\rho}(T) \right] v_k = \sum_{k=1}^{\infty} (1 - E_\rho(-\lambda_k T^\rho)) \psi_k v_k,$$

bu yerda

$$G_{k,\rho}(T) = \int_0^T (T-\eta)^\rho E_{\rho,\rho+1}(-\lambda_k (T-\eta)^\rho) g(\eta) d\eta.$$

Biz bu tenglikdan f_k koeffitsientlarni topish uchun, Furye koeffitsientlarini tenglashtiramiz va quyidagi tenglikka ega bo‘lamiz:

$$f_k \Phi_\rho(k,T) = (1 - E_\rho(-\lambda_k T^\rho)) \psi_k,$$

bu yerda

$$\Phi_\rho(k,T) = (1 - E_\rho(-\lambda_k T^\rho)) G_{k,\rho}(T) + T E_{\rho,2}(-\lambda_k T^\rho) b_{k,\rho}(T).$$

Bu tenglikka ko‘ra, f_k koeffitsientlarni topish uchun $\Phi_\rho(k,T)$ ifodaning noldan farqli bo‘lishi muhim. Agar $g(t)$ funksiya ishorasi almashmaydigan funksiya bo‘lsa, $\Phi_\rho(k,T)$ ifoda barcha k larda noldan farqli. Aks holda ya’ni $g(t)$ funksiya ishorasi almashinuvchi funksiya bo‘lsa, u holda $\Phi_\rho(k,T) = 0$ bo‘lishi mumkin. Bu holda yechimning yagonaligi yo‘qolishi mumkin. Keling, bu hol uchun quyidagi misolni keltiramiz.

2-misol *(* $g(t)$ *funksiya ishorasi almashinuvchi bo‘lganda). Aytaylik,* $0 < \rho \leq 1$ *bo‘lsin. Quyidagi teskari masalani ko‘raylik:*

$$\begin{cases} D_t^\rho u(x,t) - u_{xx}(x,t) = f(x) g(t), & (x,t) \in (0,\pi) \times (0,1], \\ u(0,t) = u(\pi,t) = 0, & t \in [0,1], \\ u(x,0) = u(x,1), & x \in [0,\pi], \\ \int_0^1 u(x,t) = 0, & x \in [0,\pi], \end{cases} \qquad (21)$$

Ma’lumki, (21) teskari masalaning $(u, f) = (0,0)$ *trivial yechimi mavjud. Bundan tashqari (21) teskari masalani qanoatlantiruvchi* $u(x,t) = \omega(t)\sin(x)$ *va* $f(x) = \sin(x)$ *notrivial yechimi ham mavjud. Bu yerda*

$$g(t) = D_t^\rho \omega(t) + \omega(t), \qquad va \qquad \omega(t) = t^2 - t + \frac{1}{6}.$$

$g(t)$ *funksiya*ni *quyidagi ko‘rinishda yozish mumkin:*

$$g(t) = \frac{t^{2-\rho} - (2-\rho) t^{1-\rho}}{\Gamma(2-\rho)} + t^2 - t + \frac{1}{6}.$$

Ushbu misolda yechimning yagonaligining buzilganligini sababi shundaki, $g(t)$ *ishorasi almashinuvchi funksiya:*

$$g(0) = \frac{1}{6} > 0, \qquad va \qquad g\left(\frac{1}{2}\right) = -\frac{\rho - 1{,}5}{2^{1-\rho}\Gamma(2-\rho)} - \frac{1}{12} < 0.$$

Keling, N natural sonlar to‘plamini ikkita guruhga ajrataylik $N = K_\rho \cup K_{0,\rho}$: Agar $\Phi_\rho(k,T) = 0$ bo‘lsa, u holda $k \in K_{0,\rho}$, agar $\Phi_\rho(k,T) \neq 0$ bo‘lsa, $k \in K_\rho$.

Endi (18)-(19) teskari masalani yechish uchun $g(t)$ funksiyaning ishorasi almashmaydigan va almashinuvchi hollarini alohida qaraymiz.

(18)-(19) teskari masalaning yechimining mavjudligi va yagonaligini isbotlash uchun 1, 2- lemmalardagi kabi maxraj uchun quyi baholarni olishimiz kerak. Xuddi shunday, $\Phi_\rho(k,T)$ ifoda uchun quyi baholarni olib, quyidagi natijalarni olishimiz mumkin.

$g(t)$ funksiya ishorasi almashmaydigan holda, quyidagi natijani keltiramiz.

6-teorema. *Aytaylik,* $\rho \in (0,1]$, $\psi \in D(A)$ *bo‘lsin. Bundan tashqari,* $g(t) \in C[0,T]$, $g(t) \geq 0$ *(yoki* $g(t) \leq 0$*),* $t \in [0,T]$ *va* $\exists \tau \in [0,T]$, $g(\tau) \neq 0$ *bo‘lsin. U holda (18)-(19) teskari masalaning yagona yechimi mavjud:*

$$u(t) = \sum_{k=1}^{\infty} \frac{\psi_k}{\Phi_\rho(k,T)} \left[(1 - E_\rho(-\lambda_k T^\rho)) b_{k,\rho}(t) + E_\rho(-\lambda_k t^\rho) b_{k,\rho}(T) \right] v_k, \tag{22}$$

$$f = \sum_{k=1}^{\infty} \frac{1 - E_\rho(-\lambda_k T^\rho)}{\Phi_\rho(k,T)} \psi_k v_k, \tag{23}$$

va ψ_k *lar* ψ *elementning Furye koeffitsientlaridir.*

$g(t)$ funksiya ishorasi almashinuvchi bo‘lgan holda, quyidagi natijani keltiramiz.

7-teorema. *Aytaylik,* $\rho \in (0,1]$, $\psi \in D(A)$, $g(t) \in C[0,T]$ *va* $\left| \int_0^T g(t)dt \right| \geq g_0 > 0$ *bo‘lsin.*

1) Agar $K_{0,\rho}$ *to‘plam bo‘sh bo‘lsa, u holda (18)-(19) teskari masalaning yagona yechimi mavjud va u (22)-(23) qator ko‘rinishda aniqlanadi.*

2) Agar $K_{0,\rho}$ *to‘plam bo‘sh bo‘lmasa, u holda teskari masalaning yechimining mavjudligi uchun quyidagi shartlarning bajarilishi zarur va yetarli:*

$$\psi_k = (\psi, v_k) = 0, \quad k \in K_{0,\rho}.$$

Bu holda, (18)-(19) teskari masalaning yechimi mavjud, lekin yagona emas:

$$f = \sum_{k \in K_\rho} \frac{\psi_k (1 - E_\rho(-\lambda_k T^\rho))}{\Phi_\rho(k,T)} v_k + \sum_{k \in K_{0,\rho}} b_k v_k,$$

$$u(t) = \sum_{k=1}^{\infty} \frac{f_k}{1 - E_\rho(-\lambda_k T^\rho)} \left[(1 - E_\rho(-\lambda_k T^\rho)) b_{k,\rho}(t) + E_\rho(-\lambda_k t^\rho) b_{k,\rho}(T) \right] v_k,$$

bu yerda b_k, $k \in K_{0,\rho}$ *ixtiyoriy haqiqiy sonlar.*

Shuni ta’kidlash kerakki, 6-7-teoremalar diffuziya tenglamalari uchun ham mutlaqo yangidir.

Dissertatsiyaning to‘rtinchi bobi **“Kasr tartibli Shredinger va subdiffuziya tenglamalari uchun teskari masalalar”** deb nomlanadi.

Ushbu bobning birinchi bo‘limida Riman-Liuvill kasr hosilali Shredinger tenglamasining o‘ng tomonidagi vaqtga bog‘liq qismini topish teskari masalasi o‘rganildi.

5-masala. Aytaylik, $\rho\in(0,1)$ va $\Omega=(0,\pi)\times(0,T]$ bo‘lsin. Kasr tartibli Shredinger tenglamasi uchun quyidagi boshlang‘ich-chegaraviy masalani

$$\begin{cases} i\partial_t^{\rho}u(x,t)-u_{xx}(x,t)=g(t)f(x)+p(x,t), \quad (x,t)\in\Omega, \\ u(0,t)=u(\pi,t)=0, \quad 0\le t\le T, \\ \lim\limits_{t\to 0} J_t^{\rho-1}u(x,t)=\varphi(x), \quad 0\le x\le\pi, \end{cases} \tag{24}$$

va

$$B[u(\cdot,t)]=\psi(t), \quad 0\le t\le T, \tag{25}$$

qo‘shimcha shartni qanoatlantiruvchi, hamda $\partial_t^{\rho}u(x,t), u_{xx}(x,t)\in C(\Omega)$ xossalarga ega bo‘lgan $t^{1-\rho}u(x,t)\in C(\overline{\Omega})$ va $t^{1-\rho}g(t)\in C[0,T]$ funksiyalar juftligini toping.
Bu yerda $B:C[0,\pi]\to R$ berilgan chiziqli funksional ya’ni $\| B[h(\cdot,t)]\|_{C[0,T]}\le b\| h(x,t)\|_{C(\overline{\Omega})}$, va $\psi(t)$ berilgan uzluksiz funksiya. Masalan, B funksional sifatida $B[u(\cdot,t)]=u(x_0,t)$, $x_0\in[0,\pi]$, $B[u(\cdot,t)]=\int_0^{\pi}u(x,t)dx$, yoki bu ikkita funksionalning chiziqli kombinatsiyasini olishimiz mumkin. $t^{1-\rho}p(x,t)$ va $\varphi(x)$, $f(x)$ esa $\overline{\Omega}$ yopiq sohada berilgan uzluksiz funksiyalar.

