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MOMENTLI ELASTIKLIK NAZARIYASI SISTEMASI UCHUN KOSHI MASALASI

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DISSERTATSIYA

Ish ko’rib chiqildi va himoyaga
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MUNDARIJA

K i r i sh

I. Ishning umumiy tavsifi. Bu ishda momentli elastiklik nazariyasi tenglamalari sistemasi yechimini tekislikda uning chegaraning qismida berilgan qiymatlari va uning kuchlanishi qiymatlari bo'yicha davom ettirish masalasi, ya'ni momentli elastiklik nazariyasi tenglamalari sistemasi uchun Koshi masalasi o'rganiladi [1,2].

Momentli elastiklik nazariyasi masalalari yechimi uchun sohaning butun chegarasida u yoki bu chegaraviy shartlarning berilishi talab etiladi.Klassik masalalarda bu ko'chish vektorning berilishi, sohaning butun chegarasida kuchlanish vektorining berilishi yoki chegaraning bir qismida ko'chish, ikkinchi qismida esa kuchlanish vektorining berilishidan iborat. Boshqa masalalarda chegaraning har bir qismida ko'chish va kuchlanishlar komponentalarining kerakli miqdori kombinatsiyasi berilgan. Ammo ko'plab masalalarda chegaraning qismi na ko'chishni na kuchlanishni o'lchashga qodir yoki faqat ba'zi integral xarakteristikalar ma'lum bo'ladi. Tajribali izlanishda tabiiy jismning kuchlanishli deformatsiyalangan holatini o'rganishda o'lchashlar sirtning faqat mumkin bo'lgan qismida o'tkazilishi mumkin.Bunday masalalar yechimi tajribaviy ma'lumotlar bo'lmaganligi sababli ma'lum qiyinchiliklarni keltirib chiqaradi.Shunga o'xshash vaziyatlar, masalan, geomexanika masalalarida vujudga keladi [3,4].

$x=(x_1,x_2)$ va $y=(y_1,y_2)$ nuqtalar E^2 - ikki o'lchovli evklid fazosidan olingan bo'lsin va D elastik muhit E^2 da bo'lakli - silliq ∂D chiziq bilan chegaralangan sohadan iborat bo'lsin, $S-\partial D$ ning silliq qismi.

I sohada bir jinsli momentli elastiklik nazariyasi tenglamalari sistemasi[5]

$$\begin{cases}(\mu+\alpha)\Delta u+(\lambda+\mu-\alpha)graddivu+2\alpha\, rotw+\rho\theta^2 u=0,\\(\nu+\beta)\Delta w+(\varepsilon+\nu-\beta)graddivw+2\alpha\, rotu-4\alpha w+j\theta^2 w=0,\end{cases} \qquad (0.1)$$

berilgan bo'lsin. Bu yerda $U(x)=(u_1(x),u_2(x),w_1(x),w_2(x))=(u(x),w(x))$ sistemaning yechimi, Δ- Laplas operatori, $\lambda,\mu,\nu,\beta,\varepsilon,\alpha$ elastik muhitni xarakterlaydigan sonlar bo'lib, quyidagi shartlarni qanoatlantiradi $\mu>0, 3\lambda+2\mu>0,\ \alpha>0,\ \varepsilon>0,\ 3\varepsilon+2\nu>0,\ \beta>0,\ j>0,\ \rho>0,\ \theta\in R^1.$

$U(x)-D$ sohada (0.1) sistemaning regulyar yechimi bo'lsin va S da quyidagi Koshi shartlarini qanoatlantirsin:

$$\left.\begin{array}{ll} U(y)=f(y), & y\in S \\ T(\partial_y,n)U(y)=g(y), & y\in S \end{array}\right\} \qquad (0.2)$$

Bunda $T(\partial_y,n(y))$-kuchlanish operatori deb atalib u quyidagi ko'rinishda ifodalanadi

$$T(\partial_y,n(y))=\left\|\begin{matrix} T^{(1)}(\partial_y,n) & T^{(2)}(\partial_y,n) \\ T^{(3)}(\partial_y,n) & T^{(4)}(\partial_y,n) \end{matrix}\right\|,$$

$$T^{(i)}(\partial_y,n)=\left\|T_{kj}^{(i)}(\partial_y,n)\right\|_{2x2},\quad i=1,2,3,4,$$

$$T_{kj}^{(1)}(\partial_y,n)=\lambda n_k\frac{\partial}{\partial y_j}+(\mu-\alpha)n_j(y)\frac{\partial}{\partial y_k}+(\mu+\alpha)\delta_{kj}\frac{\partial}{\partial n(y)},\quad k,j=1,2,$$

$$T_{kj}^{(2)}(\partial_y,n)=T_{kj}^{(3)}(\partial_y,n)=0,\ k,j=1,2,$$

$$T_{kj}^{(4)}(\partial_y,n)=\varepsilon n_k(y)\frac{\partial}{\partial y_j}+(\nu-\beta)n_j(y)\frac{\partial}{\partial y_k}+(\nu+\beta)\,\delta_{kj}\frac{\partial}{\partial n(y)},\ k,j=1,2.$$

$n(y)=(n_1(y),\ n_2(y))-\partial D$-chegaraning y nuqtasidagi tashqi birklik normal vektori, $f(y)=(f_1(y),\ f_2(y))$ и $g(y)=(g_1(y),\ g_2(y))-S$ da berilgan uzluksiz vektor – funksiyalar, δ_{kj} - Kroneker simvoli.

Berilgan f va g lardan kelib chiqib, $U(y)$ ni D da davom ettirish talab etiladi.

Garmonik funksiyalar nazariyasida Laplas operatori qanday ahamiyatga aga bo'lsa,

$$\mu\Delta + (\lambda + \mu)\, grad\ div$$

operatorimomentli elastiklik nazariyasida xuddi shunday ahamiyatga aga va xususiy holda, $\mu = 1,\ \lambda = -1$ da Laplas operatoriga aylanadi. Ixtiyoriy $f(y)$ va $g(y)$ larda (0.1) masala yechimga ega emas. Agar $f(y)$ va $g(y)$ lar S (S - analitik yoy) da analitik va D da analitik davom ettiriladigan bo'lsa, u holda xususiy hosilali tenglamalar nazariyasining fundamental teoremasiga asosan davom ettirish mumkin va yagonadir. Biroq (0.1) va (0.2) masala Adamar bo'yicha nokorrekt, ya'ni nokorrektligi xuddi Laplas tenglamasi uchun qo'yilgan Koshi masalasidagidek [7].

f va g faqat deyarli yaqin qilib berilishi mumkinligi sababli masalaning yechimiga yaqin yechim to'g'risida gap ketish mumkin. Lekin yechimda turg'unlikning bo'lmaganligidan qo'shimcha ma'lumotlarsiz yaqin yechimning bo'lishi mumkin emas.

A.N.Tixonovningishlarida [3], keyin M.M.Lavrentyevning ishlarida [1,4] rivojlangan bunday masalalar qo'yilishining shartli korrektligi, nokorrekt masalalarni yechishning yangi usullarini berdi. Nokorrekt masalalarning shartli korrektligi izlanishida o'rnatilgan yagonalik va turg'unlik teoremalaridan so'ng yechimni qurishning effektiv metodi, ya'ni regulyarizatsiyalanuvchi operatorlarni qurish tug'iladi.

M.M.Lavrentyevdan kelib chiqib, $\varepsilon > 0$ parametrdan bog'liq $\Phi_\varepsilon(y, x)$ Laplas tenglamasining fundamental yechimini $x \in D$ nuqta, S chegara qismining Karleman funksiyasi deb ataymiz, agar quyidagi tengsizlik o'rinli bo'lsa;

$$\int_{\partial D\setminus S} \left[|\Phi_\varepsilon(y, x)| + |T(\partial_y, n)\Phi_\varepsilon(y, x)| \right] dS_y \le \varepsilon .$$

Karleman formulasining ko'p o'lchovli analogi bilan A.A.Ayzenberg, elliptik sistemalar uchun Karleman matritsasini qurish bilan N.N.Tarxanov shug'ullangan.

M.M.Lavrentyev ideyasidan foydalanib, Sh.Ya.Yarmuxammedov Laplas tenglamasi [2] uchun Koshi masalasining regulyarizatsiyalangan yechimini aniq ko'rinishda qurdi.

Ma'lumki, momentli elastiklik nazariyasining uch o'lchovli fazoda qo'yilgan masalalari yaxshi o'rganilgan. Kompleks o'zgaruvchili analitik funksiyalarning ma'lum metodidan tashqari elastiklik nazariyasining qator statistik masalalari potensiallar nazariyasi va integral tenglamalar metodi bilan ham Fredgolm va Laurachel tomonidan qarab chiqilgan. Ular bilan korrekt masalalar o'rganilgan edi.

f va g o'rniga $\delta, \delta \in (0,1)$ aniqlikda (C metrikada) f_δ, g_δ berilgan bo'lsin, bular yechim mavjud bo'lgan sinfga qarashli bo'lmasligi mumkin. Ushbu ishda σ parametrdan bog'liq $U(x, f_\delta, g_\delta) = U_{\sigma\delta}$ vektor funksiyalar oilasi quriladi va biror shart va maxsus $\sigma(\delta)$, $\delta \to 0$ parametr tanlanganda $U_{\sigma\delta}(x)$ oila odatdagi ma'noda (0.1) – (0.2) masalaning yechimiga yaqinlashishi isbotlanadi.

A.N.Tixonovdan kelib chiqib, $U_{\sigma\delta}(x)$ funksiyani (0.1) – (0.2) masala, ya'ni elastiklik nazariyasi sistemasi uchun Koshi masalasining regulyarizatsiyalangan yechimi deb ataymiz.Regulyarizatsiyalangan yechim masalaning metodining turg'unligini aniqlaydi.

Maxsus sohalarda chegaralangan analitik funksiyalarni davom ettirish masalasi chegaraning qismida aniq berilgan holni T.Karleman qarab chiqqan edi. T.Karlemanning izlanishlarini G.M.Goluzin va V.I.Krilovlar davom ettirishdi. Soha yo'lakdan iborat bo'lganda Laplas tenglamasi uchun Koshi masalasining regulyarizatsiyalangan yechimi V.K.Ivanov tomonidan qurilgan.Laplas tenglamasi uchun Koshi masalasining regulyarizatsiyalangan yechimini qurish uchun foydalanilgan.

Ushbu ishning maqsadi ikki o'lchamli bo'lgan hol uchun nazariyani ishlab chiqishdan iborat. Ikkala nazariya ko'plab umumiylikka ega va parallel rivojlanadi.

Ishning maqsadi. Ushbu dissertatsion ishda momentli elastiklik nazariyasi tenglamalari sistemasi uchun qo'yilgan Koshi masalasining yechimi ma'lum sohalarda aniq ko'rinishda quriladi. Buning uchun Karleman matritsasining sohadan bog'liq ravishda ko'rinishidan foydalaniladiva uning asosida (0.1) – (0.2) masalaning regulyarizatsiyalangan yechimi quriladi. Bunda (0.1) sistema chegaralangan sohalarda va yo'lak tipidagi chegaralanmagan sohalarda qaraladi, soha chegarasi ikkita qismdan tashkil topgan, Koshi shartlari esa chegaraning bir qismida beriladi.

Shunday qilib, Karleman matritsasining effektiv qurilishi (0.1) – (0.2) masalaning regulyarizasiyalangan yechimini aniq ko'rinishda yozishga imkon beradi [8-13].

Ilmiy yangiligi. Tekis sohalarda momentli elastiklik nazariyasi tenglamalari sistemasiuchun Koshi masalasi oldin qo'yilmagan va o'rganilmagan. Dissertatsiyada maxsus sohalar sinfi uchun (0.1) – (0.2) masalaning regulyarizatsiyalangan yechimi aniq ko'rinishda topilgan [14,15].

