O'ZBEKISTON RESPUBLIKASI

OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

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AYIRMALI SXEMALAR NAZARIYASI ELEMENTLARI

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Mazkur uslubiy qo'llanmada ayirmali sxemalar nazariyasining asosiy elementlari − matematik fizika tenglamalarini diskret (ayirmali) approksimatsiyasini qurish, approksimatsiya xatoligini tekshirish, ayirmali sxema yechimi turg'unligi va berilgan differensial masala yechimining aniq yechimiga yaqinlashishi masalalari yoritilgan.

Ushbu uslubiy qo'llanma matematika, tadbiqiy matematika va informatika, mexanika hamda informatika va axborot texnologiyalari yo'nalishlari bo'yicha ta'lim olayotgan bakalavriat, shuningdek, ko'rsatilgan bakalavriat yo'nalishlariga mos keluvchi magistratura mutaxassisliklari talabalariga mo'ljallangan. Qo'llanmadan «Hisoblash matematikasi», «Hisoblash usullari», «Sonli usullar» fanlaridan dars beruvchi o'qituvchilar ham foydalanishlari mumkin.

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MUNDARIJA

K I R I SH

Matematik fizika masalalarini sonli yechishda ko'p hollarda chekli ayirmali usullar yoki to'rlar usuli qo'llaniladi. Sonli usullar nazariyasida 2 ta bosh masala mavjud:

1) Matematik fizika tenglamalarini diskret (ayirmali) approksimatsiyasini qurish, approksimatsiya xatoligini tekshirish, ayirmali sxema yechimini turg'unligini va berilgan differensial masala yechimining aniq yechimiga yaqinlashishini hamda hosil bo'lgan ayirmali sxema aniqligini tadqiq qilish;

2) Hisoblash algoritmini tejamkorligini e'tiborga olgan holda hosil bo'lgan ayirmali tenglamalar sistemasini aniq (to'g'ri) yoki iterasion usullar bilan yechish.

Ma'lumki, malakali milliy kadrlar tayyorlashda Davlat tilida yaratilgan uslubiy qo'llanmalari va darsliklar juda katta ahamiyatga ega. Hozirgi kunda hisoblash matematikasi fani bo'yicha Davlat tilida tayyorlangan o'quv qo'llanmalar va darsliklarda sonli usullar nazariyasining birinchi bosh masalasi yetarlicha yoritilmagan.

Mazkur uslubiy qo'llanmada yuqorida ta'kidlangan bo'shliqni imkon qadar to'ldirish maqsad qilib qo'yilgan. Shu sababli uslubiy qo'llanmada differensial tenglamalarni diskret modellarini qurishga, ayirmali sxemalarni qurish usullariga, approksimatsiya xatoligini baholashga, ayirmali sxema yechimini turg'unligini va berilgan differensial masalaning aniq yechimiga yaqinlashishini hamda hosil bo'lgan ayirmali sxema aniqligini tadqiq qilishga katta e'tibor qaratilgan.

Ushbu uslubiy qo'llanmadan oliy o'quv yurti talabalari, magistrantlari, aspirantlari foydalanishlari mumkin. Shuningdek, mazkur uslubiy qo'llanma oliy o'quv yurti o'qituvchilari va amaliy matematika sohasidagi mutaxassislar uchun ham foydali o'quv adabiyoti sifatida xizmat qilishi mumkin.

I-BOB. AYIRMALI SXEMALAR NAZARIYASINING BOSHLANG'ICH TUSHUNCHALARI

§ 1.1. Ayirmali tenglamalar.

Differensial tenglamalarni taqribiy usullar bilan yechishda quyidagi chiziqli algebraik tenglamalar sistemasini yechishga keltiriladi:

$$Au = f ,$$

bu yerda $A = (a_{ij})$ - tartibi N bo'lgan kvadrat matrisa, $u = (u_1, u_2, ..., u_N)^T$ - noma'lum vektor, $f = (f_1, f_2, ..., f_N)^T$ - berilgan vektor.

Chiziqli algebraik tenglamalar sistemasini yechishning ikki xil usuli mavjud:

1) to'g'ri yoki "aniq" usullar;

2) iterasion usullar yoki ketma-ket yaqinlashishlar usuli.

Matematik fizika tenglamalarini approksimatsiya qilish natijasida ayirmali tenglamalar hosil bo'ladi. Bunda to'rda, ya'ni diskret nuqtalar to'plamida berilgan ikki yoki uch o'zgaruvchili funksiyani aniqlashga to'g'ri keladi. Hisoblash to'ri o'n minglab, hattoki yuz minglab tugun nuqtalardan tashkil topgan bo'lishi mumkin. To'r funksiya qiymatlarini aniqlash uchun hosil bo'lgan chiziqli algebraik tenglamalar (ayirmali tenglamalar) sistemasi 2 holat orqali alohida xarakterlanadi:

1) chiziqli algebraik tenglamalar sistemasining A matrisasi maxsus ko'rinishga (nol qiymat qabul qiluvchi ko'pgina elementlarga) ega;

2) tenglamalar sistemasini tashkil etuvchi tenglamalar soni juda katta (o'rtacha $10^4 - 10^5$ ga teng).

Ayirmali tenglamalarga misollar. Arifmetik progressiya formulasi hadlari uchun o'rinli bo'lgan $a_{k+1} = a_k + d$ yoki $a_{k-1} - 2a_k + a_{k+1} = 0$ tenglamalar ayirmali tenglamalardir. Bu yerda $a_k = a(k)$, $k = 1,2,3,...$ ya'ni, argument k butun qiymatlarni qabul qiladi.

Endi, argumenti butun qiymatlarni qabul qiluvchi funksiyani qaraymiz

$$y(i), \quad i = 0, \pm 1, \pm 2, ...$$

i nuqtada quyidagi ayirmalarni yozamiz:

o'ng ayirma $\Delta y_i = y(i+1) - y(i)$,

chap ayirma $\nabla y_i = y(i) - y(i-1)$.

Odatda $y_i = y(i)$ belgilash qabul qilingan. U holda

$$\Delta y_i = y_{i+1} - y_i, \quad \nabla y_i = y_i - y_{i-1}.$$

Bu ifodalarni birinchi tartibli hosilani formal analogi sifatida qarash mumkin. Ikkinchi tartibli ayirmani qaraymiz

$$\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_{i+1} - y_i) = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) = y_{i+2} - 2y_{i+1} + y_i .$$

$\Delta y_{i-1} = \nabla y_i$ ekanligini qayd etish lozim. Haqiqatdan ham, tenglikning har ikki tomoni ham $y_i - y_{i-1}$ ga teng. Chap ayirmali operatorni qo'llash o'ng ayirmali operatorni bir birlik chap nuqtaga qo'llash bilan teng kuchli, ya'ni

$$\Delta\nabla y_i = \Delta^2 y_{i-1} = y_{i+1} - 2y_i + y_{i-1}.$$

Xuddi shuningdek, $\Delta^m y_i$ aniqlanadi

$$\Delta^m y_i = \Delta\left(\Delta^{m-1} y_i\right).$$

Δ operatorni har bir marta qo'llaganda o'ng tomondan yana bitta nuqta ayirmali ifodada ishtirok etadi. Δ operatorni m marta qo'llab, funksiyaning $i, i+1, ..., i+m$ nuqtalardagi $y_i, y_{i+1}, ..., y_{i+m}$ qiymatlaridan tashkil topgan $\Delta^m y_i$ ni hosil qilish mumkin.

Turli tartibdagi ayirmalar ishtirok etuvchi ayirmali tenglamani quyidagicha yozish mumkin:

$$\alpha_0 \Delta^m y_i + \alpha_1 \Delta^{m-1} y_i + ... + \alpha_{m-1} \Delta y_i + \alpha_m y_i = f_i,$$

Bu yerda $\alpha_0, \alpha_1, ..., \alpha_m$ - koeffitsiyentlar bo'lib, $\alpha_0 \neq 0$. Bu tenglama butun argumentning funksiyasi bo'lgan noma'lum funksiya -y_i ga nisbatan *m – chi tartibli ayirmali tenglama deyiladi*. Bu ayirmali tenglama m – chi tartibli quyidagi

$$\alpha_0 \frac{d^m u}{dx^m} + \alpha_1 \frac{d^{m-1} u}{dx^{m-1}} + ... + \alpha_{m-1} \frac{du}{dx} + \alpha_m u = f, \qquad \alpha_0 \neq 0.$$

differensial tenglamaning analogidir. Differensial tenglamaning koeffitsiyentlari x argumentning funksiyasi bo'lgani kabi ayirmali tenglamaning koeffitsiyentlari $\alpha_m = \alpha_m(i)$ lar i ga bog'liq bo'ladi.

Ayirmali tenglamalar qayerdan paydo bo'ladi degan haqli savol paydo bo'ladi. Differensial tenglamalar bilan ifodalanuvchi texnik va matematik masalalar mavjud. Bunday masalalarni ayirmali usullar bilan yechish ayirmali tenglamalarga olib keladi.

Oddiy differensial tenglamaga soddagina misol keltiramiz.

$$\frac{du}{dx} = f(x)$$

differensial tenglamani yechish talab etilsin. Bu tenglamadagi hosilani taqriban ayirmali ifoda bilan almashtirish mumkin:

$$\left(\frac{du}{dx}\right)_{x=x_i} \approx \frac{u(x_i + h) - u(x_i)}{h},$$

bu yerda $h > 0$ - x_i va $x_i + h$ nuqtalar orasidagi masofa. Agar $x_i + h = x_{i+1}$, $u(x_i) = u_i$, $u(x_{i+1}) = u_{i+1}$ belgilashlar kiritsak, u holda

$$\left(\frac{du}{dx}\right)_{x=x_i} \approx \frac{\Delta u_i}{h} = \frac{u_{i+1} - u_i}{h}.$$

$h \to 0$ da bu ayirmali ifoda du/dx ga intiladi. Shuni ta'kidlash lozimki, bunday almashtirish yagona, ya'ni bir qiymatli emas – chap ayirmani ham ishlatish mumkin edi:

$$\left(\frac{du}{dx}\right)_{x=x_i} \approx \frac{\nabla u_i}{h} = \frac{u_i - u_{i-1}}{h}.$$

O'ng va chap ayirmalarni yig'indisining yarmi markaziy ayirmani ifodalaydi:

$$\left(\frac{du}{dx}\right)_{x=x_i} \approx \frac{\Delta u_i + \nabla u_i}{2h} = \frac{u_{i+1} - u_{i-1}}{2h}.$$

Hamma yerda $\approx$ belgisi moslik yoki approksimatsiyani bildiradi. $\frac{\Delta u_i}{h} = \frac{u_{i+1} - u_i}{h}$ ifoda $\frac{du}{dx}$ hosilani approksimatsiya qiladi deyiladi.

Shunday qilib,

$$\frac{\Delta y_i}{h} = f_i, \ f_i = f(x_i)$$

tenglamani qaraymiz. Ta'rifga ko'ra bu birinchi tartibli ayirmali tenglamadir. Uni quyidagicha yozish mumkin

$$\Delta y_i = hf_i \quad \text{yoki} \quad y_{i+1} = y_i + hf_i.$$

Birinchi tartibli differensial tenglamani almashtirishda ikkinchi tartibli ayirmali tenglamaga ham ega bo'lish mumkin. Masalan,

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + 0{,}5h^2u''(x_i) + \frac{h^3}{6}u'''(x_i) + O(h^4),$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + 0{,}5h^2u''(x_i) - \frac{h^3}{6}u'''(x_i) + O(h^4).$$

Bu ikki ifodani qo'shib,

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u_i'' + O(h^2)$$

ifodaga ega bo'lish mumkin. Bu yerda $O(h^2)$ ni tashlab yuborib, u_i'' uchun taqribiy

$$\left(\frac{d^2u}{dx^2}\right)_{x=x_i} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \frac{\Delta^2 u_{i-1}}{h^2} = \frac{\Delta\nabla u_i}{h^2}$$

ifoda hosil qilish mumkin.

u_{i+1} ni x_i nuqta atrofida Teylor qatoriga yoyilmasi $u_{i+1} = u_i + hu_i' + 0{,}5h^2u_i'' + O(h^3)$ da u_i'' ni ikkinchi tartibli ayirmali ifoda bilan almashtirib,

$$u_i' = \frac{u_{i+1} - u_i}{h} - \frac{h}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

ya'ni ikkinchi tartibli ayirmali tenglamani hosil qilish mumkin. Buni ikkinchi tartibli ayirmali tenglama ekanligini quyidagicha isbotlash mumkin. Oxirgi formulada u_i' ni f_i ga almashtirib, $O(h^2)$ hadni tashlab yuborib, hosil bo'lgan tenglamani $2h$ ga ko'paytiramiz. U holda birinchi tartibli differensial tenglama $du/dx = f$ o'rniga quyidagi ikkinchi tartibli ayirmali tenglamaga ega bo'lamiz

$$\Delta\nabla y_i - 2\Delta y_i = 2hf_i.$$

§ 1.2. Birinchi tartibli ayirmali tenglamalar va tengsizliklar.

Birinchi tartibli ayirmali tenglamani qaraymiz:

$$b\Delta y_i + ay_i = f_i. \qquad (1.1)$$

Bu tenglama birinchi tartibli

$$b\frac{du}{dt} + au = f$$

differensial tenglamaga mos keladi. (1.1) tenglamani quyidagicha yozish mumkin:

$$b(y_{i+1} - y_i) + ay_i = f_i \text{ yoki } by_{i+1} = cy_i + f_i, \quad c = b - a.$$

Umumiy holda $b = b_i$, $a = a_i$, $c = c_i$, ya'ni bu koeffitsiyentlar i argumentning ma'lum funksiyalari. $b_i \neq 0$ bo'lsin, u holda

$$y_{i+1} = q_i y_i + \varphi_i ,$$

bu yerda $q_i = c_i / b_i$, $\varphi_i = f_i / b_i$, $b_i \neq 0$. Ko'rinib turibdiki, qandaydir i da y funksiyaning qiymati berilgan bo'lsa, yechim bir qiymatli aniqlanadi. Faraz qilaylik, $i = 0$ da y_0 berilgan bo'lsin. U holda $y_1, y_2, ...$ va h.k. aniqlash mumkin. $q_i = q = const$ bo'lsin. Agar $\varphi_i = 0$ bo'lsa, u holda y_i qiymatlar geometrik progressiyani tashkil qiladi. Agar $\varphi_i \neq 0$ bo'lsa, u holda

$$y_{i+1} = qy_i + \varphi_i = q(qy_{i-1} + \varphi_{i-1}) + \varphi_i = q^2 y_{i-1} + \varphi_i q y_{i-1} .$$

Bu jarayonni davom ettirib, quyidagi formulani hosil qilish mumkin:

$$y_{i+1} = q^{i+1} y_0 + \varphi_i + q\varphi_{i-1} + ... + q^{i-1}\varphi_1 + q^i \varphi_0 = q^{i+1} y_0 + \sum_{k=0}^{i} q^{i-k} \varphi_k . \quad (1.2)$$

Quyidagi

$$y_{i+1} = q_i y_i + \varphi_i , \; i = 0,1,2,...$$

tenglamaning yechimini yuqoridagi mulohazalardan foydalanib, topish mumkin.

Ayrim hollarda birinchi tartibli

$$y_{i+1} \leq qy_i + f_i , \; i = 0,1,2,... \; (y_0 \text{ berilgan, } q, f_i \text{ lar ma'lum}) \quad (1.3)$$

tengsizliklar bilan ishlashga to'g'ri keladi. Bu tengsizlikni yechish uchun quyidagicha mulohaza yuritamiz. Biz

$$\vartheta_{i+1} = q\vartheta_i + f_i , \; \vartheta_0 = y_0 \quad (1.4)$$

tenglamani yechishimiz mumkin. $y_i \le \vartheta_i$ ekanligini ko'rsatamiz. (1.3) tengsizlikdan (1.4) tenglikni ayiramiz:

$$y_{i+1} - \vartheta_{i+1} \le q\left(y_i - \vartheta_i\right) \le q^2\left(y_{i-1} - \vartheta_{i-1}\right) \le \ldots \le q^{i+1}\left(y_0 - \vartheta_0\right) = 0. \qquad (1.5)$$

Bundan esa ixtiyoriy q uchun $y_{i+1} \le \vartheta_{i+1}$ ekanligi kelib chiqadi. ϑ_i esa (1.2) formuladagi kabi q, ϑ_0, f_i lar orqali oshkor holda ifodalanishi mumkin.

Misol. Quyidagi bir jinsli o'zgarmas koeffitsiyentli ikkinchi tartibli ayirmali tenglamani qaraymiz

$$ay_{i-1} - cy_i + by_{i+1} = 0, \qquad (1.6)$$

bu yerda a, b, c koeffitsiyentlar i ga bog'liq bo'lmagan haqiqiy sonlardir.

(1.6) tenglamaning xususiy yechimini

$$y_i = q^i \qquad (1.7)$$

ko'rinishda (q aniqlanishi lozim bo'lgan son) izlaymiz. (1.7) tenglikni (1.6) tenglamaga qo'yib, (1.6) *ayirmali tenglamaning xarakteristik tenglamasi deb ataluvchi* quyidagi kvadrat tenglamaga ega bo'lamiz:

$$bq^2 - cq + a = 0. \qquad (1.8)$$

$c^2 - 4ab$ diskriminantning ishorasiga bog'liq holda (1.8) tenglama ildizlari uchun uchta har xil hol bo'lishi mumkin. Agar $c^2 > 4ab$ bo'lsa, tenglama ildizlari haqiqiy va har xil:

$$q_1 = \frac{c + \sqrt{c^2 - 4ab}}{2b}, \quad q_2 = \frac{c - \sqrt{c^2 - 4ab}}{2b}. \qquad (1.9)$$

Bu holda (1.6) ayirmali tenglama

$$y_i^{(1)} = q_1^i, \qquad y_i^{(2)} = q_2^i \tag{1.10}$$

xususiy yechimlarga ega bo’ladi. Agar $c^2 < 4ab$ bo’lsa, q_1, q_2 ildizlar kompleks qo’shma sonlar bo’ladi. (1.10) funksiyalar ham bu holda (1.6) ayirmali tenglamaning yechimlari bo’ladi. Ammo, bu holda (1.6) ayirmali tenglamaning yechimlarini

$$q_1 = r(cos\varphi + i\,sin\varphi)$$

trigonometrik shakldа bergan ma’qul. Bu yerda

$$r = \sqrt{\frac{a}{b}}, \quad sin\varphi = \frac{\sqrt{4ab - c^2}}{2\sqrt{ab}}, \quad cos\varphi = \frac{c}{2\sqrt{ab}}. \tag{1.11}$$

(1.6) ayirmali tenglamaning yechimlari sifatida quyidagi funksiyalarni olish mumkin:

$$y_i^{(1)} = r^i\,cos(i\varphi), \qquad y_i^{(2)} = r^i\,sin(i\varphi).$$

Nihoyat, $c^2 = 4ab$ bo’lsa, u holda (1.8) tenglama $q = c/(2b)$ karrali ildizga ega bo’ladi, (1.6) ayirmali tenglama esa

$$y_i^{(1)} = q^i, \qquad y_i^{(2)} = iq^i \tag{1.12}$$

xususiy yechimlarga ega bo’ladi.