8-teorema. *Aytaylik,* $a>\frac{1}{2}$ *va* $t^{1-\rho}p(x,t)\in C_x^a(\overline{\Omega})$, $\varphi(x)\in C^a[0,\pi]$, $t^{1-\rho}\psi(t),\ t^{1-\rho}\partial_t^{\rho}\psi(t)\in C[0,T]$, $f(x)\in C_2^a[0,\pi]$, $B[f(x)]\ne 0$ *shartlar o‘rinli bo‘lsin. U holda (24)-(25) teskari masalaning yagona yechimi mavjud.*

Agar biz qo‘shimcha ravishda boshlang‘ich funksiyaga $\varphi(x)\in C_2^a[0,\pi]$ shartni talab qilsak, teskari masalaning yechimining turg‘unligi uchun quyidagi natijani olishimiz mumkin.

9-teorema. *Aytaylik, 8-teoremaning shartlari bajarilsin va* $\varphi(x)\in C_2^a[0,\pi]$ *bo‘lsin. U holda (24)-(25) teskari masalaning yechimi uchun quyidagi turg‘unlik baholari o‘rinli:*

$$\left\|t^{1-\rho}\partial_t^{\rho}u\right\|_{C(\overline{\Omega})}+\left\|t^{1-\rho}u_{xx}\right\|_{C(\overline{\Omega})}+\left\|t^{1-\rho}g\right\|_{C[0,T]}\le$$
$$C_{\rho,f,B}\left[\left\|\varphi_{xx}\right\|_{C^a[0,\pi]}+\left\|t^{1-\rho}\psi\right\|_{C[0,T]}+\left\|t^{1-\rho}\partial_t^{\rho}\psi\right\|_{C[0,T]}+\left\|t^{1-\rho}p(x,t)\right\|_{C_x^a(\overline{\Omega})}\right].$$

Bu yerda $C_{\rho,f,B}$ *o‘zgarmas son* ρ, f *va* B *larga bog‘liq.*

Ushbu bobning ikkinchi bo‘limida Kaputo kasr hosilali subdiffuziya tenglamasining o‘ng tomonidagi vaqtga bog‘liq qismini topish teskari masalasi o‘rganildi.

6-masala. Aytaylik, $\rho\in(0,1)$ bo‘lsin. Subdiffuziya tenglamasi uchun (1) boshlang‘ich-chegaraviy masalani va

$$u(x_0,t)=\psi(t), \quad 0\le t\le T, \tag{26}$$

qo‘shimcha shartni qanoatlantiruvchi, hamda $D_t^\rho u(x,t), \Delta u(x,t) \in C(\bar{\Omega} \times (0,T])$ xossalarga ega bo‘lgan $u(x,t) \in AC([0,T]; C(\bar{\Omega}))$ va $g(t) \in C[0,T]$ funksiyalar juftligini toping. Bu yerda $\psi(t) \in C[0,T]$ va $x_0 \in \bar{\Omega}$ ixtiyoriy nuqta.

10-teorema. *Aytaylik,* $f(x_0) \neq 0$ *va* $D_t^\rho \psi(t) \in C[0,T]$ *bo‘lsin. Bundan tashqari,* $\varphi(x)$ *va* $f(x)$ *funksiyalar (2) shartlarni qanoatlantirsin. U holda (1), (26) teskari masalaning yagona yechimi mavjud.*

11-teorema. *Aytaylik, 10-teoremaning shartlari bajarilsin. Bundan tashqari,* $\varphi(x)$ *funksiya (3) shartlarni qanoatlantirsin. U holda (1), (26) teskari masalaning yechimi uchun quyidagi turg‘unlik baholari o‘rinli:*

$$\|D_t^\rho u\|_{C(\bar{\Omega}\times[0,T])} + \|\Delta u\|_{C(\bar{\Omega}\times[0,T])} + \|g\|_{C[0,T]} \leq C_{\rho,f,x_0}\left[\|\varphi\|_\tau + \|\psi\|_{C[0,T]}\right],$$

bu yerda $\tau = \left[\frac{N}{2}\right] + 3$ *va* C_{ρ,f,x_0} *o‘zgarmas son* ρ, f *va* x_0 *ga bog‘liq.*

XULOSA

Ushbu dissertatsiya kasr tartibli Shredinger tipidagi va subdiffuziya tenglamalari uchun to‘g‘ri va teskari masalalarni o‘rganishga bag‘ishlangan. Ushbu masalalarni yechishning asosiy usuli klassik Furye usulidir.

Tadqiqotning asosiy natijalari quyidagilardan iborat:

1. Elliptik qismi Laplas operatori bo‘lgan subdiffuziya tenglamasi uchun boshlang‘ich-chegaraviy masala N o‘lchamli sohada o‘rganilgan. Furye usulidan foydalanib, boshlang‘ich-chegaraviy masalaning klassik yechimining mavjudligi va yagonaligi isbotlangan.

2. Subdiffuziya tenglamasining o‘ng tomonining vaqtga bog‘liq bo‘lmagan qismini aniqlash bo‘yicha teskari masalalar o‘rganilgan. Bu teskari masalalarda tenglamaning o‘ng tomoni $f(x)g(t)$ ko‘rinishda bo‘lib, bu erda $f(x)$ noma’lum va $g(t)$ berilgan funksiya. Shuning uchun, yechimning yagonaligi va mavjudligi $g(t)$ funksiyaga bog‘liq. Xususan:

• Agar $g(t)$ ishorasi almashmaydigan funksiya bo‘lsa, u holda ko‘rib chiqilgan teskari masalalarning yechimlarining mavjudligi va yagonaligi isbotlangan. Har bir bo‘limda ba’zi ishorasi o‘zgaruvchi $g(t)$ funksiyalar uchun yechimning yagonaligi buzilganligini ko‘rsatadigan misollar keltirilgan.

• Agar $g(t)$ ishorasi almashinuvchi funksiya bo‘lsa, u holda teskari masalalarning yechimlarining mavjudligi uchun zarur va yetarli shartlar topilgan.

Ushbu teskari masalalar bo‘yicha olingan natijalar nafaqat subdiffuziya tenglamalari, balki diffuziya tenglamalari uchun ham mutlaqo yangidir.

3. Kasr tartibli Shredinger va subdiffuziya tenglamalarining o‘ng tomonining vaqtga bog‘liq qismini topish bo‘yicha teskari masalalar o‘rganilgan. Bunday teskari masalalar integral tenglamani yechishga tenglashtiriladi. Tenglamaning o‘ng tomonidagi vaqtga bog‘liq qismini topishning bunday teskari masalalari biroz murakkabroq bo‘lib, kasr tartibli Shredinger tenglamalari uchun ilgari mualliflar tomonidan o‘rganilmagan. Ushbu bobda ko‘rib chiqilgan teskari masalalarning yechimining korrektligi isbotlangan.

SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES DSc.02/30.12.2019.FM.86.01 AT V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS

INSTITUTE OF MATHEMATICS

SHAKAROVA MARJONA DILSHOD QIZI

FORWARD AND INVERSE PROBLEMS FOR FRACTIONAL SHRÖDINGER AND SUBDIFFUSION EQUATIONS

01.01.02 – Differential equations and mathematical physics

ABSTRACT OF DISSERTATION OF THE DOCTOR OF PHILOSOPHY (PhD) ON PHYSICAL AND MATHEMATICAL SCIENCES

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INTRODUCTION

Actuality and demand of the theme of the dissertation. In recent decades, fractional differential equations have emerged as a powerful mathematical tool for modeling various anomalous phenomena at the world. In this context, many scientific are actively exploring new methods for solving fractional order differential equations and discovering new properties of their solutions. In addition to studying the solutions of fractional order differential equations, it is highly important to investigate inverse problems related to determining the coefficients of the equation, its right-hand side, the boundary functions, and the order of the fractional derivative. Studying inverse problems enables us to analyze and control processes described by fractional order equations. Therefore, many mathematicians, in addition to solving initial-boundary value problems for fractional order differential equation, have been studying various inverse problems for finding the order of fractional derivative, the initial and the source functions. This dissertation is dedicated to studying initial-boundary value problem for fractional order differential equation, as well as inverse problems of determining the source function for diffusion, subdiffusion, and fractional Schrödinger type equations.

