**V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI
HUZURIDAGI ILMIY DARAJALAR BERUVCHI
DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH**

O‘ZBEKISTON MILLIY UNIVERSITETI

TUREMURATOVA ARIUXAN ABATBAYEVNA

METRIK GRAFLARDA KASR TARTIBLI PARABOLIK TENGLAMALARNING UMUMLASHGAN YECHIMLARI

01.01.02 – Differensial tenglamalar va matematik fizika

FIZIKA-MATEMATIKA FANLARI BO‘YICHA FALSAFA DOKTORI (PhD) DISSERTATSIYASI AVTOREFERATI

TOSHKENT – 2025 yil

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Contents of dissertation abstract of doctor of philosophy (PhD) on physical – mathematical sciences

Оглавление автореферата диссертации доктора философии (PhD) по физико – математическим наукам

V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

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Fizika-matematika fanlari bo‘yicha falsafa doktori (PhD) dissertatsiyasi mavzusi O‘zbekiston Respublikasi Oliy ta’lim, Fan va Innovatsiyalar Vazirligi huzuridagi Oliy attestatsiya komissiyasida B2025.1.PhD/FM1001 raqam bilan ro‘yxatga olingan.

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Dissertatsiya himoyasi V.I.Romanovskiy nomidagi Matematika Instituti huzuridagi DSc.02/30.12.2019.FM.86.01 raqamli Ilmiy kengashning 2025 yil « 25 » noyabr kuni soat 16:00 dagi majlisida bo‘lib o‘tadi. (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871)-207-91-40, e-mail: uzbmath@umail.uz, Website: www.mathinst.uz).

Dissertatsiya bilan V.I.Romanovskiy nomidagi Matematika Institutining Axborot-resurs markazida tanishish mumkin (213-raqami bilan ro‘yxatga olingan). (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko‘chasi, 9-uy. Tel.: (+99871)-207-91-40.

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KIRISH (falsafa doktori (PhD) dissertatsiyasi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zarurati. Metrik graflarda differentsial operatorlar va xususiy hosilali differensial tenglamalarni oʻrganish matematika, fizika, muhandislik va biologiyani bogʻlaydigan juda dolzarb va fanlararo tadqiqot sohasi sifatida paydo boʻldi. Metrik graflar tarmoqlangan tuzilmalarga ega murakkab tizimlarni, masalan, tarmoqlangan tizimlar, kvant graflari va nano-miqyosdagi yoki biologik tizimlardagi diffuziya jarayonlarini modellashtirish uchun kuchli asos yaratadi. Kvant mexanikasi va suyuqliklar dinamikasidan tortib boshqaruv nazariyasi va teskari masalalargacha boʻlgan tadbiqlari sonining ortib borishi ushbu sohaning amaliy ahamiyatini ta’kidlaydi.

Soʻnggi yillardagi tadqiqotlar metrik graflarda, xususan, mezoskopik tarmoqlarda, molekulyar simlarda va boshqa diskret tuzilmalarda anomal diffuziya va subdiffuziya jarayonlarini tavsiflashda kasr tartibli differensial tenglamalarning muhimligini ta’kidladi. Kasr hosilalar va bu hodisalar oʻrtasidagi bogʻliqlik nazariy va amaliy tadqiqotlar uchun yangi yoʻllarni ochdi, buni kasr Shturm-Liouvil masalalari va vaqt boʻyicha kasr tartibli diffuziya tenglamalari boʻyicha ishlar tasdiqlaydi. Ushbu ishlanmalar real dunyo tizimlarida hamma joyda mavjud boʻlgan mahalliy boʻlmagan va xotiraga bogʻliq jarayonlarni tushunish uchun juda muhimdir.

Respublikamizda “Differensial tenglamalar va matematik fizika”[1] fanining ustuvor yoʻnalishlari boʻyicha xalqaro standartlar darajasida ilmiy-tadqiqot ishlarini olib borish matematikaning asosiy vazifalari va faoliyat yoʻnalishlari etib belgilangan. Shuning uchun metrik graflarda berilgan differensial tenglamalar uchun toʻgʻri va teskari masalalarni yechish zamonaviy matematikaning muhim yoʻnalishlaridan biridir. Ushbu tadqiqotning asosi boʻlgan teskari masalalar tizimni identifikatsiyalash, tibbiy tasvirlash, masofadan zondlash va boshqa ilmiy va muhandislik ilovalari uchun juda muhimdir. Bilvosita oʻlchovlardan differensial tenglamalarda noma`lum parametrlarni yoki manbalarni qayta qurish qobiliyati texnologiya va sanoat uchun chuqur ta’sir koʻrsatadi. Yaqinda oʻtkazilgan tadqiqotlar graflardagi kasrli tartibli xususiy hosilali differensial tenglamalar uchun teskari masalalarni hal qildi, integral qayta aniqlash shartlari va spektral usullar kabi usullarni qoʻlladi, bu sohaning dinamikligi va qoʻllanilishini yanada koʻrsatdi. Ushbu yutuqlarga qaramay, ayniqsa, graflardagi kasrli differensial tenglamalar uchun teskari masalalarni yechish, yechimlarning oʻziga xosligi va barqarorligini oʻrnatish va murakkab tarmoqlar uchun raqamli usullarni ishlab chiqishda qiyinchiliklar saqlanib qolmoqda. Dissertatsiyaning ushbu hal etilmagan savollarga qaratilishi hozirgi ilmiy talablarga mos keladi va nazariy tushunish va amaliy qoʻllash uchun potentsial hissa qoʻshadi.

Mazkur dissertatsiya ishining predmeti va obyekti Oʻzbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-sonli “Oʻzbekiston Respublikasini yanada rivojlantirish boʻyicha harakatlar strategiyasi toʻgʻrisida”gi farmoni,

[1] O’zbekiston Respublikasi Vazirlar Mahkamasining 2017 yil 18 maydagi «O’zbekiston Respublikasi Fanlar akademiyasining yangidan tashkil etilgan ilmiy-tadqiqot muassasalari faoliyatini tashkil etish to’g’risida» gi № 292-sonli qarori.

2017-yil 17-fevraldagi PQ-2789-sonli “Fanlar Akademiyasi faoliyatini yanada takomillashtirish, ilmiy-tadqiqot faoliyatini tashkil etish, boshqarish va moliyalashtirish chora-tadbirlari to‘g‘risida”gi qarori, 2018-yil 27-apreldagi PQ 3682-sonli “Innovatsion g‘oyalar, texnologiyalar va loyihalarni amaliyotga joriy etish tizimini yanada takomillashtirish chora-tadbirlari to‘g‘risida”gi qarori, 2019-yil 9-iyuldagi PQ 4387-sonli “Matematika ta’limi va fanini yanada rivojlantirish, O‘zbekiston Respublikasi Fanlar akademiyasi V. I. Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida”gi qarori, hamda 2020-yil 7-maydagi PQ-4708-sonli “Matematika fanida ta’lim va ilmiy-tadqiqotlar sifatini oshirish chora-tadbirlari to‘g‘risida”gi qarorida belgilangan dolzarb yo‘nalishlarga muvofiq tanlangan. Shuningdek, mazkur dissertatsiya ishini ushbu faoliyat bilan bog‘liq bo‘lgan tegishli normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishga qaratilgan tarzda kengaytirish ham mumkin.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga bog‘liqligi. Mazkur dissertatsiya Respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishi doirasida bajarilgan.

Muammoning o‘rganilganlik darajasi. Graflardagi differensial tenglamalar uchun chegaraviy masalalarni o‘rganish rus matematiklari Yu. V. Pokorniy, O.M. Penkin va ularning shogirdlarining ishlaridan boshlangan. 1980-yillardan boshlab ular tarmoqlardagi ikkinchi tartibli tenglamalar uchun chegaraviy masalalarni yechish imkoniyatini o‘rgandilar. Ular bu masalalar uchun Grin funksiyalar usulini ishlab chiqdilar, Shturm-Liuvil masalasini geometrik graflarda ko‘rib chiqdilar va graflar uchun Shturm-Liuvil nazariyasining to‘liq analogini yaratdilar. Chex olimlari P. Exner va P. Seba deyarli bir vaqtning o‘zida graflardagi klassik tenglamalar uchun chegaraviy masalalarni o‘rganishga kirishdilar. Graflardagi differensial tenglamalar nazariyasining rivojlanishi ushbu muammolarning kvant fizikasi, mexanika va amaliy fanlarning boshqa sohalarida yangi qo‘llanilishi bilan yanada rag‘batlantirildi. 2000-yillarning boshida U. Smilanskiy, T. Kottos va S. Gnutzmann kvant graflari tushunchasini kiritdilar va grafda aniqlangan Laplasiandan foydalanib, bu graflarda kvant xaos, sochilish xossalari va spektral statistikani o‘rganishga kirishdilar. Kasr hisob murakkab jarayonlarni modellashtirish uchun kuchli vositaga aylandi, bu esa kasr tartibli Shturm-Liuvil masalalarini graflarda o‘rganishga olib keldi. Ushbu masalalar klassik Shturm-Liuvil nazariyasini kasr-tartibli hosilalarga kengaytiradi, bu esa xos sonlar va xos funksiyalar xossalari haqida yangi tushunchalarni taklif qiladi. M. Klimek va boshqalar kasr tartibli Shturm-Liuvil masalasining spektral xossalarini o‘rganib, o‘ziga xos chegaraviy shartlarda diskret, sanoqli spektr mavjudligini ko‘rsatgan. Graflarda spektral nazariyani o‘rganish tadqiqotning muhim va ta’sirli sohasini ifodalaydi. Joel Fridman va Jan-Pyer Tillich grafdagi Laplasian uchun xos funksiyalar sistemasining to‘laligini masalasini tadqiq qildilar. V. Yurko Shturm-Liouvil operatorlari uchun teskari spektral masalani graf ichki uchlarida standart shartlar bilan ixtiyoriy kompakt graflarda o‘rgandi. R.Kulaev grafdagi Shturm-Liouvil operatorining xos qiymatlari xossalarini, jumladan ildiz funksiyalar to’laligi, birinchi xos qiymatni baholash va tebranish xossalarini o’rganib chiqdi.

Kuzatuvlar natijasida olingan ma’lumotlardan noma’lum parametrlarni yoki manbalarni tiklash bilan bog‘liq teskari masalalar metrik graflar nazariyasida markaziy o‘rinni egallaydi. Ushbu masalalar tizimni identifikatsiyalash, tibbiy tasvirlash, geofizika va boshqa sohalarda tadbiqini topadi. P.Kurasov, S.Avdonin, G.Lyugering kabi tadqiqotchilar graflarga teskari masalalarni yechishda turli yondashuvlarni ishlab chiqdilar. V. Kamynin, A. Prilepko, D. Tkachenko, R. Ashurov, D. Durdiev kabi tadqiqotchilar bu muammolarni hal qilish usullarini ishlab chiqdilar, ko‘pincha qo‘shimcha integral shartlari yoki spektral ma’lumotlardan foydalanganlar. Umumlashgan kasr hosilalar, kasr tartibli Sobolev-Slobodeskii fazolari kasr tartibli differensial tenglamalarning kuchsiz yoki umumlashgan yechimlarini tadqiq qilish uchun juda muhimdir. Xususiy hosilali differensial tenglamalar nazariyasiga o‘xshab, bu muammoni hal qilish uchun turli xil yondashuvlardan foydalanish mumkin. Bu sohaga muhim hissa D. Idjak va S. Ualjaklarning ishi bo‘lib, ular Riman-Liuvil kasr hosilasidan kasr tartibli Sobolev fazolarini aniqlashgan va turli xossalarini tavsiflashgan. Joylashtirish teoremasi bo’yicha natijalarini olgandan so‘ng, ular o‘z natijalarini o‘ng va chap Riman-Liuvil kasr hosilalari tarkibi nuqtai nazaridan ifodalangan Shturm-Liouvil kasr tenglamalariga qo‘llash mumkinligini ko‘rsatdilar. Bundan tashqari, A.Kubika, K.Ryjevska va M.Yamamoto tomonidan so‘nggi paytlarda olib borilgan ishlar Sobolev fazolarida vaqt bo‘yicha Kaputo kasr hosilalari uchun qat’iy usullarni, shuningdek, vaqt bo‘yicha xususiy kasr hosilali differensial tenglamalar uchun boshlang‘ich-chegaraviy masalalar yechimlarini taqdim etadi.

Ko‘pgina mutaxassislar vaqt bo‘yicha kasr tartibli differensial tenglamalar bilan bog‘liq turli masalalarni o‘rgandilar. Xususan, ayrim yurtdoshlarimiz, jumladan, Sh. Alimov, R. Ashurov, S. Umarov, B. Kadirkulov, E. Karimov, Z. Sobirov, O. Abdullaev, T. Yo‘ldoshevlar salmoqli hissa qo‘shgan. 21-asrning boshlarida kasr hosilalaridan foydalanishga bag‘ishlangan bir qancha ilmiy ishlar nashr etildi. A.Alixanov tomonidan olingan tengsizlik kasr tartibli differensial tenglamalar uchun aprior baho olish uchun katta ahamiyatga ega hisoblanadi. Rus matematigi A. V. Psxu Grin funksiya usuli yordamida kasr tartibli differensial tenglamalar uchun boshlang'ich va boshlang‘ich-chegaraviy masalalarni chuqur o‘rgandi. So‘nggi yillarda G. Leugering, V. Mehandiratta va M. Mehra metrik graflarda turli kasrli differensial tenglamalarni o‘rganishga katta hissa qo‘shdilar. Bundan tashqari, P.S. Fosting, G. Mofu va boshqalar graflarda kasr tartibli differensial tenglamalar uchun optimal boshqarish masalalarini o‘rgandilar. Shuningdek, J. Xo‘jaqulov “Metrik graflarda kasr tartibli diffuziya-to‘lqin tenglamalari uchun to‘g‘ri va teskari masalalar” mavzusida nomzodlik dissertatsiyasini himoya qildi.

Dissertatsiya tadqiqotining dissertatsiya bajarilgan Oliy ta’lim muassasasining ilmiy-tadqiqot ishlari rejalari bilan bog‘liqligi. Ushbu dissertatsiya tadqiqoti O‘zbekiston Milliy universitetining “Kasr va butun tartibli xususiy hosilali differensial tenglamalar uchun chegaraviy masalalar” dasturi bo‘yicha ilmiy tadqiqot ishlari doirasida bajarilgan va V.I. Romanovskiy nomidagi matematika instituti rejalashtirgan tadqiqotlar rejasiga muvofiq amalga oshirildi.

