FV.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

MATEMATIKA INSTITUTI

NURALIYEVA NAVBAHOR SHAYMARDON QIZI

BUTUN VA KASR TARTIBLI DIFFERENSIAL TENGLAMALAR UCHUN KO‘PNUQTALI VA CHIZIQSIZ NOLOKAL MASALALAR

01.01.02 – Differensial tenglamalar va matematik fizika

FIZIKA-MATEMATIKA FANLARI BO‘YICHA FALSAFA DOKTORI (PhD) DISSERTATSIYASI AVTOREFERATI

TOSHKENT – 2025 yil

UDK: 517.927 517.95

Fizika – matematika fanlari bo'yicha falsafa doktori (PhD) dissertatsiyasi avtoreferati mundarijasi

Contents of dissertation abstract of doctor of philosophy (PhD) on physical – mathematical sciences

Оглавление автореферата диссертации доктора философии (PhD) по физико – математическим наукам

V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

MATEMATIKA INSTITUTI

NURALIYEVA NAVBAHOR SHAYMARDON QIZI

BUTUN VA KASR TARTIBLI DIFFERENSIAL TENGLAMALAR UCHUN KO‘PNUQTALI VA CHIZIQSIZ NOLOKAL MASALALAR

01.01.02 – Differensial tenglamalar va matematik fizika

FIZIKA-MATEMATIKA FANLARI BO‘YICHA FALSAFA DOKTORI (PhD) DISSERTATSIYASI AVTOREFERATI

TOSHKENT – 2025

Fizika-matematika fanlari bo‘yicha falsafa doktori (PhD) dissertatsiyasi mavzusi O‘zbekiston Respublikasi Oliy ta’lim, Fan va Innovatsiyalar Vazirligi huzuridagi Oliy attestatsiya komissiyasida № B2025.2.PhD/FM1288 raqam bilan ro‘yxatga olingan.

Dissertatsiya V.I. Romanovskiy nomidagi Matematika institutida bajarilgan.

Dissertatsiya avtoreferati uch tilda (o‘zbek, ingliz, rus (резюме)) Ilmiy kengash veb-sahifasi (http://kengash.mathinst.uz) va «Ziyonet» ta’lim axborot tarmog‘ida (www.ziyonet.uz) joylashtirilgan.

Ilmiy rahbar:	**Ashurov Ravshan Radjabovich** fizika-matematika fanlari doktori, professor
Rasmiy opponentlar:	**Durdiyev Durdimurod Qalandarovich** fizika-matematika fanlari doktori, professor
	Yaxshiboyev Maxmadiyor Umirovich fizika-matematika fanlari doktori, dotsent
Yetakchi tashkilot:	**O‘zbekiston Milliy universiteti**

Dissertatsiya himoyasi V.I.Romanovskiy nomidagi Matematika Instituti huzuridagi DSc.02/30.12.2019.FM.86.01 raqamli Ilmiy kengashning 2025 yil « 18 » noyabr kuni soat 16:00 dagi majlisida bo‘lib o‘tadi. (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871)-207-91-40, e-mail: uzbmath@umail.uz, Website: www.mathinst.uz).

Dissertatsiya bilan V.I.Romanovskiy nomidagi Matematika Institutining Axborot-resurs markazida tanishish mumkin (211-raqami bilan ro‘yxatga olingan). (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871)-207-91-40.

Dissertatsiya avtoreferati 2025 yil « 29 » oktyabr kuni tarqatildi.
(2025 yil « 29 » oktyabrdagi 2-raqamli reestr bayonnomasi).

U.A.Roziqov
Ilmiy darajalar beruvchi
Ilmiy kengash raisi,
f.-m.f.d., akademik

J.K.Adashev
Ilmiy darajalar beruvchi
Ilmiy kengash ilmiy kotibi,
f.-m.f.d., katta ilmiy xodim

A.A.Azamov
Ilmiy darajalar beruvchi
Ilmiy kengash huzuridagi
Ilmiy seminar raisi,
f.-m.f.d., akademik

KIRISH (falsafa doktori (PhD) dissertatsiyasi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zaruriyati. So‘nggi o‘n yil davomida kasr tartibli differensial tenglamalar sohasi butun dunyo matematiklari orasida katta qiziqish uyg‘otdi. Hozirgi vaqtda kasr tartibli xususiy hosilali tenglamalar fizik, biologik va kimyoviy jarayonlarning keng doirasini aniq va ishonchli modellashtirishda tadqiqotchilar uchun muhim vositaga aylanmoqda. Shu munosabat bilan ko‘plab olimlar kasr tartibli differensial tenglamalarni yechishning yangi usullarini ishlab chiqish va ularning yechimlari xossalarini o‘rganish bilan faol shug‘ullanmoqdalar. So‘nggi yillarda bunday tenglamalarni klassik Koshi yoki chegaraviy shartlardan farqli turli xil nolokal shartlar ostida o‘rganish differensial tenglamalar sohasida muhim burilish nuqtasiga aylandi. Bu yo‘nalish, odatda nolokal masalalar deb atalib, tajribali tadqiqotchilar hamda yosh olimlar orasida tobora ortib borayotgan qiziqishni uyg‘otmoqda. Ushbu dissertatsiya subdiffuziya, diffuziya, kasr tartibli to‘lqin tarqalishi tenglamalari va ikkinchi tartibli xususiy hosilali tenglamalarning yechimlarini turli xil parametrga bog‘liq chiziqli va nochiziqli nolokal shartlar ostida o‘rganishga bag‘ishlangan. Bunday masalalarni tadqiq etish bugungi kunda nazariy va amaliy jihatdan katta ahamiyat kasb etadi. Chiziqli va nochiziqli nolokal shartlarga parametrlarning kiritilishi klassik boshlang‘ich va chegaraviy masalalarni o‘z ichiga oluvchi umumiy sxemani taqdim etadi hamda natijalarning qo‘llanish doirasini kengaytiradi. Xususan, subdiffuziya va diffuziya tenglamalari g‘ovak muhitlarda zarrachalar transportini, biologik to‘qimalarda moddalar diffuziyasini, polimerlar ichida moddalarning tarqalishini tavsiflashda qo‘llaniladi. Kasr tartibli to‘lqin tenglamalari esa yurak to‘qimalarida elektr signallarining tarqalishini, bioelektrik impulslarni, materiallardagi tebranishlarni hamda zo‘riqishlarning taqsimlanishini modellashtirishda keng qo‘llaniladi.

Parametrik shakldagi nolokal shartlarning kiritilishi muammoning moslashuvchanligini oshiradi va xotiraga ega tizimlarni, uzoq masofali o‘zaro ta’sirlarni yoki gibrid chegaraviy xatti-harakatlarni modellashtirish imkonini beradi. Natijada, qo‘llanish sohasi sezilarli darajada kengayadi. Bunga quyidagilar kiradi (ammo ular bilan cheklanmaydi): Fizika: anomaliyali diffuziya, g‘ovak muhitlarda issiqlik va massa almashinuvi;Tibbiyot va biofizika: hujayralar ichida moddalar transporti, yurak va asab tizimlarida signal uzatilishi; Muhandislik: elastik tuzilmalarni tebranishi va deformatsiyasi; Axborot texnologiyalari: tarmoqlashtirilgan tizimlarda signal tarqalishi; Iqtisodiyot va moliya: xotiraga asoslangan dinamik tizimlar va kechikkan ta’sirli bozor modellari. Ushbu dissertatsiyada ko‘rib chiqilgan masalalar va olingan natijalar turli sohalarda yuzaga keladigan murakkab jarayonlarni tahlil qilish uchun mustahkam nazariy asos yaratadi.

Mamlakatimizda matematika, fizika, biologiya va geologiya kabi aniq va tabiiy fanlar yo‘nalishlarini rivojlantirishga alohida e’tibor qaratilmoqda. Xususan, mexanika, elektronika, boshqaruv tizimlari, fiziologiya va biologik jarayonlardagi turli hodisalarni chuqur anglashda muhim o‘rin tutgani uchun kasr va xususiy hosilali tenglamalar nazariyasini rivojlantirish ustuvor yo‘nalish sifatida belgilangan. Ushbu muhim sohalarda xalqaro miqyosda tadqiqotlar olib borish fundamental ilmiy izlanishlarning asosiy ustuvor yo‘nalishi sifatida e’tirof etilgan. Hozirgi vaqtda kasr va xususiy hosilali tenglamalar nazariyasi bo‘yicha erishilayotgan yutuqlar mazkur vazifalarning muvaffaqiyatli amalga oshirilishida hal qiluvchi ahamiyat kasb etmoqda. Mamlakatimiz olimlari ushbu sohalarga salmoqli hissa qo‘shib, farmon va qarorlarning amalda ro‘yobga chiqarilishiga sezilarli darajada yordam bermoqdalar. Xususan, differensial tenglamalar va matematik fizika sohalaridagi ilmiy izlanishlarning, ayniqsa xalqaro darajadagi tadqiqotlarning fundamental fanlarni rivojlantirishda ustuvor yo‘nalish sifatida qadrlanayotganini mamnuniyat bilan qayd etish lozim.

Mazkur dissertatsiya ishining predmeti va tadqiqot ob’yekti O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-sonli "O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi"[1] haqidagi farmonlarida belgilangan vazifalariga mos keladi, 2017 yil 17-fevraldagi PQ-2789 "Fanlar Akademiyasi faoliyati, ilmiy-tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora-tadbirlari to‘g‘risida"gi Prezident qarori, 2018 yil 27 apreldagi "Innovatsion g‘oyalar, texnologiyalar va loyihalarni amaliy joriy qilish tizimini yanada takomillashtirish chora-tadbirlari to‘g‘risida"gi PQ-3682-sonli qarori va 2019 yil 9 iyuldagi "Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab quvvatlash, shuningdek, O‘zbekiston Respublikasi Fanlar Akademiyasining V.I.Romanovskiy nomidagi matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida"gi PQ-4387 sonli Prezident qarori hamda 2020 yil 7 maydagi "Matematika sohasidagi ta’lim sifatini oshirish va ilmiy tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida"gi PQ-4708 sonli Prezident qarori hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga bog‘liqligi. Mazkur dissertatsiya Respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishi doirasida bajarilgan.

Muammoning o‘rganilganlik darajasi. Kasr tartibli xususiy hosilali tenglamalar va ular bilan bog‘liq nolokal masalalar zamonaviy matematikada faol tadqiqot yo‘nalishiga aylangan. Kasr tartibli differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalar turli mualliflar tomonidan keng o‘rganilgan. Ushbu sohada Sh.O. Alimov, R.R. Ashurov, S.R. Umarov, M. Yamamoto, Z. Li, A.V. Pskhu, M. Kirane, M. Ruzhanskiy, D.K. Durdiev, Yu.E. Fayziyev, Z.A. Sobirov, E.T. Karimov, B.X. Turmetov, Y. Zhang, H.T. Nguen, A.S. Malik va boshqa olimlar tomonidan muhim ilmiy natijalar olingan.

Ko‘plab tadqiqotchilar chiziqli va nochiziqli nolokal shartlarning turli ko‘rinishlari ostida differensial tenglamalarning yechimlarini topish masalasini

[1] O’zbekiston Respublikasi Vazirlar Mahkamasining 2017 yil 18 maydagi «O‘zbekiston Respublikasi Fanlar akademiyasining yangidan tashkil etilgan ilmiy-tadqiqot muassasalari faoliyatini tashkil etish to’g’risida » gi №292-sonli qarori

o‘rganib kelmoqda. O‘zgaruvchi parametrlarning qo‘llanilishi bunday masalalarni bir vaqtning o‘zida tadqiq etishda samarali bo‘lib, nafaqat bitta masalani, balki shu turdagi masalalarning butun bir sinfini tahlil qilish imkonini beradi. A.N. Kochubei, K. Diethelm, J. Liu, M. Yamamoto, K. Sakamoto, G. Florida, Z. Li, S.A. Alimov, R.R. Ashurov, Yu.E. Fayziyev, A. Boutiara, K. Guerbati, M. Benbachir, F. Jarad, A. Abdeljawad, A. Yacine, B. Nouredine, A. Salim, M. Benchohra va boshqa tadqiqotchilarning ishlarida diffuziya va subdiffuziya tenglamalari uchun nuqtali va integral nolokal shartlarga ega turli masalalar o‘rganilgan. Xususan, A. Boutiara, S. Krim va boshqalarning ishlarida oddiy differensial tenglamalar uchun uch parametrli nuqtali va integral nolokal masalalar tadqiq etilgan.

Mazkur dissertatsiya ishida ushbu masalalarni xususiy hosilali differensial tenglamalarga kengaytiralgan. M.O. Mamchuev klassik boshlang‘ich-chegaraviy shartlarni qanoatlantiruvchi subdiffuziya tenglamasining yechimini Grin funksiyasi usuli yordamida qurdi. Ushbu yondashuv asosida yangi nochiziqli nolokal shartli subdiffuziya tenglamasi ko‘rib chiqilgan. R.R. Ashurov va Yu.E. Fayziyev vaqt bo‘yicha kasr hosilali subdiffuziya tenglamasi uchun (Riemann–Liouville va Caputo hosilalari bilan) bitta parametrli nolokal masalani tadqiq etdilar. Ular parametr bo‘yicha muammo to‘g‘ri qo‘yilganligini ta’minlovchi ortogonal shartlarni topdilar hamda ushbu shartlar bajarilmaganda ham yechimning mavjud bo‘lishini ko‘rsatdilar. D.K. Durdiev va A.A. Raxmonov vaqt bo‘yicha kasr hosilali to‘lqin tenglamasi uchun ikki parametrli nolokal masalani o‘rgandilar. R.R. Ashurov va D.K. Durdiev ishlarida ilgari surilgan g‘oyalarni rivojlantirib, vaqt bo‘yicha kasr hosilali to‘lqin tenglamasi uchun to‘rt parametrli yangi nolokal masalani kiritilgan. Ikkinchi tartibli xususiy hosilali differensial tenglamalar uchun parametrli ko‘p nuqtali nolokal masalalar A.O. Ashiraliev, P.E. Sobolevskiy, A. Hanaliev, E. Karimov, M. Mamchuev, E. Uzturk, D.G. Orlovskiy, S.I. Piskarev va K.B. Sabitovlarning ishlarida tadqiq etilgan. Bundan tashqari, ikkinchi tartibli xususiy hosilali differensial tenglamalar uchun parametrli ko‘p nuqtali nolokal masalalarni o‘rganilgan.