Tadbiqi. Qo'yilgan masalaning yechimini aniq ko'rinishda qurish katta nazariy va amaliy qiziqish uyg'otadi. Dissertatsiyada olingan natijalar momentli elastiklik nazariyasining matematik masalalari izlanilishida, geomexanikaning masalalarida foydalanilishi mumkin.

§ 1.1 da o'rganiladigan Koshi masalasining qo'yilishi, yordamchi tasdiqlar, mulohazalar keltirilgan. Qaraladigan masala uchun Adamar misoli berilgan.

1.2 , 2.1, 2.2 paragraflar ishning asosiy qismi hisoblanadi, bu yerda elastiklik nazariyasi sistemasining fundamental yechimlari matritsasini qurish keltirilgan. Maxsus sohalar sinfi uchun yuqorida ta'kidlangan xossalarga ega momentli elastiklik nazariyasi sistemasining fundamental yechimlari matritsasi aniq ko'rinishda qurilgan. Bu yerda olingan **Somilian - Bettining** umumlashgan formulasi [14] dissertatsiyaning asosiy ishlovchi apparati hisoblanadi. § 2.2 da maxsus ko'rinishdagi chegaralanmagan sohalar uchun umumlashgan Somilian – Betti formulasi keltirilgan.

§§ 3.1 – 3.2 da maxsus sohalar sinfi uchun (0.1) – (0.2) masalaning regulyarizatsiyalangan yechimi keltirilgan. Bu yerda isbotlangan davom ettirish formulalari Somilian – Betti formulasi va Mittag – Leffler funksiyasining xossalariga asos bo'ladi [15].

I – bob.Somilian - Betti formulasi

§ 1.1.Tekislikda momentli elastiklik nazariyasi sistemasi echimi uchun Somilian - Betti formulasi

$x=(x_1,x_2)$ va $y=(y_1,y_2)$ nuqtalar E^2 - ikki o'lchovli evklid fazosidan olingan bo'lsin va D elastik muhit E^2 da bo'lakli - silliq ∂D chiziq bilan chegaralangan sohadan iborat bo'lsin, $S-\partial D$ ning silliq qismi.

I sohada bir jinsli momentli elastiklik nazariyasi tenglamalari sistemasi

$$\begin{cases}(\mu+\alpha)\Delta u+(\lambda+\mu-\alpha)graddivu+2\alpha\, rotw+\rho\theta^2u=0,\\ (\nu+\beta)\Delta w+(\varepsilon+\nu-\beta)graddivw+2\alpha\, rotu-4\alpha w+j\theta^2w=0,\end{cases} \tag{1}$$

berilgan bo'lsin. Bu yerda $U(x)=(u_1(x),u_2(x),w_1(x),w_2(x))=(u(x),w(x))$ sistemaning yechimi, Δ- Laplas operatori, $\lambda,\mu,\nu,\beta,\varepsilon,\alpha$ elastik muhitni xarakterlaydigan sonlar bo'lib, quyidagi shartlarni qanoatlantiradi $\mu>0, 3\lambda+2\mu>0,\ \alpha>0,\ \varepsilon>0,\ 3\varepsilon+2\nu>0,\ \beta>0,\ j>0,\ \rho>0,\ \theta\in R^1.$

1-ta'rif. D da aniqlangan φ funksiyani D da regulyar deb ataymiz, agar $\varphi\in C^1(\overline{D})\cap C^2(D)$ va φ- funksiyaning dekart koordinatalari bo'yicha barcha ikkinchi tartibli hosilalari D sohada integrallanuvchi bo'lsa.

Bundan keyin ∂D ni chekli D sohani chegaralovchi yopiq silliq egri chiziq deb hisoblaymiz.

1-teorema. D sohada (1) tenglamaning har qanday regulyar yechimi ushbu

$$U(x)=\int_{\partial D}(\Psi(y,x)\{T(\partial_y,n)U(y)\}-\{T(\partial_y,n)\Psi(y,x)\}^{*}U(y))ds_y,\ x\in D,\qquad (2)$$ f

ormula bilan beriladi. Bu yerda «*» belgi transponirlangan matritsani bildiradi.

Formulada keltirilgan $\Psi(y,x)$- momentli statik elastiklik nazariyasi tenglamalari sistemasining fundamental yechimlar matritsasi bo'lib, u quyidagicha beriladi [1]:

$$\Psi(y,x)=\begin{Vmatrix}\Psi^{(1)}(y,x) & \Psi^{(2)}(y,x)\\ \Psi^{(3)}(y,x) & \Psi^{(4)}(y,x)\end{Vmatrix},$$

bunda $\Psi^{(i)}(y,x)=\left\|\Psi^{(i)}_{kj}(y,x)\right\|_{2\times 2},\quad i=1,2,3,4,$

$$\Psi^{(1)}_{kj}(y,x)=\sum_{l=1}^{4}(\delta_{kj}\alpha_l+\beta_l\frac{\partial^2}{\partial x_k\partial x_j})\frac{e^{i\lambda_m|y-x|}}{|y-x|},\quad k,j=1,2,$$

$$\Psi^{(2)}_{kj}(y,x)=\Psi^{(3)}_{kj}(y,x)=\frac{2\alpha}{\mu+\alpha}\sum_{l=1}^{4}\ \sum_{p=1}^{2}\varepsilon_l\varepsilon_{kjp}\frac{\partial}{\partial x_p}\frac{e^{i\lambda_m|y-x|}}{|y-x|},\quad k,j=1,2,$$

$$\Psi^{(4)}_{kj}(y,x)=\sum_{l=1}^{4}(\delta_{kj}\gamma_l+\delta_l\frac{\partial^2}{\partial x_k\partial x_j})\frac{e^{i\lambda_m|y-x|}}{|y-x|},\quad k,j=1,2,$$

$r=|x-y|,$

$$\alpha_l=\frac{(-1)^l(\sigma_2^2-k_l^2)(\delta_{3l}+\delta_{4l})}{2\pi(\mu+\alpha)(k_3^2-k_4^2)},\quad \beta_l=-\frac{\delta_{1l}}{2\pi\rho\theta^2}+\frac{\alpha_l}{k_l^2},\ l=1,2,3,4,\quad \sum_{l=1}^{4}\beta_l=0,$$

$$\gamma_l=\frac{(-1)^l(\sigma_1^2-k_l^2)(\delta_{3l}+\delta_{4l})}{2\pi(\beta+\nu)(k_3^2-k_4^2)},\quad \delta_l=-\frac{\delta_{2l}}{2\pi(j\theta^2-4\alpha)}+\frac{\gamma_l}{k_l^2},\quad l=1,2,3,4,\quad \sum_{l=1}^{4}\delta_l=0,$$

$$\varepsilon_l = \frac{(-1)^l(\delta_{3l}+\delta_{4l})}{2\pi(\beta+\nu)(k_3^2-k_4^2)}, \quad l=1,2,3,4, \ \sum_{l=1}^{4}\varepsilon_l = 0, \ \ k_1^2 = \frac{\rho\theta^2}{\lambda+2\mu}, \ k_2^2 = \frac{j\theta^2-4\alpha}{\varepsilon+2\nu},$$

$$k_3^2+k_4^2 = \sigma_1^2+\sigma_2^2+\frac{4\alpha^2}{(\mu+\alpha)(\beta+\nu)}, \ \ k_3^2k_4^2 = \sigma_1^2\sigma_2^2.$$

$T(\partial_y, n)$-kuchlanish operatori deyilib,

$$T(\partial_y,n) = T(\partial_y,n(y)) = \begin{Vmatrix} T^{(1)}(\partial_y,n) & T^{(2)}(\partial_y,n) \\ T^{(3)}(\partial_y,n) & T^{(4)}(\partial_y,n) \end{Vmatrix},$$

$$T^{(i)}(\partial_y,n) = \left\| T_{kj}^{(i)}(\partial_y,n) \right\|_{2\times2}, \quad i=1,2,3,4,$$

$$T_{kj}^{(1)}(\partial_y,n) = \lambda n_k \frac{\partial}{\partial y_j} + (\mu-\alpha)n_j(y)\frac{\partial}{\partial y_k} + (\mu+\alpha)\delta_{kj}\frac{\partial}{\partial n(y)}, \ \ k,j=1,\ 2,$$

$$T_{kj}^{(2)}(\partial_y,n) = T_{kj}^{(3)}(\partial_y,n) = 0, \ k,j=1,\ 2,$$

$$T_{kj}^{(4)}(\partial_y,n) = \varepsilon\, n_k(y)\frac{\partial}{\partial y_j} + (\nu-\beta)n_j(y)\frac{\partial}{\partial y_k} + (\nu+\beta)\frac{\partial}{\partial n(y)}, \ k,j=1,\ 2.$$

$n(y) = (n_1(y), n_2(y)) - \partial D$-chegaraning y nuqtasidagi tashqi normal vektori.

(2) formulani Somilian – Betti formulasi deb yuritamiz.

1.2§. (1.1) sistema uchun Karleman matritsasi va integral formula

Sh. Yarmuxamedov tomonidan Laplas tenglamasi uchun qo'yilgan Koshi masalasini yechishda qo'llanilgan usuldan foydalanib Karleman funksiyasini quramiz va u asosida qaralayotgan sistema uchun Karleman matritsasini quramiz [2-6].

D - chegaralangan bir bog'lamli soha, ∂D - uning chegarasi, $S - \partial D$ ning silliq qismi.

2-ta'rif. $y \neq x$ da aniqlangan, $\sigma > 0$ parametrdan bog'liq, (4×4) o'lchamli $\Pi_\sigma(y,x)$ matritsa $x \in D$ nuqtada va $\partial D \setminus S$ qism uchun Karleman matritsasi deyiladi, agar u quyidagi shartlarni qanoatlantirsa:

1) $\Pi_\sigma(y,x) = \Psi(y,x) + G_\sigma(y,x)$

bunda $G_\sigma(y,x) - (4 \times 4)$ o'lchamli matritsa y bo'yicha (1) sistemani qanoatlantiradi,

2) $\forall x \in D$ da $\Pi_\sigma(y,x)$ matritsa ushbu

$$\int\limits_{\partial D \setminus S} \left[|\Pi_\sigma(y,x)| + |T(\partial_y, n)\Pi_\sigma(y,x)| \right] dS_y \leq \alpha(\sigma),$$

tengsizlikni qanoatlantiradi, bu yerda $\alpha(\sigma) \to 0,\ \sigma \to \infty$.

D soha va $\partial D \setminus S$ qismi uchun Karleman matritsasi mavjud bo'lsin. Bu holda quyidagi teorema o'rinli

2-teorema. D da regulyar bo'lgan (1) sistemaning yechimi quyidagi ko'rinishga ega:

$$u(x) = \int\limits_{\partial D} \left[\Pi_\sigma(y,x)\{T(\partial_y,n)u(y)\} - u(y)\{T(\partial_y,n)\Pi_\sigma(y,x)\} \right] dS_y,\ x \in D, \qquad (3)$$

bu yerda $\Pi_\sigma(y,x)$ - D soha uchun Karleman matritsasi.

Endi elastiklik nazariyasi sistemasining maxsus ko'rinishdagi fundamental yechimlarini qurishga o'tamiz. $K(w)$, $w = u + iv$(u,v - haqiqiy) – butun funksiya, quyidagi shartlarni qanoatlantiruvchi w da haqiqiy bo'lsin:

$$K(u) \neq 0,\ \sup_{v \geq 1} \left| v^p K^{(\rho)}(u+iv) \right| \leq M(\rho, u) < \infty,\ \rho = 0,1,2,...\ ,\ \ u \in R^1$$

$\alpha = |y_1 - x_1|$ bo'lsin.

