

**V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI  
HUZURIDAGI ILMIY DARAJALAR BERUVCHI  
DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH**

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**MATEMATIKA INSTITUTI**

**MIZOMOV INOMJON ERKIN O'G'LI**

**ELLIPTIK ALGEBRALARNING KALABI-YAU XOSSALARI HAQIDA**

**01.01.06 – Algebra**

**FIZIKA-MATEMATIKA FANLARI BO'YICHA FALSAFA DOKTORI (PhD)  
DISSERTATSIYASI AVTOREFERATI**

**TOSHKENT - 2025 – yil**

**Fizika-matematika fanlari bo'yicha falsafa doktori (PhD) dissertatsiyasi  
avtoreferati mundarijasi**

**Contents of dissertation abstract of doctor of philosophy (PhD) on  
physical-mathematical sciences**

**Оглавление автореферата диссертации  
доктора философии (PhD) по физико-математическим наукам**

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Dissertatsiya Matematika institutida bajarilgan.

Dissertatsiya avtoreferati uch tilda (o'zbek, ingliz, rus (rezyume)) Ilmiy kengash veb-sahifasi (<https://kengash.mathinst.uz>) va "ZiyoNet" ta'lim axborot tarmog'ida (<http://www.ziynet.uz>) joylashtirilgan.

**Ilmiy rahbar:**

**Eshmatov Farxod Xasanovich**

fizika-matematika fanlari doktori, katta ilmiy xodim

**Rasmiy opponentlar:**

**Omirov Baxrom Abdazovich**

fizika-matematika fanlari doktori, professor

**Jumaniyozov Doston Erkaboy o'g'li**

fizika-matematika fanlari bo'yicha falsafa doktori

**Yetakchi tashkilot:**

**Andijon davlat universiteti**

Dissertatsiya himoyasi V.I. Romanovskiy nomidagi Matematika instituti huzuridagi DSc.02/30.12.2019.FM.86.01 raqamli Ilmiy kengashning 2025-yil « 25 » noyabr kuni soat 17:00 dagi majlisida bo'lib o'tadi. (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko'chasi, 9-uy. Tel.: (+998 71) 207 91 40, e-mail: [uzbmath@umail.uz](mailto:uzbmath@umail.uz), Website: [www.mathinst.uz](http://www.mathinst.uz)).

Dissertatsiya bilan V.I. Romanovskiy nomidagi Matematika institutining Axborot-resurs markazida tanishish mumkin (214-raqami bilan ro'yhatga olingan). (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko'chasi, 9-uy. Tel.: (+998 71) 207 91 40.

Dissertatsiya avtoreferati 2025-yil « 10 » noyabr kuni tarqatildi.  
(2025-yil « 10 » noyabrdagi 2-raqamli reestr bayonnomasi).

**U.A. Rozikov**

Ilmiy darajalar beruvchi

Ilmiy kengash raisi,

f.-m.f.d., akademik

**J.K. Adashev**

Ilmiy darajalar beruvchi

Ilmiy kengash ilmiy kotibi,

f.-m.f.d., katta ilmiy xodim

**B.A. Omirov**

Ilmiy darajalar beruvchi

Ilmiy kengash huzuridagi

Ilmiy seminar raisi,

f.-m.f.d., professor

## KIRISH (falsafa doktori (PhD) dissertatsiyasi annotatsiyasi)

**Dissertatsiya mavzusining dolzarbligi va zarurati.** Jahon miqyosida olib borilayotgan ko‘plab ilmiy va amaliy tadqiqotlar aksariyat hollarda statistik fizika va nokommutativ geometriyani tadqiq qilish kabi masalalariga keltiriladi. Koshul Kalabi-Yau algebralari zamonaviy matematikaning bir necha sohalarida, jumladan, nokommutativ geometriya, tasvirlar nazariyasi, algebraik topologiya va matematik fizika kabi yo‘nalishlarda tabiiy ravishda uchraydigan graduirlangan algebralarning muhim sinfini tashkil etadi. Ushbu algebralalar klassik Kalabi-Yau ko‘pxilliklari nokommutativ muhitga umumlashtiradi hamda algebraik geometriyadagi Serre qo‘shmaligiga o‘xshash tarzda Hochschild (ko)gomologiyasidagi qo‘shmalik xossasini saqlab qoladi. Ushbu ishda biz ayniqsa Kalabi-Yau algebralarning elliptik algebralari deb nomlanuvchi kichik sinfiga e‘tibor qaratamiz, ular elliptik egri chiziqlar bilan bog‘liq bo‘lgan proyektiv sirtlarning nokommutativ analoglaridir. Bu algebralalar nafaqat boy algebraik strukturalarga ega, balki integrallanuvchi sistemalar, kvant algebralari va kvant Yang-Bakster tenglamasini tadqiq etishda ham muhim o‘rin tutadi. Kalabi-Yau geometriyasini ham kommutativ hamda nokommutativ shakllarda o‘rganishning asosiy ilmiy motivatsiyalaridan biri uning ko‘zgu simmetriya nazariyasidagi muhim ahamiyatidir.

Hozirgi kunda elliptik algebralarning Kalabi-Yau xossasini tadqiq etish hamda kvadratik algebralarning Koshul ma‘nosidagi qo‘shma algebralarda kvant Yang-Bakster tenglamasi yechimlarini aniqlash orqali ularning Kalabi-Yau xossasiga ega ekanligini ko‘rsatish dolzarb masalalardan biridir. Oxirgi yigirma yil davomida elliptik algebralari nazariyasi jadal rivojlangan bo‘lsa-da,  $n$ -o‘lchamli elliptik algebralarning Kalabi-Yau xossasiga egaligi masalasi hali ham yetarli darajada o‘rganilmagan. R. Boklandt, T. Shedler va M. Vemisslarning natijalari orqali barcha Koshul Kalabi-Yau algebralarning superpotensialdan hosil qilinishi isbotlangan bo‘lsa-da, superpotensialni Koshul ma‘nosidagi qo‘shma koalgebra orqali hisoblash maqsadli ilmiy tadqiqotlardan hisoblanadi.

Mamlakatimizda fundamental fanlarning ilmiy va amaliy ahamiyatga ega yo‘nalishlari bo‘lgan matematika, fizika, geologiya hamda biologiya sohalariga alohida e‘tibor qaratilgan. Jumladan, assotsiativ va noassotsiativ algebralarga oid fundamental tadqiqotlarni rivojlantirish masalasiga katta ahamiyat berilgan. Ushbu fundamental izlanishlar doirasida elliptik algebralari va ularning xossalarini o‘rganishda muhim ilmiy natijalarga erishilgan. “Algebra va funksional analiz”ning ustuvor yo‘nalishlari bo‘yicha xalqaro standartlar darajasida tadqiqotlar olib borish matematika fanining asosiy vazifalaridan biri hamda uning faoliyat yo‘nalishi sifatida belgilandi<sup>1</sup>. Mazkur vazifani amalga oshirishda, tegishli ilmiy sohalarida olingan natijalarning qo‘llanishini ta‘minlash maqsadida Kalabi-Yau algebralari tavsiflash muhim ilmiy masala hisoblanadi.

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<sup>1</sup> O‘zbekiston Respublikasi Vazirlar mahkamasining 2017-yil 18-maydagi “O‘zbekiston Respublikasi Fanlar akademiyasining yangidan tashkil etilgan ilmiy tadqiqotlar muassasalari faoliyatini tashkil etish to‘g‘risida”gi 292-sonli qarori.

O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PQ-4947-son “O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi to‘g‘risida”gi, 2017-yil 17-fevraldagi PQ-2789-son “Fanlar akademiyasi faoliyati, ilmiy-tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora-tadbirlari to‘g‘risida”gi, 2017-yil 20-apreldagi PQ-2909-son “Oliy ta’lim tizimini yanada rivojlantirish chora-tadbirlari to‘g‘risida”gi, 2019-yil 9-iyuldagi PQ-4387-son “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab-quvvatlash, shuningdek O‘zbekiston Respublikasi Fanlar akademiyasining V.I.Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida”gi, 2020-yil 7-maydagi PQ-4708-son “Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida”gi, 2022-yil 28-yanvardagi PF-60-son “2022-2026-yillarga mo‘ljallangan Yangi O‘zbekistonning taraqqiyot strategiyasi to‘g‘risida”gi qarorlari, hamda mazkur faoliyatga tegishli boshqa me‘yoriy-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishga ushbu dissertatsiya ishi muayyan darajada xizmat qiladi.

**Tadqiqotning respublika fan va texnologiyalari rivojlanishi ustuvor yo‘nalishlariga bog‘liqligi.** Mazkur tadqiqot respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustivor yo‘nalishi doirasida bajarilgan.

**Muammoning o‘rganilganlik darajasi.** So‘nggi yigirma yil davomida Kalabi-Yau algebralari nazariyasi jadal sur‘atlar bilan rivojlandi. V. Ginzburg tomonidan potensialga ega yo‘naltirilgan graflardan 3 – Kalabi-Yau differensial gradiurlangan algebralari qurilishi hamda R. Boklandt tomonidan graduirlanlangan 3 – Kalabi-Yau algebralar uchun klassifikatsiya teoremasining isbotlanishi mazkur yo‘nalishda muhim nazariy asosni shakllantirdi. M. Artin, J. Teyt va M. Van den Berglarning ishi 3 – o‘lchamli Artin-Shelter regulyar algebralarini tasniflashga bag‘ishlangan bo‘lib, ular muayyan tasniflash tizimini beradi. M. Van den Berg tomonidan isbotlangan asosiy strukturaviy teorema shuni ko‘rsatadiki, Koshul algebra Kalabi-Yau algebra bo‘lishi uchun uning Koshul ma‘nosidagi qo‘shma algebrasining simmetrik Frobenius algebra bo‘lishi zarur va yetarli. Ushbu muvofiqlik nafaqat Kalabi-Yau shartini tekshirish uchun amaliy mezon bo‘lib xizmat qiladi, balki nokommutativ qo‘shmalikni klassik Frobenius juftliklari bilan bog‘lab, deformatsiya nazariyasi va hosil qilingan ekvivalentliklarni o‘rganishda yo‘naltiruvchi omil bo‘lib xizmat qiladi.

1989-yilda B.L. Feigen va A.V. Odesskiy  $Q_{n,k}(\mathcal{E}, \tau)$  elliptik algebrani ta’riflab berdilar. Elliptik algebralar ilk bor ularning ilmiy ishlarlarida paydo bo‘lgan.  $Q_{n,1}(\mathcal{E}, \tau)$  algebralar haqida esa hozirga qadar anchagina ma’lumotlar to‘plangan. Ushbu algebralar  $n$  o‘zgaruvchili ko‘phadlar halqasi bilan bir xil Gilbert qatori hamda gomologik xossalari ega b‘lgan Nyoter sohasi ekanligini J. Tate va M. Van den Berglar tomonidan ko‘rsatilgan. Sklyanin algebralari M. Artin, J. Tate, M. Van den Berg, S.P. Smit va J.T. Staffordlarning tadqiqotlarida yetarlicha o‘rganilgan. Shuningdek, A. Chirvasitu, R. Kanda va S.P. Smit  $Q_{n,k}(\mathcal{E}, \tau)$  ning Koshul algebrasi

hamda o'lchami  $n$  bo'lgan Artin-Shelter regulyar algebrasi ekanligini ko'rsatishgan.

Mazkur strukturaviy yutuqlar bilan bir qatorda, hosil qilingan noassotsiativ Puasson nuqtayi nazari ham shakllanib, u Kalabi-Yau algebralarini tasvir nazariyasi va topologik invariantlar bilan bog'lovchi yondashuvni taqdim etmoqda. Y. Berest, X. Chen, F. Eshmatov va A. Ramadoslar noassotsiativ Puasson strukturalarini differensial-graduirlangan algebralarda ishlab chiqib, ular hosil qiladigan (siljigan) Puasson ko'paytmalar hosil qilingan tasvir sxemalarida namoyon bo'lishini isbotladilar. Kalabi-Yau algebralari uchun esa mazkur ko'paytmalar Hochschild/siklik (ko)gomologiya hamda shoda Li algebralari bilan uzviy bog'lanadi. Koshul Kalabi-Yau shartida, X. Chen, A. Eshmatov, F. Eshmatov va S. Yanglar Koshul qo'shmalikka mos keluvchi va hosil qilingan Morita ekvivalentliklari ostida funktorial bo'lgan hosil qilingan noassotsiativ Puasson ko'paytmasini qurdilar. Ushbu yondashuv kvadratik holda Kalabi-Yau xossasini aniqlash uchun maxsus moslashtirilgan gomologik vositalarni taqdim etadi. Kvantizatsiya nuqtayi nazaridan X. Chen, F. Eshmatov va W.L. Gan tor topologiyasining Li bialgebrasi (erkin sirtmoq fazolarining ekvivalent gomologiyasidagi Chas-Sullivan strukturasini)ni kvantlashtirdilar. Ular Li bialgebra turidagi tuzilmalar uchun deformatsiya-kvantizatsiya modelini taklif etdilar. Ushbu yondashuv Kalabi-Yau algebralardan kelib chiquvchi noassotsiativ Puasson ko'paytmalarini kvantlashtirish bilan mazmunan mos keladi.

Mazkur rivojlanishlar shuni ko'rsatadiki, elliptik algebralar allaqachon Kalabi-Yau turidagi obyektlardan kutiladigan ko'plab gomologik regulyar xossalariga ega. Ushbu dissertatsiyada hal qilinadigan markaziy masala shundan iboratki, Van den Berg qo'shmaligi bilan yuqorida keltirilgan hosil qilingan noassotsiativ Puasson va kvantizatsiya texnikalarini birlashtirish hamda batafsil Hochschild/Ext hisob-kitoblarini amalga oshirish orqali, Feigen-Odesskiy algebralari uchun qaysi parametr rejimlari Kalabi-Yau xossasini ta'minlaydiganini aniq belgilashdan iborat.

**Dissertatsiya tadqiqotining dissertatsiya bajarilgan ilmiy tekshirish instituti ilmiy-tadqiqot ishlari rejalari bilan bog'liqligi.** Dissertatsiya ishi V.I.Romanovskiy nomidagi Matematika institutining ilmiy-tadqiqot ishlari rejalari doirasida bajarilgan.

**Tadqiqot maqsadi** elliptik algebralarning Kalabi-Yau xossalarini tadqiq qilishdan iborat bo'lib, bunda Sklyanin algebralari va ularning umumlashmalari kabi aniq misollarga e'tibor qaratish, shuningdek, Koshul algebralarning Kalabi-Yau algebra bo'lishi uchun zarur va yetarli shartlarni aniqlashdan iborat.

**Tadqiqotning vazifalari:**

Van den Berghning natijasidan foydalanib, kvadratik Koshul algebralarining Kalabi-Yau xossasini o'rganish;

Uch va to'rt o'lchamli Sklyanin algebralari va ba'zi Koshul algebralarining Kalabi-Yau xossasiga ega ekanligini isbotlash;

to'rtinchi va beshinchi tartibli elliptik zeta funksiyalar orasidagi ayniyatlarni topish;

to'rt o'lchamli Sklyanin va elliptik algebralar o'rtasida izomorfizm qurish;

besh o'ldamli elliptik algebralar uchun superpotensialni topish.

