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Qo‘lyozma huquqida

MAVZU: TAQSIMOT QONUNLARINI TOPISHGA DOIR BA’ZI MASALALAR

**5A 130102- “Ehtimollar nazariyasi va matematik statistika” kafedrasi
Matematika bo‘yicha magistr akademik darajasini olish uchun**

MAGISTRLIK DISSERTATSIYASI

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***Masalaning qo`yilishi.*** Aytaylik, $X_1, X_2, ..., X_n$ tasodifiy miqdorlar berilgan bo`lsin, bulardan $Z_{\min} = \min(X_1, X_2, ..., X_n)$ va $Z_{\max} = \max(X_1, X_2, ..., X_n)$ tasodifiy miqdorlarni tuzamiz. $Z_{\min}$, $Z_{\max}$ va $(Z_{\min}, Z_{\max})$ sistemaning taqsimot qonunlarini va asosiy sonli xarakteristikalarini topish; normal taqsimot qonun bilan bog'langan ayrim muhim taqsimot qonunlari va ularning sonli xarakteristikalarini topish; tasodifiy sondagi tasodifiy miqdorning taqsimot qonuni va sonli xarakteristikalarini topish talab qilinadi [1,3,4,5].

Masalaning dolzarbligi. Ko`pchilik sohalarda masalan, harbiy ishda, sog'liqni saqlash sohasida, texnika sohasida, ayniqsa matematik statistikada qaralayotgan tasodifiy miqdorlarning taqsimot qonunlarini ayrim hollarda asosiy sonli xarakteristikalarini topish zarur bo`ladi.

Ishning maqsadi. $X_1, X_2, ..., X_n$ tasodifiy miqdorlarning taqsimotli hamda normal tasodifiy miqdorlar bilan bog'langan tasodifiy miqdorlarning va tasodifiy qo`shiluvchilari soni tasodifiy bo`lgan tasodifiy miqdorlarning taqsimot qonunlarini umumiy holda topishdan iborat.

Ilmiy tadqiqot usullari. Tasodifiy miqdorlarning taqsimot qonunlarini va ularning sonli xarakteristikalarini topishning to`g'ridan-to`g'ri usullaridan, ya'ni yig'indining, ayirmaning va bo`linmaning taqsimot qonunlarini topish usullaridan foydalaniladi.

Ishning ilmiyligi. Ehtimollar nazariyasining markaziy tushunchalaridan biri bo`lgan tasodifiy miqdorlar keng o`rganiladi.

Ishda ana shunday ishlar doirasida bir nechta yangi natijalar olingan, qaysikim ular ehtimollar nazariyasidagi tadqiqot ishlarda katta ahamiyatga ega hisoblanadi.

Ishning nazariy va amaliy ahamiyati. Ishning nazariy ahamiyati shundaki, o`rganilayotgan tasodifiy miqdorlarning taqsimot qonunlari va sonli xarakteristikalarini topish mumkin bo`ladigan formulalar keltirib chiqarilgan.

Amaliy ahamiyati esa topilgan formulalar amaliy mashg'ulotlarda turli sohadagi masalalarni yechish imkoniyatini beradi.

Ishning tuzulishi. Dissertatsiya ishi kirish qism, uchta bob va xulosa qismdan tashkil topgan.

Birinchi bob $Z_{\max} = \max(X_1, X_2)$ va $Z_{\min} = \min(X_1, X_2)$ tasodifiy miqdorlarni o`rganishga bag'ishlangan. Bu bob u to`rtta paragrafdan iborat bo`lib, birinchi paragrafda $Z_{\max}$ tasodifiy miqdorning taqsimot qonuni o`rganilgan; ikkinchi paragrafda $Z_{\min}$ tasodifiy miqdorning taqsimot qonuni o`rganilgan; uchinchi paragrafda $Z_{\max}$ va $Z_{\min}$ tasodifiy miqdorlarning sonli xarakteristkalari o`rganilgan; to`rtinchi paragrafda $Z_{\max}$ va $Z_{\min}$ tasodifiy miqdorlarning birgalikdagi, ya'ni $(Z_{\max}, Z_{\min})$ sistemaning taqsimot qonuni o`rganilgan.

Ikkinchi bob $Z_{\max} = \max(X_1, X_2, \ldots, X_n)$ va $Z_{\min} = \min(X_1, X_2, \ldots, X_n)$ tasodifiy miqdorlarni o`rganishga bag'ishlangan. Bu bob ikkita paragrafdan iborat bo`lib, birinchi paragrafda $Z_{\max}$ tasodifiy miqdorning taqsimot qonuni o`rganilgan; ikkinchi paragrafda $Z_{\min}$ tasodifiy miqdorning taqsimot qonuni o`rganilgan.

Uchinchi bob normal tasodifiy miqdorlar bilan bog'langan tasodifiy miqdorlarni o`rganishga bag'ishlangan. Bu bob to`rtta paragrafdan iborat bo`lib, birinchi paragrafda matematik statistikaning bir qator maxsus bo`limlarida, ayniqsa, dispersion analizda katta ahamiyatga ega bo`lgan taqsimot haqida so`z yuritiladi, ya'ni

$$F_{pq} = \frac{\frac{1}{p}\sum_{i=1}^{p} X_i^2}{\frac{1}{q}\sum_{i=1}^{q} Y_j^2}$$

tasodifiy miqdorning taqsimot qonuni hamda sonli xarakteristikalari o`rganilgan; ikkinchi paragrafda

$$Z_{nm} = \frac{\sum_{s=1}^{n} X_s^2}{\sum_{s=1}^{n} X_s^2 + \sum_{i=1}^{m} Y_j^2}$$

tasodifiy miqdorning taqsimot qonuni va sonli xarakteristikalari o`rganilgan; uchinchi paragrafda

$$Z_n = \frac{Y_1 - Y_2}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}}$$

bo`linmaning taqsimot qonuni o`rganilgan; to`rtinchi paragrafda tasodifiy sondagi tasodifiy miqdorlar yig'indisi bo`lgan

$$Z = \sum_{i=1}^{Y} X_i$$

tasodifiy miqdorning taqsimot qonuni va sonli xarakteristikalri o`rganilgan, bunda $X_1, X_2, \ldots$ lar o`zaro bog'liq bo`lmagan, bir xil taqsimlangan tasodifiy miqdorlar, $Y$ esa ularga bog'liq bo`lmagan butun qiymatli tasodifiy miqdor deb faraz qilinadi.

Dissertatsiyada olingan asosiy natijalar [11-12] ishlarda e'lon qilingan va magistrantlarning XIV ilmiy konferensiyasida (Samarqand, 25-aprel 2014) va Respublika ilmiy-nazariy anjumanida (Qarshi, 28-29 mart 2014) hamda "Ehtimollar nazariyasi va matematik statistika" kafedrasining ilmiy-uslubiy seminarida ma'ruza qilingan.

I BOB. TASODIFIY MIQDORLARDAN MAKSIMALINING VA MINIMALINING TAQSIMOT QONUNLARI

1.1-§. Ikkita tasodifiy miqdordan maksimalining taqsimot qonuni

Faraz qilaylik, X va Y uzluksiz tasodifiy miqdorlar bo`lib, $p(x, y)$ ularning birgalikdagi taqsimot zichligi bo`lsin. X va Y tasodifiy miqdorlardan maksimalining taqsimot funksiyasini va taqsimot zichligini topishni maqsad qilib qo`yamiz. Quyidagi belgilashlarni kiritamiz:

$$F(x, y) = P(X < x, Y < y) = \int_{-\infty}^{x}\int_{-\infty}^{y} p(u, v)dudv$$

$$Z_{\max} = \max(X, Y) \quad G(z) = P(Z_{\max} < z)$$

$$g(z) = \frac{d}{dz}G(z)$$

1.1.1-teorema. Ushbu

$$G(z) = F(z, z) \tag{1.1.1}$$

$$g(z) = \int_{-\infty}^{z} p(z, y)\,dy + \int_{-\infty}^{z} p(x, z)\,dx \tag{1.1.2}$$

munosabatlar o`rinli.

Isbot. X va Y tasodifiy miqdorlardan maksimali z dan kichik bo`lishi uchun ularning har biri $z$ dan kichik bo`lishi lozim:

$$P\{\max(X, Y) < z\} = P(X < z,\ Y < z) \tag{1.1.3}$$

Bu tenglikning o`ng tomonidagi ehtimol $(X, Y)$ tasodifiy nuqtaning (miqdorning) uchi $(z, z)$ nuqtada bo`lib, bu o`ngdan chapda va pastda joylashgan cheksiz kvadratga (1-chizma) tushish ehtimolidir.

Tasodifiy nuqtaning yuqorida ta'rif qilingan $P(z)$ sohaga tushish ehtimoli umumiy formulaga ko`ra

$$P\{(X, Y) \subset D(z)\} = \int_{-\infty}^{x}\int_{-\infty}^{y} p(u, v)dudv \tag{1.1.4}$$

bo`ladi. ([9], XIV bob, 10-§) Natijada (1.1.3) va (1.1.4) tengliklardan

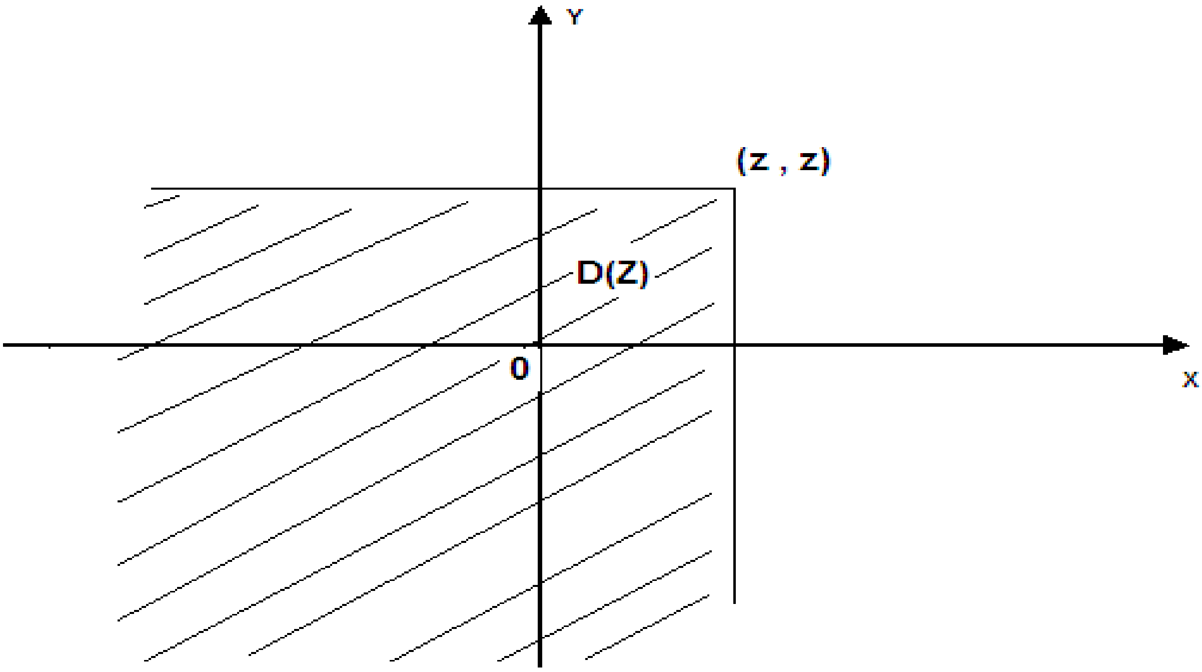
$$P\{\max(X,Y)<z\}=\int_{-\infty}^{x}\int_{-\infty}^{y}p(u,v)dudv$$

bo`lishi kelib chiqadi.

Shunday qilib,

$$G(z)=F(z,z)$$

(1.1.1) tenglik isbot bo`ldi.



1.1.1-chizma

Endi ikkinchi tenglikni isbot qilamiz. Ta'rifga ko`ra, taqsimot zichlik taqsimot funksiyadan olingan birinchi tartibli hosilaga teng. Demak, $g(z)$ taqsimot zichlikni topish uchun $G(z)$ taqsimot funksiyani ikki karrali integral chegaralariga kiruvchi $z$ o`zgaruvchi bo`yicha differensiallaymiz. Har biri $z$ ga bog'liq bo`lgan ikki z_1 va z_2 $(z_1=z,\ z_2=z)$ o`zgaruvchining murakkab funksiyasidek differensiallab quyidagini hosil qilamiz: ([8], XVIII bob, 6-§)

$$g(z)=\frac{dG(z)}{dz}=\frac{d}{dz}\left(\int_{-\infty}^{z}\int_{-\infty}^{z}p(x,y)\,dx\,dy\right)=\frac{d}{dz}\left\{\int_{-\infty}^{z_1}\left(\int_{-\infty}^{z_2}p(x,y)dy\right)dx\right\}=$$

$$=\frac{\partial G(z)}{\partial z_1}\cdot\frac{dz_1}{dz}+\frac{\partial G(z)}{\partial z_2}\cdot\frac{dz_2}{dz}=\int_{-\infty}^{z_2}p(z_2,y)\,dy+\int_{-\infty}^{z_1}p(x,z_2)\,dx=$$

$$= \int_{-\infty}^{z} p(z,y)\,dy + \int_{-\infty}^{z} p(x,z)\,dx$$

Demak,

$$g(z) = \int_{-\infty}^{z} p(z,y)dy + \int_{-\infty}^{z} p(x,z)dx\,.$$

Teorema isbot bo`ldi.

1.1.1-natija. Agar X va Y tasodifiy miqdorlar o`zaro bog'liq bo`lmasa, u holda

$$G(z) = F_X(z) \cdot F_Y(z) \qquad (1.1.5)$$

$$g(z) = P_X(z)F_Y(z) + P_Y(z)F_X(z) \qquad (1.1.6)$$

bo`ladi, bu yerda $F_X(z)$ va $F_Y(z)$ funksiyalar mos ravishda X va Y tasodifiy miqdorlarning taqsimot funksiyalari, $P_X(z)$ va $P_Y(z)$ lar esa taqsimot zichliklari.

Isbot. X va Y tasodifiy miqdorlar o`zaro bog'liq bo`lmasin. U holda $X < z$ va $Y < z$ hodisalar o`zaro bog'liqmas, binobarin, bu hodisalarning birgalikda ro`y berish ehtimoli $P(X < z, Y < z)$ ularning ehtimollari ko`paytmasiga teng ([9], XIV bob 16-§).

$$P(X < z,\, Y < z) = P(X < z) \cdot P(Y < z)$$

yoki

$$F(z,z) = F_x(z) \cdot F_Y(z)\,.$$

Demak,

$$G(z) = F_X(z) \cdot F_Y(z)$$

bo`ladi.

Endi o`zaro bog'liq bo`lmagan tasodifiy miqdorlar uchun

$$p(x,y) = p_X(z) \cdot p_Y(z)$$

bo`lishligidan ([9], XIV bob 16-§) foydalansak, unda (1.1.2) tenglikdan

$$g(z) = \int_{-\infty}^{z} p_X(z)\,p_Y(y)\,dy + \int_{-\infty}^{z} p_X(z)\,p_Y(y)\,dx = p_X(z)\int_{-\infty}^{z} p_Y(y)\,dy + p_Y(y)\int_{-\infty}^{z} p_X(z)\,dx$$

ni hosil qilamiz.

$$F_X(z) = \int_{-\infty}^{z} p_X(x)dx\,,\;\; F_Y(z) = \int_{-\infty}^{z} p_Y(y)dy$$

bo`lganligidan

$$g(z) = p_X(z)F_Y(z) + p_Y(z)F_X(z)$$

bo`ladi.

1.1.2-natija. Agar X va Y o`zaro bog'liq bo`lmagan, bir xil taqsimlangan tasodifiy miqdorlar bo`lsa, u holda

$$G(z) = F^2(z)$$

$$g(z) = 2p(z)F(z)$$

bo`ladi. Bu yerda $F(z)$ funksiya X va Y larning taqsimot funksiyasi, $p(x)$ esa taqsimot zichligi.

Isbot. Shartga ko`ra X va Y miqdorlar bir xil taqsimlangan, demak, ular bir xil taqsimot qonunlariga ega:

$$p_X(x) = p_Y(x) = (x)$$

$$F_X(x) = F_Y(x) = F(x)$$

Unda (1.1.5) va (1.1.6) tengliklardan

$$G(z) = F_X(z)\ F_Y(z) = F^2(z)$$

$$g(z) = p_X(z)F_Y(z) + p_Y(z)F_X(z) = 2p(z)F(z)$$

bo`lishi kelib chiqadi.

1.1.1-misol. Ikkita tasodifiy miqdor sistemasi (X,Y) ushbu

$$p(x,y) = \begin{cases} \frac{1}{2}\sin(x+y), \ 0 \le x \le \frac{\pi}{2}, \ 0 \le y \le \frac{\pi}{2}, & kvadratda \\ 0, & kvadratdan\ tashqarida \end{cases}$$

taqsimot zichlik orqali berilgan.

$$Z_{\max} = \max(X,Y)$$

tasodifiy miqdorning taqsimot zichligini topamiz.