Endi (1.10) xususiy yechimlardan foydalanib, quyidagi

$$ay_{i-1} - cy_i + by_{i+1} = 0, \; i = 1,2,... \tag{1.13}$$

$$y_0 = \mu_1, \qquad y_1 = \mu_2 \tag{1.14}$$

Koshi masalasini yechimlarini quramiz. (1.6) tenglama chiziqli va bir jinsli bo’lganligi bois ularning ixtiyoriy chiziqli kombinasiyalari

$$y_i = \alpha_1 q_1^i + \alpha_2 q_2^i \tag{1.15}$$

ham yechim bo'ladi. α_1 va α_2 koeffitsiyentlarni shunday tanlaymizki, natijada (1.14) boshlang'ich shartlar qanoatlantirilsin:

$$\alpha_1 + \alpha_2 = \mu_1, \quad \alpha_1 q_1 + \alpha_2 q_2 = \mu_2 . \qquad (1.16)$$

(1.16) sistemani yechib, quyidagilarga ega bo'lish mumkin:

$$\alpha_1 = \frac{\mu_1 q_2 - \mu_2}{q_2 - q_1}, \qquad \alpha_2 = \frac{\mu_2 - \mu_1 q_1}{q_2 - q_1}. \qquad (1.17)$$

(1.17) tenglikni (1.15) ifodaga qo'yib va μ_1, μ_2 sonlarning oldidagi koeffitsiyentlarni yig'ib, $c^2 > 4ab$ bo'lsa, (1.13)-(1.14) Koshi masalasini yechimi

$$y_i = \frac{q_1 q_2 \left(q_1^{i-1} - q_2^{i-1}\right)}{q_2 - q_1}\mu_1 + \frac{q_2^i - q_1^i}{q_2 - q_1}\mu_2, \qquad i = 0,1,2,... \qquad (1.18)$$

ko'rinishda bo'ladi. Bu yerda q_1, q_2 lar (1.10) tengliklarga asosan aniqlanadi.

$c^2 < 4ab$ bo'lganda ham (1.13)-(1.14) Koshi masalasining yechimi yuqoridagi tarzda aniqlanadi. Bu holda r va φ o'zgaruvchilar (1.11) tengliklarga asosan aniqlanadi va quyidagi munosabatlar o'rinli bo'ladi:

$$\frac{q_1 q_2 \left(q_1^{i-1} - q_2^{i-1}\right)}{q_2 - q_1} = -r^i \frac{\sin\left((i-1)\varphi\right)}{\sin\varphi},$$

$$\frac{q_2^i - q_1^i}{q_2 - q_1} = r^{i-1} \frac{\sin\left(i\varphi\right)}{\sin\varphi}.$$

Shu sababli Koshi masalasini yechimini quyidagicha yozish mumkin:

$$y_i = -r^i \frac{\sin\left((i-1)\varphi\right)}{\sin\varphi}\mu_1 + r^{i-1}\frac{\sin\left(i\varphi\right)}{\sin\varphi}\mu_2. \qquad (1.19)$$

$c^2 = 4ab$ bo'lgan holda (1.12) xususiy yechimlardan foydalanib, (1.13)-(1.14) Koshi masalasini yechimlarini

$$y_i = -(i-1)q^i \mu_1 + i q^{i-1}\mu_2 \qquad q = c/(2b) \qquad (1.20)$$

ko'rinishda yozish mumkin. Xuddi shunga o'xshash

$$a y_{i-1} - c y_i + b y_{i+1} = 0, \qquad i = 1,2,...,N-1, \qquad (1.21)$$

$$y_0 = \mu_1, \quad y_N = \mu_2 \qquad (1.22)$$

chegaraviy masalaning yechimi ham quriladi. Agar $c^2 \neq 4ab$ bo'lsa, u holda

$$y_i = \frac{(q_1 q_2)^i (q_2^{N-i} - q_1^{N-i})}{q_2^N - q_1^N}\mu_1 + \frac{q_2^i - q_1^i}{q_2^N - q_1^N}\mu_2 \quad , \tag{1.23}$$

bu yerda q_1, q_2 lar (1.9) tengliklarga asosan aniqlanadi.

$c^2 = 4ab$ bo'lsa, u holda chegaraviy masalaning yechimlari

$$y_i = \left(1 - \frac{i}{N}\right) q^i \mu_1 + \frac{i}{N} q^{-(N-i)} \mu_2 \quad ,$$

formula bilan aniqlanadi, bu yerda $q = c/(2b)$.

Misol. $u_{k+1} - 2pu_k + u_{k-1} = 0$ tenglamaning umumiy yechimini toping.

Yechish. Masalani yechish uchun quyidagi hollarni qaraymiz:

a) $p < 1$ bo'lsin. U holda $p = \cos\alpha, \alpha \neq 0$ deb olish mumkin. Natijada tenglama $u_{k+1} - 2\cos\alpha\, u_k + u_{k-1} = 0$ ko'rinshga keladi. $u_k = q^k$ deb faraz qilib, $q^2 - 2\cos\alpha\, q + 1 = 0$ kvadrat tenglamani hosil qilamiz. Uning diskriminanti $D = \cos^2\alpha - 1 = -\sin^2\alpha < 0$ bo'lib, ildizlari $q_{1,2} = e^{\pm i\alpha}, q^k{}_{1,2} = e^{\pm ik\alpha}$, xususiy yechimlari esa $u_k^{(1)} = \cos(k\alpha), u_k^{(2)} = \sin(k\alpha)$ bo'ladi.

b) $p > 1$ bo'lsin. Bunda $p = ch\,\alpha$ deb olamiz. $u_k = q^k$ deb faraz qilib, $q^2 - 2ch\,\alpha\, q + 1 = 0, D = ch^2\alpha - 1 > 0$ kvadrat tenglamani hosil qilamiz. Uning ildizlari $q_{1,2} = ch\alpha \pm sh\alpha = e^{\pm\alpha}, q^k{}_{1,2} = e^{\pm k\alpha}$ bo'ladi. Ayirmali tenglamaning xususiy yechimlari esa $u_k^{(1)} = ch(k\alpha), u_k^{(2)} = sh(k\alpha)$ funksiyalar bo'ladi.

c) $p = 1$. Bu holda $q^2 - 2q + 1 = 0,\ q_{1,2} = 1$ bo'lib, xususiy yechimlari $u_k^{(1)} = 1, u_k^{(2)} = k$ bo'lib, umumiy yechimi $u_k = C_1 + C_2 k$ chiziqli funksiya bo'ladi.

§ 1.3. Ikkinchi tartibli ayirmali tenglamalar. Koshi masalasi. Chegaraviy masalalar.

Endi ikkinchi tartibli ayirmali tenglamalarni ko'rib chiqamiz. Ikkinchi tartibli ayirmali tenglamalarni quyidagi qulayroq ko'rinishda yozish qabul qilingan:

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i \qquad (1.24)$$

$$i = 1,2,3... \quad A_i \neq 0, \quad B_i \neq 0.$$

Bu tenglama ikkinchi tartibli tenglama ekanligini ko'rsatamiz. $\Delta y_i = y_{i+1} - y_i$ belgilashdan foydalanamiz. U holda (1.24) tenglama

$$B_i \Delta y_i - A_i \Delta y_{i-1} - (C_i - B_i - A_i) y_i = -F_i \qquad (1.25)$$

ko'rinishga keladi.

$$\Delta y_i - \nabla y_i = \Delta y_i - \Delta y_{i-1} = \Delta^2 y_{i-1} = y_{i+1} - 2y_i + y_{i-1},$$

$$\Delta y_{i-1} = -\Delta^2 y_{i-1} + \Delta y_i$$

ekanligi tushunarli. Bu tenglikka asosan (1.25) formulani

$$A_i \Delta^2 y_{i-1} + (B_i - A_i)\Delta y_i - (C_i - B_i - A_i) y_i = -F_i, \qquad A_i \neq 0.$$

(1.24) tenlamani Δ^2 ni oldida B_i koeffisiyent turadigan ko'rinishda yozish mumkin, $B_i \neq 0$ bo'lganligi sababli hosil bo'lgan tenglama ikkinchi tartibli bo'ladi:

$$B_i \Delta^2 y_{i-1} + (B_i - A_i)\Delta y_{i-1} - (C_i - B_i - A_i) y_i = -F_i.$$

Shunday qilib, (1.24) ayirmali tenglama ikkinchi tartibli differensial tenglamaning analogidir. Uni yechish uchun ikkita qo'shimcha shart berilishi kerak. Bu shartlar sifatida y funksiya va uning birinchi tartibli ayirmasi Δy qiymatlari xizmat qilishi mumkin. Agar ikki shart (y funksiya va uning birinchi tartibli ayirmasi Δy qiymatlari) ham bir nuqtada yoki qo'shni nuqtalarda berilgan bo'lsa, u holda *Koshi masalasi*ga ega bo'lamiz. Agar qo'shimcha shartlar qo'shni bo'lmagan turli xil nuqtalarda berilsa u holda masala *chegaraviy masala deyiladi*.

Koshi masalasi yechilayotgan bo'lsin, ya'ni $i=0$ da y_0, $\Delta y_0 = y_1 - y_0$ yoki y_0 va y_1 berilgan bo'lsin. U holda y_0 va y_1 larni bilgan holda $i=1,2,3,...$ lar uchun

$$y_{i+1} = \frac{C_i y_i - A_i y_{i-1} - F_i}{B_i}, \; B_i \neq 0$$

qiymatlarni aniqlash mumkin. Shunday qilib, y_0 va y_1 qiymatlar berilganda masalaning yechimi mavjud va yagonadir. Ammo, ikkinchi tartibli tenglama uchun matematik fizikada chegaraviy masalalar, ya'ni qo'shimcha shartlar qo'shni bo'lmagan $i=0$ va $i=N$ da

$$y_0 = \mu_1, \qquad y_N = \mu_2$$

qiymatlar berilganda $0 < i < N$ nuqtalar uchun y_i qiymatlarni aniqlash ancha qiziqarlidir (μ_1, μ_2 - berilgan sonlar).

$i=0$ va $i=N$ nuqtalarda nafaqat funksiyaning qiymatlari va birinchi tartibli ayirmaning qiymati, balki funksiya qiymati va birinchi tartibli ayirmaning chiziqli kombinasiyasi berilishi ham mumkin. Umumiy holda bunday chegaraviy shartlarni

$$y_0 = \chi_1 y_1 + \mu_1, \;\; y_N = \chi_2 y_{N-1} + \mu_2 \tag{1.26}$$

ko'rinishda yozish mumkin. (1.26) tenglamaning birinchisiga

$$y_1 = y_0 + \Delta y_0$$

tenglikni qo'ysak, quyidagi munosabatni hosil qilamiz:

$$\chi_1 \Delta y_0 - (1 - \chi_1) y_0 = \mu_1 . \tag{1.27}$$

$\chi_1 = 0$ bo'lgan hol $i = 0$ nuqtada funksiyaning qiymati y_0 berilganligini anglatadi (bu esa *birinchi turdagi chegaraviy shart*). Agar $\chi_1 = 1$ bo'lsa, u holda Δy_0 ning qiymati berilgan bo'ladi (bu esa *ikkinchi turdagi chegaraviy shart*). Agar $\chi_1 \neq 0$, $\chi_1 \neq 1$ bo'lsa, $i = 0$ nuqtada funksiya va birinchi tartibli ayirmaning chiziqli kombinasiyasi berilgan bo'ladi (bu esa *uchinchi turdagi chegaraviy shart*).

Amaliyotda ayirmali chegaraviy masalalar katta ahamiyatga ega. Hisoblash matematikasining eng katta yutug'i matematik fizikaning ko'pchilik masalalarini hisoblashda har bir qadamda (1.6) ayirmali tenglamalar sistemasini (1.26) chegaraviy shart bilan yechishdan iborat. Bu masala klassik masala bo'lib, hisoblash usullari nazariyasining ko'pgina murakkab masalalari (1.24), (1.26) chegaraviy masalaga keltiriladi. Bunday tenglamalar sistemasining matrisasi uch diagonallidir. Bu matrisa quyidagi ko'rinishga ega.

$$
\begin{bmatrix}
1 & -\chi_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
A_1 & -C_1 & B_1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & A_i & -C_i & B_i & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & A_{N-1} & -C_{N-1} & B_{N-1} \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -\chi_2 & 1
\end{bmatrix}
$$

Agar ikkinchi yoki uchinchi turdagi chegaraviy shartlar berilgan bo’lsa, bu matrisaning tartibi $N+1$ ga teng bo’ladi. Birinchi turdagi chegaraviy shartlar berilgan bo’lsa, matrisa $(N-1)$ - tartibga ega bo’ladi. Bu matrisaning faqatgina uchta diagonalida, ya’ni bosh diagonalda hamda bosh diagonalning pastki va yuqorisidagi qo’shni diagonallarda elementlar noldan farqli bo’ladi. Bunday matrisaga ega bo’lgan chiziqli algebraik tenglamalar sistemasini yechishning samarali usuli – Gauss usuli asosida (1.24), (1.26) ayirmali chegaraviy masalani haydash usuli bilan samarali yechish mumkin.

Misol. $y'' + x\,y' - 0{,}5\,\dfrac{y}{x} = 1$ tenglamaning $\begin{cases} y(2) + 2y'(2) = 1, \\ y(2{,}3) = 2{,}15 \end{cases}$ shartlarni qanoatlantiruvchi yechimini haydash usuli yordamida toping.

Yechish. $[2;\,2{,}3]$ kesmani $h = 0{,}05$ qadam bilan bo’lib, to’r hosil qilamiz va to’rtta $x_0 = 2;\ x_1 = 2{,}05;\ x_2 = 2{,}1;\ x_3 = 2{,}15;\ x_4 = 2{,}2;\ x_5 = 2{,}25;\ x_6 = 2{,}3$ tugun nuqtalarni hosil qilamiz. $x_0 = 2$ va $x_6 = 2{,}3$ nuqtalar chegaraviy qolgan nuqtalar esa ichki nuqtalar deb ataladi. Berilgan tenglamani ichki nuqtalarda ayirmali tenglama bilan almashtiramiz:

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+x_i\,\frac{y_{i+1}-+y_{i-1}}{2h}-0{,}5\,\frac{y_i}{x_i}=1,\ i=\overline{1,5}.$$

Chegaraviy shartlardan esa

$$\begin{cases} y_0+2\dfrac{-y_0+4y_1-3y_0}{2h}=1, & (=0), \\ y_6=2{,}15, & (=6) \end{cases}$$

tenglamalarni hosil qilamiz va quyidagi belgilashlarni kiritamiz:

$$\alpha_0=1, \alpha_1=2, A=1, \beta_0=1, \beta_1=0, B=2{,}15,\ p_i=x_i,\quad q_i=-0{,}5/x_i, f_i=1, i=\overline{0,6}.$$

Haydash usulining to'g'ri yo'lida hisoblanadigan koeffitsiyentlar

$$m_i=\frac{2h^2q_i-4}{2+hp_i},\quad n_i=\frac{2-hp_i}{2+hp_i},\quad F_i=\frac{2f_i}{2+hp_i}\ (=\overline{1,5}),$$

$$c_0=\frac{\alpha_1}{\alpha_0 h-\alpha_1},\quad d_0=\frac{Ah}{\alpha_1}, c_i=\frac{1}{m_i-n_ic_{i-1}}, d_i=F_ih^2-n_ic_{i-1}d_{i-1}\quad (=\overline{1,5}).$$

To'g'ri yo'l bajarilib, yuqoridagi koeffitsiyentlar topilgandan so'ng, teskari yo'lda noma'lum funksiya qiymatlari quyidagi formula bilan hisoblanadi:

$$y_6=\frac{Bh+\beta_1c_5d_5}{\beta_0h+\beta_1(c_5+1)},\quad y_i=c_i(d_i-y_{i+1})\quad (=\overline{5,0}).$$

Bu yerda

$$m_i=-\frac{4+\dfrac{0{,}0025}{x_i}}{2+0{,}05x_i},\ n_i=\frac{2-0{,}05x_i}{2+0{,}05x_i},\quad F_i=\frac{2}{2+0{,}05x_i}\ (=\overline{1,5}),$$

$$c_0=\frac{2}{0.05-2}=-1.02564;\quad d_0=\frac{0{,}05}{2}=0{,}025.$$

Hisoblashlarni quyidagi jadvalda keltiramiz:

i	x_i	m_i	n_i	h^2F_i	c_i	d_i	y_i

0	2.00	-	-	-	-1.02564	0.025000	2.2490
1	2.05	-1.903077	0.902497	0.002378	-1.02308	0.095519	2.2178
2	2.10	-1.900803	0.900238	0.002375	-1.02063	0.025878	2.1933
3	2.15	-1.898535	0.897983	0.002372	-1.01830	0.026090	2.1748
4	2.20	-1.896273	0.895734	0.002370	-1.01611	0.026167	2.1618
5	2.25	-1.894017	0.893491	0.002367	-1.01406	0.026123	2.1537
6	2.30	-	-	-	-	-	2.15

Javob:

x_i	y_i	x_i	y_i
2.00	2.249	2.20	2.162
2.05	2.218	2.25	2.154
2.10	2.193	2.30	2.150
2.15	2.175		

§ 1.4. To'r va to'r funksiyalar.

Har xil sohalarda to'r qurish.

Berilgan differensial tenglamani taqribiy tavsiflovchi ayirmali sxema tuzish uchun quyidagi ikki bosqichni amalga oshirish kerak.

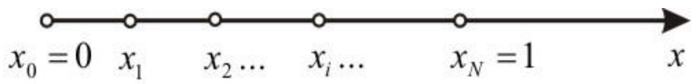
1. Argumentning uzluksiz o'zgarish sohasini diskret sohaga almashtirish kerak.

2. Differensial operatorni biror ayirmali operator bilan, shuningdek, chegaraviy va boshlang'ich shartlarni ularning ayirmali anologi bilan almashtirish kerak.

Ushbu prosedura amalga oshirilgandan keyin algebraik tenglamalar sistemasiga ega bo'lamiz. Shunday qilib, dastlabki berilgan (chiziqli) differensial tenglamani sonli yechish masalasi algebraik tenglamalar sistemasining yechimini topish masalasiga keltiriladi.