In recent years, the study of the forward and inverse problems for fractional equations has made it possible to analyze and control physical processes. The interest in the study of source (right-hand side $F(x,t)$ of the equation) identification inverse problems is caused primarily in connection with practical requirements in various branches of mechanics, seismology, medical tomography, and geophysics. The identification of $F(x,t)=g(t)$ is appropriate, for example, in cases of accidents at nuclear power plants, when it can be assumed that the location of the source is known, but the decay of the radiation power over time is unknown and it is important to estimate it. On the other hand, one example of the identification of $F(x,t)=f(x)$ can be the detection of illegal wastewater discharges, which is a serious problem in some countries. Naturally, considering the physical significance of such problems, examining similar issues for various fields generates great interest. Therefore, constructing solutions to the forward and inverse problems for fractional order equations is a targeted scientific research.

In our country, great importance is placed on the advancement of disciplines within the exact and natural sciences, such as mathematics, physics, biology and geology. Particularly, the development of fractional and partial differential equation theory has been prioritized due to its essential role in understanding various phenomena in mechanics, electronics, control systems, physiology, and biological processes. Conducting research at the international level in these critical areas has been identified as a key priority in fundamental research[2]. Currently, progress in the theory of fractional and partial differential equations is pivotal to

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[2] Decree of Cabinet of Ministers of the Republic of Uzbekistan at the 2017 year 18 May « On measures on the organization of activities of the first created scientific research institutions of the Academy of Sciences of the Republic of Uzbekistan» № 292.

the successful implementation of this directive. Researchers in our country have made significant contributions to these fields, contributing significantly to the advancement of the decree. It is highly encouraging that the study of differential equations and mathematical physics, particularly at an international level, is recognized as a primary focus of fundamental scientific research in our nation.

The subject and object of research of this thesis are in line with tasks identified in the Decrees of the President of the Republic of Uzbekistan UP-4947 of February 7, 2017 “On the strategy of action for the further development Of the Republic of Uzbekistan”, PP-2789 dated February 17, 2017 “On measures to further improve of the activities of the Academy of Sciences, organization, management and financing of research activities”, PP-3682 from April 27, 2018 “On measures to further improve the system of practical implementation of innovative ideas, technologies and projects” and PP-4387 from July 9, 2019 “On measures to further development of mathematical education and science, total improvement of the activity of the Uzbekistan Academy of Sciences V.I.Romanovskiy Institute of Mathematics” andalso PP-4708 from May 7, 2020 “On measures to improve the quality of education and research in mathematics” as well as in other regulations related to basic science.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the priority areas of science and technology of Republic of Uzbekistan IV, «Mathematics, Mechanics and Computer Science».

The degree of scrutiny of the problem. Fractional partial differential equations and their associated inverse problems have become an area of active research in contemporary mathematics. Initial-boundary value problems for fractional differential equations have been extensively studied by various authors. Notable contributions in this field have been made by researchers such as Sh.O. Alimov, R.R. Ashurov, S.R. Umarov, M. Yamamoto, Z. Li, A.V. Pskhu, M. Kirane, M. Ruzhansky, D.K. Durdiev, Yu.E. Fayziev, Z.A. Sobirov, E.T. Karimov, B.X. Turmetov, Y. Zhang, H.T. Nguyn, A.S. Malik, and others. A lot of researchers is also devoted to inverse problems of determining the right-hand side of various diffusion and subdiffusion equations due to the importance of such problems for applications. However, it should be immediately noted that for the abstract case of the source function $F(x,t)$ there is no general theory yet. In all known works, the split source function $F(x,t) \equiv f(x)g(t)$ is considered and the methods of investigation depend on whether $f(x)$ or $g(t)$ is unknown.

The inverse problem of finding the space-dependent part of the source function $f(x)$, it was considered differently in two cases: $g(t) \equiv 1$ and $g(t) \not\equiv 1$. For diffusion and subdiffusion equations, the case $g(t) \equiv 1$ and $f(x)$ is unknown has been studied by R.R. Ashurov, M. Kirane, M. Ruzhansky, N.A. Asl, K. Furati, S. Liu and others in the papers. The case $g(t) \not\equiv 1$ is more complicated and the solvability of such problems depends on the behavior of the function $g(t)$. S.I. Kabanikhin, A.I. Prilepko, D.G. Orlovskii, K.B. Sabitov, V.E. Fedorov, M.

Slodicka, K. Van Bockstal and others studied such inverse problems. When studying such problems, the authors mainly considered one of the following two conditions $u(x,T)=\psi(x)$ and $\int_0^T u(x,t)dt=\psi(x)$ as an over-determination condition. For classical diffusion equations, such inverse problems with the first condition has been studied in the work of S.I. Kabanikhin, A.I. Prilepko, D.G. Orlovskii, K.B. Sabitov, I.V. Tikhonov and others. As for the subdiffusion equation, P. Niu, D.G. Orlovskii, M. Slodicka and K. Van Bockstal and others analyzed such inverse problems in the works. The inverse problem with second condition is also studied by many authors. The study of such inverse problems was conducted in two directions: in some the diffusion equation was considered, and in others the subdiffusion equations.

Inverse problems for determining the time-dependent part $g(t)$ of the source function are somewhat more difficult to study and have been studied by relatively few researchers. For diffusion equations, contributions have been made by A. Ashyralyev, E. Azizbayov, M.J. Damirchi, A. Hazanee, and M. Slodicka. For the subdiffusion equations, key results have been provided by M. Yamamoto, K. Sakamoto, Y. Liu, Z. Li, M. Kirane, and T. Wei.

Connection of the theme of the dissertation with the research works of higher education, where the dissertation is carried out. The dissertation research is done in accordance with the planned theme of scientific research grant no. F-FA-2021-424 Ministry of higher education, science and innovations of the Republic of Uzbekistan, at the Institute of Mathematics after named V.I. Romanovskiy.

The aim of research work is to prove unique solvability in the classical sense of initial-boundary value and inverse problems for diffusion, subdiffusion and fractional Shrödinger type equations.

Research problems:

to prove unique solvability in the classical sense of the initial-boundary value problem for the subdiffusion equation with the Caputo fractional derivative whose elliptic part is the Laplace operator in an arbitrary N dimensional domain;

to prove unique solvability in the classical sense of the inverse problems on identifying the space-dependent part of the right-hand side of diffusion and subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives;

to prove unique solvability in the classical sense of the inverse problems on identifying the time-dependent part of the right-hand side of Schrödinger and subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives.

The research object. The classical diffusion equation, Schrödinger and subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives.

The research subject. The initial-boundary value problem and the inverse problems on determining the right-hand side of the equations.

Research methods. In the research the methods of functional analysis, spectral theory and the Fourier methods are used.

Scientific novelty of the research work consists of the following:

it is proved unique solvability in the classical sense of initial-boundary value problem for the subdiffusion equation with the Caputo fractional derivative whose elliptic part is a Laplace operator in an arbitrary N dimensional domain;

it is proved unique solvability in the classical sense of the inverse problems on finding the space-dependent part, of the right-hand side of the fractional Schrödinger and subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives;

it is proved unique solvability in the classical sense of the inverse problems on finding the time-dependent part of the right-hand side of the fractional Schrödinger and subdiffusion equations with the Caputo and Riemann-Liouville fractional derivatives.

Practical results of the research. The results and methodologies presented in this dissertation can be incorporated into graduate-level courses designed for master's and doctoral students at higher education institutions.

The reliability of the results of the study. The results were derived using the techniques from functional analysis, spectral theory, and the Fourier method. All obtained results are mathematically correct.

Scientific and practical significance of the research results. The scientific significance of the research results is that they can be used for further study of the initial-boundary value and inverse problems for diffusion, subdiffusion and fractional Shrödinger type equations. The practical significance of the results of the dissertation is that its results can be used in mathematical modeling of technical, physical, and biological processes.

Implementation of the research results. Based on the results obtained on the problem of forward and inverse problems for fractional Schrödinger and subdiffusion equations:

the solutions to inverse problems for the Shrödinger and subdiffusion equations with Caputo and Riemann-Liouville fractional derivatives were used in the international project 22-11-00064 on the topic “Modeling of dynamic processes in geospheres considering heredity”, in the study of the forward and inverse problems for the subdiffusion equations (Reference from the Institute of Cosmophysical Research and Radio Wave Propagation dated May 12, 2025, № 211, Russian Federation). The application of the scientific result made it possible to determine an unknown source of radon emanation in the subdiffusion equation based on experimental data;

the solutions to the forward and inverse problems for the subdiffusion equation with Caputo fractional derivative when the elliptic part of the equation is the Laplace operator were used in the mathematical modeling of various physical and biological processes in the international project 122041800013-4 on the topic “Study of boundary value problems for equations with generalized fractional order differential operators, their application to modeling physical and socio-economic processes” (Reference from the Institute of Applied Mathematics and Automation of the Kabardino-Balkarian Scientific Center of the Russian Academy of Sciences dated April 16, 2025, № 01-13/48). The application of the scientific result made it possible to solve local and non-local boundary value problems for evolutionary equations involving fractional derivatives, which in recent decades have been effectively used in the mathematical modeling of various physical and biological processes.