Tadqiqotning maqsadi metrik graflarda vaqt va fazoga bog‘liq bo‘lgan kasr tartibli differensial tenglamalar uchun to‘g‘ri va teskari masalalarni va kasr tartibli Shturm-Liuvil masalasini yechishdan iborat.

Tadqiqotning vazifalari

metrik grafda fazo va vaqt o’zgaruvchilari bo’yicha kasr tartibli parabolik tenglama uchun boshlang‘ich-chegaraviy masalalarning bir qiymatli yechilishini tadqiq qilish;

yulduzsimon metrik grafda vaqt bo‘yicha turli kasr tartibli subdiffuziya tenglamasi uchun boshlang‘ich-chegaraviy masalalarni o‘rganish;

metrik grafda subdiffuziya tenglamasi va kasr tartibli parabolik tenglamalar uchun teskari masalalarni yechish usulini ishlab chiqish;

metrik graflarda Shturm-Liouvil operatorini tadqiq qilish va uning xos funksiyalar sistemasining to‘laligini ko‘rsatish.

Tadqiqotning obyekti. Chekli, bog‘lamli metrik graflar, xususan, yulduzsimon graflarda differensial tenglamalar.

Tadqiqotning predmeti. Kasr tartibli boshlang‘ich-chegaraviy masala va tenglamalarning o‘ng tomonini aniqlashga oid teskari masalalar, Shturm-Liuvil masalasi.

Tadqiqotning usullari. Ushbu tadqiqot ishida funksional usul, aprior baholar, rezolventa usuli, variatsion usul, o‘zgaruvchilarni ajratish usullari qo‘llanilgan.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

kasr tartibli tenglamalar uchun masalalarni yechishning aprior baholarga asoslangan yangi operatorlar usuli taklif qilingan;

Sobolev-Slobodeskii fazolarida metrik graflar va yulduzsimon graflarda kasr hosilali boshlang‘ich-chegaraviy masalalar uchun kuchli yechimlar olindi;

Kaputo va Riman-Liuvilning umumlashgan (kuchsiz) kasr hosilalaridan tuzulgan simmetrik operator qatnashgan kasr parabolik va subdiffuziya tenglamalari bir qiymatli yechilishi isbotlangan;

Metrik graflarda kasr tartibli operator uchun Shturm-Liuvill masalasi qo‘yilgan hamda xos sonlarga teskari sonlar yig‘indisidan tuzilgan sonli qator yaqinlashishi va xos funksiyalar sistemasi to‘laligiga oid yangi natijalar olingan.

Tadqiqotning amaliy natijalari. Dissertatsiyada olingan natijalar asosan nazariy ahamiyatga ega bo‘lib, oliy o‘quv yurtlari magistrantlari va doktorantlari uchun ixtisoslashtirilgan kurs sifatida o‘qitilishi mumkin. Bundan tashqari, bu natijalar diffuziya jarayonlarining matematik modellarini ishlab chiqishda, xususan, metrik graflarda vaqt bo‘yicha xususiy kasr hosilali differensial tenglamalar uchun boshlang‘ich-chegaraviy va teskari masalalarni o‘rganishda qo‘llanilishi mumkin.

Tadqiqot natijalarining ishonchliligi. Olingan barcha natijalar to‘la matematik isbotlar bilan ta’minlangan. Natijalar bir qancha seminarlar va konferensiyalarda soha olimlari muhokamasidan o‘tgan.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy ahamiyati, olingan ilmiy natijalardan xususiy kasr hosilali differensial tenglamalar uchun qo‘yilgan to‘g‘ri va teskari masalalarni to‘liqroq tushunish uchun foydalanish mumkin.

Tadqiqot natijalarining amaliy ahamiyati, aprior baholar va muhim tengsizliklar Sobolev-Slobodeskii fazolarida xususiy kasr hosilali differensial tenglamalarni o‘z ichiga olgan boshlang‘ich-chegaraviy masalalarni yechimining yagonaligini ko‘rsatishga yordam beradi. Bundan tashqari, ushbu tadqiqot natijalari turli tenglamalar vositasida metrik graflarda issiqlik tarqalishi va tarmoqlangan tuzilmalarda diffuziya jarayonlarini tadqiq qilish uchun qo‘llanilishi mumkin. Ushbu tadqiqot natijalarining muhimligi, bu natijalar diffuziya jarayonlarining tarmoqlanish nuqtalarida o‘tkazuvchanlik va qaytish koeffisiyentlarini aniqlash, tarmoqlangan sohalarda diffuzion jarayonlarni tahlil qilish, nerv sistemalarida impulslarning tarqalishini matematik modellashtirish uchun asos bo‘lib xizmat qiladi.

Tadqiqot ishlarning joriy qilinishi. Metrik graflarda kasr tartibli parabolik tenglamalar uchun olingan natijalar asosida:

Sobolev fazolarida metrik graflar yordamida modellashtirilgan tarmoqlangan tuzilmalarda kasr tartibli parabolik tenglamalar uchun teskari masalalarga oid natijalar 22-11-00064 raqamli “Geosferadagi dinamik jarayonlarni irsiyatni hisobga olgan holda modellashtirish” mavzusidagi xorijiy loyihada dinamik jarayonlarni modellashtirishda foydalanilgan (Kosmofizik tadqiqotlar va radioto‘lqinlarning tarqalishi institutining 2025 yil 25-sentabrdagi №377-sonli ma’lumotnoma, Rossiya Federatsiyasi). Bu natijalar tuproq-atmosfera tizimida radon almashinuvini sonli modellashtirish va bu jarayonni tahlil qilish imkonini bergan;

kasr tartibli parabolik tenglamalar uchun to‘g‘ri va teskari masalalar hamda kasr tartibli Shturm-Liuvil masalalariga oid natijalardan 075-02-2024-1447 raqamli “Bog’langan graflarda kasr-tartibli tenglamalarni tadqiq qilish” mavzusidagi xalqaro loyihada graflarda berilgan chegaraviy masalalarni yechishda foydalanilgan (K.L. Xetagurov nomidagi Shimoliy Osetiya davlat universitetining 2025 yil 23-sentabrdagi №4087-sonli ma’lumotnoma, Rossiya Federatsiyasi). Bu natijalarning qo’llanilishi bog’langan graflarda Shturm-Liuvil masalalarida birinchi xos son uchun quyidan baho olish imkonini bergan.

Tadqiqot natijalarining aprobatsiyasi. Ushbu tadqiqot asosiy natijalari 10 ta ilmiy anjumanlarda, shu jumladan 6 ta xalqaro va 4 ta respublika ilmiy anjumanlarida muhokama qilindi.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya mavzusi bo‘yicha 15 ta ilmiy ishlar chop etilgan bo‘lib, shundan, 5 ta maqola O‘zbekiston Respublikasi Oliy attestatsiya komissiyasining falsafa doktorlik dissertatsiyalarining asosiy ilmiy natijalarini chop etish uchun tavsiya etilgan ilmiy nashrlarda chop etilgan, shulardan 2 tasi xorijiy va 3 tasi respublika jurnallarida chop etilgan bo‘lib, ulardan 3 tasi Scopus va WoS ma’lumotlar bazalarida indekslangan va 10 tasi xalqaro va respublika miqyosidagi konferensiyalar tezislaridir.

Dissertatsiyaning tuzilishi va hajmi. Dissertatsiya kirish, uchta bob, xulosa va foydalanilgan adabiyotlar ro'yxatidan iborat. Dissertatsiya umumiy hajmi 83 bet.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismida dissertasiya mavzusining dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo'nalishlariga mosligi ko'rsatilgan, mavzu bo'yicha xorijiy ilmiy-tadqiqotlar sharhi, muammoning o'rganilganlik darajasi keltirilgan, tadqiqot maqsadi, vazifalari, ob'ekti va predmeti tavsiflangan, tadqiqotning ilmiy yangiligi va amaliy natijalari bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertasiya tuzilishi bo'yicha ma'lumotlar keltirilgan.

Dissertatsiyaning **"Metrik graflarda fazo-vaqt bo'yicha kasr tartibli differensial tenglamalar uchun boshlang'ich-chegaraviy masalalar"** deb nomlangan birinchi bobining birinchi va ikkinchi paragraflarida dastlabki ma'lumotlar, uchinchi va to'rtinchi paragraflarida esa yangi natijalar keltirilgan.

Birinchi bobdagi ba'zi kerakli ma'lumotlarni keltiramiz.

$\mathcal{G}$ bog'langan, chegarasi bo'sh bo'lmagan metrik graf bo'lsin. Bu graf qirralarini $e_i,\ i=\overline{1,N},$ bilan belgilaymiz va har bir e_i qirraga $(0,l_i)$ intervalni mos qo'yamiz.

$G_\tau=\{(x,t): x\in\mathcal{G},\ 0<t\le\tau\},\ 0<\tau\le T,$ bo'lsin.

1-ta'rif. $L_2(\mathcal{G})$ fazo har bir $e_i,\ i=\overline{1,N}$ qirrada o'lchovli va kvadrati bilan integrallanuvchi funksiyalardan iborat fazo bo'lib, quyidagi skalyar ko'paytma va norma bilan aniqlangan:

$$\big(u(x),v(x)\big)_{L_2(\mathcal{G})}=\int_{\mathcal{G}}u(x)\cdot v(x)d\mathcal{G},$$

$$\|u\|^2_{L_2(\mathcal{G})}=\sum_i\|u\|^2_{L_2(e_i)}.$$

Boshqacha qilib aytganda, $L_2(\mathcal{G})$ fazo $L_2(e_i),\ i=\overline{1,N}$ fazolarning to'g'ri yig'indisidan iborat.

Biz $\mathcal{H}_+^\beta(\mathcal{G})=\left\{u\in\bigoplus_{i=1}^N\mathcal{H}_+^\beta(0,l_i),\ I_{0,x}^{1-\beta}u\in C(\mathcal{G})\right\}$ fazoni quyidagi norma bilan aniqlaymiz

$$\|u\|^2_{\mathcal{H}_+^\beta(\mathcal{G})}=\sum_{i=1}^N\left\|u^{(i)}\right\|^2_{\mathcal{H}_+^\beta(0,l_i)}=\sum_{i=1}^N\left(\left\|u^{(i)}\right\|^2_{L_2(0,l_i)}+\left\|D_{0,x}^\beta u^{(i)}\right\|^2_{L_2(0,l_i)}\right).$$

Biz yana $\mathcal{H}_{+,Dir}^\beta(\mathcal{G})=\left\{u\in\mathcal{H}_+^\beta(\mathcal{G}): I_{0,x}^{1-\beta}u\big|_{\partial\mathcal{G}}=0\right\},\quad \mathcal{H}_{+,Dir}^\beta[\mathcal{G}]=\bigoplus_{i=1}^N\mathcal{H}_{+,Dir}^\beta(0,l_i)$ fazolarni kiritamiz.

2-ta'rif. $L_2(G_\tau)=L_2(0,\tau;L_2(\mathcal{G}))$ deb aniqlaymiz. $L_2(G_\tau)$ fazoda skalyar ko'paytma va norma quyidagicha aniqlangan:

$$\left(u(x,t),v(x,t)\right)_{L_2(G_\tau)}=\int_0^\tau\int_{\mathcal{G}}u\cdot vd\mathcal{G}dt=\sum_{i=1}^{N}\int_0^\tau\int_0^{l_i}u^{(i)}(x,t)\cdot v^{(i)}(x,t)dxdt,$$
$$\|u\|_{L_2(G_\tau)}=\sqrt{(u,u)_{L_2(G_\tau)}}.$$

3-ta'rif. $W_2^{2,\alpha}(G_\tau)$ fazo Gilbert fazosi bo‘lib, $L_2(G_\tau)$ga $\partial_{0,t}^{\alpha}u$, u_x va u_{xx} tegishli umumlashgan hosilalarga ega va deyarli barcha $t\in(0,\tau)$ da $I_{0,x}^{1-\beta}u\in C(\mathcal{G})$ bo‘lgan $L_2(G_\tau)$fazoning barcha elementlaridan iborat fazo. Undagi skalyar ko‘paytma quyidagicha aniqlanadi

$$(u,v)_{W_2^{2,\alpha}(G_\tau)}=\int_0^\tau\int_{\mathcal{G}}\left(uv+u_xv_x+\partial_{0,t}^{\alpha}u\partial_{0,t}^{\alpha}v+u_{xx}v_{xx}\right)d\mathcal{G}dt$$

va norma $\|\cdot\|_{W_2^{2,\alpha}(G_\tau)}$ bilan belgilanadi.

Biz $W_{2,0}^{2,\alpha}(G_\tau)=\left\{u\in W_2^{2,\alpha}(G_\tau):u\big|_{\partial\mathcal{G}}=0\right\}$ fazoni kiritamiz.

Ushbu bobning uchinchi bo‘limida metrik grafdagi fazo-vaqt bo‘yicha kasr tartibli parabolik tenglama uchun boshlang'ich-chegaraviy masala o‘rganilgan.

1-to‘g‘ri masala. Biz quyidagi

$$\partial_{0,t}^{\alpha}u^{(i)}(x,t)+\partial_{x,l_i}^{\beta}(p^{(i)}(x)\mathrm{D}_{0,x}^{\beta}u^{(i)}(x,t))=h^{(i)}(x,t),$$
$$x\in e_i,\quad t\in(0,T],\quad i=\overline{1,N}, \tag{1}$$

fazo-vaqt bo‘yicha kasr tartibli parabolik tenglamani

$$u^{(i)}(x,0)=0,\quad x\in\overline{e}_i,\quad i=\overline{1,N}, \tag{2}$$

boshlang‘ich shartlar

$$\begin{cases}I_{0,x}^{1-\beta}u^{(i)}(v,t)=I_{0,x}^{1-\beta}u^{(j)}(v,t),\forall i,j\in I(v),\ t\in[0,T],\\ \sum\limits_{e_i\sim v}\sigma_{e_i,v}p^{(i)}(v)D_{0,x}^{\beta}u^{(i)}(v,t)=0,\quad v\in V\setminus\partial\mathcal{G},\end{cases} \tag{3}$$

ulanish shartlari va

$$I_{0,x}^{1-\beta}u(v,t)=0,\quad v\in\partial\mathcal{G},\ t\in[0,T]. \tag{4}$$

chegarviy shartlar bilan qaraymiz.

Haqiqiy o‘zgaruvchili $p^{(i)}(x)$ funksiyalar $(0,l_i)$ da absolyut uzluksiz va $0<a_0\le p^{(i)}\le a_1<+\infty$, $h^{(i)}(x,t), i=\overline{1,N}$, berilgan funksiyalar. $I(v)$ v uchga insident bo‘lgan indekslar to‘plami, $\sigma_{e_i,v}=1$, agar v uch e_i qirraning o‘ng uchi bo‘lsa, $\sigma_{e_i,v}=-1$, agar v uch e_i qirraning chap uchi bo‘lsa.