Dissertatsiya tadqiqotining dissertatsiya bajarilgan Oliy ta’lim muassasasining ilmiy-tadqiqot ishlari rejalari bilan bog‘liqligi. Dissertatsiya tadqiqoti V.I. Romanovskiy nomidagi Matematika institutida O‘zbekiston Respublikasi Oliy ta’lim, fan va innovatsiyalar vazirligining F-FA-2021-424-sonli ilmiy tadqiqot grantining rejalashtirilgan mavzusiga muvofiq amalga oshirildi.

Tadqiqotning maqsadi subdiffuziya, diffuziya, kasr tartibli to‘lqin tenglamalari va ikkinchi tartibli xususiy hosilali tenglamalarda uchraydigan parametrli chiziqli va nochiziqli nolokal masalalarning turli ko‘rinishlari uchun klassik yechimlarning mavjud va yagonaligini isbotlashdan iborat.

Tadqiqotning vazifalari:

uch parametrli chiziqli nolokal masalalar hamda subdiffuziya tenglamasi uchun nochiziqli nolokal masalalarning yagona yechimga ega ekanligini isbotlash;

kasr tartibli to‘lqin tarqalish tenglamalari uchun ikki va to‘rt parametrli nolokal masalalarning yagona yechimga ega ekanligini isbotlash;

ikkinchi tartibli xususiy hosilali tenglamalar uchun ko‘p nuqtali nolokal masalalarning yagona yechimga ega ekanligini isbotlash.

Tadqiqotning obyekti. Klassik diffuziya tenglamasi, Kaputo kasr tartibli subdiffuziya tenglamasi, Kaputo kasr tartibli to‘lqin tarqalish tenglamasi va ikkinchi tartibli xususiy hosilali differensial tenglamalar.

Tadqiqotning predmeti. Parametrlarni o‘z ichiga olgan chiziqli va nochiziqli nolokal shartlar ostida subdiffuziya, diffuziya, kasr tartibli to‘lqin tarqalish tenglamalari hamda ikkinchi tartibli xususiy hosilali tenglamalarning yechimlarini topish masalalari.

Tadqiqotning usullari. Tadqiqotda funksional analiz usullari, spektral nazariya va Furye usullari qo‘llaniladi.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

uchta parametrli chiziqli nolokal masalalar va subdiffuziya tenglamasi uchun chiziqsiz nolokal masalalarning klassik ma’noda yagona yechimliligi isbotlangan;

kasr tartibli to‘lqin tarqalishi tenglamalari uchun ikki va to‘rt parametrli nolokal masalalarning klassik ma’noda yagona yechimga ega ekanligi isbotlangan;

xususiy hosilali differensial tenglamalar uchun ko‘pnuqtali nolokal masalalarning klassik ma’noda yagona yechimliligi isbotlangan.

Tadqiqotning amaliy natijalari. Ushbu dissertatsiyada olingan natijalar va qo‘llanilgan usullar oliy o‘quv yurtlari magistrantlari va doktorantlari uchun bitiruv kursi sifatida o‘qitilishi mumkin.

Tadqiqot natijalarining ishonchliligi. Natijalar funksional analiz, spektral nazariya va Furye usuli yordamida olingan. Olingan barcha natijalar matematik jihatdan to‘g‘ri.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy va amaliy ahamiyati shundaki, parametrlarni o‘z ichiga olgan chiziqli va nochiziqli nolokal shartlar ostida turli differensial tenglamalar uchun yechimlarning mavjudligi va yagonaligi isbotlandi. Ushbu natijalar kasr tartibli differensial tenglamalar nazariyasini kengaytiradi hamda mavjud ilmiy yutuqlarning umumlashtirilgan shaklini beradi. Olingan natijalar fizika, biologiya va muhandislikdagi real jarayonlarni modellashtirishda qo‘llanishi mumkin. Xususan, ular anomal diffuziya, to‘qimalarda moddalar taqsimlanishi va elastik muhitlardagi tebranish jarayonlariga mos matematik modellarni qurishda nazariy asos bo‘lib xizmat qiladi.

Tadqiqot ishlarning joriy qilinishi. Butun va kasr tartibli differensial tenglamalar uchun ko‘pnuqtali va chiziqsiz nolokal masalalar bo‘yicha olingan natijalar asosida:

subdiffuziya, diffuziya tenglamalarning yechimlarini turli xil parametrlarga bog‘liq chiziqli va nochiziqli nolokal shartlar ostida topilgan yechimlaridan 22-11-00064 raqamli “Geosferadagi dinamik jarayonlarni irsiyatni hisobga olgan holda modellashtirish” mavzusidagi xorijiy loyihada subdiffuziya tenglamalari uchun nolokal masalalarni tasniflashda foydalanilgan (Kosmofizika tadqiqotlari va radioto‘lqinlar tarqalish institutining 2025-yil 8-sentyabrdagi № 346-sonli ma’lumotnoma, Rossiya Federatsiyasi). Ilmiy natijani qo‘llanilishi hujayra ichida dori moddalarining sekin diffuziyalanishini modellashtirib, optimal dozalash rejimini aniqlash imkonini bergan;

kasr tartibli to‘lqin tarqalishi tenglamalari hamda xususiy hosilali differensial tenglamalarning turli parametrga bog‘liq chiziqli va nochiziqli nolokal shartlar ostida yechimlaridan 122041800013-4 raqamli “Umumlashgan kasr tartibli differensial operatorli tenglamalar uchun chegaraviy masalalarni o‘rganish, ularni fizik va ijtimoiy-iqtisodiy jarayonlarni modellashtirishda qo‘llash” mavzusidagi xorijiy loyihada turli fizik va biologik jarayonlarni matematik modellashtirishda qo‘llanilgan (Kabardin-Balkar ilmiy markazi Amaliy matematika va avtomatlashtirish institutining 2025-yil 16-apreldagi №01-13/48-sonli ma’lumotnoma, Rossiya Federatsiyasi). Ilmiy natijalarning qo‘llanilishi turli fizikaviy va biologik jarayonlarni matematik modellashtirishda samarali qo‘llanilayotgan kasr va butun tartibli hosila qatnashgan evolyutsion tenglamalar uchun lokal va nolokal chegaraviy masalalarni yechish imkonini bergan.

Tadqiqot natijalarining aprobatsiyasi. Asosiy ilmiy natijalar 4 ta xalqaro, 2 ta milliy ilmiy konferensiya va 1 ta xalqaro kongressda muhokama qilindi.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya mavzusi bo‘yicha 15 ta ilmiy ishlar chop etilgan bo‘lib, shundan, 8 ta maqola O‘zbekiston Respublikasi Oliy attestatsiya komissiyasining falsafa doktorlik dissertatsiyalarining asosiy ilmiy natijalarini chop etish uchun tavsiya etilgan ilmiy nashrlarda chop etilgan, shulardan 1 tasi xorijiy va 7 tasi respublika jurnallarida chop etilgan bo‘lib, ulardan 3 tasi SCOPUS ma’lumotlar bazalarida indekslangan va 7 tasi tezisdir.

Dissertatsiyaning tuzilishi va hajmi. Dissertatsiya kirish, to‘rtta bob, xulosa va foydalanilgan adabiyotlar ro‘yxatidan iborat. Dissertatsiya hajmi 128 bet.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismida dissertatsiya mavzusining dolzarbligi va uni tanlash zarurati asoslab berilgan. Tadqiqotning O‘zbekiston Respublikasi fan va texnologiyalar rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan. Mavzu bo‘yicha xorijiy va mahalliy ilmiy-tadqiqotlar tahlil qilinib, muammoning o‘rganilganlik darajasi yoritilgan. Tadqiqotning maqsadi, vazifalari, ob’ekti va predmeti aniq bayon qilingan. Ishning ilmiy yangiligi va olingan amaliy natijalari, shuningdek, ularning nazariy hamda amaliy ahamiyati ochib berilgan. Tadqiqot natijalarining amaliyotga joriy etilishi, mavzu bo‘yicha nashr etilgan ilmiy ishlar hamda dissertatsiya tuzilishi haqida ma’lumot keltirilgan.

Dissertatsiyaning **“Dastlabki ma’lumotlar”** nomli birinchi bobi yordamchi xarakterga ega bo‘lib, asosan ishning keyingi boblarini o‘qish va tushunishni qulaylashtirish maqsadida tuzilgan. Ushbu bobda yangi ilmiy natijalar keltirilmagan, balki faqat tadqiqot uchun zarur bo‘lgan ta’riflar va oldindan ma’lum tasdiqlar jamlangan. Birinchi bobdan ba’zi zarur ma’lumotlarni keltiramiz:

Hilbert fazosida abstrakt operator. $(\cdot,\cdot)$ skalyar ko‘paytma va norma $\|\cdot\|$ aniqlangan H separabel Hilbert fazosi bo‘lsin. Aytaylik, $A: H \to H$ operator H fazoda aniqlangan bo‘lib, o‘z-o‘ziga qo‘shma, quyidan chegaralangan, musbat aniqlangan bo‘lsin. Faraz qilaylik, A operator kompakt A^{-1} teskari operatorga ega. U holda u to‘la ortonormal $\{v_k\}$ xos vektorlar sistemasiga va ularga mos $\{\lambda_k\}$ musbat xos sonlar to‘plamiga ega bo‘ladi, ya’ni v_k vektorlar va λ_k sonlar $Av_k = \lambda_k v_k$ tenglikni qanoatlantiradi. Xos sonlarni qayta nomerlash yordamida ularni kamaymaydigan qilib nomerlab olamiz, ya’ni $0 < \lambda_1 \le \lambda_2 \cdots \to +\infty$.

Keling, H Hilbert fazosida A operatorning darajasi tushunchasini eslatib o‘taylik. τ ixtiyoriy haqiqiy son bo‘lsin. Biz H da A operatorning darajasini quyidagicha kiritamiz:

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

bu yerda $h_k = (h, v_k)$ lar $h \in H$ elementning Furye koeffitsiyentlari. Shubhasiz, ushbu operatorning aniqlanish sohasi quyidagi ko‘rinishga ega bo‘ladi:

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

Bundan tashqari biz uchun muhim ikki tushunchani ta’riflaymiz:
Ikki parametrli Mittag-Leffler funktsiyasi

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z, \mu \in \mathrm{C}, \quad \rho > 0,$$

hamda Kaputo kasr hosilasi

$$D_t^\rho h(t) = \frac{1}{\Gamma(n-\rho)} \int_0^t \frac{h^{(n)}(\xi)}{(t-\xi)^{\rho+1-n}} d\xi, \quad t > 0, \quad n = [\rho] + 1.$$

Oxirgi formulada tenglikning o‘ng tomoni mavjud bo‘lsin deb faraz qilamiz.

Dissertatsiyaning asosiy natijalari ikkinchi bobdan boshlanadi. Ushbu bob **“Kaputo kasr hosilali subdiffuziya tenglamasi uchun chiziqli va chiziqsiz nolokal masalalar”** deb nomlanadi.

Ushbu bobning **birinchi bo‘limida** H Gilbert fazosida elliptik qismi A abstrak operator bo‘lgan Kaputo kasr hosilali subdiffuziya tenglamasi uchun uchta parametr qatnashgan nuqtaviy nolokal masala o‘rganildi.

Aytaylik, H Gilbert fazosi va $AC[0,T]$ to‘plam $[0,T]$ da aniqlangan absolyut uzluksiz funksiyalar to‘plami bo‘lsin. Biz $AC([0,T];H)$ orqali $[0,T]$ da absolyut uzluksiz va qiymatlari H dan bo‘lgan funksiyalar fazosini belgilaymiz. $C([0,T];H)$ fazo ham xuddi shunday aniqlanadi.

1-masala. Aytaylik, $\rho \in (0,1)$ bo‘lsin. Quyidagi boshlang‘ich-chegaraviy masalani qaraylik:

$$\begin{cases} D_t^\rho u(t) + Au(t) = f(t), \quad 0 < \rho \le 1, \quad 0 < t < T; \\ \alpha u(0) + \beta u(T) = \gamma u(\xi) + \varphi, \quad 0 < \xi < T, \end{cases} \tag{1}$$

bu yerda $f(t) \in C((0,T);H)$ berilgan funksiya, $\varphi \in H$ berilgan element, α, β, γ lar haqiqiy o‘zgarmaslar, $\xi \in (0,T)$ fiksirlangan nuqta. Bu masalaning

$u(t)\in AC([0,T];H)$, $D_t^\rho u(t), Au(t)\in C((0,T);H)$ shartlarni qanoatlantiruvchi yechimini toping.