Approbation of the research results. The main results of the research have been discussed at 12 international and 2 national scientific conferences.

Publications of the research results. On the topic of the dissertation 20 research papers have been published in the scientific journals, 6 of them are included in the list of journals proposed by the Higher Attestation Commission of the Republic of Uzbekistan for defending the PhD thesis and 2 and 4 of them were published in international and national mathematical journals, respectively. 5 of them are included the SCOPUS information database, and 14 theses.

The structure and volume of the dissertation. The dissertation consists of an introduction, four chapters, conclusion and bibliography. The total volume of the dissertation is 105 pages.

MAIN CONTENT OF THE DISSERTATION

The introduction substantiates the actuality and demand of the theme of the dissertation, determines the correspondence of the study to priority areas of development of science and technology, provides an overview of foreign scientific research on the dissertation topic and the degree of study of the problem, formulates goals and objectives, identifies the object and subject of the study, outlines the scientific novelty and practical results of the study, the theoretical and practical significance of the results obtained is disclosed, information is given on the implementation of the research results, on published works and information on the structure of the dissertation.

The first chapter of the dissertation, titled **"Preliminary information"**, is auxiliary in nature and it was created for the convenience of reading the dissertation. There are no new results here, and only the necessary definitions and assertions are collected.

Abstract operator in Hilbert space. Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $||\cdot||$. Suppose that the operator $A: H \to H$ is defined in H, is self-adjoint, bounded from below, and positive definite. Assume that an operator A has a compact inverse operator A^{-1}. Then it has a complete system of orthonormal eigenvectors $\{v_k\}$ and a set of corresponding positive eigenvalues $\{\lambda_k\}$, i.e. the vectors $\{v_k\}$ and values $\{\lambda_k\}$ satisfy the following equality:

$$Av_k = \lambda_k v_k.$$

It is assumed that the eigenvalues are ordered such that $0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty$.

In order to formulate the main results of this dissertation, we introduce the Hilbert space of "smooth" functions related to the degree of operator A.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^{\tau}h=\sum_{k=1}^{\infty}\lambda_k^{\tau}h_k v_k,$$

where h_k is the Fourier coefficient of the element $h: h_k=(h, v_k)$.
Obviously, the domain of definition of this operator has the form

$$D(A^{\tau})=\{h\in H:\sum_{k=1}^{\infty}\lambda_k^{2\tau}|h_k|^2<\infty\}.$$

Self-adjoint extension of the Laplace operator. Note that we considered an abstract self-adjoint operator A in a Hilbert space H. Since the operator A is only required to have a complete orthonormal system of eigenvectors, then as A one can consider any of the elliptic operators given in the work by Ruzhansky et al. For example, let us take $L_2(\Omega)$, $\Omega\subset R^N$, as the Hilbert space H. If A stands for the operator acting in $L_2(\Omega)$ as $Ag(x)=-\Delta g(x)$ with the domain of definition $D(A)=\{g\in C^2(\Omega): g(x)=0,\ x\in\partial\Omega\}$, then $\hat{A}\equiv\hat{A}^1$ is a self-adjoint extension of A in $L_2(\Omega)$.

Let σ be an arbitrary real number. Consider an operator $\hat{A}^{\sigma}$ acting in $L_2(\Omega)$ as:

$$\hat{A}^{\sigma}g(x)=\sum_{k=1}^{\infty}\lambda_k^{\sigma}g_k v_k(x),\quad g_k=(g,v_k),$$

with the domain of definition

$$D(\hat{A}^{\sigma})=\{g\in L_2(\Omega):\sum_{k=1}^{\infty}\lambda_k^{2\sigma}|g_k|^2<\infty\}.$$

Theorem of V.A. Il'in. In order to prove the existence of solutions of initial-boundary value problems by the Fourier method when the elliptic part of the equation is Laplace operator Δ, it is necessary to study the convergence of the following series:

$$\sum_{k=1}^{\infty}\lambda_k^{\tau}|h_k|^2,\quad \tau>\frac{N}{2},\tag{1}$$

where h_k are the Fourier coefficients of function $h(x)$. In the case of integers τ, in the fundamental paper by V.A. Il'in, conditions are obtained for the convergence of such series in terms of the membership of function $h(x)$ in the classical Sobolev spaces $W_2^k(\Omega)$, $\Omega\subset R^N$. To formulate these conditions, we introduce the class $\overset{\circ}{W}{}_2^1(\Omega)$ as the closure in the $W_2^1(\Omega)$ norm of the set of all functions that are continuously differentiable in Ω and vanish near the boundary of Ω.

The theorem of V.A. Il'in states that, if function $h(x)$ satisfies the conditions

$$h(x)\in W_2^{\left[\frac{N}{2}\right]+1}(\Omega)\quad and\quad h(x),\ \Delta h(x),....,\ \Delta^{\left[\frac{N}{4}\right]}h(x)\in\overset{\circ}{W}{}_2^1(\Omega),\tag{2}$$

then the number series (1) with $\tau = \left[\frac{N}{2}\right] + 1$ converges. Here $[a]$ denotes the integer part of the number a. Similarly, if in (1) we replace τ with $\left[\frac{N}{2}\right] + 3$, then the convergence conditions will have the form:

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+3}(\Omega) \quad and \quad h(x),\ \Delta h(x), ..., \ \Delta^{\left[\frac{N}{4}\right]+1} h(x) \in \overset{\circ}{W}{}_2^1(\Omega). \qquad (3)$$

The Hölder classes. In one dimensional case it is convenient for us to introduce the Hölder classes as follows. Let $\omega_g(\delta)$ be the modulus of continuity of function $g(x) \in C[0,\pi]$, i.e.

$$\omega_g(\delta) = \sup_{|x_1 - x_2| \le \delta} |g(x_1) - g(x_2)|, \quad x_1, x_2 \in [0,\pi].$$

If

$$\omega_g(\delta) \le C\delta^a \qquad (4)$$

is true for some $a > 0$, where C does not depend on δ and $g(0) = g(\pi) = 0$, then $g(x)$ is said to belong to the Hölder class $C^a[0,\pi]$. Let us denote the smallest of all such constants C by $\| g \|_{C^a[0,\pi]}$.

Now we define $C_2^a[0,\pi]$ as the class of functions $g(x)$ for which $g''(x)$ satisfy the estimate (4), i.e., $\omega_{g''}(\delta) \le C\delta^a$, and such that $g(0) = g(\pi) = 0$ and $g''(0) = g''(\pi) = 0$.

The main results of the dissertation begin with the second chapter, titled **“Forward and inverse problems for the diffusion and subdiffusion equations when the elliptic part is a Laplace operator”.**

The first paragraph of this chapter studies the initial-boundary problem for the subdiffusion equation with the Caputo fractional derivative, whose elliptic part is the Laplace operator, in an arbitrary N dimensional domain.

Let B be a Banach space and $AC[0,T]$ be the set of absolutely continuous functions defined on $[0,T]$. Throughout what follows, we denote by $AC([0,T];B)$ the space of functions that are absolutely continuous on $[0,T]$ and take values in B. The space $C([0,T];B)$ is defined similarly.

Problem 1. Let $\rho \in (0,1)$ be a fixed number. Find a function $u(x,t) \in AC([0,T]; C(\overline{\Omega}))$ with properties $D_t^\rho u(x,t) \in C(\overline{\Omega} \times (0,T])$, $\Delta u(x,t) \in C(\overline{\Omega} \times (0,T])$, that satisfies the following initial-boundary value problem

$$\begin{cases} D_t^\rho u(x,t) - \Delta u(x,t) = F(x,t) \equiv f(x)g(t), \quad x \in \Omega, \quad t \in (0,T], \\ u(x,t)|_{\partial\Omega} = 0, \\ u(x,0) = \varphi(x), \quad x \in \overline{\Omega}, \end{cases} \qquad (5)$$

where $g(t) \in C[0,T]$, $f(x)$ and $\varphi(x)$ are continuous functions in the domain $\Omega \subset R^N$.