U yoki bu matematik fizika masalasini sonli yechishda argumentning aniqlanish sohasidagi barcha qiymatlar uchun sonli yechimni aniqlashning imkoni yo'q. Shu sababli argumentning aniqlanish sohasidan qandaydir chekli nuqtalar to'plamini ajratib olish va taqribiy yechimlarni mana shu nuqtalardagina izlash lozim bo'ladi. Bunday chekli nuqtalar to'plami *to'r deyiladi*. Ajratib olingan nuqtalar *to'rning tugun nuqtalari deyiladi*. Demak, to'rning tugun nuqtalari hisoblash to'rini tashkil etuvchi nuqtalardir.

To'rning tugun nuqtalarida aniqlangan funksiyalar *to'r funksiyalar deyiladi*. Shunday qilib, argumentning uzluksiz o'zgarish sohasini to'r bilan, ya'ni argumentning diskret o'zgarish sohasi bilan almashtirdik. Boshqacha qilib aytganda, biz differensial tenglama yechimi yotgan fazoni to'r funksiyalar fazosi bilan approksimatsiya qildik.



1-rasm.

1-misol. Kesmada tekis to'r qurish. Birlik kesma [0,1] ni N ta teng bo'lakka bo'lamiz (1-rasm).

Ikkita qo'shni tugun nuqtalar orasidagi masofa *to'r qadami deyiladi*. Bo'linish nuqtalari $x_i = ih$ *to'rning tugun nuqtalari deyiladi*. Kesmadagi tugun nuqtalar (to'rning faqatgina ichki tugun nuqtalari) to'plami $\omega_h = \{x_i = ih, i = 1, 2, ..., N-1\}$ ushbu kesmadagi to'rni tashkil qiladi. Bu to'plamga $x_0 = 0$, $x_N = 1$ chegaraviy nuqtalarni ham kiritsak, hosil bo'lgan to'r $\overline{\omega}_h = \{x_i = ih, i = 1, 2, ..., N-1\}$ kabi belgilanadi.

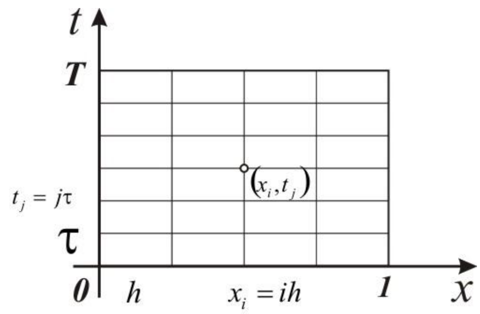
$[0,1]$ kesmada uzluksiz argumentning funksiyasi $y(x)$ o'rniga diskret argument funksiyasi $y_h(x_i)$ ni qaraymiz. Bu funksiyaning qiymati to'rning x_i tugun nuqtalarida hisoblanadi, funksiyaning o'zi esa to'r qadami h ga nisbatan parametr sifatida bog'liq bo'ladi.

2-misol. Tekislikda tekis to'r. Ikki argumentli bo'lgan $u(x,t)$ funksiyalar to'plamini qaraymiz. Aniqlanish sohasi sifatida $\overline{D}=\{0\le x\le 1,\ t\le x\le T\}$ to'g'ri to'rtburchakni tanlaymiz (2-rasm). x va t o'qlaridagi [0,1] kemalarni mos holda N_1 va N_2 qismlarga bo'lamiz. Bo'linish nuqtalaridan bu o'qlarga mos holda parallel to'g'ri chiziqlar o'tkazamiz. Bu o'qlarni kesishishi natijasida $\overline{\omega}_{h\tau}=\{(x_i,t_j)\in\overline{D}\}$ hisoblash to'rini tashkil qiluvchi (x_i,t_j) tugun nuqtalarga ega bo'lamiz.

Hosil bo'lgan to'r x va t o'qlari bo'yicha mos holda h va τ qadamlarga ega. Oralaridagi masofa h yoki τ ga teng bo'lgan bir to'g'ri (gorizontal yoki vertikal) chiziqda yotuvchi nuqtalar to'rning qo'shni tugun nuqtalari deyiladi.

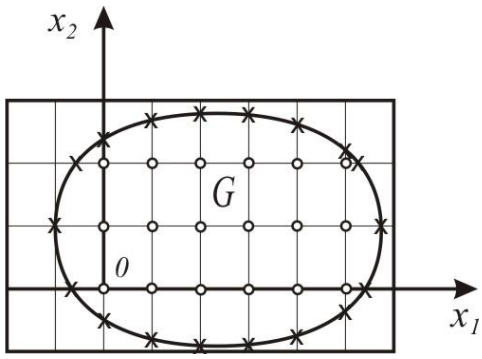
3-misol. Kesmada notekis to'r. $0\le x\le 1$ kesmani qaraymiz. $0<x_1<x_2<\ldots<x_{N-1}<1$ ixtiyoriy nuqtalarni kiritib, bu kesmani N qismga bo'lamiz. $\{x_i, i=0,\ldots, x_0=0, x_N=1\}$ nuqtalar to'plami $\overline{\omega}_h[0,1]$ notekis to'rni hosil qiladi. Qo'shni tugun nuqtalar orasidagi masofa to'rning qadami bo'lib, u tugun nuqtaning nomeri i ga bog'liq, ya'ni to'r qadami ham o'z navbatida to'r funksiyadir. To'rning qadamlari quyidagi shartni qanoatlantiradi:

$$\sum_{i=1}^{N} h_i = 1.$$



2-rasm.

4-misol. Ikki o'lchovli sohada to'r qurish.



3-rasm.

$x = (x_1, x_2)$ tekislikda chegarasi Γ bo'lgan murakkab shaklla ega bo'lgan G soha berilgan bo'lsin. $x_1^{(i_1)} = i_1 h_1, i_1 = 0, \pm1, \pm2, ...$, $h_1 > 0$ va $x_2^{(i_2)} = i_2 h_2, i_1 = 0, \pm1, \pm2, ...$, $h_1 > 0$ to'g'ri chiziqlar o'tkazamiz. U holda $x = (x_1, x_2)$ tekislikda tugun nuqtalari $(i_1 h_1, i_2 h_2)$, $i_1, i_2 = 0, \pm1, \pm2, ...$ bo'lgan to'r hosil bo'ladi. Bu to'r Ox_1 va Ox_2 o'qlarining har biri bo'yicha notekisdir. Bizni faqatgina Γ chegarali $\overline{G} = G + \Gamma$ sohaga tegishli bo'lgan nuqtalargina qiziqtiradi. G sohaning ichiga tushgan $(i_1 h_1, i_2 h_2)$ tugun nuqtalar *ichki tugun nuqtalar deyiladi*, ichki nuqtalar to'plami ω_h bilan belgilaymiz (3-rasm).

Γ chegara bilan $x_1^{(i_2)} = i_1 h_1$ va $x_2^{(i_2)} = i_2 h_2, i_1, i_2 = 0, \pm1, \pm2, ...$ to'g'ri chiziqlarning kesishidan hosil bo'lgan nuqtalar *chegaraviy* *nuqtalar deyiladi,* barcha chegaraviy nuqtalar to'plamini γ_h bilan belgilaymiz. 3-rasmda $\times$ belgi bilan chegaraviy nuqtalar, $\circ$ belgi bilan ichki tugun nuqtalar belgilangan. 3-rasmdan ko'rinib turibdiki, shunday chegaraviy nuqtalar ham mavjudki, ular o'zlariga qo'shni bo'lgan ichki nuqtalardan h_1 va h_2 dan kichik bo'lgan masofalarda turibdi. To'r tekislikda x_1 va x_2 o'qlarning har biri bo'yicha tekis bo'lsa ham, ammo $\overline{G}$ soha uchun $\overline{\omega} = \omega_h + \gamma_h$ to'r chegaralar atrofida notekisdir.

Shunday qilib, x argumentning o'zgarish sohasi $\overline{G}$ ni $\overline{G}$ sohaga tegishli x_i chekli nuqtalar to'plami $\overline{\omega}_h$ to'r bilan almashtirdik. Endi $x \in \overline{G}$ uzluksiz argumentning funksiyasi $u(x)$ o'rniga $\overline{\omega}_h = \{x_i\}$ to'rning x_i nuqtalari funksiyasi bo'lgan $y(x_i)$ to'r funksiyani qaraymiz. Uzluksiz $x \in \overline{G}$ argumentning $u(x)$ funksiyasi biror H_0 funksional fazoning elementi bo'lsa, $y(x_i)$ to'rli funksiya esa H_h diskret funksiyalar fazosining elementi bo'ladi. Shunday qilib, ayirmali sxemalar usulini qo'llab H_0 fazoni $y(x_i)$ to'rli funksiyalar fazosi H_h bilan almashtirdik. $\overline{\omega}_h$ to'rlar to'plamini qarab, h parametrga bog'liq bo'lgan $\{H_h\}$ to'rli funksiyalar fazosini hosil qilamiz. H_h chiziqli fazoda dastlabki H_0 funksional fazodagi $\|\cdot\|_0$ normaning analogi bo'lgan $\|\cdot\|_h$ norma kiritiladi. $0 \le x \le 1$ kesmada qurilgan $\omega_h = \{x_i = ih\}$ to'rli H_h fazoda normalarni quyidagicha tanlash mumkin:

1) S fazodagi normaning to'rli analogi

$$\|y\|_C = \max_{x \in \omega_h} |y(x)| \quad \text{yoki} \quad \|y\|_C = \max_{0 \le i \le N} |y_i|.$$

2) L_2 fazodagi normaning to'rli analogi

$$\|y\|_h = \left(\sum_{i=1}^{N-1} y_i^2 h \right)^{1/2} \quad \text{yoki} \quad \|y]|_h = \left(\sum_{i=1}^{N} y_i^2 h \right)^{1/2}.$$

Kelgusida biz u_h to'rli funksiyaga ega bo'lib, H_h fazoning elementi bo'lgan $y_h - u_h$ ayirmani tadqiq qilamiz. y_h ayirmali sxema yechimining dastlabki differensial tenglama yechimi u ga yaqinlashishi $\|y_h - u_h\|_h$ son bilan bog'liq bo'ladi, bu yerda $\|\cdot\|_h$ H_h fazodagi norma. Shuning uchun $\|\cdot\|_h$ norma $\|\cdot\|_0$ normani H_0 fazoning ixtiyoriy u elementi uchun $\lim_{h\to 0}\|u_h\|_h = \|u\|_0$ ma'noda approksimatsiya qilishi lozim. Ushbu shartni H_h va H_0 fazodagi normalarning moslashuv (kelishuv) sharti deb ataymiz.

§ 1.5. Oddiy differensial operatorlarning ayirmali approksimatsiyasi.

L operator $\vartheta = \vartheta(x)$ funksiyaga ta'sir etuvchi differensial operator bo'lsin. $L\vartheta$ da ishtirok etuvchi hosilalarni ayirmali hosilalar bilan almashtirib, $L\vartheta$ ifoda o'rniga ϑ_h to'r funksiya shablonini tashkil etuvchi tugun nuqtalardagi qiymatlarining chiziqli kombinasiyasidan iborat bo'lgan $L_h\vartheta_h$ ayirmali ifodaga ega bo'lamiz:

$$L_h\vartheta_h(x) = \sum_{\xi \in Ш(x)} A_h(x,\xi)\vartheta_h(\xi)$$

yoki

$$(L_h\vartheta_h)_i = \sum_{x_j \in Ш(x_i)} A_h(x_i, x_j)\vartheta_h(x_j)$$

bu yerda $A_h(x,\xi)$ - koeffitsiyentlar, h - to’r qadami, $Ш(x)$ - x nuqtadagi shablon. $L\vartheta$ ni $L_h\vartheta_h$ bilan bunday almashtirish *differensial operatorni ayirmali operator bilan approksimatsiya qilish (yoki* L *operatorni ayirmali approksimatsiyasi) deyiladi.*

L operatorni ayirmali approksimatsiyasi odatda avval lokal, ya’ni fazoning fiksirlangan ixtiyoriy x nuqtasi uchun o’tkaziladi. Agar $\vartheta(x)$ uzluksiz funksiya bo’lsa, $\vartheta_h(x)=\vartheta(x)$ bo’ladi. Differensial operator L ni ayirmali approksimatsiya qilishdan oldin shablonni tanlash lozim bo’ladi.

1-misol. $L\vartheta = d\vartheta/dx$.

Ox o’qida qandaydir x nuqtani fiksirlaymiz. $x-h$ va $x+h$ ($h>0$) nuqtalarni olamiz. $L\vartheta$ approksimatsiya qilish uchun quyidagi ifodalardan ixtiyoriy bittasidan foydalanish mumkin:

$$L_h^+\vartheta \equiv \frac{\vartheta(x+h)-\vartheta(x)}{h} \equiv \vartheta_x, \tag{1.28}$$

$$L_h^-\vartheta \equiv \frac{\vartheta(x)-\vartheta(x-h)}{h} \equiv \vartheta_{\bar{x}}. \tag{1.29}$$

(1.28) ifoda o’ng ayirmali hosila (uni ϑ_x bilan belgilaymiz), (1.29) ifoda esa chap ayirmali hosila (uni $\vartheta_{\bar{x}}$ bilan belgilaymiz) deb ataladi. $L_h^+\vartheta$ va $L_h^-\vartheta$ ayirmali ifodalar ikkita nuqtada aniqlangan (ya’ni ikki nuqtali x, $x+h$ va $x-h$, x shablonlardan foydalanilgan).

Bundan tashqari $d\vartheta/dx$ hosilani ayirmali approksimatsiyasi sifatida (1.28) va (1.29) ifodalarning chiziqli kombinasiyasidan ham foydalanish mumkin:

$$L_h^{(\sigma)}\vartheta \equiv \sigma\vartheta_x + (1-\sigma)\vartheta_{\bar{x}}, \tag{1.30}$$

bu yerda σ - ixtiyoriy haqiqiy son. Xususan, $\sigma = 0{,}5$ da markaziy (ikki tomonlama) ayirmali hosilaga ega bo'lish mumkin:

$$\vartheta_{\overset{\circ}{x}} = \frac{1}{2}\left(\vartheta_x - \vartheta_{\bar{x}}\right) = \frac{\vartheta(x+h) - \vartheta(x-h)}{2h}. \tag{1.31}$$

Shunday qilib, $L\vartheta = \vartheta'$ hosilani approksimatsiya qiluvchi ayirmali tenglamalar to'plamini yozish mumkin ekan. U yoki bu ayirmali approksimatsiyadan foydalanilganda qanday xatolikka yo'l qo'yish mumkin va $h \to 0$ da x nuqtada $\psi(x) = L_h\vartheta(x) - L\vartheta(x)$ ayirma o'zini qanday tutadi degan savol paydo bo'lishi tabiiy.

$\psi(x) = L_h\vartheta(x) - L\vartheta(x)$ miqdorga x nuqtada $L\vartheta$ ning *ayirmali approksimatsiya xatoligi deyiladi.* x nuqtaning $(x - h_0, x + h_0)$ atrofida $\vartheta(x)$ funksiya yetarlicha silliq va $h < h_0$ deb hisoblab (h_0 - fiksirlangan son), $\vartheta(x)$ ni Teylor qatoriga yoyamiz

$$\vartheta(x \pm h) = \vartheta(x) \pm h\vartheta'(x) + \frac{h^2}{2}\vartheta''(x) + O(h^3).$$

Bu yoyilmalarni (1.28), (1.29) va (1.31) ifodalarga qo'yib, quyidagilarga ega bo'lish mumkin:

$$\vartheta_x = \frac{\vartheta(x+h) - \vartheta(x)}{h} = \vartheta'(x) + \frac{h}{2}\vartheta''(x) + O(h^2),$$

$$\vartheta_{\bar{x}} = \frac{\vartheta(x) - \vartheta(x-h)}{h} = \vartheta'(x) + \frac{h}{2}\vartheta''(x) + O(h^2), \tag{1.32}$$

$$\vartheta_{\overset{\circ}{x}} = \frac{1}{2}\left(\vartheta_x - \vartheta_{\bar{x}}\right) = \frac{\vartheta(x+h) - \vartheta(x-h)}{2h} = \vartheta'(x) + O(h^2).$$

Bulardan ko'rinib turibdiki,

$$\psi = \vartheta_x - \vartheta'(x) = O(h),$$

$$\psi = \vartheta_{\bar{x}} - \vartheta'(x) = O(h),$$

$$\psi = \vartheta_{\overset{0}{x}} - \vartheta'(x) = O(h^2).$$

V - x nuqtaning $h < h_0$ bo'lganda L_h operatorni o'z ichiga oluvchi $Ш(x,h_0)$ atrofida berilgan va yetarlicha silliq $\vartheta \in V$ funksiyalar sinfi bo'lsin.

L_h operator L differensial operatorni x nuqtada m - $(m > 0)$ *tartib bilan approksimatsiya qiladi deyiladi*, agarda

$$\psi(x) = L_h\vartheta(x) - L\vartheta(x) = O(h^m)$$

tenglik o'rinli bo'lsa.

Shunday qilib, chap va o'ng ayirmali hosilalar $L\vartheta = \vartheta'$ hosilani birinchi tartib bilan, markaziy ayirmali hosila esa ikkinchi tartib bilan approksimatsiya qilar ekan.

2-misol. $L\vartheta = \vartheta'' = \dfrac{d^2\vartheta}{dx^2}$.

Ikkinchi tartibli hosilani ayirmali approksimatsiyalashda $(x-h, x, x+h)$ nuqtadan, ya'ni uch nuqtali shablondan foydalanish mumkin. U holda

$$L_h\vartheta = \frac{\vartheta(x+h) - 2\vartheta(x) + \vartheta(x-h)}{h^2}. \qquad (1.33)$$

x nuqtada o'ng ayirmali hosila $x+h$ nuqtadagi chap ayirmali hosilaga teng ekanligini, ya'ni $\vartheta_x(x) = \vartheta_{\bar{x}}(x+h)$ ni e'tiborga olsak, (1.33) ni quyidagicha yozish mumkin

$$L_h\vartheta = \frac{\vartheta_x(x)-\vartheta_{\bar{x}}(x)}{h} = \frac{1}{h}\left[\vartheta_{\bar{x}}(x+h)-\vartheta_{\bar{x}}(x)\right] = \vartheta_{\bar{x}x}(x). \qquad (1.34)$$

$\vartheta(x)$ funksiyani Teylor qatoriga yoyib, approksimatsiya xatoligining tartibi ikkiga tengligini, ya'ni

$$\vartheta_{\bar{x}x} - \vartheta''(x) = O(h^2)$$

ekanligini ko'rsatish mumkin.

3-misol. $L\vartheta = \vartheta^{(IV)}$.