Yechish. (1.1.2) formulaga asosan

$$g(z) = \int_{-\infty}^{z} p(x,y)\,dx + \int_{-\infty}^{z} p(z,y)\,dy = \frac{1}{2}\int_{0}^{z}\sin(x+z)\,dx + \frac{1}{2}\int_{0}^{z}\sin(z+y)\,dy =$$

$$= -\frac{1}{2}\cos(x+z)\Big|_0^z - \frac{1}{2}\cos(z+y)\Big|_0^z = --\frac{1}{2}(\cos 2z - \cos z) -$$

$$-\frac{1}{2}(\cos 2z - \cos z) = \cos z - \cos 2z,$$

bunda $0 \le z \le \frac{\pi}{2}$.

Demak, kutilayotgan taqsimot zichlik

$$g(z) = \cos z - \cos 2z, \qquad 0 \le z \le \frac{\pi}{2}$$

ko`rinishga ega. So`nggi ifodani quyidagicha ham yozish mumkin:

$$g(z) = -2\sin\frac{z-2z}{2}\ \sin\frac{z+2z}{2} = 2\sin\frac{z}{2}\ \sin\frac{3z}{2}$$

1.1.2-misol. X va Y o`zaro bog'liq bo`lmagan tasodifiy miqdorlar

$$p_X(x) = \lambda e^{-\lambda x}, \qquad x > 0$$
$$p_Y(y) = \mu e^{-\mu y}, \qquad y > 0$$

taqsimot zichliklar bilan berilgan. $Z_{\max} = \max(X, Y)$ tasodifiy miqdorning taqsimot zichligini toping.

Yechish. Shartga ko`ra, $X$ va $Y$ tasodifiy miqdorlar o`zaro bog'liqmas. Shuning uchun (1.1.5) formuladan foydalanamiz:

$$g(z) = p_Y(z)F_X(z) + p_X(z)F_Y(z) = p_Y(z)\int_{-\infty}^{z} p_X(x)\,dx + p_X(z)\int_{-\infty}^{\infty} p_Y(y)dy =$$

$$= \mu e^{-\mu z}\int_0^z \lambda e^{-\lambda x}dx + \lambda e^{-\lambda z}\int_0^z \mu e^{-\mu y}dy =$$

$$= \mu e^{-\mu z}(1 - e^{-\lambda z}) + \lambda e^{-\lambda z}(1 - e^{-\mu z}) =$$

$$= \mu e^{-\mu z} + \lambda e^{-\lambda z} - (\mu + \lambda)e^{-(\mu+\lambda)z} \qquad (z > 0).$$

Shunday qilib, $(0;\infty)$ intervalda

$$g(z) = \mu e^{-\mu z} + \lambda e^{-\lambda z} - (\lambda + \mu)e^{-(\lambda+\mu)}\ e^{-(\lambda+\mu)z}$$

bu intervaldan tashqarida $g(z) = 0$

Bu taqsimot ko`rsatkichli taqsimot bo`lmas ekan. $\lambda = \mu = \theta$ bo`lganda u

$$g(z) = 2\theta e^{-\theta z} - 2\theta e^{-2\theta z} = 2\,\theta e^{-\theta z}(1 - e^{-\theta z}), \qquad z > 0$$

ko`rinishga ega bo`ladi.

1.2-§. Ikkita tasodifiy miqdordan minimalining taqsimot qonuni

Aytaylik, X va Y uzluksiz tasodifiy miqdorlar bo`lib, $p(x,y)$ ularning birgalikdan taqsimot zichligi bo`lsin. Bu paragrafda X va Y tasodifiy miqdorlardan minimalining taqsimot zichligini va taqsimot funksiyasini topishni maqsad qilib qo`yamiz.

Ushbu

$$Z_{\min} = \min(X,Y), \quad \Phi(z) = P(Z_{\min} < z), \quad \varphi(z) = \frac{d}{dz}\Phi(z)$$

belgilashlarni kiritamiz. Quyidagi tasdiqlar o`rinli.

1.2.1-teorema. Ushbu

$$\Phi(z) = F_X(z) + F_Y(z) - F(z,z) \tag{1.2.1}$$

$$\varphi(z) = p_X(z) + p_Y(z) - \int_{-\infty}^{\infty} p(z,y)\,dy - \int_{-\infty}^{z} p(x,z)\,dx \tag{1.2.2}$$

munosabatlar o`rinli bo`ladi, bu yerda $F_X(z)$ va $F_Y(z)$ funksiyalar mos ravishda X va Y tasodifiy miqdorlarning taqsimot funksiyalari, $p_X(z)$ va $p_Y(z)$ funksiyalar taqsimot funksiyalari, $F(z,z)$ esa 1-§ da aniqlangan funksiya.

Isbot. $(Z_{\min} < z)$ va $(Z_{\min} > z)$ hodisalar qarama-qarshi hodisalar bo`lgani sababli bu hodisalar ehtimollarining yig'indisi birga teng:

$$P(Z_{\min} < z) + P(Z_{\min} > z) = 1$$

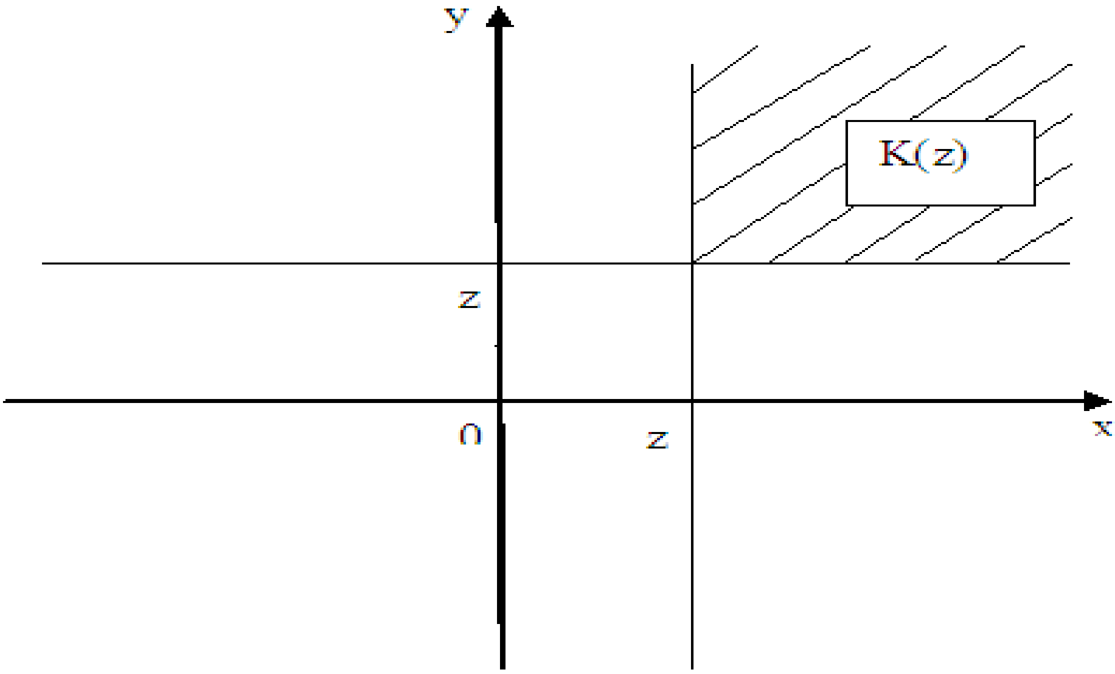
bundan esa

$$1 - \Phi(z) = P(Z_{\min} > z)$$

ga ega bo`lamiz.

X va Y tasodifiy miqdorlardan minimali z dan katta bo`lishi uchun ularning har biri $z$ dan katta bo`lishi lozim, demak,

$$1 - \Phi(z) = P(Z_{mion} > z) = P\{(X > x)(Y > z)\}.$$



2.1.1-chizma

$P\{(X>x)\ \ (Y>z)\}$ ehtimol (X,Y) tasodifiy nuqtaning uchi (z,z) nuqtada bo`lib, bu uchdan o`ngda va yuqorida joylashgan cheksiz kvadratga (2.1.1-chizma) tushish ehtimolidir.

U holda, (X,Y) tasodifiy nuqtaning 2.1.1-chizmada ko`rsatilgan $K(z)$ sohaga tushish ehtimoli

$$P\{(X,Y)\subset K(z)\}=F(\infty,\infty)-F(z,\infty)-F(\infty,z)+F(z,z)$$

ga teng (masalan, [6], VIII bob, 2-§ ga qarang), bu yerda

$$F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}p(u,v)\,du\,dv$$

Demak,

$$1-\Phi(z)=F(\infty,\infty)-F(z,\infty)-F(\infty,z)+F(z,z) \qquad (1.2.3)$$

Ikki o`lchovli taqsimot funksiyaning xossalariga ko`ra

$$F(\infty,\infty)=1,\quad F(z,\infty)=P(X<z)=F_X(z),\quad F(\infty,z)=P(Y<z)=F_Y(z)$$

ekanligini e'tiborga olsak, u holda (1.2.3) tenglikni bunday yozish mumkin:

$$1-\Phi(z)=1-F_X(z)-F_Y(z)+F(z,z)$$

bundan

$$\Phi(z) = F_X(z) + F_Y(z) - F(z,z)$$

ni hosil qilamiz. (1.2.1) tenglik isbot bo`ldi. Endi (1.2.2) tenglikni isbotlaymiz. 1.1-§ da ta'kidlanganidek taqsimot zichlik

$$\varphi(z) = \frac{d\,\Phi(z)}{d\,z}$$

ekani bizga ma'lum. (1.2.1) tenglikning har ikkala tomonini z bo`yicha differensillab,

$$\varphi(z) = \frac{d\,F_X(z)}{d\,z} + \frac{d\,F_Y(z)}{d\,z} - \frac{d}{d\,z}F(z,z) = p_X(z) + p_Y(z) - \frac{d}{d\,z}F(z,z)$$

ga ega bo`lamiz. Bu yerdan $\frac{d}{d\,z}F(z,z) = g(z)$ ni uning 1.1-§ (1.1.2) tenglikdagi ifodasi bilan almashtirib,

$$\varphi(z) = p_X(z) + p_Y(z) - \int_{-\infty}^{z} p(z,y)\,dy - \int_{-\infty}^{z} p(x,z)\,dx$$

munosabati hosil qilamiz, shuni isbotlash talab qilingan edi.

Teorema isbot bo`ladi.

1.2.1-natija. Agar X va Y tasodifiy miqdorlar o`zaro bog'liq bo`lmasa, u holda

$$\Phi(z) = F_X(z) + F_Y(z) - F_X(z)\,F_Y(z)$$

$$\varphi(z) = p_X(z)[1 - F_Y(z)] + p_Y(z)[1 - F_X(z)]$$

bo`ladi, bu yerda $F_X(z)$, $F_Y(z)$, $p_X(z)$ va $p_Y(z)$ lar 1.1-§ da aniqlangan funksiyalar (1.1.1-natijaga qarang)

Isbot. X va Y tasodifiy miqdorlar bog'liq bo`lmaganligidan ([9], XIV bob, 16-§ dagi teoremaga asosan)

$$F(z,z) = F_X(z)\ F_Y(z)\,,\ \ p(x,y) = p_X(x)\ p_Y(y)$$

ga ega bo`lamiz. Unda (1.1.1) tenglikdan

$$\Phi(z) = F_X(z) + F_Y(z) - F(z,z) = F_X(z) + F_Y(z) - F_X(z)\,F_Y(z)$$

ni, (1.2.2) tenglikdan esa

$$\varphi(z) = p_X(z) + p_Y(z) - \int_{-\infty}^{z} p_X(z)\,p_Y(y)\,dy - \int_{-\infty}^{z} p_X(x)\,p_Y(z)\,dx =$$

$$= p_X(z) + p_Y(z) - p_X(z)\int_{-\infty}^{z} p_Y(y)\,dy - p_Y(z)\int_{-\infty}^{z} p_X(x)\,dx =$$

$$= p_X(z) + p_Y(z) - p_X(z)F_Y(z) - p_Y(z)F_X(z) =$$

$$= p_X(z)[1 - F_Y(z)] + p_Y(z)\,[1 - F_X(z)]$$

ni hosil qilamiz.

1.2.2-natija. X va Y o`zaro bog'liq bo`lmagan, bir xil taqsimlangan tasodifiy miqdorlar bo`lib, ularning taqsimot funksiyasi $F(x)$ taqsimot zichligi esa $p(x)$ bo`lsin. U holda

$$\Phi(z) = 2F(z) - F^2(z),\ \varphi(z) = 2\,p(z)[1 - F(z)]$$

bo`ladi.

Bu formulalar bevosita (1.2.3) va (1.2.4) formulalardan kelib chiqadi.

1.2.1-misol. X va Y o`zaro bog'liq bo`lmagan tasodifiy miqdorlar

$$p_X(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

$$p_Y(y) = \mu e^{-\mu y}, \qquad y > 0$$

taqsimot zichliklar bilan berilgan. $Z_{\min} = \min(X, Y)$ tasodifiy miqdorning taqsimot zichligini toping.

Yechish. X va Y tasodifiy miqdorlar o`zaro bog'liq bo`lmaganligi sababli (1.2.4) formuladan foydalanamiz:

$$g(z) = p_X(z)[1 - F_Y(z)] + p_Y(z)[1 - F_X(z)] = \lambda e^{-\lambda z}[1 - (1 - e^{-\mu z})] +$$

$$+ \mu e^{-\mu z}[1 - (1 - e^{-\lambda z})] = \lambda e^{-\lambda z} \cdot \mu e^{-\mu z} +$$
$$+ \mu e^{-\mu z} \cdot e^{-\lambda z} = (\lambda + \mu)e^{-(\lambda+\mu)z}, \qquad z > 0$$

Demak,

$$g(z) = (\lambda + \mu)e^{-(\lambda+\mu)z} \qquad (z > 0)$$

ya'ni ko`rsatkichli qonun bo`yicha taqsimlangan tasodifiy miqdorning minimali ham ko`rsatkichli qonun bo`yicha taqsimlangan bo`lib, uning parametri dastlabki qonunlar parametrlari yig'indisiga teng bo`lar ekan.

1.2.2-misol. Ikkita tasidifiy miqdor sxemasi (X, Y) ushbu

$$p(x,y) = \begin{cases} \dfrac{1}{2}\sin(x+y),\ 0 \le x \le \dfrac{\pi}{2},\ 0 \le y \le \dfrac{\pi}{2}, & \textit{kvadratda} \\ 0, & \textit{kvadratdan tashqarida} \end{cases}$$

taqsimot zichlik bilan berilgan. $Z_{\min} = \min(X,Y)$ tasodifiy miqdorning taqsimot zichligini toping?

Yechish. (1.2.2) formuladan foydalanamiz. X tashkil etuvchining taqsimot zichligini topamiz:

$$p_X(x) = \int_{-\infty}^{\infty} p(x,y)\,dy = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(x+y)\,dy = -\frac{1}{2}\cos(x+z)\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}(\sin x + \cos x) \qquad (1.2.4)$$

Shunga o`xshash, Y tashkil etuvchining taqsimot zichligini hosil qilamiz:

$$p_Y(y) = \frac{1}{2}(\sin y + \cos y)\,. \qquad (1.2.5)$$

Endi

$$\int_{-\infty}^{z} p(z,y)\ dy$$

integralni baholaymiz:

$$\int_{-\infty}^{z} p(z,y)\,dy = \frac{1}{2}\int_0^{z} \sin(z+y)\,dy = -\frac{1}{2}\cos(z+y)\Big|_0^{z} = \frac{1}{2}(\cos z - \cos 2z) \qquad (1.2.6)$$

Shunga o`xshash ,

$$\int_{-\infty}^{z} p(x,z)\,dx = \frac{1}{2}\int_0^{z} \sin(x+z)\,dx = \frac{1}{2}(\cos z - \cos 2z) \qquad (1.2.7)$$

ni hosil qilamiz.

(1.2.4), (1.2.5), (1.2.6) va (1.2.7) ni (1.2.1) ga qo`yib, quyidagiga ega bo`lamiz:

$$\varphi(z) = \sin z + \cos z - (\cos z - \cos 2z) = \sin z + \cos 2z$$

Shunday qilib, izlanayotgan taqsimot zichlik $\left(0, \frac{\pi}{2}\right)$ intervalda

$$\varphi(z) = \sin z + \cos 2z$$

bu intervaldan tashqarida $\varphi(z) = 0$.

1.3-§. Z_{max} va Z_{min} tasodifiy miqdorlarning sonli xarakteristikalari

Tasodifiy miqdorning taqsimot qonuning bilish ehtimollik nuqtai nazaridan tasodifiy miqdor haqida to`liq ma'lumot berishini bilamiz. Ammo har bir masalada ham taqsimot hammasini bilish shart emas.

Bir qator hollarda taqsimot qonunining eng muhim xususiyatlarini belgilovchi bir yoki bir nechta sonlar bilan kifoyalanish mumkin. Masalan, tasodifiy miqdorning "o`rta qiymati" ma'nosiga ega bo`lgan son, yoki tasodifiy miqdor o`zining o`rtacha qiymatidan chetlanishining o`rtacha qiymatini xarakterlovchi son va hokazo. Bunday turdagi sonlarni tasodifiy miqdorning sonli xarakteristikalari deb ataladi. Ularning ehtimollar nazariyasidagi ahamiyati nihoyatda katta, juda ko`plab masalalar taqsimot qonunini chetda qoldirib faqat sonli xarakteristikalarga tayangan holda oxirigacha yetkazilishi mumkin.