Theorem 1. *Let* $g(t)\in C[0,T]$. *Moreover, let functions* $\varphi(x)$ *and* $f(x)$ *satisfy condition (2). Then problem (5) has a unique solution and it has the form of series*

$$u(x,t)=\sum_{k=1}^{\infty}\left[\varphi_k E_\rho(-\lambda_k t^\rho)+f_k\int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)g(t-\eta)d\eta\right]v_k(x), \quad (6)$$

which absolutely and uniformly converges on $x\in\overline{\Omega}$ *and for each* $t\in[0,T]$. *Here* φ_k *and* f_k *are the Fourier coeffitcients of the functions* $\varphi(x)$ *and* $f(x)$, *respectively.*

In the second section of this chapter, the inverse problem of finding the time-independent part of the right-hand side of the subdiffusion equation in the initial-boundary problem (5) is studied.

Problem 2. Let $\rho\in(0,1]$ be a fixed number. Find a pair of functions $u(x,t)\in AC([0,T];C(\overline{\Omega}))$ and $f(x)\in C(\overline{\Omega})$ with properties $D_t^\rho u(x,t)\in C(\overline{\Omega}\times(0,T])$, $\Delta u(x,t)\in C(\overline{\Omega}\times(0,T])$, that satisfies the initial-boundary-value problem (5) and the following the over-determination condition:

$$u(x,t_0)=\psi(x),\quad x\in\overline{\Omega}, \quad (7)$$

where $\psi(x)$ is a given function and t_0 is a given fixed point of the segment $(0,T]$.

We apply the additional condition (7) to solution (6) of the initial-boundary problem (5) and denote by ψ_k the Fourier coefficients of function $\psi(x):\psi_k=(\psi,v_k)$. Then

$$\sum_{k=1}^{\infty} f_k b_{k,\rho}(t_0)v_k(x)=\sum_{k=1}^{\infty}\psi_k v_k(x)-\sum_{k=1}^{\infty}\varphi_k E_\rho(-\lambda_k t_0{}^\rho)v_k(x),$$

where

$$b_{k,\rho}(t)=\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(-\lambda_k(t-s)^\rho)g(s)ds.$$

From here, to find f_k, we obtain the following equation

$$f_k b_{k,\rho}(t_0)=\psi_k-\varphi_k E_\rho(-\lambda_k t_0{}^\rho).$$

Of course the case $b_{k,\rho}(t_0)=0$ is critical. This can happen when $g(t)$ changes sign. The following example shows that for such $g(t)$ the uniqueness of the unknowns f_k can be violated.

Example 1. Consider the following homogeneous inverse problem

$$\begin{cases} D_t^\rho u(x,t)-\Delta u(x,t)=f(x)g(t), & (x,t)\in\Omega\times(0,T], \\ u(x,t)|_{\partial\Omega}=0, & \\ u(x,0)=0, & x\in\overline{\Omega}, \\ u(x,t_0)=0, & x\in\overline{\Omega}. \end{cases} \quad (8)$$

Take any eigenfunction v of the Laplace operator subject to homogeneous Dirichlet boundary conditions, i.e. $-\Delta v=\lambda v$ with $v(x)|_{\partial\Omega}=0$ and set $t_0=1$, $T(t)=t^\rho(1-t^b)$, $b>0$. Then, $u(x,t)=T(t)v(x)$ satisfies problem (8) with

$$f(x) = v(x) \quad \text{and} \quad g(t) = D_t^\rho T(t) + \lambda T(t).$$

Then, besides the trivial solution $(u, f) = (0,0)$ to problem (8), we also have the following non-trivial solution

$$u(x,t) = T(t)v(x), \qquad f(x) = v(x).$$

It can be easily shown that, for example, for the parameters $b = 0.1$ and $\rho = 0.5$, the function $g(t)$ changes its sign. Indeed, one has

$$g(t) = \frac{\rho B(\rho, 1-\rho)}{\Gamma(1-\rho)} - \frac{(b+\rho)t^\rho B(b+\rho, 1-\rho)}{\Gamma(1-\rho)} + \lambda t^\rho (1 - t^b),$$

and

$$g(0) = 0.5\Gamma(0.5) = \frac{\sqrt{\pi}}{2} > 0,$$

$$g(1) = 0.5\Gamma(0.5) - \frac{0.6B(0.6, 0.5)}{\Gamma(0.5)} = \frac{\sqrt{\pi}}{2} - \frac{0.6\Gamma(0.6)}{\Gamma(1.1)} < 0.$$

Let us divide the set of natural numbers N into two groups $K_{0,\rho}$ and K_ρ: $N = K_\rho \cup K_{0,\rho}$, while the number k is assigned to $K_{0,\rho}$, if $b_{k,\rho}(t_0) = 0$, and if $b_{k,\rho}(t_0) \neq 0$, then this number is assigned to K_ρ. Note that for some t_0 the set $K_{0,\rho}$ may be empty, then $K_\rho = N$. For example, if $g(t)$ is sign-preserving, then $K_\rho = N$, for all t_0.

If the function $g(t)$ is sign-preserving, we obtain the following lemma.

Lemma 1. *Let* $g(t) \in C[0,T]$, $g(t) \geq 0$ *(or* $g(t) \leq 0$*),* $t \in [0,T]$ *and* $\exists \tau \in [0, t_0]$, $g(\tau) \neq 0$. *Then there is constant* $C_0 > 0$*, depending on* t_0*, such that for all* k*:*

$$|b_{k,\rho}(t_0)| \geq \frac{C_0}{\lambda_k}.$$

Now, according to Lemma 1, we present the following result for the case where function $g(t)$ does not change sign.

Theorem 2. *Let* $g(t) \in C[0,T]$, $g(t) \geq 0$ *(or* $g(t) \leq 0$*),* $t \in [0,T]$ *and* $\exists \tau \in [0, t_0]$, $g(\tau) \neq 0$. *Moreover, let function* $\varphi(x)$ *satisfy condition (2) and* $\psi(x)$ *satisfy condition (3). Then there exists a unique solution of the inverse problem (5), (7):*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0{}^\rho) \right] v_k(x), \tag{9}$$

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x) + \sum_{k=1}^{\infty} \frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0{}^\rho) \right] v_k(x), \tag{10}$$

which absolutely and uniformly converges on $x \in \overline{\Omega}$ *and for each* $t \in [0,T]$. *Here* φ_k *and* ψ_k *are the Fourier coeffitcients of the functions* $\varphi(x)$ *and* $\psi(x)$*, respectively.*

If function $g(t)$ changes sign, we obtain the following lemma.

Lemma 2. *Let* $\rho \in (0,1]$, $g(t) \in C^1[0,T]$ *and* $g(t_0) \neq 0$. *Then there exists a number* k_0 *such that, starting from the number* $k \geq k_0$, *the following estimate holds:*

$$|b_{k,\rho}(t_0)| \geq \frac{C_0}{\lambda_k}, \tag{11}$$

where constant $C_0 > 0$ *depends on* ρ, k_0 *and* t_0.

According to Lemma 2, we can get the following corollaries when function $g(t)$ changes sign.

Corollary 1. *If conditions of Lemma 2 are satisfied, then estimate (11) holds for all* $k \in K_\rho$.

Corollary 2. *If conditions of Lemma 2 are satisfied and set* $K_{0,\rho}$ *is not empty, then set* $K_{0,\rho}$ *has a finite number elements.*

Now we introduce the following result when function $g(t)$ changes sign.

Theorem 3. *Let* $\rho \in (0,1]$, $g(t) \in C^1[0,T]$, $g(t_0) \neq 0$. *Moreover, let function* $\varphi(x)$ *satisfy condition (2) and* $\psi(x)$ *satisfy condition (3).*

1) If set $K_{0,\rho}$ *is empty, then there exists a unique solution of inverse problem (5), (7) and this solution has the form (9), (10).*

2) If set $K_{0,\rho}$ *is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions*

$$\psi_k = \varphi_k E_\rho(-\lambda_k t_0^\rho), \quad k \in K_{0,\rho}, \tag{12}$$

be satisfied. In this case, the solution to the problem (5), (7) exists, but is not unique:

$$f(x) = \sum_{k \in K_\rho} \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x) + \sum_{k \in K_{0,\rho}} b_k v_k(x),$$

$$u(x,t) = \sum_{k=1}^{\infty} [\varphi_k E_\rho(-\lambda_k t^\rho) + f_k b_{k,\rho}(t)] v_k(x),$$

where b_k, $k \in K_{0,\rho}$, *are arbitrary real numbers.*

Remark 1. *For conditions (12) to be satisfied, it suffices that the following orthogonality conditions hold:*

$$\varphi_k = (\varphi, v_k) = 0, \quad \psi_k = (\psi, v_k) = 0, \quad k \in K_{0,\rho}.$$

It should be noted that the results presented for the inverse problem in this paragraph are also completely new for the diffusion equation.

The third chapter of the dissertation is titled **"The strong solutions of the inverse problems for the diffusion and subdiffusion equations"**.