$\Phi_\sigma(y,x,\lambda)$ funksiyani $\alpha > 0$ va $y \neq x$ larda quyidagi tenglik bilan aniqlaymiz:

$$2\pi K(\sigma x_2)\Phi_\sigma(y,x,\lambda) = \int_0^\infty \operatorname{Im} \frac{K(\sigma w)}{w - x_2} \cdot \frac{u J_0(\lambda u) du}{\sqrt{u^2+\alpha^2}}, \tag{4}$$

bu yerda $w = i\sqrt{u^2+\alpha^2} + y_2$, $J_0(u)$-Besselning nolinchi tartibli funksiyasi.

[7] da quyidagi lemma isbotlanadi

Lemma.(4) formula bilan aniqlangan $\Phi_\sigma(y,x,\lambda)$ funksiya quyidagi ko'rinishga ega:

$$\Phi_\sigma(y,x,\lambda) = \frac{\exp r}{2\pi r} + g_\sigma(y,x,\lambda)$$

bu yerda $g_\sigma(y,x,\lambda)$ - x, y ning barcha qiymatlarida aniqlangan va butun E^2 da y o'zgaruvchi bo'yicha Gelmgolts tenglamasining reguylyar yechimi, $\mathbf{r} = |\mathbf{x} - \mathbf{y}|$.

$\Phi_\sigma(y,x,\lambda)$ funksiya yordamida quyidagi matritsani quramiz:

$$\Pi_\sigma(y,x) = \Pi(y,x,\sigma) = \begin{Vmatrix} \Pi^{(1)}(y,x,\sigma) & \Pi^{(2)}(y,x,\sigma) \\ \Pi^{(3)}(y,x,\sigma) & \Pi^{(4)}(y,x,\sigma) \end{Vmatrix},$$

$$\Pi^{(i)}(y,x,\sigma) = \|\Pi^{(i)}_{kj}(y,x,\sigma)\|_{2\times 2}, \quad i=1,2,3,4,$$

$$\Pi^{(1)}_{kj}(y,x,\sigma) = \sum_{m=1}^{4}\left(\delta_{kj}\alpha_m + \beta_m \frac{\partial^2}{\partial y_k \partial y_j}\right)\cdot \Phi_\sigma(y,x,i\lambda_m),$$

$$\Pi^{(2)}_{kj}(y,x,\sigma) = \Pi^{(3)}_{kj}(y,x,\sigma) = 0,$$

$$\Pi^{(4)}_{kj}(y,x,\sigma) = \sum_{m=1}^{4}\left(\delta_{kj}\gamma_m + \delta_m \frac{\partial^2}{\partial y_k \partial y_j}\right)\cdot \Phi_\sigma(y,x,i\lambda_m)\,. \tag{5}$$

3-teorema. (5) formula bilan aniqlangan $\Pi(y,x,\sigma)$ matritsa Karleman matritsasi bo'ladi.

Isbot. Karleman matitsasining 1)-sharti lemmadan osongina kelib chiqadi, haqiqattan ham

$$\Pi(y,x,\sigma) = \Psi(y,x) + G(y,x,\sigma),$$

bunda

$$G(y,x,\sigma) = \begin{Vmatrix} G^{(1)}(y,x,\sigma) & G^{(2)}(y,x,\sigma) \\ G^{(3)}(y,x,\sigma) & G^{(4)}(y,x,\alpha\sigma) \end{Vmatrix},$$

$$G^{(i)}(y,x,\sigma) = \|G^{(i)}_{kj}(y,x,\sigma)\|_{2\times 2}, \quad i=1,2,3,4,$$

$$G^{(1)}_{kj}(y,x,\sigma) = \sum_{l=1}^{4}(\delta_{kj}\alpha_l + \beta_l \frac{\partial^2}{\partial x_k \partial x_j})g_\sigma(y,x,\lambda_l), \quad k,j=1,2,$$

$$G^{(2)}_{kj}(y,x,\lambda_l) = G^{(3)}_{kj}(y,x,\lambda_l) = 0, \quad k,j=1,2,$$

$$G^{(4)}_{kj}(y,x,\lambda_l) == \sum_{l=1}^{4}(\delta_{kj}\gamma_l + \delta_l \frac{\partial^2}{\partial x_k \partial x_j})g_\sigma(y,x,\lambda_l), \quad k,j=1,2.$$

Karleman matitsasining 2)-sharti, (4) dan kelib chiqadi.

I bob bo'yicha xulosa

Bu bobda asosan chegaralangan sohalar uchun Somilian-Betti formulasi va momently elastiklik nazariyasi sistemasining fundamental echimini qurishga bag'ishlangan. Bu integral formulalar yechimni soha chegarasining biror qismidagi qiymatlariga ko'ra yechimni soha ichidagi qiymatini toppish imkonini beradi.Bu integral formulalar umumiy kursdan ma'lum bo'lgan Grin funksiyalari va Grin formulalaridek ahamiyatga ega.Bu bobda sohadan bog'liq ravishda qurilib, yechimlar mos ravishda aniq ko'rinishda yoziladi.

II-bob. Maxsus sohalarda Karleman matritsasi

2.1§. Maxsus sohalarda (1) sistema uchun Karleman matritsasi

(4) va (5) formulalarda $K(w)$ funksiyani aniq ko'rinishda tanlab, momentli elastiklik nazariyasi sistemasi uchun Karleman matritsasini maxsus sohada aniq ko'rinishda quramiz.

1. Delastik muhit E^2 da chegaralangan bir bog'lamli, chegarasi $y_1 = 0$ haqiqiy o'qning l bo'lagi va $y_2 > 0$ yarim tekislikda yotuvchi silliq S egri chiziqdan iborat bo'lsin, ya'ni D qalpoq shaklidagi soha.

$\sigma > 0$ da (4) formulada

$$K(w) = \exp \sigma w, \; w = i\sqrt{u^2 + \alpha^2}$$

ifodani qo'yamiz va quyidagiga ega bo'lamiz:

$$2\pi \exp(\sigma x_2)\Phi_\sigma(y, x, \lambda) = \int_0^\infty \mathrm{Im} \frac{\exp(\sigma w)}{w - x_2} \cdot \frac{uJ_0(\lambda u)du}{\sqrt{u^2+\alpha^2}} \tag{6}$$

(6) formuladan ko'rinadiki, l da $\Phi_\sigma(y, x, \lambda)$ funksiya, uning $\nabla\Phi_\sigma(y, x, \lambda)$ gradiyenti va uning ikkinchi tartibli xususiy xosilalari

$$\frac{\partial^2 \Phi_\sigma(y, x, \lambda)}{\partial y_i \partial y_j}, \quad i, j = 1,2,$$

$\sigma \to \infty$ da barcha y va $\mathbf{x} \in \mathbf{E}^2$ larda eksponensial nolga intiladi.

(5) ga (6) ni qo'yib, $\Pi(y, x, \sigma)$ hosil qilamiz.Quyidagi teorema o'rinli

4-teorema.Qurilgan $\Pi(y, x, \sigma)$ matritsa D soha va l qism uchun Karleman matritsasidan iborat bo'ladi.

Isbot. $\Pi_\sigma(y,x) = \Psi(y,x) + G_\sigma(y,x)$ ifoda va $G_\sigma(y,x)$ ning (1) sistema yechimi ekanligi lemmadan birdan kelib chiqadi.

$\int\limits_{l} \left[\left|\Pi(y,x,\sigma)\right| + \left|T(\partial_y, n)\Pi(y,x,\sigma)\right| \right] dS_y \leq c(\lambda,\mu,x,\sigma)\exp(-\sigma x_2)$ (7) ekanligini

isbotlaymiz.

$$\frac{\partial\Phi_\sigma(y,x)}{\partial y_1} = \frac{\exp\sigma(y_2-x_2)\left[(y_2-x_2)\sin\sigma|y_1-x_1| + |y_1-x_1|\cos\sigma(y_2-x_2)\right]}{2\pi|x-y|^2},$$

$$\frac{\partial\Phi_\sigma(y,x)}{\partial y_2} = \frac{\exp\sigma(y_2-x_2)\left[(y_2-x_2)\cos\sigma|y_1-x_1| + |y_1-x_1|\sin\sigma(y_2-x_2)\right]}{2\pi|x-y|^2}$$

va

$$\left|\Phi_\sigma(y,x)\right| \leq \frac{const\left(\dfrac{1}{|x-y|}\right)}{\exp\sigma(y_2-x_2)}$$

bo'lgani uchun quyidagiga ega bo'lamiz:

$$\left|\Pi(y,\lambda,\sigma)\right| \leq \frac{c_1(\lambda,\mu)}{\ln\left(\dfrac{1}{|x-y|}\right)}\exp\sigma(y_2-x_2)$$

$$\left|T(\partial,n)\Pi(y,x,\sigma)\right| \leq \frac{c_2(\lambda,\mu)}{\left(\dfrac{\sigma}{|x-y|}\right)}\exp\sigma(y_2-x_2)$$

Ushbu oxirgi tengsizliklardan (7) ni olamiz. Teorema isbotlandi.

Endi $D_p \subset E^2$ -chegaralangan bir bog'lamli soha bo'lib chegarasi quyidagi konusning yon sirti

$$\Sigma : \alpha_1 = \tau y_2,\ \alpha_1 = |y_1|,\ \tau = tg\frac{\pi}{2\rho},\quad y_2 > 0,\quad \rho > 1,$$

va konus ichida yotgan S- silliq chiziqdan iborat bo'lsin.Quyidagi belgilashlarni kiritamiz

$$\tilde{\beta} = \tau y_2 - \alpha_0,\quad \gamma = \tau x_2 - \alpha_0,\quad \alpha_0^2 = x_1^2,\ s = \tilde{\alpha}^2 = (y_1 - x_1)^2\ ,$$

$$\omega = i\tau\sqrt{u^2 + s} + \tilde{\beta},\qquad \omega_0 = i\tau\tilde{\alpha} + \tilde{\beta}.$$

Parametr $\sigma > 0$da (5),(6) formulalar yordamida $\Pi(y, x, \sigma)$ni quyidagicha qurib olamiz.

$$K(\omega) = E_\rho\left(\sigma(\omega - x_2)\right),\qquad \rho > 1.$$

deb, bunda $E_p(\omega)$ - Mittag-Lefler funksiyasini olamiz. (4) formulada

$$\Phi(y, x, \lambda) = \Phi_\sigma(y - x, \lambda),\quad \lambda > 0$$

$$2\pi\Phi_\sigma(y - x, \lambda) = \int_0^\infty Im\left[\frac{E_p\left(\sigma(i\sqrt{u^2 + s} + y_2 - x_2)\right)}{i\sqrt{u^2 + s} + y_2 - x_2}\right]\frac{J_0(\lambda u)du}{\sqrt{u^2 + s}} \tag{8}$$

ni olamiz. (8) formula bilan aniqlangan funksiya uchun ham yuqorida keltirilgan lemma o'rinli va bundan hosil bo'lgan matritsa Karleman matritsasi ekanligini yuqorida 3,4-teoremalardagidek isbotlash mumkin.

Bizga bundan keyin $E_\rho(w)$ funksiyaning ayrim xossalari kerak bo'ladi. Ularning ayrimlarini keltiramiz.

Mittag-Lefler turidagi funksiya deb, quyidagi darajali qator ko'rinishida tasvirlanga butun funksiyani tushunamiz.

$$E_\rho(w) = \sum_{n=0}^{\infty}\Gamma^{-1}\left(1 + \frac{n}{\beta}\right)w^n,\quad w = u + iv,$$

bu yerda $\Gamma(s)$ – Eylerning Gamma funksiyasini ifodalaydi.

Mittag-Lefler turidagi funksiyasi $E_\rho(w)$ moduli bo'yicha etarli katta kompleks w larda ajoyib assimtotik ko'rinishlarga ega.