**Tadqiqot ob'ekti:** Kvadratik va Koshul algebra, elliptik zeta funksiyalar, elliptik algebralar.

**Tadqiqot predmeti.** Koshul algebralar nazariyasi, kvadratik algebralar nazariyasi, elliptik algebralar nazariyasi.

**Tadqiqot usullari.** Dissertatsiyada assosiativ algebralar metodlari, induksiya metodi, shuningdek Koshul algebra usullari qo'llanilgan.

**Tadqiqotning ilmiy yangiligi** quyidagilardan iborat:

Koshul algebralarining Kalabi-Yau xossasiga ega bo'lishi uchun zarur bo'lgan kriteriya aniqlangan;

to'rtinchi va beshinchi tartibli elliptik zeta funksiyalar orasida ayniyatlar topilgan;

to'rt o'ldamli elliptik va Sklyanin algebralari orasida izomorfizm oshkor ko'rinishda qurilgan;

to'rt o'ldamli elliptik algebralar uchun superpotensial topilgan;

besh o'ldamli elliptik algebralar uchun superpotensial  $\tau \in \mathcal{E}$  ning ba'zi qiymatlari uchun topilgan.

**Tadqiqotning amaliy natijalari.** Olingan natijalar va dissertatsiyada qo'llanilgan usullar oliy o'quv yurtlarida magistratura talabalari, tayanch doktorantlar uchun o'quv kurs sifatida o'qitilishi mumkin. Shuningdek, dissertatsiya natijalari, Koshul algebralarining Kalabi-Yau xossasiga oid bo'limlari,  $n$  - o'ldamli elliptik algebralarining Kalabi-Yau xossasiga ega ekanligi haqidagi gipotezalarni tekshirish imkonini beradi.

**Tadqiqot natijalarining ishonchliligi.** Mazkur tadqiqot natijalari invariantlar nazariyasi hamda algebralar nazariyasining asosiy qoidalaridan foydalanilgan holda olindi. Olingan natijalar isbotlari matematik jihatdan to'g'ri hisoblanadi. To'rt va besh o'ldamli elliptik algebralar uchun topilgan superpotensiallar va ularning xususiy hosilalari Mathematica 12 dasturining standart buyruqlari hamda NCAgebra paketi yordamida tekshirish mumkin. Shuningdek, to'rtinchi va beshinchi tartibli elliptik zeta funksiyalari orasida topilgan ayniyatlar ham Mathematica 12 dasturining standart buyruqlari orqali tekshirish mumkin.

**Tadqiqot natijalarining ilmiy va amaliy ahamiyati.** Tadqiqotning amaliy ahamiyati uning sof va amaliy matematikadagi qo'llanishi bilan belgilanadi. Sof algebra doirasida u nokommutativ sirtlar va ularning modullarini o'rganishga hissa qo'shadi. Matematik fizika sohasida esa mazkur algebralar kvant simmetriyalarning modellarini yaratadi hamda integrallanuvchi sistemalarda uchraydigan algebraik strukturalar asosini tashkil etadi. Tadqiqotda qo'llanilgan usullar-gomologik algebra, nokommutativ proyektiv geometriya va invariantlar nazariyasi mustaqil ilmiy qiziqish uyg'otuvchi bo'lib, keng qo'llanish imkoniyatiga ega. Dissertatsiyada olingan natijalar va foydalanilgan metodlar oliy ta'lim muassasalarining magistratura va doktorantura bosqichlari uchun maxsus kurs sifatida o'qitilishi mumkin.

**Tadqiqot natijalarining joriy qilinishi.** Elliptik algebralarining Kalabi-Yau xossalari bo'yicha olingan natijalar asosida:

to‘rt va besh o‘lchamli elliptik algebra uchun topilgan superpotensiallardan NSFC12271377 raqamli “Nokommutativ algebraik geometriya va hosil qilingan algebraik geometriyaning ayrim masalalari” mavzusidagi xorijiy loyihada kompakt golomorf Puasson ko‘pxillarida golomorf Koshul–Brilinski gomologiyalari uchun kengaytirish formulasini hosil qilishda foydalanilgan (Sichuan universitetining 2025 yil 5-sentyabrdagi ma‘lumotnomasi, Xitoy Xalq Respublikasi). Ilmiy natijaning qo‘llanilishi holomorfik Puasson tuzilmalarga ega Pezzo sirtlari hamda ikki xil kompleks nilko‘pxilligi uchun golomorfik Koshul–Brilinski gomologiyalarini hisoblash imkonini bergan;

Koshul algebra larning Kalabi-Yau xossasiga ega bo‘lishi uchun zarur bo‘lgan kiriteriyadagi kvadratik algebra larning Koshul ma‘nosidagi qo‘shma algebra lari uchun kvant Yang–Bakster tenglamasining yechimlaridan F-FA-2021-423 raqamli “Operator algebra larning avtomorfizmlari, cheksiz o‘lchamli noassosiativ algebra lar va superalgebra larning klassifikatsiyasi” mavzusidagi fundamental loyihada Li algebra larning Rota-Bakster operatorlarini aniqlashda foydalanilgan (Matematika institutining 2025 yil 2-sentyabrdagi №2/337-sonli ma‘lumotnomasi). Ilmiy natijaning qo‘llanilishi kichik o‘lchamli Li algebra larning Rota-Bakster va anti-Rota-Bakster operatorlarini tasniflash imkonini bergan.

**Tadqiqot natijalarining aprotatsiyasi.** Mazkur tadqiqot natijalari 7 ta ilmiy-amaliy anjumanlarda, jumladan 2 ta xalqaro va 5 ta respublika ilmiy-amaliy anjumanlarida muhokamadan o‘tkazilgan.

**Tadqiqot natijalarining e‘lon qilinganligi.** Dissertatsiya tadqiqoti mavzusi bo‘yicha jami 12 ta ilmiy ish chop etilgan, shulardan O‘zbekiston Respublikasi Oliy Attestatsiya komissiyasining falsafa doktorlik dissertatsiyalari asosiy ilmiy natijalarini chop etish tavsiya etilgan ilmiy nashrlarda 5 ta maqola, jumladan 2 tasi xorijiy va 3 tasi respublika jurnallarida nashr etilgan.

**Dissertatsiyaning tuzilishi va hajmi.** Dissertatsiya kirish qismi, uchta bob, to‘qqizta paragraf, xulosa va foydalanilgan adabiyotlar ro‘yxatidan tashkil topgan. Dissertatsiyaning umumiy hajmi 73 betni tashkil etgan.

## DISSERTATSIYANING ASOSIY MAZMUNI

**Kirish** qismida dissertatsiya mavzusining dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustivor yo‘nalishlariga mosligi ko‘rsatilgan, muammoning o‘rganilganlik darajasi keltirilgan, tadqiqot maqsadi, vazifalari, ob‘ekti va predmeti tavsiflangan, tadqiqotning ilmiy yangiligi va amaliy natijalari bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertatsiya tuzilishi bo‘yicha ma‘lumotlar keltirilgan.

Dissertatsiyaning “**Dastlabki ma‘lumotlar va natijalar**” deb nomlanuvchi birinchi bobida dissertatsiya mavzusini to‘la yoritish uchun zarur bo‘lgan asosiy ta‘riflar va muhim tushunchalar keltirilgan. Dastlab, assosiativ algebra, kvadratik algebra, koassosiativ koalgebra va kvadratik, assosiativ algebra larning Koshul ma‘nosidagi qo‘shma algebra va koalgebra tushunchalari eslatib o‘tilgan. Keyin,

ushbu dissertatsiyada muhim ahamiyat kasb etuvchi Koshul va Kalabi–Yau algebralari ko‘rib chiqiladi. Ushbu bobning oxirgi paragrafi kvant Yang-Bakster tenglamasi haqidagi asosiy ma’lumotlarni o‘z ichiga oladi.

Aytaylik,  $\mathbb{K}$  biror maydon bo‘lsin.  $\mathbb{K}$  maydon ustidagi chiziqli fazolar ( $A$  algebra ustidagi modullarga o‘xshash tarzda) va  $\mathbb{K}$  –chiziqli akslantirishlar ketma-ketligi

$$(V, d) = \cdots \rightarrow V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \rightarrow \cdots, \quad n \in \mathbb{Z}$$

har bir  $n \in \mathbb{Z}$  uchun  $d_n \circ d_{n+1} = 0$  bo‘lsa, u holda  $(V, d)$  ga (zanjir) kompleks deyiladi.  $(V, d)$  kompleksning  $n$  – gomologiyasi

$$H_n(V, d) = \ker d_n / \text{Im } d_{n+1}$$

nol bo‘lsa, unga *aniq* yoki *siklsiz* deyiladi.

Aytaylik,  $V$   $\mathbb{K}$  ustidagi chekli o‘lchamli chiziqli fazo bo‘lsin.

**1-ta’rif.** Agar  $V$  chiziqli fazo orqali hosil qilingan

$$T(V) := \mathbb{K} \oplus V \oplus (V \otimes V) \cdots \oplus V^{\otimes n} \oplus \cdots$$

tenzor modulda  $T(V) \otimes T(V) \rightarrow T(V)$  ko‘paytirishni

$$v_1 \cdots v_p \otimes v_{p+1} \cdots v_{p+q} \mapsto v_1 \cdots v_p v_{p+1} \cdots v_{p+q}$$

kabi aniqlasak, unga  $V$  ustidagi *tenzor algebra* deyiladi.

Bu amal assosiativ amal va birlik element sifatida 1 olinadi.

**2-ta’rif.**  $V$  chekli o‘lchamli chiziqli fazoning tenzor algebrasini  $R \subset V \otimes V$  qism fazo orqali hosil qilingan ikki tomonlama ideali bo‘yicha faktor algebrasiga *kvadratik algebra* deyiladi va  $A(V, R) = T(V) / (R)$  kabi belgilanadi.

**3-ta’rif.**  $A = A(V, R)$  kvadratik algebraning *Koshul ma’nosidagi qo‘shma algebrasi* deb,  $A^1 = T(V^*) / (R^\perp)$  algebra ga aytiladi, bu yerda

$$R^\perp = \{r \in V^* \otimes V^* \simeq (V \otimes V)^* \mid r(R) = 0\}.$$

**4-ta’rif.**  $A = A(V, R)$  kvadratik algebraning *Koshul ma’nosidagi qo‘shma koalgebrasi* deb, quyidagi koalgebra ga aytiladi.

$$A^i = \mathbb{K} \oplus sV \oplus s^2R \oplus \cdots \oplus \left( \bigcap_{i+2+j=n} sV^{\otimes i} \otimes s^2R \otimes sV^{\otimes j} \right) \oplus \cdots,$$

bu yerda  $s^2R$  fazo  $R$  ning  $V^{\otimes 2} \rightarrow (sV)^{\otimes 2}, vw \mapsto svsw$  akslantirishning obrazi.

$V$  vektor fazoning  $\{e_i\}$  bazisini olaylik va uning  $V^*$  dagi qo‘shma bazisi  $\{e_i^*\}$  bo‘lsin hamda  $V^*$  ning graduirovkasini birga pasaytirishni  $s^{-1}V^*$  orqali belgilaylik. U holda  $A$  kvadratik algebra orqali tuzilgan

$$\cdots \xrightarrow{\delta} A \otimes A_{i+1}^i \xrightarrow{\delta} A \otimes A_i^i \xrightarrow{\delta} \cdots A \otimes A_0^i \xrightarrow{\delta} \mathbb{K}, \quad (1)$$

kompleksga *Koshul kompleksi* deyiladi. Bu yerda  $r \otimes f \in A \otimes A^i$ ,

$$\delta(r \otimes f) = \sum_i e_i r \otimes s^{-1} e_i^* f.$$

**5-ta’rif.** Agar (1) Koshul kompleks aniq bo‘lsa, u holda  $A = TV / (R)$  kvadratik algebra ga *Koshul* deyiladi.

**6-ta’rif.** Agar ixtiyoriy  $M \rightarrow N$   $A$  – modul epimorfizm uchun

$$\text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$$

morfizm syuryektiv bo'lsa,  $P \in A - \text{Mod}$  modulga *proyektiv modul* deyiladi.  $M \in A - \text{Mod}$  ning *proyektiv rezolyutsiyasi* deb quyidagi aniq ketma-ketlikka aytiladi.

$$P := \cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

bu yerda har bir  $i$  uchun  $P_i$  – proyektiv  $A$  – modul.

$P_M := \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow 0$  ketma-ketlik  $M \in A - \text{Mod}$  ning kesilgan proyektiv rezolyutsiyasi bo'lsin. Unga  $\text{Hom}_A(-, N)$  funktorni ta'sir qildirsak, quyidagi kompleksga ega bo'lamiz:

$$\text{Hom}_A(P_M, N) = 0 \rightarrow \text{Hom}_A(P_0, N) \xrightarrow{d_1^*} \text{Hom}_A(P_1, N) \rightarrow \cdots$$

$\text{Hom}_A(P_M, N)$  ning  $n$  – kogomologiyasiga

$$\text{Ext}_A^n(M, N) := H^n(\text{Hom}_A(P_M, N)) = \ker d_{n+1}^* / \text{Im } d_n^*.$$

deyiladi. Bu yerda  $d^*(\varphi) = d \circ \varphi$ .

$A$  assosiativ algebra bo'lsin. U holda  $A \otimes A$  fazo ikkita *tashqi* va *ichki* deb nomlangan  $A$  – bimodul strukturaga ega. Ular mos ravishda quyidagicha beriladi:

$$a(b_1 \otimes b_2)c = ab_1 \otimes b_2c, \quad a(b_1 \otimes b_2)c = b_1c \otimes ab_2.$$

Aytaylik,  $A$  bimodullar kategoriyasi  $A - \text{Bimod}$  bo'lsin. U holda,  $A \otimes A$  ni tashqi bimodul struktura deb qarab, qo'shma funksiyani

$$\text{Hom}_{A-\text{Bimod}}(-, A \otimes A) : A - \text{Bimod} \rightarrow A - \text{Bimod}, \quad M \mapsto \text{Hom}_{A-\text{Bimod}}(M, A \otimes A).$$

kabi aniqlaymiz.

$\text{Hom}_{A-\text{Bimod}}(M, A \otimes A)$  ustidagi bimodul struktura  $A \otimes A$  ustidagi ichki bimodul struktura orqali olinadi.