$(x-2h,\ x-h,\ x,\ x+h,\ x+2h)$ nuqtalardan iborat shablonni tanlaymiz va $L_h\vartheta = \vartheta_{\bar{x}x\bar{x}x}$ ni aniqlaymiz. $\vartheta_{\bar{x}x}$ ni (1.33) formuladagi ifodasidan foydalanib, $\vartheta_{\bar{x}x\bar{x}x}$ uchun quyidagiga ega bo'lish mumkin:

$$L_h\vartheta = \vartheta_{\bar{x}x\bar{x}x} = \frac{1}{h^2}\left[\vartheta_{\bar{x}x}(x+h) - 2\vartheta_{\bar{x}x}(x) + \vartheta_{\bar{x}x}(x-h)\right] = .$$

$$= \frac{1}{h^4}\left[\vartheta(x+2h) - 4\vartheta(x+h) + 6\vartheta(x) - 4(x-h) + \vartheta(x-2h)\right].$$

L_h ayirmali operator L differensial operatorni

$$\vartheta_{\bar{x}x\bar{x}x} - \vartheta^{(4)} = \frac{h^2}{6}\vartheta^{(6)} + O(h^4).$$

ikkinchi tartib bilan approksimatsiya qilishini ko'rsatish mumkin.

Buning uchun Teylor qatorining

$$\vartheta(x \pm kh) = \vartheta(x) + \sum_{s=1}^{7} \frac{(\pm 1)^s k^s h^s}{s!}\,\frac{d^s\vartheta(x)}{dx^s} + O(h^8)$$

yoyilmasidan $k=1,2$ lar uchun foydalanib va $\vartheta(x+kh)+\vartheta(x-kh)$ yig'indi faqat juft darajalardan iborat ekanligini hisobga olinsa, $\vartheta_{\bar{x}x\bar{x}x}$ uchun yuqorida keltirilgan formulaga ega bo'lish mumkin.

Approksimatsiya xatoligi $\psi = L_h\vartheta - L\vartheta$ ni h darajalari bo'yicha Teylor qatoriga yoyishdan approksimatsiya xatoligi tartibini oshirishda foydalanish mumkin. Haqiqatdan ham,

$$\vartheta_{\bar{x}x} - \vartheta'' = \frac{h^2}{12}\vartheta^{(4)} + O(h^4) = \frac{h^2}{12}\vartheta_{\bar{x}x\bar{x}x} + O(h^4).$$

Agar $(x-2h,\ x-h,\ x,\ x+h,\ x+2h)$ nuqtalardan iborat shablonda

$$L_h'\vartheta = \vartheta_{\bar{x}x} - \frac{h^2}{12}\vartheta_{\bar{x}x\bar{x}x}$$

ayirmali operatordan foydalanilsa, bu operator $L\vartheta = \vartheta''$ ni to'rtinchi tartib bilan approksimatsiya qiladi.

4-misol. $\dfrac{\partial u(x,t)}{\partial t} - a\dfrac{\partial u(x,t)}{\partial x} = f(x,t), \quad -\infty < x < +\infty, t > 0$ tenglamani $t=0$ da $u(x,0) = \psi(x), -\infty < x < +\infty$ boshlang'ich shartni qanoatlantiruvchi $u(x,t)$ yechimini topish uchun ayirmali sxema quring va approksimatsiya xatoligini baholang.

Yechish. $\dfrac{\partial u}{\partial t}$ hosilani quyidagi ayirmali nisbatlarning biri bilan almashtirish mumkin:

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u^{(h)}(x,t+\tau) - u^{(h)}(x,t)}{\tau}; \quad \frac{\partial u(x,t)}{\partial t} \approx \frac{u^{(h)}(x,t) - u^{(h)}(x,t-\tau)}{\tau};$$

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u^{(h)}(x,t+\tau) - u^{(h)}(x,t-\tau)}{2\tau}.$$

Xuddi shuningdek $\frac{\partial u}{\partial x}$ hosilani

$$\frac{\partial u(x,t)}{\partial x} \approx \frac{u^{(h)}(x+h,t)-u^{(h)}(x,t)}{h}, \quad \frac{\partial u(x,t)}{\partial x} \approx \frac{u^{(h)}(x,t)-u^{(h)}(x-h,t)}{h};$$

$$\frac{\partial u(x,t)}{\partial x} \approx \frac{u^{(h)}(x+h,t)-u^{(h)}(x-h,t)}{2h}$$

ifodalarning biri bilan almashtirish mumkin.

Ayirmali sxemaning muhim xossalaridan biri, to'r nuqtalarida ayirmali sxema yechimining differensial masala yechimiga yaqinligidir. Buning uchun ayirmali masala differensial masalaga «yaqin» bo'lishi lozim. Ushbu «yaqin»lik $\left\|\delta f^{(h)}\right\|_{F_h} = \left\|L_h[u]_h - f^{(h)}\right\|_{F_h}$ miqdor bilan baholanadi, bu yerda $[u]_h$- differensial masala yechimining to'r tugun nuqtalaridagi qiymati.

Endi berilgan tenglamani quyidagi ayirmali sxema bilan almashtiramiz:

$$\frac{u(x_m,t_{n+1})-u(x_m,t_n)}{\tau} - a\frac{u(x_{m+1},t_n)-u(x_m,t_n)}{h} = f_n, \quad m=0,\pm1,\pm2,\ldots.,n=0,1,2,\ldots...$$

Approksimatsiya tartibini aniqlash uchun quyidagi tenglikdan foydalanamiz:

$$\delta f^{(h)} = \begin{cases} \frac{u(x_m,t_{n+1})-u(x_m,t_n)}{\tau} - a\frac{u(x_{m+1},t_n)-u(x_m,t_n)}{h} - f(x_m,t_n), \\ u(x_m,t_0)-\psi(x_m), \quad m=0,\pm1,\pm2,\ldots., n=0,1,\ldots \end{cases}$$

Agar $-\infty < x < +\infty, t \geq 0$ sohada $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ hosilalar mavjud bo'lsa,

$$u(x_m,t_{n+1}) = u(x_m,t_n) + \frac{\tau}{1!}\frac{\partial u(x_m,t_n)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 u(x_m,\tilde{t}_n)}{\partial t^2},$$

$$u(x_{m+1},t_n)=u(x_m,t_n)+\frac{h}{1!}\frac{\partial u(x_m,t_n)}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u(\tilde{x}_m,t_n)}{\partial x^2}, \quad t_n\le\tilde{t}_n\le t_{n+1},\ x_m\le\tilde{x}_m\le x_{m+1}$$

tengliklardan foydalanib

$$\delta f^{(h)}=\begin{cases}\dfrac{\partial u(x_m,t_n)}{\partial t}+\dfrac{\tau}{2!}\dfrac{\partial u(x_m,\tilde{t}_n)}{\partial t}-a\dfrac{\partial u(x_m,t_n)}{\partial x}-a\dfrac{h}{2}\dfrac{\partial^2 u(\tilde{x}_m,t_n)}{\partial x^2}-f(x_m,t_n),\\ u(x_m,t_0)-\psi(x_m), \quad m=0,\pm1,\pm2,....,n=0,1,...\end{cases}$$

hosil qilamiz.

$$\frac{\partial u(x_m,t_n)}{\partial t}-a\frac{\partial u(x_m,t_n)}{\partial x}=f(x_m,t_n), \quad u(x_m,t_0)=\psi(x_m)$$

ekanligini e'tiborga olib, quyidagi tenglikni olamiz:

$$\delta f^{(h)}=\begin{cases}\dfrac{\tau}{2}\dfrac{\partial^2 u(x_m,\tilde{t}_n)}{\partial t^2}-a\dfrac{h}{2}\dfrac{\partial^2 u(\tilde{x}_m,t_n)}{\partial x^2},\\ 0,\\ m=0,\pm1,\pm2,....,n=0,1,...\end{cases}$$

$$f^{(h)}=\begin{cases}\varphi(x_m,t_n),\ m=0,\pm1,\pm2,.....,n=1,2,...,\\ \psi(x_m,t_n),\ m=0,\pm1,\pm2,....\end{cases}$$

to'rli funksiya uchun $\left\|\delta f^{(h)}\right\|_{F_h}=\max\limits_{m,n}\left|\varphi(x_m,t_n)\right|+\max\limits_{m}\left|\psi(x_m)\right|$ normani kiritib

va $-\infty<x<+\infty, t\ge0$ da $\left|\dfrac{\partial^2 u}{\partial x^2}\right|\le M_x^{(2)}, \left|\dfrac{\partial^2 u}{\partial t^2}\right|\le M_t^{(2)}$ deb faraz qilib quyidagi

munosabatni hosil qilamiz:

$$\left\|\delta f^{(h)}\right\|_{F_h}=\max_{m,n}\left|\frac{\tau}{2}\frac{\partial^2 u(x_m,\tilde{t}_n)}{\partial t^2}-a\frac{h}{2}\frac{\partial^2 u(\tilde{x}_m,t_n)}{\partial x^2}\right|\le\frac{\tau}{2}M_t^{(2)}+|a|\frac{h}{2}M_x^{(2)}.$$

Bu tenglik ayirmali sxema differensial masalani τ va h bo'yicha birinchi tartib bilan approksimatsiya qilishini ko'rsatadi.

5-**misol.** Ikki o'lchovli ko'chish tenglamasini

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \qquad (1.1.1)$$

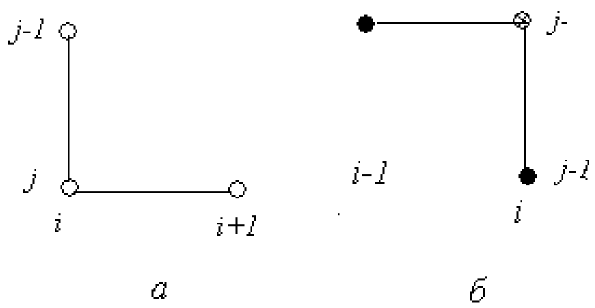
approksimatsiya qiluvchi Mak-Kormak sxemasi quyidagi ko'rinishda yozilishi mumkin:

$$\tilde{u}_{ij} = u_{ij}^n - \frac{\tau}{h_1} a\left(u_{i+1,j}^n - u_{i,j}^n\right) - \frac{\tau}{h_2} b\left(u_{i,j+1}^n - u_{i,j}^n\right), \qquad (1.1.2)$$

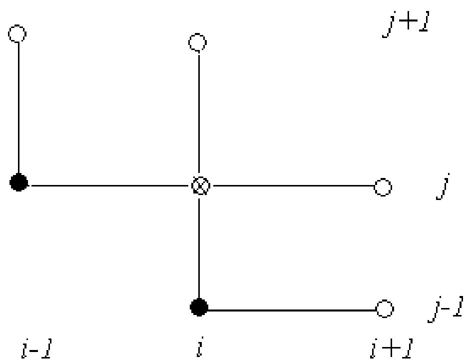
$$u_{ij}^{n+1} = \frac{1}{2}\left(u_{i,j}^n + \tilde{u}_{ij}\right) - \frac{1}{2}\frac{\tau}{h_1} a\left(\tilde{u}_{ij} - \tilde{u}_{i-1j}\right) - \frac{1}{2}\frac{\tau}{h_2} b\left(\tilde{u}_{ij} - \tilde{u}_{ij-1}\right), \qquad (1.1.3)$$

bu yerda $u_{ij}^n = u(ih_1, jh_2, n\tau)$, h_1, h_2 – mos ravishda x, y o'qlari bo'yicha, τ- vaqt bo'yicha qadam kattaligi. (1.1.2)-(1.1.3) sxemaning n- qatlam shablonini quring. Ushbu shablon $x = x_i = ih_1$, $y = y_j = jh_2$ chiziqlarining birortasiga nisbatan simmetrik bo'ladimi?

Yechish. Ushbu masalani ikki usul bilan yechish mumkin. Birinchisi geometrik qurishga asoslanadi. Buni qaralayotgan sxema uchun bajaramiz. n- qatlam (1.1.2) tenglamasining shabloni (bu tenglamani "prediktor" sxema deb ataladi) 4,a-rasmdagi kabi bo'ladi. (1.1.3) tenglama (bu tenglamani "korrektor" sxema deb ataladi) to'rda aniqlangan funksiyaning qiymati tilda bilan belgilangan, uning shabloni 4,b-rasmda ko'rsatilgan. Endi 4,a-rasmda ko'rsatilgan markaziy (i,j) nuqtani 4, b-rasmning barcha nuqtalariga joylashtirib, Mak-Kormak sxemasining 5-rasmda ko'rsatilgan n- qatlam shablonini hosil qilish mumkin. Ushbu rasmdan ko'rinadiki (1.1.2)-(1.1.3) Mak–Kormak sxemasi $x = x_i$ chiziqqa ham, $y = y_j$ chiziqqa nisbattan ham simmetrik emas.



4-rasm. n - qatlam shabloni.



5-rasm. (1.1.2)-(1.1.3) Mak-Kormak sxemasi, n - qatlam shabloni.

Ikkinchi uslub matematik amallarni bajarishga asoslangan. (1.1.2) tenglamadan $\tilde{u}_{ij}$ miqdor u^n to'r yechimning $(i,j),(i+1,j),(i,j+1)$ tugun nuqtadagi qiymatlariga bog'liqligi kelib chiqadi. Buni quyidagi formulani qo'llab, matematik ifodalashimiz mumkin:

$$\tilde{u}_{ij} = F_1\left((i,j),(i+1,j),(i,j+1)\right) \tag{1.1.4}$$

U holda (1.1.3) tenglamadagi $\tilde{u}_{i-1j}$ yechimning qiymati (1.1.4) tenglamadan i indeksni minus 1 ga siljitib topiladi:

$$\tilde{u}_{i-1j} = F_1\left((i-1,j),(i+1,j),(i-1,j+1)\right) \tag{1.1.5}$$

Shunga o'xshash

$$\tilde{u}_{ij-1} = F_1\left((i,j-1),(i+1,j-1),(i,j+1)\right). \tag{1.1.6}$$

(1.1.3),(1.1.4), (1.1.5) ifodaning o'ng tomondagi barcha to'r nuqtalarini yig'ib, (1.1.2), (1.1.3) Mak-Kormak sxemasining $\mathrm{St}_{\mathrm{MC}}$ shablonini hosil qilamiz:

$$\mathrm{St}_{\mathrm{MC}} = \{(i-1, j), (i, j), (i+1, j), (i, j+1), (i-1, j+1), (i, j-1), (i+1, j-1)\} \qquad (1.1.7)$$

(1.1.7) ko'rinib turibdiki (1.1.2), (1.1.3) Mak-Kormak sxemasining n-qatlam shabloni 7 ta nuqtadan iborat (5-rasmga qarang).

Misollar

1. y va z $a_i\varphi_{i+1} + b_i\varphi_i + c\varphi_{i-1} = 0$, $a_i \neq 0$, $c_i \neq 0$ tenglamaning ikki xususiy yechimi bo'lsin. $\begin{vmatrix} y_i & y_{i+1} \\ z_i & z_{i+1} \end{vmatrix}$ ditermenant hamma i nuqtalarda nolga teng yoki teng emasligini isbotlang.

2. $a_{i+k}\varphi_{i+k} + a_{i+k-1}\varphi_{i+k-1} + ... + a_{i-k}\varphi_{i-k} = 0$ ayirmali tenglama uchun 1-masalaga o'xshash masala tuzing va tuzilgan masalani yeching.

Quyidagi ayirmali tenglamalarni umumiy yechimlarini toping.

3. $6\varphi_{i+1} - 5\varphi_i + \varphi_{i-1} = 0$ 4. $2\varphi_{i+1} - 5\varphi_i + 2\varphi_{i-1} = 0$ 5. $\varphi_{i+1} - 4\varphi_i + 4\varphi_{i-1} = 0$.

6. $9\varphi_{i+1} - 6\varphi_i + \varphi_{i-1} = 0$ 7. $\varphi_{i+1} - 4\varphi_i - 5\varphi_{i-1} = 0$ 8. $5\varphi_{i+1} - 6\varphi_i + 5\varphi_{i-1} = 0$

9. $i \geq 0$ larda

$$\varphi_{i+1} - \frac{10}{3}\varphi_i + \varphi_{i-1} = 0$$

ayirmali tenglamaning $\varphi_0 = 1$ shartni qanoatlantiruvchi chekli yechimlarini toping.

10. Quyidagi

0,1,1,2,3,5,8,13,21,34,55,...

Fibonachchi sonlar ketma-ketligining millioninchi hadini toping.

11. $$\varphi_{i+1} - 2\varphi_i + \varphi_{i-1} = 0,\ i = \overline{1, n-1},$$
$$\varphi_0 = a,\ \varphi_n = b.$$

ayirmali tenglamalar sistemasining yechimini toping.

12. $$-\varphi_{i+1} + 2\varphi_i - \varphi_{i-1} = h^2 \sin\ ih\ ,\ i = \overline{1, n-1}$$
$$\varphi_0 - \varphi_1 = -h,\ \varphi_n = 1,\ nh = \pi / 2$$

ayirmali tenglamalar sistemasining yechimini toping.

II BOB. AYIRMALI SXEMALAR QURISh USULLARI.

Berilgan shablonda ayirmali sxema qurishning 3 xil usuli mavjud: ayirmali approksimatsiya usuli, integro-interpolyatsion usul va nomalum koeffitsiyentlar usuli.

§ 2.1. Ayirmali approksimatsiya usuli.

Ayirmali approksimatsiya usulida differensial tenglama va qo'shimcha shartlarga kiruvchi har bir hosila faqatgina shablonni tashkil qiluvchi tugun nuqtalarda ifodalangan ayirmali ifodalar bilan almashtiriladi. Ushbu usul juda sodda bo'lganligi bois qo'shimcha izohlarga hojat yo'q.

To'g'ri to'rtburchakli to'rda uzluksiz (va yetarlicha silliq) koeffitsiyentli differensial tenglamalar uchun ayirmali approksimatsiya usuli birinchi va ikkinchi tartibli approksimatsiyaga ega bo'lgan ayirmali sxemalarni oson tuzish imkonini beradi. Ammo, ushbu usulni murakkabroq bo'lgan hollar uchun qo'llash ancha mushkul yoki qo'llashni imkoni bo'lmaydi. Masalan, uzilishli koeffisiyentga ega bo'lgan differensial tenglamalar uchun, hisoblash sohasi to'g'ri to'rtburchak bo'lmasa, yuqori tartibli differensial tenglamalar uchun notekis to'rda va boshqa hollarda.