$0<\beta\le\pi,\quad \varepsilon_0>0$ sonlari uchun quyidagi belgilashlarni kritamiz. $\gamma(\varepsilon_0,\beta_0)$ deb ζ kompleks tekisligida shunday konturni belgilaymizki u $arg(\zeta)$ ning o'sishi yo'nalishida harakatlanib, quyidagi qismlardan iborat:

1) $arg(\zeta)=-\beta_0,\quad |\zeta|\ge\varepsilon_0$-nurdan

2) $|\zeta|=\varepsilon_0$ aylananing $-\beta_0\le arg(\zeta)\le\beta_0$ yoyidan

3) $arg(\zeta)=\beta_0,\quad |\zeta|\ge\varepsilon_0$ nurdan.

$0<\beta_0<\pi$ holida $\gamma(\varepsilon_0,\beta_0)$ kontur ζ tekislikni ikkita: D^- va D^+ cheksiz sohalarga ajratadiki, bunda D^- $\gamma(\varepsilon_0,\beta_0)$ dan chapda, D^+ esa o'ngda.

$\rho>1,\quad \dfrac{\pi}{2\rho}<\beta_0<\dfrac{\pi}{\rho}$ uchun quyidagi belgilashni olamiz:

$$\Psi_\rho(w)=\frac{\rho}{2\pi i}\int\limits_{\gamma(\varepsilon_0,\beta_0)}\frac{\exp\zeta^\rho}{\zeta-w}d\zeta.$$

Bu holda quyidagi integral tasvir o'rinlidir:

$$E_\rho(w)=\Psi_\rho(w),\qquad w\in D^-,$$

$$E_\rho(w)=\rho\exp w^\rho+\Psi_\rho(w),\qquad w\in D^+.$$

Bu formulalardan ayrim $\eta_0>0$ uchun quyidagi baholashlarni olishimiz mumkin.

$$|E_\rho(w)|\le\rho\exp(Rew^\rho)+|\Psi_\rho(w)|,\quad |argw|\le\frac{\pi}{2\rho}+\eta_0,$$

$$|E_\rho(w)|=|\Psi_\rho(w)|, \quad \frac{\pi}{2\rho}+\eta_0 \le |argw| \le \pi,$$

$$|\Psi_\rho(w)| \le \frac{const}{1+|w|}, \, E_\rho(w) : \rho \exp w^\rho, \quad w>0, \quad w \to \infty.$$

2.2. Somilian – Betti formulasi va uning umumlashmasi

$D - E^2$ chekli bog'lamli soha, ∂D chegarasi bo'lakli – silliq bo'lsin. $A(D)$ orqali (1) momentli elastiklik nazariyasi sistemalarining barcha yechimlari fazosini belgilaymiz.

Agar D chegaralangan va $U(x) \in A(D)$ bo'lsa, u holda Somilian – Betti formulasi o'rinli:

$$\int_{\partial D} \left\{ \Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\} \right\} dS_y = \begin{cases} u(x), & x \in D \\ 0, & x \notin \overline{D} \end{cases}$$

bu yerda $T(\partial_y, n)$ - kuchlanish operatori.

$U(x) \in A(D)$ bo'lsin, bunda D - chekli bog'lamli chegaralanmagan soha.

$$D_R = \{y \in D : |y| < R\},\ D_R^\infty = D \setminus D_R,\ R > 0$$

belgilashni kiritamiz.

Agar har bir $x \in D$ da

$$\lim_{x\to\infty} \int_{\partial D_R^\infty} \left\{ \Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\} \right\} ds_y = 0 \quad (9)$$

bo'lsa, u holda $x \in D$ da

$$\int_{\partial D} \left\{ \Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\} \right\} ds_y = U(x). \quad (10)$$

Haqiqatan, $x \in D$ $(|x| < R)$ da quyidagiga egamiz:

$$\int_{\partial D} \left\{ \Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\} \right\} dS_y =$$

$$= \int_{\partial D_R} \left[\Pi(y,x)\, T(\partial_y, n)U(y) - U(y)\, T(\partial_y, n)\Pi(y,x) \right] dS_y =$$

$$= \int_{\partial D_R^{\infty}} \left[\Pi(y,x)\, T(\partial_y, n)U(y) - U(y)\, T(\partial_y, n)\Pi(y,x) \right] dS_y =$$

$$= U(y) + \int_{\partial D_R^{\infty}} \left[\Pi(y,x)\, T(\partial_y, n)U(y) - U(y)\, T(\partial_y, n)\Pi(y,x) \right] dS_y$$

Endi $R \to \infty$ da limitga o'tsak, (10) ni olamiz.

1. (10) formulaning xususiy hollarini qarab chiqamiz, chegaralanmagan D soha $0 < y_2 < h$, $h = \pi / \rho$, $\rho > 0$ tengsizlik bilan aniqlanuvchi qatlam ichida yotsin, bunda ∂D uzunligi quyidagi shartini qanoatlantirsin:

$$\int_{\partial D} \exp\left(-b_0 ch \rho_0 |y_1|\right) dS_y < \infty, \; 0 < \rho_0 < \rho. \qquad (11)$$

Bundan tashqari $U(x) \in A(D)$ quyidagi o'sish shartini qanoatlantirsin:

$$|U(y)| + |T(\partial_y, n)U(y)| \le c \exp(\exp \rho_2 |y_2|) \; (12)$$

$$\rho_2 < \rho, \; y \in D.$$

(4) dagi $\Phi_\sigma(y,x,\lambda)$ ning ifodasiga

$$K(w) = \exp\left[-bchi\rho_1 \left(w - \frac{h}{2} \right) - b_1 chi\rho_0 \left(w - \frac{h}{2} \right) \right]$$

ni qo'yamiz, bu yerda $w = i\sqrt{u^2 + \alpha^2} + y_2$, $0 < \rho_1 < \rho$, $0 < x_2 < h$, $b > 0$,

$$b_1 > b_0 \left[\cos\left(\frac{\rho_0 h}{2} \right) \right]^{-1}, \; \alpha = |y_1 - x_1|.$$

$\Phi_\sigma(y,x,\lambda) = \Phi(y,x,\lambda)$ ni quramiz:

$$-2\pi K(x_2)\Phi(y,x,\lambda)=\int_0^\infty \mathrm{Im}\frac{K(w)}{w-x_2}\cdot\frac{uJ_0(\lambda u)du}{\sqrt{u^2+\alpha^2}}$$

Endi $\Pi(y,x)$ ni $\Phi(y,x,\lambda)$ bo'yicha (5) dagidek quramiz.

(10) ni isbotlash uchun (9) ni ko'rsatish yetarli.

$$\int_{\partial D_R^\infty}\left[\Pi(y,x)T(\partial_y,n)U(y)-U(y)T(\partial_y,n)\Pi(y,x)\right]ds_y\le$$

$$\le\int_{\partial D_R^\infty}\left(|\Pi(y,x)|+|T(\partial_y,n)\Pi(y,x)|\right)\left(|U(y)|+|T(\partial_y,n)U(y)|\right)ds_y\,.\quad(1.13)$$

bo'lgani uchun $\Phi(y,x,\lambda)$, $\frac{\partial\Phi}{\partial y_i}$, $\frac{\partial^2\Phi}{\partial y_i\partial y_j}$, $i,j=1,2,...$ larni baholashimiz lozim:

$$\left|\exp\left[-bchi\rho_1\left(w-\frac{h}{2}\right)-b_1chi\rho_0\left(w-\frac{h}{2}\right)\right]\right|=$$

$$=\left|\exp\mathrm{Re}\left[-bchi\rho_1\left(w-\frac{h}{2}\right)-b_1chi\rho_0\left(w-\frac{h}{2}\right)\right]\right|=$$

$$=\exp\left[-bch\rho_1\sqrt{u^2+\alpha^2}\cos\rho_1\left(y_2-\frac{h}{2}\right)-\right.$$

$$\left.-bch\rho_0\sqrt{u^2+\alpha^2}\cos\left(y_2-\frac{h}{2}\right)\right],$$

$$-\pi|2<-\rho_1h|2\le\rho_1\left(y_2-\frac{h}{2}\right)\le\rho_1\frac{h}{2}<\frac{\pi}{2},$$

$$-\frac{\pi}{2}<\rho_0\left(y_2-\frac{h}{2}\right)<\frac{\pi}{2}.$$

Shuningdek, $\cos\rho_1\left(y_2-\frac{h}{2}\right)>0$, $\cos\rho_0\left(y_2-\frac{h}{2}\right)>0$ ya’ni D sohada nolga aylanmaydi, u holda quyidagi baho o’rinli:

$$|\Phi(y,x,\lambda)|+\left|\frac{\partial\Phi(y,x,\lambda)}{\partial y_i}\right|+\left|\frac{\partial^2\Phi(y,x,\lambda)}{\partial y_i\partial y_j}\right|=$$

$$=o\left(\exp\left[-\varepsilon ch\rho_1|y_1|-\delta ch\rho_0|y_1|\right]\right)\quad(14)$$

bu yerda $\varepsilon=b\cos\rho_1\left(y_2-\frac{h}{2}\right)$, $\delta=b_1\cos\rho_0\left(y_2-\frac{h}{2}\right)$, $y\to\infty$, $y\in\partial D$. (13) va (14) baholardan foydalanib, (9) shartning bajarilishini olamiz. Bundan esa, (10) formula D sohada to’g’ri.

(12) shartni kuchsizlantirish mumkin. Quyidagicha belgilaymiz:

$$A_\rho(D)=\{U(y):U(y)\in A(D),\ |U(y)|+|T(\partial_y,n)U(y)|\le$$

$$\le\exp(o(\exp\rho|y_1|)),\ y\to\infty,\quad \rho>0\}.$$

5-teorema **.** $U(x)\in A_\rho(D)$ o’sishning chegaraviy shartini qanoatlantirsin:

$$|U(y)|+|T(\partial_y,n)U(y)|\le c\exp\left[a\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\rho_1|y_1|\right].\quad(15)$$

Agar $0<\rho_1<\rho$ bo’lsa, u holda quyidagi tengsizlik o’rinli:

$$U(x)=\int_{\partial D_R^\infty}\left[\Pi(y,x)\{T(\partial_y,n)U(y)\}-U(y)\{T(\partial_y,n)\Pi(y,x)\}\right]ds_y,\ x\in D$$

bunda $a=b$.

D *soha* $y_2 > 0$ yarim fazoda yotuvchi, ∂D chegarasi silliq, bir bog'lamli chegaralanmagan soha bo'lsin.

Faraz qilamizki, ∂D ushbu $y_2 = f(y_1)$ tenglama bilan berilgan, bunda

$|f'(y)| \le C < \infty$

$K(w) = (w + x_2)^{-k} \exp\{-\varepsilon (w+1)^{\rho_1}\}$

$w = i\sqrt{u^2 + \alpha^2} + y_2$, $\varepsilon > 0$, $y_2 \ge 0$, $0 < \rho_1 < 1$, $k \in N$ ni olamiz. Bu yerda w^{ρ} analitik funksiyaning manfiy yarim o'qi olib tashlanganda regulyar tarmog'i butun tekislikda olingan bo'lib, u haqiqiy w da haqiqiydir.