1-teorema. *Biror* $\varepsilon\in(0,1)$ *soni uchun quyidagi shartlardan biri bajarilsin:*

(N1) $f(t)\in C([0,T];D(A^\varepsilon))$, $\varphi\in H$ *va* $\alpha>0,\quad \beta\ge 0,\quad \gamma\le 0$ *(yoki* $\alpha<0,\quad \beta\le 0,\quad \gamma\ge 0$*);*

(N2) $f(t)\in C([0,T];D(A^{\varepsilon+1}))$, $\varphi\in D(A)$, $\alpha=0$ *va* $\beta\gamma\le 0$, *yoki bir vaqtning o'zida* $\beta\gamma>0$ *va* $|\beta|\le|\gamma|$ *shartlar o'rinli.*

U holda (1) masalaning yechimi mavjud, yagona va quyidagi ko'rinishga ega:

$$u(t)=\sum_{k=1}^{\infty}\left[\frac{\varphi_k-\beta v_k(T)+\gamma v_k(\xi)}{\Delta_k}E_{\rho,1}(-\lambda_k t^\rho)+v_k(t)\right]V_k,$$

$$\Delta_k=\alpha+\beta E_{\rho,1}(-\lambda_k T^\rho)-\gamma E_{\rho,1}(-\lambda_k\xi^\rho),\ v_k(t)=\int_0^t\eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta.$$

Shu bilan birga, shunday $C_\varepsilon, C>0$ *sonlar mavjudki quyidagi koersitiv tengsizliklar bajariladi:*

(N1) hol:

$$\|D_t^\rho u(t)\|^2+\|u(t)\|_1^2\le t^{-2\rho}(C_\varepsilon\max_{t\in[0,T]}\|f\|_\varepsilon^2+C\|\varphi\|^2);$$

(N2) hol:

$$\|D_t^\rho u(t)\|^2+\|u(t)\|_1^2\le t^{-2\rho}(C_\varepsilon\max_{t\in[0,T]}\|f\|_{\varepsilon+1}^2+C\|\varphi\|_1^2).$$

2-teorema. *Biror* $\varepsilon\in(0,1)$ *soni uchun quyidagi shartlardan biri bajarilsin:*

(L1) $f(t)\in C([0,T];D(A^\varepsilon))$ $\varphi\in H$ $\alpha\ne 0$, *parametrlar 1-teoremaning (N1) shartini qanoatlantirmaydi ;*

(L2) $f(t)\in C([0,T];D(A^{\varepsilon+2}))$ $\varphi\in D(A^2)$, $\alpha=0$, $\beta\gamma>0$, $|\beta|\ge|\gamma|$ *va* β,γ,T,ξ *lar uchun* $\frac{\beta}{\gamma}=\left(\frac{T}{\xi}\right)^\rho$ *shart o'rinli;*

(L3) $f(t)\in C([0,T];D(A^{\varepsilon+1}))$ $\varphi\in D(A)$, $\alpha=0$, $\beta\gamma>0$, $|\beta|\ge|\gamma|$ *va* $\beta,\gamma,T,$ ξ *lar uchun* $\frac{\beta}{\gamma}\ne\left(\frac{T}{\xi}\right)^\rho$ shart *shart o'rinli.*

U holda (1) masala yechimining mavjudligi uchun quyidagi ortogonallik shartlari bajarilishi zarur va yetarlidir:

$$\begin{cases}(\varphi,V_k)=0,\ k\in K_0;\\ (f(t),V_k)=0,\ t\ge 0,\quad k\in K_0,\end{cases}$$

bu yerda

$$K_0:=K_0(\Omega\subset R^3)=\{k:\Delta_k(\alpha,\beta,\gamma)=0,\quad k\in\mathrm{N},\quad (\alpha,\beta,\gamma)\in\Omega\}.$$

Bu holda (1) masalaning yechimi yagona emas va ixtiyoriy a_k, $k\in K_0$

koeffitsientlar bilan quyidagi ko'rinishga ega bo'ladi

$$u(t)=\sum_{k\in K_0} a_k E_{\rho,1}(-\lambda_k t^{\rho})V_k+\sum_{k\notin K_0}\left[\frac{\varphi_k-\beta v_k(T)+\gamma v_k(\xi)}{\Delta_k}E_{\rho,1}(-\lambda_k t^{\rho})+v_k(t)\right]V_k.$$

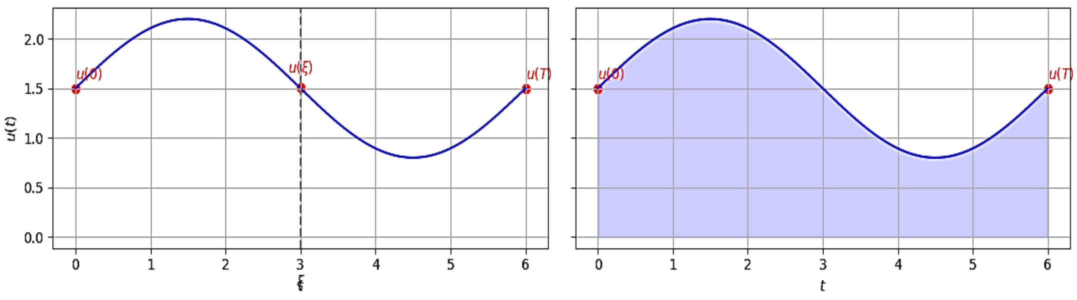
Ushbu bobning **ikkinchi bo‘limida** (1) masalaga o‘xshash bo‘lgan quyidagi integral shartli nolokal masala ko’rib chiqiladi:

2-masala. Quyidagi

$$\begin{cases} D_t^{\rho}u(t)+Au(t)=f(t), \quad 0<\rho<1, \quad 0<t<T, \\ \alpha u(0)+\beta u(T)+\gamma\int_0^T u(\eta)d\eta=\varphi, \end{cases} \tag{2}$$

$u(t)\in AC([0,T];H)$, $D_t^{\rho}u(t), Au(t)\in C((0,T);H)$ shartlarni qanoatlantiruvchi yechimini toping.

Bu bo‘limda, (2) masala uchun ham xuddi yuqoridagi 1- va 2-teoremalar kabi teoremalar isbotlangan. Ammo (1) va (2) masalalarda foydalanilgan nolokal shartlar bir biridan ancha farq qiladi. Bu farqni quyidagi rasm orqali osonroq tushunib olish mumkin:



$\alpha u(0)+\beta u(T)=\gamma u(\xi)+\varphi$ $\qquad$ $\alpha u(0)+\beta u(T)+\gamma\int_0^T u(\eta)\,d\eta=\varphi$

Ushbu bobning **uchinchi bo‘limida** $C([0,1]\times[0,T])$ fazoda Kaputo kasr hosilali subdiffuziya tenglamasi uchun quyidagi nochiziqli nolokal masala qaralgan:

3-masala. Kasr subdiffuziya tenglamasining $(0,1)\times(0,T]$ sohada

$$D_{0t}^{\alpha}u(x,t)-u_{xx}(x,t)=f(x,t), \tag{3}$$

quyidagi chiziqsiz nolokal shart hamda

$$u(x,0)=g(x,u(x,T)), \quad x\in[0,1], \tag{4}$$

Dirixle chegara shartlarini

$$u(0,t)=u(1,t)=0, \quad t\in[0,T], \tag{5}$$

qanoatlantiruvchi yechimini toping. Bu yerda $f(x,t)$, $g(x,w)$lar berilgan funksiyalar.

3-teorema. *Aytaylik* $f(x,t)\in C([0,1]\times[0,T])$ *hamda* $g(x,w)\in C\big([0,1]\times \mathrm{R}\big)($

funksiya ikkinchi argumenti bo‘yicha quyidagi Lipshits shartini qanoatlantirsin:

$$|g(x,w)| \le L|w|, \quad 0 < L < \frac{1}{\sqrt{E_\alpha(T^\alpha)}}.$$

U holda (3)–(5) masalaning yechimi uchun quyidagi Apirior tengsizlik obajariladi:

$$\|u(\cdot,t)\|^2 \le \left[\frac{L^2 E_{\alpha,\alpha}(T^\alpha)}{1-L^2 E_\alpha(T^\alpha)} E_\alpha(t^\alpha) + E_{\alpha,\alpha}(t^\alpha)\right] \frac{T}{\alpha} \max_{[0,T]} \|f(\cdot,t)\|,$$

bu yerda $\|u(\cdot,t)\|^2 = \int_0^1 u^2(x,t)dx$ *va* $E_\alpha(\cdot), E_{\alpha,\tau}(\cdot)$ *Mittag–Leffler funsiyalari.*

Qo‘yilgan masala chiziqsiz bo‘lgani sababli uning yechimini biroz noodatiy usulda topamiz. (3)–(5) masalaning formal yechimini Grin funksiyasi yordamida quyidagicha ifodalaymiz:

$$u(x,t) = \int_0^1 g(\xi, v(\xi)) D_{0t}^{\alpha-1} G(x,t,\xi,0)d\xi + \int_0^t\int_0^1 G(x,t,\xi,\tau) f(\xi,\tau)d\xi d\tau.$$

Bu yerda $v(\cdot)$ quyidagi integral tenglamani yechimi bo‘lib,

$$v(x) = \int_0^1 D_{0T}^{\alpha-1} G(x,T,\xi,0) g(\xi, v(\xi))d\xi + F(x),$$

integral ichidagi Grin funksiyasi uchun quyidagi tengsizlik bajariladi:

$$|G(x,t,\xi,\tau)| \le C|t-\tau|^{\frac{\alpha}{2}-1}, \quad 0 \le \tau < t \le T. \qquad (6)$$

4-teorema *Faraz qilaylik quyidagi shartlar bajarilsin:*

1. $f(x,t) \in C([0,1]\times[0,T])$*;*

2. $g(\xi,v) \in C([0,1]\times \mathbb{R})$ *funksiya ikkinchi argumenti bo‘yicha quyidagi Lipshits shartini qanoatlantiradi:*

$$|g(\xi,v_1) - g(\xi,v_2)| \le L|v_1 - v_2|, \quad \textit{barcha} \quad \xi \in [0,1], v_1, v_2 \in \mathbb{R} \; \textit{lar uchun};$$

3. $L < \dfrac{\Gamma\left(1-\frac{\alpha}{2}\right)}{C\Gamma\left(\frac{\alpha}{2}\right)T^{1-\frac{\alpha}{2}}}$, *bu yerda* C *soni (6) bahodagi maxsus konstanta.*

U holda (3)–(5) masalaning yechimi mavjud va yagona.

Uchinchi bob **“Kaputo kasr tartibli to‘lqin tenglamasi uchun bir nechta parametrga ega nolokal masalalar”** deb nomlanadi.

Ushbu bobning birinchi bo‘limida H separabel Hilbert fazosida elliptik qismi abstrakt operator bo‘lgan holda Kaputo kasr hosilali to‘lqin tenglamasining ikki parametrga bog‘liq nolokal shartni qanoatlantiruvchi yechimini topish masalasi o’rganilgan.

4-masala. Aytaylik, $\rho \in (1,2)$ bo‘lsin. Quyidagi nolokal masalani qaraylik

$$\begin{cases} D_t^\rho u(t) + Au(t) = f, & 0 < t \le T; \\ \alpha u(0) + \beta u(\xi) = \varphi, & 0 < \xi \le T; \\ u'(0) = \psi, \end{cases} \tag{7}$$

bu yerda $f(t) \in C([0,T];H)$, φ, ψ esa H Hilbert fazosida berilgan ma'lum elementlar va ξ esa $(0,T]$ da ixtiyoriy fiksirlangan nuqta. Bu masalaning $u(t) \in AC([0,T];H)$, $D_t^\rho u(t), Au(t) \in C((0,T);H)$ shartlarni qanoatlantiruvchi yechimini toping.

5-teorema. *Aytaylik, $f, \varphi, \psi \in D(A)$, α va β parametrlar quyidagi shartni qanoatlantirsin:*

$$|\alpha| > |\beta|.$$

U holda (7) masalaning yagona yechimi mavjuda va quyidagi ko'rinishga ega:

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{1}{\Delta_k} (\varphi_k - \beta v_k(\xi) - \beta \xi E_{\rho,2}(-\lambda_k \xi^\rho) \psi_k) E_{\rho,1}(-\lambda_k t^\rho) + \psi_k t E_{\rho,2}(-\lambda_k t^\rho) \right] V_k,$$

$$+ \sum_{k=1}^{\infty} f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) V_k,$$

bu yerda $\Delta_k = \alpha + \beta E_{\rho,1}(-\lambda_k \xi^\rho).$

Shuningdek, shunday $C > 0$ son mavjudki, quyidagi koersitiv tengsizlik o'rinli:

$$\| D_t^\rho u(t) \|^2 + \| u(t) \|_1^2 \le C(\| f \|^2 + \| \varphi \|^2 + \| \psi \|^2), \quad t > 0.$$

Ushbu bobning ikkinchi bo'limida (7) masalani qamrab oladigan, jarayonning dastlabki va keying vaqtlardagi holatlarini bog'lovchi to'rtta parametr qatnashgan umumiy ko'rinishdagi murakkab nolokal masala to'la o'rganilgan.

5-masala. Quyidagi nolokal masalani qaraylik

$$\begin{cases} D_t^\rho u(t) + Au(t) = f, & 0 < t \le T; \\ \alpha_1 u(0) + \alpha_2 u(\xi) = \varphi, & 0 < \xi \le T; \\ \beta_1 u'(0) + \beta_2 u'(\xi) = \psi, \end{cases} \tag{8}$$

bu yerda, $\alpha_1, \alpha_2, \beta_1, \beta_2$ $(\alpha_1^2 + \alpha_2^2 \ne 0, \beta_1^2 + \beta_2^2 \ne 0)$ sonlar haqiqiy parametrlar. Bu masalaning $u(t) \in AC([0,T];H)$, $D_t^\rho u(t), Au(t) \in C((0,T);H)$ shartlarni qanoatlantiruvchi yechimini toping.

(8) masala haqida asosiy natijalarni keltirishdan oldin Mittag-Leffler funksiyasining muhim bir xossasini eslatib o'tamiz.