Tasodifiy miqdorlarning hozir biz o`rganadigan asosiy sonli xarakteristika lari, ularning matematik kutilishi (yoki o`rtacha qiymati) va dispersiyasidir.

Aytaylik, X va Y o`zaro bog'liq bo`lmagan uzluksiz tasodifiy miqdorlar bo`lib, ularning taqsimot zichliklari ma'lum va mos ravishda $p_X(x)$ va $p_Y(y)$ ga teng bo`lsin.

1. Bu tasodifiy miqdorlardan maksimumi $Z_{max} = \max(X, Y)$ ning, ya'ni

$$Z_{max} = \begin{cases} X, & X \geq Y \\ Y, & X < Y \end{cases}$$

munosabat bilan aniqlanadigan Z_{max} tasodifiy miqdorning sonli xarakterestikalarini topamiz.

$y = x$ to`g'ri chiziq $xOy$ koordinatalar tekisligini ikkita sohaga bo`ladi (3-chizmaga qarang). D_1 soha, bunda $Z_{max} = X$, D_2 soha bunda $Z_{max} = Y$. ($X = Y$ hol nol ehtimolga ega bo`lganligi uchun qaralmaydi). Matematik kutilishning ifodasiga ko`ra

$$M\,Z_{max} = \iint_{(D_1)} x p_X(x)\, p_Y(y)\; dx\, dy + \iint_{(D_2)} y p_X(x)\, p_Y(y)\; dx\, dy$$

ga ega bo`lamiz.

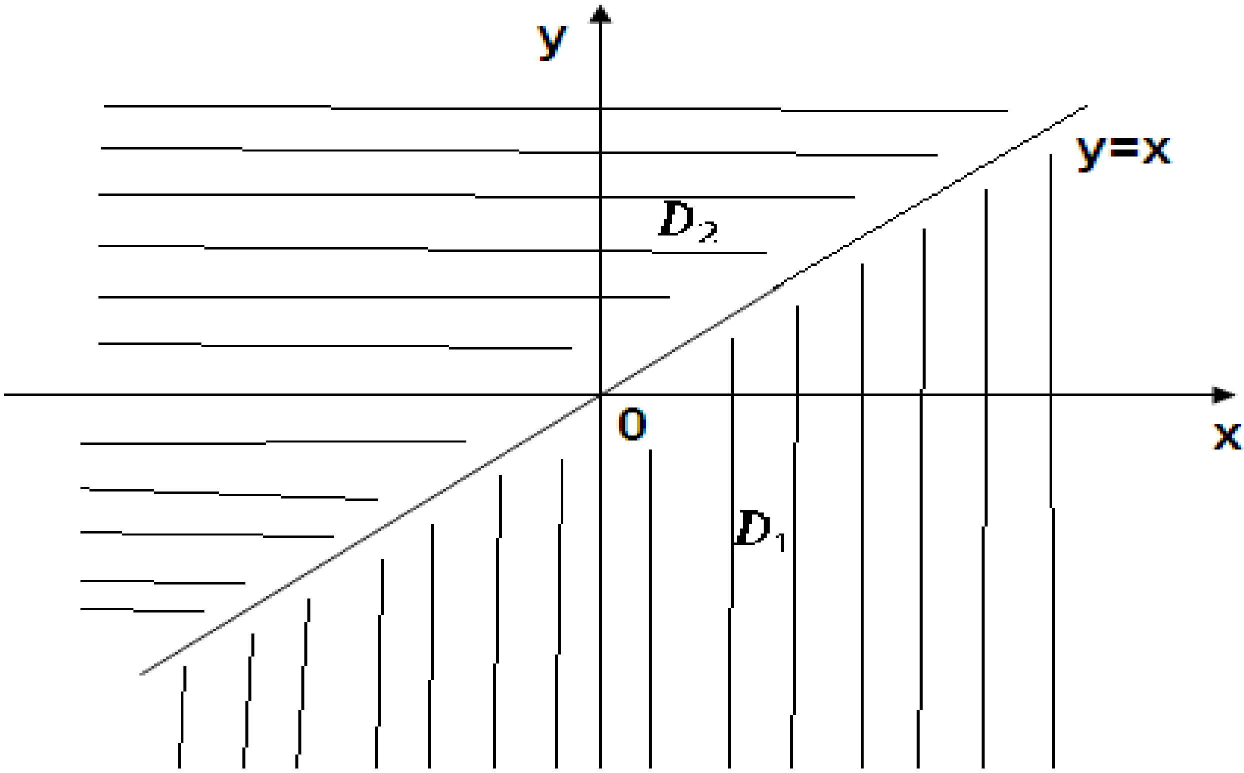
Bu ikki karrali integralni hisoblab natijani topamiz:

$$M\,Z_{\max} = \int_{-\infty}^{\infty} x p_X(x) \left(\int_{-\infty}^{x} p_Y(y)\,dy \right) dx + \int_{-\infty}^{\infty} y p_Y(x) \left(\int_{-\infty}^{y} {}_X(x)\,dx \right) dy$$

Buni ixchamroq bunday ko`rinishda ham yozish mumkin:

$$M\,Z_{\max} = \int_{-\infty}^{\infty} x p_X(x)\; F_Y(x)\,dx + \int_{-\infty}^{\infty} y p_Y(y)\; F_X(y)\,dy\,. \qquad (1.3.1)$$

Bu yerda $F_X(z)$ va $F_Y(y)$ lar mos ravishda X va Y tasodifiy miqdorlarning taqsimot funksiyalari.



1.3.1-chizma

Dispersiyani uning ta’rifidan foydalanib hisoblash mumkin, lekin bu maqsadga tezroq olib boradigan

$$DZ_{\max} = MZ_{\max}^2 - (MZ_{\max})^2$$

formuladan foydalanamiz. Bu formulaga $MZ_{\max}$ ning (1.3.1) tenglikdagi ifodasini qo`yib quyidagini hosil qilamiz:

$$DZ_{\max} = MZ_{\max}^2 - \left(\int_{-\infty}^{\infty} x p_X F_Y(x)\,dx + \int_{-\infty}^{\infty} y p_Y F_Y(y)\,dy \right)^2$$

$MZ_{\max}^2$ ni topamiz.

$$MZ_{\max}^2 = \iint_{(D_1)} x^2 p_X(x) p_Y(y)\,dx\,dy + \iint_{(D_1)} y^2 p_X(x) p_Y(y)\,dx\,dy =$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x) \left(\int_{-\infty}^{x} p_Y(y)\,dy \right) dx + \int_{-\infty}^{\infty} y^2 p_Y(y) \left(\int_{-\infty}^{y} p_Y(x)\,dx \right) dy$$

yoki ixchamroq ko`rinishda

$$MZ_{\max}^2 = \int_{-\infty}^{\infty} x^2 p_X(x)\, F_Y(x)\,dx + \int_{-\infty}^{\infty} y^2 p_Y(y) F_X(y)\,dy$$

bo`ladi.

Shunday qilib,

$$DZ_{\max} = \int_{-\infty}^{\infty} x^2 p_X(x)\, F_Y(x)\,dx + \int_{-\infty}^{\infty} y^2 p_Y(y) F_X(y)\,dy -$$

$$- \left(\int_{-\infty}^{\infty} x p_X(x)\, F_Y(x)\,dx + \int_{-\infty}^{\infty} y p_Y(y) F_X(y)\,dy \right)^2$$

1.3.1-Misol. X va Y tasodifiy miqdorlar sistemasi (X,Y) normal taqsimlangan shu bilan birga har birining matematik kutilishi nolga, o`rtacha kvadratik chetlanishi birga teng, ya’ni $MX = MY = 0$, $\sigma_X = \sigma_Y = 1$ $\rho$ $X$ va $Y$ tasodifiy miqdorlarning korrelyatsiya koeffisienti bo`lsa, $Z_{\max} = \max(X,Y)$ tasodifiy miqdorning parametrik ko`rinishini toping.

Yechish. Shartga ko`ra

$$p(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)}$$

Bunday holda, ya’ni X va Y tasodifiy miqdorlar ikki o`lchovli taqsimot zichlik bilan berilganda $MZ_{\max}$ matematik kutilmasini

$$MZ_{\max} = \int_{-\infty}^{\infty} z\, g(z)\,dz \qquad (1.3.2)$$

formula bo`yicha topish mumkin, bu yerda $g(z)$ funksiya $Z_{\max}$ tasodifiy miqdorning taqsimot zichligi (1.3.2) formulaga qarang).

Demak, $MZ_{\max}$ ni topish uchun avval $g(z)$ ni topamiz:

$$g(z)=\int_{-\infty}^{z}p(z,y)\,dy+\int_{-\infty}^{z}p(x,z)\,dx=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{z}e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}\,dy+$$

$$+\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{z}e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xz+z^2)}\,dx=\frac{1}{\pi\sqrt{1-\rho^2}}\int_{-\infty}^{z}e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xz+x^2)}\,dx$$

Integral ostida turgan ko`rsatkichli funksiyaning daraja ko`rsatkichini to`la kvadratga to`ldirib, integrallash o`zgaruvchisi $x$ ga bog'liq bo`lmagan $e^{-\frac{z^2}{2}}$ ko`paytuvchini integral belgisidan tashqariga chiqarib yozamiz. U holda

$$g(z)=\frac{1}{\pi\sqrt{1-\rho^2}}e^{-\frac{z^2}{2}}\int_{-\infty}^{z}e^{-\frac{(x-\rho z)^2}{2\sqrt{1-\rho^2}}}\,dx$$

bo`ladi. Bu integralda $\frac{x-\rho z}{\sqrt{1-\rho^2}}=t$ almashtirishni bajaramiz. Unda

$$x=t\sqrt{1-\rho^2}+\rho z,$$

$$dx=\sqrt{1-\rho^2}\;dt$$

bo`lib,

$$g(z)=\frac{1}{\pi}e^{-\frac{z^2}{2}}\int_{-\infty}^{z\sqrt{\frac{1-\rho^2}{1+\rho^2}}}e^{-\frac{t^2}{2}}\,dt=\frac{1}{\pi}e^{-\frac{z^2}{2}}\Phi\left(z\sqrt{\frac{1-\rho^2}{1+\rho^2}}\right)\qquad(1.3.3)$$

bo`ladi, bu yerda $\Phi(x)=\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$ - Laplas funksiyasi.

(1.3.3) ni (1.3.2) ga qo`yib,

$$MZ_{\max}=\frac{1}{\pi}\int_{-\infty}^{\infty}z\,e^{-\frac{z^2}{2}}\,\Phi\left(z\sqrt{\frac{1-\rho^2}{1+\rho^2}}\right)$$

ni hosil qilamiz. Ushbu

$$\int_{-\infty}^{\infty} u\,dv = u\,v\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du$$

formula bo`yicha bo`laklab integrallaymiz, bunda

$$u = \Phi\left(z\sqrt{\frac{1-\rho}{1+\rho}}\right), \quad dv = z\, e^{-\frac{z^2}{2}}$$

deymiz, u holda

$$du = \sqrt{\frac{1-\rho}{1+\rho}}\; e^{-\frac{1}{2}\left(\frac{1-\rho}{1+\rho}\right)z^2}\, dz, \qquad v = -e^{-\frac{z^2}{2}}$$

bo`lib,

$$MZ_{\max} = \frac{1}{\pi}\left[-e^{-\frac{z^2}{2}}\;\Phi\left(z\sqrt{\frac{1-\rho}{1+\rho}}\right)\Bigg|_{-\infty}^{\infty} + \sqrt{\frac{1-\rho}{1+\rho}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}-\frac{1}{2}\left(\frac{1-\rho}{1+\rho}\right)z^2}\, dz\right] =$$

$$= \frac{1}{\pi}\sqrt{\frac{1-\rho}{1+\rho}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}\left(1+\frac{1-\rho}{1+\rho}\right)}\, dz = \frac{1}{\pi}\sqrt{\frac{1-\rho}{1+\rho}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{(1+\rho)}}\, dz = \frac{1}{\pi}\sqrt{\frac{1-\rho}{1+\rho}}\int_{-\infty}^{\infty} e^{-\left(\frac{z}{\sqrt{1+\rho}}\right)^2}\, dz =$$

$$= \frac{1}{\pi}\sqrt{1-\rho}\int_{-\infty}^{\infty} e^{-\left(\frac{z}{\sqrt{1+\rho}}\right)^2}\, d\left(\frac{z}{\sqrt{1+\rho}}\right) = \frac{\sqrt{1-\rho}}{\pi}\int_{-\infty}^{\infty} e^{-t^2}\, dt$$

bo`ladi.

Tenglikning o`ng tomonida turgan Puasson integrali $\sqrt{\pi}$ ga tengligini hisobga olib uzil-kesil quyidagiga ega bo`lamiz:

$$MZ_{\max} = \sqrt{\frac{1-\rho}{\pi}}\,.$$

2. Endi $Z_{\min} = \min(X,Y)$ tasodifiy miqdorning, ya'ni

$$Z_{\min} = \begin{cases} X, & X \le Y \\ Y, & X > Y \end{cases}$$

munosabat bilan aniqlanadigan $Z_{\min}$ tasodifiy miqdorning sonli xarakteristikalarini topamiz.

Yuqoridagidek, $y = x$ to`g'ri chiziq $xOy$ koordinatalar tekisligini ikkiga $D_1$ va $D_2$ sohalarga bo`ladi. D_1 sohada $x > y$, demak, $Z_{\min} = Y$, D_2 sohada $x < y$, $Z_{\min} = X$. Bu holda birinchi punktdagiga o`xshash, $M\,Z_{\min}$ uchun

$$MZ_{\min} = \iint\limits_{(D_2)} x\, p_X(x)\ p_Y(y)\, dx\, dy + \iint\limits_{(D_1)} y\, p_X(x)\ p_Y(y)\, dx\, dy =$$

$$= \int_{-\infty}^{\infty} xp_X(x)\left(\int_x^{\infty} p_Y(y)\, dy\right)dx + \int_{-\infty}^{\infty} yp_Y(y)\left(\int_y^{\infty} p_X(x)\, dx\right)dy =$$

$$= \int_{-\infty}^{\infty} xp_X(x)\left(\int_{-\infty}^{\infty} p_Y(y)\, dy - \int_{-\infty}^{x} p_Y(y)\, dy\right)dx + \int_{-\infty}^{\infty} yp_Y(y)\left(\int_{-\infty}^{\infty} p_X(x)\, dx - \int_{-\infty}^{y} p_X(x)\, dx\right)dy =$$

$$= \int_{-\infty}^{\infty} xp_X(x)\left(1 - \int_{-\infty}^{x} p_Y(y)\, dy\right)dx + \int_{-\infty}^{\infty} yp_Y(y)\left(1 - \int_{-\infty}^{y} p_X(x)\, dx\right)dy$$

yoki

$$M\ Z_{\min} = \int_{-\infty}^{\infty} xp_X(x)\big(1 - F_Y(x)\big)dx + \int_{-\infty}^{\infty} yp_Y(y)\big(1 - F_X(y)\ \big)dy \qquad (1.3.4)$$

ni hosil qilamiz. Bu yerda $F_X(x)$ va $F_Y(y)$ funksiyalar mos ravishda X va Y tasodifiy miqdorlarning taqsimot funksiyalari. $Z_{\min}$ tasodifiy miqdorning dispersiyasini

$$DZ_{\min} = MZ^2_{\min} - (MZ_{\min})^2 \qquad (1.3.5)$$

formula bo`yicha topamiz. Buning uchun $MZ^2_{\min}$ ni topamiz.

$$MZ^2_{\min} = \iint\limits_{(D_2)} x^2\ p_X(x)\ p_Y(y)\, dxdy + \iint\limits_{(D_2)} y^2\ p_X(x)\ p_Y(y)\, dxdy =$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x)\left(\int_x^{\infty} p_Y(y)\, dy\right)\ dx + \int_{-\infty}^{\infty} y^2 p_Y(y)\left(\int_y^{\infty} p_X(x)\, dx\right)\ dy =$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x)(1 - F_Y(x))\ dx + \int_{-\infty}^{\infty} y^2 p_Y(y)(1 - F_X(y))\ dy \qquad (1.3.6)$$

(1.3.3) va (1.3.6) ni (1.3.5) ga qo`yib, izlanayotgan dispersiyani topamiz:

$$DZ_{\min} = \int_{-\infty}^{\infty} x^2 p_X(x)(1 - F_Y(x))\ dx + \int_{-\infty}^{\infty} y^2 p_Y(y)(1 - F_X(y))\ dy -$$

$$-\left(\int_{-\infty}^{\infty} xp_X(x)(1 - F_Y(x))\ dx + \int_{-\infty}^{\infty} yp_Y(y)(1 - F_X(y))\ dy\right)^2$$

Xususan, X va Y tasodifiy miqdorlar bir xil taqsimlanganda , ya’ni , $F_X(x) = F_Y(x) = F(x)$ bo`lganda $MZ_{\min}$, $MZ_{\max}$, $DZ_{\max}$, $DZ_{\min}$ larning ifodalari

$$MZ_{\max} = \int_{-\infty}^{\infty} xp_X(x)\,F_Y(x)\,dx + \int_{-\infty}^{\infty} yp_Y(y)\,F_X(y)\,dy = 2\int_{-\infty}^{\infty} xp(x)\,F(x)\,dx$$

$$MZ_{\min} = \int_{-\infty}^{\infty} xp_X(x)(1-F_Y(x))\,dx + \int_{-\infty}^{\infty} yp_Y(y)(1-F_X(y))dy = 2\int_{-\infty}^{\infty} xp(x)(1-F(x))\,dx$$

$$DZ_{\max} = 2\int_{-\infty}^{\infty} x^2 p(x)\,F(x)\,dx - \left(2\int_{-\infty}^{\infty} xp(x)\,F(x)\,dx\right)^2$$

$$DZ_{\min} = 2\int_{-\infty}^{\infty} x^2 p(x)(1-F(x))\,dx - \left(2\int_{-\infty}^{\infty} xp(x)(1-F(x))\,dx\right)^2$$

ko`rinishga ega bo`ladi.