**7-ta'rif.** Agar  $A$  algebra quyidagi ikki shartni qanoatlantirsa unga  $d$  – *Kalabi-Yau* algebra deyiladi. Birinchidan,  $A$  gomologik silliq, ya'ni  $A$  chekli uzunlikdagi chekli hosil qilingan  $A^e$  – bimodullarning proyektiv rezolyutsiyasi ega. Ikkinchidan,  $A$  – bimodullarning quyidagi izomorfizmi o'rinli.

$$\text{Ext}_{A-\text{Bimod}}^k(A, A \otimes A) \cong \begin{cases} A, & \text{if } k = d \\ 0, & \text{if } k \neq d \end{cases}.$$

Kalabi-Yau algebralarining katta sinfi *superpotensialdan* kelib chiqadi. Faraz qilaylik,  $\mathbb{C}\langle x_1, \dots, x_n \rangle$   $n$  ta hosil qiluvchilarning erkin algebrasi bo'lsin. U holda,  $\mathbb{C}\langle x_1, \dots, x_n \rangle$  monomlarning uzunligi  $|a|$  bo'yicha tabiiy graduirovkaga ega.  $a_i \in \{x_1, \dots, x_n\}$  bo'lsin,  $a = a_1 a_2 \dots a_m \in \mathbb{C}\langle x_1, \dots, x_n \rangle$  monom uchun,  $x_k$  ( $k = 1, \dots, n$ ) bo'yicha *nokommutativ xususiy hosilani* aniqlaymiz.

$$\partial_{x_k}(a) := \sum_{\{i \in [1, m] | a_i = x_k\}} a_{i+1} a_{i+2} \dots a_m a_1 \dots a_i.$$

Chiziqlilik xossasidan uni  $\partial_{x_k} : \mathbb{C}\langle x_1, \dots, x_n \rangle \rightarrow \mathbb{C}\langle x_1, \dots, x_n \rangle$  akslantirishga kengaytirish mumkin. Keyin, ushbu akslantirishlarning kompozitsiyasini olish orqali, yuqori tartibli hosilani aniqlaymiz. Boshqacha qilib aytganda,  $p = x_{i_1} \dots x_{i_s}$  monom uchun  $\partial_p := \partial_{x_{i_1}} \circ \dots \circ \partial_{x_{i_s}}$  deb belgilaymiz. Keyin,  $\omega \in \mathbb{C}\langle x_1, \dots, x_n \rangle$  uchun  $k$  – tartibli hosilaviy-faktor algebrani quyidagicha aniqlashimiz mumkin

$$\mathcal{D}(\omega, k) := \frac{\mathbb{C}\langle x_1, \dots, x_n \rangle}{(\partial_p \omega : |p| = k)}$$

bu yerda  $p$  uzunligi  $k$  bo'lgan  $\mathbb{C}\langle x_1, \dots, x_n \rangle$  ning barcha monomlari bo'yicha o'zgaradi.

Agar  $\omega \in \mathbb{C}\langle x_1, \dots, x_n \rangle$  quyidagi ko'rinishdagi elementlar yig'indisidan iborat bo'lsa, unga *superpotensial* deyiladi.

$$\sum_{i=1}^n (-1)^{(i-1)(n+1)} a_i a_{i+1} \dots a_n a_1 a_2 \dots a_{i-1}, \quad a_i \in \{x_1, \dots, x_n\} \quad (2)$$

Faraz qilaylik,  $A = \mathcal{D}(\omega, k)$  (2) ko'rinishdagi  $\omega$  superpotensialning hosilaviy-faktor algebra bo'lsin.  $A$  kvadratik algebra bo'lganligi uchun biz  $|\omega| = k + 2$  deb faraz qilamiz.

Quyidagi kompleksni qaraylik

$$\mathcal{V} = 0 \rightarrow A \otimes V_{|\omega|} \otimes A \xrightarrow{d_{|\omega|}} A \otimes V_{|\omega|-1} \otimes A \xrightarrow{d_{|\omega|-1}} \dots \xrightarrow{d_1} A \otimes V_0 \otimes A \rightarrow 0, \quad (3)$$

bu yerda,  $V_{n-k} := \langle \partial_p \omega : |p| = k \rangle$  va  $v_1, \dots, v_i \in V, a, a' \in A$  lar uchun

$$\begin{aligned} d_i &= \varepsilon_i (\text{split}_L + (-1)^i \text{split}_R) |_{A \otimes V_i \otimes A}, \\ \text{split}_L (a \otimes v_1 v_2 \dots v_i \otimes a') &= a v_1 \otimes v_2 \dots v_i \otimes a', \\ \text{split}_R (a \otimes v_1 v_2 \dots v_i \otimes a') &= a \otimes v_1 \dots v_{i-1} \otimes v_i a', \\ \varepsilon_i &:= \begin{cases} (-1)^{i(|\omega|-i)}, & \text{agar } i < \frac{|\omega|+1}{2}, \\ 1, & \text{aks holda.} \end{cases} \end{aligned}$$

Yuqoridagini kompleks hosil qilishini ya'ni  $d_i \circ d_{i+1} = 0$  ekanligini tekshirish oson. Bundan tashqari,  $A$  ning o'zidan tashqari barcha hadlar proyektiv bimodullar bo'lib, akslantirishlar  $A$  – bimodul akslantirishlardir.

1997-yilda R. Boklandt, T. Shedler va M. Vemisslar quyidagi teoremani isbotlagan.

**1-teorema.**  $T(V)/(R)$  algebra Koshul Kalabi-Yau algebra bo'lishi uchun uning  $\mathcal{D}(\omega, k)$  ko'rinishda bo'lishi hamda unga mos keluvchi (3) kompleks musbat darajalarda aniq bo'lishi va  $H^0(\mathcal{V}) = A$  bo'lishi zarur va yetarli.

Faraz qilaylik,  $V \subset \mathbb{C}$  – chiziqli fazo va  $R \in \text{End}(V \otimes V)$  bo'lsin. Agar  $R$

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12},$$

qanoatlantirsa, unga kvant Yang-Bakster tenglamasini yechimi deyiladi.

bu yerda  $R^{12} = R \otimes \text{id}, R^{23} = \text{id} \otimes R$ , va hokazolar  $V^{\otimes 3}$  ning endomorfizmlari. Agar  $R$

ni  $R = \sum_{i=1}^n a_i \otimes b_i \in A \otimes A$  kabi olsak, u holda ushbu yig'indi  $A \otimes A$  ga chapdan

ko'paytirish orqali ta'sir qiladi va  $R^{12} = \sum_{i=1}^n a_i \otimes b_i \otimes 1$  va hokazo. U holda kvant

Yang-Bakster tenglamasini quyidagicha yozishimiz mumkin.

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}.$$

Dissertatsiyaning “Kalabi-Yau algebralari va kvant Yang-Bakster tenglamasi” deb nomlangan ikkinchi bobida Koshul Kalabi-Yau algebralarning yangi xarakterizatsiyasi o‘rganilgan va bir qancha Koshul algebralarning Kalabi-Yau xossasini qanoatlantirishi isbotlangan.

Faraz qilaylik,  $A$  chekli o‘lchamli  $\mathbb{K}$  – algebra bo‘lsin. Aning  $\mathbb{K}$  – chiziqli qo‘shmasi  $A^* = \text{Hom}_{\mathbb{K}}(A, \mathbb{K})$  quyidagi skalyar ko‘paytirish bilan chap  $A$  – modul bo‘ladi.

$$a.f(x) = f(xa), \quad a, x \in A, \quad f \in A^*.$$

**8-ta’rif.**  $A$  chekli o‘lchamli  $\mathbb{K}$  – algebra bo‘lsin. Agar chap  $A$  – modullar sifatida  $A \cong A^*$  bo‘lsa,  $A$  ga *Frobenius algebra* deyiladi.

Bu,  $\langle -, - \rangle: A \times A \rightarrow \mathbb{C}$  xosmas bichiziqli forma mavjud bo‘lib, barcha  $a, b, c \in A$  lar uchun  $\langle ab, c \rangle = \langle a, bc \rangle$  niq qanoatlantirishiga ekvivalent.

Quyidagi natija, Beydar, Fong va Stolinlar tomonidan olingan bo‘lib, Frobenius algebralarni kvant Yang-Bakster tenglamasining yechimlari yordamida tavsiflaydi.

**2-teorema.** Faraz qilaylik,  $Q = \sum_{i=1}^n e_i \otimes e^i$  bo‘lsin. U holda quyidagilar ekvivalent:

1.  $Q^{12}Q^{13} = Q^{23}Q^{12}$  va  $Q^{12}Q^{23} = Q^{23}Q^{13}$ ,
2.  $Q^{12}Q^{13} = Q^{13}Q^{23}$  va  $Q^{12}Q^{23} = Q^{13}Q^{12}$ ,
3.  $Q^{23}Q^{12} = Q^{13}Q^{23}$  va  $Q^{23}Q^{13} = Q^{13}Q^{12}$ ,
4. barcha  $a \in A$  uchun  $(a \otimes b)Q = Q(b \otimes a)$ ,
5. element  $Q$  kvant Yang-Bakster tenglamasining yechimi bo‘ladi va  $\tau_{A,A} \circ Q = Q$  bo‘ladi.

Bundan tashqari, M. Van den Bergning ishida muallif quyidagi teoremani isbotlagan.

**3-teorema.** Faraz qilaylik,  $A$  Koshul algebra bo‘lsin. U holda  $A$  ning  $d$  – Kalabi-Yau bo‘lishi uchun  $A^!$  ning graduirlangan simmetrik Frobenius algebra bo‘lishi zarur va yetarli.

Dissertatsiyaning 2.1-paragrafida isbotlangan quyidagi teorema 2-teoremaning graduirlangan holatidir.

**4-teorema.** Faraz qilaylik,  $A$  Koshul algebra bo‘lsin. U holda  $A$  algebraning Kalabi-Yau algebra bo‘lishi uchun uning Koshul ma’nosidagi qo‘shma algebra  $A^!$  da  $e_1, \dots, e_n$  va  $e^1, \dots, e^n$  bir jinsli bazislar mavjud bo‘lib,  $Q = \sum_{i=1}^n e_i \otimes e^i$  element quyidagi ekvivalent shartlarni qanoatlantirishi zarur va yetarli.

1. barcha  $a, b \in A^!$  lar uchun  $(a \otimes b)Q = (-1)^{|a||b| + (|a|+|b|)(|Q|+1)} Q(b \otimes a)$ ,
2.  $Q^{12} \left( (-1)^{|e_j|+|Q|} Q^{13} \right) = Q^{23}Q^{12}$  va  $Q^{12}Q^{23} = \left( (-1)^{|e_j|+|Q|} Q^{23} \right) Q^{13}$ .

**5-teorema.** Faraz qilaylik,  $A$   $\mathbb{K}$  – algebra va  $Q = e_i \otimes e^i \in A^! \otimes A^!$  bo‘lsin. Barcha  $a, b \in A^!$  lar uchun  $(a \otimes b)Q = (-1)^{|a||b| + (|a|+|b|)(|Q|+1)} Q(b \otimes a)$  bajarilsin. U holda quyidagilar o‘rinli:

1.  $Q^{12}Q^{13} = \left((-1)^{|e_j|+|Q|} Q^{23}\right) Q^{12}$  va  $Q^{12}Q^{23} = \left((-1)^{|e_j|+|Q|} Q^{13}\right) Q^{12}$ ,
2.  $Q^{12}Q^{13}Q^{23} = \left((-1)^{|e_j|} Q^{23}\right) \left((-1)^{|e_j|} Q^{13}\right) Q^{12}$ .

Aytaylik,  $a, b, c \in \mathbb{C}$  sonlari quyidagi ikkita shartni qanoatlantirsin:

- 1)  $[a:b:c] \in \mathbb{P}_{\mathbb{C}}^2 \setminus D$ , bu yerda

$$D = \{[1:0:0], [0:1:0], [0:0:1]\} \cup \{[a:b:c] \mid a^3 = b^3 = c^3 = 1\}.$$

- 2)  $abc \neq 0$  va  $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ .

Uch o'lchamli Sklyanin algebra  $S(a, b, c)$  ni 1-darajali hosil qiluvchilari  $x, y, z$  va munosabatlari

$$ax^2 + byz + czy = 0, \quad ay^2 + bzx + cxz = 0, \quad az^2 + bxy + cyx = 0.$$

bo'lgan graduirlangan  $\mathbb{C}$ -algebra kabi aniqlash mumkin.

To'rt o'lchamli Sklyanin algebra  $S(\alpha, \beta, \gamma)$  1-darajali hosil qiluvchilari  $x_0, x_1, x_2, x_3$  lar bo'lgan graduirlangan  $\mathbb{C}$ -algebra bo'lib, ular quyidagi munosabatlarni qanoatlantiradi.

$$\begin{aligned} x_0x_1 - x_1x_0 - \alpha(x_2x_3 + x_3x_2) &= 0, & x_0x_1 + x_1x_0 - (x_2x_3 - x_3x_2) &= 0, \\ x_0x_2 - x_2x_0 - \beta(x_3x_1 + x_1x_3) &= 0, & x_0x_2 + x_2x_0 - (x_3x_1 - x_1x_3) &= 0, \\ x_0x_3 - x_3x_0 - \gamma(x_1x_2 + x_2x_1) &= 0, & x_0x_3 + x_3x_0 - (x_1x_2 - x_2x_1) &= 0. \end{aligned}$$

bu yerda  $\alpha, \beta$  va  $\gamma$  lar  $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$  ni qanoatlantiruvchi kompleks sonlar.

**6-teorema.** Uch va to'rt o'lchamli Sklyanin algebralari  $S(a, b, c)$  va  $S(\alpha, \beta, \gamma)$  Koshul Kalabi-Yau algebrasi bo'ladi.

Koso ko'phadlar halqasi  $A_1 := \mathbb{C}_{q_i}[x_0, x_1, x_2, x_3]$  graduirlangan  $\mathbb{C}$ -algebra bo'lib,  $x_0, x_1, x_2, x_3$  lar orqali hosil qilingan va quyidagi munosabatlarni qanoatlantiradi.

$$x_i x_{i+1} = \left(\frac{q_{i+3}}{q_{i+2}}\right)^{(-1)^i} x_{i+1} x_i, \quad x_i x_{i+2} = \left(\frac{q_{i+1}}{q_{i+3}}\right)^{(-1)^i} x_{i+2} x_i$$

bu yerda  $q_0, \dots, q_3 \in \mathbb{C}^*$  bo'lib,  $\prod_{i=0}^3 q_i = 1$  shartni qanoatlantiradi va indekslar 4 modul bo'yicha olinadi.

**7-teorema.**  $A_1$  algebra Koshul Kalabi-Yau algebrasi bo'ladi.

Uch o'lchamli Sklyanin algebrasining markaziy kengaytmasi  $B$  hosil qiluvchilari  $x_0, x_1, x_2, x_3$  lar bo'lib, quyidagi munosabatlarni qanoatlantiradi.

$$\begin{aligned} [x_0, x_1] &= 0, & [x_2, x_1] &= \lambda(x_2x_1 + x_1x_2) - \nu x_3^2 - \sum_{j=1}^3 b_{1j} x_j x_0, \\ [x_0, x_2] &= 0, & [x_3, x_2] &= \lambda(x_3x_2 + x_2x_3) - \nu x_1^2 - \sum_{j=1}^3 b_{2j} x_j x_0, \\ [x_0, x_3] &= 0, & [x_1, x_3] &= \lambda(x_1x_3 + x_3x_1) - \nu x_2^2 - \sum_{j=1}^3 b_{3j} x_j x_0. \end{aligned}$$

bu yerda  $\lambda, \nu \in \mathbb{C}$  - o'zgarmas sonlari va  $b = (b_{ij}) \in \mathbb{C}^{3 \times 3}$  - simmetrik matritsa.