Tasdiq. *Agar $\rho \in (1,2)$ bo'lsa, u holda shunday $c_1 > 0$ son mavjudki, quyidagi tengsizlik bajariladi:*

$$|E_{\rho,\mu}(-t)| \le \frac{c_1}{1+t}. \tag{9}$$

6-teorema. *Aytaylik $f, \varphi, \psi \in H$, $\alpha_1 \beta_1 \ne 0$ va (9) baxodagi $c_1 > 0$ soni uchun* *quyidagi shartlardan biri bajarilsin:*

(a) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}\geq 0,\quad 1>c_1^2\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}+c_1\left|\dfrac{\alpha_2}{\alpha_1}+\dfrac{\beta_2}{\beta_1}\right|;$

(b) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}<0,\quad 1<-c_1^2\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}+c_1\left|\dfrac{\alpha_2}{\alpha_1}-\dfrac{\beta_2}{\beta_1}\right|.$

U holda (8) masalaning yagona yechimi mavjuda va quyidagi ko‘rinishga ega:

$$u(t)=\sum_{k=1}^{\infty}\Bigg[\frac{\varphi_k-(\alpha_1 v_k(0)+\alpha_2 v_k(\xi))}{\Delta_k}\Big((\beta_1+\beta_2E_{\rho,1}(-\lambda_k\xi^\rho))E_{\rho,1}(-\lambda_k t^\rho)$$

$$+\beta_2\lambda_k\xi^{\rho-1}E_{\rho,\rho}(-\lambda_k\xi^\rho)tE_{\rho,2}(-\lambda_k t^\rho)\Big)$$

$$+\frac{\psi_k-(\beta_1 v_k'(0)+\beta_2 v_k'(\xi))}{\Delta_k}\Big(-\alpha_2\xi E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,1}(-\lambda_k t^\rho)$$

$$+(\alpha_1+\alpha_2E_{\rho,1}(-\lambda_k\xi^\rho))tE_{\rho,2}(-\lambda_k t^\rho)\Big)+v_k(t)\Bigg]V_k, \tag{10}$$

bu yerda

$$v_k(t)=f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)v_k.$$

$$\Delta_k=\alpha_1\beta_1+(\alpha_1\beta_2+\alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho)+\alpha_2\beta_2(E_{\rho,1}(-\lambda_k\xi^\rho))^2$$

$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho).$$

Shuningdek, shunday $C>0$ son mavjudki, quyidagi koersitiv tengsizlik o‘rinli:

$$\|D_t^\rho u(t)\|^2+\|u(t)\|_1^2\leq C(\|f\|^2+\|\varphi\|^2+\|\psi\|^2),\quad t>0. \tag{11}$$

Quyidagi to’plamni aniqlab olaylik:

$$K_0=\{k\in \mathrm{N}:\Delta_k=0\}.$$

7-teorema. *Aytaylik $\alpha_1,\alpha_2,\beta_1,\beta_2$ parametrlar va f,φ,ψ elementlar uchun quyidagi shartlardan biri bajarilsin:*

(1) *$f,\varphi,\psi\in H$ va (9) tengsizlikdagi $c_1>0$ soni uchun quyidagi tengsizliklar bajarilmaydi:*

(1a) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}\geq 0,\quad 1>c_1^2\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}+c_1\left|\dfrac{\alpha_2}{\alpha_1}+\dfrac{\beta_2}{\beta_1}\right|;$

(1b) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}<0,\quad 1<-c_1^2\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1}+c_1\left|\dfrac{\alpha_2}{\alpha_1}-\dfrac{\beta_2}{\beta_1}\right|;$

(2) *$f,\varphi,\psi\in D(A)$ hamda quyidagi shartlardan biri bajariladi:*

(2a) $\alpha_1\beta_1=0,\quad |\alpha_1|+|\beta_1|\neq 0,\quad \alpha_2\beta_2\neq 0,$

(2b) $\alpha_1\beta_1=0,\quad \alpha_2\beta_2=0,\quad |\alpha_1\beta_2|+|\alpha_2\beta_1|\neq 0;$

(3) *$f,\varphi,\psi \in D(A^2)$ va quyidagi shart bajariladi:*

(3) $|\alpha_1|+|\beta_1|=0, \quad \alpha_2\beta_2 \neq 0.$

Agar K_0 to‘plam bo‘sh bo‘lsa, u holda (8) masala (11) formula bilan berilgan yagona yechimga ega.

Bundan tashqari, C o‘zgarmas son mavjud bo'lib, biz har bir holat uchun mos keladigan quyidagi koersitiv tengsizliklarni ham olamiz:

(1) *hol uchun (11) tengsizlik;*

(2) *hol uchun*

$$\| D_t^\rho u(t)\|^2 + \|u(t)\|_1^2 \le C(\|f\|_1^2 + \|\varphi\|_1^2 + \|\psi\|_1^2), \quad t>0;$$

(3) *hol uchun*

$$\| D_t^\rho u(t)\|^2 + \|u(t)\|_1^2 \le C(\|f\|_2^2 + \|\varphi\|_2^2 + \|\psi\|_2^2), \quad t>0.$$

Agar K_0 to'plam bo'sh bo'lmasa, u holda (8) masala yechimining mavjudligi uchun quyidagi ortogonallik shartlari bajarilishi zarur va yetarlidir:

$$\begin{cases} (\varphi,V_k)=0, \quad (\psi,V_k)=0, \quad k\in K_0, \\ (f,V_k)=0, \quad k\in K_0. \end{cases}$$

Bu holda (1) masalaning yechimi yagona emas va ixtiyoriy b_k, $k\in K_0$ koeffitsientlar bilan quyidagi ko'rinishga ega bo'ladi

$$u(t)=\sum_{k\in K_0} b_k\left[-\frac{\alpha_2\xi E_{\rho,2}(-\lambda_k\xi^\rho)}{\alpha_1+\alpha_2 E_{\rho,1}(-\lambda_k\xi^\rho)}E_{\rho,1}(-\lambda_k t^\rho)+tE_{\rho,2}(-\lambda_k t^\rho)\right]V_k$$

$$+\sum_{k\notin K_0}\left[\frac{\varphi_k-(\alpha_1 v_k(0)+\alpha_2 v_k(\xi))}{\Delta_k}\Big((\beta_1+\beta_2 E_{\rho,1}(-\lambda_k\xi^\rho))E_{\rho,1}(-\lambda_k t^\rho)\right.$$

$$+\beta_2\lambda_k\xi^{\rho-1}E_{\rho,\rho}(-\lambda_k\xi^\rho)tE_{\rho,2}(-\lambda_k t^\rho)\Big)$$

$$+\frac{\psi_k-(\beta_1 v_k'(0)+\beta_2 v_k'(\xi))}{\Delta_k}\Big(-\alpha_2\xi E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,1}(-\lambda_k t^\rho)+$$

$$\left.(\alpha_1+\alpha_2 E_{\rho,1}(-\lambda_k\xi^\rho))tE_{\rho,2}(-\lambda_k t^\rho)\Big)\right]V_k+\sum_{k=1}^{\infty}v_k(t)V_k.$$

Dissertatsiyaning to‘rtinchi bobi **“Xususiy hosilali differensial tenglamalar uchun ko‘pnuqtali nolokal masalalar”** deb nomlanadi.

Ushbu bobning birinchi bo‘limida giperbolik tipga tegishli bo‘lgan tenglama uchun haqiqiy qiymatli ikkita parameter qatnashgan nolokal masala o‘rganildi.

6-masala. Aytaylik, $\Omega=(0,\pi)\times(0,T]$ sohada quyidagi nolokal masala berilgan bo‘lsin:

$$\begin{cases} u_{tt} - u_{xx} = f(x,t), \quad 0 < x < 1, \quad 0 < t \le T; \\ u(0,t) = u(1,t) = 0, \quad 0 \le t \le T; \\ u(x,\xi) = \alpha u(x,0) + \varphi(x), \quad 0 < \xi \le T; \\ u_t(x,\xi) = \beta u_t(x,0) + \psi(x), \end{cases} \tag{12}$$

bu yerda $\varphi(x), \psi(x) \in C[0,1]$, $f(x,t) \in C([0,1]\times[0,T])$, va α, β lar haqiqiy sonlar. Bu masalaning $u(x,t) \in C([0,1]\times(0,T])$, $u_{tt}, u_{xx} \in C([0,1]\times(0,T])$ shartlarni qanoatlantiruvchi yechimini toping.

8-teorema. *Aytaylik, $f(x,t)$ funksiya $\Omega = (0,\pi)\times(0,T]$ da uzluksiz, x o'zgaruvchisi bo'yicha ikki marta uzluksiz differensiallanuvchi va $f(0,t) = f(1,t) = 0$ shart barcha $t \in [0,T]$ da bajarilsin, hamda $\varphi(x) \in C^3[0,1]$, $\psi(x) \in C^2[0,1]$ funksiyalar quyidagi shartlarni qanoatlantirsin:*

$$\varphi(0) = \varphi(1) = 0, \quad \varphi''(0) = \varphi''(1) = 0, \quad \psi(0) = \psi(1) = 0.$$

Agar α, β parametrlar $\left|\dfrac{1+\alpha\beta}{\alpha+\beta}\right| > 1$ shartni qanoatlantirsa, *u holda, (1) masala yagona yechimga ega bo'ladi va bu yechim quyidagi ko'rinishga ega bo'ladi:*

$$u(x,t) = \sum_{k=1}^{\infty}\left[T_k(t) + (\varphi_k - T_k(\xi))\frac{(\cos\pi k(\xi - t) - \beta\cos\pi kt)}{1+\alpha\beta-(\alpha+\beta)\cos\pi k\xi}\right.$$

$$\left. -(\psi_k - T_k'(\xi))\frac{1}{\pi k}\frac{(\sin\pi k(\xi - t) + \alpha\sin\pi kt)}{1+\alpha\beta-(\alpha+\beta)\cos\pi k\xi}\right]\sin\pi kx,$$

bu yerda $T_k(t) = \dfrac{1}{\pi k}\int_0^t f_k(\eta)\sin\pi k(t-\eta)d\eta.$

Ushbu bobning ikkinchi bo'limida H separabel Hilbert fazosida elliptik qismi $-A$ abstrakt operator bo'lgan ikkinchi tartibli differensial tenglamaning cheksiz sohada m parametrga bog'liq ko'pnuqtali nolokal shartni qanoatlantiruvchi yechimini topish masalasi o'rganilgan.

7-masala. Aytaylik, A^{-1} kompakt operator bo'lsin. nolokal masala berilgan bo'lsin:

$$\begin{cases} u_{tt}(t) - Au(t) = F(t), \quad t > 0; \\ u(0) = \sum_{j=1}^{m}\alpha_j u(\xi_j) + \varphi; \\ \|u(t)\| \ \textit{chegaralangan}, \ \textit{agar} \ t \to \infty \ \textit{bo'lsa}, \end{cases} \tag{13}$$

bu yerda $F(t) \in C([0,\infty), H)$ berilgan, $\varphi \in H$ berilgan element, $\alpha_j, j = 1,2,..m;$ lar haqiqiy o'zgarmaslar, $\xi_j, j = 1,2,..m$ lar $(0,\infty)$ yarim o'qdagi $0 < \xi_1 < \xi_2 < \cdots < \xi_m$. fiksirlangan nuqtalar. Bu masalaning $u(t) \in C([0,\infty); H)$, $u_{tt}(t), Au(t) \in C((0,\infty); H)$ shartlarni qanoatlantiruvchi yechimini toping.

$C([0,\infty), H)$ sifatida $[0,\infty)$ da uzluksiz va qiymatlari H fazoda yotuvchi funksiyalar fazosini tushuniladi.

9-teorema. *Aytaylik* $\varphi \in H$, $F(t) \in C([0,\infty), D(A^{\frac{1}{4}}))$ *va quyidagi shart bajarilsin:*

$$\int_0^\infty \| F(t) \|_{\frac{1}{4}}^2 \, dt < \infty.$$

Bundan tashqari parametrlar va fiksirlangan nuqtalar quyidagi shartlardan birini qanoatlantirsin:

a) $\sum_{j=1}^{m} |\alpha_j| e^{-\sqrt{\lambda_1}\xi_j} < 1,$

b) $\alpha_j \le 0, j = 1,2,\dots,m.$

U holda (13) masalaning yechimi mavjud, yagona va quyidagi ko'rinishga ega:

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k + \sum_{j=1}^{m} \alpha_j v_k(\xi_j)}{1 - \sum_{j=1}^{m} \alpha_j e^{-\sqrt{\lambda_k}\xi_j}} e^{-\sqrt{\lambda_k}t} + v_k(t) \right] V_k. \quad (14)$$

Bu yerda

$$v_k(t) = -\frac{1}{\sqrt{\lambda_k}} \left[e^{-\sqrt{\lambda_k}t} \int_0^t \sinh(\sqrt{\lambda_k}\eta) F_k(\eta) d\eta + \sinh(\sqrt{\lambda_k}t) \int_t^\infty e^{-\sqrt{\lambda_k}\eta} F_k(\eta) d\eta \right],$$

φ_k, $F_k(\eta)$ *lar* φ, $F(\eta)$ larning mos Furye koeffitsiyentlari.

Shuningdek, shunday $C > 0$ *son mavjudki, quyidagi koersitiv tengsizlik o'rinli:*

$$\| u_{tt}(t) \|^2 + \| u(t) \|_1^2 \le C \left[\| F(t) \|^2 + \int_0^\infty \| F(t) \|_{\frac{1}{4}}^2 \, dt + t^{-4} \left(\int_0^\infty \| F(t) \|^2 \, dt + \| \varphi \|^2 \right) \right], \quad t > 0.$$

Quyidagi to'plamni aniqlab olaylik:

$$K_0 = \{ k : k \in \mathrm{N}, \quad 1 - \sum_{j=1}^{m} \alpha_j e^{-\sqrt{\lambda_k}\xi_j} = 0 \}.$$

10-teorema. *Aytaylik* $\varphi \in H$, $F(t) \in C([0,\infty), D(A^{\frac{1}{4}}))$ *va* $\int_0^\infty \| F(t) \|_{\frac{1}{4}}^2 \, dt < \infty$ *bo'lsin. U holda*

agar K_0 *to'plam bo'sh bo'lsa, () masala yagona yechimga ega va bu yechim (14) formula bilan aniqlanadi;*

agar K_0 *to'plam elementlarga ega bo'lsa, (8) masala yechimining mavjudligi uchun quyidagi ortogonallik shartlari bajarilishi zarur va yetarlidir:*

$$(\varphi, V_k) = 0, (F(t), V_k) = 0, k \in K_0, t > 0.$$

Bu holda (1) masalaning yechimi yagona emas va ixtiyoriy a_k, $k \in K_0$ *koeffitsientlar bilan quyidagi ko'rinishga ega bo'ladi:*

$$u(t)=\sum_{k\in K_0}a_k e^{-\sqrt{\lambda_k}t}V_k+\sum_{k\notin K_0}\frac{\varphi+\sum_{j=1}^{m}\alpha_j v(\xi_j)}{1-\sum_{j=1}^{m}\alpha_j e^{-\sqrt{\lambda_k}\xi_j}}e^{-\sqrt{\lambda_k}t}V_k+\sum_{k=1}^{\infty}v_k(t)V_k.$$

XULOSA

Mazkur dissertatsiya subdiffuziya, diffuziya, kasr tartibli to‘lqin tenglamalari hamda ikkinchi tartibli tenglamalar uchun parametrli nolokal masalalarni o‘rganishga bag‘ishlangan. Ushbu masalalarni yechishning asosiy usuli klassik Furye usulidir.