Quyidagicha belgilaymiz:

$B_\rho(D) = \{U(y) : U(y) \in A(D), |U(y)| + |grad\, U(y)| \le \exp|y|^{p}, \rho > 0\}$

ρ_1 ni $\rho < \rho_1 < 1$ shart bo'yicha tanlaymiz. U holda $U(x) \in B_\rho(D)$ uchun (9) shart bajariladi va (10) formula o'rinli. Haqiqatan, ham

$|K(w)| = |(w + x_2)^{-k}| \exp \mathrm{Re}\{-\varepsilon (w+1)^{\rho_1}\}$

$w + 1 = i\sqrt{u^2 + \alpha^2} + y_2 + 1 = \sqrt{u^2 + \alpha^2 + (y_2+1)^2}\,(\cos\varphi + i\sin\varphi)$

bu yerda

$$\cos\varphi = \frac{(y_2+1)}{\sqrt{u^2 + \alpha^2 + (y_2+1)^2}}, \quad \sin\varphi = \frac{\sqrt{u^2+\alpha^2}}{\sqrt{u^2 + \alpha^2 + (y_2+1)^2}}$$

$|K(w)|$ da $w+1$ ifodani qo'yib, quyidagini olamiz:

$|K(w)| = |w|^{-k} \exp\{-\varepsilon |w|^{\rho_1} \cos \rho_1 \varphi\}$

$-\pi<\arg w<\pi$ bo'lgani uchun $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$ va $-\frac{\pi}{2}<-\frac{\pi\rho_1}{2}<\rho_1\varphi<\frac{\pi\rho_1}{2}<\frac{\pi}{2}$ va

shuning uchun $\cos\rho_1\ge\delta_0>0$, bulardan esa quyidagini olish mumkin:

$$2\pi\left|K(x_2)\Phi(y,x)\right|\le\int_0^\infty\left|\operatorname{Im}\frac{K(w)}{w-x_2}\right|\cdot\frac{udu}{\sqrt{u^2+\alpha^2}}=$$

$$=\int_\alpha^\infty\frac{\exp\left[-\varepsilon_0\left(t^2+(y_2+1)^2\right)^{\frac{\rho_1}{2}}\right]dt}{\left(t^2+\left(y_2+x_2\right)^2\right)^{\frac{k}{2}}\left(t^2+\left(y_2-x_2\right)^2\right)^{\frac{1}{2}}}\le$$

$$\le\exp\left[-\varepsilon_0\left(\alpha^2+(y_2+1)^2\right)^{\frac{\rho_1}{2}}\right]\cdot\int_\alpha^\infty\frac{dt}{\left(t^2+\left(y_2-x_2\right)^2\right)^{\frac{k+1}{2}}}$$

Agar $\displaystyle\int_\alpha^\infty\frac{dt}{\left(t^2+\left(y_2-x_2\right)^2\right)^{\frac{k+1}{2}}}$ integralda $t=(y_2-x_2)tgu$ almashtirish kiritsak,

u holda

$$dt=\frac{(y_2+x_2)}{\cos^2u}du \text{ va } \alpha_0=arctg\,\frac{\alpha}{(y_2-x_2)}\le t\le\frac{\pi}{2}$$

shuning uchun

$$\int_\alpha^\infty\frac{dt}{\left(t^2+\left(y_2-x_2\right)^2\right)^{\frac{k+1}{2}}}=\frac{1}{\left(y_2-x_2\right)^k}\int_0^{\frac{\pi}{2}}\cos^{k-1}udu$$

Shuningdek,

$$\left|\Phi(y,x,\lambda)\right|\le\frac{\exp\left[-\varepsilon\left(\alpha^2+\left(y_2+1\right)^2\right)^{\frac{\rho_1}{2}}\right]}{\left(y_2-x_2\right)^k}$$

bu yerda $\varepsilon_0=\varepsilon\delta_0$.

$\dfrac{\partial\Phi}{\partial y_i},\ \dfrac{\partial^2\Phi}{\partial y_i\partial y_j},\ i,j=1,2$ ni baholaymiz. Tekshirish qiyinmaski,

$$\frac{\partial\Phi}{\partial y_i}=\frac{sign(y_1-x_1)}{2\pi K(x_2)}\operatorname{Im}\frac{K(i\alpha+y_2)}{i\alpha+y_2-x_2}$$

$$\frac{\partial\Phi}{\partial y_2}=\frac{1}{2\pi K(x_2)}\operatorname{Re}\frac{K(i\alpha+y_2)}{i\alpha+y_2-x_2}=\frac{1}{2\pi K(x_2)}\cdot\operatorname{Im}\frac{K(i\alpha+y_2)}{i(y_2-x_2)-\alpha}$$

$$\frac{\partial^2\Phi}{\partial y_1^2}=\frac{1}{2\pi K(x_2)}\operatorname{Re}\left[\frac{K'(i\alpha+y_2)}{i\alpha+y_2-x_2}-\frac{K(i\alpha+y_2)}{(i\alpha+y_2-x_2)^2}\right]$$

$$\frac{\partial^2\Phi}{\partial y_2^2}=-\frac{1}{2\pi K(x_2)}\operatorname{Re}\left[\frac{K'(i\alpha+y_2)}{i\alpha+y_2-x_2}-\frac{K(i\alpha+y_2)}{(i\alpha+y_2-x_2)^2}\right]$$

$$\frac{\partial^2\Phi}{\partial y_1\partial y_2}=\frac{1}{2\pi K(x_2)}\operatorname{Im}\left[\frac{K'(i\alpha+y_2)}{i\alpha+y_2-x_2}-\frac{K(i\alpha+y_2)}{(i\alpha+y_2-x_2)^2}\right]$$

Ushbu oxirgi tengliklardan quyidagini olamiz:

$$|\Pi(y,x)|+|T(\partial,n)\Pi(y,x)|\le\frac{C}{|x-y|^{k+2}}\exp\left[-\varepsilon_0\left(\alpha^2+(y_2+1)^2\right)^{\frac{1}{2}}\right]$$

Endi agar $U(x)\in B_\rho(D)$ bo'lsa, u holda

$$\lim_{n\to\infty}\int_{\partial D_e^\infty}\left[\Pi(y,x)\{T(\partial_y,n)U(y)\}-U(y)\{T(\partial_y,n)\Pi(y,x)\}\right]dS_y=0.$$

Va shu bilan tasdiq isbotlandi.

$\Phi(y,x,\lambda)$ funksiya hosilalarining formulalaridan ko'rinadiki,

$$|\Pi(y,x)|\le o\left(\frac{1}{y^2}\right),\ y\to\infty,\ y\in\partial D$$

$$|T(\partial,n)\Pi(y,x)|\le o\left(\frac{1}{y^3}\right),\ y\to\infty.(16)$$

6-Teorema. $U(x)\in B_\rho(D)$ bo'lsin, bu yerda D *soha* $y_2>0$ yarimtekislikdan olingan chegarasi cheksizlikkacha cho'ziladigan va $y_2=f(y_1)$ tenglama bilan berilgan soha; bunda f, $|f'(y)|\le C<\infty$ va $0\le f\le y_0<\infty$ shartlarni qanoatlantiruvchi differensiallanuvchi funksiya. Bundan tashqari U(y)funksiya quyidagi chegaraviy shartlarni qanoatlantirsin

$$\int_{\partial D}\frac{|U(y)|}{1+|y|^2}dS_y<\infty,\quad \int_{\partial D}\frac{|T(\partial_y,n)U(y)|}{1+|y|}dS_y<\infty .$$

Agar $\rho<1$ bo'lsa, u holda

$$U(x)=\int_{\partial D}\left[\Pi(y,x)\,T(\partial_y,n)U(y)-U(y)\,T(\partial_y,n)\Pi(y,x)\right]dS_y,\quad x\in D$$

bu yerda $\Pi(y,x)$ (5) formula bo'yicha aniqlanadi.

$$\Phi(y,x)=\frac{1}{4\pi}\ln\left[(y_1-x_1)^2+(y_2-x_2)^2\right]-\frac{1}{2\pi}\ln r,\ r=|x-y|$$

Isbot. (16) dan (10) ning chap qismidagi integralning $\varepsilon\ge 0$ parametrga nisbatan tekis yaqinlashishi kelib chiqadi. $K(w)$ da $\varepsilon=0$ ni qo'yamiz. U holda quyidagiga ega bo'lamiz:

$$|\Phi(y,x,\lambda)|\le\left(\frac{x_2}{\pi}\right)\left|\int_0^\infty \mathrm{Im}\frac{uJ_0(\lambda u)du}{\left(\sqrt{u^2+\alpha^2}+y_2+x_2\right)\left(\sqrt{u^2+\alpha^2}+y_2-x_2\right)\sqrt{u^2+\alpha^2}}\right|=$$

$$\le C\left(\frac{x_2}{\pi}\right)\left|\int_0^\infty \mathrm{Im}\frac{dt}{(t+y_2+x_2)(t+y_2-x_2)}\right|=$$

$$=\left(\frac{x_2y_2}{\pi}\right)\int_0^\infty\frac{tdt}{\left(t^2+(y_2+x_2)^2\right)\left(t^2+(y_2-x_2)^2\right)}=\frac{1}{4\pi}\ln\left[(y_1-x_1)^2+(y_2+x_2)^2\right]-\frac{1}{2\pi}\ln r$$

va shu bilan teorema isbotlandi.

$D: y_2 > 0$ yarimtekislik bo'lsin. U holda quyidagi teorema o'rinli:

7-Teorema. $U(x) \in B_\rho(D)$ ushbu

$$\int_{y_2=0} \frac{|U(y)|}{1+|y|^2} dS_y < \infty \quad , \quad \int_{y_2=0} \frac{|T(\partial_y, n)U(y)|}{1+|y|} dS_y < \infty \qquad (17)$$

chegaraviy shartlarni qanoatlantirsin. Agar $\rho < 1$ bo'lsa, u holda $x_2 > 0$ da

$$U(x) = \int_{y_i=0} \left[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}\right] dS_y, \quad x \in D$$

formula to'g'ridir.

Isbot. (17) shartni ushbu tarzda qayta yozish mumkin:

$$\int_{-\infty}^{\infty} \frac{|U(y_1,0)|}{1+|y_1|^2} dy_1 < \infty, \quad \int_{-\infty}^{\infty} \frac{|T(\partial_y,n)U(y_1,0)|}{1+|y_1|} dy_1 < \infty .$$

$K(w)$ ni yuqoridagidek olamiz. $\Phi(y,x,\lambda)$ va $\Pi(y,x)$ ni quramiz.

$$D_R = \{x \in D : |x| < R\}, \partial D = \{|x| = R\}, D_R^\infty = \frac{D}{D_k}$$

doirani qaraymiz, u holda $x \in D_R$ da

$$\int_{\partial D} \left[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}\right] dS_y =$$

$$= \int_{\partial D_R} \left[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}\right] dS_y +$$

$$+ \int_{\partial D_R^\infty} \left[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}\right] dS_y =$$

$$= U(x) + \int_{\partial D_R^\infty} \left[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}\right] dS_y$$

Lekin (15) va (17) shartlarda ∂D_R^{∞} bo'yicha oxirgi integral $R \to \infty$ da nolga tekis yaqinlashadi. Shu bilan teorema isbotlanadi.

II bob bo'yicha xulosa

Bu bobda asosan chegaralangan sohalar uchun Karleman matritsalari va ular yordamida momently elastiklik nazariyasi sistemasi echimini integral ko'rinishida tasviri olingan. Bu integral formulalar yechimni soha chegarasining biror qismidagi qiymatlariga ko'ra yechimni soha ichidagi qiymatini toppish imkonini beradi. Bu integral formulalar umumiy kursdan ma'lum bo'lgan Grin funksiyalari va Grin formulalaridek ahamiyatga ega, ya'ni Karleman funksiyasi, Karleman matritsasi bu ishda sohadan bog'liq ravishda qurilib, yechimlar mos ravishda aniq ko'rinishda yoziladi.

III-**Bob**

§3.1. Koshi masalasini tekislikdagi chegaralangan soxalarda yechish

1. D elastik muhit E^2 da chegaralangan bir bog'lamli, chegarasi $y_1 = 0$ haqiqiy o'qning l bo'lagi va $y_2 > 0$ yarim tekislikda yotuvchi silliq S egri chiziqdan iborat bo'lsin, ya'ni D qalpoq shaklidagi soha.