1.4-§. ($Z_{\min}$, $Z_{\max}$) sistemaning taqsimot qonuni

Oldingi paragraflarda (X,Y) sistemaning taqsimot zichligini bilgan holda $Z_{\min} = \min(X,Y)$ va $Z_{\max} = \max(X,Y)$ tasodifiy miqdorlarning taqsimot qonunlarini topish masalasi qaralgan edi. Bu paragrafda esa ularning birgalikdagi taqsimot qonunini topish masalasini qaraymiz.

1.4.1-teorema: (X,Y)-taqsimot zichligi $p(x,y)$ bo`lgan uzluksiz tasodifiy miqdorlar sistemasi bo`lsin. Agar $G(u,v)$ va $g(u,v)$ funksiyalar mos ravishda ($Z_{\min}$, $Z_{\max}$) sistemaning taqsimot funksiyasi va taqsimot zichligi bo`lsa, u holda

$$G(u,v) = \begin{cases} \int_{-\infty}^{v}\int_{-\infty}^{v} p(x,y)\,dx\,dy, & agar\ v \le u \\ \int_{-\infty}^{u} dx \int_{x}^{v} p(x,y)\,dy + \int_{-\infty}^{u} dy \int_{y}^{v} p(x,y)\,dx, & agar\ \ v > u \end{cases}$$

$$g(u,v) = \begin{cases} \int_{-\infty}^{v} p(v,y)dy + \int_{-\infty}^{v} p(x,v)\,dx, & agar\ \ v \le u \\ p(u,v) + p(v,u), & agar\ \ v > u \end{cases}$$

munosabatlar o`rinli bo`ladi.

Isbot. Ikkita tasodifiy miqdor sistemasining taqsimot funksiyasi ta’rifiga ko`ra

$$G(u,v) = P(Z_{\min} < u, Z_{\max} < v)$$

mumkin bo`lgan ikki holni qaraymiz.

1. $v \le u$ bo`lsin. U holda $(Z_{\min} < u,\ Z_{\max} < v)$ hodisa $\left(Z_{\max} < v\right)$ hodisaga teng kuchli bo`ladi. Bundan

$$P(Z_{\min} < u, Z_{\max} < v) = P(Z_{\max} < v)$$

bo`lishi kelib chiqadi.

Tasodifiy miqdorlarning maksimali v dan kichik bo`lganda ularning har biri v dan kichik ekanligini hisob olib, $G(u,v)$ taqsimot funksiya uchun

$$G(u,v) = P(\max(X,Y) < v) = P(X < v, Y < v)$$

tenglikni hosil qilamiz. Shunday qilib, $v \le u$ bo`lganda

$$G(u,v) = P(X < v, Y < v) = \int_{-\infty}^{v}\int_{-\infty}^{v} p(x,y)\ dx\,dy$$

tenglik o`rinli bo`ladi.

2. $v > u$ bo`lsin. Bu holda $\left(Z_{\min} < u,\ Z_{\max} < v\right)$ hodisa birgalikda bo`lmagan $(X < Y,\ X < u,\ Y < v)$ va $(X \ge Y, Y < u,\ X < v)$ hodisalarning yig'indisi ko`rinishida ifodalash mumkin:

$$(Z_{\min} < u,\ Z_{\max} < v) = (X < Y,\ X < u,\ Y < v) \cup (Y \le X,\ Y < u,\ X < v)$$

Birgalikda bo`lmagan hodisalar ehtimollarining qo`shish teoremasiga ko`ra

$$P(Z_{\min} < u,\ Z_{\max} < v) = P(X < Y,\ X < u,\ Y < v) + P(Y \le X,\ Y < u,\ X < v)$$

ni hosil qilamiz. U holda bu tenglikning o`ng tomonidagi ehtimollarni $p(x,y)$ taqsimot zichlik yordamida yozib, quyidagiga ega bo`lamiz: (4-chizmaga qarang)

$$P\left(Z_{\min} < u, Z_{\max} < v\right) = \iint_{(D_1)} p(x,y)\,dx\,dy + \iint_{(D_2)} p(x,y)\,dx\,dy =$$

$$= \iint_{\substack{x<y \\ x<u,\, y<v}} p(x,y)\,dx\,dy + \iint_{\substack{y\le x \\ y<u,\, x<v}} p(x,y)\,dx\,dy \qquad (1.4.1)$$

(1.4.1) tenglikda ikki karrali integraldan takroriy integralga o`tib, quyidagini topamiz:

$$P\left(Z_{\min} < u, Z_{\max} < v\right) = \int_{-\infty}^{u} dx \int_{x}^{v} p(x,y)\ dy + \int_{-\infty}^{u} dy \int_{y}^{v} p(x,y)\ dx\,.$$

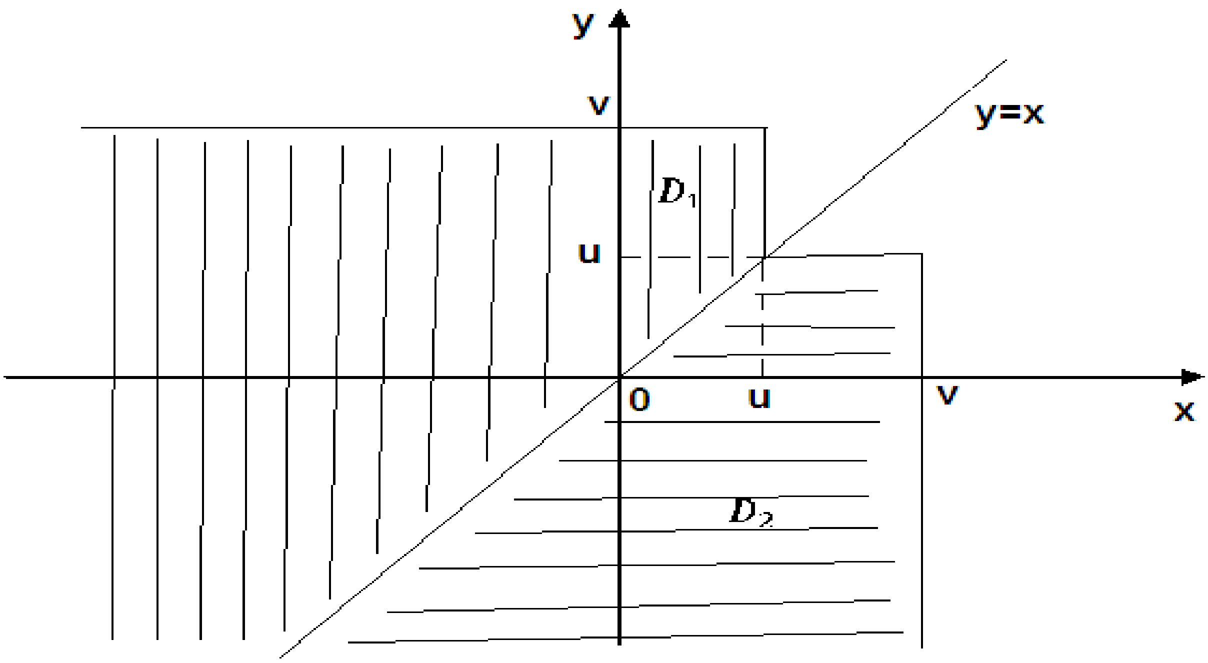
Shunday qilib, taqsimot funksiya $v \le u$ bo`lganda

$$G(u,v) = \int_{-\infty}^{v}\int_{-\infty}^{v} p(x,y)\ dx\,dy \tag{1.4.2}$$

$v > u$ bo`lganda esa,

$$G(u,v) = \int_{-\infty}^{u} dx \int_{x}^{v} p(x,y)\ dy + \int_{-\infty}^{u} dy \int_{y}^{v} p(x,y)\ dx \tag{1.4.3}$$

bo`ladi.



1.4.1-chizma

Ma'lumki, ikki o`lchovli uzluksiz tasodifiy miqdorning taqsimot zichligi taqsimot funksiyadan olingan ikkinchi tartibli aralash xususiy hosilaga teng.

Demak,

$$g(u,v) = \frac{\partial^2 G(u,v)}{\partial u\,\partial v}$$

bo`ladi.

$v \le u$ bo`lsin. U holda (1.4.2) formulani $v$ bo`yicha differensiallab, topamiz:

$$g(u,v) = \frac{d\,G(u,v)}{d\,v} = \frac{d}{d\,v}\left\{\int_{-\infty}^{v}\int_{-\infty}^{v} p(x,y)\,dx\,dy\right\} = \frac{d}{d\,v}\left\{\int_{-\infty}^{v_1=v}\left(\int_{-\infty}^{v_2=v} p(x,y)\,dy\right)dx\right\} =$$

$$=\frac{\partial G(u,v)}{\partial v_1}\cdot\frac{d\,v_1}{dv}+\frac{\partial G(u,v)}{\partial v_2}\cdot\frac{d\,v_2}{dv}=\int_{-\infty}^{v}p(x,y)dy+\int_{-\infty}^{v}p(x,v)\;dx$$

Endi $v>u$ bo\`lsin. Bu holda (1.4.3) formuladan avval $u$ bo\`yicha olingan xususiy hosilani keyin hosil bo\`lgan natijadan $v$ bo\`yicha xususiy hosilani topamiz. Natijada,

$$g(u,v)=\frac{\partial^2\,G(u,v)}{\partial u\;\partial v}=\frac{\partial}{\partial v}\left(\frac{\partial G(u,v)}{\partial u}\right)=$$

$$\frac{\partial}{\partial v}\left\{\frac{\partial}{\partial u}\left(\int_{-\infty}^{u}dx\int_{x}^{v}p(x,y)\;dy+\int_{-\infty}^{u}dy\int_{y}^{v}p(x,y)\;dx\right)\right\}=$$

$$=\frac{\partial}{\partial v}\left(\int_{u}^{v}p(u,y)dy+\int_{u}^{v}p(x,u)\;dx\right)=p(u,v)+p(v,u)$$

ni hosil qilamiz.

Shunday qilib, taqsimot zichlik $v\le u$ bo\`lganda

$$g(u,v)=\int_{-\infty}^{v}p(v,y)dy+\int_{-\infty}^{v}p(x,v)\;dx \qquad (1.4.4)$$

$v>u$ bo\`lganda esa

$$g(u,v)=p(u,v)+p(v,u) \qquad (1.1.5)$$

bo\`ladi.

1.4.1-natija. Agar X va Y tasodifiy miqdorlar o\`zaro bog’liq bo\`lmay, X tasodifiy miqdor $p_X(x)$ taqsimot zichligiga va

$$F_X(x)=\int_{-\infty}^{x}p_X(u)\,du$$

taqsimot funksiyaga, Y tasodifiy miqdor esa $P_Y(y)$ taqsimot zichligiga va

$$F_Y(y)=\int_{-\infty}^{y}p_Y(u)\,du$$

taqsimot funksiyaga ega bo\`lsa, u holda

$$G(u\,,v)=\begin{cases}F_X(v)\cdot F_Y(v), & v\le u\\ \int_{-\infty}^{u}(F_Y(v)-F_Y(x))\,p_X(x)\,dx+\int_{-\infty}^{u}(F_X(v)-F_X(u))\,p_Y(y)\,dy, & v>u\end{cases}$$

va

$$g(u,v)=\begin{cases} p_X(v)\,F_Y(v)+p_Y(v)\,F_X(v), & v\le u\\ p_X(u)\,P_Y(v)+p_X(v)\,P_Y(u), & v>u\end{cases}$$

bo`ladi.

Isbot. X va Y tasodifiy miqdorlar o`zaro bog’liq bo`lmasin. U holda ularning taqsimot zichligi

$$p(x,y)=p_X(x)\,p_Y(y)$$

ga, taqsimot funksiya esa

$$F(x,y)=F_X(x)\;F_Y(y)$$

ga teng. Bu qiymatlarni $G(u,v)$ taqsimot funksiyaning (1.4.2), (1.4.3) va $g(u,v)$ taqsimot zichlikning (1.4.4) , (1.4.5) ifodalariga qo`yamiz. U holda

$$G(u,v)=\int_{-\infty}^{v}\int_{-\infty}^{v}p(x,y)\;dx\,dy=\int_{-\infty}^{v}p_X(x)\;dx\int_{-\infty}^{v}p_Y(y)\;dy=F_X(v)\cdot F_Y(v),\qquad (v\le u)\qquad (1.4.6)$$

$$G(u,v)=\int_{-\infty}^{u}dx\int_{x}^{v}p(x,y)\;dy\;+\int_{-\infty}^{u}dy\int_{y}^{v}p(x,y)\,dx=\int_{-\infty}^{u}\left(\int_{x}^{v}p_X(y)\;dy\right)p_X(x)\,dx+$$

$$+\int_{-\infty}^{u}\left(\int_{y}^{v}p_x(x)\;dx\right)p_Y(y)\,dy=\int_{-\infty}^{u}(F_Y(v)-F_Y(x))\;p_X(x)dx+$$

$$+\int_{-\infty}^{u}(F_X(v)-F_X(y))\;p_Y(y)\,dy\qquad v>u\qquad (1.4.7)$$

(bunda $\int_a^b p(x)dx=P(a<X<b)=F(b)-F(a)$ formuladan foydalandik) va

$$g(u,v)=\int_{-\infty}^{v}p(v,y)\,dy+\int_{-\infty}^{v}p(x,v)\,dx=p_X(v)\int_{-\infty}^{v}p_Y(y)\,dy+p_Y(v)\int_{-\infty}^{v}p_X(x)\,dx=$$

$$=p_X(v)\,F_Y(v)+p_Y(v)\,F_X(v)\qquad (v\le u)\qquad (1.4.8)$$

$$g(u,v)=p(u,v)+p(v,u)=p_X(u)p_Y(v)+p_X(v)\;p_Y(u),\qquad (v>u)$$

bo`ladi.

1.4.2-natija. Agar X va Y o`zaro bog’liq bo`lmagan, bir xil taqsimlangan tasodifiy miqdorlar bo`lib, $P(x)$ ularning taqsimot zichligi, $F(x)$ esa taqsimot funksiyasi bo`lsa, u holda

$$G(u,v)=\begin{cases} F^2(v), & v\le u \\ 2\int\limits_{-\infty}^{u}(F(v)-F(x))\ P(x)dx, & v>u \end{cases}$$

$$g(u,v)=\begin{cases} 2p(v)F(v), & v\le u \\ 2p(u)\ p(v), & v>u \end{cases}$$

bo`ladi.

Isbot. X va Y miqdorlar o`zaro bog'liqmas va bir xil taqsimlanganligi sababli $p_X(z)=p_Y(z)$, $F_X(z)=F_Y(z)$ tengliklarga ega bo`lamiz.

Ularni $p(z)$ va $F(z)$ deb belgilab, (1.4.6) formuladan

$$G(u,v)=F^2(v), \qquad (v\le u)$$

ni (1.4.7) formuladan,

$$G(u,v)=2\int\limits_{-\infty}^{u}(F(v)-F(x))p(x)\ dx, \qquad (v>u)$$

ni (1.4.8) formuladan,

$$g(u,v)=2\,p(v)F(v), \qquad (v\le u)$$

ni (1.4.9) formuladan esa

$$g(u,v)=2\,p(u)\ p(v), \qquad (v>u)$$

ni hosil qilamiz.