Berilgan masala operator tenglamaga olib kelinib, aprior baholarga asoslangan takomillashtirilgan funksional usul yordamida yechilgan.

Quyidagi fazoni kiritamiz:

$$V(G_T):=\left\{\begin{array}{c} u\in L_2(G_T): u(\cdot,t)\in H_\alpha(0,T),\ \partial_{0,t}^{\alpha}u\in L_2(G_T),\\ u\in L_2(0,T;\bigoplus_{i=1}^{N}\mathcal{V}_i(0,l_i)), I_{0,x}^{1-\beta}u^{(i)}(v,t)=I_{0,x}^{1-\beta}u^{(j)}(v,t), \forall i,j\in I(v),\\ \sum_{e_i\sim v}\sigma_{e_i,v}\,p^{(i)}(v)D_{0,x}^{\beta}u^{(i)}(v,t)=0,\quad v\in V\setminus\partial\mathcal{G},\\ I_{0,x}^{1-\beta}u(v,t)=0,\quad v\in\partial\mathcal{G},\ t\in[0,T] \end{array}\right\},$$

bu fazoda norma quyidagicha

$$\|u\|_{V(G_T)}^2=\|u\|_{L_2(G_T)}^2+\left\|\partial_{0,t}^{\alpha}u\right\|_{L_2(G_T)}^2+\left\|D_{+}^{\beta}u\right\|_{L_2(G_T)}^2+\left\|\partial_{-}^{\beta}(pD_{+}^{\beta}u)\right\|_{L_2(G_T)}^2.$$

Biz u uchun to‘g‘ri masalani operator tenglamaga keltiramiz:

$$Au=h.$$

Bunda A operatorning aniqlanish sohasi $D(A)=V(G_T)$ va

$$Au\,|_{e_i}=\partial_{0,t}^{\alpha}u^{(i)}(x,t)+\partial_{x,l_i}^{\beta}(p^{(i)}(x)\mathrm{D}_{0,x}^{\beta}u^{(i)}(x,t)),\qquad i=\overline{1,N}.$$

Bu operatorning qiymatlar sohasi $R(A)\subset L_2(G_T)$ bo‘ladi.

1-teorema. $h\in L_2\left(G_T\right)$ bo‘lsin. U holda 1-to‘g‘ri masala $V(G_T)$ fazoda bir qiymatli yechiladi.

Bu teorema quyidagi ikkita tasdiq va bitta lemmadan kelib chiqadi.

1-tasdiq. $A:D(A)\to L_2(G_T)$ operator uzluksiz.

A operatorning uzluksizligi quyidagi tengsizlikdan kelib chiqadi

$$\|Au\|_{L_2(G_T)}=\left\|\partial_{0,t}^{\alpha}u+\partial_{-}^{\beta}(pD_{+}^{\beta}u)\right\|_{L_2(G_T)}$$

$$\le\|\partial_{0,t}^{\alpha}u\|_{L_2(G_T)}+\left\|\partial_{-}^{\beta}(pD_{+}^{\beta}u)\right\|_{L_2(G_T)}\le\|u\|_{V(G_T)}.$$

2-tasdiq. $A^{-1}:R(A)\to V(G_T)$ teskari operator korrekt aniqlangan va uzluksiz.

$$\int_0^t d\tau\int_{\mathcal{G}}\left[\left(\partial_{0,\tau}^{\alpha}u\right)^2+\left(\partial_{-}^{\beta}(pD_{+}^{\beta}u)\right)^2\right]d\mathcal{G}+a_0\cdot\partial_{0,t}^{\alpha-1}\int_{\mathcal{G}}\left(D_{+}^{\beta}u\right)^2d\mathcal{G}\le\int_0^t d\tau\int_{\mathcal{G}}\left(Au\right)^2d\mathcal{G}.$$

$$\|u\|_{V(G_T)}\le C\|Au\|_{L_2(G_T)}.$$

1-natija. A operatorning $R(A)$ qiymatlar sohasi $L_2(G_T)$ fazoning yopiq chiziqli qism fazosidir.

Endi to‘g‘ri masalaning yechimga ega ekanligini ko‘rsatish uchun A operatorning $R(A)$ qiymatlar sohasining $L_2(G_T)$ fazoda ortogonal to‘ldiruvchisi yo‘qligini ko‘rsatish yetarli.

1-lemma. Agar ba’zi $\omega\in L_2(G_T)$ uchun barcha $v\in D(A)$ da $(Av,\omega)=0$ bo‘lsa, u holda $\omega=0$.

$$\int_0^t d\tau\int_{\mathcal{G}}\left(\partial_{0,t}^{\alpha}v+\partial_{-}^{\beta}(pD_{+}^{\beta}v)\right)\omega d\mathcal{G}=0.$$

$Lv = \partial_-^\beta (pD_+^\beta v)$ bo‘lsin. Biz $0 < t_1 < t < T$ da $v(x,t) = \partial_{t_1,t}^{-\alpha} L^{-1}\omega$ va $0 < t < t_1$ da $v = 0$ bo‘ladigan $v(x,t)$ funksiyasini tanlaymiz, bu yerda L^{-1} operator L ga teskari operator.

$$0 = \int_{t_1}^{t} d\tau \int_{\mathcal{G}} \left(\partial_{0,t}^\alpha v + \partial_-^\beta (p\mathrm{D}_+^\beta v) \right) \cdot \partial_{0,t}^\alpha \left(\partial_-^\beta (p\mathrm{D}_+^\beta v) \right) d\mathcal{G} =$$

$$= \int_{t_1}^{t} d\tau \int_{\mathcal{G}} \partial_{0,t}^\alpha \left(\partial_-^\beta (pD_+^\beta v) \right) \cdot \partial_{0,t}^\alpha v d\mathcal{G} + \int_{t_1}^{t} d\tau \int_{\mathcal{G}} \partial_{0,t}^\alpha \left(\partial_-^\beta (pD_+^\beta v) \right) \cdot \partial_-^\beta (pD_+^\beta v) d\mathcal{G} \geq$$

$$\geq a_0 \cdot \int_{t_1}^{t} d\tau \int_{\mathcal{G}} \left[D_+^\beta \left(\partial_{0,t}^\alpha v \right) \right]^2 d\mathcal{G} + \frac{1}{2} \partial_{t_1,t}^{\alpha-1} \int_{\mathcal{G}} \left[\partial_-^\beta \left(pD_+^\beta v \right) \right]^2 d\mathcal{G}.$$

Oxirgi tengsizlikdan $\omega = \partial_{0,t}^\alpha \left(\partial_-^\beta (pD_+^\beta v) \right) = 0$ ga ega bo‘lamiz.

Shuni ta’kidlash kerakki, ushbu paragrafda 1-to‘g‘ri masala uchun keltirilgan natijalar kasr tartibli parabolik tenglamalar uchun eng oddiy graf, kesma uchun ham yangidir.

Ushbu bobning to‘rtinchi paragrafida biz yulduzsimon metrik grafda vaqt bo‘yicha kasr hosilaning tartibi grafning har bir qirrasiga bog‘liq bo‘lgan subdiffuziya tenglamasi uchun boshlang‘ich-chegaraviy masalani ko‘rib chiqamiz.

2-to‘g‘ri masala. Biz $\mathcal{G}$ grafning har bir qirrasida

$$\partial_{0,t}^{\alpha_i} u^{(i)}(x,t) - u_{xx}^{(i)}(x,t) = h^{(i)}(x,t), \ x \in e_i, t \in (0,T], i = \overline{1,N}, \tag{5}$$

subdiffuziya tenglamasini tadqiq qilamiz. Bunda $0 < \alpha_i < 1, i = \overline{1,N}$. Biz (5) tenglamani quyidagi

$$u^{(i)}(x,0) = 0, \quad x \in \overline{e}_i, \quad i = \overline{1,N}, \tag{6}$$

boshlang‘ich shartlar

$$\sum_{i=1}^{N} u_x^{(i)}(0,t) = 0, \ I_{0,t}^{1-\alpha_i} u^{(i)}(0,t) = I_{0,t}^{1-\alpha_j} u^{(j)}(0,t), i \neq j, \ i, j = \overline{1,N}, \ t \in [0,T], \tag{7}$$

ulanish shartlari va

$$u^{(i)}(l_i,t) = 0, \quad t \in [0,T], \tag{8}$$

chegaraviy shartlar bilan qaraymiz, bu yerda $h^{(i)}(x,t), i = \overline{1,N},$ berilgan funksiyalar.

2-teorema. $h \in L_2(G_T)$ bo‘lsin. U holda 2-to‘g‘ri masala $W_{2,0}^{2,\alpha}(G_T)$ fazoda yagona kuchli yechimga ega bo‘ladi.

Teorema isboti quyidagi tasdiqlar va lemmadan kelib chiqadi.

3-tasdiq. $A : D(A) \to L_2(G_T)$ operator uzluksiz.

4-tasdiq. $A^{-1} : R(A) \to W_{2,0}^{2,\alpha}(G_T)$ teskari operator korrekt aniqlangan va uzluksiz.

2-natija. A operatorning $R(A)$ qiymatlar sohasi $L_2(G_T)$ fazoning yopiq chiziqli qism fazosidir.

2-lemma. Agar biror $\varphi \in L_2(G_T)$ uchun barcha $\omega \in D(A)$ da $(A\omega, \varphi) = 0$ bo‘lsa, u holda $\varphi = 0$.

Dissertatsiyaning **ikkinchi bobi "Metrik graflarda subdiffuziya tenglamalari uchun teskari masalalar"** deb nomlanadi.

Ushbu bobning birinchi paragrafida metrik grafda fazo-vaqt bo'yicha kasr tartibli parabolik tenglama uchun manbani aniqlashga oid teskari masala o'rganilgan.

1-teskari masala. Biz quyidagi

$$\partial_{0,t}^{\alpha}u^{(i)}(x,t)+\partial_{x,l_i}^{\beta}(p^{(i)}(x)\mathrm{D}_{0,x}^{\beta}u^{(i)}(x,t))= f(t)g^{(i)}(x,t),$$
$$x\in e_i,\quad t\in(0,T],\quad i=\overline{1,N}, \tag{9}$$

fazo-vaqt bo'yicha kasr tartibli parabolik tenglamani (2) boshlang'ich shartlar, (3) ulanish shartlari va (4) chegaraviy shartlar bilan birgalikda qaraymiz. (9) tenglamada $g^{(i)}(x,t),\, i=\overline{1,N},$ berilgan funksiyalar, $f(t)$ noma'lum funksiya. Biz $V(G_T)\times L_2(0,T)$ fazoda $\{u(x,t), f(t)\}$ funksiyalar juftligini topishimiz kerak. Buning uchun biz integral qayta aniqlash sharti deb ataladigan quyidagi qo'shimcha shartni kiritamiz

$$\int_{\mathcal{G}}\eta(x)u(x,t)d\mathcal{G}=\psi(t),\quad t\in[0,T], \tag{10}$$

bu yerda $\eta^{(i)}(x)$, $i=\overline{1,N}$ va $\psi(t)$ ma'lum funksiyalar.

Quyidagi shartlar bajarilsin va biz uni $(K1)$ shartlar deb belgilab olamiz:

$$g(x,t)\in L_{\infty}\left(0,T,L_2(\mathcal{G})\right),$$
$$\eta(x)\in\mathcal{H}_{+}^{\beta}(\mathcal{G}),\;\; I_{0,x}^{1-\beta}\eta^{(i)}(v)=I_{0,x}^{1-\beta}\eta^{(j)}(v),\forall i,j\in I(v),\;\; v\in V\setminus\partial\mathcal{G},$$
$$\left\|\mathrm{D}_{0,x}^{\beta}\eta(x)\right\|_{L_2(\mathcal{G})}=m>0,\quad \psi(t)\in H^{\alpha}(0,T),\quad |g^{*}(t)|\geq q>0,$$

bu yerda

$$g^{*}(t)=\int_{\mathcal{G}}\eta(x)g(x,t)d\mathcal{G},\qquad t\in(0,T].$$

(9) tenglamaning ikkala tomonini $\eta(x)$ funksiyaga ko'paytirib, $\mathcal{G}$ bo'yicha integrallab quyidagi tenglamaga ega bo'lamiz

$$f(t)=(Bf)(t)+\frac{\partial_{0,t}^{\alpha}\psi(t)}{g^{*}(t)},$$

bu yerda

$$(Bf)(t)=\frac{1}{g^{*}(t)}\int_{\mathcal{G}}p(x)D_{0,x}^{\beta}u(x,t)D_{0,x}^{\beta}\eta(x)d\mathcal{G},$$
$$B:L_2(0,T)\to L_2(0,T).$$

3-teorema. (K1) shartlar o'rinli bo'lsin. Agar $h(x,t)\in L_2(G_T)$ bo'lsa, u holda 1-teskari masala yagona $\{u,f\}\in V(G_T)\times L_2(0,T)$ umumlashgan yechimga ega bo'ladi va $\|f(t)\|_{L_2(0,T)}\leq C\|\psi(t)\|_{H^{\alpha}(0,T)}$.

Ushbu teorema rezolventa usulida isbotlangan.

Ushbu bobning ikkinchi paragrafi yulduzsimon metrik grafda vaqt bo'yicha Kaputo kasr hosilasining tartibi grafning har bir qirrasiga bog'liq bo'lgan subdiffuziya tenglamasi uchun teskari manba masalasining bir qiymatli yechilishini ko‘rsatishga qaratilgan.

2-teskari masala. Quyidagi tenglamani

$$\partial_{0,t}^{\alpha_i} u^{(i)}(x,t) - u_{xx}^{(i)}(x,t) = f(t) g^{(i)}(x,t), \; x \in e_i, t \in (0,T], \; i = \overline{1, \mathrm{N}}, \tag{11}$$

va (6) boshlang’ich shartlar, (7) ulanish shartlari, (8) chegarviy shartlar va quyidagi

$$\int_{\mathcal{G}} \eta(x) I_{0,t}^{1-\alpha} u(x,t) d\mathcal{G} = \psi(t), \quad t \in [0,T], \tag{12}$$

qo’shimcha integral qayta aniqlash shartlarini qanoatlantiruvchi $\{u(x,t), f(t)\}$ funksiyalar juftligini $W_{2,0}^{2,\alpha}(G_T) \times L_2(0,T)$ fazoda aniqlash masalasini qaraymiz, bu yerda $g^{(i)}(x,t)$, $\eta_i(x)$, $i = \overline{1,N}$ va $\psi(t)$ ma’lum funksiyalar.