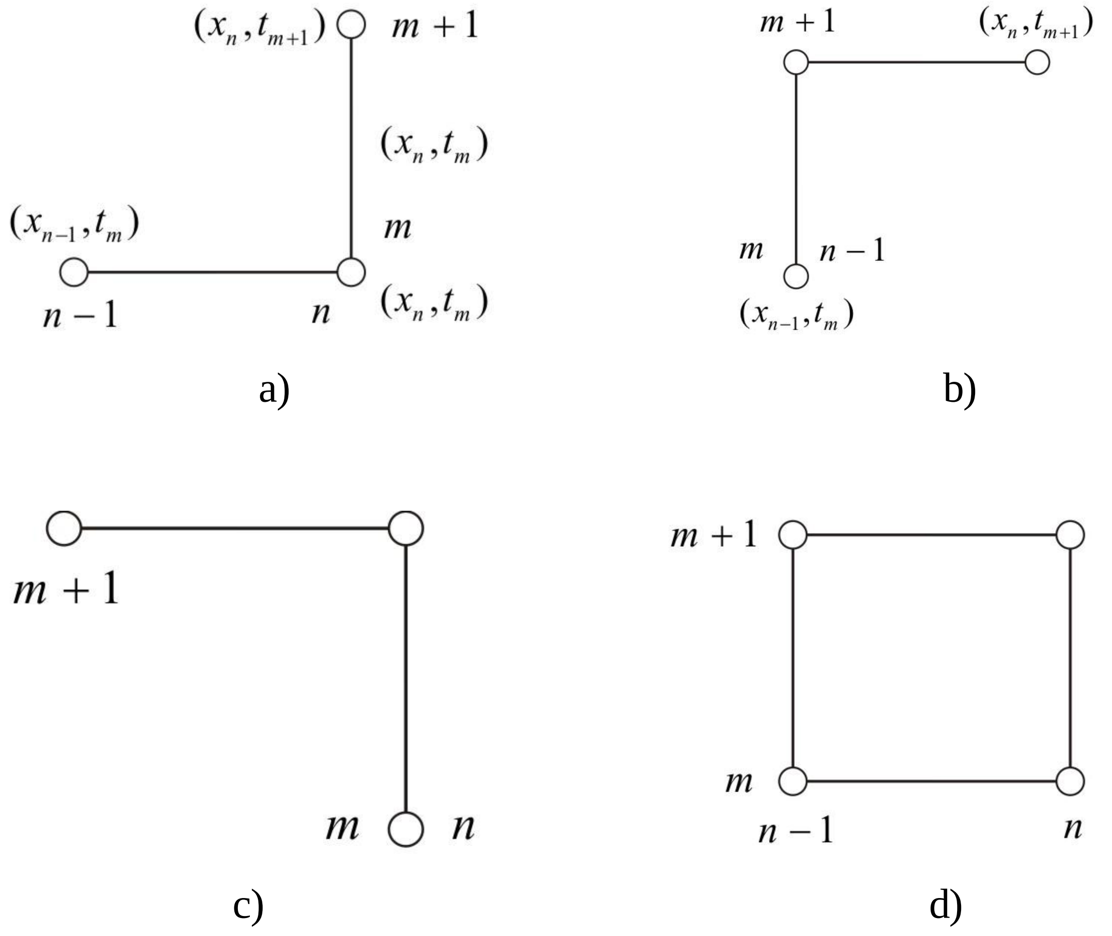
Misol. Quyidagi differensial masala uchun ayirmali sxema tuzish talab etilsin:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = f(x,t), \; 0 < x \le 1, \tag{2.1}$$

$$u(x,0) = \mu_1(x), \; 0 < x \le 1, \tag{2.2}$$

$$u(0,t)=\mu_2(t)\,,\quad 0\le t\le T\,. \qquad (2.3)$$

Ayirmali approksimatsiya usuli bo’yicha ayirmali sxema tuzish uchun shablon tanlaymiz. Buning uchun 4-(a, b, c, d) rasmlarda keltirilgan shablonlardan foydalanamiz. Ushbu shablonlardan (2.1) differensial tenglamani quyidagicha approksimatsiya qilish mumkin:



6-rasm.

a) shablon uchun:

$$\frac{y_n^{m+1}-y_n^m}{\tau}+c\,\frac{y_n^m-y_{n-1}^m}{n}=f(x_n,t_m)$$

b) shablon uchun:

$$\frac{y_{n-1}^{m+1}-y_{n-1}^m}{\tau}+c\,\frac{y_n^{m+1}-y_{n-1}^{m+1}}{n}=f(x_n-0{,}5h,\,t_m+0{,}5\tau)$$

c) shablon uchun:

$$\frac{y_n^{m+1}-y_n^m}{\tau}+c\frac{y_n^{m+1}-y_{n-1}^{m+1}}{h}=f(x_n-0,5h,\,t_m+0,5\tau)$$

d) shablon uchun:

$$\frac{y_n^{m+1}+y_{n-1}^{m+1}-y_n^m+y_{n-1}^m}{2\tau}+c\frac{y_n^{m+1}+y_n^m-y_{n-1}^{m+1}-y_{n-1}^m}{2h}=f(x_n-0,5h,\,t_m+0,5\tau)$$

Qo'shimcha shartlar barcha hollar uchun quyidagicha approksimatsiya qilinadi:

$$y_n^0=\mu_1(nh),\quad n=\overline{0,N}\;,h=\frac{a}{N}.$$

$$y_0^m=\mu_2(\tau m),\quad m=\overline{0,M}\;,\;\tau=\frac{T}{M}.$$

§ 2.2. Integro-interpolyatsion usul.

Ushbu usulni balans usuli ham deb nomlashadi. $D=\{0\le x\le 1, 0\le t\le T\}$ to'g'ri to'rtburchakda

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,t), \qquad (2.4)$$

differensial tenglamani va

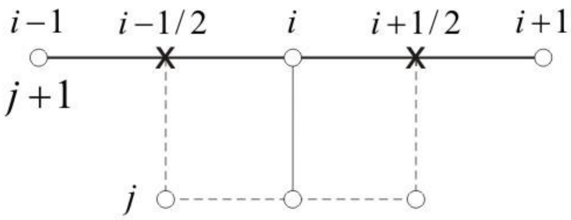
$$u(x,0)=u_0(x),\;0\le x\le 1, \qquad (2.5)$$

$$u(0,t)=u_1(t),\;u(1,t)=u_2(t),\quad 0\le t\le T, \qquad (2.6)$$

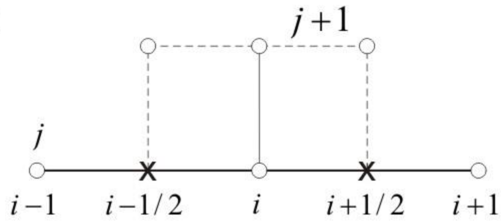
qo'shimcha shatrlarni qanoatlantiruvchi $u=u(x,t)$ funksiyani aniqlash talab etilgan bo'lsin. Ushbu masalani sonli yechish uchun $D=\{0\le x\le 1, 0\le t\le T\}$ sohada tekis to'r quramiz.

$\overline{\omega_h} = \{x_i = ih,\ i = \overline{0,N},\ h = 1/N\}$ $0 \le x \le 1$ kesmada h qadamli tekis to'r bo'lsin va $\overline{\omega_\tau} = \{t_j = j\tau,\ j = \overline{0,M},\ \tau = T/M\}$ $0 \le t \le T$ kesmada τ qadamli to'r bo'lsin. U holda $\overline{\omega}_{h\tau} = \overline{\omega}_h \cdot \overline{\omega}_\tau = \{(x_i t_j);\ x_i \in \overline{\omega}_h,\ t_j \in \overline{\omega}_\tau\}$ - $D = \{0 \le x \le 1, 0 \le t \le T\}$ to'g'ri to'rtburchakda h va τ qadamlar bilan qurilgan to'rni anglatadi.

Integro-interpolyatsion usul yordamida (2.4) differensial tenglamani ayirmali sxema bilan approksimatsiya qilish uchun (2.4) tenglamani $x_{i-0,5} \le x \le x_{i+0,5}$, $t_{i-0,5} \le t \le t_{i+0,5}$ to'g'ri to'rtburchakda integrallaymiz:



7-rasm



8-rasm

$$\frac{1}{h\tau}\int_{x_{i-0,5}}^{x_{i+0,5}} \left[u(x,t_{j+1}) + u(x,t)\right]dx = \frac{1}{h\tau}\int_{t_j}^{t_{j+1}} \left[\frac{\partial u}{\partial x}(x_{i+\frac{1}{2}},t) - \frac{\partial u}{\partial x}(x_{i-\frac{1}{2}},t)\right]dt +$$

$$+\frac{1}{h\tau}\int_{t_j}^{t_{j+1}} dt \int_{x_{i-0,5}}^{x_{i+0,5}} f(x,t)dx \qquad (2.7)$$

(2.7) tenglikka kiruvchi integrallarni quyidagicha approksimatsiyalaymiz:

$$\int_{x_{i-0,5}}^{x_{i+0,5}} u(x,t)dx \approx hu(x_i,t)$$

$$\int_{t_j}^{t_{j+1}} \frac{\partial u(x_{i+0,5},t)}{\partial x} dx \approx \tau\, u_{\bar{x},j+1}^{j+1}$$

$$\int_{t_j}^{t_{j+1}} dt \int_{x_{i-0,5}}^{x_{i+0.5}} \frac{\partial u(x_{i+0,5},t)}{\partial x} dx \approx \tau\, u_{\bar{x},i+1}^{j+1}$$

$$\int_{t_j}^{t_{j+1}} dt \int_{x_{i-0,5}}^{x_{i+0.5}} f(x,t)dx \approx h\tau\, 0,5\left(f(x_i,t_j) + f(x_i,t_{j+1})\right)$$

U holda (2.7) tenglikdan quyidagi oshkormas ayirmali sxemaga ega bo'lish mumkin:

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \frac{1}{h}\left[u_{\bar{x},i+1}^{j+1} - u_{\bar{x},i}^{j+1}\right] + \frac{1}{2}\left[f_i^{\,j} + f_i^{\,j+1}\right]$$

yoki

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \frac{1}{2}\left[f_i^{\,j} + f_i^{\,j+1}\right].$$

Endi (2.4) tenglamani 8–rasmda ko'rsatilgan yacheyka bo'yicha integrallaymiz:

$$\frac{1}{h\tau}\int_{x_{i-0,5}}^{x_{i+0,5}} \left[u(x,t_{j+1}) + u(x,t_j)\right] dx = \frac{1}{h\tau}\int_{t_j}^{t_{j+1}} \left[\frac{\partial u(x_{i+\frac{1}{2}},t)}{\partial x} - \frac{\partial u(x_{i-\frac{1}{2}},t)}{\partial x}\right] dt +$$

$$+\frac{1}{h\tau}\int_{t_j}^{t_{j+1}} dt \int_{x_{i-0,5}}^{x_{i+0,5}} f(x,t)dx \qquad (2.8)$$

(2.8) tenglamaga kiruvchi integrallardan $\int\limits_{t_j}^{t_{j+1}} \frac{\partial u(x_{i+0,5},t)}{\partial x} \approx \tau u_{\bar{x},i+1}^{j}$ va $\frac{1}{h\tau}\int\limits_{t_j}^{t_{j+1}} dt \int\limits_{x_{i-0,5}}^{x_{i+0,5}} f(x,t)dx \approx \frac{h\tau}{3}(f(x_{i-1},t_j)+f(x_i,t_j)+f(x_{i+1},t_j))$ approksimatsiya qilsak oshkor ayirmali sxemaga ega bo'lish mumkin:

$$\frac{u_i^{j+1}-u_i^j}{\tau} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \frac{1}{3}(f_{i-1}^j + f_i^j + f_{i+1}^j)$$

qo'shimcha shartlar ikkala holda ham

$$u_i^0 = u_0(x_i) \quad , \quad y_0^j = \mu_1\ t_j \quad , \quad u_N^j = \mu_2\ t_j$$

ko'rinishda approksimatsiya qilinadi.

§ 2.3. Nomalum koeffitsiyentlar usuli.

Nomalum koeffitsiyentlar usulida ayirmali sxema sifatida nomalum to'r funksiyaning shablonni tashkil qiluvchi tugun nuqtalardagi qiymatlarining chiziqli kombinasiyasi olinadi. Ushbu chiziqli kombinasiyaning koeffitsiyentlari ayirmali sxema berilgan differensial tenglamani to'r qatlamlari bo'yicha iloji boricha yuqori tartibda approksimatsiya qilish shartidan topiladi.

Masalan,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

tenglama uchun

$$Ш(x,t) = \quad x_{i-1},t_j \ ; \ x_i,t_j \ ; \quad x_{i+1},t_j \ ; \ x_i,t_{j+1} \ ;$$

shablonda ayirmali sxema qurish talab etilgan bo’lsin. Demak ayirmali sxema

$$\alpha u_{i-1}^{j} + \beta u_{i}^{j} + \gamma u_{i+1}^{j} + \mu u_{i}^{j+1} = 0. \qquad (2.9)$$

ko’rinishda ekan. u_{i-1}^{j} , u_{i+1}^{j} va u_{i}^{j+1} to’r funksiyalarni (x_i, t_j) nuqta atrofida Teylor qatoriga yoyib, (2.9) tenglamaga qo’yamiz.

$$\alpha u_{i-1}^{j} + \beta u_{i}^{j} + \gamma u_{i+1}^{j} + \mu u_{i}^{j+1} =$$

$$\alpha\left(u_i^j - h\frac{\partial u(x_i,t_j)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial x^3} + \ldots\right) +$$

$$+\beta\left(u_i^j + \gamma(u_i^j + h\frac{\partial u(x_i,t_j)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial x^3} + \ldots\right) +$$

$$+\mu\left(u_i^j + \tau\frac{\partial u(x_i,t_j)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial t^2} + \frac{h^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial t^3} + \ldots\right) =$$

$$\alpha + \beta + \gamma + \mu \;\; u_i^j + \;\; \gamma - \alpha \;\; h\frac{\partial u \;\; x_i, t_j}{\partial x} + \frac{h^2}{2} \;\; \alpha + \gamma \;\; \frac{\partial^2 u \;\; x_i, t_j}{\partial x^2}$$

$$+\mu\tau\frac{\partial u \;\; x_i, t_j}{\partial t} + O \;\; h^3 + \tau^2 \;\; . \qquad (2.10)$$

(2.10) tenglikdan $\alpha u_{i-1}^{j} + \beta u_{i}^{j} + \gamma u_{i+1}^{j} + \mu u_{i}^{j+1} = \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)_{(x_i,t_j)} + O(h^3 + \tau^2)$

bo’lishi uchun α, β, γ va μ koeffitsiyentlar quyidagi tenglamalar sistemasini qanoatlantirishi kerak:

$$\begin{cases} \alpha+\beta+\gamma+\mu=0 \\ \gamma-\alpha=0 \\ \alpha+\gamma=-\dfrac{2}{h^2} \\ \mu\tau=1 \end{cases} \qquad . \qquad (2.11)$$

(2.11) tenglamalar sistemasini yechib, $\alpha=\gamma=-\frac{1}{h^2}$, $\mu=\frac{1}{\tau}$ va $\beta=\frac{2}{h^2}=-\frac{1}{\tau}$ ekanligini aniqlaymiz. Koeffitsiyentlarni bu qiymatlarini (2.10) ga qo'yamiz:

$$-\frac{u_{i-1}^{j}}{h^2}+\frac{2u_i^j}{h^2}-\frac{u_i^j}{\tau}-\frac{u_{i+1}^{j}}{h^2}+\frac{u_i^{j+1}}{\tau}=0 \quad \text{yoki}$$

$$\frac{u_i^{j+1}-u_i^j}{\tau}-\frac{u_{i-1}^{j}-2u_i^j+u_{i+1}^{j}}{h^2}=0 \qquad (2.12)$$

Demak,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

differensial tenglamani $Ш(x_i,t_i)=\{(x_{i-1},t_j);\ (x_i,t_j);\ (x_{i+1},t_j);\ (x_i,t_{j+1})\}$ shablonda approksimatsiya qiluvchi ayirmali sxema quyidagi ko'rinishga ega ekan:

$$\frac{u_i^{j+1}-u_i^j}{\tau}=\frac{u_{i-1}^{j}-2u_i^j+u_{i+1}^{j}}{h^2}.$$

Endi $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ differensial tenglamani

$$Ш(x_i,t_j)=\{(x_i,t_j);\ (x_{i-1},t_{j+1});\ (x_i,t_{j+1});\ (x_{i+1},t_{j+1})\}$$

shablonda approksimatsiya qiluvchi ayirmali tenglamani topamiz. Bu holda ayirmali sxema

$$\alpha u_i^j + \beta u_{i-1}^{j+1} + \gamma u_i^{j+1} + \mu u_{i+1}^{j+1} = 0. \qquad (2.13)$$

ko'rinishda bo'ladi. (2.13) da u_{i-1}^{j+1}, u_i^{j+1}, u_{i+1}^{j+1} to'r funksiyalarni (x_i,t_j) nuqta atrofida Teylor qatoriga yoyamiz:

$$u_i^{j+1} = u_i^j + \tau\frac{\partial u(x_i,t_j)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial t^2} + \frac{\tau^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial t^3} + \frac{\tau^4}{24}\frac{\partial^4 u(x_i,t_j)}{\partial t^4} + \ldots$$

$$u_{i-1}^{j+1} = u_i^j - h\frac{\partial u(x_i,t_j)}{\partial x} + \tau\frac{\partial u(x_i,t_j)}{\partial t} + \frac{h^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial x^2} + \frac{\tau^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial t^2} -$$

$$- h\tau\frac{\partial^2 u(x_i,t_j)}{\partial x\partial t} - \frac{h^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial x^3} + \frac{\tau^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial t^3} + \frac{h^2\tau}{2}\frac{\partial^3 u(x_i,t_j)}{\partial x^2\partial t} - \frac{h\tau^2}{2}\frac{\partial^3 u(x_i,t_j)}{\partial x\partial t^2} + \ldots$$

$$u_{i+1}^{j+1} = u_i^j + h\frac{\partial u(x_i,t_j)}{\partial x} + \tau\frac{\partial u(x_i,t_j)}{\partial t} + \frac{h^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial x^2} + \frac{\tau^2}{2}\frac{\partial^2 u(x_i,t_j)}{\partial t^2} +$$

$$+ h\tau\frac{\partial^2 u(x_i,t_j)}{\partial x\partial t} + \frac{h^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial x^3} + \frac{\tau^3}{6}\frac{\partial^3 u(x_i,t_j)}{\partial t^3} + \frac{h^2\tau}{2}\frac{\partial^3 u(x_i,t_j)}{\partial x^2\partial t} + \frac{h\tau^2}{2}\frac{\partial^3 u(x_i,t_j)}{\partial x\partial t^2} + \ldots$$