II BOB. BIR NECHTA TASODIFIY MIQDORLARDAN MAKSIMALINING VA MINIMALINING TAQSIMOT QONUNLARI

2.1-§. n ta o`zaro bog'liq bo`lmagan tasodifiy miqdorlardan maksimalining taqsimot qonuni

Faraz qilaylik, $X_1, X_2, \ldots, X_n$ - o`zaro bog'liq bo`lmagan uzluksiz tasodifiy miqdorlar bo`lsin. Ulardan maksimalining taqsimot qonunini topishni maqsad qilib qo`yaylik. Quyidagi belgilashlarni kiritamiz:

$$Z_{\max} = \max(X_1, X_2, \ldots, X_n), \quad G^*(z) = P(Z_{\max} < z), \quad g^*(z) = \frac{d}{dz} G^*(z)$$

2.1.1-teorema. $X_1, X_2, \ldots, X_n$ - o`zaro bog'liq bo`lmagan, mos ravishda $P_{X_1}(x_1), P_{X_2}(x_2), \ldots, P_{X_n}(x_n)$ taqsimot zichliklarga ega bo`lgan tasodifiy miqdorlar bo`lsin. U holda

$$G^*(z) = \prod_{k=1}^{n} F_{X_K}(z),$$

$$g^*(z) = \sum_{i=1}^{n} \frac{P_{X_i}(z)}{F_{X_i}(z)} \cdot \prod_{k=1}^{n} F_{X_k}(z)$$

munosabatlar o`rinli bo`ladi, bu yerda $F_{X_k}(z)$ funksiya X_k tasodifiy miqdorning taqsimot funksiyasi.

Isbot. $Z_{\max} < z$ bo`lishi uchun $X_k$ tasodifiy miqdorning har biri $z$ dan kichik bo`lishi kerak, u holda $Z_{\max} < z$ tengsizlikning ehtimoli $X_k < z$ hodisalarning birgalikda ro`y berish ehtimoli sifatida topiladi:

$$P(Z_{\max} < z) = P(X_1 < x_1, X_2 < x_2, \ldots, X_n < x_n)$$

$X_1, X_2, \ldots, X_n$ tasodifiy miqdorlar o`zaro bog'liqmas deb faraz qilingan edi. U holda $x_1 < z, x_2 < z, \ldots, x_n < z$ hodisalar ham o`zaro bog'liq bo`lmaydi, binobarin, bu hodisalarning birga ro`y berish ehtimoli $P(x_1 < z, x_2 < z, \ldots, x_n < z)$ ularning ehtimollari ko`paytmasiga teng:

$$P(X_1 < z, X_2 < z, ..., X_n < z) = P(X_1 < z).....P(X_n < z) = \prod_{k=1}^{n} P(X_n < z)$$

yoki

$$P(X_1 < z, X_2 < z, ..., X_n < z) = \prod_{k=1}^{n} F_{X\,k}(z)$$

Shunday qilib,

$$G^*(z) = P(X_1 < z, X_2 < z, ..., X_n < z) = \prod_{k=1}^{n} F_{X_k}(z) \qquad (2.1.1)$$

Endi, $Z_{\max}$ tasodifiy miqdorning taqsimot zichligini topamiz. Taqsimot zichlik ta'rifiga ko`ra

$$g^*(z) = \frac{d}{dz} G^*(z)$$

U holda (2.1.1) formulaning z bo`yicha differensiallab, quyidagiga ega bo`lamiz:

$$g^*(z) = \frac{d}{dz}\left(\prod_{k=1}^{n} F_{X_k}(z)\right) = \frac{d\,F_{X_1}(z)}{dz}\prod_{k=1}^{n} F_{X_k}(z) + F_{X_1}(z)\frac{d\,F_{X_2}(z)}{dz}\prod_{k=3}^{n} F_{X_k}(z) +$$

$$+ F_{X_1}(z)\;F_{X_2}(z)\frac{d\,F_{X_3}(z)}{dz}\prod_{k=4}^{n} F_{X_k}(z) + ... +$$

$$+\prod_{k=1}^{n-2} F_{X_k}(z)\frac{d\,F_{X_{n-1}}(z)}{dz}F_{X_n}(z) + \prod_{k=1}^{n-1} F_{X_k}(z)\frac{d\,F_{X_n}(z)}{dz}$$

yoki

$$g^*(z) = p_{X_1}(z)\prod_{k=2}^{n} F_{X_k}(z) + F_{X_1}(z)\,p_{X_2}(z)\prod_{k=3}^{n} F_{X_k}(z) +$$

$$+ F_{X_1}(z)F_{X_2}(z)\,p_{X_3}(z)\prod_{k=4}^{n} F_{X_k}(z) + ... + \prod_{k=1}^{n-2} F_{X_k}(z)\,p_{X_n}(z)F_{X_n}(z) + \prod_{k=1}^{n-1} F_{X_k}(z)\,p_{X_n}(z) =$$

$$= \frac{p_{X_1}(z)}{F_{X_1}(z)}\prod_{k=1}^{n} F_{X_k}(z) + \frac{p_{X_2}(z)}{F_{X_2}(z)}\prod_{k=1}^{n} F_{X_k}(z) + \frac{p_{X_3}(z)}{F_{X_3}(z)}\prod_{k=1}^{n} F_{X_k}(z) + ... +$$

$$+ \frac{p_{X_{n-1}}(z)}{F_{X_{n-1}}(z)}\prod_{k=1}^{n} F_{X_k}(z) + \frac{p_{X_n}(z)}{F_{X_n}(z)}\prod_{k=1}^{n} F_{X_k}(z) =$$

$$= \prod_{k=1}^{n} F_{X_k}(z)\left[\frac{p_{X_1}(z)}{F_{X_1}(z)} + \frac{p_{X_2}(z)}{F_{X_2}(z)} + \frac{p_{X_3}(z)}{F_{X_3}(z)} + ... + \frac{p_{X_{n-1}}(z)}{F_{X_{n-1}}(z)} + \frac{p_{X_n}(z)}{F_{X_n}(z)}\right] =$$

$$= \prod_{k=1}^{n} F_{X_k}(z) \sum_{i=1}^{n} \frac{p_{X_i}(z)}{F_{X_i}(z)} .$$

Shunday qilib,

$$g^*(z) = \sum_{i=1}^{n} \frac{p_{X_i}(z)}{F_{X_i}(z)} \cdot \prod_{k=1}^{n} F_{X_k}(z) \qquad (2.1.2)$$

Teorema isbot bo`ldi.

Eslatma. Agar o`zaro bog'liq bo`lmagan $X_1, X_2, ..., X_n$ tasodifiy miqdorlar bir xil taqsimlangan bo`lsa, u holda

$$G^*(z) = F^n(z)$$

$$g^*(z) = n\, p(z) F^{n-1}(z)$$

bo`ladi, bu yerda $p(z)$ va $F(z)$ funksiyalar mos ravishda $X_1, X_2, ..., X_n$ tasodifiy miqdorlarning taqsimot zichligi va taqsimot funksiyasi.

Haqiqatan ham barcha X_k miqdorlar bir xil taqsimlanganligi sababli, ular bir xil taqsimot zichliklarga (taqsimot funksiyalarga) ega, ya'ni

$$p_{X_1}(z) = ... = p_{X_n}(z), \quad F_{X_1}(z) = ... = F_{X_n}(z)$$

X_k miqdorlarning taqsimot zichliklarini $p(z)$ orqali, taqsimot funksiyalarini $F(z)$ orqali belgilasak, u holda (1) formuladan

$$G^*(z) = \prod_{k=1}^{n} F_{X_k}(z) = F^n(z)$$

ni, (2.1.2) formuladan esa

$$g^*(z) = \sum_{i=1}^{n} \frac{p_{X_i}(z)}{F_{X_i}(z)} \cdot \prod_{k=1}^{n} F_{X_k}(z) = n \frac{p(z)}{F(z)} F^n(z) = n\, p(z)\, F^{n-1}(z)$$

ni hosil qilamiz.

Hosil qilingan formulalar tadbiq qilinadigan bir misolni ko`rib o`tamiz.

2.1.1-misol. O`zaro bog'liq bo`lmagan $X_1, X_2, ..., X_n$ tasodifiy miqdorlar

$$F_{X_i}(x_i) = \begin{cases} 0, & x_i \le 0 \\ x_i, & 0 < x_i < 1 \\ 1, & x_i > 1 \end{cases} \qquad i = \overline{1, n}$$

taqsimot funksiyalar bilan berilgan. $Z_{\max} = \max(x_1, x_2, ..., x_n)$ tasodifiy miqdorning taqsimot funksiyasini va taqsimot zichligini toping?

Yechish. taqsimot funksiyani (2.1.1) formula bo`yicha topamiz. Unga $F_{X_i}(x_i)$ larning masala shartiga berilgan ifodalarini qo`yib quyidagini hosil qilamiz:

$$G^*(z) = \prod_{i=1}^{n} F_{X_i}(z) = \prod_{i=1}^{n} z^{k_i} = z^k, \quad (0 < z \le 1)$$

Bu yerda $k = \sum_{i=1}^{n} k_i$.

Shunday qilib, izlanayotgan taqsimot funksiya quyidagicha

$$G^*(z) = \begin{cases} 0, & z \le 0 \\ z^k, & 0 < z \le 1 \\ 1, & z > 1 \end{cases} \tag{2.1.3}$$

ya’ni $(0;1)$ intervalda darajali qonun bo`yicha taqsimlangan bir necha tasodifiy miqdorlarning maksimumi ham darajali qonun bo`yicha taqsimlangan bo`ladi, uning daraja ko`rsatkichi alohida qonunlar daraja ko`rsatkichlari yig’indisiga teng.

Taqsimot zichlikni topish uchun avval $p_{X_i}(x_i)$ taqsimot zichliklarini topamiz:

$$p_{X_i}(x_i) = F'_{X_i}(x_i) = \begin{cases} 0, & x_i \le 0 \\ x_i\, x_i^{k_i - 1}, & 0 < x_i \le 1 \\ 0, & x_i > 1 \end{cases}$$

(2.1.2) formulaga $p_{X_i}(z)$ va $F_{X_i}(z)$ funksiyalarning ifodalarini qo`yib, quyidagini hosil qilamiz:

$$g^*(z) = \sum_{j=1}^{n} \frac{p_{X_j}(z)}{F_{X_j}(z)} \cdot \prod_{i=1}^{n} F_{X_i}(z) = \sum_{j=1}^{n} \frac{k_j\, z^{k_j - 1}}{z^{k_j}} \prod_{i=i}^{n} Z^{k_i} =$$

$$= \sum_{j=1}^{n} k_j z^{-1}\, z^{\sum_{i=1}^{n} k_i} = \sum_{j=1}^{n} k_j z^{\sum_{i=1}^{n} k_i - 1} = k\, z^{k-1} \qquad (0 < z \le 1).$$

Bu yerda $k = \sum_{i=1}^{n} k_i$

Shunday qilib, izlanayotgan taqsimot zichlik (0;1) intervalda

$$g^*(z) = k\ z^{k-1}, \qquad k = \sum_{i=1}^{n} k_i$$

bu intervaldan tashqarida $g(z) = 0$

Izoh. Taqsimot zichlikning bu ifodasini to`g'ridan-to`g'ri (2.1.3) tenglikdan hosila olish bilan ham hosil qilish mumkin.

2.2-§. n ta o`zaro bog'liq bo`lmagan tasodifiy miqdorlardan minimalining taqsimot qonuni

Faraz qilaylik, o`zaro bog'liq bo`lmagan n ta $X_1, X_2, \ldots, X_n$ uzluksiz tasodifiy miqdorlar berilgan bo`lsin. Ulardan minimalining taqsimot qonunini topishni maqsad qilib qo`yamiz. Quyidagi belgilashlarni kiritamiz:

$$Z_{\min} = \min(X_1, X_2, \ldots, X_n), \ \Phi^*(z) = P(Z_{\min} < z), \ \varphi^*(z) = \frac{d}{dz}\Phi^*(z)$$

2.2.1-teorema. Agar o`zaro bog'liq bo`lmagan $X_1, X_2, \ldots, X_n$ uzluksiz tasodifiy miqdorlar mos ravishda $p_{X_1}(x_1), p_{X_2}(x_2), \ldots, p_{X_n}(x_n)$ taqsimot zichliklar bilan berilgan bo`lsa, u holda

$$\Phi^*(z) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(z))$$

$$\varphi^*(z) = \sum_{k=1}^{n} \frac{p_{X_k}(z)}{1 - F_{X_k}(z)} \cdot \prod_{i=1}^{n}(1 - F_{X_i}(z))$$

munosabatlar o`rinli bo`ladi, bu yerda $F_{X_i}(z)$ funksiya X_i tasodifiy miqdorning taqsimot funksiyasi.

Isbot. $\Phi^*(z)$ taqsimot funksiyani birgacha to`ldiruvchisini topamiz:

$$1 - \Phi^*(z) = P(Z_{\min} > z) = P\ (X_1 > z,\ X_2 > z, \ldots X_n > z)$$

Shartga ko`ra $X_1, X_2, \ldots, X_n$ tasodifiy miqdorlar o`zaro bog'liqmas. U holda $X_1 > z,\ X_2 > z, \ldots X_n > z$ hodisalar o`zaro bog'liq bo`lmaydi, binobarin, bu hodisalarning birgalikda ro`y berish ehtimoli $P\ (X_1 > z,\ X_2 > z,....X_n > z)$ ularning ehtimollari ko`paytmasiga teng.

$$P\ (X_1 > z,\ X_2 > z,....X_n > z) = \prod_{i=1}^{n} P(X_i > z)$$

yoki

$$P\ (X_1 > z,\ X_2 > z,....X_n > z) = \prod_{i=1}^{n}(1 - P(X_i > z)) = \prod_{i=1}^{n}(1 - F_{X_i}(z))$$

Shunday qilib,

$$1 - \Phi^*(z) = \prod_{i=1}^{n}(1 - F_{X_i}(z))$$

bundan

$$\Phi^*(z) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(z)) \qquad (2.2.1)$$

ni hosil qilamiz.

Endi, $\varphi^*(z)$ taqsimot zichlikni topishimiz mumkin. Buning uchun (2.2.1) formulani z bo`yicha differensiallaymiz:

$$\varphi^*(z) = \frac{d}{dz}\Phi(z) = -\frac{d}{dz}\left(\prod_{i=1}^{n}(1 - F_{X_i}(z))\right) =$$

$$= -\frac{d}{dz}(1 - F_{X_1}(z))\prod_{i=2}^{n}(1 - F_{X_i}(z)) -$$

$$-\frac{d}{dz}(1 - F_{X_2}(z))\cdot(1 - F_{X_1}(z))\prod_{i=3}^{n}(1 - F_{X_i}(z)) -$$

$$-(1 - F_{X_1}(z))\cdot(1 - F_{X_2}(z))\cdot\frac{d}{dz}(1 - F_{X_3}(z))\prod_{i=4}^{n}(1 - F_{X_i}(z)) - ...$$

$$... - \prod_{i=1}^{n-2}(1 - F_{X_i}(z))\cdot\frac{d}{dz}(1 - F_{X_{n-1}}(z))\cdot(1 - F_{X_n}(z)) -$$

$$-\prod_{i=1}^{n-1}(1 - F_{X_i}(z))\cdot\frac{d}{dz}(1 - F_{X_n}(z)) \qquad (2.2.2)$$

Taqsimot zichlik ta’rifiga ko`ra $F'_{X_i}(z) = p_{X_i}(z)$ ekanligini e’tiborga olib, (22.2) tenglikdan

$$\varphi^*(z) = p_{X_1}(z)\cdot\prod_{i=2}^{n}(1-F_{X_i}(z)) + (1-F_{X_1}(z))\,p_{X_2}(z)\prod_{i=3}^{n}(1-F_{X_i}(z)) +$$

$$+(1-F_{X_1}(z))(1-F_{X_2}(z))\,p_{X_3}(z)\prod_{i=4}^{n}(1-F_{X_i}(z)) + ...$$

$$\ldots + \prod_{i=4}^{n-2}(1-F_{X_i}(z))\,p_{X_{n-1}}(z)(1-F_{X_n}(z)) + \prod_{i=1}^{n-1}(1-F_{X_i}(z))\,p_{X_n}(z) \qquad (2.2.3)$$

ni hosil qilamiz.

(2.2.3) tenglikning o`ng tomonida turgan birinchi qo`shiluvchini $1-F_{X_1}(z)$ ga, ikkinchi qo`shiluvchini $1-F_{X_2}(z),\ldots,$ oxirgi qo`shiluvchini $1-F_{X_m}(z)$ ga ham ko`paytirib ham bo`lamiz. Natijada,

$$\varphi^*(z) = \frac{p_{X_2}(z)}{1-F_{X_1}(z)}\prod_{i=1}^{n}(1-F_{X_i}(z)) + \frac{p_{X_2}(z)}{1-F_{X_2}(z)}\prod_{i=1}^{n}(1-F_{X_i}(z)) +$$

$$+\frac{p_{X_3}(z)}{1-F_{X_3}(z)}\prod_{i=1}^{n}(1-F_{X_i}(z)) + ... + \frac{p_{X_n}(z)}{1-F_{X_n}(z)}\prod_{i=1}^{n}(1-F_{X_i}(z)) =$$

$$=\sum_{k=1}^{n}\frac{p_{X_k}(z)}{1-F_{X_k}(z)}\cdot\prod_{i=1}^{n}(1-F_{X_i}(z))$$

tenglikka ega bo`lamiz.

Shunday qilib, izlanayotgan taqsimot zichlik

$$\varphi^*(z) = \sum_{k=1}^{n}\frac{p_{X_k}(z)}{1-F_{X_k}(z)}\cdot\prod_{i=1}^{n}(1-F_{X_i}(z)) \qquad (2.2.4)$$

bo`ladi. Teorema isbot bo`ldi.