Quyidagi $(K1)$ shartlar bajarilsin:

$$g(x,t) \in L_\infty\left(0,T, L_2(\mathcal{G})\right),$$

$$\eta(x) \in W_2^1(\mathcal{G}), \; I_{0,x}^{1-\beta}\eta^{(i)}(l_i) = 0, \; I_{0,x}^{1-\beta}\eta^{(i)}(0) = I_{0,x}^{1-\beta}\eta^{(j)}(0), \quad i \neq j, \; i,j = \overline{1,N},$$

$$\left\|\eta_x(x)\right\|_{L_2(\mathcal{G})} = m > 0, \; \psi(t) \in W_2^1(0,T), \; |\, g^*(t)\,| \geq q > 0,$$

bu yerda

$$g^*(t) = \int_{\mathcal{G}} \eta(x) g(x,t) d\mathcal{G}, t \in (0,T].$$

(11) tenglamaning ikkala tomonini $\eta(x)$ funksiyaga ko‘paytirib, $\mathcal{G}$ bo‘yicha integrallab quyidagi tenglamaga ega bo‘lamiz

$$f(t) = (Bf)(t) + \frac{\psi_t(t)}{g^*(t)},$$

bu yerda

$$(Bf)(t) = \frac{1}{g^*(t)} \left\{ \int_{\mathcal{G}} \eta_x(x) u_x(x,t) d\mathcal{G} \right\}, \; B : L_2(0,T) \to L_2(0,T).$$

4-teorema. (K1) shartlar o‘rinli bo‘lsin. Agar $h(x,t) \in L_2(G_T)$ bo‘lsa, u holda 2-teskari masala yagona $\{u, f\} \in W_{2,0}^{2,\alpha}(G_T) \times L_2(0,T)$ umumlashgan yechimga ega bo‘ladi. Bundan tashqari, $\| f(t)\|_{L_2(0,T)} \leq C \|\psi(t)\|_{H^1(0,T)}$.

Ushbu bobning uchinchi paragrafida biz yulduzsimon metrik grafda subdiffuziya tenglamasi uchun teskari manba masalasini o‘rganamiz.

3-teskari masala. Biz $\mathcal{G}$ yulduzsimon metrik grafda quyidagi

$$\partial_{0,t}^{\alpha} u^{(i)}(x,t) - u_{xx}^{(i)}(x,t) = f^{(i)}(x) g^{(i)}(x,t), \; x \in e_i, t \in (0,T], \; i = \overline{1,N}, \tag{13}$$

tenglamaning

$$u^{(i)}(x,0) = 0, \; x \in \overline{e}_i, \; i = \overline{1,N},$$

boshlang‘ich shartlar

$$\sum_{i=1}^{N} u_x^{(i)}(0,t) = 0, \; u^{(1)}(0,t) = u^{(2)}(0,t) = \ldots = u^{(n)}(0,t), \; t \in [0,T],$$

ulanish shartlari,

$$u^{(i)}(l_i,t)=0,\ t\in[0,T],\ i=\overline{1,N},$$

chegaraviy shartlar va

$$\int_0^T \eta(t)u^{(i)}(x,t)dt=\psi^{(i)}(x),\ i=\overline{1,N},$$

integral qayta aniqlash shartini qanoatlantiruvchi $\{u(x,t),f(x)\}$ yechimini topish masalasini qaraymiz, bu yerda $\eta(t)$, $\psi^{(i)}(x)$, $i=\overline{1,N}$, berilgan funksiyalar.

Faraz qilaylik, ushbu masalaga berilgan funksiyalar quyidagi shartlarni qanoatlantirsin:

$$\sup_{1\le i\le N}\operatorname{ess}\sup_{0\le x\le l_i} g^{(i)}(x,t)=\gamma(t),\quad \operatorname{ess}\sup_{0<t\le T}|\gamma(t)|\le s, \qquad \text{(K1)}$$

$$\eta(t), D^{\alpha}_{t,T}\eta(t)\in L_2(0,T),\ \ I^{1-\alpha}_{t,T}\eta(T)=0,\ \int_0^T\left(D^{\alpha}_{t,T}\eta(t)\right)^2dt=q^2, \qquad \text{(K2)}$$

$$\max_{1\le i\le N}\left|g_*^{(i)}(x)\right|\ge m>0, \qquad \text{(K3)}$$

bu yerda

$$g_*^{(i)}(x)=\int_0^T g^{(i)}(x,t)\eta(t)dt,\ i=\overline{1,N}.$$

(13) tenglamaning ikkala tomonini $\eta(t)$ funksiyaga ko‘paytirib, t bo‘yicha $[0,T]$ kesmada integrallab quyidagi tenglamaga ega bo‘lamiz

$$f^{(i)}(x)=Bf^{(i)}(x)-\frac{\psi^{(i)}_{xx}(x)}{g_*^{(i)}(x)},$$

bu yerda

$$Bf^{(i)}(x)=\frac{1}{g_*^{(i)}(x)}\int_0^T u^{(i)}(x,t)D^{\alpha}_{tT}\eta(t)dt,\ i=\overline{1,N},$$

$$Bf=\left(Bf^{(1)},Bf^{(2)},\ldots,Bf^{(N)}\right),\ B:L_2(\mathcal{G})\to L_2(\mathcal{G}).$$

5-teorema. $(K1)-(K3)$ shartlar o‘rinli bo‘lsin va

$$\frac{q^2s^2}{m^2\alpha(\alpha+1)}T^{\alpha+1}E_{\alpha,\alpha}(2T^{\alpha})<1.$$

U holda 3-teskari masalaning umumlashgan $\{u(x,t),f(x)\}$ yechimi mavjud bo‘ladi va $u(x,t)\in V(G_T)$, $f(x)\in L_2(\mathcal{G})$.

Dissertatsiyaning uchinchi bobi “Metrik graflarda kasr Shturm-Liuvil masalasi va uning qo‘llanilishi” deb nomlanadi.

Ushbu bobning birinchi paragrafida kasr Shturm-Liuvil operatori sanoqli sondagi xos sonlar to’plamiga ega ekanligi ko’rsatilgan. Shu bilan bir qatorda, xos funksiyalar sistemasi to‘laligi isbotlangan.

Biz $\mathcal{G}$ grafda quyidagi

$$\mathcal{L}u=\lambda ru,\quad x\in e_i,\quad \frac{1}{2}<\beta<1, \qquad (14)$$

kasr tartibli differensial tenglamani qaraymiz, bu yerda $u:\mathcal{G}\to\mathbb{R}$, $r:\mathcal{G}\to\mathbb{R}$, va

$$\mathcal{L}u|_{e_i} = \partial^{\beta}_{x,l_i}\left(p^{(i)}(x)D^{\beta}_{0,x}u^{(i)}(x)\right),$$

$0 < a_0 \le p^{(i)} \le a_1 < +\infty$, $0 < R_0 \le r^{(i)} \le R_1 < +\infty$ va $r, p \in C[\mathcal{G}]$ deb olamiz. Grafning har bir ν ichki uchida uzluksizlik va oqim saqlanish shartlari:

$$\begin{cases} I^{1-\beta}_{0,x}u^{(i)}(x)\ \ \nu\ \ da\ uzluksiz, \\ \sum\limits_{e_i\sim\nu}\sigma_{e_i,\nu}\, p^{(i)}(\nu)D^{\beta}_{0,x}u^{(i)}(\nu) = 0, \end{cases} \tag{15}$$

va har bir chegaraviy nuqtada

$$I^{1-\beta}_{0,x}u(\nu) = 0, \quad \nu \in \partial\mathcal{G},$$

Dirixle shartlari berilgan bo‘lsin.

6-teorema. (14)-(16) masalaning xos sonlari diskret va quyidagi tasdiqlar o’rinli:

1) $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$,

2) $\lambda_k \to \infty$,

3) u_k xos funksiyalar $\|\cdot\|_{1,L_2(\mathcal{G})}$ norma bilan $L_2(\mathcal{G})$ fazoda ortonormal bazis tashkil qiladi.

Ushbu teorema variatsion usul yordamida isbotlangan.

Ushbu bobning ikkinchi paragrafida xos sonlarga teskari miqdorlar yig‘indisidan iborat qatorning yaqinlashishi ko‘rsatilgan va xos funksiyalar uchun yuqori baho olingan.

Maks-min prinsipiga asosan

$$\lambda_k = \sup_{A_{k-1}\subset\mathcal{H}^{\beta}_{+}(\mathcal{G})}\ \inf_{\substack{u\in A^{\perp}_{k-1}\\ \|u\|=1}} \mathcal{R}(u)$$

bu yerda supremum $\mathcal{H}^{\beta}_{+}(\mathcal{G})$ fazoning $k-1$ o’lchamli barcha A_{k-1} qism fazolari bo’yicha olinadi.

7-teorema. $\sum\limits_{k=1}^{\infty}\frac{1}{\lambda_k}$ qator yaqinlashuvchi va

$$|u_k| \le C\left(1+\sqrt{\lambda_k}\right),$$

bu yerda C o’zgarmas R_0, a_0, β va $\mathcal{G}$ grafning qirralarining uzunliklaridan bog’liq.

Ushbu bobning uchinchi paragrafida fazo-vaqt bo‘yicha kasr tartibli diffuziya tenglamasi uchun boshlang‘ich-chegaraviy masala yechimga ega ekanligi ko‘rsatilgan va bitta teskari masalaning yagona yechimi mavjud bo‘lishi ham ko‘rsatilgan.

Biz $\mathcal{G}$ grafda quyidagi

$$\partial^{\alpha}_{0,t}u^{(i)}(x,t) + \mathcal{L}u^{(i)} = F^{(i)}(x,t) = f^{(i)}(x)g(t),\ x \in e_i,\ t \in (0,T],\ i = \overline{1,n}, \tag{17}$$

kasr tartibli diffuziya tenglamasini qaraymiz, bu yerda

$$\mathcal{L}u^{(i)} = \frac{1}{r^{(i)}(x)}\partial^{\beta}_{x,l_i}\left(p^{(i)}(x)D^{\beta}_{0,x}u^{(i)}(x,t)\right),$$

$F^{(i)}(x,t)$ berilgan funksiyalar, $0<a_0\le p^{(i)}(x)\le a_1<+\infty$, $0<R_0\le r^{(i)}(x)\le R_1<+\infty$ va $r,p\in C[\mathcal{G}]$ bo'lsin. $f^{(i)}(x)$ noma'lum funksiyalar va $g(t)$ berilgan funksiya. (17) tenglamaning $G_T=\mathcal{G}\times(0,T]$ sohada

$$u^{(i)}(x,0)=\varphi^{(i)}(x),\quad x\in\overline{e}_i,\ i=\overline{1,n}, \tag{18}$$

boshlang'ich shartlar

$$\begin{cases} I_{0,x}^{1-\beta}u^{(i)}(\nu,t)=I_{0,x}^{1-\beta}u^{(j)}(\nu,t), \forall i,j\in I(\nu), t\in[0,T], \\ \sum\limits_{e_i\sim\nu}\sigma_{e_i,\nu}p^{(i)}(\nu)D_{0,x}^{\beta}u^{(i)}(\nu,t)=0,\quad \nu\in V\setminus\partial\mathcal{G}, \end{cases} \tag{19}$$

ulanish shartlar

$$I_{0,x}^{1-\beta}u(\nu,t)=0,\quad \nu\in\partial\mathcal{G},\ t\in[0,T], \tag{20}$$

Dirixle shartlarini va

$$u^{(i)}(x,T)=\psi^{(i)}(x),\quad x\in\overline{e}_i,\ i=\overline{1,n}, \tag{21}$$

oxirgi-vaqt sharti deb ataluvchi shartlarni qanoatlantiradigan $\{u(x,t),f(x)\}$ yechimini topish masalasini qaraymiz, bu yerda $\varphi^{(i)}(x)$, $\psi^{(i)}(x)$, $i=\overline{1,n}$ ma'lum funksiyalar.

Birinchi navbatda (17) tenglamaning o'ng tomonidagi funksiya ma'lum deb qarab, (17)-(20) boshlang'ich chegaraviy masala (to'g'ri masala) yechimining mavjudligi va yagonaligi ko'rsatiladi.

(17)-(20) masalani yechish uchun o'zgaruvchilarni ajratish usulidan foydalanib, har bir e_i qirrada yechim uchun quyidagi formulalarni olamiz

$$\begin{aligned} u^{(i)}(x,t)&=\sum_{k=1}^{\infty}\varphi_k^{(i)}w_k^{(i)}(x)E_{\alpha,1}(-\lambda_k t^{\alpha}) \\ &+\sum_{k=1}^{\infty}w_k^{(i)}(x)\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})F_k^{(i)}(\tau)d\tau. \end{aligned} \tag{22}$$

3-lemma. $\mathcal{L}F(x,t)\in C([0,T],L_2(\mathcal{G}))$, $\mathcal{L}\varphi(x)\in L_2(\mathcal{G})$ bo'lsin va

$$F(x,t)=\sum_{k=1}^{\infty}F_k(t)w_k(x),\quad \varphi(x)=\sum_{k=1}^{\infty}\varphi_k w_k(x).$$

U holda quyidagi tengliklar o'rinli:

$$F_k(t)=\left(F(x,t),w_k(x)\right)_{L_2(\mathcal{G})}=\frac{\left(\mathcal{L}F(x,t),w_k(x)\right)_{L_2(\mathcal{G})}}{\lambda_k}=\frac{q_k(t)}{\lambda_k};$$

$$\varphi_k=\left(\varphi(x),w_k(x)\right)_{L_2(\mathcal{G})}=\frac{\left(\mathcal{L}\varphi(x),w_k(x)\right)_{L_2(\mathcal{G})}}{\lambda_k}=\frac{s_k}{\lambda_k},$$

bu yerda $\left(s_1,s_2,\ldots\right)\in l_2$, $\forall t\in[0,T]$ uchun $\left(q_1(t),q_2(t),\ldots\right)\in l_2$.

8-teorema. 3-lemma o'rinli bo'lsin va $I_{0,x}^{1-\beta}F(x,t)\in C([0,T];C(\mathcal{G}))$, $D_{0,x}^{\beta}F(x,t),\partial_{0,t}^{\alpha}F(x,t)\in C\left([0,T];C(\mathcal{G})\right)$, $\varphi(x)\in C(\mathcal{G})$, $D_{0,x}^{\beta}\varphi(x)\in C(\mathcal{G})$ va $F(x,t)$,

$\varphi(x)$ funksiyalar (15),(16) shartlarni qanoatlantirsin. U holda (17)-(20) masala (22) ko'rinishda aniqlangan $u: G_T \to \mathbb{R}$ yagona yechimga ega.