Koshi masalasini qaraylik. I sohada bir jinsli momentli elastiklik nazariyasi tenglamalari sistemasi

$$\begin{cases} (\mu+\alpha)\Delta u + (\lambda+\mu-\alpha) graddivu + 2\alpha\, rotw + \rho\theta^2 u = 0, \\ (\nu+\beta)\Delta w + (\varepsilon+\nu-\beta) graddivw + 2\alpha\, rotu - 4\alpha w + j\theta^2 w = 0, \end{cases} \qquad (18)$$

berilgan bo'lsin. Bu yerda $U(x) = (u_1(x), u_2(x), w_1(x), w_2(x)) = (u(x), w(x))$ sistemaning yechimi, Δ- Laplas operatori, $\lambda, \mu, \nu, \beta, \varepsilon, \alpha$ elastik muhitni xarakterlaydigan sonlar bo'lib, quyidagi shartlarni qanoatlantiradi

$\mu > 0, 3\lambda + 2\mu > 0,\ \alpha > 0,\ \varepsilon > 0,\ 3\varepsilon + 2\nu > 0,\ \beta > 0,\ j > 0,\ \rho > 0,\ \theta \in R^1.$

$U(x) - D$ sohada (18) sistemaning regulyar yechimi bo'lsin va S da quyidagi Koshi shartlarini qanoatlantirsin:

$$\left.\begin{aligned} U(y) &= f(y), & y \in S \\ T(\partial_y, n)U(y) &= g(y), & y \in S \end{aligned}\right\} \qquad (19)$$

Bunda $T(\partial_y, n(y))$-kuchlanish operatori, $n(y)=(n_1(y), n_2(y))-\partial D$-chegaraning y nuqtasidagi tashqi birklik normal vektori, $f(y)=(f_1(y), f_2(y))$ и $g(y)=(g_1(y), g_2(y))-S$ da berilgan uzluksiz vektor – funksiyalar.

Berilgan f va g lardan kelib chiqib, $U(y)\in D$ ni davom ettirish talab etiladi.

Yuqorida eslatganimizdek bu masala nokorrekt bo'lib, uni yechish maqsadida biz yechimni ma'lum sinfda va Koshining berilganlari yechimni mavjudligi va yagonaligini ta'minlagan holda uni yechish yo'lini izlaymiz.

$\sigma>0$ da (4) formulada

$$K(w)=\exp\sigma w,\ w=i\sqrt{u^2+\alpha^2}$$

ifodani qo'yamiz va quyidagiga ega bo'lamiz:

$$2\pi\exp(\sigma x_2)\Phi_\sigma(y,x,\lambda)=\int_0^\infty \operatorname{Im}\frac{\exp(\sigma w)}{w-x_2}\cdot\frac{uJ_0(\lambda u)du}{\sqrt{u^2+\alpha^2}} \qquad (20)$$

(20) formuladan ko'rinadiki, l da $\Phi_\sigma(y,x,\lambda)$ funksiya, uning $\nabla\Phi_\sigma(y,x,\lambda)$ gradiyenti va uning ikkinchi tartibli xususiy xosilalari

$$\frac{\partial^2\Phi_\sigma(y,x,\lambda)}{\partial y_i\partial y_j},\quad i,j=1,2,$$

$\sigma\to\infty$ da barcha y va $\mathbf{x}\in\mathbf{E}^2$ larda eksponensial nolga intiladi.

Quyidagi funksiyani kritamiz

$$U_\sigma(x)=\int_S[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\}-\{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y.\ (21)$$

Quyidagi teorema o'rinli

8-теорема. $U(x)-$ *(18) sistemaning* D *-sohadagi regulyar yechimi quyidagi shartni qanoatlantirsin*

$$|U(y)|+|T(\partial_y,n)U(y)|\le M,\ y\in\partial D\setminus S. \qquad (22)$$

Bu holda $\sigma\ge 1$*da*

$$|U(y)-U_\sigma(y)|\le MC_2(x)\sigma\exp(-\sigma x_2),$$

tengsizlik o'rinlidir.

Isbot. (3) va (21) formulalarga asosan

$$|U(x)-U_\sigma(x)|=\int\limits_{\partial D\setminus S}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\}-\{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y.$$

(22) va (7) tengsizliklarga asosan teoremaning isboti kelib chiqadi. Teorema isbot bo'ldi.

$U(x)$ ni taqribiy topish formulasini keltiramiz. Faraz qilamiz S da $U(y)$ va $T(\partial_y,n)U(y)$ o'rniga ularning δ aniqlikdagi qiymatlari ma'lum bo'lsin: $f_\delta(y)$ va $g_\delta(y)$:

$$\max_S|f(y)-f_\delta(y)|+\max_S|T(\partial_y,n)U(y)-g_\delta(y)|\le\delta,\ 0<\delta<1. \qquad (23)$$

Quyidagi funksiyani qaraymiz:

$$U_{\sigma\delta}(x)=\int\limits_S[\Pi(y,x,\sigma)g_\delta(y)-\{T(\partial_y,n)\Pi(y,x,\sigma)\}^*f_\delta(y)]ds_y,$$

bunda

$$\sigma=\frac{1}{x_2^0}ln\frac{M}{\delta},\quad x_2^0=\max_D x_2,\quad x_2>0.$$

9-теорема. $U(x)$ – *(18) sistemaning* D *-sohadagi regulyar yechimi* $\partial D \setminus S$ *da (22) shartni qanoatlantirsin. U holda quyidagi baho o'rinlidir.*

$$|U(x) - U_{\sigma\delta}(x)| \le C(x)\delta^{\frac{x_2}{x_2^0}}\left(ln\frac{M}{\delta}\right), \quad x \in D$$

Isbot. (3) va (21) formulalarga asosan

$$U(x) - U_{\sigma\delta}(x) = \int_S [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y) - g_\delta(y)\} -$$

$$-\{T(\partial_y,n)\Pi(y,x,\sigma)\}(U(y) - f_\delta(y))\,]ds_y +$$

$$+\int_{\partial D\setminus S}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^* U(y)]ds_y.$$

Endi teoremaning shartidan va (22), (23) tengsizliklardan $\forall x \in D$ da

$$|U(x) - U_{\sigma\delta}(x)| = C_2(x)\delta\sigma\exp\sigma(x_2^0 - x_2) + C_1(x)\sigma\exp(-\sigma x_2) \le$$

$$\le C(x)\sigma(M + \delta\exp\sigma x_2^0)\exp(-\sigma x_2)$$

ga ega bo'lamiz. $\sigma = \dfrac{1}{x_2^0} ln\dfrac{M}{\delta}$, dan teoremaning tasdiqi kelib chiqadi.

Turg'unlik bahosini keltiramiz.

10-теорема. $U(x)$ – *(18) sistemaning* D *-sohadagi regulyar yechimi quyidagi shartlarni qanoatlantirsin*

$$|U(y)| + |T(\partial_y,n)U(y)| \le M,\ y \in \partial D \setminus S.$$

$$|U(y)| + |T(\partial_y,n)U(y)| \le \delta, \quad 0 < \delta < 1, \quad y \in S.$$

Bu holda

$$|U(x)| \le C_4(x)\delta^{\frac{x_2}{x_2^0}}\left(ln(\frac{M}{\delta})\right),$$

Bunda $C_4(x) = C\int_{\partial D}\left(\frac{1}{r}+\frac{1}{r^2}\right)ds_y$, $C-\lambda,\mu,\varepsilon,\beta,\nu$ *o'zgarmaslardan bog'liq koeffisiyent.*

Isbot.8 –teoremaga asosan

$$|U(x)| \le |U_\sigma(x)| + MC_2(x)\sigma\exp(-\sigma x_2).$$

Teorema shartlaridan va (23) dan:

$$|U_{\sigma\delta}(x)| = \left|\int_S [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*(U(y))ds_y\right| \le$$

$$\le \int_S \left[|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|\right]\left[|U(y)| + |T(\partial_y,n)U(y)|\right]ds_y \le$$

$$\le \delta\int_S \left[|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|\right]ds_y \le C_3(x)\delta\sigma\exp(\sigma x_2^0 - \sigma x_2).$$

Bundan

$$|U(x)| \le C_4(x)\sigma\exp(-\sigma x_2)(M + \delta\exp\sigma x_2^0).$$

Endi $\sigma = \frac{1}{x_2^0}ln\frac{M}{\delta}$ ligidan

$$|U(x)| \le C_4(x)\delta^{\frac{x_2}{x_2^0}}\left(ln\frac{M}{\delta}\right).$$

Turg'unlik bahosini olamiz.Teorema isbotbo'ldi.

Yuqoridagi teoremalardan quyidagi natijani yozamiz.

Natija.

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x), \quad \lim_{\delta\to 0} U_{\sigma\delta}(x) = U(x)$$

limitlar D ning har bir ichki nuqtasida tekis bajariladi.

2. Endi $D_p \subset E^2$ -quyidagi doiraviy sektorsimon soha bo'lib, u quyidagi tenglama orqali ifodalansin:

$$\Sigma : \alpha_1 = \tau\, y_2, \;\; \alpha_1 = |\, y_1 \,|, \;\; \tau = tg\frac{\pi}{2\rho}, \quad y_2 > 0, \quad \rho > 1,$$

S -bu burchak ichida yotgan silliq chiziq.

Quyidagi belgilashlarni kritamiz:

$$\tilde{\beta} = \tau\, y_2 - \alpha_0, \;\; \gamma = \tau\, x_2 - \alpha_0, \;\; \alpha_0^2 = x_1^2, s = \tilde{\alpha}^2 = (y_1 - x_1)^2 \;,$$

$$\omega = i\tau\sqrt{u^2 + s} + \tilde{\beta}, \quad \omega_0 = i\tau\tilde{\alpha} + \tilde{\beta}.$$

D_p sohada (18)-(19) Koshi masalasini qaraymiz. Buning uchun yuqoridagi qalpoq ko'rinishidagi sohalar uchun qilingan usullardan foydalanamiz. Karleman matritsasini quramiz:

$\sigma > 0$ da (4) formulaga

$$K(\omega) = E_\rho(\sigma(\omega - x_2)), \;\; \rho > 1$$

ni qo'yib, bunda $E_p(\omega)$ -Mittag-Lefler funksiyasi, (5) formulaga

$$2\pi\Phi_\sigma(y - x, \lambda) = \int_0^\infty Im\left[\frac{E_p\left|\sigma(i\sqrt{u^2+s} + y_2 - x_2)\right|}{i\sqrt{u^2+s} + y_2 - x_2}\right]\frac{\psi(\lambda u)\,du}{\sqrt{u^2+s}}$$

ni qo’yib matrisa $\Pi(y,x,\sigma)$-Karleman matritsasini quramiz.

Quyidagi funksiyalarni quramiz

$$U_\sigma(x)=\int_S [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\}-\{T(\partial_y,n)\Pi(y,x,\sigma)\}^* U(y)]ds_y \ ,$$
$$U_{\sigma\delta}(x)=\int_S [\Pi(y,x,\sigma)g_\delta(y)-\{T(\partial_y,n)\Pi(y,x,\sigma)\}^* f_\delta(y)]ds_y,$$

buyerda $f_\delta(y)$ va $g_\delta(y)$ vektor-funksiyalar $U(y)$ va $T(\partial_y,n)U(y)$ funksiyalarning S dagi taqribiy qiymatlari, ya’ni

$$\max_S |f(y)-f_\delta(y)|+\max_S |T(\partial_y,n)U(y)-g_\delta(y)|\le\delta,\ 0<\delta<1.$$

Quyidagi teorema o’rinli

11-теорема. $U(x)$ – *(18) sistemaning* D_p*-sohadag regulyar yechimi quyidagi shartlarni qanoatlantirsin*

$$|U(y)|+|T(\partial_y,n)U(y)|\le M,\ y\in\partial D_\rho\setminus S.$$

Bu holda $x\in D_\rho,\ \sigma\ge\sigma_0>0$ *lar uchun*

$$|U(x)-U_\sigma(x)|\le MC(x)\sigma^2\exp(-\sigma\gamma^\rho),$$

bahoo’rinlidir. Bunda

$$C(x)=C(\rho)\int_{\partial D_\rho}\frac{ds_y}{r^2},\quad C(\rho)-\textit{faqat}\ \rho\ \textit{va sohadan bog’liq son}.$$

Теорема: $U(x)$ – *(18)sistemaning* D_p*-sohadagi regulyar yechimi quyidagi shartlarni qanoatlantirsin.*

$$|U(y)|+|T(\partial_y,n)U(y)|\le M,\ y\in\partial D_\rho\setminus S.$$

$$\max_S |f(y)-f_\delta(y)| + \max_S |T(\partial_y,n)U(y)-g_\delta(y)| \le \delta,\ 0<\delta<1.$$

Bu holda quyidagi baho o'rinli

$$|U(x)-U_{\sigma\delta}(x)| \le C_p(x)\delta^q (ln\frac{M}{\delta})^2, \quad x\in D_p,$$

bunda

$$\sigma=(\tau R)^{-p} ln\frac{M}{\delta}, \quad R^p=\max_S \mathrm{Re}(i\sqrt{s}+y_2)^p,$$

$$q=(\frac{x_2}{R})^p, \quad C_p(x)=C_p\int_\Sigma\left[\frac{1}{r}+\frac{1}{r^2}\right]ds_y.$$

§3.2.Koshi masalasini tekislikdagi chegaralanmagan soxada yechish

E^2 ikki o'lchovli evlkid fazosi, D- E^2 dagi bo'lakli silliq ∂D chegarali soha bo'lsin. ∂Dniikki qismga, S-silliq qism va $\partial D\backslash S$ ga ajratamiz.