Dissertatsiyaning har bir bobidagi mazmun va asosiy natijalar quyidagicha umumlashtiriladi:

1. Hilbert fazosida Kaputo kasr hosilasiga ega subdiffuziya tenglamasi uchun parametrlar uchligi bilan bog‘liq ikki xil chiziqli nolokal shartlar ostida yechim topish masalasi o‘rganildi va ushbu shartlar $L_2[0,1]$ fazosidagi $u(x,0)=g(x,u(x,T)),\quad x\in[0,1],$ nochiziqli nolokal shart bilan almashtirilib, Banachning qisuvchi operatorlar prinsipi yordamida yechimning mavjudligi va yagonaligi isbotlanadi. Olingan natijalar subdiffuziya tenglamalari nazariyasini rivojlantiradi hamda qisman diffuziya tenglamalari nazariyasini umumlashtiradi.

2. Kaputo kasr tartibli to‘lqin tenglamalari uchun ikki va to‘rt parametrli nolokal masalalar parametrlarning barcha haqiqiy qiymatlari uchun to‘liq o‘rganildi. Parametrlar muayyan shartlarni qanoatlantirganda yechim mavjud va yagona bo‘lishi ko‘rsatildi. Ushbu shartlar buzilganda esa qo‘shimcha ortogonallik shartlari ostida masala yechimga ega bo‘lishi mumkin, biroq yagonalik saqlanmasligi aniqlangan.

3. Klassik to‘lqin tenglamasi uchun vaqt bo‘yicha nolokal masalalar tadqiq qilindi. Bundan tashqari, chegaralanmagan sohada $u_{tt}(t)-Au(t)=F(t),\ t>0$ tenglama uchun Bitsadze–Samarskiy masalasi qaralgan. Bu yerda qo‘yilgan $u(0)=\sum_{j=1}^{m}\alpha_j u(\xi_j)+\varphi,$ nolokal shart noma’lum funksiyaning soha chegarasidagi qiymatini uning ichki nuqtalardagi diskret qiymatlari bilan bog‘laydi.

Dissertatsiyada keltirilgan barcha masalalar uchun yechimlarning yagonaligini ta’minlovchi mezonlar aniqlangan. Bundan tashqari, masalalarning korrekt bo‘lishi uchun yetarli shartlar keltirilgan va koersitiv tengsizliklar isbotlangan.

Ko‘rib chiqilgan barcha nolokal masalalarning umumiy jihati, nolokal shartlarda parametrlarning ishtiroki bo‘lib, bu mavjud natijalarni umumlashtirish va kengaytirish imkonini berdi. Shuningdek, turli tipdagi tenglamalar uchun nolokal masalalarning o‘rganilgani natijasida keng qamrovli va yangicha ilmiy xulosalar olindi.

SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES DSc.02/30.12.2019.FM.86.01 AT V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS

INSTITUTE OF MATHEMATICS

NURALIEVA NAVBAKHOR SHAYMARDON QIZI

MULTIPOINT AND NONLINEAR NONLOCAL PROBLEMS FOR INTEGER AND FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

01.01.02 – Differential equations and mathematical physics

ABSTRACT OF DISSERTATION OF THE DOCTOR OF PHILOSOPHY (PhD) ON PHYSICAL AND MATHEMATICAL SCIENCES

TASHKENT– 2025

The theme of thesis of doctor of philosophy (PhD) on physical and mathematical sciences was registered at the Supreme Attestation Commission at the Ministry of Higher education, Science and Innovations of the Republic of Uzbekistan under number № B2025.2.PhD/FM1288

Dissertation has been prepared at the Institute of Mathematics named after V.I.Romanovskiy.

The abstract of the thesis is posted in three languages (Uzbek, English, Russian (summary)) on the website http://kengash.mathinst.uz and in the website of "ZiyoNet" Information and educational portal http://www.ziyonet.uz.

Scientific supervisor:	**Ashurov Ravshan Radjabovich** Doctor of Physical and Mathematical Sciences, Professor
Official opponents:	**Durdiyev Durdimurod Qalandarovich** Doctor of Physical and Mathematical Sciences, Professor
	Yaxshiboyev Maxmadiyor Umirovich Doctor of Physical and Mathematical Sciences, Docent
Leading organization:	**National University of Uzbekistan**

Defense will take place « 18 » November 2025 at 16:00 at the meeting of Scientific Council number DSc.02/30.12.2019.FM.86.01 at the Institute of Mathematics named after V.I. Romanovsky. (Address: University str. 9, Almazar area, Tashkent city, 100174, Uzbekistan, Ph.: (99871) 207-91-40, e-mail: uzbmath@umail.uz, Website: www.mathinst.uz)

Dissertation is possible to review in Information-resource center at Institute of Mathematics named after V.I.Romanovskiy (is registered № 211). (Address: University str. 9, Almazar area, Tashkent city, 100174, Uzbekistan, Ph.: (99871)-207-91-40).

Abstract of the dissertation sent out on « 29 » October 2025 year
(Mailing report № 2 on « 29 » October 2025 year)

U.A.Rozikov
Chairman of Scientific Council
on award of scientific degrees,
D.F.M.S., Academician

J.K.Adashev
Scientific secretary of Scientific Council
on award of scientific degrees,
D.F.M.S., Senior researcher

A.A.Azamov
Chairman of Scientific seminar under
Scientific Council on award of scientific degrees,
D.F.M.S., Academician

INTRODUCTION

Actuality and demand of the theme of the dissertation. Over the past decade, the field of fractional-order differential equations has attracted significant interest among mathematicians worldwide. At present, fractional partial differential equations have become an essential tool for researchers in accurately and reliably modeling a wide range of physical, biological, and chemical processes. In this regard, many scientists are actively engaged in developing new methods for solving fractional-order differential equations and studying new properties of their solutions. More recently, the study of such equations under various types of nonlocal conditions – distinct from classical Cauchy or boundary conditions – has marked a turning point in the field of differential equations. This direction, often referred to as nonlocal problems, has sparked growing interest among both experienced researchers and young scholars. This dissertation covers the study of problems involving the determination of solutions to subdiffusion, diffusion, fractional wave propagation equations, and second-order partial differential equations under various forms of parameter-dependent linear and nonlinear nonlocal conditions. The investigation of such problems is of considerable theoretical and practical importance today. The inclusion of parameters in linear and nonlinear nonlocal conditions provides a general framework that encompasses classical initial and boundary value problems, thereby broadening the applicability of the results. In particular, subdiffusion and diffusion equations are used to describe particle transport in porous media, the diffusion of substances in biological tissues, and the spread of materials within polymers. Fractional-order wave equations, on the other hand, are widely applied in modeling the propagation of electrical signals in cardiac tissue, bioelectrical impulses, vibrations in materials, and stress distribution.

Introducing nonlocal conditions in a parametric form enhances the flexibility of the problem and enables the modeling of systems with memory, long-range interactions, or hybrid boundary behaviors. As a result, the scope of applications is significantly extended. These include, but are not limited to: *Physics*: anomalous diffusion, heat and mass transfer in porous media; Medicine and biophysics: transport of substances within cells, signal transmission in cardiac and neural systems; Engineering: vibration and deformation of elastic structures; Information technology: signal propagation in networked systems; Economics and finance: memory-driven dynamic systems and delayed-response market models. The problems addressed and the results obtained in this dissertation provide a solid theoretical foundation for analyzing complex processes arising in these diverse fields. In our country, great importance is placed on the advancement of disciplines within the exact and natural sciences, such as mathematics, physics, biology, and geology. Particularly, fractional and partial differential equation theory development has been prioritized due to its essential role in understanding various phenomena in mechanics, electronics, control systems, physiology, and biological processes. Conducting research at the international level in these critical areas has been identified as a key priority in fundamental research. Currently, progress in the theory of fractional and partial differential equations is pivotal to the successful implementation of this directive. Researchers in our country have made significant contributions to these fields, contributing significantly to the advancement of the decree. It is gratifying that research in the field of differential equations and mathematical physics, particularly at the international level, is valued as one of the priority areas for the development of fundamental science in our country.

The subject and object of research of this thesis are in line with tasks identified in the Decrees of the President of the Republic of Uzbekistan UP-4947[1] of February 7, 2017 “On the strategy of action for the further development Of the Republic of Uzbekistan”, PP-2789 dated February 17, 2017 “On measures to further improve of the activities of the Academy of Sciences, organization, management and financing of research activities”, PP-3682 from April 27, 2018 “On measures to further improve the system of practical implementation of innovative ideas, technologies and projects” and PP-4387 from July 9, 2019 “On measures to further development of mathematical education and science, total improvement of the activity of the Uzbekistan Academy of Sciences V.I. Romanovskiy Institute of Mathematics” andalso PP-4708 from May 7, 2020 “On measures to improve the quality of education and research in mathematics” as well as in other regulations related to basic science.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the priority areas of science and technology of Republic of Uzbekistan IV, «Mathematics, Mechanics and Computer Science».

The degree of scrutiny of the problem. Fractional partial differential equations and their associated inverse problems have become an area of active research in contemporary mathematics. Initial-boundary value problems for fractional differential equations have been extensively studied by various authors. Notable contributions in this field have been made by researchers such as Sh.O. Alimov, R.R. Ashurov, S.R. Umarov, M. Yamamoto, Z. Li, A.V. Pskhu, M. Kirane, M. Ruzhansky, D.K. Durdiev, Yu.E. Fayziev, Z.A. Sobirov, E.T. Karimov, B.X. Turmetov, Y. Zhang, H.T. Nguyn, A.S. Malik, and others.

Many researchers have been studying the problem of finding solutions to differential equations under various forms of linear and nonlinear nonlocal conditions. The use of variable parameters has proven to be effective in simultaneously investigating multiple such problems, as it allows for the analysis of not just a single problem but an entire class of related problems.

In the works of A.N. Kochubei, K. Diethelm, J. Liu, M. Yamamoto, K. Sakamoto, G. Florida, Z. Li, S.A. Alimov, R.R. Ashurov, Yu.E. Fayziyev, A. Boutiara, K. Guerbati, M. Benbachir, F. Jarad, A. Abdeljawad, A. Yacine, B. Nouredine, A. Salim, M. Benchohra, and others various nonlocal problems with pointwise and integral conditions for diffusion and subdiffusion equations have

---

[1] Decree of Cabinet of Ministers of the Republic of Uzbekistan at the 2017-year 18 May «On measures on the organization of activities of the first created scientific research institutions of the Academy of Sciences of the Republic of Uzbekistan» No. 292 dated May 17, 2017.

been studied. In particular, nonlocal problems involving three parameters with pointwise and integral conditions for ordinary differential equations have been investigated by A. Boutiara, S. Krim, and others in their works. We extend the analysis of such problems to partial differential equations. M.O. Mamchuev constructed a solution of the subdiffusion equation satisfying classical initial-boundary conditions using the Green's function method. We consider a subdiffusion equation with a new nonlinear nonlocal condition. R.R. Ashurov and Yu.E. Fayziyev studied a nonlocal problem with a single parameter for the time-fractional subdiffusion equation with Riemann–Liouville and Caputo derivatives. They found orthogonality conditions that guarantee that the problem is well-posed under certain conditions with respect to the parameter and that solutions exist when these conditions are not met. D.K. Durdiyev and A.A. Rakhmonov examined a nonlocal problem involving two parameters for the time-fractional wave equation. We introduce and analyze a new nonlocal problem with four parameters for the time-fractional wave equation, extending the ideas developed in papers of R.R. Ashurov and D.K Durdiyev. Nonlocal multipoint problems involving parameters for second-order partial differential equations have been investigated in the works of A.O. Ashyralyev, P.E. Sobolevskii, A. Hanalyev, E. Karimov, M. Mamchuev, E. Uzturk, D.G. Orlovskii, S.I. Piskarev, and K.B. Sabitov. We study multipoint nonlocal problems with parameters for second-order partial differential equations.

Connection of the theme of the dissertation with the research works of higher education, where the dissertation is carried out. The dissertation work was carried out at the V.I. Romanovsky Institute of Mathematics under the research grant no.F-FA-2021-424 of the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan.

The aim of research work: The main objective of this research is to prove the existence and uniqueness of classical solutions for various forms of linear and nonlinear nonlocal problems with parameters, arising in subdiffusion, diffusion, fractional order wave equations, and second-order partial differential equations.

Research problems:

to prove the unique solvability of linear nonlocal problems with three parameters and nonlinear nonlocal problems for the subdiffusion equation;

to prove the unique solvability of nonlocal problems with two and four parameters for fractional wave propagation equations;

to prove the unique solvability of multipoint nonlocal problems for second-order partial differential equations.

The research object. The classical diffusion equation, the Caputo fractional subdiffusion equation, the fractional-order wave propagation equation, and second-order partial differential equations.

The research subject. Problems of finding solutions to subdiffusion, diffusion, fractional-order wave propagation equations, and second-order partial differential equations under linear and nonlinear nonlocal conditions involving parameters.

Research methods. In the research the methods of functional analysis, spectral theory and the Fourier methods are used.

Scientific novelty of the research work consists of the following:

it is proved unique solvability in the classical sense of linear nonlocal problems involving three parameters and nonlinear nonlocal problems for the subdiffusion equation;

it is proved unique solvability in the classical sense of two-parameter and four-parameter nonlocal problems for fractional-order wave propagation equations;

it is proved unique solvability in the classical sense of multipoint nonlocal problems for second-order partial differential equations.

Practical results of the research. The results and methodologies presented in this dissertation can be incorporated into graduate-level courses designed for master's and doctoral students at higher education institutions.

The reliability of the results of the study. The results were derived using the techniques from functional analysis, spectral theory, and the Fourier method. All obtained results are mathematically correct.

Scientific and practical significance of the research results is that the existence and uniqueness of solutions for various differential equations with linear and nonlinear nonlocal conditions involving parameters have been established. These results extend the theory of fractional-order differential equations and provide a generalized form of existing scientific developments. The obtained results can be applied to the modeling of real processes in physics, biology, and engineering. In particular, they serve as a theoretical foundation for constructing mathematical models corresponding to processes such as anomalous diffusion, substance distribution in tissues, and vibrations in elastic media.