To'g'ri masala yechimining yagonaligi.

4-lemma. 8-teoremaning shartlari o'rinli bo'lsin. U holda (17)-(20) masala yagona yechimga ega va quyidagi aprior baho o'rinli:

$$\| u \|^2_{L_2(\mathcal{G})} \le E_{\alpha,1}(t^\alpha) \cdot \| \varphi \|^2_{L_2(\mathcal{G})} + \Gamma(\alpha) \cdot E_{\alpha,\alpha}(t^\alpha) \cdot I^\alpha_{0,t} \| F \|^2_{L_2(\mathcal{G})} .$$

Teskari masala. $B_{k,\alpha}(t) = \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) g(\tau) d\tau$ bo'lsin.

$$f(x) = \sum_{k=1}^{\infty} f_k w_k(x), \quad \text{yoki} \quad F(x,t) = f(x) g(t) = g(t) \sum_{k=1}^{\infty} f_k w_k(x).$$

deb olamiz. Bundan

$$\begin{aligned} u(x,t) = & \sum_{k=1}^{\infty} \varphi_k w_k(x) E_{\alpha,1}(-\lambda_k t^\alpha) \\ & + \sum_{k=1}^{\infty} f_k w_k(x) \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) g(\tau) d\tau. \end{aligned} \tag{23}$$

ega bo'lamiz.

(23) tenglamada t o'zgaruvchining o'rniga T ni qo'yib

$$u(x,T) = \psi(x) = \sum_{k=1}^{\infty} \psi_k w_k(x),$$

va qo'shimcha (21) oxirgi-vaqt shartini qo'llash orqali f_k ni quyidagicha topishimiz mumkin:

$$f_k = \frac{\psi_k - \varphi_k E_{\alpha,1}(-\lambda_k T^\alpha)}{B_{k,\alpha}(T)}.$$

Bundan

$$f(x) = \sum_{k=1}^{\infty} \left[\frac{\psi_k - \varphi_k E_{\alpha,1}(-\lambda_k T^\alpha)}{B_{k,\alpha}(T)} \right] w_k(x).$$

topamiz.

9-teorema. $g(t) \in C[0,T]$ va $g(t) \neq 0$, $|g(t)| < m$, $t \in [0,T]$ bo'lsin. Bundan tashqari, 5-lemma shartlari o'rinli bo'lsin va $\psi(x) \in C(\mathcal{G})$, $\mathcal{L}\psi(x), \mathcal{L}^2\psi(x) \in L_2(\mathcal{G})$, $\mathcal{L}\psi(x)$ (14)-(16) shartlarni qanoatlantirsin. U holda (17)-(21) masala regular yechimga ega.

XULOSA

Ushbu dissertatsiya metrik graflarda kasr tartibli differensial tenglamalar uchun boshlang‘ich-chegaraviy, teskari va spektral masalalar o‘rganilgan. Masalalarni yechishda berilgan tenglamalarni operator tenglamalariga olib kelib, aprior baholarga asoslangan takomillashtirilgan funksional usul qo‘llanildi.

Ushbu tadqiqotning asosiy natijalari quyidagicha umumlashtiriladi.

1. Metrik graflarda kasr tartibli parabolik tenglamalar uchun boshlang‘ich-chegaraviy masalalarning Sobolev-Slobodeskii fazolarida kuchli yechimlari olingan;

2. Vaqt va fazo o‘zgaruvchilari bo‘yicha kasr hosilali parabolik tenglamalar va subdiffuziya tenglamalari uchun manba funksiyalarni aniqlashga oid integral shartli teskari masalalar tadqiq qilingan. Masalalarni yechishda teskari masalalar uchun rezolventalar usuli birinchi marotaba qo‘llanilgan;

3. Metrik graflarda kasr tartibli Shturm-Liuvil masalasi tadqiq qilindi. Xos funksiyalar sistemasi to‘laligi isbotlandi;

4. Metrik graflarda kasr tartibli Shturm-Liuvil masalasi xos sonlarga teskari miqdorlar yig‘indisidan iborat qator yaqinlashuvchiligi isbotlandi. Xos funksiyalar uchun yuqori baholar olindi;

5. Fazo va vaqt o‘zgaruvchilari bo‘yicha kasr tartibli parabolik tenglama uchun vaqtning oxirgi momentidagi ma’lumot bo‘yicha manba funksiyani aniqlash bo‘yicha teskari masala yechimining mavjudligi va yagonaligi isbotlanib, qator ko‘rinishidagi aniq yechim topildi.

SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES DSc.02/30.12.2019.FM.86.01 AT V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS

NATIONAL UNIVERSITY OF UZBEKISTAN

TUREMURATOVA ARIUKHAN ABATBAYEVNA

GENERALIZED SOLUTIONS OF FRACTIONAL PARABOLIC EQUATIONS ON METRIC GRAPHS

01.01.02 – Differential equations and mathematical physics

ABSTRACT OF DISSERTATION OF THE DOCTOR OF PHILOSOPHY (PhD) ON PHYSICAL AND MATHEMATICAL SCIENCES

Tashkent– 2025

The theme of thesis of doctor of philosophy (PhD) on physical and mathematical sciences was registered at the Supreme Attestation Commission at the Ministry of Higher education, Science and Innovations of the Republic of Uzbekistan under number B2025.1.PhD/FM1001.

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Dissertation is possible to review in Information-resource center at Institute of Mathematics named after V.I.Romanovskiy (is registered № 213). (Address: University str. 9, Almazar area, Tashkent city, 100174, Uzbekistan, Ph.: (99871)-207-91-40).

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INTRODUCTION

Actuality and demand of the theme of the dissertation. The study of differential operators and partial differential equations on metric graphs has emerged as a highly relevant and interdisciplinary research area, bridging mathematics, physics, engineering, and biology. Metric graphs provide a powerful framework for modeling complex systems with branched structures, such as networks of strings, quantum graphs, and diffusion processes in nanoscale or biological systems. The increasing number of applications - ranging from quantum mechanics and fluid dynamics to control theory and inverse problems-underscores the practical significance of this field.

Recent advancements have highlighted the importance of fractional differential equations on metric graphs, particularly in describing anomalous diffusion and subdiffusion processes in mesoscopic networks, molecular wires, and other discrete structures. The connection between fractional derivatives and these phenomena has opened new avenues for theoretical and applied research, as evidenced by works on fractional Sturm-Liouville problems and time-fractional diffusion equations. These developments are critical for understanding non-local and memory-dependent processes, which are ubiquitous in real-world systems.

Conducting scientific research at the level of international standards in the priority areas of “Differential Equations and Mathematical Physics”[1] has been defined as the main tasks and areas of activity of the mathematics. Therefore, solving the direct and inverse problems for differential equations given on metric graphs is one of the important tasks of modern mathematics. Inverse problems, a cornerstone of this research, are essential for system identification, medical imaging, remote sensing, and other scientific and engineering applications. The ability to reconstruct unknown parameters or sources in differential equations from indirect measurements has profound implications for technology and industry. Recent studies have tackled inverse problems for fractional partial differential equations on graphs, leveraging techniques such as integral overdetermination conditions and spectral methods, further demonstrating the field’s dynamism and applicability. Despite these advancements, challenges remain, particularly in solving inverse problems for fractional differential equations on graphs, establishing uniqueness and stability of solutions, and developing numerical methods for complex networks. The dissertation’s focus on these unresolved questions aligns with current scientific demands and offers potential contributions to both theoretical understanding and practical applications.

The subject and object of this dissertation work is has been chosen according to actual directions, mentioned in the Decree of the President of the Republic of Uzbekistan UP-4947 of February 7, 2017 “On the strategy of actions for the further development of the Republic of Uzbekistan”, PP-2789 dated February 17, 2017 “On measures to further improve of the activities of the Academy of

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[1] Decree of Cabinet of Ministers of the Republic of Uzbekistan at the 2017 year 18 May « On measures on the organization of activities of the first created scientific research institutions of the Academy of Sciences of the Republic of Uzbekistan» № 292.

Sciences, organization, management and financing of research activities", PP-3682 dated April 27, 2018 "On measures to further improve the system of practical implementation of innovative ideas, technologies and projects" and PP-4387 from July 9, 2019 "On measures to further development of mathematical education and sciences, total improvement of the activity of the Uzbekistan Academy of Sciences V. I. Romanovsky Institute of Mathematics" and also PP-4708 dated May 7, 2020 "On measures to improve the quality of education and research in mathematics". This dissertation could also be extended to implement tasks defined in all relevant legal documents related to this activity.

Connection of research to priority directions of development of science and technologies of the Republic. This study was conducted according to the priority areas of science and technology of Republic of Uzbekistan IV, "Mathematics, Mechanics and Computer Science".

The degree of scrutiny of the problem. The study of boundary value problems for differential equations on graphs has its origins in the work of Russian mathematicians Yu. V. Pokorny, O. M. Penkin, and their students. Beginning in the 1980s, they explored the solvability of boundary value problems for second-order equations on networks. They developed Green's functions for these problems, examined the Sturm-Liouville problem on geometric graphs, and established a complete analogue of Sturm-Liouville theory for graphs. Czech scientists P. Exner and P. Seba began to study boundary value problems for classical equations on graphs around the same time. The development of the theory of differential equations on graphs has been further stimulated by new applications of these problems in quantum physics, mechanics, and other fields of applied sciences. In the early 2000s, U. Smilansky, T. Kottos, and S. Gnutzmann introduced the concept of quantum graphs and began studying quantum chaos, scattering properties, and spectral statistics on these graphs using the Laplacian defined on the graph. Fractional calculus has become a powerful tool for modeling complex processes, leading to the study of fractional Sturm-Liouville problems (FSLP) on graphs. These problems extend classical Sturm-Liouville theory to fractional-order derivatives, offering new insights into eigenvalue distributions and eigenfunction properties. Works by M. Klimek et al. and others have explored the spectral properties of FSLP, demonstrating the existence of discrete, countable spectra under specific boundary conditions. The study of spectral theory on graphs represents a significant and impactful area of research. Joel Friedman and Jean-Pierre Tillich investigated the completeness of edge-based Laplacian eigenfunctions. V. Yurko studied the inverse spectral problem for the Sturm–Liouville operators on arbitrary compact graphs with standard gluing conditions at internal vertices. R. Kulaev examined the properties of the eigenvalues of the operator S-L on the graph, including root completeness, estimation of the first eigenvalue, and oscillatory properties.

Inverse problems, which involve reconstructing unknown parameters or sources from observed data, are a central focus in metric graph theory. These problems have applications in system identification, medical imaging, geophysics, and more. Researchers like P. Kurasov, S. Avdonin, and G. Leugering have developed various approaches to tackle inverse problems on graphs. [45-48, 8, 14-17, 51]. Researchers like V. Kamynin, A. Prilepko, D. Tkachenko, A. Ashurov, D. Durdiev have developed methods to tackle these problems, often using integral overdetermination conditions or spectral data. The application of fractional derivatives in fractional Sobolev-Slobodeskii spaces is crucial for introducing weak or generalized solutions to fractional differential equations. Similar to the theory of partial differential equations, various approaches can be employed to tackle this issue. A significant contribution in this field is the work by D. Idczak and S. Walczak, who utilized the Riemann-Liouville fractional derivative to define and characterize fractional Sobolev spaces. By establishing embedding results, they demonstrated the applicability of their findings to Sturm-Liouville fractional equations, expressed in terms of the composition of right and left Riemann-Liouville fractional derivatives. Additionally, recent work by A. Kubica, K. Ryszewska, and M. Yamamoto provides rigorous treatments for time-fractional derivatives in Sobolev spaces, as well as solutions to initial boundary value problems for time-fractional partial differential equations.

Numerous specialists have studied various problems related to time fractional differential equations. Notably, some of our compatriots, including Sh. Alimov, R. Ashurov, S. Umarov, B. Kadirkulov, E. Karimov, Z. Sobirov, O. Abdullaev, T. Yuldashev, and others, have made significant contributions in this field. At the beginning of the 21st century, several works were published on the use of fractional derivatives. The inequality derived by A. Alikhanov is significant for obtaining an a priori estimate for fractional derivative differential equations. Russian mathematician A.V. Pskhu conducted a thorough study of initial and initial-boundary problems for fractional differential equations using the method of Green’s functions. In recent years, G. Leugering, V. Mehandiratta, and M. Mehra have made significant contributions to the study of various fractional differential equations on metric graphs. Additionally, P. S. Fotsing, G. Mophou, and others explored optimal control problems for fractional equations on graphs. Also, J. Khujakulov defended his PhD thesis on the topic “Direct and inverse problems for fractional order diffusion-wave equations on metric graphs”.

Connection of the theme of the dissertation with the research works of higher education, where the dissertation is carried out. This PhD dissertation was conducted as part of the scientific research activities at the National University of Uzbekistan. It aligns with the program “Solution of the boundary value problem for the fractional order and whole order partial differential equations” and follows the research plan of the V. I. Romanovskiy Institute of Mathematics.

The aim of this research work is to study direct and inverse problems for the space-time fractional differential equations, as well as to investigate the fractional Sturm-Liouville problem on metric graphs.

Research problems:

to prove the existence and uniqueness of the solution of initial-boundary value problems for the space-time fractional parabolic equation on a metric graph;

to prove the unique solvability of initial-boundary value problems for the subdiffusion equation with edge-dependent order of time-fractional derivative on a metric star graph;

to solve time-dependent and space-dependent inverse problems for subdiffusion and space-time fractional parabolic equations on a metric graphs;

to investigate the Sturm–Liouville operator on metric graphs and to establish the completeness of its system of eigenfunctions.

The research object. Differential equations on finite connected metric graphs, in particular on star graphs.

The research subject. The fractional initial-boundary value problems and inverse problems focus on recovering time-dependent and space-dependent source terms, the fractional Sturm-Liouville problem.

Research methods. In this research work, the functional method, a priori estimates, the resolvent method, the variational method and the method of separation of variables are utilized.

Scientific novelty of the research work consists of the following:

a new operator method for solving problems involving fractional-order equations, based on a priori estimates, is proposed;

strong solutions are obtained for initial-boundary value problems with fractional derivatives on metric graphs and star graphs in Sobolev-Slobodeskii spaces;

the unique solvability of the fractional parabolic equation with a symmetric operator constructed from generalized (weak) fractional derivatives of Caputo and Riemann-Liouville and subdiffusion equations is proven;

the Sturm-Liouville problem is posed for a fractional-order operator on metric graphs, and new results are obtained on the convergence of a numerical series constructed from the sum of reciprocals of the eigenvalues and the completeness of the system of eigenfunctions.