**8-teorema.**  $B$  algebra Koshul Kalabi-Yau algebra bo‘ladi.

Dissertatsiyaning “**Elliptik algebraarning Kalabi-Yau xossasi**” deb nomlangan uchinchi bobida, elliptik zeta funksiyalar va elliptik algebralarning tadqiq qilingan. Shuningdek, to‘rt o‘lchamli Sklyanin va elliptik algebraarning izomorf ekanligi keltirilgan. Bundan tashqari, to‘rt va besh o‘lchamli elliptik algebraarning Kalabi-Yau xossasiga ega ekanligi isbotlangan.

Faraz qilaylik,  $\Lambda \neq 1$  va  $\eta \in \mathbb{C}$  orqali hosil qilingan butun panjara bo‘lsin, bu yerda  $\text{Im } \eta > 0$  va  $\mathcal{E} := \mathbb{C} / \Lambda$  unga mos bo‘lgan elliptik egri chiziq bo‘lsin. Feigen va Odesskii maqolasida  $\Theta_n(\Lambda)$  orqali bir o‘zgaruvchili kvazi-preoidiklik shartlarini qanoatlantiruvchi butun funksiyalar fazosini belgilashgan.

$$f(z+1) = f(z), \quad f(z+\eta) = -e^{-2\pi iz} f(z).$$

$\Theta_n(\Lambda)$  dagi funksiyalar  $\Lambda$  panjaraga mos bo‘lgan  $n$ -tartibli zeta funksiyalar deb ataladi. Tekshirish osonki,

$$\theta(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi i \left( nz + \frac{n(n-1)}{2} \eta \right)}$$

funksiya  $\Theta_1(\Lambda)$  fazoning bazisi bo‘ladi.

$\alpha \in \mathbb{Z} / n\mathbb{Z}$  uchun

$$\theta_\alpha(z) := e^{2\pi i(\alpha z + [\alpha])} \prod_{m=0}^{n-1} \theta\left(z + \frac{m}{n} + \frac{\alpha}{n} \eta\right), \quad [\alpha] := \frac{\alpha(\alpha - n)}{2n} \eta + \frac{\alpha}{2n}.$$

funksiyani aniqlaymiz. U holda  $\{\theta_0, \theta_1, \dots, \theta_{n-1}\}$  lar  $\Theta_n(\Lambda)$  fazoning bazisi bo‘ladi.  $n = 4$  bo‘lsin.

**1-xossa.** Barcha  $\tau \in \mathbb{C} - \frac{1}{4}\Lambda$  uchun to‘rtinchi tartibli elliptik zeta funksiyalar quyidagi ayniyatni qanoatlantiradi.

$$\frac{\theta_0^2(\tau) + \theta_2^2(\tau)}{\theta_1(\tau)\theta_3(\tau)} = \frac{\theta_1^2(\tau) + \theta_3^2(\tau)}{\theta_0(\tau)\theta_2(\tau)}.$$

Beshinchi tartibli elliptik zeta funksiyalar quyidagi kabi aniqlanadi.

$$\begin{aligned} \theta_0(z) &= \theta(z) \theta\left(z + \frac{1}{5}\right) \theta\left(z + \frac{2}{5}\right) \theta\left(z + \frac{3}{5}\right) \theta\left(z + \frac{4}{5}\right), \\ \theta_1(z) &= \theta\left(z + \frac{1}{5}\eta\right) \theta\left(z + \frac{1}{5} + \frac{1}{5}\eta\right) \cdots \theta\left(z + \frac{4}{5} + \frac{1}{5}\eta\right) e^{2\pi i \left(z + \frac{1}{10} - \frac{2}{5}\eta\right)}, \\ \theta_2(z) &= \theta\left(z + \frac{2}{5}\eta\right) \theta\left(z + \frac{1}{5} + \frac{2}{5}\eta\right) \cdots \theta\left(z + \frac{4}{5} + \frac{1}{5}\eta\right) e^{2\pi i \left(2z + \frac{1}{5} - \frac{3}{5}\eta\right)}, \\ \theta_3(z) &= \theta\left(z + \frac{3}{5}\eta\right) \theta\left(z + \frac{1}{5} + \frac{3}{5}\eta\right) \cdots \theta\left(z + \frac{4}{5} + \frac{3}{5}\eta\right) e^{2\pi i \left(3z + \frac{3}{10} - \frac{3}{5}\eta\right)}, \\ \theta_4(z) &= \theta\left(z + \frac{4}{5}\eta\right) \theta\left(z + \frac{1}{5} + \frac{4}{5}\eta\right) \cdots \theta\left(z + \frac{4}{5} + \frac{4}{5}\eta\right) e^{2\pi i \left(4z + \frac{2}{5} - \frac{2}{5}\eta\right)}. \end{aligned}$$

**2-xossa.** Barcha  $\tau \in \mathbb{C} - \frac{1}{5}\Lambda$  lar uchun yuqoridagi beshinchi tartibli elliptik zeta funksiyalar quyidagi o'nta ayniyatni qanoatlantiradi.

$$\begin{aligned}
m_1 &= \theta_0^2 \theta_1^3 \theta_3 + \theta_0 \theta_1 \theta_2^2 \theta_3^2 - \theta_1^3 \theta_2^2 \theta_4 - \theta_0 \theta_3^4 \theta_4 - \theta_0^2 \theta_1 \theta_2 \theta_4^2 + \theta_2 \theta_3 \theta_4^4 = 0, \\
m_2 &= -\theta_0^2 \theta_1^2 \theta_2 \theta_3 - \theta_0 \theta_2^3 \theta_3^2 + \theta_1^2 \theta_2^3 \theta_4 + \theta_0^4 \theta_3 \theta_4 + \theta_1 \theta_2 \theta_3^2 \theta_4^2 - \theta_0 \theta_1 \theta_4^4 = 0, \\
m_3 &= \theta_0^4 \theta_1 \theta_2 - \theta_0 \theta_2^2 \theta_3^3 - \theta_0 \theta_1^4 \theta_4 + \theta_1^2 \theta_2^2 \theta_3 \theta_4 - \theta_0^2 \theta_2 \theta_3 \theta_4^2 + \theta_1 \theta_3^3 \theta_4^2 = 0, \\
m_4 &= \theta_0 \theta_1 \theta_2^4 - \theta_1^4 \theta_2 \theta_3 + \theta_0^2 \theta_1^2 \theta_3 \theta_4 - \theta_0 \theta_2^2 \theta_3^2 \theta_4 - \theta_0^2 \theta_2 \theta_4^3 + \theta_1 \theta_3^2 \theta_4^3 = 0, \\
m_5 &= \theta_0^3 \theta_1^2 \theta_3 - \theta_1 \theta_2 \theta_3^4 - \theta_0 \theta_1^2 \theta_2^2 \theta_4 + \theta_2^4 \theta_3 \theta_4 - \theta_0^3 \theta_2 \theta_4^2 + \theta_0 \theta_1 \theta_3^2 \theta_4^2 = 0, \\
k_1 &= \theta_0^2 \theta_1 \theta_2^3 - \theta_0 \theta_1^4 \theta_3 - \theta_0^2 \theta_2 \theta_3^2 \theta_4 + \theta_1 \theta_3^4 \theta_4 + \theta_0 \theta_1^2 \theta_2 \theta_4^2 - \theta_2^3 \theta_3 \theta_4^2 = 0, \\
k_2 &= \theta_0^2 \theta_1 \theta_2^2 \theta_3 + \theta_1^2 \theta_2 \theta_3^3 - \theta_1 \theta_2^4 \theta_4 - \theta_0^2 \theta_3^3 \theta_4 - \theta_0 \theta_1^2 \theta_3 \theta_4^2 + \theta_0 \theta_2 \theta_4^4 = 0, \\
k_3 &= \theta_0^4 \theta_1 \theta_3 - \theta_0 \theta_2 \theta_3^4 - \theta_0^2 \theta_1 \theta_2^2 \theta_4 + \theta_1^2 \theta_2 \theta_3^2 \theta_4 - \theta_0 \theta_1^2 \theta_4^3 + \theta_2^2 \theta_3 \theta_4^3 = 0, \\
k_4 &= -\theta_0^3 \theta_1 \theta_2^2 - \theta_0 \theta_1^2 \theta_2 \theta_3^2 + \theta_1^4 \theta_2 \theta_4 + \theta_0^3 \theta_3^2 \theta_4 + \theta_0 \theta_2^2 \theta_3 \theta_4^2 - \theta_1 \theta_3 \theta_4^4 = 0, \\
k_5 &= -\theta_0 \theta_2^4 \theta_3 + \theta_1^3 \theta_2 \theta_3^2 + \theta_0^4 \theta_2 \theta_4 - \theta_0^2 \theta_1 \theta_3^2 \theta_4 - \theta_0 \theta_1^3 \theta_4^2 + \theta_1 \theta_2^2 \theta_3 \theta_4^2 = 0.
\end{aligned}$$

Barcha  $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$  lar va  $1 \leq k \leq n$  bo'ladigan o'zaro tub sonlar juftligi uchun Feigin va Odesski  $Q_{n,k}(\mathcal{E}, \tau)$  graduirlangan algebra oilasini kiritgan.

**9-ta'rif.**  $\mathbb{C}\langle x_0, x_1, \dots, x_{n-1} \rangle$  erkin algebra ning quyidagi munosabatlar bo'yicha faktor algebrasiga *elliptik algebra* deyiladi va  $Q_{n,k}(\mathcal{E}, \tau)$  kabi yoziladi.

$$r_{ij} = \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(\tau) \theta_{j-i-r}(-\tau)} x_{j-r} x_{i+r}, (i, j) \in \mathbb{Z}^2.$$

$n=4, k=1$  deylik, to'rt o'lchamli elliptik algebra  $Q_{4,1}(\mathcal{E}, \tau)$   $\mathbb{C}\langle x_0, x_1, x_2, x_3 \rangle$  erkin algebra ning quyidagi munosabatlar bo'yicha olingan faktor algebrasi  $Q(a, b, c, d)$  ga izomorf

$$\begin{aligned}
r_1 &= a^2 x_1 x_0 - a b x_0 x_1 + a c x_2 x_3 + b c x_3 x_2, & r_2 &= a^2 x_3 x_2 - a b x_2 x_3 + a c x_0 x_1 + b c x_1 x_0, \\
r_3 &= d^2 (x_2 x_0 - x_0 x_2) + i b c (x_1^2 - x_3^2), & r_4 &= d^2 (x_1 x_3 - x_3 x_1) + i b c (x_0^2 - x_2^2), \\
r_5 &= a^2 x_0 x_3 - a b x_3 x_0 + a c x_1 x_2 + b c x_2 x_1, & r_6 &= a^2 x_2 x_1 - a b x_1 x_2 + a c x_3 x_0 + b c x_0 x_3.
\end{aligned}$$

bu yerda  $a, b, c, d$  lar kompleks sonlar bo'lib, 4-tartibli zeta funksiyalar orqali quyidagicha aniqlanadi.

$$a := \frac{i}{\theta_0(\tau) \theta_3(\tau)}, b := \frac{1}{\theta_0(\tau) \theta_1(\tau)}, c := \frac{1}{\theta_2(\tau) \theta_3(\tau)}, d := \frac{1}{\theta_0(\tau) \theta_2(\tau)}.$$

**3-xossa.** Yuqoridagi  $a, b, c, d \in \mathbb{C} - \{0\}$  sonlari quyidagi ayniyatni qanoatlantiradi.

$$a^2 b^3 c - b^3 c^3 - a^2 d^4 + b^2 d^4 = 0.$$

**9-teorema.** Faraz qilaylik,  $a, b, c, d \in \mathbb{C} - \{0\}$  sonlari yuqoridagi ayniyatni qanoatlantirsin va

$$\alpha = \left( \frac{a^2 - ab + ac + bc}{a^2 + ab - ac + bc} \right)^2, \quad \beta = -\frac{b^2 c^2}{d^4}, \quad \gamma = -\left( \frac{a^2 - ab - ac - bc}{a^2 + ab + ac - bc} \right)^2.$$

bo'lsin. U holda  $\alpha, \beta, \gamma$  lar  $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$  ayniyatni qanoatlantiradi va  $Q(a, b, c, d)$  va  $S(\alpha, \beta, \gamma)$  algebrai izomorf bo'ladi.

**10-teorema.**  $Q(a, b, c, d)$  algebra  $\mathcal{D}(\omega, 2)$  hosilaviy-faktor algebraga izomorf, bu yerda  $\omega \in \mathbb{C}\langle x_0, x_1, x_2, x_3 \rangle$  superpotential

$$\omega := k_1(r_1 r_2 + r_2 r_1) + k_2(r_3 r_4 + r_4 r_3) + k_3(r_5 r_6 + r_6 r_5)$$

kabi aniqlanadi va  $(k_1, k_2, k_3) \neq (0, 0, 0)$  bo'lib quyidagicha aniqlanadi.

$$k_1 = -k_3 = ib(b-c)(b+c)(a^2 bc + d^4), \quad k_2 = -ia(b-c)(b+c)(a^2 - bc)(a^2 + bc).$$

Xulosa qilamizki, ushbu algebra Koshul Kalabi-Yau algebra bo'ladi.

**10-ta'rif.** Besh o'lchamli elliptik algebra  $Q_{5,1}(\mathcal{E}, \tau)$  graduirlangan  $\mathbb{C}$ -algebra bo'lib, 1-darajali hosil qiluvchilari  $x_0, x_1, x_2, x_3, x_4$  lar quyidagi munosabatlarni qanoatlantiradi.

$$r_j = f_0 x_j^2 + f_1 x_{j-1} x_{j+1} + f_2 x_{j-2} x_{j+2} + f_3 x_{j-3} x_{j+3} + f_4 x_{j-4} x_{j+4},$$

$$s_j = \frac{f_2 f_3}{f_0} x_j^2 + \frac{f_3 f_4}{f_1} x_{j-1} x_{j+1} + \frac{f_0 f_4}{f_2} x_{j-2} x_{j+2} + \frac{f_0 f_1}{f_3} x_{j-3} x_{j+3} + \frac{f_1 f_2}{f_4} x_{j-4} x_{j+4}.$$

bu yerda  $f_j = \frac{\omega^j}{\theta_{j+1}(\tau)\theta_{j-1}(\tau)}$ ,  $\omega = e^{\frac{2\pi i}{5}}$  va  $j \in \mathbb{Z}_5$ .

$f_4 = f_1, f_3 = f_2$  holni qaraylik bu esa  $\theta_3(\tau) = \omega^3 \theta_2(\tau)$ ,  $\theta_4(\tau) = \omega^4 \theta_1(\tau)$  ekanligini anglatadi. U holda biz quyidagi  $A_5$  algebrani hosil qilamiz.

**11-ta'rif.**  $A_5$  algebra 1-darajali hosil qiluvchilari  $x_0, x_1, x_2, x_3, x_4$  va munosabatlari

$$\begin{aligned} R_1 &= ax_0^2 + b(x_2 x_3 + x_3 x_2), & S_1 &= cx_0^2 - b(x_1 x_4 + x_4 x_1), \\ R_2 &= ax_1^2 + b(x_3 x_4 + x_4 x_3), & S_2 &= cx_1^2 - b(x_2 x_0 + x_0 x_2), \\ R_3 &= ax_2^2 + b(x_4 x_0 + x_0 x_4), & S_3 &= cx_2^2 - b(x_3 x_1 + x_1 x_3), \\ R_4 &= ax_3^2 + b(x_0 x_1 + x_1 x_0), & S_4 &= cx_3^2 - b(x_4 x_2 + x_2 x_4), \\ R_5 &= ax_4^2 + b(x_1 x_2 + x_2 x_1), & S_5 &= cx_4^2 - b(x_0 x_3 + x_3 x_0). \end{aligned}$$

kabi bo'lgan graduirlangan  $\mathbb{C}$ -algebradir, bu yerda  $a = f_2^2(f_0^2 - f_1 f_2)$ ,  $b = f_0(f_2^3 - f_0 f_1^2)$  va  $c = f_0^3 f_1 - f_2^4$ .