Implementation of the research results. Based on the results obtained for multipoint and nonlinear nonlocal problems of integer- and fractional-order differential equations:

the results on the solutions of subdiffusion and diffusion equations under various parameter-dependent linear and nonlinear nonlocal conditions were applied in the international project No. 22-11-00064 titled “Modeling of Dynamic Processes in the Geosphere Considering Heredity” (Certificate No. 346, dated September 8, 2025, from the Institute of Cosmophysical Research and Radio Wave Propagation, Russian Federation). The application of this scientific result makes it possible to model the slow diffusion of drugs inside a cell and to determine the optimal dosing regimen;

The results obtained for fractional-order wave propagation equations and partial differential equations with parameter-dependent linear and nonlinear nonlocal conditions were applied in the international project No. 122041800013-4 titled “Investigation of Boundary Value Problems for Generalized Fractional-Order Differential Operator Equations and Their Applications in Modeling Physical and Socio-Economic Processes” (Certificate No. 01-13/48, dated April 16, 2025, from the Institute of Applied Mathematics and Automation, Kabardino-Balkarian Scientific Center, Russian Federation). The application of these scientific results has made it possible to effectively solve local and nonlocal boundary value problems for evolutionary equations involving integer- and fractional-order derivatives, which are successfully used in mathematical modeling of various physical and biological processes.

Approbation of the research results. The main results of the research were discussed at 4 international, 2 national scientific conferences, and 1 international congress.

Publications of the research results. A total of 15 scientific articles on the topic of the dissertation were published in academic publications included in the list of scientific journals recommended for the defense of PhD dissertations by the Higher Attestation Commission. 1 of them was published in an international mathematical journal, and the remaining 7 were published in national mathematical journals. 7 conference abstracts were also published.

The structure and volume of the dissertation. The dissertation consists of an introduction, four chapters, conclusion, and bibliography. The total volume of the dissertation is 128 pages.

MAIN CONTENT OF THE DISSERTATION

The **Introduction** substantiates the relevance of the dissertation topic and the necessity of its selection. A review of foreign and national scientific studies on the subject is provided, highlighting the degree of investigation of the problem. The objectives, tasks, object, and subject of the research are clearly stated. The first chapter of the dissertation, entitled *"***Preliminary Information***"*, is of an auxiliary nature and is primarily intended to facilitate the reading and understanding of the subsequent chapters. This chapter does not present new scientific results; rather, it contains only the definitions and previously known statements necessary for the research. Below, we provide some essential information from the first chapter:

Abstract operator in Hilbert space. Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Suppose that the operator $A: H \to H$ is defined in H, is self-adjoint, bounded from below, and positive definite. Assume that an operator A has a compact inverse operator A^{-1}. Then it has a complete system of orthonormal eigenvectors $\{v_k\}$ and a set of corresponding positive eigenvalues $\{\lambda_k\}$, i.e. the vectors $\{v_k\}$ and values $\{\lambda_k\}$ satisfy the following equality:

$$Av_k = \lambda_k v_k.$$

It is assumed that the eigenvalues are ordered such that $0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty$.

In order to formulate the main results of this dissertation, we introduce the Hilbert space of "smooth" functions related to the degree of operator A.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

where h_k is the Fourier coefficient of the element $h: h_k = (h, v_k)$.

Obviously, the domain of definition of this operator has the form

$$D(A^{\tau}) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

In addition, we define two concepts that are important for our study:
The two-parameter Mittag-Leffler function

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z, \mu \in C, \quad \rho > 0,$$

and the Caputo fractional derivative

$$D_t^{\rho} h(t) = \frac{1}{\Gamma(n-\rho)} \int_0^t \frac{h^{(n)}(\xi)}{(t-\xi)^{\rho+1-n}} d\xi, \quad t > 0, \quad n = [\rho] + 1.$$

We assume that the right-hand side of the equation in the last formula exists.

The main results of the dissertation begin with the second chapter, titled **"Linear and nonlinear nonlocal problems for subdiffusion equation".**

In the **first section of this chapter**, a three-parameter pointwise nonlocal problem for the subdiffusion equation with the Caputo fractional derivative and elliptic part given by an abstract operator A in a Hilbert space H was studied.

Let H be a Hilbert space and $AC[0,T]$ be the set of absolutely continuous functions defined on $[0,T]$. Throughout what follows, we denote by $AC([0,T];H)$ the space of functions that are absolutely continuous on $[0,T]$ and take values in H The space $C([0,T];B)$ is defined similarly.

Problem 1. Let $\rho \in (0,1)$. Consider the following three-parameter non-local problem

$$\begin{cases} D_t^{\rho} u(t) + Au(t) = f(t), \quad 0 < \rho \le 1, \quad 0 < t < T; \\ \alpha u(0) + \beta u(T) = \gamma u(\xi) + \varphi, \quad 0 < \xi < T, \end{cases} \tag{1}$$

where $f(t) \in C((0,T);H)$ is a given function, $\varphi \in H$ is a known element of H, α, β, γ are given constants, $\xi \in (0,T)$ is a fixed point. Find a function $u(t) \in AC([0,T];H)$ with properties $D_t^{\rho} u(t), Au(t) \in C((0,T);H)$, that satisfies the initial-boundary value problem (1).

Theorem 1. *Let one of the conditions be metfor some* $\varepsilon \in (0,1)$:

(N1) $f(t) \in C([0,T];D(A^{\varepsilon}))$, $\varphi \in H$ *and* $\alpha > 0, \ \beta \ge 0, \ \gamma \le 0$ *(or* $\alpha < 0, \ \beta \le 0, \ \gamma \ge 0$*);*

(N2) $f(t) \in C([0,T];D(A^{\varepsilon+1}))$, $\varphi \in D(A)$, $\alpha = 0$ *and* $\beta\gamma \le 0$, *or simultaneously the inequalities* $\beta\gamma > 0$ *and* $|\beta| \le |\gamma|$ *hold. Then problem (1) has a unique solution and this solution has the form*

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - \beta v_k(T) + \gamma v_k(\xi)}{\Delta_k} E_{\rho,1}(-\lambda_k t^{\rho}) + v_k(t) \right] V_k,$$

$$\Delta_k = \alpha + \beta E_{\rho,1}(-\lambda_k T^{\rho}) - \gamma E_{\rho,1}(-\lambda_k \xi^{\rho}), \ v_k(t) = \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^{\rho}) f_k(t-\eta) d\eta.$$

Moreover, there are constant C_{ε}, C *such that in case (N1) the following* *coercive type inequality holds:*

$$\| D_t^\rho u(t)\|^2 + \| u(t)\|_1^2 \le t^{-2\rho}(C_\varepsilon \max_{t\in[0,T]} \| f\|_\varepsilon^2 + C\|\varphi\|^2).$$

and in case (N2) we have

$$\| D_t^\rho u(t)\|^2 + \| u(t)\|_1^2 \le t^{-2\rho}(C_\varepsilon \max_{t\in[0,T]} \| f\|_{\varepsilon+1}^2 + C\|\varphi\|_1^2).$$

Theorem 2. *Let one of the following conditions be met for some* $\varepsilon \in (0,1)$ *:*

(L1) $f(t)\in C([0,T];D(A^\varepsilon))$ $\varphi \in H$ $\alpha \ne 0$ *and* the parameters do not satisfy condition (N1) of Theorem 1;

(L2) $f(t)\in C([0,T];D(A^{\varepsilon+2}))$ $\varphi\in D(A^2)$ *and* $\alpha = 0$, *at the same time* $\beta\gamma>0$ *and* $|\beta|\ge|\gamma|$, *for some* β,γ,T *and* ξ, *condition* $\dfrac{\beta}{\gamma} = \left(\dfrac{T}{\xi}\right)^\rho$ *holds;*

(L3) $f(t)\in C([0,T];D(A^{\varepsilon+1}))$ $\varphi\in D(A)$ *and* $\alpha=0$, *at the same time* $\beta\gamma>0$ *and* $|\beta|\ge|\gamma|$, *for some* β,γ,T *and* ξ *condition* $\dfrac{\beta}{\gamma} \ne \left(\dfrac{T}{\xi}\right)^\rho$ *holds.*

Then for the existence of a solution to problem (1) it is necessary and sufficient that the orthogonality conditions

$$\begin{cases} (\varphi, V_k) = 0, \ k\in K_0; \\ (f(t), V_k) = 0, \ t\ge 0, \quad k\in K_0, \end{cases}$$

where

$$K_0 := K_0(\Omega \subset R^3) = \{k : \Delta_k(\alpha,\beta,\gamma) = 0, \quad k\in \mathrm{N}, \quad (\alpha,\beta,\gamma)\in\Omega\}.$$

are satisfied. In this case, the solution to problem (1) is not unique and has the form

$$u(t) = \sum_{k\in K_0} a_k E_{\rho,1}(-\lambda_k t^\rho)V_k + \sum_{k\notin K_0}\left[\frac{\varphi_k - \beta v_k(T) + \gamma v_k(\xi)}{\Delta_k} E_{\rho,1}(-\lambda_k t^\rho) + v_k(t)\right]V_k.$$

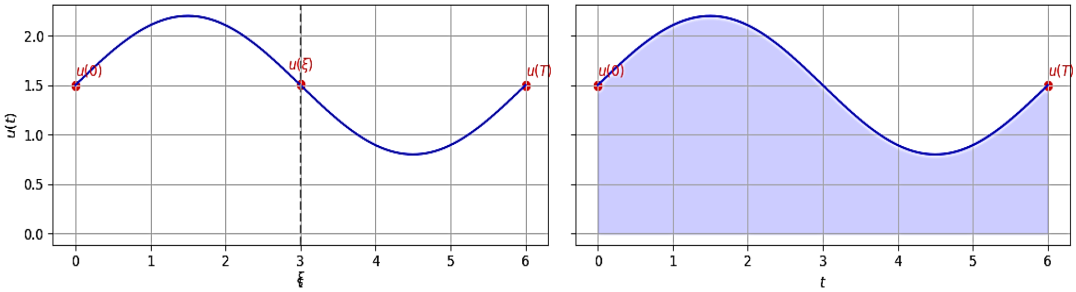
with arbitrary coefficients a_k, $k\in K_0$.

In **the second section of this chapter**, the following nonlocal problem with an integral condition, similar to problem (1), is considered:

Problem 2. Find a function $u(t)\in AC([0,T];H)$ with properties $D_t^\rho u(t), Au(t)\in C((0,T);H)$, that satisfies the following initial-boundary value problem:

$$\begin{cases} D_t^\rho u(t) + Au(t) = f(t), \quad 0<\rho<1, \quad 0<t<T, \\ \alpha u(0) + \beta u(T) + \gamma\int_0^T u(\eta)d\eta = \varphi, \end{cases} \tag{2}$$

In this **section**, theorems similar to **Theorems 1** and **2** above are proved for problem (2) as well. However, the nonlocal conditions used in problems (1) and (2) differ significantly from each other. This difference can be more easily understood through the following figure:



$\alpha u(0)+\beta u(T)=\gamma u(\xi)+\varphi$ $\qquad$ $\alpha u(0)+\beta u(T)+\gamma\int_0^T u(\eta)\,d\eta=\varphi$

In **the third section of this chapter**, the following nonlinear nonlocal problem for the subdiffusion equation with the Caputo fractional derivative in the space $C([0,1]\times[0,T])$ is considered:

Problem 3. Let us study the following initial-boundary-value problem for the fractional subdiffusion equation set in the domain $(0,1)\times(0,T]$:

$$D_{0t}^{\alpha}u(x,t)-u_{xx}(x,t)=f(x,t), \tag{3}$$

subject to the nonlocal initial condition

$$u(x,0)=g(x,u(x,T)),\quad x\in[0,1], \tag{4}$$

and the Dirichlet boundary conditions

$$u(0,t)=u(1,t)=0,\quad t\in[0,T], \tag{5}$$

where ∂_{0t}^{α} denotes the Caputo fractional derivative of order $0<\alpha<1$ with respect to the time variable t, and $f(x,t)$, $g(x,w)$ are given functions.

Theorem 3. *Let* $f(x,t)\in C([0,1]\times[0,T])$, $g(x,w)$ *be continuous functions of their arguments* $(x,w)\in[0,1]\times\mathrm{R}$, *satisfying the following Lipschitz condition*

$$|g(x,w)|\le L|w|,\quad 0<L<\frac{1}{\sqrt{E_{\alpha}(T^{\alpha})}}.$$

Then the solution of (2.3.1)–(2.3.3) satisfies the following a priori estimate

$$\|u(\cdot,t)\|^2\le\left[\frac{L^2E_{\alpha,\alpha}(T^{\alpha})}{1-L^2E_{\alpha}(T^{\alpha})}E_{\alpha}(t^{\alpha})+E_{\alpha,\alpha}(t^{\alpha})\right]\frac{T}{\alpha}\max_{[0,T]}\|f(\cdot,t)\|,$$

where $\|u(\cdot,t)\|^2=\int_0^1u^2(x,t)dx$ *and* $E_{\alpha}(\cdot),E_{\alpha,\tau}(\cdot)$ *are the Mittag–Leffler functions.*

Since the stated problem is nonlinear, we will find its solution by a somewhat nonstandard method. The formal solution of problem (3)–(5) can be expressed using the Green's function as follows:

$$u(x,t)=\int_0^1 g(\xi,v(\xi))D_{0t}^{\alpha-1}G(x,t,\xi,0)d\xi+\int_0^t\int_0^1G(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau.$$

Here, $v(\cdot)$ is the solution of the following integral equation,

$$v(x)=\int_0^1D_{0T}^{\alpha-1}G(x,T,\xi,0)g(\xi,v(\xi))d\xi+F(x),$$

for the Green's function under the integral, the following inequality holds:

$$|G(x,t,\xi,\tau)|\le C|t-\tau|^{\frac{\alpha}{2}-1},\quad 0\le\tau<t\le T. \quad (6)$$

Theorem 4. *Assume that:*

1. $f(x,t) \in C([0,1]\times[0,T])$;
2. $g(\xi,v) \in C([0,1]\times \mathrm{R})$ *satisfies the Lipschitz condition*

$$|g(\xi,v_1)-g(\xi,v_2)| \le L|v_1-v_2| \quad \text{for} \quad \text{all} \quad \xi\in[0,1], v_1,v_2\in \mathrm{R};$$

3. $L < \dfrac{\Gamma\left(1-\frac{\alpha}{2}\right)}{C\Gamma\left(\frac{\alpha}{2}\right)T^{1-\frac{\alpha}{2}}}$ *for* C*, which is in (6).*

Then the problem (3)–(5) admits a unique solution.