Practical results of the research consists of the following:

The results obtained and the methods used in this dissertation can be taught as a specialized course for master's and doctoral students. Additionally, these results can be applied to the development of mathematical models related to diffusion processes, particularly in the study of initial-boundary and inverse problems for time-fractional partial differential equations, with time defined on metric graphs.

The reliability of the results of the study. All the results obtained are supported by full mathematical proofs. The results have been discussed by scientists in the field at several seminars and conferences.

Scientific and practical significance of the research results. The scientific significance of this research lies in the fact that the results obtained in the dissertation can be utilized for more comprehensive studies on both direct and inverse problems related to fractional order partial differential equations. Specifically, the a priori estimates and important inequalities derived from this work can help establish the uniqueness of solutions for the initial-boundary problems associated with fractional differential equations in Sobolev-Slobodeskii spaces. Furthermore, the findings from this research can be applied to investigate the properties of heat propagation in metric graphs and diffusion processes in branched structures across various equations. The practical implications of this research are substantial, as the results provide a foundation for determining the coefficients of passage and return at branching points in diffusion processes, analyzing diffusion in branched fields, and modeling impulse propagation in nervous systems.

Implementation of the research results. Based on the results obtained for fractional parabolic equations on metric graphs:

the results concerning inverse problems for fractional parabolic equations on branched structures modeled by metric graphs in Sobolev spaces were applied in the framework of the international project No. 22-11-00064 “Modeling of dynamic processes in the geosphere with consideration of heredity”. These results were utilized for modeling dynamic processes (as confirmed by Certificate No. 377 dated September 25, 2025, issued by the Institute of Cosmophysical Research and Radio Wave Propagation, Russian Federation). The obtained outcomes made it possible to numerically model and analyze the radon exchange processes in the soil–atmosphere system.

the results related to direct and inverse problems for fractional parabolic equations and to fractional Sturm-Liouville problems were employed in solving boundary value problems on graphs within the framework of the international project No. 075-02-2024-1447 “Investigation of fractional order equations on coupled graphs” (as confirmed by Certificate No. 4087 dated September 23, 2025, issued by the North Ossetian State University named after K.L. Khetagurov, Russian Federation). The application of these results made it possible to obtain a lower bound for the first eigenvalue in the Sturm-Liouville problems defined on connected graphs.

Approbation of the research results. The results of this research were discussed at 10 scientific conferences, including 6 international and 4 national scientific conferences.

Publications of the research results. On the topic of the dissertation 15 research papers have been published in the scientific journals, 5 of them are included in the list of journals proposed by the Higher Attestation Commission of the Republic of Uzbekistan for defending the PhD thesis and 3 of them were published in international journals, 3 of them are included the Scopus and WoS information database, 2 papers published in national mathematical journals and 10 theses.

The structure and volume of the dissertation. The dissertation includes an introduction, three chapters, a conclusion, and a bibliography. The total length of the thesis is 83 pages.

MAIN CONTENT OF THE DISSERTATION

In the introduction, the actuality and demand of the theme of the dissertation are substantiated, and the alignment of the research with the priority directions of the development of science and technology in the Republic is determined. A review of international scientific studies on the dissertation topic and the degree of problem exploration are presented. The objectives and tasks are formulated, the object and subject of the research are identified, the scientific novelty and practical results of the study are set out, and the theoretical and practical significance of the obtained results is disclosed. Information is provided on the implementation of the research results, the published works, and the structure of the dissertation.

The first chapter of the dissertation, titled **“Initial-boundary value problems for the space-time fractional differential equations on metric graphs”** provides preliminary information in the first and second paragraphs, and new results are given in the third and fourth paragraphs.

Let $\mathcal{G}$ is a connected graph with a non-empty set of boundary vertices. We denote each edge as $e_i, i=\overline{1,N}$, and each edge e_i is assigned the interval $(0,l_i), i=\overline{1,N}$. Let $G_\tau=\{(x,t): x\in\mathcal{G}, 0<t\le\tau\}, 0<\tau\le T$.

Definition 1. The space $L_2(\mathcal{G})$ on $\mathcal{G}$ consists of functions that are measurable and square-integrable on each edge $e_i, i=\overline{1,N}$ with the scalar product and the norm:

$$\left(u(x),v(x)\right)_{L_2(\mathcal{G})}=\int_{\mathcal{G}} u(x)\cdot v(x)d\mathcal{G},$$

$$\|u\|^2_{L_2(\mathcal{G})}=\sum_i \|u\|^2_{L_2(e_i)}.$$

In other words, $L_2(\mathcal{G})$ is the orthogonal direct sum of spaces $L_2(e_i), i=\overline{1,N}$.

We put the space $\mathcal{H}^\beta_+(\mathcal{G})=\left\{u\in\bigoplus_{i=1}^N \mathcal{H}^\beta_+(0,l_i),\ I^{1-\beta}_{0,x}u\in C(\mathcal{G})\right\}$ equipped with the norm by

$$\|u\|^2_{\mathcal{H}^\beta_+(\mathcal{G})}=\sum_{i=1}^N \left\|u^{(i)}\right\|^2_{\mathcal{H}^\beta_+(0,l_i)}.$$

We put $\mathcal{H}^\beta_{+,Dir}(\mathcal{G})=\left\{u\in\mathcal{H}^\beta_+(\mathcal{G}): I^{1-\beta}_{0,x}u\Big|_{\partial\mathcal{G}}=0\right\}$, $\mathcal{H}^\beta_{+,Dir}[\mathcal{G}]=\bigoplus_{i=1}^N \mathcal{H}^\beta_{+,Dir}(0,l_i)$.

Definition 2. Let $L_2(G_\tau)=L_2(0,\tau;L_2(\mathcal{G}))$. The space $L_2(G_\tau)$ consists of functions that are measurable and square-integrable on each edge $e_i, i=\overline{1,N}$ with the scalar product and the norm:

$$\left(u(x,t),v(x,t)\right)_{L_2(G_\tau)}=\int_0^\tau\int_{\mathcal{G}} u\cdot v d\mathcal{G}dt=\sum_{i=1}^N\int_0^\tau\int_0^{l_i} u^{(i)}(x,t)\cdot v^{(i)}(x,t)dxdt,$$

$$\|u\|_{L_2(G_\tau)}=\sqrt{(u,u)_{L_2(G_\tau)}}.$$

Definition 3. $W_2^{2,\alpha}(G_\tau)$ is the Hilbert space consisting of all elements of $L_2(G_\tau)$ that have generalized derivatives $\partial^\alpha_{0,t}u$, u_x and u_{xx} from $L_2(G_\tau)$. The scalar product in it is defined by the equality

$$(u,v)_{W_2^{2,\alpha}(G_\tau)} = \int_0^\tau \int_{\mathcal{G}} \left(uv + u_x v_x + \partial_{0,t}^\alpha u \partial_{0,t}^\alpha v + u_{xx} v_{xx} \right) d\mathcal{G} dt$$

and the norm is denoted as follows: $\|\cdot\|_{W_2^{2,\alpha}(G_\tau)}$.

We put the space $W_{2,0}^{2,\alpha}(G_\tau) = \left\{ u \in W_2^{2,\alpha}(G_\tau) : u\big|_{\partial\mathcal{G}} = 0 \right\}$.

The third paragraph of this chapter studies the initial-boundary problem for the space-time fractional parabolic equation on metric graph $\mathcal{G}$.

Direct problem 1. We consider the following space-time fractional parabolic equations

$$\partial_{0,t}^\alpha u^{(i)}(x,t) + \partial_{x,l_i}^\beta (p^{(i)}(x) \mathrm{D}_{0,x}^\beta u^{(i)}(x,t)) = h^{(i)}(x,t),$$
$$x \in e_i, \quad t \in (0,T], \quad i = \overline{1,N}, \tag{1}$$

with the initial conditions

$$u^{(i)}(x,0) = 0, \quad x \in \overline{e}_i, \quad i = \overline{1,N}, \tag{2}$$

the vertex conditions

$$\begin{cases} I_{0,x}^{1-\beta} u^{(i)}(\nu,t) = I_{0,x}^{1-\beta} u^{(j)}(\nu,t), \forall i,j \in I(\nu),\ t \in [0,T], \\ \sum\limits_{e_i \sim \nu} \sigma_{e_i,\nu} p^{(i)}(\nu) D_{0,x}^\beta u^{(i)}(\nu,t) = 0, \quad \nu \in V \setminus \partial\mathcal{G}, \end{cases} \tag{3}$$

the boundary conditions

$$I_{0,x}^{1-\beta} u(\nu,t) = 0, \quad \nu \in \partial\mathcal{G},\ t \in [0,T]. \tag{4}$$

The real-valued functions $p^{(i)}(x)$ are absolutely continuous on $[0,l_i]$ and $0 < a_0 \le p^{(i)}(x) \le a_1 < +\infty$, $h^{(i)}(x,t),\, i = \overline{1,N},$ are given functions. $I(\nu)$ is the index set of the edges incident to a vertex ν, $\sigma_{e_i,\nu} = 1$ if ν is the right end of the edge e_i, $\sigma_{e_i,\nu} = -1$ if ν is the left end of the edge e_i.

We solve the problem by reducing the given equation to an operator equation and using a priori estimates, which can be classified as a functional method.

We introduce the following space

$$V(G_T) := \left\{ \begin{array}{c} u \in L_2(G_T) : u(\cdot,t) \in H_\alpha(0,T),\ \partial_{0,t}^\alpha u \in L_2(G_T), \\ u \in L_2(0,T; \bigoplus_{i=1}^N \mathcal{V}_i(0,l_i)), I_{0,x}^{1-\beta} u^{(i)}(\nu,t) = I_{0,x}^{1-\beta} u^{(j)}(\nu,t), \forall i,j \in I(\nu), \\ \sum\limits_{e_i \sim \nu} \sigma_{e_i,\nu} p^{(i)}(\nu) D_{0,x}^\beta u^{(i)}(\nu,t) = 0, \quad \nu \in V \setminus \partial\mathcal{G}, \\ I_{0,x}^{1-\beta} u(\nu,t) = 0, \quad \nu \in \partial\mathcal{G},\ t \in [0,T] \end{array} \right\},$$

with the following norm

$$\|u\|_{V(G_T)}^2 = \|u\|_{L_2(G_T)}^2 + \left\|\partial_{0,t}^\alpha u\right\|_{L_2(G_T)}^2 + \left\|D_+^\beta u\right\|_{L_2(G_T)}^2 + \left\|\partial_-^\beta (p D_+^\beta u)\right\|_{L_2(G_T)}^2.$$

We solve the direct problem for u by bringing it into the operator equation*:*

$$Au = h.$$

The domain of operator A $D(A) = V(G_T),$ and

$$Au|_{e_i} = \partial_{0,t}^{\alpha} u^{(i)}(x,t) + \partial_{x,l_i}^{\beta}(p^{(i)}(x) \mathrm{D}_{0,x}^{\beta} u^{(i)}(x,t)), \qquad i = \overline{1,N}.$$

The range of operator A defines as $R(A) \subset L_2(G_T)$.

Theorem 1. Let $h \in L_2(G_T)$. Then Direct problem 1 is uniquely solvable in $V(G_T)$.

This theorem follows from the following two propositions and one lemma.

Proposition 1. Operator $A: D(A) \to L_2(G_T)$ is continuous.

Continuity of the operator A follows from the following inequality

$$\| Au \|_{L_2(G_T)} = \left\| \partial_{0,t}^{\alpha} u + \partial_{-}^{\beta}(pD_{+}^{\beta} u) \right\|_{L_2(G_T)}$$

$$\le \| \partial_{0,t}^{\alpha} u \|_{L_2(G_T)} + \left\| \partial_{-}^{\beta}(pD_{+}^{\beta} u) \right\|_{L_2(G_T)} \le \| u \|_{V(G_T)}.$$

Proposition 2. The inverse operator $A^{-1}: R(A) \to V(G_T)$ is well-defined and continuous.

$$\int_0^t d\tau \int_{\mathcal{G}} \left[\left(\partial_{0,\tau}^{\alpha} u \right)^2 + \left(\partial_{-}^{\beta}(pD_{+}^{\beta} u) \right)^2 \right] d\mathcal{G} + a_0 \cdot \partial_{0,t}^{\alpha-1} \int_{\mathcal{G}} \left(D_{+}^{\beta} u \right)^2 d\mathcal{G}$$

$$\le \int_0^t d\tau \int_{\mathcal{G}} \left(Au \right)^2 d\mathcal{G}.$$

$$\|u\|_{V(G_T)} \le C \|Au\|_{L_2(G_T)}.$$

Corollary 1. The range $R(A)$ of the operator A is a closed linear subspace of $L_2(G_T)$.

Now to show the solvability of the direct problem it is sufficient to prove that there is no orthogonal complement to the range $R(A)$ of the operator A in $L_2(G_T)$.

Lemma 1. If for some $\omega \in L_2(G_T)$ it holds $(Av, \omega) = 0$ for all $v \in D(A)$, then $\omega = 0$.

$$\int_0^t d\tau \int_{\mathcal{G}} \left(\partial_{0,t}^{\alpha} v + \partial_{-}^{\beta}(pD_{+}^{\beta} v) \right) \omega d\mathcal{G} = 0.$$

Let $Lv = \partial_{-}^{\beta}(pD_{+}^{\beta} v)$. We choose $v(x,t) = \partial_{t_1,t}^{-\alpha} L^{-1} \omega$ for $0 < t_1 < t < T$ and $v = 0$ for $0 < t < t_1$, where L^{-1} – inverse operator of L.

$$0 = \int_{t_1}^t d\tau \int_{\mathcal{G}} \left(\partial_{0,t}^{\alpha} v + \partial_{-}^{\beta}(p\mathrm{D}_{+}^{\beta} v) \right) \cdot \partial_{0,t}^{\alpha} \left(\partial_{-}^{\beta}(p\mathrm{D}_{+}^{\beta} v) \right) d\mathcal{G} =$$

$$= \int_{t_1}^t d\tau \int_{\mathcal{G}} \partial_{0,t}^{\alpha} \left(\partial_{-}^{\beta}(pD_{+}^{\beta} v) \right) \cdot \partial_{0,t}^{\alpha} v d\mathcal{G} + \int_{t_1}^t d\tau \int_{\mathcal{G}} \partial_{0,t}^{\alpha} \left(\partial_{-}^{\beta}(pD_{+}^{\beta} v) \right) \cdot \partial_{-}^{\beta}(pD_{+}^{\beta} v) d\mathcal{G} \ge$$

$$\ge a_0 \cdot \int_{t_1}^t d\tau \int_{\mathcal{G}} \left[D_{+}^{\beta} \left(\partial_{0,t}^{\alpha} v \right) \right]^2 d\mathcal{G} + \frac{1}{2} \partial_{t_1,t}^{\alpha-1} \int_{\mathcal{G}} \left[\partial_{-}^{\beta} \left(pD_{+}^{\beta} v \right) \right]^2 d\mathcal{G}.$$

From the last inequality, we get $\omega = \partial_{0,t}^{\alpha} \left(\partial_{-}^{\beta}(pD_{+}^{\beta} v) \right) = 0$.