**11-teorema.**  $A_5$  algebra  $\mathcal{D}(\Phi, 3)$  algebraga izomorf, bu yerda superpotensial  $\Phi \in \mathbb{C}\langle x_0, x_1, x_2, x_3, x_4 \rangle$  quyidagicha aniqlanadi.  $V = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4$  fazo uchun  $Q_1, Q_2, Q_3, Q_4, Q_5 \in V^{\otimes 4}$  larni

$$\begin{aligned}
Q_i = & \frac{a^3b + b^2c^2 + ac^3}{b} (R_i^2 - R_{i+3}S_{i+2} - S_{i+2}R_{i+3} - R_{i+2}S_{i+3} - S_{i+3}R_{i+2}) \\
& + \frac{a^2b^2 + a^3c - bc^3}{b} (S_i^2 - R_{i+1}S_{i+4} - S_{i+4}R_{i+1} - S_{i+1}R_{i+4} - R_{i+4}S_{i+1}) \\
& - \frac{b^4 + 3ab^2c + a^2c^2}{b} (R_iS_i + S_iR_i - S_{i+2}S_{i+3} - S_{i+3}S_{i+2} - R_{i+1}R_{i+4} - R_{i+4}R_{i+1}) \\
& + \frac{ab^3 + 2a^2bc + c^4}{b} (R_{i+2}R_{i+3} + R_{i+3}R_{i+2}) + \frac{a^4 - b^3c - 2abc^2}{b} (S_{i+4}S_{i+1} + S_{i+1}S_{i+4})
\end{aligned}$$

kabi aniqlaymiz, bu yerda indekslar  $\{i, i+1, i+2, i+3, i+4\} \in \{1, 2, 3, 4, 5\}$  5 modul bo'yicha olinadi. U holda quyidagi tenzor ushbu algebra uchun superpotensial bo'ladi:

$$\Phi := x_0Q_1 + x_1Q_2 + x_2Q_3 + x_3Q_4 + x_4Q_5$$

Xususan,  $A_5$  algebra Kalabi-Yau xossasiga ega.

### XULOSA

Dissertatsiya elliptik algebralarining Kalabi-Yau xossalarini o'rganishga bag'ishlangan.

Tadqiqotning asosiy natijalari quyidagilardan iborat:

1. Koshul algebralarining Kalabi-Yau xossasini qanoatlantirish kriteriyasi kvant Yang-Bakster tenglamasining yechimlari orqali aniqlangan;
2. Uch va to'rt o'lchamli Sklyanin algebralarining Kalabi-Yau xossasining boshqacha isboti berilgan;
3. Bir qancha Koshul algebralarining Kalabi-Yau xossasiga ega ekanligi kvant Yang-Bakster tenglamasining yechimlarini topish orqali isbotlangan;
4. To'rtinchi tartibli elliptik zeta funksiyalar orasidagi ayniyat topilgan. Bundan tashqari, beshinchi tartibli elliptik zeta funksiyalar o'nta ayniyatni qanoatlantirishi isbotlangan;
5. To'rt o'lchamli Sklyanin va elliptik algebralar orasida izomorfizm oshkor holda qurilgan;
6. To'rt o'lchamli elliptik algebralari uchun superpotensial aniqlangan;
7.  $\tau \in \mathcal{E}$  ning ba'zi qiymatlari uchun besh o'lchamli elliptik algebralarning superpotensialni aniqlanib, bu algebralarning Calabi-Yau xossasiga ega ekanligini ko'rsatilgan.

**SCIENTIFIC COUNCIL AWARDING OF THE SCIENTIFIC DEGREES  
DSc.02/30.12.2019.FM.86.01 INSTITUTE OF MATHEMATICS NAMED  
AFTER V.I. ROMANOVSKIY**

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**INSTITUTE OF MATHEMATICS**

**MIZOMOV INOMJON ERKIN O'G'LI**

**ON CALABI-YAU PROPERTIES OF ELLIPTIC ALGEBRAS**

**01.01.06-Algebra**

**ABSTRACT OF THESIS OF THE DOCTOR OF PHILOSOPHY (PhD)  
ON PHYSICAL AND MATHEMATICAL SCIENCES**

**TASHKENT-2025**

**The theme of dissertation of doctor of philosophy (PhD) on physical and mathematical sciences was registered at the Supreme Attestation Commission at the of Ministers of Higher education, Science and Innovations of the Republic of Uzbekistan under number B2023.4.PhD/FM960.**

Thesis has been prepared at Institute of Mathematics.

The abstract of the thesis is posted in three languages (Uzbek, English, Russian (summary)) on the website <http://kengash.mathinst.uz> and in the website of “ZiyoNet” Information and educational portal <http://www.ziyo.net>.

**Scientific supervisor:** **Eshmatov Farkhod Khasanovich**  
doctor of physical and mathematical sciences, senior researcher

**Official opponents:** **Omirov Bakhrom Abdazovich**  
doctor of physical and mathematical sciences, Professor

**Jumaniyozov Doston Erkaboy o‘g‘li**  
doctor of philosophy (PhD) on physical and mathematical sciences

**Leading organization:** **Andijan State University**

Defense will take place « 25 » November 2025 at 17:00 at the meeting of Scientific Council number DSc.02/30.12.2019.FM.86.01 at Institute of Mathematics named after V.I. Romanovskiy. (Address: University str. 9, Almazar area, Tashkent city, 100174, Uzbekistan, Ph.: (99871) 207-91-40, e-mail: [uzbmath@umail.uz](mailto:uzbmath@umail.uz), Website: [www.mathinst.uz](http://www.mathinst.uz))

Dissertation is possible to review in Information-resource center at Institute of Mathematics named after V.I.Romanovskiy (is registered №214). (Address: University str. 9, Almazar area, Tashkent city, 100174, Uzbekistan, Ph.: (99871)-207-91-40).

Abstract of the thesis sent out on « 10 » November 2025 year  
(Mailing report № 2 on « 10 » November 2025 year)

**U.A. Rozikov**  
Deputy Chairman of Scientific Council  
on award of scientific degrees,  
D.F.-M.S., Academician

**J.K. Adashev**  
Scientific secretary of Scientific Council  
on award of scientific degrees,  
D.F.-M.S., Senior researcher

**B.A. Omirov**  
Chairman of Scientific seminar under  
Scientific Council on award of scientific degrees,  
D.F.-M.S., Professor

## INTRODUCTION

**Actuality and demand of the theme of the dissertation.** Many scientific and practical types of research conducted around the world are mostly focused on the analysis of statistical physics and noncommutative geometry. Koszul Calabi-Yau algebras form an important class of graded algebras appearing naturally in several areas of modern mathematics, including noncommutative geometry, representation theory, algebraic topology, and mathematical physics. These algebras generalize the classical notion of Calabi-Yau varieties to a noncommutative setting, preserving a duality property in Hochschild (co)homology analogous to Serre duality in algebraic geometry. In this work, we are particularly interested in the subclass of Calabi-Yau algebras known as elliptic algebras-noncommutative analogues of projective surfaces associated with elliptic curves. These algebras not only exhibit rich algebraic structures but also arise in the context of integrable systems, quantum algebras, and the study of the quantum Yang-Baxter equation. One of the driving motivations for the study of Calabi-Yau geometry, both commutative and noncommutative, is its central role in mirror symmetry.

Nowadays, one of the urgent tasks is to investigate the Calabi-Yau property of elliptic algebras, as well as to find solutions to the quantum Yang-Baxter equation in the Koszul dual algebras of quadratic algebras in order to demonstrate that they possess the Calabi-Yau property. Despite the rapid development of the theory of elliptic algebras in the last two decades, the question of whether  $n$ -dimensional elliptic algebras possess the Calabi-Yau property remains insufficiently explored. Although it has been proved by the results of R. Bocklandt, T. Schedler, and M. Wemyss that all Koszul Calabi-Yau algebras were derived from superpotential, computing the superpotential through the Koszul dual coalgebra is considered a purposeful scientific research.

In our country much attention has been paid to mathematics, physics, geology and biological sciences, which have scientific and practical applications of fundamental sciences. In particular, special attention was paid to the development of fundamental research on associative and non-associative algebras. Within the frame of this fundamental research, significant results have been obtained in the study of the elliptic algebras and their properties. Conducting research at the level of international standards on the priority areas “Algebra and Functional Analysis” is the main task and direction of activity of mathematics<sup>2</sup>. To ensure the implementation of the task, it is important to describe the Calabi-Yau algebras for the application of scientific results in the relevant field of science.

The subject and object of this dissertation align with the tasks outlined in the Decrees of the President of the Republic of Uzbekistan UP-4947 dated February 7, 2017 “On the strategy of action for the further development of the Republic of Uzbekistan”, UP-2789 dated February 17, 2017 “On measures to further

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<sup>2</sup>Decree of the President of the Republic of Uzbekistan dated July 9, 2019 № PQ-4387 “On state support for the further development of mathematics education and subjects, as well as measures to fundamentally improve the activities of the Institute of Mathematics named after V.I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan”.

improvement of the activities of the Academy of Sciences, organization, management, and financing of research activities, PP-4387 dated July 9, 2019 “On measures to further development of mathematical education and science, and also root improvement of the activity of the Uzbekistan Academy of Sciences V.I.Romanovsky Institute of Mathematics”, UP-4708 dated May 7, 2020 “Quality of education in the field of mathematics”, PP-60 dated January 28, 2022, “On the Development Strategy of New Uzbekistan for 2022-2026”, and furthermore, regulations that encompass this research.

**Connection of research to priority directions of development of science and technologies of the Republic.** This study was performed in accordance with the priority areas of science and technology of Republic of Uzbekistan IV, “Mathematics, Mechanics and Computer Science.”

**The degree of scrutiny of the problem.** The theory of Calabi-Yau algebras has developed rapidly in the last two decades. V. Ginzburg’s construction of 3–Calabi-Yau differential graded algebras from quivers with potential, and R. Bocklandt’s classification theorem for graded 3–Calabi-Yau algebras, a foundational structure has been established. The works of M. Artin and J. Tate and M. Van den Bergh which classify 3–dimensional Artin-Schelter regular algebras, provide a classification framework. A major structural theorem due to M. Van den Bergh states that a Koszul algebra is Calabi-Yau algebra if and only if its Koszul dual algebra is a *symmetric Frobenius algebra*. This correspondence not only provides a practical criterion for verifying the CY condition but also links noncommutative duality to classical Frobenius pairings, guiding deformation theory and the study of derived equivalences.

In 1989, B.L. Feigen and A.V. Odesskii defined elliptic algebras  $Q_{n,k}(\mathcal{E}, \tau)$ . The elliptic algebras appear first in their manuscript. A lot is known about the algebras  $Q_{n,1}(\mathcal{E}, \tau)$ . J. Tate and M. Van den Bergh showed that  $Q_{n,1}(\mathcal{E}, \tau)$  is a noetherian domain having the same Hilbert series and the same homological properties as the polynomial ring on  $n$  variables. The Sklyanin algebras are well understood due to the works of M. Artin, J. Tate and M. Van den Bergh, S.P. Smith and J.T. Stafford. Moreover, A. Chirvasitu, R. Kanda and S. P. Smith showed that  $Q_{n,k}(\mathcal{E}, \tau)$  is Koszul algebra and Artin-Schelter regular of dimension  $n$ .

Parallel to these structural advances, a derived noncommutative Poisson viewpoint has emerged that interfaces Calabi-Yau algebras with representation-theoretic and topological invariants. Y. Berest, X. Chen, F. Eshmatov, A. Ramadoss developed noncommutative Poisson structures on (differential graded) algebras and showed that they induce (shifted) Poisson brackets on *derived representation schemes*; for Calabi-Yau algebras, these brackets interact naturally with Hochschild/cyclic (co)homology and necklace Lie algebras. Specializing to the Koszul CY context, X. Chen, A. Eshmatov, F. Eshmatov and S. Yang constructed a *derived noncommutative Poisson bracket* compatible with Koszul duality and functorial under derived Morita equivalences, providing homological tools tailored to Calabi-Yau detection in quadratic settings. From a quantization perspective, X. Chen, F. Eshmatov and W.L. Gan quantized the Lie bialgebra of string topology (the

Chas–Sullivan structure on equivariant homology of free loop spaces), offering a deformation–quantization model for Lie bialgebra-type structures that conceptually aligns with quantizing noncommutative Poisson brackets arising from Calabi-Yau algebras.

These developments show that elliptic algebras already satisfy many homological regularity properties expected of Calabi-Yau-type objects. The central challenge addressed in this dissertation is to pinpoint, by combining Van den Bergh duality with the derived noncommutative Poisson and quantization techniques above (together with detailed Hochschild/Ext computations), exactly which parameter regimes for Feigin–Odesskii algebras yield the Calabi-Yau property.

**Connection of the theme of the dissertation with the research works of higher education, where the dissertation is carried out.** The dissertation work was carried out within the framework of the research plans of the Institute of Mathematics named after V.I.Romanovsky.

**The aim of research work** is to investigate the Calabi-Yau properties of elliptic algebras, focusing on concrete examples like Sklyanin algebras and their generalizations and to find the necessity and sufficiency conditions for Koszul algebras to be Calabi-Yau algebras.

**Research problems:**

to study of the Calabi-Yau property of quadratic Koszul algebras using Van den Bergh’s result;

to prove that the three- and four-dimensional Sklyanin algebras and some Koszul algebras has the Calabi-Yau property;

to find some identities between elliptic theta functions of order four and five;

to construct an isomorphism between four-dimensional Sklyanin and elliptic algebras;

to find the superpotential for the five-dimensional elliptic algebras.

**The research object:** Quadratic and Koszul algebras, elliptic theta functions, elliptic algebras.

**The research subject:** The theory of Koszul algebras, theory of the quadratic algebras, the theory of the elliptic algebras.

**Research methods:** In the dissertation the methods of the theory of associative algebras, inductive method, as well as, the methods of Koszul algebras are applied.

**Scientific novelty of the research work** consists of the following:

the criteria for Koszul algebras to satisfies Calabi-Yau property is obtained;  
several Koszul algebras possess Calabi-Yau property is proved with this criteria;

the identities between elliptic theta functions of order four and five are found;  
an explicit isomorphism between four dimensional elliptic and Sklyanin algebras is constructed;

the superpotential for the four-dimensional elliptic algebras are identified;

the superpotential for the five-dimensional elliptic algebras for special values of  $\tau \in \mathcal{E}$  are identified;

**Practical results of the research.** The taken results and used methods in the dissertation can be taught as a graduate course for masters and doctoral students of higher education institutions. In addition, the results of the dissertation concerning the Calabi–Yau property of Koszul algebras can make it possible to verify hypotheses about the  $n$ –dimensional elliptic algebras possess the Calabi–Yau property.