The third chapter is entitled **"Nonlocal problems for the caputo fractional order wave equation with some parametrs".**

In the first section of this chapter, the problem of finding a solution to the Caputo fractional wave equation in a separable Hilbert space H, with elliptic part given by an abstract operator, subject to a two-parameter nonlocal condition, is studied.

Problem 4. Let $\rho\in(1,2)$. Consider the following non-local problems, depending on real numbers α,β:

$$\begin{cases} D_t^{\rho}u(t)+Au(t)=f, & 0<t\le T; \\ \alpha u(0)+\beta u(\xi)=\varphi, & 0<\xi\le T; \\ u'(0)=\psi, \end{cases} \tag{7}$$

where $f(t)\in C([0,T];H)$ is a given function. $\varphi,\psi\in H$ are known elements of H, and ξ is a fixed number. Find a function $u(t)\in AC([0,T];H)$ with properties $D_t^{\rho}u(t), Au(t)\in C((0,T);H)$, that satisfies the problem.

Theorem 5. *Let* $f,\varphi,\psi\in D(A)$*. Futher let the parametrs* α *and* β *satisfy conditions*

$$|\alpha|>|\beta|.$$

Then problem (7) has a unique solution and this solution has the form

$$u(t)=\sum_{k=1}^{\infty}\left[\frac{1}{\Delta_k}(\varphi_k-\beta v_k(\xi)-\beta\xi E_{\rho,2}(-\lambda_k\xi^{\rho})\psi_k)E_{\rho,1}(-\lambda_k t^{\rho})+\psi_k tE_{\rho,2}(-\lambda_k t^{\rho})\right]V_k,$$

$$+\sum_{k=1}^{\infty}f_k t^{\rho}E_{\rho,\rho+1}(-\lambda_k t^{\rho})V_k,$$

where $\Delta_k=\alpha+\beta E_{\rho,1}(-\lambda_k\xi^{\rho})$.

Moreover, with some constant C *we have*

$$\|D_t^{\rho}u(t)\|^2+\|u(t)\|_1^2\le C(\|f\|^2+\|\varphi\|^2+\|\psi\|^2),\quad t>0.$$

In **the second section of this chapter**, a general complex nonlocal problem involving four parameters, which encompasses problem (7) and connects the states of the process at the initial and subsequent times, is thoroughly studied.

Problem 5. Let $\rho\in(1,2)$ be a fixed number. Consider the following class of non-local boundary value problems involving real parameters: $\alpha_1, \alpha_2, \beta_1, \beta_2$ $(\alpha_1^2 + \alpha_2^2 \neq 0, \beta_1^2 + \beta_2^2 \neq 0)$:

$$\begin{cases} D_t^\rho u(t) + Au(t) = f, \quad 0 < t \leq T; \\ \alpha_1 u(0) + \alpha_2 u(\xi) = \varphi, \quad 0 < \xi \leq T; \\ \beta_1 u'(0) + \beta_2 u'(\xi) = \psi, \end{cases} \tag{8}$$

where $f, \varphi, \psi \in H$ are prescribed elements of H, and $\xi \in (0,T)$ is a fixed number. Find a function $u(t) \in AC([0,T];H)$ with properties $D_t^\rho u(t), Au(t) \in C((0,T);H)$, that satisfies the problem (8).

Before presenting the main results concerning problem (8), we recall an important property of the Mittag-Leffler function.

Proposition. If $\rho \in (1,2)$, then there exists a constant $c_1 > 0$ such that the following inequality holds:

$$|E_{\rho,\mu}(-t)| \leq \frac{c_1}{1+t}. \tag{9}$$

Theorem 6. *Let $f, \varphi, \psi \in H$, $\alpha_1\beta_1 \neq 0$ and one of the following conditions hold for $c_1 > 0$, which in (9)*

(a) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} \geq 0, \quad 1 > c_1^2 \dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \dfrac{\alpha_2}{\alpha_1} + \dfrac{\beta_2}{\beta_1} \right|$;

(b) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} < 0, \quad 1 < -c_1^2 \dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \dfrac{\alpha_2}{\alpha_1} - \dfrac{\beta_2}{\beta_1} \right|$;

Then problem (8) has a unique solution, and this solution has the form

$$u(t) = \sum_{k=1}^{\infty} \Bigg[\frac{\varphi_k - (\alpha_1 v_k(0) + \alpha_2 v_k(\xi))}{\Delta_k} \Big((\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho)) E_{\rho,1}(-\lambda_k t^\rho)$$

$$+ \beta_2 \lambda_k \xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) t E_{\rho,2}(-\lambda_k t^\rho) \Big)$$

$$+ \frac{\psi_k - (\beta_1 v_k'(0) + \beta_2 v_k'(\xi))}{\Delta_k} \Big(-\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,1}(-\lambda_k t^\rho)$$

$$+ (\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)) t E_{\rho,2}(-\lambda_k t^\rho) \Big) + v_k(t) \Bigg] V_k, \tag{10}$$

where

$$v_k(t) = f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) v_k.$$

$$\Delta_k = \alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1) E_{\rho,1}(-\lambda_k \xi^\rho) + \alpha_2\beta_2 (E_{\rho,1}(-\lambda_k \xi^\rho))^2$$
$$+ \alpha_2\beta_2 \lambda_k \xi^\rho E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,\rho}(-\lambda_k \xi^\rho).$$

Moreover, there is a constant C, we also obtain a coercive inequality:

$$\| D_t^\rho u(t) \|^2 + \| u(t) \|_1^2 \leq C(\| f \|^2 + \| \varphi \|^2 + \| \psi \|^2), \quad t > 0; \tag{11}$$

Let us define the following set:

$$K_0 = \{k \in \mathrm{N} : \Delta_k = 0\}.$$

Theorem 7. *Let one of the following conditions hold for parameters* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *and elements* f, φ, ψ *:*

(1) $f, \varphi, \psi \in H$ *and the parameters do not satisfy any of the following conditions for any* $c_1 > 0$*, which in (9):*

(1a) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} \geq 0, \quad 1 > c_1^2 \dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \dfrac{\alpha_2}{\alpha_1} + \dfrac{\beta_2}{\beta_1} \right|$;

(1b) $\dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} < 0, \quad 1 < -c_1^2 \dfrac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \dfrac{\alpha_2}{\alpha_1} - \dfrac{\beta_2}{\beta_1} \right|$;

(2) $f, \varphi, \psi \in D(A)$ *and one of the following conditions hold:*

(2a) $\alpha_1\beta_1 = 0, \quad |\alpha_1| + |\beta_1| \neq 0, \quad \alpha_2\beta_2 \neq 0$,

(2b) $\alpha_1\beta_1 = 0, \quad \alpha_2\beta_2 = 0, \quad |\alpha_1\beta_2| + |\alpha_2\beta_1| \neq 0$;

(3) $f, \varphi, \psi \in D(A^2)$ *and the following condition hold:*

(3a) $|\alpha_1| + |\beta_1| = 0, \quad \alpha_2\beta_2 \neq 0.$

If the set K_0 *is empty, then the problem (8) has a unique solution, given by the formula (10).*

Moreover, there is a constant C*, we also obtain the corresponding coercive inequalities for each case:*
in case ***(1)*** *the inequality (11)*
in case ***(2)***

$$\| D_t^\rho u(t) \|^2 + \| u(t) \|_1^2 \leq C(\| f \|_1^2 + \| \varphi \|_1^2 + \| \psi \|_1^2), \quad t > 0;$$

in case ***(3)***

$$\| D_t^\rho u(t) \|^2 + \| u(t) \|_1^2 \leq C(\| f \|_2^2 + \| \varphi \|_2^2 + \| \psi \|_2^2), \quad t > 0;$$

If the set K_0 *is non-empty and the orthogonality condition*

$$\begin{cases} (\varphi, V_k) = 0, \quad (\psi, V_k) = 0, \quad k \in K_0, \\ (f, V_k) = 0, \quad k \in K_0. \end{cases}$$

is satisfied for indices $k \in K_0$*, then the problem (8) has a solution, given in the form*

$$u(t) = \sum_{k \in K_0} b_k \left[-\frac{\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho)}{\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)} E_{\rho,1}(-\lambda_k t^\rho) + t E_{\rho,2}(-\lambda_k t^\rho) \right] V_k$$

$$+ \sum_{k \notin K_0} \left[\frac{\varphi_k - (\alpha_1 v_k(0) + \alpha_2 v_k(\xi))}{\Delta_k} \Big((\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho)) E_{\rho,1}(-\lambda_k t^\rho) \right.$$

$$\left. + \beta_2 \lambda_k \xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) t E_{\rho,2}(-\lambda_k t^\rho) \Big) \right.$$

$$+ \frac{\psi_k - (\beta_1 v_k'(0) + \beta_2 v_k'(\xi))}{\Delta_k} \Big(-\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,1}(-\lambda_k t^\rho) +$$

$$(\alpha_1+\alpha_2 E_{\rho,1}(-\lambda_k\xi^\rho))tE_{\rho,2}(-\lambda_k t^\rho)\big)\Big]V_k+\sum_{k=1}^{\infty}v_k(t)V_k.$$

with arbitrary coefficients b_k.

The fourth chapter of the dissertation is entitled **“Multipoint nonlocal problems for partial differential equations”.**

In the first section of this chapter, a nonlocal problem with two real-valued parameters for an equation of hyperbolic type is studied.

Problem 6. Let the following nonlocal problem be given in the domain $\Omega=(0,\pi)\times(0,T]$:

$$\begin{cases} u_{tt}-u_{xx}=f(x,t), \quad 0<x<1, \quad 0<t\le T; \\ u(0,t)=u(1,t)=0, \quad 0\le t\le T; \\ u(x,\xi)=\alpha u(x,0)+\varphi(x), \quad 0<\xi\le T; \\ u_t(x,\xi)=\beta u_t(x,0)+\psi(x), \end{cases} \tag{12}$$

Here $\varphi(x),\psi(x)\in C[0,1]$, $f(x,t)\in C([0,1]\times[0,T])$, α, β are constant real numbers, and $\xi\in(0,T]$ is a fixed point. Find a function $u(x,t)\in C([0,1]\times(0,T])$ with properties $u_{tt},u_{xx}\in C([0,1]\times(0,T])$, that satisfies the problem (12).

Theorem 8. Let the function $f(x,t)$ be continuous on $\Omega=(0,\pi)\times(0,T]$, twice continuously differentiable with respect to the variable x, and satisfy the condition $f(0,t)=f(1,t)=0$ for all $t\in[0,T]$. Suppose that the functions $\varphi(x)\in C^3[0,1]$, $\psi(x)\in C^2[0,1]$ satisfy the conditions

$$\varphi(0)=\varphi(1)=0, \quad \varphi''(0)=\varphi''(1)=0, \quad \psi(0)=\psi(1)=0.$$

Then, if for the numbers α, β the following estimate holds:

$$\left|\frac{1+\alpha\beta}{\alpha+\beta}\right|>1,$$

problem (12) has a unique solution, and it is given by:

$$u(x,t)=\sum_{k=1}^{\infty}\Bigg[T_k(t)+(\varphi_k-T_k(\xi))\frac{(\cos\pi k(\xi-t)-\beta\cos\pi kt)}{1+\alpha\beta-(\alpha+\beta)\cos\pi k\xi}$$

$$-(\psi_k-T_k'(\xi))\frac{1}{\pi k}\frac{(\sin\pi k(\xi-t)+\alpha\sin\pi kt)}{1+\alpha\beta-(\alpha+\beta)\cos\pi k\xi}\Bigg]\sin\pi kx,$$

where $\quad T_k(t)=\dfrac{1}{\pi k}\displaystyle\int_0^t f_k(\eta)\sin\pi k(t-\eta)d\eta.$

In **the second section of this chapter**, the problem of finding a solution to a second-order differential equation in a separable Hilbert space H, whose elliptic part is represented by an abstract operator $-A$, is studied under a multipoint nonlocal condition with m parameters in an infinite domain.

Problem 7. Let A^{-1} be a compact operator. Consider the following nonlocal problem:

$$\begin{cases} u_{tt}(t) - Au(t) = F(t), \quad t > 0; \\ u(0) = \sum_{j=1}^{m} \alpha_j u(\xi_j) + \varphi; \\ \|u(t)\| \text{ is bounded, if } t \to \infty, \end{cases} \tag{13}$$

where $F(t) \in C([0,\infty), H)$ is a given function, $\varphi \in H$ is known element, $\alpha_j, j = 1,2,..m;$ are real constants, $\xi_j, j = 1,2,..m$ are fixed points of interval $(0,\infty)$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_m$. Find a function $u(t) \in C([0,\infty); H)$ with properties $u_{tt}(t), Au(t) \in C((0,\infty); H)$, that satisfies the problem (13).

$C([0,\infty), H)$ is the space of the function that is continious in $[0,\infty)$ and its values from H.