U holda

$$\alpha u_i^j + \beta u_{i-1}^{j+1} + \gamma u_i^{j+1} + \mu u_{i+1}^{j+1} = \alpha u(x_i,t_j) + \beta\big[u(x_i,t_j) - hu'_x(x_i,t_j) + u'_t(x_i,t_j) +$$

$$+ 0{,}5h^2 u''_{xx}(x_i,t_j) + 0{,}5\tau^2 u''_{tt}(x_i,t_j) - h\tau u''_{xt}(x_i,t_j) - \frac{h^3}{6}u'''_{xxx}(x_i,t_j) + \frac{\tau^3}{6}u'''_{ttt}(x_i,t_j) +$$

$$+ \frac{h^2\tau}{2}u'''_{xxt}(x_i,t_j) - \frac{h\tau^2}{2}u'''_{xtt}(x_i,t_j) + \frac{h^4}{24}u''''_{xxxx}(x_i,t_j) - \frac{h^3\tau}{6}u''''_{xxxt}(x_i,t_j) +$$

$$+ \frac{h^2\tau^2}{4}u''''_{xxtt}(x_i,t_j) - \frac{h\tau^3}{6}u''''_{xttt}(x_i,t_j) + \frac{\tau^4}{24}u''''_{tttt}(x_i,t_j) + \ldots\big] +$$

$$+\gamma\left[u(x_i,t_j)+\tau u'_t(x_i,t_j)+0{,}5\tau^2u''_{tt}(x_i,t_j)+\frac{\tau^3}{6}u'''_{ttt}(x_i,t_j)+\frac{\tau^4}{24}u'''_{ttt}(x_i,t_j)+\dots\right]+$$

$$+\mu\left[u(x_i,t_j)+hu'_x(x_i,t_j)+u'_t(x_i,t_j)+\right.$$

$$+0{,}5h^2u''_{xx}(x_i,t_j)+0{,}5\tau^2u''_{tt}(x_i,t_j)+h\tau u''_{xt}(x_i,t_j)+\frac{h^3}{6}u'''_{xxx}(x_i,t_j)+\frac{\tau^3}{6}u'''_{ttt}(x_i,t_j)+$$

$$+\frac{h^2\tau}{2}u'''_{xxt}(x_i,t_j)+\frac{h\tau^2}{2}u'''_{xtt}(x_i,t_j)+\frac{h^4}{24}u''''_{xxxx}(x_i,t_j)+\frac{h^3\tau}{6}u''''_{xxxt}(x_i,t_j)+$$

$$\left.+\frac{h^2\tau^2}{4}u''''_{xxtt}(x_i,t_j)+\frac{h\tau^3}{6}u''''_{xttt}(x_i,t_j)+\frac{\tau^4}{24}u''''_{tttt}(x_i,t_j)+\dots\right]$$

O'xshash hadlarni ixchamlab, quyidagiga ega bo'lamiz:

$$\alpha u_i^j+\beta u_{i-1}^{j+1}+\gamma u_i^{j+1}+\mu u_{i+1}^{j+1}=(\alpha+\beta+\gamma+\mu)u(x_i,t_j)+h(\mu-\beta)u'_x(x_i,t_j)+$$

$$+\tau(\beta+\gamma+\mu)u'_t(x_i,t_j)+0{,}5h^2(\beta+\mu)u''_{xx}(x_i,t_j)+0{,}5\tau^2(\beta+\gamma+\mu)u''_{tt}(x_i,t_j)+$$

$$+h\tau(\mu-\beta)u''_{xt}(x_i,t_j)+\frac{h^3}{6}(\mu-\beta)u'''_{xxx}(x_i,t_j)+\frac{\tau^3}{6}(\beta+\gamma+\mu)u'''_{ttt}(x_i,t_j)+$$

$$+0{,}5h^2\tau(\beta+\mu)u'''_{xxt}(x_i,t_j)+0{,}5h^2\tau(\mu-\beta)u'''_{xtt}(x_i,t_j)+\frac{h^4}{24}(\beta+\mu)u''''_{xxxx}+$$

$$+\frac{h^3\tau}{6}(\mu-\beta)u''''_{xxxt}(x_i,t_j)+\frac{h^2\tau^2}{4}(\beta+\mu)u''''_{xxxt}(x_i,t_j)+\frac{h\tau^3}{6}(\mu-\beta)u''''_{xttt}(x_i,t_j)+$$

$$+\frac{h\tau^3}{6}(\mu-\beta)u''''_{xttt}(x_i,t_j)+\frac{\tau^4}{24}(\beta+\mu)u''''_{tttt}(x_i,t_j)+\dots.$$

Oxirgi tenglikda quyidagilarni bajarilishini talab qilamiz:

$$\begin{cases} \alpha + \beta + \gamma + \mu = 0 \\ \beta + \gamma + \mu = \dfrac{1}{\tau} \\ \beta + \mu = -\dfrac{2}{h^2} \\ \mu - \beta = 0 \end{cases}.$$

Bu tenglamalar sistemasini yechib, noma’lum α,β,γ,μ parametrlar qiymatini aniqlaymiz:

$$\alpha = -\frac{1}{\tau}, \quad \beta = \mu = -\frac{1}{h^2}, \quad \gamma = \frac{1}{\tau} + \frac{1}{h^2}.$$

Parametrlarning ushbu qiymatlarida

$$\alpha u_i^j + \beta u_{i-1}^{j+1} + \gamma u_i^{j+1} + \mu u_{i+1}^{j+1} = \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)_{(x_i,t_j)} - 0{,}5\tau u''_{tt}(x_i,t_j) + \frac{\tau^2}{6}u'''_{ttt}(x_i,t_j) -$$

$$-\frac{h^2}{12}u''''_{xxxx}(x_i,t_j) - \frac{\tau^2}{2}u''''_{xxxx}(x_i,t_j) + -\frac{h^2\tau^4}{12}u''''_{tttt}(x_i,t_j) + \ldots.. \qquad (2.13')$$

α,β,γ,μ parametrlarning topilgan qiymatlarini (2.13) tenglamaga qo’ysak, biz qurgan ayirmali sxemaning ko’rinishi kelib chiqadi:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} = \frac{\partial u(x_i,t_j)}{\partial t} - \frac{\partial^2 u(x_i,t_j)}{\partial t^2} + O(\tau,h^2).$$

Demak, $Ш(x_i,t_j) = \{(x_i,t_j), (x_{i-1},t_{j+1}), (x_i,t_{j+1}), (x_{i+1},t_{j+1})\}$ shablonda qurilgan ayirmali sxema oshkormas bo’lib, berilgan differensial masalani τ bo’yicha birinchi tartib bilan va h bo’yicha ikkinchi tartib bilan approksimatsiya qilar ekan.

§ 2.4. Chegaraviy shartlarni ayirmali approksimatsiya qilish.

Faraz qilaylik, $D = \{0 \le x \le 1,\ 0 \le t \le T\}$ sohada quyidagi differensial masala berilgan bo'lsin:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ 0 \le t \le T, \tag{2.14}$$

$$u(x,0) = \mu(x),\ u_x(0,t) = \mu_1(t),\ u_x(1,t) = \mu_2(t)$$

Bu yerda $u_x(0,t) = \mu_1(t)$ chegaraviy shartni

$$\frac{y_1^{j+1} - y_0^{j+1}}{h} = \mu_1(t_{j+1})$$

ayirmali tenglama bilan approksimatsiya qilish mumkin. Malumki, ushbu ayirmali hosila berilgan chegaraviy shartni $O(h)$ bilan approksimatsiya qiladi. Bu esa masalani yechishdagi umumiy aniqlikni pasayishiga olib keladi. Bunday holatdan chiqish uchun chegaraviy shartlarni ayirmali approksimatsiya qilish usullari bilan tanishamiz.

Fiktiv nuqtalar usuli. $0 \le x \le 1$ kesma tashqarisida $x_{-1} = x_0 - h$ tugun nuqta kiritamiz va ushbu x_{-1} nuqtada ham berilgan (2.14) tenglama o'rinli deb hisoblaymiz. $i = 0$ nuqtada (2.14) tenglamani approksimatsiya qiluvchi ayirmali tenglamani yozamiz.

$$\frac{y_0^{j+1} - y_0^j}{\tau} = \frac{y_{-1}^j - 2y_0^j + y_1^j}{h^2} \tag{2.15}$$

Chap chegaraviy shartni markaziy ayirmali hosila bilan approksimatsiya qilamiz.

$$\frac{y_1^j - y_{-1}^j}{2h} = \mu_1(t_j). \tag{2.16}$$

Endi (2.16) dan $y_{-1}^{j} = y_1^{j} - 2h\mu_1$ ni aniqlab, uni (2.15) tenglikga qo'ysak va soddalashtirsak,

$$\frac{y_1^j - y_0^j}{h} = \mu_1 \; t_j \; + \frac{h}{2\tau} \; y_0^{j+1} - y_0^j \qquad (2.17)$$

ayirmali tenglamaga ega bo'lish mumkin. Bu tenglamadan y_0^{j+1} ni oshkor holda aniqlash mumkin.

Approksimatsiya xatoligini kamaytirish usuli.

$u(x_1,t)$ funksiyani (x_0,t) nuqta atrofida Teylor qatoriga yoyamiz:

$$u(x_1,t) = u(x_0,t) + hu_x(x_0,t_j) + \frac{h^2}{2}u_{xx} + ...$$

Chegaraviy shartga ko'ra $u_x(0,t) = \mu_1(t)$ va $u_{xx} = u_t$ ekanligini hisobga olib, ularni Teylor qatoriga qo'yamiz:

$$u(x_1,t) = u(x_0,t) + h\mu_1(t) + \frac{h^2}{2}u_t(x_0,t) + ...$$

Bu yerda $u_t \approx y_0^{j+1} - y_0^j / \tau$ ekanligini hisobga olsak, yana (2.17) chegaraviy shartga ega bo'lish mumkin. O'ng chegaraviy shartga nisbatan ham bayon etilgan amallarni qo'llash mumkin.

Misol. (3.1.1)-(3.1.2) ko'chish tenglamasini approksimatsiya qiluvchi (qarang §3.5) (3.1.3) Laks sxemasini tadqiq qilishdan tushunarliki, vaqt bo'yicha τ qadamni $\tau = O(h)$ kabi tanlansa mazkur sxema turg'un bo'ladi. Vaqt bo'yicha τ qadamni $\tau = O(h^2)$ kabi tanlansa ham sxema turg'un bo'ladi. Agar $h \to 0$ da τ ni

$$\tau = h^2/\mu, \quad \mu = const > 0 \qquad (2.1.1)$$

ko'rinishda olinsa (3.1.3) Laks sxemasi (3.1.1)-(3.1.2) ko'chirish tenglamasini approksimatsiya qiladimi?

Yechish. Dastlab Laks sxemasi approksimatsiya xatoligining bosh hadini topamiz. Buning uchun (3.1.3) tenglikka quyidagi Teylor qatoriga $(jh, n\tau)$ nuqta atrofida yoyilgan quyidagi ifodalarni qo'yamiz:

$$u_j^{n+1} = u_j^n + \tau u_t + \frac{\tau^2}{2} u_{tt} + O(\tau^3); \; u_{j\pm1}^n = u_j^n \pm \tau u_x + \frac{h^2}{2} u_{xx} + O(h^3).$$

Natijada Laks sxemasining birinchi differensial yaqinlashishini topamiz:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2} + \frac{h^2}{2\tau}\frac{\partial^2 u}{\partial x^2}. \qquad (2.1.2)$$

Ushbu tenglikka τ uchun yozilgan (2.1.1) ifodani qo'yamiz va quyidagi tenglikni hosil qilamiz:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{h^2}{2\mu}\frac{\partial^2 u}{\partial t^2} + \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2}. \qquad (2.1.3)$$

(2.1.3.) tenglikdan ko'rinadiki, $h \to 0$da Laks sxemasi (3.1.1) tenglamani emas, balki

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2}$$

parabolik tenglamani approksimatsiya qiladi. Laks sxemasi shartli approksimatsiya qiluvchi sxemaga misol bo'ladi.

Misollar

1. $$\lim_{h\to 0}\frac{u(x+h)-2u(x)+u(x-h)}{h^2}=$$

$$=\lim_{h\to 0}\frac{\dfrac{u(x+2h)-u(x)}{2}-2u(x)+\dfrac{u(x)-u(x-2h)}{2}}{h^2}=$$

$$=\lim_{h\to 0}\frac{u(x+h)-u(x-h)}{2h}=\lim_{h\to 0}\frac{\dfrac{u(x+2h)+u(x)}{2}-\dfrac{u(x)-u(x-2h)}{2}}{2h},$$

Agar $u(x)\in C^4$ bo'lsa, keltirilgan tengliklar o'rinlimi?

2. α,β,γ ning qanday qiymatlarida

$$\frac{-\varphi_{i+1}+2\varphi_i-\varphi_{i-1}}{h^2}+(\alpha\varphi_{i+1}+\beta\varphi_i+\gamma\varphi_{i-1})=f(x_i)+\frac{h^2}{12}f''(x_i),$$

$$\varphi_0=0,\varphi_n=0,\ i=\overline{1,n-1},\ x_i=ih,\ h=1/n$$

ayirmali sxema

$$-\frac{d^2u}{dx^2}+u=f(x),\ x\in[0,1]$$

$$u(0)=0,\ \ u(1)=0$$

masalani to'rtinchi tartib bilan approksimatsiya qiladi?

3. $$\frac{du}{dx}+2u\cos x=\cos x+\sin(2x),\ x\in[0,1],$$

$$u(0)=0$$

differensial masala

$$\frac{\varphi_{i+1}-\varphi_i}{h}+a_i\frac{\varphi_{i+1}+\varphi_i}{2}=f_i,$$

$$\varphi_0 = 0,\ i = \overline{0,\ n-1},\ h = 1/n$$

ayirmali sxema orqali nechanchi tartibda approksimatsiya qilinishini aniqlang. Bu yerda

$$a_i = \cos x_i + \cos x_{i+1},\ f_i = \frac{1}{2}a_i + \frac{1}{2}\ \sin\ 2x_i\ + \sin\ 2x_{i+1}\ ,$$

aniqlanish sohasi f^h sifatida

$$Df^h = \ x_{i+1/2},\ i = \overline{0,\ n-1}$$

$$x_i = ih,\ \ x_{i+1/2} = ih + h/2$$

ni oling.

4. $Df^h = \ x_i,\ i = \overline{0,\ n-1}\ ,\ x_i = ih$

aniqlanish sohasi f^h uchun 3-masalani yeching.

5. $a_i = 2\cos x_i,\ f_i = \cos x_{i+1} + \sin\ 2x_{i+1}$

$$Df^h = \ x_{i+1/2},\ i = \overline{0,\ n-1}\ ,\ x_i = ih,\ x_{i+1/2} = ih + h/2.$$

bo'lganda 3-masalani yeching.

6. $a_i = 2\cos x_i,\ f_i = \cos x_i + \sin\ 2x_i$

$$Df^h = \ x_i,\ i = \overline{0,\ n-1}\ ,\ x_i = ih$$

bo'lganda 3-masalani yeching.

7. $a_i = 2\cos x_{i+1/2},\ f_i = \cos x_{i+1/2} + \sin\ 2x_{i+1/2}\ ,$

$$Df^h = \ x_{i+1/2},\ i = \overline{0,\ n-1}\ ,\ x_i = ih,\ x_{i+1/2} = ih + h/2.$$

bo'lganda 3-masalani yeching.

8. $$\frac{du}{dx} + a(x)u(x) = f(x), \quad x \in (0,1),$$

$$u(0) = c$$

differensial masala va

$$\frac{\varphi_{i+1}-\varphi_i}{h} + \left(\alpha_1 a(x_i) + \alpha_2 a(x_{i+1})\right)\left(\beta_1\varphi_i + \beta_2\varphi_{i+1}\right) = \gamma_1 f(x_1) + \gamma_2 f(x_{i+1}),$$

$$i = \overline{0, n-1}, \ \varphi_0 = c, \ h = 1/n, \ x_i = ih$$

ayirmali sxema berilgan. $\alpha_i, \beta_i, \gamma_i$ koeffitsiyentlarni qanday tanlasa, approksimatsiya tartibi ikkiga teng bo'ladi?

9. $$\frac{\varphi_{i+1}-\varphi_i}{2h} + \varphi_i = ih + 1, \ i = \overline{1, n-1}, \ h = \frac{1}{n},$$

$$\varphi_0 = 0, \ \varphi_1 = 0$$

ayirmali sxema

$$\frac{du}{dx} + u = x + 1, \ x \in [0,1]$$

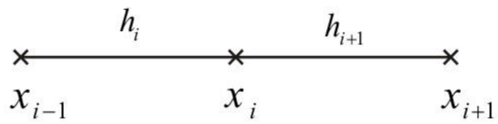
$u(0) = 0$ differensial masalani h ga nisbatan ikkinchi tartib bilan approksimatsiya qiladimi? Agar unday bo'lmasa, ayirmali sxemani ko'rinishini shunday o'zgartiringki, u ikkinchi tartib bilan approksimatsiya qilsin.

10. $$-\frac{d^2u}{dx^2} + au = \cos x, \ x \in (0,\pi), \ a > 0,$$

$$u(0) = 0, \ u(\pi) = 1$$

differensial masala uchun uch nuqtali shablonda o'ninchi tartibli approksimatsiya qiluvchi ayirmali sxemani quring.

11. Noma'lum koeffitsiyentlar usuli bilan notekis to'rda



9-rasm.

$$-\frac{d^2u}{dx^2} = f(x),$$

$$u(0) = a,\ u(1) = b,\ u \in C^4$$

masalani birinchi va ikkinchi tartib bilan approksimatsiya qiluvchi ayirmali sxemani quring. Bunda 9–rasmdagi shablondan foydalaning.

12. $\dfrac{du}{dx} + cu = f(x),\ \ u(0) = a,\ c = const,$

masala uchun o'zgarmas qadam bilan integro-interpolyatsion usul yordamida uch nuqtali shablonda to'rtinchi tartib bilan approksimatsiya qiluvchi ayirmali sxemani quring.

13. $-\dfrac{d^2u}{dx^2} + cu = f(x),\ c \geq 0,\ x \in [0,1],$

$$u(0) = a,\ u(1) = b$$

masalani 12-masala shartlari bilan yeching.

III BOB. AYIRMALI SXEMALAR NAZARIYASINING ASOSIY TUSHUNCHALARI.

Differensial masalaning asosiy tenglamasini va qo'shimcha shartlarini approksimatsiya qiluvchi ayirmali tenglamalar sistemasi ayirmali sxema deyiladi.

Ayirmali sxemaning approksimatsiya xatoligi, turg'unligi, yaqinlashishi va aniqligi ayirmali sxemalar nazariyasining asosiy tushunchalaridir.

Ayirmali sxemalar nazariyasining asosiy masalasi - ayirmali sxemaning aniqligi uning approksimatsiya xatoligi, yaqinlashishi va turg'unligini o'rganishga olib keladi.

§ 3.1. Approksimatsiya xatoligi.