**The reliability of the results of the study.** Our results have been obtained by using the invariant theory, as well as fundamental results of the theory of algebras. The proofs of obtained results are mathematically correct. Found the superpotentials for the four and five-dimensional elliptic algebras and their derivations can be checked by using standart procedures and the "NCAAlgebra" package of Mathematica 12. Moreover, the identities found between elliptic theta functions of orders four and five can be verified by using standart procedures of Mathematica 12.

**Scientific and practical significance of the research results.** The practical significance of this research lies in its applicability to both pure and applied mathematics. In pure algebra, it contributes to the understanding of noncommutative surfaces and their moduli. In mathematical physics, these algebras model quantum symmetries and underlie the algebraic structures appearing in integrable systems. The methods employed-homological algebra, noncommutative projective geometry, and invariant theory-are of independent interest and have broad relevance. The taken results and used methods in the dissertation can be taught as a graduate course for masters and doctoral students of higher education institutions.

**Implementation of the research results.** Based on the results obtained from Calabi-Yau properties of elliptic algebras:

The superpotentials found for four- and five-dimensional elliptic algebras were used to derive a blow-up formula for holomorphic Koszul-Brylinski homologies of compact holomorphic Poisson manifolds within the foreign research project named “Certain problems of noncommutative algebraic geometry and derived algebraic geometry” with number NSFC12271377, (Reference of Sichuan University dated September 5, 2025, People’s Republic of China). Application of the obtained results was possible to compute the holomorphic Koszul-Brylinski homologies for del Pezzo surfaces and two complex nilmanifolds with holomorphic Poisson structures.

The solutions of the quantum Yang-Baxter equation for the Koszul dual algebras of quadratic algebras in the criterion necessary for Koszul algebras to possess the Calabi-Yau property were used in the fundamental project No. F-FA-2021-423 on the topic “Automorphisms of operator algebras, classification of infinite-dimensional nonassociative algebras and superalgebras” to identify Rota–Baxter operators on Lie algebras (the reference of number 2/337 of the Institute of Mathematics dated September 2, 2025). The application of this scientific result enabled the classification of Rota–Baxter and anti-Rota–Baxter operators on low-dimensional Lie algebras.

**Approbation of the research results.** The main results of the research have been discussed at 2 international and 5 national scientific conferences.

**Publications of the research results.** On the topic of the dissertation, 12 scientific papers were published, 5 of which are included in the list of scientific publications proposed by the Higher Attestation Commission of the Republic of Uzbekistan for the defense of theses of the Doctor of Philosophy, including 2 of them published in foreign journals and 3 in national scientific journals and 7 abstracts.

**The structure and volume of the dissertation.** This dissertation consists of an introduction, three chapters divided into nine paragraphs, conclusion and bibliography. The volume of the dissertation is 73 pages.

## THE MAIN CONTENT OF THE THESIS

The **introduction** of the thesis includes the motivation of the research, the relevance of the research to the priorities of science and technology, the review of foreign research on the topic, the degree of scrutiny of the problem, the aim, research problems, object and subject of research, scientific novelty and practical results, theoretical and practical significance of the results obtained, the statement of research results, published works and information on the structure of the thesis.

In the first chapter of the thesis, titled “**Preliminaries and results**” we present some necessary definitions and preliminary results which are used throughout the dissertation. We first review the notions of associative algebras, quadratic algebras, coassociative coalgebras and Koszul duality for the quadratic associative algebras. Then we deal with Koszul and Calabi-Yau algebras which are going to play an essential role in this dissertation. The last section of the chapter covers basic facts about the quantum Yang-Baxter equation.

Let  $\mathbb{K}$  be a field. A (*chain*) *complex* of  $\mathbb{K}$ -vector spaces (analogously of modules over an algebra  $A$ )  $(V, d)$  is a sequence of vector spaces and  $\mathbb{K}$ -linear maps

$$(V, d) = \cdots \rightarrow V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \rightarrow \cdots, \quad n \in \mathbb{Z}$$

with  $d_n \circ d_{n+1} = 0$  for all  $n \in \mathbb{Z}$ . A complex  $(V, d)$  is *exact* or *acyclic* if its  $n^{\text{th}}$  *homology group*

$$H_n(V, d) = \ker d_n / \text{Im } d_{n+1}$$

vanishes.

Let  $V$  be a finite-dimensional vector space over  $\mathbb{K}$ .

**Definition 1.** The *tensor algebra* over the vector space  $V$  is the tensor module

$$T(V) := \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

equipped with the concatenation product  $T(V) \otimes T(V) \rightarrow T(V)$  given by

$$v_1 \cdots v_p \otimes v_{p+1} \cdots v_{p+q} \mapsto v_1 \cdots v_p v_{p+1} \cdots v_{p+q}.$$

This operation is clearly associative and  $1$  is taken as a unit.

**Definition 2.** A *quadratic algebra* is a quotient algebra of a tensor algebra  $T(V)$  of a finite-dimensional vector space  $V$  by the two-sided ideal generated in  $T(V)$  by a vector subspace  $R \subset V \otimes V$  and denoting by  $A(V, R) = T(V) / (R)$ .

**Definition 3.** The *Koszul dual algebra* of a quadratic algebra  $A = A(V, R)$  is  $A' = T(V^*) / (R^\perp)$ , where

$$R^\perp = \{r \in V^* \otimes V^* \simeq (V \otimes V)^* \mid r(R) = 0\}.$$

**Definition 4.** The *Koszul dual coalgebra* of the quadratic algebra  $A = A(V, R)$  is the coalgebra

$$A^i = \mathbb{K} \oplus sV \oplus s^2R \oplus \dots \oplus \left( \bigcap_{i+2+j=n} sV^{\otimes i} \otimes s^2R \otimes sV^{\otimes j} \right) \oplus \dots,$$

where  $s^2R$  is the image of  $R$  in  $(sV)^{\otimes 2}$  under the map  $V^{\otimes 2} \rightarrow (sV)^{\otimes 2}, vw \mapsto svsw$ .

Choose a set of basis  $\{e_i\}$  for  $V$ , and let  $\{e_i^*\}$  be their duals in  $V^*$  and shift grading of  $V^*$  down by one, denoted by  $s^{-1}V^*$ . There is a chain complex associated to  $A$ , called the *Koszul complex*:

$$\dots \xrightarrow{\delta} A \otimes A_{i+1}^i \xrightarrow{\delta} A \otimes A_i^i \xrightarrow{\delta} \dots \xrightarrow{\delta} A \otimes A_0^i \xrightarrow{\delta} \mathbb{K}, \quad (1)$$

where for any  $r \otimes f \in A \otimes A^i, \delta(r \otimes f) = \sum_i e_i r \otimes s^{-1}e_i^* f$ .

**Definition 5.** A quadratic algebra  $A = TV / (R)$  is called *Koszul* if the Koszul chain complex (1) is acyclic.

**Definition 6.** A module  $P \in A - \text{Mod}$  is *projective* if for any epimorphism  $M \rightarrow N$  in  $A - \text{Mod}$  the induced morphism

$$\text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$$

is also surjective.

*Projective resolution* of a module  $M \in A - \text{Mod}$  is an exact sequence

$$P := \dots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

where  $P_i$  - projective  $A$  - module, for all  $i$ .

Let  $P_M := \dots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow 0$  be a truncated complex of projective resolution of  $M \in A - \text{Mod}$ . By applying the functor  $\text{Hom}_A(-, N)$ , we can obtain the following complex:

$$\text{Hom}_A(P_M, N) = 0 \rightarrow \text{Hom}_A(P_0, N) \xrightarrow{d_1^*} \text{Hom}_A(P_1, N) \rightarrow \dots$$

the  $n^{\text{th}}$  cohomology of  $\text{Hom}_A(P_M, N)$  is called

$$\text{Ext}_A^n(M, N) := H^n(\text{Hom}_A(P_M, N)) = \ker d_{n+1}^* / \text{Im } d_n^*.$$

where  $d^*(\varphi) = d \circ \varphi$ .

Let  $A$  be an associative algebra. Then the space  $A \otimes A$  has two  $A$  - bimodule structures, called *outer* and *inner* structures. They are given respectively as follows:

$$a(b_1 \otimes b_2)c = ab_1 \otimes b_2c, \quad a(b_1 \otimes b_2)c = b_1c \otimes ab_2.$$

Let  $A - \text{Bimod}$  be the category of  $A$  bimodules. Then, by viewing  $A \otimes A$  as a bimodule via outer structure, we define a duality functor

$$\text{Hom}_{A - \text{Bimod}}(-, A \otimes A) : A - \text{Bimod} \rightarrow A - \text{Bimod}, \quad M \mapsto \text{Hom}_{A - \text{Bimod}}(M, A \otimes A).$$

A bimodule structure on  $\text{Hom}_{A - \text{Bimod}}(M, A \otimes A)$  is induced from the inner bimodule structure on  $A \otimes A$ .

**Definition 7.**  $A$  is called a  $d$  – Calabi-Yau algebra if it satisfies the following two conditions. First,  $A$  is homologically smooth, i.e.,  $A$  is viewed as an  $A^e$  – bimodule with a finitely-generated projective resolution of finite length. Second, there is an isomorphism of  $A$  – bimodules

$$\text{Ext}_{A\text{-Bimod}}^k(A, A \otimes A) \cong \begin{cases} A, & \text{if } k = d \\ 0, & \text{if } k \neq d \end{cases}.$$

There is a large class of Calabi-Yau algebras which are derived from *superpotentials*.

Let  $\mathbb{C}\langle x_1, \dots, x_n \rangle$  be the free algebra with  $n$  generators. Then  $\mathbb{C}\langle x_1, \dots, x_n \rangle$  has the natural grading by the length of monomials, which denote by  $|a|$ . For a monomial  $a = a_1 a_2 \dots a_m \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ , where each  $a_i \in \{x_1, \dots, x_n\}$ , we define the *noncommutative partial derivative* with respect to  $x_k$  ( $k = 1, \dots, n$ )

$$\partial_{x_k}(a) := \sum_{\{i \in [1, m] | a_i = x_k\}} a_{i+1} a_{i+2} \dots a_m a_1 \dots a_i.$$

By linearity, it can be extended to a map  $\partial_{x_k} : \mathbb{C}\langle x_1, \dots, x_n \rangle \rightarrow \mathbb{C}\langle x_1, \dots, x_n \rangle$ . Next, taking composition of such maps, we can define higher order derivatives. More precisely, for  $p = x_{i_1} \dots x_{i_s}$ , we set  $\partial_p := \partial_{x_{i_1}} \circ \dots \circ \partial_{x_{i_s}}$ . Then, for  $\omega \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ , we can define the *derivation-quotient algebra* of order  $k$  as

$$\mathcal{D}(\omega, k) := \frac{\mathbb{C}\langle x_1, \dots, x_n \rangle}{(\partial_p \omega | p | = k)}$$

where  $p$  runs over all monomials of  $\mathbb{C}\langle x_1, \dots, x_n \rangle$  of length  $k$ .

$\omega \in \mathbb{C}\langle x_1, \dots, x_n \rangle$  is called *superpotential* if it is the sum of elements of the following form

$$\sum_{i=1}^n (-1)^{(i-1)(n+1)} a_i a_{i+1} \dots a_n a_1 a_2 \dots a_{i-1}, \quad a_i \in \{x_1, \dots, x_n\}. \quad (2)$$

Let  $A = \mathcal{D}(\omega, k)$  be a derivation-quotient algebra of the superpotential  $\omega$  of the form (2). We assume that  $|\omega| = k + 2$ , so that  $A$  is quadratic.

Consider the complex

$$\mathcal{V}^* = 0 \rightarrow A \otimes V_{|\omega|} \otimes A \xrightarrow{d_{|\omega|}} A \otimes V_{|\omega|-1} \otimes A \xrightarrow{d_{|\omega|-1}} \dots \xrightarrow{d_1} A \otimes V_0 \otimes A \rightarrow 0, \quad (3)$$

where,  $V_{n-k} := \langle \partial_p \omega | p | = k \rangle$  and for  $v_1, \dots, v_i \in V, a, a' \in A$ ,

$$\begin{aligned} d_i &= \varepsilon_i (\text{split}_L + (-1)^i \text{split}_R) |_{A \otimes V_i \otimes A}, \\ \text{split}_L(a \otimes v_1 v_2 \dots v_i \otimes a') &= a v_1 \otimes v_2 \dots v_i \otimes a', \\ \text{split}_R(a \otimes v_1 v_2 \dots v_i \otimes a') &= a \otimes v_1 \dots v_{i-1} \otimes v_i a', \\ \varepsilon_i &:= \begin{cases} (-1)^{i(|\omega|-i)}, & \text{if } i < \frac{|\omega|+1}{2}, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to check that the above yields a complex, i.e.,  $d_i \circ d_{i+1} = 0$ . Moreover, the terms, aside from  $A$  itself, are projective bimodules, and the maps are  $A$  – bimodule maps.

In 1997 R. Bocklandt, T. Schedler, and M. Wemyss proved the following theorem.

**Theorem 1.** The algebra  $T(V) / (R)$  is Koszul and Calabi–Yau if and only if it is of the form  $\mathcal{D}(\omega, k)$  and the corresponding complex (3) is exact in positive degree and  $H^0(\mathcal{V}) = A$ .

Let  $V$  be a  $\mathbb{C}$  – vector space, and let  $R \in \text{End}(V \otimes V)$ . Then we say  $R$  is a solution of the *quantum Yang-Baxter equation* if

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12},$$

where  $R^{12} = R \otimes \text{id}, R^{23} = \text{id} \otimes R$ , etc., are endomorphisms of  $V^{\otimes 3}$ . If

$R = \sum_{i=1}^n a_i \otimes b_i \in A \otimes A$ , then  $R$  acts on  $A \otimes A$  via left multiplication and

$R^{12} = \sum_{i=1}^n a_i \otimes b_i \otimes 1$  etc. Then we can write the quantum Yang-Baxter equation in the form

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}.$$

The second chapter of the dissertation entitled “**Calabi-Yau algebras and the quantum Yang-Baxter equation**” studied a new characterization of Koszul Calabi-Yau algebras and proved that several Koszul algebras satisfy Calabi-Yau property.

Let  $A$  be a finite-dimensional  $\mathbb{K}$  – algebra. The  $\mathbb{K}$  – linear dual of  $A$ ,  $A^* = \text{Hom}_{\mathbb{K}}(A, \mathbb{K})$ , is a left  $A$  – module with scalar multiplication . given by

$$a.f(x) = f(xa), \quad a, x \in A, \quad f \in A^*.$$

**Definition 8.** The finite-dimensional  $\mathbb{K}$  – algebra  $A$  is called *Frobenius* if  $A \cong A^*$  as left  $A$  – modules.

It is equivalent to existence of a non-degenerate bilinear form  $\langle -, - \rangle : A \times A \rightarrow \mathbb{C}$  satisfying  $\langle ab, c \rangle = \langle a, bc \rangle$  for all  $a, b, c \in A$ .

The following result, due to Beidar, Fong, and Stolin describes Frobenius algebras using solutions of the quantum Yang-Baxter equation.