Theorem 9. *Let* $\varphi \in H$, $F(t) \in C([0,\infty), D(A^{\frac{1}{4}}))$ *, and suppose that*

$$\int_0^\infty \| F(t) \|_{\frac{1}{4}}^2 \, dt < \infty.$$

Additionally, assume that one of the following conditions holds:

a) $\sum_{j=1}^{m} |\alpha_j| e^{-\sqrt{\lambda_1}\xi_j} < 1,$

b) $\alpha_j \le 0, j = 1,2,\ldots,m.$

Then the solution to problem (13) exists, is unique, and has the following form:

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k + \sum_{j=1}^{m} \alpha_j v_k(\xi_j)}{1 - \sum_{j=1}^{m} \alpha_j e^{-\sqrt{\lambda_k}\xi_j}} e^{-\sqrt{\lambda_k} t} + v_k(t) \right] V_k \tag{14}$$

where

$$v_k(t) = -\frac{1}{\sqrt{\lambda_k}} [e^{-\sqrt{\lambda_k} t} \int_0^t \sinh(\sqrt{\lambda_k}\eta) F_k(\eta) d\eta + \sinh(\sqrt{\lambda_k} t) \int_t^\infty e^{-\sqrt{\lambda_k}\eta} F_k(\eta) d\eta]$$

and φ_k , $F_k(\eta)$ *are the Fourier coefficients of* φ *and* $F(\eta)$*, respectively. Moreover, the following coercive inequality holds:*

$$\| u_{tt}(t) \|^2 + \| u(t) \|_1^2 \le C[\| F(t) \|^2 + \int_0^\infty \| F(t) \|_{\frac{1}{4}}^2 \, dt + t^{-4} (\int_0^\infty \| F(t) \|^2 \, dt + \| \varphi \|^2)], \quad t > 0.$$

Let us define the following set

$$K_0 = \{k : k \in \mathrm{N}, \quad 1 - \sum_{j=1}^{m} \alpha_j e^{-\sqrt{\lambda_k}\xi_j} = 0\}.$$

Theorem 10. *Let* $\varphi \in H$, $F(t) \in C([0,\infty), D(A^{\frac{1}{4}}))$ *and* $\int_0^\infty \| F(t) \|_{\frac{1}{4}}^2 \, dt < \infty$ *hold. Then,*

if K_0 *is an empty set, the problem (13) has a unique solution, which is given by (14);*

if K_0 is not empty and the orthogonality conditions

$$(\varphi,V_k)=0,(F(t),V_k)=0,k\in K_0,t>0,$$

hold for indices $k\in K_0$, then the problem (13) has a solution, which is given by

$$u(t)=\sum_{k\in K_0}a_k e^{-\sqrt{\lambda_k}t}V_k+\sum_{k\notin K_0}\frac{\varphi+\sum_{j=1}^{m}\alpha_j v(\xi_j)}{1-\sum_{j=1}^{m}\alpha_j e^{-\sqrt{\lambda_k}\xi_j}}e^{-\sqrt{\lambda_k}t}V_k+\sum_{k=1}^{\infty}v_k(t)V_k.$$

with arbitrary coefficients a_k.

CONCLUSION

The main results of the dissertation can be summarized as follows:

1. In a Hilbert space, the problem of finding a solution to a subdiffusion equation with Caputo derivative under a nonclassical nonlocal condition involving two parameters was investigated. It was shown that under the conditions $u(x,0)=g(x,u(x,T)),\quad x\in[0,1],$ in the space $L_2[0,1]$, the nonclassical nonlocal condition is equivalent to the classical one. By applying the principle of contraction mappings for Banach's bounded linear operators, the existence and uniqueness of the solution were established. The obtained results contribute to the theory of subdiffusion equations and serve as further development of the general theory of diffusion-type equations.

2. For fractional-order wave equations, nonlocal problems with two and four parameters were studied, and the conditions for the existence and uniqueness of solutions were obtained. It was shown that if the parameter values satisfy certain constraints, then the unique solvability of the problem holds. It was also proved that if these conditions are violated, then despite the loss of strict orthogonality of the system, the uniqueness of the solution may still be preserved under certain circumstances.

3 For the classical wave equation, a nonlocal problem with two parameters was studied. In the unbounded domain, the problem $u_{tt}(t)-Au(t)=F(t),\ t>0$ with Bitsadze–Samarskii type conditions was considered. Here the condition was given as $u(0)=\sum_{j=1}^{m}\alpha_j u(\xi_j)+\varphi,$ where the nonlocal condition connects the unknown function with its internal values and boundary conditions.

For all the investigated nonlocal problems, the uniqueness of solutions was rigorously established by the method of contraction mappings. Furthermore, sufficient conditions for solvability were obtained.

The common feature of all the studied nonlocal problems is that the parameters in the conditions are interrelated, and their precise values influence the final results. Thus, for different types of differential equations, new generalizations and extensions of known results were derived, leading to comprehensive and original scientific conclusions.

НАУЧНЫЙ СОВЕТ DSc.02/30.12.2019.FM.86.01
ПО ПРИСУЖДЕНИЮ УЧЕНЫХ СТЕПЕНЕЙ ПРИ
ИНСТИТУТЕ МАТЕМАТИКИ ИМЕНИ В.И.РОМАНОВСКОГО

ИНСТИТУТ МАТЕМАТИКИ

НУРАЛИЕВА НАВБАХОР ШАЙМАРДОН КИЗИ

МНОГОТОЧЕЧНЫЕ И НЕЛИНЕЙНЫЕ НЕЛОКАЛЬНЫЕ ЗАДАЧИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ЦЕЛОЧИСЛЕННОГО И ДРОБНОГО ПОРЯДКА

01.01.02 – Дифференциальные уравнения и математическая физика

АВТОРЕФЕРАТ ДИССЕРТАЦИИ ДОКТОРА ФИЛОСОФИИ (PhD)
ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ

ТАШКЕНТ – 2025

Тема диссертации доктора философии (PhD) по физико-математическим наукам зарегистрирована в Высшей аттестационной комиссии при Министрстве высшего образования, науки и инноваций Республики Узбекистан за № B2025.2.PhD/FM1288

Диссертация выполнена в Институте Математики имени В.И.Романовкого.

Автореферат диссертации на трех языках (узбекский, английский, русский, (резюме)) размещен на веб-странице по адресу http://kengash.mathinst.uz и на Информационно-образовательном портале «ZiyoNet» по адресу http://www.ziyonet.uz.

Научный руководитель:	**Ашуров Равшан Раджабович** доктор физико-математических наук, профессор
Официальные оппоненты:	**Дурдиев Дурдимурод Каландарович** доктор физико-математических наук, профессор
	Яхшибоев Махмадиёр Умирович доктор физико-математических наук, доцент
Ведущая организация:	**Национальный университет Узбекистана**

Защита диссертации состоится « 18 » ноября 2025 года в 16:00 на заседании Научного совета DSc.02/30.12.2019.FM.86.01 при Институте Математики имени В.И.Романовского. (Адрес: 100174, г. Ташкент, Алмазарский район, ул. Университетская, 9.Тел.: (+99871) 207-91-40, e-mail: uzbmath@umail.uz, Website: www.mathinst.uz)

С диссертацией можно ознакомиться в Информационно-ресурсном центре Института Математики имени В.И.Романовского (зарегистрирована за № 211). (Адрес: 100174, г. Ташкент, Алмазарский район, ул. Университетская, 9.Тел.: (+99871) 207-91-40).

Автореферат диссертации разослан « 29 » октября 2025 года.
(протокол рассылки № 2 от « 29 » октября 2025 года).

У.А.Розиков
Председатель Научного совета
по присуждению ученых
степеней, д.ф.-м.н., академик

Ж.К.Адашев
Ученый секретарь Научного
совета по присуждению ученых
степеней, д.ф.-м.н., старший
научный сотрудник

А.А.Азамов
Председатель Научного семинара
при Научном совете по присуждению ученых
степеней, д.ф.-м.н., академик

ВВЕДЕНИЕ (аннотация диссертации доктора философии (PhD))

Целью исследования. Основной целью данной работы является доказательство существования и единственности классических решений для различных форм линейных и нелинейных нелокальных задач с параметрами, возникающих в уравнениях субдиффузии, диффузии, волновых уравнениях дробного порядка и уравнениях в частных производных второго порядка.

Объектом исследовани. Классическое уравнение диффузии, субдиффузионное уравнение с дробной производной в смысле Капуто, волновое уравнение дробного порядка и уравнения в частных производных второго порядка.

Научная новизна исследования состоит в следующем:

доказана единственность решения в классическом смысле для линейных нелокальных задач с тремя параметрами и нелинейных нелокальных задач для уравнения субдиффузии;

доказана единственность решения в классическом смысле для двухпараметрических и четырёхпараметрических нелокальных задач для волновых уравнений дробного порядка;

доказана единственность решения в классическом смысле для многоточечных нелокальных задач для уравнений в частных производных второго порядка.

Внедрение результатов исследования.

На основе полученных результатов выводы были применены к задачам определения решений уравнений субдиффузии, диффузии, дробного волнового уравнения и уравнений в частных производных второго порядка при различных формах линейных и нелинейных нелокальных условий с параметрами:

для уравнений субдиффузии и диффузии решения, полученные при различных линейных и нелинейных нелокальных условиях, зависящих от разных параметров, были применены в зарубежном проекте № 22-11-00064 на тему «Моделирование динамических процессов в геосфере с учетом наследственности» для классификации нелокальных задач для уравнений субдиффузии (Институт космофизических исследований и распространения радиоволн, справка № 346 от 8 сентября, 2025 г., Российская Федерация). Применение данного научного результата позволяет моделировать медленную диффузию лекарственных веществ внутри клетки и определять оптимальный режим дозирования.

для уравнений Для уравнений распространения волн дробного порядка и уравнений в частных производных исследованы решения при различных параметрах в условиях линейных и нелинейных нелокальных задач. Полученные результаты были применены в рамках международного проекта № 122041800013-4 «Изучение краевых задач для обобщённых дифференциальных операторов дробного порядка и их применение при моделировании физических и социально-экономических процессов» для математического моделирования различных физических и биологических процессов (Кабардино-Балкарский научный центр, Институт прикладной математики и автоматизации, Российская Федерация, справка № 01-13/48 от 16 апреля 2025 года). Научные результаты позволили эффективно решать локальные и нелокальные краевые задачи для эволюционных уравнений с производными дробного и целого порядка, что существенно расширило возможности их применения при моделировании различных физических и биологических процессов.

Объем и структура диссертации. Диссертация состоит из введения, четырех глав, заключения и списка использованной литературы. Общий объем диссертации составляет 128 стр.

E'LON QILINGAN ISHLAR RO'YXATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo'lim (part I; I часть)

1. Ashurov R.R., Nuraliyeva N.Sh. A three-parameter problem for frac tional differential equation with an abstract operator. Lobachevskii Journal of Mathematics. 2024. Vol. 45, No. 11, pp. 5788–5801. (3. SCOPUS, IF=0.45).
2. Ashurov R.R., Saparbayev R., Nuraliyeva N.Sh. An investigation of non-local and nonlinear boundary value problems for the fractional subdiffusion equa tion. Uzbek Mathematical Journal, 2025, Vol. 69, Issue 2, pp. 29-37. (01.00.00, №6).
3. Ashurov R.R., Nuraliyeva N. Time-dependent nonlocal conditions for hyperbolic-type equations. . Bulletin of the Institute of Mathematics, 2023, Vol. 6, No. 6. pp. 46-55. (01.00.00, №17).
4. Ashurov R.R, Nuraliyeva N., A four-parameter non-local problem for a fractional wave equation. Uzbek Mathematical Journal, 2025, Volume 69, Issue 3, pp.25-38. (01.00.00, №6).
5. Nuraliyeva N.Sh. A time non-local problem for the fractional differential equation. Uzbek Mathematical Journal. 2024. Vol. 68, Issue 3, pp. 123-132. (01.00.00, №6).
6. Nuraliyeva N.Sh. A three-parameter nonlocal problem involving the integral condition for the subdiffusion equation. Bulletin of the Institute of Mathe matics. 2024. Vol. 7(4), pp. 55–70. (01.00.00, №17).
7. Nuraliyeva N.Sh. Multipoint Bitsadze–Samarskii problem for an elliptic equation in an infinite domain. Bulletin of the Institute of Mathematics. 2025. Vol. 8(2), pp. 51–65. (01.00.00, №17).
8. Nuraliyeva N.Sh., A nonlinear nonlocal problem for the Caputo fractional subdiffusion equation., A nonlinear nonlocal problem for the Caputo fractional subdiffusion equation. Bulletin of the Institute of Mathematics. 2025. Vol. 8, Issue 4, pp. 89-95. (01.00.00, №17).

II bo'lim (Part II; II часть)

9. Ashurov R.R., Nuraliyeva N.Sh., Nonlocal in time problem for hyperbolic equation, Traditional International April Mathematical Conference in honor of the Day of Science Workers of the Republic of Kazakhstan, pp. 166-167, Almaty, April 16-19, 2024.
10. Ashurov R.R., Nuraliyeva N.Sh., Bitsadze–Samarskii problem for an elliptic equation in an infinite domain, International scientific conference: **"**Ufa Autumn Mathematical School-2024**",**pp. 15-16, Ufa, October 2-5, 2024.
11. Dusanova U.Kh., Nuraliyeva N.Sh., Nonlocal problem for a parabolic-hyperbolic equation with fractional Caputo derivative, International scientific conference "Modern problems of differential equations and their applications", pp. 333-335, Tashkent, November 23-25, 2023

12. Nuraliyeva N.Sh., A time non-local problem for the fractional differential equation, Actual problems of algebra, analysis, topology and computational mathematics, pp. 176-178, Tashkent, May 30-31, 2025.
13. Nuraliyeva N.Sh., A three-parameter problem for fractional differential equation with an abstract operator, 15th ISAAC Congress, Section 17: “Modern problems of fractional order differential equations”, pp. 106, Almaty, July 21-25, 2025.
14. Nuraliyeva N.Sh., A time-dependent nonlocal boundary problem for equations of hyperbolic type, International scientific and practical conference “Actual problems of physics, mathematics and mechanics”, pp. 163-165, Bukhara, May 24-25, 2023.
15. Navbahor Nuraliyeva., Nigora Axmadova. A four-parameter non-local problem for a fractional wave equation, International Online Scientific Conference “Advances in Mathematical Physics: Methods and Applications” Ghent, August 19-20, 2025.

Avtoreferat “O‘zbekiston matematika jurnali” tahririyatida
2025 yil 6-oktyabrda tahrirdan o‘tkazilib, o‘zbek, ingliz va rus tillaridagi matnlar
o‘zaro muvofiqlashtirildi.

Bosmaxona litsenziyasi:

9338

Bichimi: 84x60 $^{1}/_{16}$. «Times New Roman» garniturasi.
Raqamli bosma usulda bosildi.
Shartli bosma tobog‘i: 2,75. Adadi 100 dona. Buyurtma № 20/25.

Guvohnoma № 851684.
«Tipograff» MChJ bosmaxonasida chop etilgan.
Bosmaxona manzili: 100011, Toshkent sh., Beruniy ko‘chasi, 83-uy.