L, L_h- aniqlanish sohalari mos holda Φ va Φ_h, qiymatlar sohalari mos holda F va F_h bo'lgan operatorlar bo'lsin. Bundan keyin L, L_h operatorlarni mos holda differenial va ayirmali operatorlar deb ataymiz.

Ayirmali L_h operator differensial L operatorni $n-$tartib bilan approksimatsiya qiladi deyiladi, agarda shunday musbat $\tilde{h}$ va C doimiylari mavjud bo'lsaki, barcha $h<\tilde{h}$ lar uchun quyidagi tengsizlik o'rinli bo'lsa

$$\left\|L_h(u)_h-(Lu)_h\right\|_{F_h}\leq Ch^n.$$

L_h operator L operatorni x_i nuqtada n chi tartib bilan approksimatsiya qiladi deyiladi, agarda shunday $\tilde{h}$ va C doimiylari mavjud bo'lib, barcha $h\leq\tilde{h}$ lar uchun

$$\left|(L_h(u)_h - (Lu)_h)_{x=x_i}\right| \le Ch^n$$

tengsizlik o'rinli bo'lsa.

Quyidagi

$$Lu = f\,,\ u \in \Phi\,,\ f \in F\,,$$

$$lu = g\,,\ g \in G\,, \tag{3.1}$$

differensial masala berilgan bo'lsin.

$$L_h\varphi^h = f^h\,,\ \varphi^h \in \Phi_h\,,\ f^h \in F_h\,,$$

$$l_h\varphi^h = g^h\,,\ g^h \in G_h \tag{3.2}$$

ayirmali sxemalar oilasini qaraylik. Bu ayirmali masalalar to'plamini kelgusida ayirmali sxemalar, ayirmali masalalar yechimlari to'plamini ayirmali sxemalar yechimi deb ataymiz.

(3.2) ayirmali sxema berilgan (3.1) differensial masalani n chi tartib bilan approksimatsiya qiladi deyiladi, agarda shunday $\tilde{h}$, C_1 va C_2 musbat doimiylari mavjud bo'lsaki, barcha $h < \tilde{h}$ lar uchun

$$\left\|L_h(u)_h - f^h\right\|_{F_h} \le C_1 h^{n_1}\,,$$

$$\left\|l_h(u)_h - g^h\right\|_{G_h} \le C_2 h^{n_2}\,,$$

$$n = \min(n_1, n_2)$$

tengsizliklar o'rinli bo'lsa,

$\psi_h = L_h(u)_h - (Lu)_h$ to'r funksiya ayirmali approksimatsiya xatoligi deyiladi. Berilgan (3.1) differensial masala va (3.2) ayirmali masalalar yechimlari ayirmasi $Z^h = \varphi^h - u$ (3.2) ayirmali sxemaning xatoligi deyiladi.

1-misol. $Lu_{\bar{x}x} = \dfrac{u_{x,i} - u_{\bar{x},i}}{h} = \dfrac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$

ayirmali operator $Lu = \dfrac{d^2u}{dx^2}$ operatorni $x = x_i$ nuqtada h bo'yicha ikkinchi tartib bilan approksimatsiya qilishini ko'rsatish mumkin. Buning uchun $u_{\bar{x}x,i}$ ikkinchi tartibli ayirmali hosiladagi u_{i-1} va u_{i+1} larni Teylor qatoriga yoysak

$$u_{\bar{x}x,i} - u''(x_i) = \frac{h^2}{12} u^{IV}(x_i) + O(h^4)$$

ekanligi tasdiqlanadi.

2-misol. $Lu = u^{IV}(x)$ differensial operatorni $L_h u = u_{\bar{x}x\bar{x}x}$ ayirmali operator bilan $(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$ shablonda approksimatsiya qilish mumkin.

$$L_h u = \frac{1}{h^2}\left(u_{\bar{x}x}(x_{i+1}) - 2u_{\bar{x}x}(x_i) + u_{\bar{x}x}(x_{i-1}) \right) =$$

$$= \frac{1}{h^4}\left(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2} \right) \qquad (*)$$

(*) dagi u_{i-2}, u_{i-1}, u_i, u_{i-2} larni $x = x_i$ nuqtada Teylor qatoriga yoysak $u_{\bar{x}x\bar{x}x,i} - u^{IV}(x_i) = \dfrac{h^2}{6} u^{IV}(x_i) + O(h^4)$, yani ayirmali operator L_h L operatorni ikkinchi tartib bilan approksimatsiya qiladi.

§ 3.2. Diskretlashtirish. Kelishilganlik.

Diskretlashtirish. Xususiy hosilali differensial tenglama (tenglamalar sistemasi) ni algebraik tenglamalar sistemasiga keltirish uchun bir necha variantlardan birini tanlash mumkin. Eng ko'p qo'llanadigan usullar - chekli ayirmali usullar, chekli elementlar usuli va spektral usul bo'lib hisoblanadi.

Diskretlashtirishda bu usullardan birini tanlash berilgan differensial tenglamada (tenglamalar sistemasida) vaqt bo'yicha hosila qatnashishi yoki qatnashmasligiga bog'liq.

Vaqt bo'yicha hosila qatnashgan hollarda chekli ayirmali usuldan foydalanadi. Faqatgina fazoviy koordinatalar bo'yicha diskretlashtirishda chekli ayirmali usuldan tashqari chekli elementlar usuli, spektral usul yoki chekli hajmlar usulini qo'llash mumkin.

Kelishilganlik. Diskretlashitirish natijasida hosil bo'lgan algebraik tenglamalar sistemasi berilgan xususiy hosilali differensial tenglama (tenglamalar sistemasi) bilan kelishilgan deyiladi, agarda to'r yacheykalari o'lchamlari nolga intilganda algebraik tenglamalar sistemasi to'rning har bir tugun nuqtasida berilgan xususiy hosilali differensial tenglamaga ekvivalent bo'lsa.

Ayirmali masalaning yechimi differensial masala yechimiga yaqinlashishi uchun kelishilganlik sharti bajarilishi zarur. Ammo, bu yetarli emas, chunki to'r yacheykalari o'lchamlari nolga intilganda algebraik tenglamalar sistemasi berilgan differensial tenglamaga ekvivalent bo'lsada, algebraik tenglamalar sistemasi yechimi berilgan differensial tenglama yechimiga intilishi kelib chiqmasligi mumkin.

Misol sifatida shartli turg'un ayirmali sxemalarni keltirish mumkin. Agar turg'unlik sharti buzilsa, algebraik tenglamalar sistemasi berilgan differensial tenglamaga ekvivalent bo'lsa-da, taqribiy yechim uzoqlashuvchi bo'ladi.

Misol. Quyidagi chegaraviy masala berilgan bo'lsin.

$$\frac{\partial \overline{T}}{\partial t} = \alpha \frac{\partial^2 \overline{T}}{\partial x^2},\ 0 < x < 1,\ 0 < t \le t_{\max} \tag{3.3}$$

$$\overline{T}(0,t) = b\,, \overline{T}(1,t) = d\,, \tag{3.4}$$

$$\overline{T}(x,0) = T_0(x)\,, 0 \le x \le 1. \tag{3.5}$$

Bu yerda T – berilgan differensial masalaning aniq yechimini bildiradi.

(3.3) tenglamani diskretlashtirish uchun hosilalarni ularga ekvivalent bo'lgan chekli ayirmali ifodalar bilan almashtirish mumkin.

$$\frac{\overline{T}_i^{n+1} - \overline{T}_i^n}{\Delta t} = \frac{\alpha(\overline{T}_{i-1}^n - 2\overline{T}_i^n + \overline{T}_{i+1}^n)}{\Delta x^2} \tag{3.6}$$

(3.6) da Δt va Δx lar mos holda vaqt bo'yicha va fazoviy koordinata x bo'yicha to'r qadamlaridir. $T_i^n - T$ ning (i,n) tugun nuqtadagi qiymatiga mos keladi.

(3.6) ni quyidagicha yozish mumkin:

$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{\Delta x^2}(T_{i-1}^n - 2T_i^n + T_{i+1}^n) \tag{3.7}$$

agar $\frac{\partial^2 \tilde{T}}{\partial x^2}$ hosila $n+1$ vaqt qatlamida diskretlashtirilsa, u holda oshkormas ayirmali sxemaga ega bo'lish mumkin:

$$sT_{i-1}^{n+1} - (1+2s)T_i^{n+1} + sT_{i+1}^{n+1} = -T_i^n, \qquad (3.8)$$

bu yerda $s = \alpha\Delta t / \Delta x^2$.

Shunday qilib, (3.3) differensial tenglamani diskretlashtirishda quyidagi oshkor va oshkormas

$$T_i^{n+1} = sT_{i-1}^n + (1-2s)T_i^n + sT_{i+1}^n \qquad (3.9)$$

$$sT_{i-1}^{n+1} - (1+2s)T_i^{n+1} + sT_{i+1}^{n+1} = -T_i^n \qquad (3.10)$$

ayirmali sxemalarga ega bo'ldik.

I. (3.9) oshkor ayirmali sxema uchun kelishilganlik shartini tekshirish uchun bu tenglamaga berilgan differensial tenglamani (i,n) tugun nuqtadagi aniq yechimini anglatuvchi $\overline{T}_i^n$ ni qo'yamiz.

$$\overline{T}_i^{n+1} = s\overline{T}_{i-1}^n + (1-2s)\overline{T}_i^n + s\overline{T}_{i+1}^n \qquad (3.11)$$

Endi (3.11) tenglamani berilgan differensial tenlama (3.3) ga mosligini (x_i, t_n) tugun nuqtada qanchalik yaqinligini aniqlashimiz zarur. (3.11) tenglamadagi ayrim hadlarni (x_i, t_n) nuqta atrofida Teylor qatoriga yoyib, soddalashtirsak quyidagi munosabatga ega bo'lish mumkin:

$$\left[\frac{\partial \overline{T}}{\partial t}\right]_i^n - \alpha\left[\frac{\partial^2 \overline{T}}{\partial t^2}\right]_i^n + \mathrm{E}_i^n = 0, \qquad (3.12)$$

bu yerda

$$\mathrm{E}_i^n = 0,5\Delta t\left[\frac{\partial^2 \overline{T}}{\partial t^2}\right]_i^n - \alpha\left(\frac{\Delta x^2}{12}\right)\left[\frac{\partial^4 \overline{T}}{\partial t^4}\right]_i^n + O(\Delta t^2, \Delta t^4)$$

(3.13)

ko'rinib turibdiki, (3.13) differensial tenglama (3.3) differensial tenglamadan approksimatsiya xatoligi deb ataluvchi E_i^n qo'shimcha had bilan farq qilib turibdi. Ushbu qo'shimcha hadning paydo bo'lishi esa $\frac{\partial^2 \overline{T}}{\partial t^2}$ va $\frac{\partial^2 \overline{T}}{\partial x^2}$ hosilalarni diskretlashtirish natijasi bilan bog'liq. (3.12) da to'r yacheykalari o'lchamlari $(\Delta x^2, \Delta t)$ kichik qilib tanlansa, approksimatsiya xatoligi E_i^n fiksirlangan qandaydir (x_i, t_n) nuqtada nolga intiladi. $\Delta x \to 0$, $\Delta t \to 0$ dagi limitda (3.9) tenglama (3.3) differensial tenglamaga ekvivalent bo'lib qoladi. Bu xossa esa kelishilganlik deyiladi.

(3.3) differensial tenglamaga asosan quyidagi munosabatlar o'rinli:

$$\frac{\partial^2 \overline{T}}{\partial t^2} = \alpha \frac{\partial}{\partial t}\frac{\partial^2 \overline{T}}{\partial x^2} = \alpha \frac{\partial^2}{\partial x^2}\frac{\partial \overline{T}}{\partial x} = \alpha^2 \frac{\partial^4 \overline{T}}{\partial x^4} \tag{3.14}$$

Shu sababli approksimatsiya xatoligi E_i^n ifodasini quyidagicha qayta yozish mumkin:

$$\mathrm{E}_i^n = 0,5\Delta x^2 (s - \frac{1}{6})\left[\frac{\partial^4 \overline{T}}{\partial t^4}\right]_i^n + O(\Delta t^2, \Delta t^4) \tag{3.15}$$

$s = \frac{1}{6}$ bo'lsa, (3.15) dagi birinchi had nolga teng bo'ladi va approksimatsiya xatoligi $O(\Delta t^2, \Delta t^4)$ bo'ladi.

II. Endi (3.10) oshkormas ayirmali sxemani berilgan (3.3) differensial tenglama bilan kelishilganligini tekshiramiz.

(3.10) tenglamaga (3.3) differensial tenglamani (x_i, t_n) tugun nuqtadagi aniq yechimini anglatuvchi T_i^n ni qo'yamiz:

$$\frac{\overline{T}_i^{n+1} - \overline{T}_i^n}{\Delta t} - \alpha \frac{\overline{T}_{i-1}^{n+1} - 2\overline{T}_i^{n+1} + \overline{T}_{i+1}^{n+1}}{\Delta x^2} = 0. \tag{3.16}$$

(3.16) dagi $\overline{T}_{i-1}^{n+1}$ va $\overline{T}_{i+1}^{n+1}$ larni $(i, n+1)$ tugun nuqta atrofida Teylor qatoriga yoyamiz:

$$\frac{\overline{T}_i^{n+1}-\overline{T}_i^{n}}{\Delta t}-\alpha\left\{\left[\overline{T}_{xx}\right]_i^n+\left(\frac{\Delta x^2}{12}\right)\left[\overline{T}_{x^2}\right]_i^n+\left(\frac{\Delta x^4}{360}\right)\left[\overline{T}_{x^4}\right]_i^{n+1}+...\right\}=0.$$

Endi oxirgi munosabatdagi $\overline{T}_i^{n+1}$, $\left[\overline{T}_{xx}\right]_i^{n+1}$ va hokozolarni (x_i, t_n) nuqta atrofida Teylor qatoriga yoysak, quyidagiga ega bo'lamiz:

$$\left[\overline{T}_t\right]_i^n-0,5\Delta t\left[\overline{T}_{tt}\right]_i^n+\frac{\Delta t^2}{6}\left[\overline{T}_{t^3}\right]_i^n+...-\alpha\left[\overline{T}_{xx}\right]_i^n+$$

$$\Delta t\left[\overline{T}_{xxt}\right]_i^n+0,5\Delta t^2\left[\overline{T}_{xxtt}\right]_i^n+...+$$

$$+\frac{\Delta x^2}{12}\left(\left[\overline{T}_{x^4}\right]_i^n+\Delta t\left[\overline{T}_{x^4t}\right]_i^n+...\right)+\frac{\Delta x^4}{360}\left(\left[\overline{T}_{x^4}\right]_i^n+...\right)...=0 \qquad (3.17)$$

Agar $\overline{T}_t=\alpha\overline{T}_{xx}$, $\overline{T}_{tt}=\alpha^2\overline{T}_{x^4}$, $\overline{T}_{ttt}=\alpha^3\overline{T}_{x^6}$, $s=\dfrac{\alpha\Delta t}{\Delta x^2}$, $\Delta x^2=\alpha\Delta t/s$

tengliklardan foydalansak, (3.17) tenglama quyidagi ko'rinishga keladi:

$$\left[\overline{T}_t-\alpha\overline{T}_{xx}\right]_i^n+E_i^n=0. \qquad (3.18)$$

Bu yerda approksimatsiya xatoligi

$$\mathrm{E}_i^n=-0,5\Delta t\left(1+\frac{1}{6s}\right)\ T_{tt}{}_i^n+\frac{\Delta t^2}{3}\left(1+\frac{1}{4s}+\frac{1}{120s^2}\right)\left[\overline{T}_{tttt}\right]_i^n+... \qquad (3.19)$$

Ko'rinib turibdiki $\Delta t\to 0$ da $E_i^n\to 0$, (3.18) tenglama berilgan (3.3) differensial tenglama bilan ustma-ust tushadi. Bu esa (3.10) oshkormas ayirmali sxema (3.3) differensial tenglama bilan kelishilganligini bildiradi. (3.19) ni (3.13) bilan solishtirib, shuni aytish mumkinki oshkormas ayirmali sxemada $O(\Delta x^4)$ tartib bilan taminlovchi s ni (3.19) dan topish mumkin emas.

§ 3.3. Turg'unlik.

Xususiy hosilali differensial tenglamalarni dikretlashtirishda hosil bo'ladigan algebraik tenglamalar sistemasini yechishda $(x_i,t_n),(i=\overline{1,N_1}),n=\overline{1,N_2}$ tugun nuqtadagi xatolikni ξ_i^n bilan belgilaymiz.

$$\xi_i^n = T_i^n - {*T_i^n} \qquad (3.20)$$

(3.20) da T_i^n, $*T_i^n$ lar mos xolda algebraik tenglamalar sistemasining aniq va taqribiy yechimlaridir.

Diskretlashtirishda hosil bo'ladigan chiziqli algebraik tenglamalar sistemasining xatoliklari ξ_i^n ham xuddi shu chiziqli algebraik tenglamalar sistemasini qanoatlantiradi. Masalan, (3.9) oshkor ayirmali sxemadan foydalansak, yuqoridagi fikrimiz $*T_i^{n+1}$

$$*T_i^{n+1} = s*T_i^{n+1} + (1-2s)*T_i^n + s*T_{i+1}^n \qquad (3.21)$$

tenglamani qanoatlantirishini anglatadi.

Agar algebraik tenglamalar sistemasini aniq yechimi T_i^n ham (3.9) tenglamani qanoatlantirishini eslasak va (3.21), tenglamaga (3.20) ni hisobga olsak ξ_i^n xatolikka nisbatan quyidagi bir jinsli algebraik tenglamaga ega bo'lamiz:

$$\xi_i^{n+1} = s\xi_{i-1}^n + (1-2s)\xi_i^n + s\xi_{i+1}^n \qquad (3.22)$$

Agar boshlang'ich va chegaraviy shartlar berilgan deb faraz qilsak, u holda barcha boshlang'ich xatoliklar ξ_i^0 $(i=\overline{1,N_1-1})$, shuningdek chegaraviy xatoliklar ξ_0^n va $\xi_{N_1}^n (n=0,...N_2)$ lar (3.20) tenglikka asosan nolga teng bo'ladi.

Ayirmali sxemalar turg'unligini tahlil qiluvchi matrisali usul va Neyman usullari eng ko'p qo'llaniladigan usullardir. Ushbu usullar asosida hisoblash algoritmining haqiqiy yechimi va taqribiy yechimi o'rtasidagi farq yoki xatolikni o'sishi yoki kamayishini bashorat qilish yotadi.