In the first section of this chapter, the inverse problem of finding the time-independent part of the right-hand side of the subdiffusion equation with Riemann-Liouville fractional derivative is studied in a Hilbert space H, when the elliptic part is an abstract operator.

Problem 3. Let $\rho \in (0,1)$ be a fixed number. Find a pair of function $t^{1-\rho}u(t) \in C([0,T];H)$ and element $f \in H$ with properties $\partial_t^\rho u(t),\, Au(t) \in C((0,T];H),$ that satisfies following Cauchy problem

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = g(t)f + p(t), \quad t \in (0,T], \\ \lim\limits_{t\to 0} J_t^{\rho-1} u(t) = \varphi, \end{cases} \tag{13}$$

and the over-determination condition

$$u(t_0) = \psi, \tag{14}$$

where $g(t) \in C[0,T]$ is a given scalar function, $p(t) \in C((0,T];H)$, $\varphi, \psi \in H$ and t_0 is a given fixed point of the segment $(0,T]$.

If the function $g(t)$ is sign-preserving, we obtain the following result.

Theorem 4. *Let $\varphi \in H$, $\psi \in D(A)$ and $t^{1-\rho}p(t) \in C([0,T];D(A^\epsilon))$ for some $\epsilon \in (0,1)$. Moreover let function $g(t)$ satisfy condition of Lemma 1. Then there exists a unique solution of the inverse problem (13)-(14):*

$$f = \sum_{k=1}^\infty \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - t_0^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t_0^\rho) - P_k(t_0)\right] v_k, \tag{15}$$

$$u(t) = \sum_{k=1}^\infty \left[t^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t^\rho) + P_k(t)\right] v_k \tag{16}$$

$$+\sum_{k=1}^\infty \frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)} \left[\psi_k - t_0^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t_0^\rho) - P_k(t_0)\right] v_k,$$

where

$$P_k(t) = \int_0^t (t-\eta)^{\rho-1} E_{\rho,\rho}(-\lambda_k (t-\eta)^\rho)\, p_k(\eta)\, d\eta.$$

If function $g(t)$ changes sign, we obtain the following result.

Theorem 5. *Let $\varphi \in H$, $\psi \in D(A)$ and $t^{1-\rho}p(t) \in C([0,T];D(A^\epsilon))$ for some $\epsilon \in (0,1)$. Further, we will assume that the conditions of Lemma 2 are satisfied.*

1) If set $K_{0,\rho}$ is empty, then there exists a unique solution of the inverse problem (13)-(14) and this solution has the form (15), (16).

2) If set $K_{0,\rho}$ is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions

$$\psi_k = t_0^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t_0^\rho) + P_k(t_0), \quad k \in K_{0,\rho} \tag{17}$$

be satisfied. In this case, the solution to the problem (13)-(14) exists, but is not unique:

$$f = \sum_{k\in K_\rho} \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - t_0^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t_0^\rho) - P_k(t_0)\right] v_k + \sum_{k\in K_{0,\rho}} b_k v_k,$$

$$u(t) = \sum_{k=1}^\infty [t^{\rho-1}\varphi_k E_{\rho,\rho}(-\lambda_k t^\rho) + f_k + P_k(t)] v_k,$$

where b_k, $k \in K_0$, are arbitrary real numbers.

Remark 2. Particularly, instead of condition (17), we can take orthogonality conditions that is easy to verify:

$$\varphi_k = (\varphi, v_k) = 0, \quad \psi_k = (\psi, v_k) = 0, \quad p_k(t) = (p(t), v_k) = 0, \quad k \in K_{0,\rho}.$$

In the second section of this chapter, the inverse problem of finding the time-independent part of the right-hand side of the subdiffusion equation with the non-local boundary condition is studied.

Problem 4. Let $\rho \in (0,1]$ be a fixed number. Find a pair of function $u(t) \in AC([0,T]; H)$ and element $f \in H$ with properties $D_t^\rho u(t),\ Au(t) \in C((0,T]; H)$, that satisfy the following non-local boundary value problem

$$\begin{cases} D_t^\rho u(t) + Au(t) = g(t) f, \quad t \in (0,T], \\ u(T) = u(0), \end{cases} \tag{18}$$

and the over-determination condition

$$\int_0^T u(t)dt = \psi, \tag{19}$$

where $g(t) \in C[0,T],\ \psi \in H$ are the given function and element, respectively.

To solve the inverse problem (18)-(19), we use the solution of the non-local boundary value problem (18):

If $f \in H$ and $g(t) \in C[0,T]$ then the unique solution of the forward problem has the form:

$$u(t) = \sum_{k=1}^{\infty} \frac{f_k}{1 - E_\rho(-\lambda_k T^\rho)} \Big[(1 - E_\rho(-\lambda_k T^\rho)) b_{k,\rho}(t) + E_\rho(-\lambda_k t^\rho) b_{k,\rho}(T) \Big] v_k, \tag{20}$$

where $f_k, k \ge 1,$ are Fourier coefficients of the element f.

Now we need to find the unknowns $\{u(t), f\}$ of inverse problem (18)-(19). For this, we apply additional condition (19) to equality (20) and denote by ψ_k the Fourier coefficients of the element $\psi : \psi_k = (\psi, v_k)$. Then:

$$\sum_{k=1}^{\infty} f_k \big[(1 - E_\rho(-\lambda_k T^\rho)) G_{k,\rho}(T) + T E_{\rho,2}(-\lambda_k T^\rho) b_{k,\rho}(T) \big] v_k = \sum_{k=1}^{\infty} (1 - E_\rho(-\lambda_k T^\rho)) \psi_k v_k,$$

where

$$G_{k,\rho}(T) = \int_0^T (T-\eta)^\rho E_{\rho,\rho+1}(-\lambda_k (T-\eta)^\rho) g(\eta) d\eta.$$

If we pass to the Fourier coefficients, then:

$$f_k \Phi_\rho(k,T) = (1 - E_\rho(-\lambda_k T^\rho)) \psi_k,$$

where

$$\Phi_\rho(k,T) = (1 - E_\rho(-\lambda_k T^\rho)) G_{k,\rho}(T) + T E_{\rho,2}(-\lambda_k T^\rho) b_{k,\rho}(T).$$

According to this equality, to find the coefficients f_k, it is important that expression $\Phi_\rho(k,T)$ different from zero. If function $g(t)$ does not changes sign, then expression $\Phi_\rho(k,T) \ne 0$ for all k. Otherwise, if function $g(t)$ changes sign, then $\Phi_\rho(k,T)=0$. In this case, the uniqueness of the solution may be lost. Let us give the following example for this case.

Example 2. Let $0<\rho\le 1$ and the inverse problem have the form

$$\begin{cases} D_t^\rho u(x,t)-u_{xx}(x,t)=f(x)g(t), & (x,t)\in(0,\pi)\times(0,1],\\ u(0,t)=u(\pi,t)=0, & t\in[0,1],\\ u(x,0)=u(x,1), & x\in[0,\pi],\\ \int_0^1 u(x,t)=0, & x\in[0,\pi], \end{cases} \tag{21}$$

First, note that this problem has a trivial solution $(u,f)=(0,0)$. One can easily verify that the pair of functions

$$u(x,t)=\omega(t)\sin x, \qquad f(x)=\sin x,$$

also satisfy problem (21) with

$$g(t)=D_t^\rho\omega(t)+\omega(t),$$

where

$$\omega(t)=t^2-t+\frac{1}{6}.$$

The function $g(t)$ is continuous on the interval $[0,1]$ and can be written in the form

$$g(t)=\frac{t^{2-\rho}-(2-\rho)t^{1-\rho}}{\Gamma(2-\rho)}+t^2-t+\frac{1}{6}.$$

The reason a non-unique solution exists in this example is that the function $g(t)$ is not sign-preserving. Indeed, one has

$$g(0)=\frac{1}{6}>0,$$

and

$$g\left(\frac{1}{2}\right)=-\frac{\rho-1{,}5}{2^{1-\rho}\Gamma(2-\rho)}-\frac{1}{12}<0.$$

Let us divide the set of natural numbers N into two groups $K_{0,\rho}$ and K_ρ: $N=K_\rho\cup K_{0,\rho}$.

Namely, the number k belongs to $K_{0,\rho}$, if $\Phi_\rho(k,T)=0$, and $k\in K_\rho$ if $\Phi_\rho(k,T)\ne 0$. Note that for some T the set $K_{0,\rho}$ may be empty, then $K_\rho=N$. For example, if $g(t)$ is sign-preserving, then $K_\rho=N$ for all k regardless of the value T.

Now, to solve the inverse problem (18)-(19), we consider separately the cases where function $g(t)$ does not change sign and changes sign, respectively.

To prove the existence and uniqueness of the solution to the inverse problem (18)-(19), we also need to obtain the lower bounds for the denominator obtained in Lemmas 1 and 2 in the second section. Similarly, lower bounds are established for the expression $\Phi_\rho(k,T)$ and we can obtain the following results.