2.2.1-natija. Agar $X_1, X_2, ...,X_n$ o`zaro bog'liq bo`lmagan va bir xil taqsimlangan tasodifiy miqdorlar bo`lib, ularning taqsimot zichligi $p(x)$, taqsimot funksiyasi $F(x)$ bo`lsa, u holda

$$\Phi^*(z) = 1-[1-F(z)]^n \qquad (2.2.5)$$

$$\varphi^*(z) = n[1-F(z)]^{n-1}\,p(z) \qquad (2.2.6)$$

bo`ladi.

Bu tengliklar yuqoridagi (2.2.1) va (2.2.4) munosabatlardan kelib chiqadi. Haqiqatan ham, $X_1, X_2, ...,X_n$ tasodifiy miqdorlar bir xil taqsimlangan bo`lgani-dan

$$p_{X_1}(z) = p_{X_2}(z) = ... = p_{X_n}(z) = p(z)$$

$$F_{X_1}(z) = F_{X_2}(z) = ... = F_{X_n}(z) = F(z)$$

bo`ladi. (2.2.1) va (2.2.4) munosabatlardan

$$\Phi^*(z) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(z)) = 1 - [1 - F(z)]^n$$

$$\varphi^*(z) = \sum_{k=1}^{n} \frac{P_{X_k}(z)}{1 - F_{X_k}(z)} \cdot \prod_{i=1}^{n}(1 - F_{X_i}(z)) =$$

$$= n\frac{P(z)}{1 - F(z)}(1 - F(z))^n = n\ p(z)[1 - F(z)]^{n-1}$$

bo`lishi kelib chiqadi.

2.2.1-misol. xOy tenglikka qarata bir biriga bog'liq bo`lmagan holda uchta o`q uziladi. Sochilish markazi koordinatalar boshida bo`lgan normal taqsimlangan doiraviy sochilish $(\sigma_X = \sigma_Y = \sigma)$. Uchta tushadigan nuqtadan sochilish markaziga eng yaqini topiladi. Tanlangan nuqtadan sochilish markaziga cha bo`lgan $R_{\min}$ masofaning taqsimot qonunini toping.

Yechish. Masofalarni quyidagicha belgilaymiz:

R_1 - birinchi nuqtadan sochilish markazigacha bo`lgan masofa ;

R_2 - ikkinchi nuqtadan sochilish markazigacha bo`lgan masofa;

R_3 - uchinchi nuqtadan sochilish markazigacha bo`lgan masofa.

U holda $R_{\min} = \min\ (R_1, R_2, R_3)$ ga ega bo`lamiz.

O`q uzishlar bog'liq bo`lmaganligi sababli R_1, R_2 va R_3 tasodifiy miqdorlar o`zaro bog'liq bo`lmaydi, shu bilan birga R_i miqdorlar bir xil taqsimot qonunga ega bo`ladi. U holda (6) formuladan

$$\varphi^*(r) = 3[1 - F(r)]^2\ p(r) \tag{2.2.7}$$

ga ega bo`lamiz. Bu yerda $F(r)$ va $p(r)$ lar otilgan ixtiyoriy o`q tushadigan nuqtadan sochilish markazigacha bo`lgan R masofaning taqsimot funksiyasi va taqsimot zichligi.

Tekislikning tasodifiy nuqtasidan koordinatalar boshida yotadi deb qaraluvchi sochilish markazigacha bo`lgan masofa tasodifiy miqdor bo`lib,

$$R = \sqrt{X^2 + Y^2}$$

ga teng, bu yerda X va Y lar tekislikdagi nuqtalarning koordinatalarini ifodalaydi.

R tasodifiy miqdorning taqsimot funksiyasi va taqsimot zichligini aniqlaymiz.

R tasodifiy miqdorning taqsimot funksiyasi (X,Y) tasodifiy nuqtaning radiusi r markazi koordinatalar boshida bo`lgan doiraga (D soha) tushish ehtimolidir, ya'ni

$$F(r) = P(R < r) = P\{(X,Y) \subset D\}$$

Ma'lumki, bu ehtimol

$$P\{(X,Y) \subset D\} = \iint_D p(x,y)\, dx\, dy$$

ga teng, bu yerda $p(x,y)$ funksiya (X,Y) tasodifiy miqdorning taqsimot zichligi.

Shartga ko`ra,

$$p(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}.$$

Demak,

$$P\{(X,Y) \subset D\} = \frac{1}{2\pi\sigma^2} \iint_D e^{-\frac{x^2+y^2}{2\sigma^2}}\, dx\, dy$$

bo`ladi.

Hosil qilingan bu integralni qutb koordinatalariga almashtiramiz.

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

$$x^2 + y^2 = r^2, \quad dx\, dy = r\, dr\, d\varphi$$

ni topib olamiz.

Shu sababli,

$$P\{(X,Y) \subset D\} = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\varphi \int_0^r e^{-\frac{r^2}{2\sigma^2}}\, r\, dr = -\int_0^r e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{r^2}{2\sigma^2}\right).$$

Bu integralni hisoblab, R tasodifiy miqdorning taqsimot funksiyasini hosil qilamiz:

$$F(r) = P\{(X,Y) \subset D\} = 1 - e^{-\frac{r^2}{2\sigma^2}}. \qquad (2.2.8)$$

Taqsimot zichligini topish uchun (2.2.8) formulani r bo`yicha differensiallaymiz, u holda

$$\varphi^*(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad (r > 0)$$

ni hosil qilamiz.

Endi $R_{\min}$ tasodifiy miqdorning taqsimot zichligini topish mumkin. Buning uchun $F(r)$ va $p(r)$ larning yuqorida topilgan ifodalarini (2.2.7) formulaga qo`yish kifoya, u holda

$$\varphi^*(r) = 3\left[1 - (1 - e^{-\frac{r^2}{2\sigma^2}})\right]^2 \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{3r}{\sigma^2} e^{-\frac{3r^2}{2\sigma^2}} \qquad (r > 0)$$

ni hosil qilamiz.

Bu ifodani quyidagicha ham yozish mumkin:

$$\varphi^*(r) = \frac{r}{\left(\frac{\sigma}{\sqrt{3}}\right)^2} e^{-\frac{r^2}{\left(\frac{\sigma}{\sqrt{3}}\right)^2}} \qquad (r > 0)\,.$$

Demak, uchta nuqtaning eng yaqinidan sochilish markazigacha bo`lgan masofaning taqsimot zichligi har qaysi nuqtadan sochilish markazigacha bo`lgan masofalar qanday ko`rinishdagi taqsimot zichlikka ega bo`lsa, shunday ko`rinishga ega ekan, lekin $\sigma$ parametr $\sqrt{3}$ marta kichiklashgan bo`ladi.

III BOB. NORMAL TAQSIMOT BILAN BOG'LANGAN TAQSIMOT QONUNLAR.

1-§. F taqsimot

Bu paragrafda matematik statistikaning bir qator maxsus bo`limlarida, ayniqsa, dispersial analizda katta ahamiyatga ega bo`lgan taqsimot haqida so`z yuritiladi.

3.1.1-teorema. Agar $X_1, X_2, ..., X_p$ va $Y_1, Y_2, ..., Y_q$ lar o`zaro bog'liq bo`lmagan tasodifiy miqdorlar bo`lib, ularning har biri $(0;1)$ parametrli normal qonun bo`yicha taqsimlangan bo`lsa, u holda

$$F_{pq} = \frac{\frac{1}{p}\sum_{i=1}^{p} X_i^2}{\frac{1}{q}\sum_{i=1}^{q} Y_j^2}$$

tasodifiy miqdor $\frac{p}{q}x$

$$P(F_{pq} < x) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \int_0^{\frac{p}{q}x} \frac{y^{\frac{p}{2}-1}}{(1+y)^{\frac{p+q}{2}}} dy, \quad x > 0 \qquad (3.1.1)$$

taqsimot qonunga ega bo`ladi.

(3.1.1) taqsimot Fisher-Snedekorning F taqsimoti deyiladi.

Isbot. $\frac{p}{q}F_{pq}$ tasodifiy miqdorni qaraymiz. U

$$\frac{p}{q}F_{pq} = \frac{\sum_{i=1}^{p} X_i^2}{\sum_{i=1}^{q} Y_j^2}$$

ko`rinishda bo`ladi.

[6] da ko`rsatilgan bo`yicha $\sum_{i=1}^{p} X_i^2$ va $\sum_{i=1}^{q} Y_j^2$ tasodifiy miqdorlar mos ravishda p va q ozodlik darajali χ^2 (Xi kvadrat) qonun bo`yicha taqsimlangan, shu sababli ular $\chi_p^2$ va $\chi_q^2$ orqali belgilanadi. Bu belgilashlardan so`ng $\frac{p}{q}F_{pq}$ tasodifiy miqdor

$$\frac{p}{q}F_{pq} = \frac{\chi_p^2}{\chi_q^2}$$

ko`rinishda bo`ladi.

χ_p^2 va χ_q^2 miqdorlarning taqsimot zichliklari quyidagi ko`rinishga ega:

$$\left.\begin{aligned} P_{\chi_p^2}(u) &= \frac{u^{\frac{p}{2}-1}}{2^{\frac{p}{2}}\,\Gamma\left(\frac{p}{2}\right)}\, e^{-\frac{u}{2}}, \qquad u>0 \\ P_{\chi_q^2}(v) &= \frac{v^{\frac{q}{2}-1}}{2^{\frac{q}{2}}\,\Gamma\left(\frac{q}{2}\right)}\, e^{-\frac{v}{2}}, \qquad v>0 \end{aligned}\right\} \tag{3.1.2}$$

bog’liq bo`lmagan tasodifiy miqdorlar bo`linmasining taqsimot funksiyasini topish formulasiga ko`ra

$$P\left(\frac{p}{q}F_{pq} < x\right) = P\left(\frac{\chi_p^2}{\chi_q^2} < x\right) = \iint\limits_{\left\{\substack{\frac{u}{v}<x \\ u>0,\, v>0}\right\}} P_{\chi_p^2}(u)\, P_{\chi_q^2}(v)\, du\, dv$$

Bu formulaga $P_{\chi_p^2}(u)$ va $P_{\chi_q^2}(v)$ larning (3.1.2) dagi ifodalarini qo`yib, quyidagini hosil qilamiz:

$$P\left(\frac{p}{q}F_{pq} < x\right) = \iint\limits_{\left\{\substack{\frac{u}{v}<x \\ u>0,\, v>0}\right\}} \frac{u^{\frac{p}{2}-1}\, e^{-\frac{u}{2}}}{2^{\frac{p}{2}}\,\Gamma\left(\frac{p}{2}\right)} \cdot \frac{u^{\frac{q}{2}-1}\, e^{-\frac{v}{2}}}{2^{\frac{q}{2}}\,\Gamma\left(\frac{q}{2}\right)}\, du\, dv =$$

$$= \frac{1}{2^{\frac{p+q}{2}}\,\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \iint\limits_{\left\{\substack{\frac{u}{v}<x \\ u>0,\, v>0}\right\}} u^{\frac{p}{2}-1} \cdot u^{\frac{q}{2}-1}\, e^{-\frac{u+v}{2}}\, du\, dv \tag{3.1.3}$$

Agar integralda $u = yz,\ v = z$ almashtirish bajarsak,

$$\iint\limits_{\left\{\substack{\frac{u}{v}<x \\ u>0,\, v>0}\right\}} u^{\frac{p}{2}-1} \cdot u^{\frac{q}{2}-1} e^{-\frac{u+v}{2}}\, du\, dv$$

integral quyidagi ko`rinishga keladi:

$$\iint\limits_{\left\{\substack{\frac{u}{v}<x \\ u>0,\, v>0}\right\}} u^{\frac{p}{2}-1} \cdot u^{\frac{q}{2}-1} e^{-\frac{u+v}{2}}\, du\, dv = \int_0^x y^{\frac{p}{2}-1}\, dy \int_0^{\infty} z^{\frac{p}{2}-1}\, z^{\frac{q}{2}-1}\, e^{-\frac{z(1+y)}{2}}\, zdz =$$

$$= \int_0^x y^{\frac{p}{2}-1}\, dy \int_0^{\infty} z^{\frac{p+q}{2}-1}\, e^{-\frac{z(1+y)}{2}}\, dz$$

Bu integralni hisoblash uchun $\frac{z(1+y)}{2} = t$ almashtirish bajaramiz. Unda

$$z = \frac{2t}{1+y},\ dz = \frac{2\,dt}{1+y}$$

bo`lib,

$$\iint\limits_{\left\{\substack{\frac{u}{v}<x \\ u>0,\, v>0}\right\}} u^{\frac{p}{2}-1} \cdot u^{\frac{q}{2}-1} e^{-\frac{u+v}{2}}\, du\, dv = \int_0^x y^{\frac{p}{2}-1}\, dy \int_0^{\infty} \left(\frac{2t}{1+y}\right)^{\frac{q+q}{2}-1} e^{-t}\, \frac{2\,dt}{1+y}\, dt =$$

$$= 2^{\frac{p+q}{2}} \int_0^x y^{\frac{p}{2}-1} \left(\frac{1}{1+y}\right)^{\frac{p+q}{2}} dy \int_0^{\infty} t^{\frac{p+q}{2}-1} e^{-t}\, dt = 2^{\frac{p+q}{2}}\, \Gamma\!\left(\frac{p+q}{2}\right) \int_0^x \frac{y^{\frac{p}{2}-1}}{(1+y)^{\frac{p+q}{2}}}\, dy$$

bo`ladi. Bu ifodani (3.1.3) ga qo`yib,

$$P\!\left(\frac{p}{q} F_{pq} < x\right) = \frac{1}{2^{\frac{p+q}{2}}\, \Gamma\!\left(\frac{p}{2}\right)\Gamma\!\left(\frac{q}{2}\right)}\, 2^{\frac{p+q}{2}}\, \Gamma\!\left(\frac{p+q}{2}\right) \int_0^x \frac{y^{\frac{p}{2}-1}}{\left(1+y\right)^{\frac{p+q}{2}}}\, dy =$$

$$= \frac{\Gamma\!\left(\frac{p+q}{2}\right)}{\Gamma\!\left(\frac{p}{2}\right)\ \Gamma\!\left(\frac{q}{2}\right)} \int_0^x \frac{y^{\frac{p}{2}-1}}{\left(1+y\right)^{\frac{p+q}{2}}} dy \qquad (3.1.4)$$

ni hosil qilamiz. Endi (4) yordamida F_{pq} tasodifiy miqdorning taqsimot funksiyasi

$$P\left(\frac{p}{q}F_{pq} < x\right) = P\left(\frac{p}{q}\,F_{pq} < \frac{p}{q}x\right) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\int_0^{\frac{p}{q}x}\frac{y^{\frac{p}{2}-1}}{(1+y)^{\frac{p+q}{2}}}dy, \qquad x > 0 \qquad (3.1.5)$$

ga tengligiga ishonch hosil qilamiz. Shu bilan teorema isbotlandi.