It should be noted that the results for fractional-order parabolic equations presented in this paragraph for the Direct problem 1 are new, even for the simplest graph, namely, for the interval.

The fourth section of this chapter addresses the initial-value boundary problem for the subdiffusion equation, which features an edge-dependent order of time-fractional derivative on a metric star graph.

Direct problem 2. We investigate the subdiffusion equation on each edge of the graph $\mathcal{G}$

$$\partial_{0,t}^{\alpha_i} u^{(i)}(x,t) - u_{xx}^{(i)}(x,t) = h^{(i)}(x,t),\ x \in e_i, t \in (0,T], i = \overline{1,N}. \tag{5}$$

Where $0 < \alpha_i < 1, i = \overline{1,N}$. We need to establish the following initial conditions

$$u^{(i)}(x,0) = 0, \quad x \in \overline{e}_i, \quad i = \overline{1,N}, \tag{6}$$

the vertex conditions

$$\sum_{i=1}^{N} u_x^{(i)}(0,t) = 0,\ I_{0,t}^{1-\alpha_i} u^{(i)}(0,t) = I_{0,t}^{1-\alpha_j} u^{(j)}(0,t), i \neq j,\ i, j = \overline{1,N},\ t \in (0,T], \tag{7}$$

and the boundary conditions

$$u^{(i)}(l_i,t) = 0, \quad t \in (0,T], \tag{8}$$

where $h^{(i)}(x,t), i = \overline{1,N},$ are given functions.

Theorem 2. Let $h(x,t) \in L_2(G_T)$. Then the Direct problem 2 has a unique strong solution in $W_{2,0}^{2,\alpha}(G_T)$.

The proof of the theorem follows from the following propositions and a lemma.

Proposition 3. Operator $A: D(A) \to L_2(G_T)$ is continuous.

Proposition 4. The inverse operator $A^{-1}: R(A) \to W_{2,0}^{2,\alpha}(G_T)$ is well-defined and continuous.

Corollary 2. The range $R(A)$ of the operator A is a closed linear subspace of $L_2(G_T)$.

Lemma 2. If for some $\varphi \in L_2(G_T)$ it holds $(A\omega, \varphi) = 0$ for all $\omega \in D(A)$, then $\varphi = 0$.

The second chapter of the dissertation is titled **"Inverse problems for subdiffusion equations on metric graphs".**

In the first section of this chapter, the inverse source problem for the space-time fractional parabolic equation is established on a metric graph.

Inverse Problem 1. We focus on the following space-time fractional parabolic equations

$$\partial_{0,t}^{\alpha} u^{(i)}(x,t) + \partial_{x,l_i}^{\beta}(p^{(i)}(x) \mathrm{D}_{0,x}^{\beta} u^{(i)}(x,t)) = f(t) g^{(i)}(x,t),$$
$$x \in e_i, \quad t \in (0,T], \quad i = \overline{1,N}, \tag{9}$$

which are accompanied by the initial conditions (2), the vertex conditions (3), and boundary conditions (4). In (9) $g^{(i)}(x,t), i = \overline{1,N},$ are given functions, $f(t)$ is an unknown function. We are tasked with finding the pair of functions $\{u(x,t), f(t)\}$ in $V(G_T)\times L_2(0,T)$. To achieve this, we define an additional condition, specifically an integral overdetermination condition, as follows

$$\int_{\mathcal{G}} \eta(x)u(x,t)d\mathcal{G} = \psi(t), \quad t\in[0,T], \tag{10}$$

where $\eta^{(i)}(x)$, $i=\overline{1,N}$, and $\psi(t)$ are known functions.

Let the following conditions be satisfied and we will refer to these as conditions $(K1)$:

$$g(x,t)\in L_\infty\left(0,T,L_2(\mathcal{G})\right),$$

$$\eta(x)\in \mathcal{H}_+^\beta(\mathcal{G}),\ \ I_{0,x}^{1-\beta}\eta^{(i)}(\nu) = I_{0,x}^{1-\beta}\eta^{(j)}(\nu), \forall i,j\in I(\nu),\ \ \nu\in V\setminus\partial\mathcal{G},$$

$$\left\|\mathrm{D}_{0,x}^\beta \eta(x)\right\|_{L_2(\mathcal{G})} = m>0, \quad \psi(t)\in H^\alpha(0,T), \quad |g^*(t)|\geq q>0,$$

where

$$g^*(t) = \int_{\mathcal{G}} \eta(x)g(x,t)d\mathcal{G}, \qquad t\in(0,T].$$

Multiplying both sides of equation (9) by the function $\eta(x)$ and integrating over $\mathcal{G}$, we obtain

$$f(t) = (Bf)(t) + \frac{\partial_{0,t}^\alpha \psi(t)}{g^*(t)},$$

where

$$(Bf)(t) = \frac{1}{g^*(t)}\int_{\mathcal{G}} p(x)D_{0,x}^\beta u(x,t)D_{0,x}^\beta \eta(x)d\mathcal{G},$$

$$B: L_2(0,T)\to L_2(0,T).$$

Theorem 3. Let condition (K1) hold. If $h(x,t)\in L_2(G_T)$, then the Inverse Problem 1 has a unique generalized solution $\{u,f\}\in V(G_T)\times L_2(0,T)$ and $\|f(t)\|_{L_2(0,T)} \leq C\|\psi(t)\|_{H^\alpha(0,T)}$.

This theorem is proved by the resolvent method.

The second section of this chapter is focused on establishing the unique solvability of an inverse source problem for the subdiffusion equation, which involves an edge-dependent order of the Caputo time-fractional derivative on the metric star graph.

Inverse problem 2. Find a couple of functions $\{u(x,t), f(t)\}$ in $W_{2,0}^{2,\alpha}(G_T)\times L_2(0,T)$, satisfy following equation

$$\partial_{0,t}^{\alpha_i} u^{(i)}(x,t) - u_{xx}^{(i)}(x,t) = f(t)g^{(i)}(x,t),\ \ x\in e_i, t\in(0,T], i=\overline{1,\mathrm{N}}, \tag{11}$$

and the initial conditions (6), the vertex conditions (7), the boundary conditions (8), and under an additional integral overdetermination condition

$$\int_{\mathcal{G}} \eta(x)I_{0,t}^{1-\alpha}u(x,t)d\mathcal{G} = \psi(t), \ \ t\in[0,T], \tag{12}$$

where $g^{(i)}(x,t)$, $\eta_i(x)$, $i=\overline{1,N}$, and $\psi(t)$ are known functions.

Let the following conditions be satisfied and we call they conditions $(K1)$:

$$g(x,t)\in L_\infty\left(0,T,L_2(\mathcal{G})\right),$$

$$\eta(x)\in W_2^1(\mathcal{G}),\ I_{0,x}^{1-\beta}\eta^{(i)}(l_i)=0,\ I_{0,x}^{1-\beta}\eta^{(i)}(0)=I_{0,x}^{1-\beta}\eta^{(j)}(0),\ \ i\neq j,\ \ i,j=\overline{1,N},$$

$$\left\|\eta_x(x)\right\|_{L_2(\mathcal{G})}=m>0,\ \psi(t)\in W_2^1(0,T),\ |g^*(t)|\geq q>0,$$

where

$$g^*(t)=\int_{\mathcal{G}}\eta(x)g(x,t)d\mathcal{G},t\in(0,T].$$

We multiply both sides of equation (11) by the function $\eta(x)$ and integrate over $\mathcal{G}$, we get

$$f(t)=(Bf)(t)+\frac{\psi_t(t)}{g^*(t)},$$

and

$$(Bf)(t)=\frac{1}{g^*(t)}\left\{\int_{\mathcal{G}}\eta_x(x)u_x(x,t)d\mathcal{G}\right\},B:L_2(0,T)\to L_2(0,T).$$

Theorem 4. Let conditions $(K1)$ hold. If $h(x,t)\in L_2(G_T)$, then the Inverse problem 2 has a unique generalized solution $\{u,f\}\in W_{2,0}^{2,\alpha}(G_T)\times L_2(0,T)$. Moreover, $\|f(t)\|_{L_2(0,T)}\leq C\|\psi(t)\|_{H^1(0,T)}$.

In the third section of this chapter, the inverse source problem for the subdiffusion equation on a metric star graph is investigated.

Inverse problem 3. We are looking for the solution $\{u(x,t),f(x)\}$ of the equation

$$\partial_{0,t}^{\alpha}u^{(i)}(x,t)-u_{xx}^{(i)}(x,t)=f^{(i)}(x)g^{(i)}(x,t),\ x\in e_i,t\in(0,T],\ i=\overline{1,N}, \tag{13}$$

on the metric star graph $\mathcal{G}$, that satisfy the initial conditions

$$u^{(i)}(x,0)=0,\ x\in\overline{e}_i,\ i=\overline{1,N},$$

the vertex conditions

$$\sum_{i=1}^{N}u_x^{(i)}(0,t)=0,\ u^{(1)}(0,t)=u^{(2)}(0,t)=\ldots=u^{(n)}(0,t),\ \ t\in[0,T],$$

the boundary conditions

$$u^{(\mathrm{i})}(l_i,t)=0,\ t\in[0,T],\ \ i=\overline{1,N},$$

and the overdetermination conditions given by

$$\int_0^T\eta(t)u^{(i)}(x,t)dt=\psi^{(i)}(x),\ i=\overline{1,N},$$

where $\eta(t)$, $\psi^{(i)}(x)$, $i=\overline{1,\mathrm{N}}$ are given functions.

Let us suppose that the functions appearing in the problem data are measurable and satisfy the given conditions:

$$\sup_{1\le i\le N} ess\sup_{0\le x\le l_i} g^{(i)}(x,t)=\gamma(t),\quad ess\sup_{0<t\le T}|\gamma(t)|\le s, \tag{K1}$$

$$\eta(t), D^{\alpha}_{t,T}\eta(t)\in L_2(0,T),\ \ I^{1-\alpha}_{t,T}\eta(T)=0,\ \int_0^T\left(D^{\alpha}_{t,T}\eta(t)\right)^2dt=q^2, \tag{K2}$$

$$\max_{1\le i\le N}\left|g_*^{(i)}(x)\right|\ge m>0, \tag{K3}$$

where

$$g_*^{(i)}(x)=\int_0^T g^{(i)}(x,t)\eta(t)dt,\ i=\overline{1,N}.$$

Multiplying both sides of equation (13) by the function $\eta(t)$ and integrate over t on the closed interval $[0,T]$ we get

$$f^{(i)}(x)=Bf^{(i)}(x)-\frac{\psi_{xx}^{(i)}(x)}{g_*^{(i)}(x)},$$

where

$$Bf^{(i)}(x)=\frac{1}{g_*^{(i)}(x)}\int_0^T u^{(i)}(x,t)D^{\alpha}_{tT}\eta(t)dt,\ i=\overline{1,N},$$

$$Bf=\left(Bf^{(1)},Bf^{(2)},\ldots,Bf^{(N)}\right),\ B:L_2(\mathcal{G})\to L_2(\mathcal{G}).$$

Theorem 5. Let conditions $(K1)-(K3)$ hold and

$$\frac{q^2s^2}{m^2\alpha(\alpha+1)}T^{\alpha+1}E_{\alpha,\alpha}(2T^{\alpha})<1.$$

Then there exists a generalized solution $\{u(x,t), f(x)\}$ of the Inverse problem 3 and $u(x,t)\in V(G_T)$, $f(x)\in L_2(\mathcal{G})$.

The third chapter of the dissertation is titled **"Fractional Sturm–Liouville problem on metric graphs and its applications".**

In the first section of this chapter, it is shown that the fractional Sturm-Liouville operator has a countable set of eigenvalues. At the same time, the completeness of the system of eigenfunctions is proved.We consider the following fractional differential equation on the graph $\mathcal{G}$

$$\mathcal{L}u=\lambda ru,\quad x\in e_i,\quad \frac{1}{2}<\beta<1, \tag{14}$$

where $u:\mathcal{G}\to\mathbb{R}$, $r:\mathcal{G}\to\mathbb{R}$, and

$$\mathcal{L}u|_{e_i}=\partial^{\beta}_{x,l_i}\left(p^{(i)}(x)D^{\beta}_{0,x}u^{(i)}(x)\right),$$

where $0<a_0\le p^{(i)}\le a_1<+\infty$, $0<R_0\le r^{(i)}\le R_1<+\infty$ and we suppose that $r,p\in C[\mathcal{G}]$. At each interior vertex v, continuity conditions and transmission conditions

$$\begin{cases} I_{0,x}^{1-\beta}u^{(i)}(x) \text{ are continuous in } \nu, \\ \sum_{e_i\sim\nu}\sigma_{e_i,\nu}\, p^{(i)}(\nu) D_{0,x}^{\beta}u^{(i)}(\nu) = 0, \quad \nu \in V \setminus \partial\mathcal{G}, \end{cases} \tag{15}$$

are specified, where $\sigma_{e_i,\nu} = 1$ if ν is the right end of the edge e_i, $\sigma_{e_i,\nu} = -1$ if ν is the left end of the edge e_i, and at each boundary vertex, Dirichlet conditions

$$I_{0,x}^{1-\beta}u(\nu) = 0, \quad \nu \in \partial\mathcal{G}, \tag{16}$$

are specified.

Theorem 6. For the eigenfunctions u_k and the eigenvalues λ_k of problem (14)-(16), the following statements are true:

1) $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$,

2) $\lambda_k \to \infty$,

3) the u_k form a complete orthonormal basis for $L_2(\mathcal{G})$ with respect to norm $\|\cdot\|_{1,L_2(\mathcal{G})}$.

This theorem is proven using the variational method.

The second paragraph of this chapter shows the convergence of a series consisting of the sum of inverse quantities to the eigenvalues, and a high estimate is obtained for the eigenfunctions.

According to the max-min principle,

$$\lambda_k = \sup_{A_{k-1}\subset\mathcal{H}_+^{\beta}(\mathcal{G})} \inf_{\substack{u\in A_{k-1}^{\perp} \\ \|u\|=1}} \mathcal{R}(u)$$

where the supremum is taken over all $k-1$ dimensional linear subspaces A_{k-1} of the space $\mathcal{H}_+^{\beta}(\mathcal{G})$.