If function $g(t)$ is sign-preserving, we have

Theorem 6. *Let* $\rho\in(0,1]$, $\psi\in D(A)$. *Moreover let* $g(t)\in C[0,T]$, $g(t)\ge 0$ *(or* $g(t)\le 0$*),* $t\in[0,T]$ *and* $\exists\tau\in[0,T]$, $g(\tau)\ne 0$. *Then there exists a unique solution of inverse problem (18)-(19):*

$$u(t)=\sum_{k=1}^{\infty}\frac{\psi_k}{\Phi_\rho(k,T)}\left[(1-E_\rho(-\lambda_k T^\rho))b_{k,\rho}(t)+E_\rho(-\lambda_k t^\rho)b_{k,\rho}(T)\right]v_k, \quad (22)$$

$$f=\sum_{k=1}^{\infty}\frac{1-E_\rho(-\lambda_k T^\rho)}{\Phi_\rho(k,T)}\psi_k v_k. \quad (23)$$

Here ψ_k *is the Fourier coeffitcients of the function* $\psi(x)$.

If function $g(t)$ changes sign, we obtain the following result.

Theorem 7. *Let* $\rho\in(0,1]$, $\psi\in D(A)$, $g(t)\in C[0,T]$ *and* $|\int_0^T g(t)dt|\ge g_0>0$.

1. If the set $K_{0,\rho}$ *is empty, then there exists a unique solution of the inverse problem (18)-(19) and this solution has the form (22), (23).*

2. If the set $K_{0,\rho}$ *is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions*

$$\psi_k=(\psi,v_k)=0,\quad k\in K_{0,\rho},$$

be satisfied. In this case, the~solution to the problem (18)-(19) exists, but is not unique. In this case the solution has the representation

$$f=\sum_{k\in K_\rho}\frac{\psi_k(1-E_\rho(-\lambda_k T^\rho))}{\Phi_\rho(k,T)}v_k+\sum_{k\in K_{0,\rho}}b_k v_k,$$

$$u(t)=\sum_{k=1}^{\infty}\frac{f_k}{1-E_\rho(-\lambda_k T^\rho)}\left[(1-E_\rho(-\lambda_k T^\rho))b_{k,\rho}(t)+E_\rho(-\lambda_k t^\rho)b_{k,\rho}(T)\right]v_k,$$

where b_k, $k\in K_{0,\rho}$, *are arbitrary real numbers.*

It should be noted that Theorems 6, 7 are also completely new for the diffusion equations.

The fourth chapter of the dissertation is titled **“Inverse problems for fractional order Schrödinger and subdiffusion equations”.**

Problem 5. Let $\rho\in(0,1)$ be a fixed number and $\Omega=(0,\pi)\times(0,T]$. Find a pair of functions $t^{1-\rho}u(x,t)\in C(\overline{\Omega})$ and $t^{1-\rho}g(t)\in C[0,T]$ with properties $\partial_t^\rho u(x,t),\ u_{xx}(x,t)\in C(\Omega)$, that that satisfies the following initial-boundary problem for the Shrödinger equation

$$\begin{cases} i\partial_t^\rho u(x,t)-u_{xx}(x,t)=g(t)f(x)+p(x,t), & (x,t)\in\Omega;\\ u(0,t)=u(\pi,t)=0, & 0\le t\le T;\\ \lim\limits_{t\to 0}J_t^{\rho-1}u(x,t)=\varphi(x), & 0\le x\le\pi, \end{cases} \quad (24)$$

and the over-determination condition

$$B[u(\cdot,t)]=\psi(t),\quad 0\le t\le T, \quad (25)$$

where $t^{1-\rho}p(x,t)$ and $\varphi(x)$, $f(x)$ are continuous functions in the closed domain $\overline{\Omega}$. $B:C[0,\pi]\to R$ is a given linear functional such that $\| B[h(\cdot,t)]\|_{C[0,T]}\le b\| h(x,t)\|_{C(\overline{\Omega})}$, and $\psi(t)$ is a given continuous function. For example, as the functional B one can take $B[u(\cdot,t)]=u(x_0,t)$, $x_0\in[0,\pi]$, or $B[u(\cdot,t)]=\int_0^\pi u(x,t)dx$, or a linear combination of these two functionals.

Theorem 8. *Let* $a>\frac{1}{2}$ *and the following conditions be satisfied* $t^{1-\rho}p(x,t)\in C_x^a(\overline{\Omega})$, $\varphi(x)\in C^a[0,\pi]$, $t^{1-\rho}\psi(t)$, $t^{1-\rho}\partial_t^\rho\psi(t)\in C[0,T]$, $f(x)\in C_2^a[0,\pi]$, $B[f(x)]\neq 0$. *Then the inverse problem has a unique solution* $\{u(x,t),g(t)\}$.

Theorem 9. *Let assumptions of Theorem 8 be satisfied and let* $\varphi(x)\in C_2^a[0,\pi]$.*Then the solution to the inverse problem obeys the stability estimate*

$$\left\|t^{1-\rho}\partial_t^\rho u\right\|_{C(\overline{\Omega})}+\left\|t^{1-\rho}u_{xx}\right\|_{C(\overline{\Omega})}+\left\|t^{1-\rho}g\right\|_{C[0,T]}$$
$$\le C_{\rho,f,B}\left[\left\|\varphi_{xx}\right\|_{C^a[0,\pi]}+\left\|t^{1-\rho}\psi\right\|_{C[0,T]}+\left\|t^{1-\rho}\partial_t^\rho\psi\right\|_{C[0,T]}+\left\|t^{1-\rho}p(x,t)\right\|_{C_x^a(\overline{\Omega})}\right]$$

where $C_{\rho,q,B}$ *is a constant, depending only on* ρ, f *and* B.

Problem 6. Let $\rho\in(0,1)$. Find a pair of functions $u(x,t)\in AC([0,T];C(\overline{\Omega}))$ and $g(t)\in C[0,T]$ with properties $D_t^\rho u(x,t)\in C(\overline{\Omega}\times(0,T])$, $\Delta u(x,t)\in C(\overline{\Omega}\times(0,T])$, that satisfies the initial-boundary problem (5) with the following additional condition:

$$u(x_0,t)=\psi(t),\quad 0\le t\le T, \tag{26}$$

Here $f(x)$ and $\varphi(x)$ are continuous functions in the closed domain $\Omega\subset R^N$, $\psi(t)\in C[0,T]$ and $x_0\in\overline{\Omega}$ is a given fixed point.

Theorem 10. *Let* $f(x_0)\neq 0$ *and* $D_t^\rho\psi(t)\in C[0,T]$. *Moreover let functions* $\varphi(x)$ and $f(x)$ *satisfy condition (2). Then the inverse problem has a unique solution* $\{u(x,t),g(t)\}$.

Theorem 11. *Let* $\varphi(x)$ *satisfy condition (3) and let assumptions of Theorem 10 be satisfied. Then the solution to the inverse problem obeys the stability estimate*

$$\left\|D_t^\rho u\right\|_{C(\overline{\Omega}\times[0,T])}+\left\|\Delta u\right\|_{C(\overline{\Omega}\times[0,T])}+\left\|g\right\|_{C[0,T]}\le C_{\rho,f,x_0}\left[\left\|\varphi\right\|_\tau+\left\|\psi\right\|_{C[0,T]}\right],$$

where $\tau=\left[\frac{N}{2}\right]+3$ *and* C_{ρ,f,x_0} *is a constant, depending only on* ρ, f *and* x_0.

CONCLUSION

This dissertation is devoted to the study of the direct and inverse problems for fractional order Schrödinger type and subdiffusion equations. The main method for solving these problems is the classical Fourier method.

The main results of the research are as follows:

1. Initial-boundary value problem for the subdiffusion equation were studied in arbitrary N dimensional domain, whose the elliptic part is the Laplace operator. Using the Fourier method, the existence and uniqueness of classical solutions for the initial-boundary value problem were proved.

2. The inverse problems for determining the space-dependent part of the right-hand side of the subdiffusion equation were studied. In these inverse problems, the right-hand side of the equation is expressed as $g(t)f(x)$, where $f(x)$ is unknown and $g(t)$ is a given function. Therefore, the uniqueness and existence of the solution depends on the function $g(t)$. Specifically:

- If the function $g(t)$ does not changes sign, then it is proved that the solution of the inverse problems exists and is unique. The examples are given showing the violation of the uniqueness of the solution for some sign-variable functions $g(t)$ in each inverse problem.
- If function $g(t)$ changes sign, then necessary and sufficient conditions are found for the existence of solutions to inverse problems, but in this case there is no uniqueness.

The results obtained for the inverse problems studied are completely new not only for the subdiffusion equations but also for diffusion equations.