(3.1.5) dan F_{pq} tasodifiy miqdorning taqsimot zichligini topamiz:

$$f_{pq}(x) = \frac{dP(F_{pq} < x)}{dx} = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\frac{d}{dx}\left(\int_0^{\frac{p}{q}x}\frac{y^{\frac{p}{2}-1}}{(1+y)^{\frac{p+q}{2}}}\,dy\right) =$$

$$= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{p}{q}x\right)'\frac{\left(\frac{p}{q}x\right)^{\frac{p}{2}-1}}{\left(1+\frac{px}{q}\right)^{\frac{p+q}{2}}}\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\frac{\left(\frac{p}{q}\right)^{\frac{p}{2}}q^{\frac{p+q}{2}}}{(q+px)^{\frac{p+q}{2}}}\cdot x^{\frac{p}{2}-1} =$$

$$p^{\frac{p}{2}}\,q^{\frac{q}{2}}\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\cdot\frac{x^{\frac{p}{2}-1}}{(q+px)^{\frac{p+q}{2}}}$$

Demak,

$$f_{pq}(x) = p^{\frac{p}{2}}\,q^{\frac{q}{2}}\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\,\frac{x^{\frac{p}{2}-1}}{(q+px)^{\frac{p+q}{2}}}, \qquad x > 0 \qquad (3.1.6)$$

Endi F taqsimotning asosiy xarakteristikalarini topamiz. Tasodifiy miqdorning matematik kutilmasi ta’rifiga ko`ra

$$MF_{pq} = \int_0^{\infty} x\,f_{pq}(x)\,dx$$

Bu formulaga $f_{pq}(x)$ ning (3.1.6) dagi ifodasini qo`yib,

$$MF_{pq} = p^{\frac{1}{2}}\, q^{\frac{1}{2}} \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \int_0^\infty \frac{x^{\frac{p}{2}}}{\left(1+\frac{p}{q}x\right)^{\frac{p+q}{2}}} dx \qquad (3.1.7)$$

ni hosil qilamiz. Bu tenglikdagi integralni hisoblash uchun $t = \frac{p}{q}x$ deb almashtirish bajaramiz. Unda

$$x = \frac{q}{p}t, \quad dx = \frac{q}{p}dt$$

bo`lib,

$$\int_0^\infty \frac{x^{\frac{p}{2}}}{\left(1+\frac{p}{q}\right)^{\frac{p+q}{2}}} dx = \int_0^\infty \left(\frac{q}{p}t\right)^{\frac{p}{2}} (1+t)^{-\frac{p+q}{2}} \frac{q}{p} dt = \left(\frac{q}{p}\right)^{\frac{p}{2}+1} \int_0^\infty t^{\frac{p}{2}} (1+t)^{-\frac{p+q}{2}} dt$$

bo`ladi. Beta-funksiya deb ataladigan va ushbu

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt = \int_0^\infty t^{x-1}(1+t)^{-(x+y)} dt$$

tenglik bilan ataladigan funksiyadan foydalanib

(Bu analiz kursidan ma'lum bo`lgan

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} du, \quad x>0, \; y>0$$

Beta funksiyadan kelib chiqadi. Haqiqatan ham, integralda $t = \frac{u}{1-u}$ almashtirishni bajarsak, unda $u = \frac{t}{1+t}$, $du = \frac{dt}{(1+t)^2}$ bo`ladi. Integrallashning yangi chegaralarini topamiz: $u=0$ da $t=0$ ga, $u=1$ da $t=\infty$ ga ega bo`lamiz. Demak,

$$\int_0^1 u^{x-1}(1-u)^{y-1} du = \int_0^\infty \left(\frac{t}{1+t}\right)^{x-1} \left(1-\frac{t}{1+t}\right)^{y-1} \frac{dt}{(1+t)^2} =$$

$$= \int_0^\infty t^{x-1}(1+t)^{-(x-1)-(y-1)-2} dt = \int_0^\infty t^{x-1}(1+t)^{-(x+y)} dt \, \mathbf{)}$$

$$\int_0^x \frac{x^{\frac{p}{2}}}{\left(1+\frac{p}{q}x\right)^{\frac{p+q}{2}}}dx = \left(\frac{q}{p}\right)^{\frac{p}{2}+1}\int_0^\infty t^{\frac{p}{2}}(1+t)^{-\frac{p+q}{2}}dt = \left(\frac{q}{p}\right)^{\frac{p}{2}+1}\int_0^\infty t^{\left(\frac{p}{2}+1\right)-1}(1+t)^{-\left(\frac{p}{2}+1+\frac{p+q}{2}-\frac{p}{2}-1\right)}dt =$$

$$= \left(\frac{q}{p}\right)^{\frac{p}{2}+1}\int_0^\infty t^{\left(\frac{p}{2}+1\right)-1}(1+t)^{-\left(\frac{p}{2}+1+\frac{q}{2}-1\right)}dt = \left(\frac{q}{p}\right)^{\frac{p}{2}+1} B\left(\frac{p}{2}+1, \frac{q}{2}-1\right)$$

ga ega bo`lamiz.

Agar Gamma va Beta funksiyalar o`rtasidagi ushbu

$$B(x,y) = \frac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)}$$

bog ’lanishdan foydalansak, u holda

$$\int_0^x \frac{x^{\frac{p}{2}}}{\left(1+\frac{p}{q}x\right)^{\frac{p+q}{2}}}dx = \left(\frac{q}{p}\right)^{\frac{p}{2}+1}\frac{\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{q}{2}-1\right)}{\Gamma\left(\frac{p+q}{2}\right)} \tag{3.1.8}$$

bo`ladi.

(3.1.8) ni (3.1.7) ga qo`yib, quyidagini hosil qilamiz:

$$MF_{pq} = p^{\frac{1}{2}}\ q^{\frac{1}{2}}\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}q^{-\frac{p+q}{2}}\left(\frac{q}{p}\right)^{\frac{p}{2}+1}\frac{\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{q}{2}-1\right)}{\Gamma\left(\frac{p+q}{2}\right)} =$$

$$= \frac{q}{p}\cdot\frac{\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{q}{2}-1\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} = \frac{q}{p}\cdot\frac{\frac{p}{2}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}-1\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} = \frac{q}{2}\cdot\frac{\Gamma\left(\frac{q}{2}-1\right)}{\Gamma\left(\frac{q}{2}\right)} =$$

Bu yerda biz Gamma funksiyaning $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$ xossasidan foydalandik. Shunday qilib,

$$MF_{pq} = \frac{q}{q-2}, \qquad q \neq 2$$

F_{pq} miqdorning dispersiyasining ushbu

$$DF_{pq} = MF_{pq}^2 - \left(MF_{pq}\right)^2$$

formula bo`yicha topamiz.

$$MF_{pq}=\frac{q}{q-2}$$

ni hisobga olib,

$$DF_{pq}=MF_{pq}^2-\left(\frac{q}{q-2}\right)^2 \tag{3.1.9}$$

ni hosil qilamiz.

MF_{pq}^2 ni topamiz.

$$MF_{pq}^2=\int_0^\infty x^2 f_{pq}(x)dx=p^{\frac{p}{2}}q^{\frac{q}{2}}\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\int_0^\infty\frac{x^{\frac{p}{2}+1}}{(p+qx)^{\frac{p+q}{2}}}dx \tag{3.1.10}$$

Yuqoridagiga o`xshab integrallab,

$$\int_0^\infty\frac{x^{\frac{p}{2}+1}}{(p+qx)^{\frac{p+q}{2}}}dx=\int_0^\infty\left(\frac{q}{p}t\right)^{\frac{p}{2}+1}(1+t)^{-\frac{p+q}{2}}\frac{\frac{q}{p}}{q^{\frac{p+q}{2}}}dt=\left(\frac{q}{p}\right)^{\frac{p}{2}+2}q^{-\frac{p+q}{2}}\int_0^\infty t^{\frac{p}{2}+1}(1+t)^{-\frac{p+q}{2}}dt=$$

$$=\left(\frac{q}{p}\right)^{\frac{p}{2}+2}q^{-\frac{p+q}{2}}\int_0^\infty t^{\left(\frac{p}{2}+2\right)-1}(1+t)^{-\left(\frac{p}{2}+2+\frac{q}{2}-2\right)}dt=\left(\frac{q}{p}\right)^{\frac{p}{2}+2}q^{-\frac{p+q}{2}}B\left(\frac{p}{2}+2,\frac{q}{2}-2\right)=$$

$$=\left(\frac{q}{p}\right)^{\frac{p}{2}+2}q^{-\frac{p+q}{2}}\frac{\Gamma\left(\frac{p}{2}+2\right)\Gamma\left(\frac{q}{2}-2\right)}{\Gamma\left(\frac{p+q}{2}\right)} \tag{3.1.11}$$

ni topamiz. (3.1.11) ni (3.1.10) ga qo`yib,

$$MF_{pq}^2=p^{\frac{p}{2}}q^{\frac{q}{2}}\frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{q}{p}\right)^{\frac{p}{2}+2}q^{-\frac{p+q}{2}}\frac{\Gamma\left(\frac{p}{2}+2\right)\Gamma\left(\frac{q}{2}-2\right)}{\Gamma\left(\frac{p+q}{2}\right)}=$$

$$=\left(\frac{q}{p}\right)^2\frac{\Gamma\left(\frac{p}{2}+2\right)\Gamma\left(\frac{q}{2}-2\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}=\left(\frac{q}{p}\right)^2\frac{\left(\frac{p}{2}+1\right)\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{q}{2}-2\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}=$$

$$=\left(\frac{q}{p}\right)^2\frac{\left(\frac{p}{2}+1\right)\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{q}{2}-2\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(1+\frac{q}{2}-1\right)}=\left(\frac{q}{p}\right)^2\frac{\left(\frac{p}{2}+1\right)\frac{p}{2}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}-2\right)}{\Gamma\left(\frac{p}{2}\right)\left(\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}-1\right)}=$$

$$=\left(\frac{q}{p}\right)^2\frac{\left(\frac{p}{2}+1\right)\frac{p}{2}\Gamma\left(\frac{q}{2}-2\right)}{\left(\frac{q}{2}-1\right)\Gamma\left(1+\frac{q}{2}-2\right)}=\left(\frac{q}{p}\right)^2\frac{\left(\frac{p}{2}+1\right)\frac{p}{2}\;\Gamma\left(\frac{q}{2}-2\right)}{\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)\Gamma\left(\frac{q}{2}-2\right)}=$$

$$=\left(\frac{q}{p}\right)^2\frac{\frac{p}{2}\left(\frac{p}{2}+1\right)}{\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)}=\left(\frac{q}{p}\right)^2\frac{p(p+2)}{(q-2)\,(q-4)}=\frac{q^2(p+2)}{p(q-2)(q-4)} \qquad (3.1.12)$$

ni hosil qilamiz.

Izlanayotgan dispersiyani topamiz, buning uchun (3.1.12) ni (3.1.9) ga qo`yamiz:

$$DF_{pq}=\frac{q^2(p+2)}{p(q-2)(q-4)}-\left(\frac{q}{q-2}\right)^2=\frac{q^2}{q-2}\left(\frac{p+2}{p(q-4)}-\frac{1}{q-2}\right)=$$

$$=\frac{q^2}{q-2}\cdot\frac{(p+2)(q-2)-p(q-4)}{p(q-2)(q-4)}=\frac{q^2}{q-2}\cdot\frac{(p+2)(q-2)-p(q-2)+2p}{p(q-2)(q-4)}=$$

$$=\frac{q^2}{q-2}\cdot\frac{2(p+q-2)}{p(q-2)(q-4)}=\frac{2q^2(p+q-2)}{p(q-2)^2(q-4)}$$

Shunday qilib,

$$DF_{pq}=\frac{2q^2(p+q-2)}{p(q-2)^2(q-4)},\;\; q>4$$

3.2-§. Betta taqsimot

Bog'liq bo`lmagan $X_1, X_2, X_3 ..., X_n$, $Y_1, Y_2, Y_3 ..., Y_m$ tasodifiy miqdorlar berilgan bo`lsin. Ushbu

$$Z_{nm} = \frac{\sum_{s=1}^{n} X_s^2}{\sum_{s=1}^{n} X_s^2 + \sum_{j=1}^{m} Y_j^2} \qquad (3.2.1)$$

tasodifiy miqdorni tuzamiz.

Bu yerda masala $X_1, X_2, X_3 ..., X_n$, $Y_1, Y_2, Y_3 ..., Y_m$ tasodifiy miqdorning taqsimot funksiyalarini bilgan holda (3.2.1) tasodifiy miqdorning taqsimot funksiyasini topishdan iborat.

3.2.1-teorema. Agar bog'liq bo`lmagan $X_1, X_2, X_3 ..., X_n$, $Y_1, Y_2, Y_3 ..., Y_m$ tasodifiy miqdorlar (0;1) parametrli normal taqsimot qonun bo`yicha taqsimlangan bo`lsa, u holda

$$P(Z_{nm} < z) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \int_0^{\frac{z}{1-z}} x^{\frac{n}{2}-1} (1+x)^{-\frac{n+m}{2}} dx \qquad 0 < z < 1 \qquad (3.2.2)$$

bo`ladi, bu yerda $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx$ gamma funksiya

Isbot. Ravshanki, $Z_{nm} = \frac{\chi_n^2}{\chi_n^2 + \chi_m^2}$. Bu yerda χ_n^2 va χ_m^2 lar ozodlik darajalari mos ravishda n va m bo`lgan χ^2 (xi kvadrat) tasodifiy miqdorlar.

Bizni qiziqtirayotgan

$$\frac{\chi_n^2}{\chi_n^2 + \chi_m^2} < z$$

tengsizlikning har ikkala tomonini $\chi_n^2 + \chi_m^2$ ga ko`paytirib,

$$\frac{\chi_n^2}{\chi_m^2} < \frac{z}{1-z}$$

tengsizlikka ega bo`lamiz.

Shunday qilib,

$$P(Z_{nm} < z) = P\left(\frac{\chi_n^2}{\chi_m^2} < \frac{z}{1-z}\right).$$

Endi

$$F_{nm} = \frac{\frac{1}{n}\sum_{s=1}^{n} X_s^2}{\frac{1}{m}\sum_{i=1}^{m} X_i^2}$$

tasodifiy miqdordan foydalanamiz [1]. Bundan

$$\frac{\chi_n^2}{\chi_m^2} = \frac{n}{m} F_{nm}$$

bo`lishi kelib chiqadi.

3.1-§ da $\frac{n}{m} F_{nm}$ tasodifiy miqdor quyidagi taqsimot funksiyaga ega ekanligi isbot qilingan edi:

$$P\left(\frac{n}{m} F_{nm} < t\right) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\,\Gamma\left(\frac{m}{2}\right)} \int_0^t \frac{x^{\frac{n}{2}-1}}{(1+x)^{\frac{n+m}{2}}}\, dx.$$

Bu formuladan foydalanib, izlanayotgan taqsimot funksiyani hosil qilamiz

$$P(Z_{nm} < z) = P\left(\frac{n}{m} F_{nm} < \frac{z}{1-z}\right) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\,\Gamma\left(\frac{m}{2}\right)} \int_0^{\frac{z}{1-z}} \frac{x^{\frac{n}{2}-1}}{(1+x)^{\frac{n+m}{2}}}\, dx.$$

Bu taqsimot n va m ozodlik darajali Beta taqsimot deb yuritiladi. Beta taqsimotning zichlik funksiyasi

$$g_{nm}(z) = \frac{dP(Z_{nm} < z)}{dz} = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \frac{d}{dz}\left(\int_0^{\frac{z}{1-z}} \frac{x^{\frac{n}{2}-1}}{(1+x)^{\frac{n+m}{2}}} dx\right) =$$

$$= \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \cdot \frac{\left(\frac{z}{1-z}\right)^{\frac{n}{2}-1}}{\left(1+\frac{z}{1-z}\right)^{\frac{n+m}{2}}} \frac{d}{dz}\left(\frac{z}{1-z}\right) =$$

$$= \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \cdot \frac{(1-z)^{\frac{n+m}{2}}}{(1-z)^2} \cdot \left(\frac{z}{1-z}\right)^{\frac{n}{2}-1} =$$

$$= \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \cdot z^{\frac{n}{2}-1}(1-z)^{\frac{m}{2}-1}, \quad 0 < z < 1. \qquad (3.2.3)$$

Beta va Gamma funksiyalar orasida ushbu

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

bog'lanish matematik analiz kursidan bizga ma'lum [10]. U holda (3.2.2) va (3.2.3) formulalar quyidagi ko`rinishlarga ega bo`ladi:

$$P(Z_{nm} < z) = \frac{1}{B\left(\frac{n}{2},\frac{m}{2}\right)} \int_0^{\frac{z}{1-z}} x^{\frac{n}{2}}(1+x)^{-\frac{n+m}{2}}\,dx, \qquad (0 < z < 1)$$

$$g_{nm}(z) = \frac{1}{B\left(\frac{n}{2},\frac{m}{2}\right)} z^{\frac{n}{2}-1}(1-z)^{\frac{m}{2}-1}, \qquad (0 < z < 1).$$

Teorema isbot bo`ldi.

(3.2.2) taqsimot funksiyani yig'indining va bo`linmaning taqsimot funksiyalarini topish formulalari yordamida bevosita topish mumkin, ammo bu ancha murakkab hisoblanadi. Biz bu yerda uni qisqa ham oddiy yo`l bilan topishga muvaffaq bo`ldik.

Endi asosiy sonli xarakteristikalarini topamiz.

$$M\,Z_{nm} = \int_0^1 z\,g_{nm}(z)\,dz = \frac{1}{B\left(\frac{n}{2},\frac{m}{2}\right)} \int_0^1 z^{\frac{n}{2}}(1-z)^{\frac{m}{2}-1}\,dz =$$

$$\frac{1}{B\left(\frac{n}{2},\frac{m}{2}\right)}\int_0^1 z^{\left(\frac{n}{2}+1\right)-1}(1-z)^{\frac{m}{2}-1}dz=\frac{B\left(\frac{n}{2}+1,\frac{m}{2}\right)}{B\left(\frac{n}{2},\frac{m}{2}\right)}=$$

$$=\frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}\cdot\frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}=$$

$$=\frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\frac{m+n}{2}\Gamma\left(\frac{m+n}{2}\right)}\cdot\frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}=\frac{n}{n+m},$$

$$DZ_{nm}=MZ_{nm}^2-(MZ_{nm})^2=MZ_{nm}^2-\left(\frac{n}{n+m}\right)^2=$$

$$=\frac{1}{B\left(\frac{n}{2},\frac{m}{2}\right)}\int_0^1 z^{\frac{n}{2}+1}(1-z)^{\frac{m}{2}-1}\ dz-\left(\frac{n}{n+m}\right)^2=$$

$$=\frac{1}{B\left(\frac{n}{2},\frac{m}{2}\right)}\int_0^1 z^{(\frac{n}{2}+2)-1}(1-z)^{\frac{m}{2}-1}\ dz-\left(\frac{n}{n+m}\right)^2=$$

$$=\frac{1}{B\left(\frac{n}{2},\frac{m}{2}\right)}\cdot B\left(\frac{n}{2}+2,\frac{m}{2}\right)-\left(\frac{n}{n+m}\right)^2=$$

$$=\frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\ \frac{\Gamma\left(\frac{n}{2}+2\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n+m}{2}+2\right)}-\left(\frac{n}{n+m}\right)^2=$$

$$=\frac{\Gamma\left(\frac{n+m}{2}\right)\left(\frac{n}{2}+1\right)\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+m}{2}+1\right)\frac{n+m}{2}\Gamma\left(\frac{n+m}{2}\right)}-$$

$$-\left(\frac{n}{n+m}\right)^2=\frac{\left(\frac{n}{2}+1\right)\frac{n}{2}}{\left(\frac{n+m}{2}+1\right)\frac{n+m}{2}}-\left(\frac{n}{n+m}\right)^2=\frac{2nm}{(n+m)^2(n+m+2)}.$$

Shunday qilib, Z_{nm} tasodifiy miqdorning matematik kutilishi $\frac{n}{n+m}$ ga, dispersiyasi $\frac{2nm}{(n+m)^2(n+m+2)}$ ga teng.