**Theorem 2.** Let  $Q = \sum_{i=1}^n e_i \otimes e^i$ . Then the followings are equivalent:

1.  $Q^{12} Q^{13} = Q^{23} Q^{12}$  and  $Q^{12} Q^{23} = Q^{23} Q^{13}$ ,
2.  $Q^{12} Q^{13} = Q^{13} Q^{23}$  and  $Q^{12} Q^{23} = Q^{13} Q^{12}$ ,
3.  $Q^{23} Q^{12} = Q^{13} Q^{23}$  and  $Q^{23} Q^{13} = Q^{13} Q^{12}$ ,
4.  $(a \otimes b)Q = Q(b \otimes a)$  for all  $a \in A$ ,
5.  $A$  is a symmetric Frobenius algebra with Frobenius homomorphism  $\phi : A \rightarrow \mathbb{C}$  such that  $\phi(e_i e^j) = \delta_{ij}$  for all  $i, j = 1, 2, \dots, n$ .

6. The element  $Q$  is a solution of the quantum Yang-Baxter equation and  $\tau_{A,A} \circ Q = Q$ .

Moreover, M. Van den Bergh's work the author proved the following theorem.

**Theorem 3.** Suppose that  $A$  is a Koszul algebra. Then  $A$  is  $d$ -Calabi-Yau if and only if  $A^!$  is a graded symmetric Frobenius algebra.

The following theorem proved in the Section 2.1 of the dissertation is the graded version of Theorem 2.

**Theorem 4.** Let  $A$  be a Koszul algebra. Then  $A$  is a Calabi-Yau algebra if and only if its Koszul dual algebra  $A^!$  has homogeneous bases  $e_1, \dots, e_n$  and  $e^1, \dots, e^n$  so that

the element  $Q = \sum_{i=1}^n e_i \otimes e^i$  satisfies one of the following equivalent conditions:

1.  $(a \otimes b)Q = (-1)^{|a||b| + (|a|+|b|)(|Q|+1)} Q(b \otimes a)$  for all  $a, b \in A^!$ ,
2.  $Q^{12} \left( (-1)^{|e_j|+|Q|} Q^{13} \right) = Q^{23} Q^{12}$  and  $Q^{12} Q^{23} = \left( (-1)^{|e_j|+|Q|} Q^{23} \right) Q^{13}$ .

**Theorem 5.** Let  $A$  be a  $\mathbb{K}$ -algebra and  $Q = e_i \otimes e^i \in A^! \otimes A^!$ . Suppose that  $(a \otimes b)Q = (-1)^{|a||b| + (|a|+|b|)(|Q|+1)} Q(b \otimes a)$  for all  $a, b \in A^!$ . Then:

1.  $Q^{12} Q^{13} = \left( (-1)^{|e_j|+|Q|} Q^{23} \right) Q^{12}$  and  $Q^{12} Q^{23} = \left( (-1)^{|e_j|+|Q|} Q^{13} \right) Q^{12}$ ,
2.  $Q^{12} Q^{13} Q^{23} = \left( (-1)^{|e_j|} Q^{23} \right) \left( (-1)^{|e_j|} Q^{13} \right) Q^{12}$ .

Let  $a, b, c \in \mathbb{C}$  satisfy the following two conditions:

- 1)  $[a:b:c] \in \mathbb{P}_{\mathbb{C}}^2 \setminus D$ , where

$$D = \{[1:0:0], [0:1:0], [0:0:1]\} \cup \{[a:b:c] \mid a^3 = b^3 = c^3 = 1\}.$$

- 2)  $abc \neq 0$  and  $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ .

The *three-dimensional Sklyanin algebra*  $S(a, b, c)$  can be realized as the graded  $\mathbb{C}$ -algebra with generators  $x, y, z$  of degree one, and the relations

$$ax^2 + byz + czy = 0, \quad ay^2 + bzx + cxz = 0, \quad az^2 + bxy + cyx = 0.$$

The *four-dimensional Sklyanin algebra*  $S(\alpha, \beta, \gamma)$  is the graded  $\mathbb{C}$ -algebra with generators  $x_0, x_1, x_2, x_3$  of degree one, subject to the following relations

$$\begin{aligned} x_0x_1 - x_1x_0 - \alpha(x_2x_3 + x_3x_2) &= 0, & x_0x_1 + x_1x_0 - (x_2x_3 - x_3x_2) &= 0, \\ x_0x_2 - x_2x_0 - \beta(x_3x_1 + x_1x_3) &= 0, & x_0x_2 + x_2x_0 - (x_3x_1 - x_1x_3) &= 0, \\ x_0x_3 - x_3x_0 - \gamma(x_1x_2 + x_2x_1) &= 0, & x_0x_3 + x_3x_0 - (x_1x_2 - x_2x_1) &= 0. \end{aligned}$$

where  $\alpha, \beta$  and  $\gamma$  are complex numbers such that  $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ .

**Theorem 6.** The three and four-dimensional Sklyanin algebras  $S(a, b, c)$  and  $S(\alpha, \beta, \gamma)$  are Koszul Calabi-Yau algebras.

The *skew polynomial ring*  $A_1 := \mathbb{C}_q[x_0, x_1, x_2, x_3]$  is the graded  $\mathbb{C}$ -algebra generated by  $x_0, x_1, x_2, x_3$  subject to the relations

$$x_i x_{i+1} = \begin{pmatrix} q_{i+3} \\ q_{i+2} \end{pmatrix}^{(-1)^i} x_{i+1} x_i, \quad x_i x_{i+2} = \begin{pmatrix} q_{i+1} \\ q_{i+3} \end{pmatrix}^{(-1)^i} x_{i+2} x_i$$

where  $q_0, \dots, q_3 \in \mathbb{C}^*$  satisfy  $\prod_{i=0}^3 q_i = 1$ , and again the indices are taken modulo four.

**Theorem 7.** The algebra  $A_1$  is Koszul Calabi-Yau algebra.

The central extension of three-dimensional Sklyanin algebra  $B$  is generated by  $x_0, x_1, x_2, x_3$  with the following relations

$$\begin{aligned} [x_0, x_1] &= 0, & [x_2, x_1] &= \lambda(x_2x_1 + x_1x_2) - \nu x_3^2 - \sum_{j=1}^3 b_{1j}x_jx_0, \\ [x_0, x_2] &= 0, & [x_3, x_2] &= \lambda(x_3x_2 + x_2x_3) - \nu x_1^2 - \sum_{j=1}^3 b_{2j}x_jx_0, \\ [x_0, x_3] &= 0, & [x_1, x_3] &= \lambda(x_1x_3 + x_3x_1) - \nu x_2^2 - \sum_{j=1}^3 b_{3j}x_jx_0. \end{aligned}$$

for constants  $\lambda, \nu \in \mathbb{C}$  and a symmetric matrix  $b = (b_{ij}) \in \mathbb{C}^{3 \times 3}$ .

**Theorem 8.** The algebra  $B$  is Koszul Calabi-Yau algebra.

In the third chapter of the dissertation entitled “**Calabi-Yau property of elliptic algebras**” the elliptic theta functions and elliptic algebras are considered. We also provide that the four dimensional Sklyanin and elliptic algebras are isomorphic. Moreover, we prove that the four-dimensional and five-dimensional elliptic algebras possess the Calabi-Yau property.

Let  $\Lambda$  be an integral lattice generated by 1 and  $\eta \in \mathbb{C}$ , where  $\text{Im}\eta > 0$ , and let  $\mathcal{E} := \mathbb{C} / \Lambda$  be the corresponding elliptic curve. In Feigen and Odesskii’s paper they denoted by  $\Theta_n(\Lambda)$  the space of the entire functions of one variable satisfying the quasi-periodicity conditions:

$$f(z+1) = f(z), \quad f(z+\eta) = -e^{-2\pi iz} f(z).$$

The functions in  $\Theta_n(\Lambda)$  are called *theta functions of order  $n$*  with the respect to the lattice  $\Lambda$ . It is easy to check that the function

$$\theta(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi i \left( nz + \frac{n(n-1)}{2} \eta \right)}$$

form a basis for  $\Theta_1(\Lambda)$ .

For  $\alpha \in \mathbb{Z} / n\mathbb{Z}$ , we define

$$\theta_\alpha(z) := e^{2\pi i(\alpha z + [\alpha])} \prod_{m=0}^{n-1} \theta\left(z + \frac{m}{n} + \frac{\alpha}{n} \eta\right), \quad [\alpha] := \frac{\alpha(\alpha - n)}{2n} \eta + \frac{\alpha}{2n}.$$

Then  $\{\theta_0, \theta_1, \dots, \theta_{n-1}\}$  form a basis for  $\Theta_n(\Lambda)$ . Let  $n = 4$ .

**Proposition 1.** For any  $\tau \in \mathbb{C} - \frac{1}{4}\Lambda$  the above functions satisfy the following identity

$$\frac{\theta_0^2(\tau) + \theta_2^2(\tau)}{\theta_1(\tau)\theta_3(\tau)} = \frac{\theta_1^2(\tau) + \theta_3^2(\tau)}{\theta_0(\tau)\theta_2(\tau)}.$$

The elliptic theta functions of order five are

$$\begin{aligned}
\theta_0(z) &= \theta(z)\theta\left(z + \frac{1}{5}\right)\theta\left(z + \frac{2}{5}\right)\theta\left(z + \frac{3}{5}\right)\theta\left(z + \frac{4}{5}\right), \\
\theta_1(z) &= \theta\left(z + \frac{1}{5}\eta\right)\theta\left(z + \frac{1}{5} + \frac{1}{5}\eta\right)\cdots\theta\left(z + \frac{4}{5} + \frac{1}{5}\eta\right)e^{2\pi i\left(z + \frac{1}{10} - \frac{2}{5}\eta\right)}, \\
\theta_2(z) &= \theta\left(z + \frac{2}{5}\eta\right)\theta\left(z + \frac{1}{5} + \frac{2}{5}\eta\right)\cdots\theta\left(z + \frac{4}{5} + \frac{1}{2}\eta\right)e^{2\pi i\left(2z + \frac{1}{5} - \frac{3}{5}\eta\right)}, \\
\theta_3(z) &= \theta\left(z + \frac{3}{5}\eta\right)\theta\left(z + \frac{1}{5} + \frac{3}{5}\eta\right)\cdots\theta\left(z + \frac{4}{5} + \frac{3}{5}\eta\right)e^{2\pi i\left(3z + \frac{3}{10} - \frac{3}{5}\eta\right)}, \\
\theta_4(z) &= \theta\left(z + \frac{4}{5}\eta\right)\theta\left(z + \frac{1}{5} + \frac{4}{5}\eta\right)\cdots\theta\left(z + \frac{4}{5} + \frac{4}{5}\eta\right)e^{2\pi i\left(4z + \frac{2}{5} - \frac{2}{5}\eta\right)}.
\end{aligned}$$

**Proposition 2.** For all  $\tau \in \mathbb{C} - \frac{1}{5}\Lambda$  the above elliptic theta functions of order five satisfy the following ten identities:

$$\begin{aligned}
m_1 &= \theta_0^2\theta_1^3\theta_3 + \theta_0\theta_1\theta_2^2\theta_3^2 - \theta_1^3\theta_2^2\theta_4 - \theta_0\theta_3^4\theta_4 - \theta_0^2\theta_1\theta_2\theta_4^2 + \theta_2\theta_3\theta_4^4 = 0, \\
m_2 &= -\theta_0^2\theta_1^2\theta_2\theta_3 - \theta_0\theta_2^3\theta_3^2 + \theta_1^2\theta_2^3\theta_4 + \theta_0^4\theta_3\theta_4 + \theta_1\theta_2\theta_3^2\theta_4^2 - \theta_0\theta_1\theta_4^4 = 0, \\
m_3 &= \theta_0^4\theta_1\theta_2 - \theta_0\theta_2^2\theta_3^3 - \theta_0\theta_1^4\theta_4 + \theta_1^2\theta_2^2\theta_3\theta_4 - \theta_0^2\theta_2\theta_3\theta_4^2 + \theta_1\theta_3^3\theta_4^2 = 0, \\
m_4 &= \theta_0\theta_1\theta_2^4 - \theta_1^4\theta_2\theta_3 + \theta_0^2\theta_1^2\theta_3\theta_4 - \theta_0\theta_2^2\theta_3^2\theta_4 - \theta_0^2\theta_2\theta_4^3 + \theta_1\theta_3^2\theta_4^3 = 0, \\
m_5 &= \theta_0^3\theta_1^2\theta_3 - \theta_1\theta_2\theta_3^4 - \theta_0\theta_1^2\theta_2^2\theta_4 + \theta_2^4\theta_3\theta_4 - \theta_0^3\theta_2\theta_4^2 + \theta_0\theta_1\theta_3^2\theta_4^2 = 0, \\
k_1 &= \theta_0^2\theta_1\theta_2^3 - \theta_0\theta_1^4\theta_3 - \theta_0^2\theta_2\theta_3^2\theta_4 + \theta_1\theta_3^4\theta_4 + \theta_0\theta_1^2\theta_2\theta_4^2 - \theta_2^3\theta_3\theta_4^2 = 0, \\
k_2 &= \theta_0^2\theta_1\theta_2^2\theta_3 + \theta_1^2\theta_2\theta_3^3 - \theta_1\theta_2^4\theta_4 - \theta_0^2\theta_3^3\theta_4 - \theta_0\theta_1^2\theta_3\theta_4^2 + \theta_0\theta_2\theta_4^4 = 0, \\
k_3 &= \theta_0^4\theta_1\theta_3 - \theta_0\theta_2\theta_3^4 - \theta_0^2\theta_1\theta_2^2\theta_4 + \theta_1^2\theta_2\theta_3^2\theta_4 - \theta_0\theta_1^2\theta_4^3 + \theta_2^2\theta_3\theta_4^3 = 0, \\
k_4 &= -\theta_0^3\theta_1\theta_2^2 - \theta_0\theta_1^2\theta_2\theta_3^2 + \theta_1^4\theta_2\theta_4 + \theta_0^3\theta_3^2\theta_4 + \theta_0\theta_2^2\theta_3\theta_4^2 - \theta_1\theta_3\theta_4^4 = 0, \\
k_5 &= -\theta_0\theta_2^4\theta_3 + \theta_1^3\theta_2\theta_3^2 + \theta_0^4\theta_2\theta_4 - \theta_0^2\theta_1\theta_3^2\theta_4 - \theta_0\theta_1^3\theta_4^2 + \theta_1\theta_2^2\theta_3\theta_4^2 = 0.
\end{aligned}$$

For all  $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$  and for a pair of coprime numbers  $1 \leq k \leq n$ , Feigin and Odesski have introduced a family of graded algebras  $Q_{n,k}(\mathcal{E}, \tau)$ .