3. The inverse problems studied for finding the time-dependent part of the right-hand side of the fractional Schrödinger and subdiffusion equations were studied. The inverse problem is equivalently reduced to solving an integral equation. These inverse problems of finding the time-dependent part of the right-hand side of the equation are somewhat complicated, and such inverse problems for fractional order Schrödinger equation have not been previously studied by the authors. The correctness of the solution of the inverse problems is proved.

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ИНСТИТУТ МАТЕМАТИКИ

ШАКАРОВА МАРЖОНА ДИЛШОД КИЗИ

ПРЯМЫЕ И ОБРАТНЫЕ ЗАДАЧИ ДЛЯ УРАВНЕНИЙ ШРЕДИНГЕРА И СУБДИФФУЗИИ ДРОБНОГО ПОРЯДКА

01.01.02 – Дифференциальные уравнения и математическая физика

**АВТОРЕФЕРАТ ДИССЕРТАЦИИ ДОКТОРА ФИЛОСОФИИ (PhD)
ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ**

ТАШКЕНТ – 2025

Тема диссертации доктора философии (PhD) по физико-математическим наукам зарегистрирована в Высшей аттестационной комиссии при Министрстве высшего образования, науки и инноваций Республики Узбекистан за № B2024.4. PhD/FM1178.

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ВВЕДЕНИЕ (аннотация диссертации доктора философии (PhD))

Целью исследования является доказательство однозначной разрешимости в классическом смысле начально-краевых и обратных задач для уравнений диффузии, субдиффузии и дробного типа Шредингера.

Объектом исследования являются классические уравнения диффузии, типа Шредингера и уравнения субдиффузии с дробными производными Капуто и Римана-Лиувилля.

Научная новизна исследования состоит в следующем:

Доказана однозначная разрешимость в классическом смысле начально-краевой задачи для уравнения субдиффузии с дробной производной Капуто, эллиптическая часть которого является оператором Лапласа в произвольной N - мерной области;

Доказана однозначная разрешимость в классическом смысле обратных задач по нахождению зависящей от пространства части правой части дробных уравнений Шредингера и субдиффузии с производными Капуто и Римана-Лиувилля.

Доказана однозначная разрешимость в классическом смысле обратных задач по нахождению зависящей от времени части правой части дробных уравнений Шредингера и субдиффузии с дробными производными Капуто и Римана-Лиувилля.

Внедрение результатов исследования.

Результаты, полученные в диссертационной работе по проблеме прямой и обратной задач для дробных уравнений Шредингера и субдиффузии, были внедрены в практику в рамках следующих научно-исследовательских проектах:

метод решения обратных задач для уравнений Шредингера и субдиффузии с дробными производными Капуто и Римана-Лиувилля были использованы в зарубежном проекте 22-11-00064 по теме “Моделирование динамических процессов в геосферах с учетом наследственности”, при исследовании прямых и обратных задач для уравнений субдиффузии (Института космофизических исследований и распространения радиоволн, справка № 211 от 12 мая 2025 года, Российская Федерация). Применение научного результата позволило изучить определение неизвестного источника эманации радона в уравнении субдиффузии на основе экспериментальных данных;

метод решения прямой и обратной задач для уравнения субдиффузии с дробной производной Капуто, когда эллиптическая часть уравнения является оператором Лапласа были использованы при математическом моделировании различных физических и биологических процессов в зарубежном проекте 122041800013-4 по теме “Исследование краевых задач для уравнений с обобщенными дробно-дифференциальными операторами, их применение к моделированию физических и социально-экономических процессов” (Института прикладной математики и автоматизации Кабардино-Балкарского научного центра РАН, справка 16 апреля 2025 г. № 01-13/48).

Применение научного результата позволило решить локальные и нелокальные краевые задачи для эволюционных уравнений с дробными производными, которые в последние десятилетия эффективно используются при математическом моделировании различных физических и биологических процессов.

Объем и структура диссертации. Диссертация состоит из введения, четырех глав, заключения и списка использованной литературы. Общий объем диссертации составляет 105 стр.

E’LON QILINGAN ISHLAR RO‘YXATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo‘lim (part I; I часть)

1. Ashurov R.R., Shakarova M.D., Time-dependent source identification problem for a fractional Schrödinger equation with the Riemann-Liouville derivative, Ukrainian Mathematical Journal, 75, 7, 871-887, 2023 (3. Scopus, IF=0,42).

2. Shakarova M.D., Time-dependent source identification problem for the subdiffusion equation with Caputo fractional derivative, Uzbek Mathematical Journal, 67, 3, 156-165, 2023 (Scopus).

3. Ashurov R.R., Shakarova M.D., Inverse problem for the subdiffusion equation with fractional Caputo derivative, Ufa Mathematical Journal, 16, 1, 111-126, 2024 (3. Scopus, IF=0,42).

4. Shakarova M.D., Inverse problem of the determining the right-hand side for the subdiffusion equation with Riemann-Liouville fractional derivative, Uzbek Mathematical Journal, 68, 1, 127-134, 2024 (Scopus).

5. Shakarova M.D., Inverse problems for the subdiffusion equation, Uzbek Mathematical Journal, 68, 3, 143-149, 2024 (Scopus).

6. Shakarova M.D., Space-dependent source identification problem for the subdiffusion equation with a non-local in time condition, Bulletin of the Institute of Mathematics, 8, 1, 43-51, 2025.

II bo‘lim (Part II; II часть)

7. Ashurov R.R., Shakarova M.D., Time-dependent source identification problem for fractional Schrödinger type equations, Traditional International April Mathematical Conference in honor of the Day of Science Workers of the Republic of Kazakhstan, pp. 126-127, Almaty, April 6-8, 2022.

8. Ashurov R.R., Shakarova M.D., Time-dependent source identification problem for a fractional Schrödinger equation with the Riemann-Liouville derivative, International scientific and practical conference “Modern problems of applied mathematics and information technologies”, pp. 143-144, Bukhara, May 11-12, 2022.

9. Ashurov R.R., Shakarova M.D., Inverse problem for the subdiffusion equation with fractional Caputo derivative, Traditional International April Mathematical Conference in Honor of the Day of Scientists of the Republic of Kazakhstan, pp. 126-128, Almaty, April 5-7, 2023.

10. Ashurov R.R., Shakarova M.D., Inverse problems for fractional Schrödinger and subdiffusion equations, Bulletin of Osh State University Mathematics. Physics. Technical Sciences, 1, 2, pp. 249-253, 2023.

11. Shakarova M.D., Time-dependent source identification problem for the subdiffusion equation with Caputo fractional derivative, International scientific and

practical conference “Current problems of mathematical modeling and information technologies”, pp. 228-230, Nukus, May 2-3, 2023.

12. Shakarova M.D., Inverse problem for fractional order subdiffusion equation, International mathematical conference “Functional Analysis in Interdisciplinary Applications”, pp. 101-102, Turkey, Antalya, October 2-7, 2023.

13. Ashurov R.R., Shakarova M.D., An inverse problem for the subdiffusion equation with a non-local in time condition, International scientific and practical conference “Actual problems of improving the quality of education using digital technologies”, pp. 67-69, Nukus, May 15, 2024.

14. Shakarova M.D., Inverse problem for the subdiffusion equation. Republican Scientific Conference “Actual Problems of Applied Mathematics, Mathematical Modeling and Informatics”, pp. 122-124, Nukus, May 24-25, 2024.

15. Shakarova M.D., Inverse problem for the subdiffusion equation with integral over-determination condition, International Scientific and Practical Conference “Modern Problems of Mathematics and its Teaching”, pp. 18-20, Khujand, June 21-22, 2024.

16. Ashurov R.R., Shakarova M.D., Inverse problem for the subdiffusion equation with fractional Caputo derivative, International conference “Inverse Problems: Modeling and Simulation”, p. 145, Malta, May 26- June 2, 2024.

17. Ashurov R.R., Shakarova M.D., Inverse problems for the subdiffusion equation with fractional Caputo derivative, XI International Scientific Conference “Modern Problems of Mathematics and Mechanics“, pp. 94-95, Baku, July 3-6, 2024.

18. Ashurov R.R., Shakarova M.D., Space-dependent source identification problem for subdiffusion equation, International Conference “Actual Problems Of Analysis, Differential Equations And Algebra“, pp. 82-84, Astana, January 7-11, 2025.

19. Ashurov R.R., Shakarova M.D., Space-dependent source identification problem for the subdiffusion equation, Traditional International April Mathematical Conference in honor of the Day of Science Workers of the Republic of Kazakhstan, p. 141, Almaty, April 1-4, 2025.

20. Ashurov R.R., Shakarova M.D., Space-dependent source identification problem for subdiffusion equation with a non-local in time condition, Republican Scientific Conference “Modern Methods of Mathematical Physics and their applications“, p. 23, Tashkent, April 22-24, 2025.

Avtoreferat "O'zbekiston matematika jurnali" tahririyatida
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