3.3-§. Bo`linmaning taqsimot qonuni

O`zaro bog'liq bo`lmagan $(0;1)$ parametrli normal qonun bo`yicha taqsimlangan

$$Y_1, Y_2, X_1, X_2, ..., X_n$$

tasodifiy miqdor berilgan bo`lsin.

$$Z_n = \frac{Y_1 - Y_2}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}}$$

tasodifiy miqdorning taqsimot funksiyasini topish talab qilinadi.

3.3.1-teorema. Yuqorida kiritilgan Z_n tasodifiy miqdor

$$P(Z_n < x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\,\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2}}} \left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} du, \qquad -\infty < x < \infty$$

taqsimot funksiyaga ega bo`ladi, bu yerda $\Gamma(\alpha)$ - Gamma funksiya

Isbot. Avval $Y_0 = Y_1 - Y_2$ ayirmaning taqsimot zichligini topamiz. U bizga keyin kerak bo`ladi. Ushbu formuladan foydalanamiz [6]:

$$g(z) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-z)\, dz$$

U holda

$$g(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(x-z)^2}{2}}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}-\frac{x^2}{2}+xz-\frac{z^2}{2}}\, dx =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2+xz-\frac{z^2}{4}-\frac{z^2}{4}}\, dx = \frac{e^{-\frac{z^2}{4}}}{2\,\pi}\int_{-\infty}^{\infty} e^{-\left(x-\frac{z}{2}\right)^2}\, dx$$

$$= \frac{e^{-\frac{z^2}{4}}}{2\pi\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{\left(x-\frac{z}{2}\right)^2}{2}} dx = \frac{e^{-\frac{z^2}{4}}}{2\pi\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

Agar

$$\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$$

ligini e’tiborga olsak, u holda

$$g(z) = \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} = \frac{1}{\sqrt{2} \cdot \sqrt{2\pi}} e^{-\frac{z^2}{4}} \qquad (3.3.1)$$

bo`ladi.

Demak, bog’liq bo`lmagan $Y_1, Y_2$ larning har biri $(0;1)$ parametrli normal qonun bo`yicha taqsimlangan bo`lsa, ularning ayirmasi $(0;2)$ parametrli normal qonun bo`yicha taqsimlangan tasodifiy miqdor bo`lar ekan.

Endi (3.3.1) yordamida $\frac{Y_0}{2}$ tasodifiy miqdorning zichlik funksiyasi

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

ga tengligiga ishonch hosil qilamiz, bu demak, $\frac{Y_0}{\sqrt{2}}$ tasodifiy miqdor $(0;1)$ parametrli normal qonun bo`yicha taqsimlangan bo`ladi.

Teoremani isbotlash uchun Z_n tasodifiy miqdorni

$$Z_n = \sqrt{2}\,\tau_n$$

ko`rinishda yozib olamiz, bu yerda

$$\tau_n = \frac{\frac{Y_0}{2}\sqrt{n}}{\chi_n} \qquad \left(\chi_n = \sqrt{\sum_{i=1}^{n} X_i^2}\right)$$

τ_n tasodifiy miqdorning taqsimoti bizga ma’lum, u n ozodlik darajali Styudent taqsimotga ega [1]:

$$P(\tau_n < x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\,\pi}} \int_{-\infty}^{x} \left(1 + \frac{n^2}{n}\right)^{-\frac{n+1}{2}} du \qquad (3.3.2)$$

Endi (3.3.2) yordamida Z_n tasodifiy miqdorning taqsimot funksiyasini topamiz:

$$P(Z_n < x) = P(\sqrt{2}\,\tau_n < x) = P\left(\tau_n < \frac{x}{\sqrt{2}}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2}}} \left(1+\frac{u^2}{n}\right)^{-\frac{n+1}{2}} du$$

Taqsimot zichligi

$$f(x) = \frac{d\,P(Z_n < x)}{dx} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \cdot \frac{1}{2}\left(1+\frac{x^2}{2n}\right)^{-\frac{n+1}{2}} =$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{2\,n\pi}} \left(1+\frac{x^2}{2n}\right)^{-\frac{n+1}{2}}; \quad -\infty < x < \infty.$$

Teorema isbot bo`ldi.

1.4-§. Tasodifiy sondagi tasodifiy miqdorlar yig’indisining taqsimot qonuni

Bu paragrafda biz qo`shiluvchilari soni tasodifiy bo`lgan yig’indining taqsimot qonunini topishni maqsad qilib qo`yamiz.

3.4.1-teorema. $X_1, X_2, \ldots$ bog’liq bo`lmagan $f(x)$ taqsimot zichlikka ega bo`lgan tasodifiy miqdorlar, Y ularga bog’liq bo`lmagan $P_n = P(Y = n)$ $(n = 1, 2, \ldots, N)$ taqsimot qonunga ega butun qiymatli tasodifiy miqdor bo`lsin. U holda

$$Z = \sum_{i=1}^{Y} X_i$$

tasodifiy miqdorning taqsimot zichligi

$$\varphi(z) = \sum_{n=1}^{N} f^{n}(z)\; P_n$$

tenglik bilan aniqlanadi.

Isbot. Y tasodifiy miqdor $n\ (n=1,2,...,N)$ qiymatni qabul qilgan deb faraz qilaylik. Buning ehtimoli P_n ga teng. Bu farazda

$$Z=Z_n=\sum_{i=1}^{Y}X_i$$

bo`ladi.

$f^{(n)}(x)$ orqali n ta bog'liq bo`lmagan bir xil taqsimlangan $X_1, X_2,..., X_n$ tasodifiy miqdorlar yig'indisining taqsimot zichligini belgilaymiz. Bu zichliklarni ketma-ket topish mumkin: avval $f^{(2)}(x)$-ikkita bir xil $f(x)$ va $f(x)$ qonunlar kompozitsiyasini so`ng $f^{(3)}(x)-f^{(2)}(x)$ va $f(x)$ qonunlar kompozitsiyasini topamiz va hokazo.

To`la ehtimol formulasiga ko`ra Z tasodifiy miqdorning taqsimot zichligi

$$\varphi(z)=\sum_{n=1}^{N}f^{(n)}(z)\ P_n$$

ga teng bo`ladi. Teorema isbot bo`ldi.

Endi Z tasodifiy miqdorning sonli xarakteristikalarini topamiz. Buning uchun o`sha usulning o`zidan foydalanamiz. $Y=n$ bo`lsin deb faraz qilaylik. Bu holda shartli matematik kutilma

$$MZ_n=M\left(\sum_{i=1}^{n}X_i\right)=n\ m_x$$

bo`ladi, bu yerda

$$m_x=MX_i=\int_{-\infty}^{\infty}x\ f(x)\ dx\,.$$

Unda to`la matematik kutilmani

$$m_z=MZ=\sum_{n=1}^{N}n\,m_x\,p_n=m_x\sum_{n=1}^{N}n\ p_n=m_x\cdot m_y$$

ifodadan topamiz, bu yerda

$$m_y=\sum_{n=1}^{N}n\ p_n$$

Xuddi shunday $Y=n$ sharda ikkinchi tartibli boshlang'ich momentni ham topamiz:

$$M\,Z_n^2=M\left(\sum_{i=1}^{n}X_i\right)^2=M\left[\sum_{i=1}^{n}X_i^2+2\sum_{i<j}X_i\,X_j\right]=M\left(\sum_{i=1}^{n}X_i^2\right)+$$

$$+2\sum_{i<j}M(X_i\,X_j)=M\left(\sum_{i=1}^{n}X_i^2\right)+2\sum_{i<j}MX_i\cdot M\,X_j)=$$

$$=n\alpha_{2x}+2\sum_{i<j}m_x\,m_x=n\alpha_{2x}+(n-1)\,n\,m_x^2=$$

$$=n\alpha_{2x}-n\,m_x^2+n^2m_x^2=n(\alpha_{2x}-m_x^2)+n^2\,m_x^2=n\,D_x+n^2\,m_x^2\,.$$

Bu yerda

$$D_x=DX_i=\int_{-\infty}^{\infty}(x-m_x)^2\;f(x)\,dx$$

Z tasodifiy miqdorning ikkinchi tartibli boshlang'ich momenti

$$\alpha_{2z}=MZ^2=\sum_{n=1}^{N}M\,Z_n^2\cdot P_n=\sum_{n=1}^{N}(nD_x+n^2\,m_x^2)\;P_n=$$

$$=D_x\sum_{n=1}^{N}n\,P_n+m_x^2\sum_{n=1}^{N}n^2P_n=D_x\,m_y+m_x^2\,\alpha_{2y}.$$

Bu yerda

$$\alpha_{2y}=\sum_{n=1}^{N}n^2\,P_n=D_y+m_y^2$$

Z tasodifiy miqdorning dispersiyasi

$$D_z=\alpha_{2z}-m_z^2=D_x\,m_y+m_x^2\,\alpha_{2y}-m_x^2\,m_y^2=$$
$$D_x\,m_y+m_x^2\,(D_y+m_y^2)-m_x^2\,m_y^2=D_x\,m_y+m_x^2\,D_y$$

bo`ladi.

Hosil qilingan formulalarni qo`llanilishiga doir misol keltiramiz.

4.4.1-misol. bog'liq bo`lmagan $X_1,X_2,...,X_n,...$ tasodifiy miqdorlar $\lambda$ parametrli ko`rsatkichli qonun bo`yicha bir xil taqsimlangan. $Y=Y_1+1$ tasodifiy miqdor, bu yerda $Y_1$ $a$ parametrli Puasson qonuni bo`yicha taqsimlangan tasodifiy miqdor

$$Z=\sum_{i=1}^{Y}X_i$$

tasodifiy miqdorning taqsimot qonuni va sonli xarakteristikalarini topish talab qilinadi.

$\sum_{i=1}^{n} X_i$ yig'indining taqsimot qonuni $(n-1)$-tartibli Erlang taqsimot qonunidan iborat bo`ladi [2]:

$$f^{(n)}(x) = \frac{\lambda(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \qquad (x > 0)$$

To`la ehtimol formulasiga ko`ra Z tasodifiy miqdorning taqsimot qonuni $Z > 0$ bo`lganda

$$\varphi(z) = \sum_{n=}^{\infty} f^{(n)}(z) P_n = \sum_{n=1}^{\infty} \frac{\lambda(\lambda z)^{n-1}}{(n-1)!} e^{-\lambda z} \cdot \frac{a^{n-1}}{(n-1)!} e^{-a} =$$

$$= \lambda e^{-\lambda z - a} \sum_{n=1}^{\infty} \frac{(\lambda z a)^{n-1}}{[(n-1)!]^2} = \lambda e^{-\lambda z - a} \sum_{n=1}^{\infty} \frac{(\lambda z a)^{k}}{[(k)!]^2}$$

bo`ladi.

Bu zichlikni modifikatsiya qilingan

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2}$$

silindrik funksiya orqali ifodalash mumkin: $z > 0$ bo`lganda

$$\varphi(z) = \lambda e^{-\lambda z - a} \; I_0\,(2\sqrt{\lambda z a}),$$

Z tasodifiy miqdorning matematik kutilmasini topamiz:

$$MZ = m_x \cdot m_y = M\,X_i\,M\,Y = M\,X_i \cdot M(Y_1 + 1) = MX_i\,(MY_1 + 1) \qquad (3.4.1)$$

ko`rsatkichli qonun bo`yicha taqsimlangan X_i tasodifiy miqdorning matematik kutilishi:

$$MX_i = \frac{1}{\lambda}.$$

Puasson qonuni bo`yicha taqsimlangan Y_1 tasodifiy miqdorning matematik kutilmasi:

$$M\,Y_1 = a.$$

(3.4.1) ga $MX_i = \frac{1}{\lambda}$, $M\,Y_1 = a$ ni qo`yib, quyidagi ega bo`lamiz:

$$M\,Z = \frac{1}{\lambda}(a + 1)$$

Z tasodifiy miqdorning dispersiyasini topamiz:

$$DZ = D_x\, m_y + m_x^2\, D_y = DX_i \cdot MY + (MX_i)^2\, D(Y_1 + 1) =$$

$$= DX_i \cdot M(Y_1 + 1) + (MX_i)^2\, D(Y_1 + 1) = DX_i\,(MY_1 + 1) + (MX_i)^2\, DY_1 .$$

Ko`rsatkichli taqsimotning dispersiyasi $DX_i = \frac{1}{\lambda^2}$, Puasson taqsimotining dispersiyasi $DY_1 = a$ ga tengligini hisobga olib, quyidagiga ega bo`lamiz:

$$DZ = \frac{1}{\lambda^2}(a + 1) + \left(\frac{1}{\lambda}\right)^2 a = \frac{2a + 1}{\lambda^2}$$

Xulosa

Amaliyotning ko`pgina masalalarida ayniqsa, matematik statistikada tasodifiy argument funksiyasining taqsimot qonunini topish masalasi tez-tez uchrab turadi. Masalan, agar $X$ asbob ko`rsatishlarining xatosi, Y esa ko`rsatishlar shkalasidan eng yaqin bo`linmagacha yaxlitlash xatosi bo`lsa, u holda xatolar yig'indisi $X+Y$ ning taqsimot qonunini topish masalasi yuzaga keladi; nishonga qarata ikkita o`q uziladi, agar X va Y lar mos ravishda birinchi va ikkinchi o`qning tushish nuqtasidan nishon markazigacha bo`lgan masofa (oraliq) bo`lsa, u holda masofalardan kichigi $\min(X,Y)$ ning taqsimot qonunini topish masalasi yuzaga keladi. Bunday misollarni ko`plab keltirish mumkin.

Umumiyroq qilib aytadigan bo`lsak, $X_1, X_2, ..., X_n$ tasodifiy miqdorlar bilan funksional bog`lanishda bo`lgan $Z = f(X_1, X_2, ..., X_n)$ tasodifiy miqdorning taqsimot qonunini topish zarur bo`ladi.

Ishda bu umumiy masalaning bir nechta xususiy hollarini qaradik:

1. Ikki o`lchovli $(X,Y)$ tasodifiy miqdor $p(x,y)$ taqsimot zichlik bilan berilgan bo`lsin. u holda

a) X va Y tasodifiy miqdorlardan maksimalining taqsimot zichligi

$$g(z) = \int_{-\infty}^{z} p(z,y)dy + \int_{-\infty}^{z} p(x,z)dx$$

b) minimalining taqsimot zichligi

$$\varphi(z) = p_x(z) + p_Y(z) - \int_{-\infty}^{z} p(z,y)\,dy - \int_{-\infty}^{z} p(x,z)\,dx$$

$P_x(z)$ va $P_Y(z)$ lar mos ravishda X va Y tasodifiy miqdorlarning taqsimot zichliklari

c) X va Y tasodifiy miqdorlardan maksimali va minimalining birgalikdagi taqsimoti

$$g(u,v) = \begin{cases} \int_{-\infty}^{v} p(v,y)\,dy + \int_{-\infty}^{v} p(x,v)\,dx, & v \le u; \\ p(u,v) + p(v,u), & v > u \end{cases}$$

2. $X_1, X_2, ..., X_n$ lar o`zaro bog'liq bo`lmagan va $P_{X_1}(x_1), P_{X_2}(x_2), ..., P_{X_n}(x_n)$ taqsimot zichliklar bilan berilgan tasodifiy miqdorlar bo`lsin. U holda:

a) ulardan maksimalining taqsimot zichligi

$$g^*(z) = \sum_{j=1}^{n} \frac{P_{X_j}(z)}{F_{X_j}(z)} \prod_{i=1}^{n} F_{X_i}(z)$$

b) minimalining taqsimot zichligi

$$\varphi^*(z) = \sum_{j=1}^{n} \frac{P_{X_j}(z)}{1 - F_{X_j}(z)} \prod_{i=1}^{n} [1 - F_{X_i}(z)]$$

bu yerda $F_{X_i}(z)$ funksiya X_i tasodifiy miqdorning taqsimot funksiyasi

Ishda bundan tashqari taqsimot bilan bog'langan tasodifiy miqdorlaridan ba'zi muhimlarining taqsimot qonunlari topilgan.

Yana tasodifiy sondagi tasodifiy miqdorlar yig'indisining taqsimot qonunini topish masalasi ham o`rganildi, u yerda biz asosiy e'tiborni qo`shiluvchi tasodifiy miqdorlar o`zaro bog`liq bo`lmagan, bir xil taqsimlangan holga qaratdik.

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