**Definition 9.** The *elliptic algebras*  $Q_{n,k}(\mathcal{E}, \tau)$  are the quotient of the free algebra  $\mathbb{C}\langle x_0, x_1, \dots, x_{n-1} \rangle$  by the following relations

$$r_{ij} = \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(\tau)\theta_{j-i-r}(-\tau)} x_{j-r} x_{i+r}, (i, j) \in \mathbb{Z}^2.$$

For  $n=4, k=1$ , the *four dimensional elliptic algebra*  $Q_{4,1}(\mathcal{E}, \tau)$  is isomorphic to the algebra  $Q(a, b, c, d)$ , which is the quotient of the free algebra  $\mathbb{C}\langle x_0, x_1, x_2, x_3 \rangle$  subject to relations:

$$\begin{aligned}
r_1 &= a^2 x_1 x_0 - abx_0 x_1 + acx_2 x_3 + bcx_3 x_2, & r_2 &= a^2 x_3 x_2 - abx_2 x_3 + acx_0 x_1 + bcx_1 x_0, \\
r_3 &= d^2 (x_2 x_0 - x_0 x_2) + ibc (x_1^2 - x_3^2), & r_4 &= d^2 (x_1 x_3 - x_3 x_1) + ibc (x_0^2 - x_2^2), \\
r_5 &= a^2 x_0 x_3 - abx_3 x_0 + acx_1 x_2 + bcx_2 x_1, & r_6 &= a^2 x_2 x_1 - abx_1 x_2 + acx_3 x_0 + bcx_0 x_3.
\end{aligned}$$

where  $a, b, c, d$  are complex numbers defined in terms of theta functions of order 4 as follows

$$a := \frac{i}{\theta_0(\tau)\theta_3(\tau)}, b := \frac{1}{\theta_0(\tau)\theta_1(\tau)}, c := \frac{1}{\theta_2(\tau)\theta_3(\tau)}, d := \frac{1}{\theta_0(\tau)\theta_2(\tau)}.$$

**Proposition 3.** The above numbers  $a, b, c, d \in \mathbb{C} - \{0\}$  satisfy the following identity

$$a^2 b^3 c - b^3 c^3 - a^2 d^4 + b^2 d^4 = 0.$$

**Theorem 9.** Let  $a, b, c, d \in \mathbb{C} - \{0\}$  satisfy the above identity, and let

$$\alpha = \left( \frac{a^2 - ab + ac + bc}{a^2 + ab - ac + bc} \right)^2, \quad \beta = -\frac{b^2 c^2}{d^4}, \quad \gamma = -\left( \frac{a^2 - ab - ac - bc}{a^2 + ab + ac - bc} \right)^2.$$

Then  $\alpha, \beta, \gamma$  satisfy  $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$  and the algebras  $Q(a, b, c, d)$  and  $S(\alpha, \beta, \gamma)$  are isomorphic.

**Theorem 10.** The algebra  $Q(a, b, c, d)$  is isomorphic to a derivation-quotient algebra  $\mathcal{D}(\omega, 2)$ , where  $\omega \in \mathbb{C}\langle x_0, x_1, x_2, x_3 \rangle$  is a superpotential

$$\omega := k_1(r_1 r_2 + r_2 r_1) + k_2(r_3 r_4 + r_4 r_3) + k_3(r_5 r_6 + r_6 r_5)$$

where  $(k_1, k_2, k_3) \neq (0, 0, 0)$  are defined as follows

$$k_1 = -k_3 = ib(b-c)(b+c)(a^2 bc + d^4), \quad k_2 = -ia(b-c)(b+c)(a^2 - bc)(a^2 + bc).$$

We observe that these algebras are Koszul Calabi-Yau algebra.

**Definition 10.** The five dimensional elliptic algebra  $Q_{5,1}(\mathcal{E}, \tau)$  is graded  $\mathbb{C}$ -algebra with generators  $x_0, x_1, x_2, x_3, x_4$  of degree one, subject to the following ten relations

$$\begin{aligned}
r_j &= f_0 x_j^2 + f_1 x_{j-1} x_{j+1} + f_2 x_{j-2} x_{j+2} + f_3 x_{j-3} x_{j+3} + f_4 x_{j-4} x_{j+4}, \\
s_j &= \frac{f_2 f_3}{f_0} x_j^2 + \frac{f_3 f_4}{f_1} x_{j-1} x_{j+1} + \frac{f_0 f_4}{f_2} x_{j-2} x_{j+2} + \frac{f_0 f_1}{f_3} x_{j-3} x_{j+3} + \frac{f_1 f_2}{f_4} x_{j-4} x_{j+4}.
\end{aligned}$$

where  $f_j = \frac{\omega^j}{\theta_{j+1}(\tau)\theta_{j-1}(\tau)}$ ,  $\omega = e^{\frac{2\pi i}{5}}$  and  $j \in \mathbb{Z}_5$

Let us consider the case of  $f_4 = f_1, f_3 = f_2$  which means that  $\theta_3(\tau) = \omega^3 \theta_2(\tau)$ ,  $\theta_4(\tau) = \omega^4 \theta_1(\tau)$ . Then we will obtain the algebra  $A_5$ .

**Definition 11.** The algebra  $A_5$  is a graded  $\mathbb{C}$ -algebra with generators  $x_0, x_1, x_2, x_3, x_4$  of degree one, with relations of the form:

$$\begin{aligned}
R_1 &= ax_0^2 + b(x_2x_3 + x_3x_2), & S_1 &= cx_0^2 - b(x_1x_4 + x_4x_1), \\
R_2 &= ax_1^2 + b(x_3x_4 + x_4x_3), & S_2 &= cx_1^2 - b(x_2x_0 + x_0x_2), \\
R_3 &= ax_2^2 + b(x_4x_0 + x_0x_4), & S_3 &= cx_2^2 - b(x_3x_1 + x_1x_3), \\
R_4 &= ax_3^2 + b(x_0x_1 + x_1x_0), & S_4 &= cx_3^2 - b(x_4x_2 + x_2x_4), \\
R_5 &= ax_4^2 + b(x_1x_2 + x_2x_1), & S_5 &= cx_4^2 - b(x_0x_3 + x_3x_0).
\end{aligned}$$

where  $a = f_2^2(f_0^2 - f_1f_2)$ ,  $b = f_0(f_2^3 - f_0f_1^2)$  and  $c = f_0^3f_1 - f_2^4$ .

**Theorem 11.** The algebra  $A_5$  is isomorphic to  $\mathcal{D}(\Phi, 3)$ , where superpotential  $\Phi \in \mathbb{C}\langle x_0, x_1, x_2, x_3, x_4 \rangle$  is defined as follows. We define  $Q_1, Q_2, Q_3, Q_4, Q_5 \in V^{\otimes 4}$  for  $V = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4$  by

$$\begin{aligned}
Q_i &= \frac{a^3b + b^2c^2 + ac^3}{b} (R_i^2 - R_{i+3}S_{i+2} - S_{i+2}R_{i+3} - R_{i+2}S_{i+3} - S_{i+3}R_{i+2}) \\
&+ \frac{a^2b^2 + a^3c - bc^3}{b} (S_i^2 - R_{i+1}S_{i+4} - S_{i+4}R_{i+1} - S_{i+1}R_{i+4} - R_{i+4}S_{i+1}) \\
&- \frac{b^4 + 3ab^2c + a^2c^2}{b} (R_iS_i + S_iR_i - S_{i+2}S_{i+3} - S_{i+3}S_{i+2} - R_{i+1}R_{i+4} - R_{i+4}R_{i+1}) \\
&+ \frac{ab^3 + 2a^2bc + c^4}{b} (R_{i+2}R_{i+3} + R_{i+3}R_{i+2}) + \frac{a^4 - b^3c - 2abc^2}{b} (S_{i+4}S_{i+1} + S_{i+1}S_{i+4})
\end{aligned}$$

with the indices  $\{i, i+1, i+2, i+3, i+4\} \in \{1, 2, 3, 4, 5\}$  taken modulo 5. Then the following tensor is a superpotential for these algebras:

$$\Phi := x_0Q_1 + x_1Q_2 + x_2Q_3 + x_3Q_4 + x_4Q_5$$

In particular, the algebra  $A_5$  has a Calabi-Yau property.

## CONCLUSION

The dissertation is devoted to learn the Calabi-Yau properties of the elliptic algebras.

Basic results of the research are as follows:

1. The criteria for Koszul algebras to satisfies Calabi-Yau property is obtained through solutions of the quantum Yang-Baxter equation;
2. It is given another proof of the Calabi-Yau property for three- and four-dimensional Sklyanin algebras;
3. It is proved that several Koszul algebras possess Calabi-Yau property by finding solutions of the quantum Yang-Baxter equation;
4. The identity between the elliptic theta functions of order four are found. Moreover, it is proved the elliptic theta functions of order five satisfy ten identities;
5. The explicit isomorphism between the four dimensional Sklyanin and elliptic algebras is constructed.
6. The superpotential for the four-dimensional elliptic algebras are identified;

7. the superpotential of the five-dimensional elliptic algebras for special values of  $\tau \in \mathcal{E}$  is determined which shows that these algebras possess Calabi-Yau property.

**НАУЧНЫЙ СОВЕТ DSc.02/30.12.2019.FM.86.01  
ПО ПРИСУЖДЕНИЮ УЧЕНЫХ СТЕПЕНЕЙ ПРИ  
ИНСТИТУТЕ МАТЕМАТИКИ ИМЕНИ В.И.РОМАНОВСКОГО**

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**ИНСТИТУТ МАТЕМАТИКИ**

**МИЗОМОВ ИНОМЖОН ЭРКИН УГЛИ**

**О СВОЙСТВАХ КАЛАБИ-ЯУ ДЛЯ ЭЛЛИПТИЧЕСКИХ АЛГЕБР**

**01.01.06–Алгебра**

**АВТОРЕФЕРАТ ДИССЕРТАЦИИ ДОКТОРА ФИЛОСОФИИ (PhD)  
ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ**

**ТАШКЕНТ-2025**

**Тема диссертации доктора философии (PhD) по физико-математическим наукам зарегистрирована в Высшей аттестационной комиссии при Министерстве Высшего образования, Науки и Инноваций Республики Узбекистан за №. B2023.4.PhD/FM960.**

Диссертация выполнена в Институте Математики.

Автореферат диссертации на трех языках (узбекский, английский, русский, (резюме)) размещен на веб-странице по адресу <http://kengash.mathinst.uz> и на Информационно-образовательном портале «ZiyoNet» по адресу <http://www.ziyo.net>.

**Научный руководитель:**

**Эшматов Фарход Хасанович**  
доктор физико-математических наук,

**Официальные оппоненты:**

**Омиров Бахром Абдазович**  
доктор физико-математических наук, профессор

**Жуманиёзов Достон Эркабой угли**  
доктор философии (PhD) по физико-математическим наукам

**Ведущая организация:**

**Андижанский государственный университет**

Защита диссертации состоится « 25 » ноября 2025 года в 17:00 на заседании Научного совета DSc.02/30.12.2019.FM.86.01 при Институте Математики имени В.И.Романовского. (Адрес: 100174, г. Ташкент, Алмазарский район, ул. Университетская, 9.Тел.: (+99871) 207-91-40, e-mail: [uzbmath@umail.uz](mailto:uzbmath@umail.uz), Website: [www.mathinst.uz](http://www.mathinst.uz))

С диссертацией можно ознакомиться в Информационно-ресурсном центре Института Математики имени В.И.Романовского (зарегистрирована за № 214). (Адрес: 100174, г. Ташкент, Алмазарский район, ул. Университетская, 9.Тел.: (+99871) 207-91-40).

Автореферат диссертации разослан « 10 » ноября 2025 года.  
(протокол рассылки № 2 от « 10 » ноября 2025 года).

**У.А. Розиков**  
Председатель Научного совета по  
присуждению ученых  
степеней, д.ф.-м.н., академик

**Ж.К. Адашев**  
Ученый секретарь Научного  
совета по присуждению ученых  
степеней, д.ф.-м.н., старший  
научный сотрудник

**Б.А. Омиров**  
Председатель Научного семинара  
при Научном совете по присуждению ученых  
степеней, д.ф.-м.н., профессор

## **ВВЕДЕНИЕ (аннотация диссертации доктора философии(PhD))**

**Целью исследования** является изучение свойств Калиби–Яу эллиптических алгебр с акцентом на конкретные примеры, такие как алгебры Склянина и их обобщения, а также выявление необходимых и достаточных условий для того, чтобы Кошулевы алгебры обладали свойством Калиби–Яу.

**Объект исследования:** Квадратные и Кошулевы алгебры, эллиптические зета-функции, эллиптические алгебры.

**Научная новизна исследования** состоит в следующем:

Получено критерий того, что алгебры Кошуля удовлетворяют свойству Калиби–Яу;

С помощью этого критерия доказано, что несколько алгебр Кошуля обладают свойством Калиби–Яу;

Найдены тождества между эллиптическими зета-функциями четвёртого и пятого порядков;

Построен явный изоморфизм между четырёхмерными эллиптическими и алгебрами Склянина;

Определён суперпотенциал для четырёхмерных эллиптических алгебр;

Определён суперпотенциал для пятимерных эллиптических алгебр при специальных значениях параметра  $\tau \in \mathcal{E}$ .

**Внедрение результатов исследования.** На основе результатов, полученных по свойствам Калаби–Яу эллиптических алгебр:

Суперпотенциалы, найденные для четырёх- и пятимерных эллиптических алгебр, были использованы для вывода формулы взрывного расширения (blow-up formula) для голоморфных гомотопий Кошуля–Брылинского компактных голоморфных пуассоновских многообразий в рамках зарубежного исследовательского проекта № NSFC12271377 под названием «Некокоммутативная алгебраическая геометрия и отдельные задачи производной алгебраической геометрии» (справка Сычуаньского университета от 5 сентября 2025 года, Китайская Народная Республика). Применение полученных результатов позволило вычислить голоморфные гомотопии Кошуля–Брылинского для поверхностей дель Пеццо и двух комплексных нильмногообразий, обладающих голоморфными пуассоновскими структурами.

Решения квантового уравнения Янг–Бакстера для кошулевых двойственных алгебр квадратичных алгебр, входящих в критерий необходимости наличия свойства Калаби–Яу у кошулевых алгебр, были использованы в фундаментальном проекте № F-FA-2021-423 на тему «Автоморфизмы операторных алгебр, классификация бесконечномерных неассоциативных алгебр и супералгебр» для определения операторов Рота–Бакстера на алгебрах Ли (справка Института математики № 2/337 от 2 сентября 2025 года). Применение данного научного результата обеспечило возможность классификации операторов Рота–Бакстера и анти-Рота–Бакстера на алгебрах Ли малой размерности.

**Структура и объем диссертации.** Диссертация состоит из введения, трёх глав, разбитых на десять параграфов, заключения и списка использованной литературы. Объем диссертации составляет 73 страниц.

**E'OLON QILINGAN ILMIY ISHLAR RO'YXATI**  
**LIST OF PUBLISHED WORKS**  
**СПИСОК ОПУБЛИКОВАННЫХ РАБОТ**

**I bo'lim (part 1; часть 1)**

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2025-yil 27-oktyabrda tahrirdan o‘tkazilib, o‘zbek, ingliz va rus tillaridagi matnlar  
o‘zaro muvofiqlashtirildi.

**Bosmaxona litsenziyasi:**



**9338**

Bichimi: 84x60 <sup>1</sup>/<sub>16</sub>. «Times New Roman» garniturasida.  
Raqamli bosma usulda bosildi.  
Shartli bosma tabog‘i: 2,75. Adadi 100 dona. Buyurtma № 35/25.

Guvohnoma № 851684.  
«Tipograff» MCHJ bosmaxonasida chop etilgan.  
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