Theorem 7. The series $\sum_{k=1}^{\infty}\frac{1}{\lambda_k}$ is convergent and

$$|u_k| \le C\left(1+\sqrt{\lambda_k}\right),$$

where the constant C depends on R_0, a_0, β and lengths of the edges of $\mathcal{G}$.

In the third section of this chapter, it is shown that the initial-boundary problem for the space-time fractional diffusion equation has a solution, and the unique solvability of the inverse problem is demonstrated.

We consider the following fractional diffusion equation on the graph $\mathcal{G}$

$$\partial_{0,t}^{\alpha}u^{(i)}(x,t) + \mathcal{L}u^{(i)} = F^{(i)}(x,t) = f^{(i)}(x)g(t), \; x\in e_i, \; t\in(0,T], \; i=\overline{1,n}, \tag{17}$$

where

$$\mathcal{L}u^{(i)} = \frac{1}{r^{(i)}(x)}\partial_{x,l_i}^{\beta}\left(p^{(i)}(x)D_{0,x}^{\beta}u^{(i)}(x,t)\right),$$

$F^{(i)}(x,t)$ are given functions, $0 < a_0 \le p^{(i)}(x) \le a_1 < +\infty$, $0 < R_0 \le r^{(i)}(x) \le R_1 < +\infty$ and we suppose that $r, p \in C[\mathcal{G}]$. Also, $f^{(i)}(x)$ are unknown functions and $g(t)$ is a given function. So, we are searching for the solution $\{u(x,t), f(x)\}$ to equation

(17) within a bounded domain $G_T = \mathcal{G}\times(0,T]$, while satisfying the following initial conditions

$$u^{(i)}(x,0) = \varphi^{(i)}(x), \quad x\in \overline{e}_i,\ i=\overline{1,n}, \tag{18}$$

the vertex conditions

$$\begin{cases} I_{0,x}^{1-\beta}u^{(i)}(\nu,t) = I_{0,x}^{1-\beta}u^{(j)}(\nu,t), \forall i,j\in I(\nu),\, t\in[0,T], \\ \sum\limits_{e_i\sim\nu}\sigma_{e_i,\nu}\,p^{(i)}(\nu)D_{0,x}^{\beta}u^{(i)}(\nu,t) = 0, \quad \nu\in V\setminus\partial\mathcal{G}, \end{cases} \tag{19}$$

and the Dirichlet conditions

$$I_{0,x}^{1-\beta}u(\nu,t) = 0, \quad \nu\in\partial\mathcal{G},\ t\in[0,T], \tag{20}$$

and an additional final-time condition

$$u^{(i)}(x,T) = \psi^{(i)}(x), \quad x\in \overline{e}_i,\ i=\overline{1,n}, \tag{21}$$

where $\varphi^{(i)}(x)$, $\psi^{(i)}(x)$, $i=\overline{1,n}$ are known functions.

Given that the function $F(x,t)$ on the right-hand side of equation (17) is specified, the existence and uniqueness of a solution to the initial-boundary value problem (17)-(20) (direct problem) are established.

Applying the method of separation of variables to solve problem (17)-(20), we get the following formulas for the solution on each edge e_i

$$\begin{aligned} u^{(i)}(x,t) &= \sum_{k=1}^{\infty}\varphi_k^{(i)}w_k^{(i)}(x)E_{\alpha,1}(-\lambda_k t^{\alpha}) \\ &+ \sum_{k=1}^{\infty}w_k^{(i)}(x)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^{\alpha})F_k^{(i)}(\tau)d\tau. \end{aligned} \tag{22}$$

Lemma 3. *Let* $\mathcal{L}F(x,t)\in C([0,T],L_2(\mathcal{G}))$, $\mathcal{L}\varphi(x)\in L_2(\mathcal{G})$ are given in the form of series

$$F(x,t) = \sum_{k=1}^{\infty}F_k(t)w_k(x), \quad \varphi(x) = \sum_{k=1}^{\infty}\varphi_k w_k(x).$$

Then the following equalities are hold:

$$F_k(t) = \left(F(x,t),w_k(x)\right)_{L_2(\mathcal{G})} = \frac{\left(\mathcal{L}F(x,t),w_k(x)\right)_{L_2(\mathcal{G})}}{\lambda_k} = \frac{q_k(t)}{\lambda_k};$$

$$\varphi_k = \left(\varphi(x),w_k(x)\right)_{L_2(\mathcal{G})} = \frac{\left(\mathcal{L}\varphi(x),w_k(x)\right)_{L_2(\mathcal{G})}}{\lambda_k} = \frac{s_k}{\lambda_k},$$

where $(s_1,s_2,\ldots)\in l_2$, for $\forall t\in[0,T]$ $(q_1(t),q_2(t),\ldots)\in l_2$.

Theorem 8. Let Lemma 3 holds and $I_{0,x}^{1-\beta}F(x,t)\in C([0,T];C(\mathcal{G}))$, $D_{0,x}^{\beta}F(x,t),\partial_{0,t}^{\alpha}F(x,t)\in C\left([0,T];C(\mathcal{G})\right)$, $\varphi(x)\in C(\mathcal{G})$, $D_{0,x}^{\beta}\varphi(x)\in C(\mathcal{G})$ and the functions $F(x,t)$, $\varphi(x)$ satisfy the conditions (15),(16). Then the problem (17)-(20) has a regular solution $u: G_T\to\mathbb{R}$ given in the form (22).

Uniqueness of the solution of the direct problem.

Lemma 4. Let the conditions of Theorem 8 hold. Then problem (17)-(20) has a unique solution, and the following apriori estimate is valid:

$$\| u \|^2_{L_2(\mathcal{G})} \le E_{\alpha,1}(t^\alpha) \cdot \| \varphi \|^2_{L_2(\mathcal{G})} + \Gamma(\alpha) \cdot E_{\alpha,\alpha}(t^\alpha) \cdot I^\alpha_{0,t} \| F \|^2_{L_2(\mathcal{G})} .$$

Inverse Problem. Let $B_{k,\alpha}(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) g(\tau) d\tau$.

We put

$$f(x) = \sum_{k=1}^{\infty} f_k w_k(x), \quad \text{or} \quad F(x,t) = f(x) g(t) = g(t) \sum_{k=1}^{\infty} f_k w_k(x).$$

From these, we get

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \varphi_k w_k(x) E_{\alpha,1}(-\lambda_k t^\alpha) \\ &+ \sum_{k=1}^{\infty} f_k w_k(x) \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) g(\tau) d\tau. \end{aligned} \tag{23}$$

Substituting the variable T for t in equation (23) and applying the additional final-time condition (21)

$$u(x,T) = \psi(x) = \sum_{k=1}^{\infty} \psi_k w_k(x),$$

yields the following expression for f_k:

$$f_k = \frac{\psi_k - \varphi_k E_{\alpha,1}(-\lambda_k T^\alpha)}{B_{k,\alpha}(T)}.$$

So, we can write

$$f(x) = \sum_{k=1}^{\infty} \left[\frac{\psi_k - \varphi_k E_{\alpha,1}(-\lambda_k T^\alpha)}{B_{k,\alpha}(T)} \right] w_k(x).$$

Theorem 9. Let $g(t) \in C[0,T]$ and $g(t) \ne 0$, $|g(t)| < m$, $t \in [0,T]$. Furthermore, are hold and $\psi(x) \in C(\mathcal{G})$, $\mathcal{L}\psi(x), \mathcal{L}^2\psi(x) \in L_2(\mathcal{G})$, $\mathcal{L}\psi(x)$ satisfies the conditions (14)-(16). Then the inverse source problem (17)-(21) has a regular solution.

CONCLUSION

This dissertation studies initial-boundary, inverse and spectral problems for fractional-order differential equations on metric graphs. In solving the problems, an improved functional method based on a priori estimates was used, reducing the given equations to operator equations.

The main results of this study are summarized as follows.

1. Strong solutions of initial-boundary problems for fractional-order parabolic equations on metric graphs in Sobolev-Slobodeskii spaces were obtained;
2. Inverse problems related to determining source functions for space-time fractional parabolic equations and subdiffusion equations were investigated. The method of resolvents for inverse problems was used for the first time in solving the problems;
3. The fractional Sturm-Liouville problem on metric graphs was studied. The completeness of the system of eigenfunctions was proved;
4. The convergence of the series, consisting of the sum of inverse quantities to the eigenvalues of fractional Sturm-Liouville problem, is shown. Estimates for eigenfunctions were obtained;
5. The existence and uniqueness of the solution to the inverse problem on determining the source function for a space-time fractional parabolic equation, based on the information at the last moment of time, was proved, and an exact solution in the form of a series was found.

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НАЦИОНАЛЬНЫЙ УНИВЕРСИТЕТ УЗБЕКИСТАНА

ТУРЕМУРАТОВА АРЫУХАН АБАТБАЕВНА

ОБОБЩЕННЫЕ РЕШЕНИЯ ДРОБНО-ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ НА МЕТРИЧЕСКИХ ГРАФАХ

01.01.02 – Дифференциальные уравнения и математическая физика

АВТОРЕФЕРАТ ДИССЕРТАЦИИ ДОКТОРА ФИЛОСОФИИ (PhD)
ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ

ТАШКЕНТ – 2025

Тема диссертации доктора философии (PhD) по физико-математическим наукам зарегистрирована в Высшей аттестационной комиссии при Министрстве высшего образования, науки и инноваций Республики Узбекистан за № B2025.1. PhD/FM1001.

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ВВЕДЕНИЕ (аннотация диссертации доктора философии (PhD))

Целью исследования является изучение прямых и обратных задач для дробных дифференциальных уравнений, зависящих от времени и пространства, а также исследование дробной задачи Штурма-Лиувилля на метрических графах.

Объектом исследования. Дифференциальные уравнения на конечных связных метрических графах, в частности на звёздных графах.

Научная новизна исследования состоит в следующем:

предложен новый операторный метод, основанный на априорных оценках, для решения задач, связанных с дробными дифференциальными уравнениями;

в пространствах Соболева–Слободецкого получены сильные решения начально-краевых задач с дробными производными на метрических и звездообразных графах;

доказаны существование и единственность решения для дробных параболических уравнений содержащих симметрические операторы на основе обобщённых (слабых) дробных производных Капуто и Римана-Лиувилля и субдиффузионных уравнений;

для операторов дробного порядка на метрических графах поставлена задача Штурма-Лиувилля, и получены новые результаты, касающиеся сходимости числового ряда, составленного из обратных собственных значений, а также полноты системы собственных функций.

Внедрение результатов исследования. На основе результатов, полученных для параболических уравнений дробного порядка на метрических графах:

результаты, полученные для обратных задач дробного порядка параболических уравнений на разветвлённых структурах, моделируемых с помощью метрических графов в пространствах Соболева, были использованы при моделировании динамических процессов в рамках международного проекта № 22-11-00064 «Моделирование динамических процессов в геосферах с учётом наследственности» (Справка № 377 от 25 сентября 2025 года Института космофизических исследований и распространения радиоволн, Российская Федерация). Эти результаты позволили осуществить численное моделирование и анализ процесса обмена радоном в системе почва - атмосфера.

результаты, относящиеся к прямым и обратным задачам для параболических уравнений дробного порядка, а также к задачам Штурма – Лиувилля дробного порядка, были применены при решении краевых задач на графах в рамках международного проекта № 075-02-2024-1447 «Исследование уравнений дробного порядка на связанных графах» (Справка № 4087 от 23 сентября 2025 года Северо-Осетинского государственного университета имени К. Л. Хетагурова, Российская Федерация). Применение указанных результатов позволило получить нижнюю оценку для первого собственного значения в задачах Штурма-Лиувилля, заданных на связанных графах.

Объем и структура диссертации. Диссертация состоит из введения, четырех глав, заключения и списка использованной литературы. Общий объем диссертации составляет 83 стр.

E’LON QILINGAN ISHLAR RO‘YXATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo‘lim (part I; I часть)

1. Sobirov Z. A., Turemuratova A. A. Inverse source problem for the heat equation on a metric star graph with integral over-determination condition. Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences, 2023, 6 (1), 1-15. (01.00.00, №8).
2. Ashurov R. R., Sobirov Z. A., Turemuratova A. A. Inverse source problem for the space-time fractional parabolic equation on a metric star graph with an integral overdetermination condition. Math. notes, 2024. 116 (5), 892-904. (3. SCOPUS, IF=0.5).
3. Sobirov Z. A., Turemuratova A. A. Inverse source problem for the subdiffusion equation with edge-dependent order of time-fractional derivative on the metric star graph. Nanosystems: Phys. Chem. Math., 2024. 15 (5), 586-596. (3. SCOPUS, IF=0.334).
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II bo‘lim (Part II; II часть)

6. Sobirov Z. A., Turemuratova A. A. Inverse problem for heat equation with an integral over determination condition. International scientific and practical conference “Actual problems of mathematical modeling and information technology”, pp. 232-233, Nukus, May 2-3, 2023.
7. Sobirov Z. A., Turemuratova A. A. Initial-boundary value problem for the space-time fractional parabolic equation on a metric star graph. International scientific conference “Modern problems of differential equations and their applications”, pp. 322-323, Tashkent, November 23-25, 2023.
8. Sobirov Z. A., Turemuratova A. A. Initial-boundary value problem for the diffusion equation with edge dependent order of time-fractional derivative on a metric graph. Scientific conference of young scientists “Current problems and applications of modern mathematics”, pp. 148-149, Tashkent, March 14-15, 2024.
9. Ashurov R. R., Sobirov Z. A., Turemuratova A. A. Inverse source problem for the space-time fractional parabolic equation on a metric star graph. International conference “Gibbs measures and the theory of dynamical systems”, pp. 33-34, Tashkent, May 20-21, 2024.
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11. Sobirov Z. A., Turemuratova A. A. Inverse source problem for the subdiffusion equation on a metric star graph. International scientific conference “Non-classical equations of mathematical physics and their applications”, p. 103, Tashkent, October 24-26, 2024.
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13. Sobirov Z. A., Turemuratova A. A., Kulaev R. Ch. Fractional Sturm–Liouville problem on metric graphs. International Scientific Conference “Modern methods of mathematical physics and their applications”, pp. 58-59, Uzbekistan, Tashkent, April 22-24, 2025.
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15. Sobirov Z. A., Turemuratova A. A., Narziyeva I.A. Inverse source problem for the space-time fractional diffusion equations on metric graphs. International conference “Order Analysis and Related Questions of Mathematical Modeling XVIII: Operator Theory and Differential Equations”, pp. 47-48, Dzinaga, June 29 – July 5, 2025.

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