$x=(x_1,x_2),\ y=(y_1,y_2)$ – E^2 daginuqtalar, $U(x)$ Dsohada momentli elastiklik nazariyasi sistemasini yechimi bo'lsin:

$$\begin{cases}(\mu+\alpha)\Delta u+(\lambda+\mu-\alpha)graddivu+2\alpha\, rotw+\rho\theta^2 u=0,\\ (\nu+\beta)\Delta w+(\varepsilon+\nu-\beta)graddivw+2\alpha\, rotu-4\alpha w+j\theta^2 w=0,\end{cases}$$

berilgan bo'lsin. Bu yerda $U(x)=(u_1(x),u_2(x),w_1(x),w_2(x))=(u(x),w(x))$ sistemaning yechimi, Δ- Laplas operatori, $\lambda,\mu,\nu,\beta,\varepsilon,\alpha$ elastik muhitni xarakterlaydigan sonlar bo'lib, quyidagi shartlarni qanoatlantiradi $\mu>0, 3\lambda+2\mu>0,\ \alpha>0,\ \varepsilon>0,\ 3\varepsilon+2\nu>0,\ \beta>0,\ j>0,\ \rho>0,\ \theta\in R^1.$

Quyidagi

$$U(y) = f(y),$$
$$T(\partial y, n)U(y) = g(y), \quad y \in S$$

shartlarni kanoatlantiruvchi $U(x)$ yechimni D sohada topish talab qilinadi.

Bunda $f(y) = (f_1(y), f_2(y))$ va $g(y) = (g_1(y), g_2(y))$–S daberilgan uzluksiz vektor-funksiyalar, $T(\partial_y, n)$-kuchlanish operatori deyilib,

$$T(\partial_y, n(y)) = \begin{Vmatrix} T^{(1)}(\partial_y, n) & T^{(2)}(\partial_y, n) \\ T^{(3)}(\partial_y, n) & T^{(4)}(\partial_y, n) \end{Vmatrix},$$

$$T^{(i)}(\partial_y, n) = \left\| T_{kj}^{(i)}(\partial_y, n) \right\|_{2x2}, \quad i = 1,2,3,4,$$

$$T_{kj}^{(1)}(\partial_y, n) = \lambda n_k \frac{\partial}{\partial y_j} + (\mu - \alpha) n_j(y) \frac{\partial}{\partial y_k} + (\mu + \alpha)\delta_{kj} \frac{\partial}{\partial n(y)}, \quad k, j = 1,\ 2,$$

$$T_{kj}^{(2)}(\partial_y, n) = T_{kj}^{(3)}(\partial_y, n) = 0, \ k, j = 1,\ 2,$$

$$T_{kj}^{(4)}(\partial_y, n) = \varepsilon\, n_k(y) \frac{\partial}{\partial y_j} + (\nu - \beta) n_j(y) \frac{\partial}{\partial y_k} + (\nu + \beta)\ \delta_{kj} \frac{\partial}{\partial n(y)}, \ k, j = 1,\ 2.$$

$n(y) = (n_1(y),\ n_2(y)) - \partial D$-chegaraning y nuqtasidagi tashqi birlik normal vektori, δ_{kj} - Kroneker simvoli.

Endi sohamiz $D \in E^2$ quyidagi qatlamda $0 < y_2 < h,\ h = \dfrac{\pi}{\rho}$, $\rho > 0$, chegaralari $y_2 = 0$ va qandaydir silliq S egri chizikdan iborat bo'lib, bu chiziq quyidagicha tenglama $y_2 = f(y_1), y_1 \in R^1$ va $0 < f(y_1) \le h$, $|f'(y_1)| \le c < \infty$ shartlar bilan berilgan bo'lsin. $U(x) \in A(D)$ bo'lib,

$$|U(y)| + |T(\partial y, n)U(y)| \le M, \ y \in \partial D.$$

Shartni bajarsin. Bu holda oldingi paragraflarda ko'rganimizdek integral tasvir o'rinli bo'lib, bunda

$$K(y) = (y - x_2 + 2h)^{-1} \exp \sigma\omega \,, \ \omega = i\sqrt{u^2 + \alpha^2} + y_2 ,$$

$$K(x_2) = (2h)^{-1} \exp \sigma x_2 , \ 0 < x_2 < h$$

Quyidagi vektor- funksiyani quramiz

$$U_\sigma(x) = \int_S \left\{ \Pi(y,x,\sigma)\{T(\partial y,n)U(y)\} - U(y)\{T(\partial y,n)\Pi(y,x,\sigma)\} \right\} ds_y , x \in D.$$

$$A_\rho(D) = \{U(x) \in A(D): |U(y)| + |T(\partial y,n)U(y)| \le \exp(\rho \exp|y_1|), y_1 \to \infty, \ y \in \bar{D}\}.$$

12-teorema. *$U(x) \in A_\rho(D)$ bo'lib (3) shartni qanoatlantirsin. U holda*

$$|U(x) - U_\sigma(x)| \le MC(\rho,x)\sigma^3 \exp(-\sigma x_2), x \in D, \qquad (24)$$

bunda $C(\rho,x) = C(\rho)\int\limits_{y_2=0} \frac{ds_y}{r}$.

Quyida biz qulaylik uchun ρ dan bog'lik har qanday o'zgarmasni $C(\rho)$ bilan belgilaymiz.

Isbot. Integral tasvir va belgilashimizga asosan

$$|U(x) - U_\sigma(x)| \le \int\limits_{y_2=0} \left\{ \Pi(y,x,\sigma)\{T(\partial y,n)u(y)\} - u(y)\{T(\partial y,n)\Pi(y,x,\sigma)\} \right\} ds_y \le$$

$$\le \int\limits_{y_2=0} \left[|\Pi(y,x,\sigma)| + |T(\partial y,n)\Pi(y,x,\sigma)| \right] \left[|U(y)| + |T(\partial y,n)U(y)| \right] ds_y \le$$

$$\le M \int\limits_{y_2=0} \left[|\Pi(y,x,\sigma)| + |T(\partial y,n)\Pi(y,x,\sigma)| \right] ds_y .$$

$$I = \int\limits_{y_2=0} \left[|\Pi(y,x,\sigma)| + |T(\partial y,n)\Pi(y,x,\sigma)| \right] ds_y ,$$

ni baholaymiz. Karleman matritsasi ta'rifiga ko'ra va undagi funksiyalarning qurilishiga asosan

$$\operatorname{Im}\frac{K(\theta+y_2)}{(\theta+y_2-x_2)}=\frac{\exp\sigma\, y_2}{(\theta^2+r^2)(\theta^2+4h(y_2-x_2)+4h^2+u^2)}$$

$$\cdot\left((y_2-x_2)(y_2-x_2+2h)-\theta^2\right)\sin\sigma\theta-2\theta(y_2-x_2+h)\cos\sigma\theta\,,$$

bunda $\theta=\alpha$. Endi $\Phi_\sigma(y,x,\lambda)$ ni topamiz

$$C_2\Phi_\sigma(y,x,\lambda)=2h\exp\sigma(y_2-x_2)\cdot$$

$$\cdot\int_0^\infty\frac{1}{\left(u^2+s+(y_2-x_2)^2\right)\cdot\left(s+(y_2-x_2)^2+u^2+4h(y_2-x_2)+4h^2\right)}$$

$$\cdot\left[(y_2-x_2)(y_2-x_2+2h)-s-u^2\right]\sin\sigma\sqrt{u^2+s}-$$

$$-2\sqrt{u^2+s}\,(y_2-x_2+h)\cos\sigma\sqrt{u^2+s}\,\frac{J_0(\lambda u)du}{\sqrt{u^2+s}}\,.$$

Bu yerda xosmas integrallar uchun o'rinli bo'lgan o'rta qiymat haqidagi teorema o'rinli, chunki $\dfrac{1}{s+(y_m-x_m)^2+u^2}$ monoton kamayuvchi va chegaralangan. Shuning uchun u $\exists\ \xi\geq 0$, borki,

$$C_2\phi(y,x,\sigma)=2h\exp(\sigma y_2-\sigma x_2)\frac{1}{s+(y_2-x_2)^2+\xi^2}\cdot$$

$$\cdot\int_0^\xi\left[\frac{(y_2-x_2)(y_2-x_2+2h)-s-u^2}{u^2+s+(y_2-x_2+2h)^2}\cdot\frac{\sin\sigma\sqrt{u^2+s}}{\sqrt{u^2+s}}-\frac{y_2-x_2+h}{u^2+s+(y_2-x_2)^2}\cos\sigma\sqrt{u^2+s}\right]du=$$

$$=\frac{2h\exp(\sigma y_2-\sigma x_2)}{\left(s+(y_2-x_2)^2+\xi^2\right)}\cdot\int_0^\xi\left(\frac{(y_2-x_2)(y_2-x_2+2h)-s-u^2}{u^2+s+(y_2-x_2+2h)^2}\cdot\right.$$

$$\left.\cdot\frac{\sin\sigma\sqrt{u^2+s}}{\sqrt{u^2+s}}-\frac{y_2-x_2+h}{u^2+s+(y_2-x_2)^2}\cos\sigma\sqrt{u^2+s}\right)du\leq\frac{C\sigma\exp(\sigma y_2-\sigma x_2)}{r}$$

Bu yerdan ko'rinadiki, undagi integral $x \neq y$ da yaqinlashadi. Integral ostiga differensiallashga o'tish mumkin, chunki s bo'yicha va $y_j,\ j=1,2$ bo'yicha differensiallashga o'tsak u bo'yicha nolga ketish darajasi yuqori bo'ladi. Demak,

$$\left|\frac{\partial^q \phi(y,x,\sigma)}{\partial y_r^{q_1} \ldots \partial y_2^{q_2}}\right| \leq \frac{C\sigma^3 \exp(\sigma y_2 - \sigma x_2)}{r^2}, \quad q = q_1 + q_2, \quad q = 0,1,2.$$

Unda,

$$|\Pi(y,x,\sigma)| + |T(\partial y,n)\Pi(y,x,\sigma)| \leq \frac{C\sigma^3 \exp(\sigma y_2 - \sigma x_2)}{r^2} \tag{25}$$

Demak, $I \leq MC(\rho, x)\sigma^3 \exp(-\sigma x_2)$.

§3.2 Koshi masalasi yechimini regulyarlashtirish

Endi faraz qilamiz $U(y)$ va $T(\partial y,n)U(y)$ o'rniga S da ularning taqribiy qiymatlari $f_\delta(y)$ va $g_\delta(y)$ lar berilgan bo'lsin,

$$\max_S |U(y) - f_\delta(y)| + \max_S |T(\partial y,n)U(y) - g_\delta(y)| < \delta, \qquad 0 < \delta < 1.$$

Quyidagi funksiyani qurib olamiz

$$U_{\sigma\delta}(x) = \int_S \left[\Pi(y,x,\sigma) g_\delta(y) - f_\delta(y) T(\partial y,n)\Pi(y,x,\sigma)\right] ds_y, \qquad x \in D.$$

Quyidagi teorema o'rinli

13-teorema. *$U(x) \in A_\rho(D)$ bo'lib, (3) shartni qanoatlantirsin. U holda*

$$|U(x) - U_{\sigma\delta}(x)| \leq M^{1-\frac{x_2}{h}} C_\rho(x) \delta^{\frac{x_2}{h}} \left[\ln\left(\frac{M}{\delta}\right)\right]^3, \qquad x \in D,$$

bunda $\sigma = h^{-1} \ln \dfrac{M}{\delta}$, $C_\rho(x) = \displaystyle\int_{\partial D} \frac{ds_y}{r^2}$

Isbot. Integral tasvir ifodasi, yuqoridagi funksiyaning qurilishiga ko'ra va (24), (25) baholashlarga ko'ra quyidagiga ega bo'lamiz

$$|U(x)-U_{\sigma\delta}(x)|\leq\left|\int_{y_2=0}\left[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)-g_\delta(y)\}-\{U(y)-f_0(y)\}\{T(\partial_y,n)\Pi(y,x,\sigma)\}\right]ds_y\right|+$$

$$+\left|\int_S\left[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)-g_\delta(y)\}-\{U(y)-f_0(y)\}\{T(\partial_y,n)\Pi(y,x,\sigma)\}\right]ds_y\right|\leq$$

$$\leq MC(\rho,x)\sigma^3\exp(-\sigma x_2)+\delta\int_S\left[|\Pi(y,x,\sigma)|+|T(\partial_y,n)\Pi(y,x,\sigma)|\right]ds_y\leq$$

$$MC(\rho,x)\sigma^3\exp(-\sigma x_2)+$$

$$+\delta\int_S\frac{C_2(\rho)\sigma^3\exp(\sigma y_2-\sigma x_2)}{r^2}ds_y\leq C(\rho,x)\sigma^3(M\exp(-\sigma x_2)+\delta\exp(h\sigma-\sigma x_2))=$$

$$=C(\rho,x)\sigma^3\exp(-\sigma x_2)(M+\delta\exp h\sigma).$$

Endi $M=\delta\exp h\sigma,$ deb olsak, u holda

$$|U(x)-U_{\sigma\delta}(x)|\leq C(\rho,x)M^{1-\frac{x_2}{h}}h^{-3}\left(\ln\frac{M}{\delta}\right)^3\delta^{\frac{x_2}{h}}.$$

Teorema isbot bo'ldi.

Natija. *Quyidagi limitik tengliklar*

$$\lim_{\sigma\to\infty}U_\sigma(x)=U(x),\ \lim_{\delta\to 0}U_{\sigma\delta}(x)=U(x)$$

D *dagi ixtiyoriy kompakt sohada tekis bajariladi.*

Yuqoridagi natijalarni bir muncha umumlashtirish mumkin. S bo'lak quyidagi shartni kanoatlantirsin:

$$\int_S\exp(-b_0 ch\rho_0|y_1|)ds_y<\infty,\quad 0<\rho_0<\rho$$

$U(x)\in A_\rho(D)$ bo'lib, quyidagi chegaraviy o'sish shartini qanoatlantirsin

$$|U(y)|+|T(\partial_y,n)U(y)|\le C\exp\left(a\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\rho_1|y_1|\right),\ y\in\partial D$$

S da berilgan $f_\delta(y), g_\delta(y)$ funksiyalar quyidagi shartlarni qanoatlantirsin:

$$|f_\delta(y)|+|g_\delta(y)|\le\exp\left(a\cos\rho_1\left(y_2-\frac{h}{2}\right)\cdot\exp\rho_1|y_1|\right),\ y\in S,$$

$$|U(y)-f_\delta(y)|+|T(\partial_y,n)U(y)-g_\delta(y)|\le\delta\exp\left(a\cos\rho_1\left(y_2-\frac{h}{2}\right)\cdot\exp\rho_1|y_1|\right),\ y\in S,$$

bunda $0<\delta<1,\ \ 0<\rho_1<\rho,\ \ a\ge 0$. Olamiz

$$K(\omega)=\exp\left[\sigma\omega-bchi\rho_1\left(\omega-\frac{h}{2}\right)-b_1chi\rho_0\left(\omega-\frac{h}{2}\right)\right],$$

bunda $\omega=i\sqrt{u^2+\alpha^2}+y_2,\ \ 0<x_2<h,\ \ b\ge 0,\ \ b_1\ge b_0\left(\cos\rho_0\frac{h}{2}\right)^{-1}+\varepsilon,\ \varepsilon>0,$

$$\alpha=|y_1-x_1|,\ b=2a\exp\rho_1|x_1|,\qquad \sigma=h^{-1}\ln\delta^{-1}.$$

Bu shartlar ostida quyidagi belgilashni olamiz

$$U_{\sigma\delta}(x)=\int_S\left[\Gamma(y,x,\sigma)g_\delta(y)-f_\delta(y)T(\partial_y,n)\Gamma(y,x,\sigma)\right]ds_y,\ x\in D$$

14-teorema. *Yuqoridagi shartlar bajarilganda quyidagi tengsizlik o'rinli bo'ladi*

$$|U(x)-U_{\sigma\delta}(x)|\le C(x)\delta^{\frac{x_2}{h}}\left(\ln\frac{1}{\delta}\right)^3\cdot\exp\left(2a\cos\rho_1\left(x_2-\frac{h}{2}\right)\cdot\exp\rho_1|x_1|\right),x\in D$$

bunda $C(x)=C(\rho)\int_{\partial D}\frac{1}{r^2}\exp(-b_0ch\rho_0\alpha)\exp\left[\frac{b}{2}\cos\rho_1\frac{h}{2}\left(\exp\rho_1(|y_1|-|x_1|)-\exp\rho_1|y_1-x_1|\right)\right]ds_y$

Isbot. Yuqoridagi teoremalarning isbotidagidek

$$|U(x)-U_{\sigma\delta}(x)|=\int_S \left|\left[\Pi(y,x,\sigma)\right]+\left|T(y,n)\Pi(y,x,\sigma)\right|\right]\cdot\left[\left|U(x)-f_\delta(y)\right|+\left|T(y,n)U(y)-g_\delta(y)\right|\right]ds_y+$$

$$+\int_{y_2=0}\left|\left[\Pi(y,x,\sigma)\right]+\left|T(y,n)\Pi(y,x,\sigma)\right|\right]\left[\left|U(y)\right|+\left|T(y,n)U(y)\right|\right]ds_y\le$$

$$\le\delta\int_S \exp\left(a\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\rho_1|y_1|\right)\left|\left[\Pi(y,x,\sigma)\right]+\left|T(y,n)\Pi(y,x,\sigma)\right|\right]ds_y+$$

$$+C\int_{y_2=0}\exp\left(a\cos\rho_1\frac{h}{2}\cdot\exp\rho_1|y_1|\right)\left|\left[\Pi(y,x,\sigma)\right]+\left|T(y,n)\Pi(y,x,\sigma)\right|\right]ds_y$$

$$\left|\Pi(y,x,\sigma)\right|+\left|T(y,n)\Pi(y,x,\sigma)\right|\le\frac{C(\rho)}{r^2}\cdot\sigma^3\exp\left(-\sigma(y_2-x_2)\right)\cdot$$

$$\exp\left(\frac{-b}{2}\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\rho_1\alpha-b_1\cos\rho_0\left(y_2-\frac{h}{2}\right)ch\rho_0\alpha\right)\exp\left(b\cos\rho_1\left(x_2-\frac{h}{2}\right)\right)$$

(10) dan

$$|U(x)-U_{\sigma\delta}(x)|\le C\sigma^2\delta\int_S\exp\left(a\cos\rho_1\left(y_2-\frac{h}{2}\right)\cdot\exp\rho_1|y_1|\right)\cdot\exp\left(b\cos\rho_1\left(x_2-\frac{h}{2}\right)\right)\cdot$$

$$\cdot\frac{1}{r^2}\exp\sigma(y_2-x_2)\exp\left(-\frac{b}{2}\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\rho_1\alpha-b_0ch\rho_0\alpha\right)ds_y+$$

$$+C\sigma^3\int_{y_2=0}\exp\left[a\cos\rho_1\frac{h}{2}\exp\rho_1|y_1|\right]\exp\left(-\sigma x_2+b\cos\rho_1\left(x_2-\frac{h}{2}\right)\right)\cdot$$

$$\cdot\exp\left(\frac{-b}{2}\cos\rho_1\frac{h}{2}\exp\rho_1\alpha\right)\cdot\exp\left(-b_0ch\rho_0\alpha\right)\frac{1}{r}ds_y\le C\sigma^3\cdot\delta\exp\sigma h\exp\left(-\sigma x_2\right)\cdot$$

$$\cdot\int_S\exp\left[\frac{b}{2}\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\rho\left(|y_1|-|x_1|\right)-\frac{b}{2}\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\rho_1\alpha\right]\cdot$$

$$\cdot\exp\left(-b\cos\rho_0\left(y_2-\frac{h}{2}\right)ch\rho_0\alpha\right)\exp\left[b\cos\rho_1\left(x_2-\frac{h}{2}\right)\right]\frac{1}{r}ds_y+$$

$$+C\sigma^3\exp\left(-\sigma x_2\right)\int_{y_2=0}\exp\left[\frac{b}{2}\cos\frac{\rho_1h}{2}\exp\rho_1\left(|y_1|-|x_1|\right)-\frac{b}{2}\cos\frac{\rho_1h}{2}\exp\rho_1|y_1-x_1|\right]$$

$$\cdot \exp\left[b\cos\rho_1\left(x_2-\frac{h}{2}\right)-b_0 ch\rho_0\alpha\right]\frac{1}{r}ds_y \le \sigma^3 (\exp\sigma h+1)\exp(-\sigma x_2)\exp\left(b\cos\rho_1\left(x_2-\frac{h}{2}\right)\right)\cdot$$

$$\cdot\int_{\partial D}\exp\left[\frac{b}{2}\cos\frac{\rho_1 h}{2}\left(\exp\rho_1\left(|y_1|-|x_1|\right)+\exp\rho_1|y_1-x_1|\right)\right]\cdot\exp\left(-b_0 ch\rho_0\alpha\right)\frac{-1}{r}ds_y .$$

Endi $\sigma = h^{-1}\ln\frac{1}{\delta}$ va $b = 2a\exp\rho_1|x_1|$, deb olsak teoremani isbotlagan bo’lamiz.

Xulosa

Bu ish elastiklik nazariyasi tenglamalari sistemasi yechimini tekislikdagi chegaralanmagan sohada chegaraning qismida uning berilgan qiymatlari va uning kuchlanishi qiymatlari bo’yicha davom ettirish masalasi, ya’ni elastiklik nazariyasi tenglamalari sistemasi uchun Koshi masalasi o’rganiladi.

Bu ishda asosan chegaralanmagan sohada mos ravishda chegaralanmagan yechim holida Karleman matritsasini qurish yo’llari o’rganilgan. Karleman matritsasini qurishda bundan oldin qaralgan holdan farqli ravishda maxsuslik tartibi katta bo’lgan yadro holida Karleman matritsasini mustaqil ravishda qurildi va shu holda regulyarlashgan yechim bilan aniq yechim orasidagi farq baholangan.

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