Misol. $\dfrac{\partial u}{\partial t}-\dfrac{\partial u}{\partial x}=\varphi(x,t),\ -\infty<x<+\infty,\ 0\le t\le 1,$ tenglamaga qo'yilgan $u(x,0)=\psi(x),\ -\infty<x<+\infty$ Koshi masalasini aproksimasiya qiluvchi

$$L_h u^{(h)}=\begin{cases}\dfrac{u_m^{n+1}-u_m^n}{\tau}-\dfrac{u_m^n-u_{m-1}^n}{h},\\ u_m^0,\\ m=0,\pm 1,\pm 2,...,\ n=0,1,2,...,\end{cases}\qquad f^{(h)}=\begin{cases}\varphi(x_m,t_n),\\ \psi(x_m),m=0,\pm 1,\pm 2,...,\\ n=0,1,2,...,\end{cases}$$

ayirmali sxemaning turg'unligini tekshiring.

Yechish. Dastlab sxemani $u_m^{n+1}=r\,u_{m+1}^n+(1+r)\,u_m^n,\quad r=\tau/h$ ko'rinishda yozib olamiz va quyidagi normalarni aniqlaymiz:

$$\left\|u^{(h)}\right\|_{U_h}=\max_{m,n}\left|u_m^n\right|,\ \left\|f^{(h)}\right\|_{F_h}=\max_m\left|\psi(x_m)\right|+\max_{m,n}\left|\varphi(x_m,t_n)\right|.$$ Agar $r\le 1$

bo'lsa

$$\left|u_m^{n+1}\right|\le\left|r\,u_{m+1}^n+(1-r)u_m^n+\tau\,\varphi(x_m,t_n)\right|\le r\left|u_{m+1}^n\right|+(1-r)\left|u_m^n\right|+\tau\left|\varphi(x_m,t_n)\right|$$

bo'ladi. Demak

$$\left|u_m^{n+1}\right|\le\max_m\left|u_m^n\right|+\tau\max_{m,n}\left|\varphi(x_m,t_n)\right|.$$

Hosil bo’lgan

$$\max_m \left|u_m^0\right| = \max_m \left|\psi(x_m)\right|,$$

$$\left|u_m^1\right| \le \max_m \left|u_m^0\right| + \tau \max_{m,n} \left|\varphi(x_m, t_n)\right|,$$

$$\left|u_m^2\right| \le \max_m \left|u_m^2\right| + \tau \max_{m,n} \left|\varphi(x_m, t_n)\right|,$$

$$. \; .,$$

$$\left|u_m^n\right| \le \max_m \left|u_m^{n-1}\right| + \tau \max_{m,n} \left|\varphi(x_m, t_n)\right|$$

tengsizliklarni qo’shib, $\max_m \left|u_m^n\right| \le \max_m \left|\psi(x_m)\right| + \tau\, n \max_{m,n} \left|\varphi(x_m, t_n)\right|$ tengsizlikni hosil qilamiz. Natijada

$$\max_{m,n} \left|u_m^n\right| \le \max_m \left|\psi(x_m)\right| + \tau\, N \max_{m,n} \left|\varphi(x_m, t_n)\right| \le K \left(\max_m \left|\psi(x_m)\right| + \max_{m,n} \left|\varphi(x_m, t_n)\right|\right)$$

ekanligi tushunarli, bu yerda $K = \max(1, T)$, $T = \tau N$. Shunday qilib $r \le 1$ bo’lganda ayirmali sxema turg’un va differensial masalani approksimatsiya qiladi. Demak, Laks teoremasiga asosan ayirmali masalaning yechimi differensial masala yechimiga yaqinlashadi.

§ 3.4. Oshkor ayirmali sxema turg’unligini tekshirish uchun matrisali usulni qo’llash.

Matrisali usulni mohiyati xatoliklarni aniqlovchi tenglamalar sistemasini matrisa ko’rinishiga keltiriladi. Undan so’ng mos matrisaning xos qiymatlarini aniqlash orqali turg’unlik tahlil qilinadi.

Ushbu usulni (3.9) oshkor ayirmali sxemaga nisbatan qo’llashni ko’rib o’tamiz.

(3.22) da $i=\overline{1,N_1-1}$ larni qo'yib, chegaraviy xatoliklar barcha n lar uchun $\xi_0^n=\xi_{N_1}^n=0$ ekanligini hisobga olib quyidagi tenglamalar sistemasiga ega bo'lamiz:

$$\xi_1^{n+1}=(1-2s)\xi_1^n+s\xi_2^n$$

$$\xi_2^{n+1}=s\xi_1^n+(1-2s)\xi_2^n+s\xi_3^n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3.23}$$

$$\xi_i^{n+1}=s\xi_{i-1}^n+(1-2s)\xi_i^n+s\xi_{i+1}^n$$

$$\xi_{N_{11}-1}^{n+1}=s\xi_{N_1-2}^n+(1-2s)\xi_{N_1}^n+s\xi_{i+1}^n$$

bu tenglamalar sistemasini esa quyidagi matrisali ko'rinishda yozish mumkin:

$$\xi^{n+1}=A\xi^n,\quad n=0,1,\ldots, \tag{3.24}$$

bu yerda A - N_1-1 tartibli kvadrat matrisa , ξ^n esa N_1-1 ta elementdan iborat ustun vektor.

Ularning ko'rinishi qo'yidagicha:

$$A=\begin{bmatrix}(1-2s) & s & & & \\ s & (1-2s) & s & & \\ & s & & s & \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & s & (1-2s)\end{bmatrix},\quad \xi^n=\begin{bmatrix}\xi_1^n\\ .\\ .\\ .\\ \xi_i^n\\ .\\ .\\ .\\ \xi_{N_1-1}\end{bmatrix}.$$

Agar A matrisaning xos qiymatlari λ_m lar har xil va absolyut qiymatlari birdan kichik yoki teng bo'lsa, yani

$$|\lambda_m| \leq 1 \tag{3.25}$$

barcha m lar uchun bajarilsa n ning ortishi bilan ξ^n xatoliklar chegaralanganligini ko'rsatish mumkin.

Ilmiy manbalardan ma'lumki r diagonalli A matrisaning xos qiymatlari quyidagicha aniqlanadi:

$$\lambda_m = 1 - 4s\sin^2\left(\frac{m\pi}{2(N_1 - 1)}\right), \; m = \overline{1, N_1 - 1} \tag{3.26}$$

(3.25) turg'unlik sharti bo'lib, faqatgina quyidagi tengsizlikini qanoatlantiruvchi s ni qiymatidan foydalanishni taqozo etadi!

$$-1 \leq 1 - 4s\sin^2\left(\frac{m\pi}{2(N_1 - 1)}\right) \leq 1 \tag{3.27}$$

Ushbu tengsizlikning o'ng qismi barcha m va s larda bajariladi. Ammo, tengsizlikning chap qismi bajarilishi uchun

$$s \cdot \sin^2\left(\frac{m\pi}{2(N_1 - 1)}\right) \leq \frac{1}{2},$$

bo'lishi zarur. Bu esa $s \leq \dfrac{1}{2}$ bo'lganda barcha m lar uchun bajariladi.

Yuqoridagi fikrdan (3.9) oshkor ayirmali sxema $s = \dfrac{\alpha\Delta t}{\Delta x^2} \leq \dfrac{1}{2}$ da turg'un ekan.

§ 3.5. Ikki qatlamli ayirmali sxemani turg'unligini tekshirishda matrisali usulni qo'llash.

Endi ikki qatlamli ayirmali sxemani turg'unligini tekshirishda matrisali usuldan foydalanishni ko'rib o'tamiz. Ushbu ayirmali sxema quyidagi ko'rinishda bo'lsin:

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}-\alpha\beta L_{xx}T_i^{n+1}-\alpha(1-\beta)L_{xx}T_i^n=0, \qquad (3.28)$$

bu yerda

$$L_{xx}T_i=\left(T_{i-1}-2T_i+T_{i+1}\right)/\Delta x^2 .$$

(3.28) tenglamada α ni issiqlik o'tkazuvchanlik koeffisiyenti deb, β ni oshkormaslik darajasini xarakterlovchi parametr deb talqin qilish mumkin. ξ_i^n xatolik (3.28) tenglamaga ekvivalent bo'lgan ushbu tenglama orqali aniqlanadi.

$$-s\xi_{i-1}^{n+1}+\left(1+2s\beta\right)\xi_i^n-s\beta\xi_{i+1}^{n+1}=s\left(1-\beta\right)\xi_{i-1}^n+$$

$$+\left(1-2s(1-\beta)\right)\xi_i^n+s\left(1-\beta\right)\xi_{i+1}^n . \qquad (3.29)$$

(3.29) tenglama ichki tugun nuqtalar uchun o'rinli. Agar chegaraviy shart sifatida Dirixle sharti qo'yilgan bo'lsa, u holda chegaraviy nuqtalarda bu tenglamaga ehtiyoj qolmaydi.

(3.29) tenglamani barcha $i=\overline{1,N_1-1}$ tugun nuqtalar uchun takrorlab, hosil bo'lgan tenglamalar sistemasini quyidagi matrisali ko'rinishda yozish mumkin:

$$A\xi^{n+1}=B\xi^n ,$$

bu yerda

$$A = \begin{bmatrix} 1+2s & -s\beta & & & \\ -s\beta & 1+2s\beta & -s\beta & & \\ \multicolumn{5}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ & & -s\beta & 1+2s\beta & -s\beta \\ & & & -s\beta & (1+2s\beta) \end{bmatrix},$$

$$B = \begin{bmatrix} 1-2s(1-\beta) & s(1-\beta) & & & \\ s(1-\beta) & 1-2s(1-\beta) & s(1-\beta) & & \\ \multicolumn{5}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ & & s(1-\beta) & 1-2s(1-\beta) & s(1-\beta) \\ & & & s(1-\beta) & 1-2s(1-\beta) \end{bmatrix}.$$

Agar $A^{-1}B$ matrisaning xos qiymatlari moduli birdan katta bo'lmasa, (3.28) hisoblash algoritmi turg'un bo'ladi. A va B matrisalarning tuzilishidan kelib chiqib, ushbu shart quyidagi cheklanishga ekvivalent bo'ladi:

$$\text{УСТ} = \left| (\lambda_B)_m \,/\, (\lambda_A)_m \right| \le 1. \qquad (3.30)$$

A va B matrisalar uch diagonalli simmetrik xarakterga ega ekanligini hisobga olinib, ularning xos qiymatlari adabiyotlarda quyidagi analitik ko'rinishlarda keltirilgan:

$$\lambda_A = 1 + 2s\beta - 2s\beta\cos\left(\frac{i\pi}{N_1 - 1}\right) = 1 + 4s\beta \cdot \sin^2\left(\frac{i\pi}{2(N_1 - 1)}\right),$$

$$\lambda_B = 1 - 2s(1-\beta) - 2s(1-\beta)\cos\left(\frac{i\pi}{N_1 - 1}\right) = 1 - 4s(1-\beta) \cdot \sin^2\left(\frac{i\pi}{2(N_1 - 1)}\right)$$

Xususan turg'unlik sharti quyidagicha:

$$УСТ=\left|\frac{1-4s(1-\beta)\sin^2 \ i\pi/2(N_1-1)}{1+4s\beta\sin^2 \ i\pi/2(N_1-1)}\right|\leq 1.$$

Agar $\sin\left(\frac{i\pi}{N_1-1}\right)=0$ bo'lsa $УСТ=1$,

$$\sin\left(\frac{i\pi}{N_1-1}\right)=1 \text{ bo'lsa } УСТ=\left|\frac{1-4s(1-\beta)}{1+4s\beta}\right|\leq 1 \quad .$$

Turg'unlik sharti bajarilishi uchun $1-4s(1-\beta)<1+4s\beta$ bo'lishi kerak va u bajariladi. Shuningdek $1-4s(1-\beta)\geq -1-4s\beta$ yoki $2>4s(1-2\beta)$ bo'lishi talab etiladi. Oxirgi tengsizlik $s\leq 0,5/(1-2\beta)$ tengsizlikga ekvivalent. Bu esa $\beta<0,5$ bo'lganda bajariladi.

Xususan, agar $\beta<0,5$ bo'lsa, turg'unlik uchun $s\leq 0,5/(1-2\beta)$ talab etiladi. Agar $\beta\geq 0,5$ bo'lsa, $2>4s(1-2\beta)$ tengsizlik qiyinchiliksiz bajariladi va bu xolda (3.28) hisoblash algoritmi shartsiz turg'un bo'ladi.

Quyidagi ko'rinishdagi Koshi masalasini qaraymiz

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0, \quad -\infty<x<+\infty \tag{3.1.1}$$

$$u(x,0)=u_0(x), \quad -\infty<x<+\infty, \tag{3.1.2}$$

bu yerda $a=const>0$, $u_0(x)-$berilgan funksiya. (3.1.1)-(3.1.2) masalani quyidagicha approksimatsiya qilamiz:

$$\frac{u_j^{n+1}-\left(u_{j-1}^n+u_{j+1}^n\right)/2}{\tau}+a\frac{u_{j+1}^n-u_{j-1}^n}{2h}=0; \tag{3.1.3}$$

$$u_j^0=u_0(jh),\ j=0,\pm1,\pm2,...;\ n=0,1,2....$$

(3.1.3) ayirmali sxema Laks sxemasi deb ataladi.

Misol. Ushbu ayirmali sxemaning turg'unligini tekshiring. (3.1.3) sxemani $a<0$ bo'lgan hol uchun tadqiq qiling.

Yechish. (3.1.3) sxemada u_j^n yechim o'rniga $u_j^n=\lambda^n e^{ijkh}, i=\sqrt{-1}$ ko'rinishdagi ifodani qo'yamiz. Natijada (3.1.3) tenglamadan

$$\lambda^n e^{ijkh}\left[\frac{\lambda-\left(e^{-ikh}+e^{ikh}\right)/2}{\tau}+a\frac{e^{ikh}-e^{-ikh}}{2h}\right]=0 \qquad (3.1.4)$$

tenglikni hosil qilamiz. Biz $\lambda\neq 0$ notrivial yechimni izlaganligimiz uchun (3.1.4) tenglikni $\lambda^n e^{ijkh}$ ga bo'lib,

$$\frac{\lambda-\left(e^{-ikh}+e^{ikh}\right)/2}{\tau}+a\frac{e^{ikh}-e^{-ikh}}{2h}=0 \qquad (3.1.5)$$

ega bo'lamiz. Soddalik uchun $\xi=kh$ belgilashni kiritamiz va $e^{\pm i\xi}=\cos\xi\pm i\sin\xi$ Eyler formulasidan foydalanamiz. Yana $k=a\tau/h$ belgilashni kiritamiz. Natijada (3.1.3) Laks sxemasida o'tish ko'paytuvchisi λ uchun:

$$\lambda=\cos\xi-ik\sin\xi \qquad (3.1.6)$$

tenglikni aniqlaymiz. Fon Neyman usuliga asosan $|\lambda|$ ni hisoblaymiz:

$$|\lambda|^2=cos^2\xi+k^2\sin^2\xi=1-\sin^2\xi+k^2\sin^2\xi=1-\sin^2\xi\left(1-k^2\right). \qquad (3.1.7)$$

Turg'unlik sharti $|\lambda|\leq 1$ ga ko'ra k ni aniqlaymiz:

$$1-\sin^2\xi\left(1-k^2\right)\leq 1;\ -\sin^2\xi\left(1-k^2\right)\leq 0;\quad \sin^2\xi\left(1-k^2\right)\geq 0;\quad 1-k^2\geq 0;$$

$$(1-k)(1+k)\geq 0.$$

(3.1.8) tengsizlik bajarilishi uchun k Kurant soni

$$-1\leq k\leq 1 \qquad (3.1.8)$$

tengsizlikni qanoatlantirishi kerak. (3.1.8) tengsizlikka asosan Laks sxemasi $a<0$ bo'lgan holda ham qo'llanishi mumkin, faqat berilgan a va h lar uchun τ ni shunday tanlash kerakki $k=a\tau/h$ qaralayotgan sxemaning (3.1.8) turg'unlik sohasiga tushishsin.

§ 3.6. Oshkor ayirmali sxema turg'unligini tadqiq qilishda Neyman usulini qo'llash.

Turg'unlik shartini aniqlashda eng ko'p qo'llaniladigan usul – bu Neyman usuli bo'lib, uni qo'llash uchun juda qulay. Ammo, bu usulni faqatgina o'zgarmas koeffisentli chiziqli masalalar turg'unligining zaruriy shartlarini aniqlashda qo'llash mumkin.

Neyman usuli mohiyati shundaki, hisoblash algoritmining bir vaqt qatlamidagi xatoliklari chekli Furye qatoriga yoyiladi. Bir vaqt qatlamidan ikkinchi vaqt qatlamiga o'tishda Furye qatorining alohida komponentalari o'sishi yoki kamayishiga qarab hisoblash algortmining turg'unmasligi yoki turg'unligi haqida fikr yuritiladi.

Neyman usuli faqatgina to'rning ichki nuqtalari uchun qo'llanilishini eslatish foydali.

Shunday qilib, hisoblash algorimtining boshlang'ich xatolik vektori ε_i^0 ni Furyening chekli kompleks qatori ko'rinishida ifodalaymiz:

$$\varepsilon_j^0=\sum_{m=1}^{N_1-1}a_m e^{i\theta_m j}\ ,\ j=\overline{1,N_1-1}, \qquad (3.31)$$

bu yerda

$$i=\sqrt{-1}\ \text{va}\ \theta_m=m\pi\Delta x\,.$$

(3.31) da mavhum birlik i qatnashganligi uchun xatolik ε_i dagi i indeksi j ga o’zgartirdik.

Hisoblash algoritmining xatoligi (3.22) chiziqli ekanligini hisobga olsak, (3.31) Furye qatoridagi yagona bitta hadining xatoligi $e^{i\theta_m j}$ ni o’rganish yetarli. Bundan keyin θ_m dagi m indeksni tushirib qoldiramiz.

(3.31) tenglikni hisobga olib, (3.22) hisoblash algoritmining xatoligini o’zgaruvchilarni ajratish usuli bo’yicha izlaymiz:

$$\varepsilon_j^n = (G)^n e^{i\theta j} \ , \tag{3.32}$$

bu formulada Furye komponentining vaqtga bog’liqligi kompleks koeffisent $(G)^n$ ni ichiga kiritib yuborilgan. $(G)^n$ da n indeks G miqdorni n chi darajasini bildiradi.

(3.32) ifodani (3.22) tenglamaga qo’yib, quyidagiga ega bo’lish mumkin:

$$(G)^{n+1} e^{i\theta j} = s\cdot(G)^n e^{i\theta(j-1)} + \ 1-2s\ (G)^n e^{i\theta j} + s\cdot(G)^n e^{i\theta(j+1)}$$

Oxirgi tenglikni soddalashtirib, G miqdorni qiymatini topish mumkin:

$$G = 1 - 4s\cdot\sin^2\ \theta/2 \tag{3.33}$$

G miqdorni hisoblash algoritmi xatoligini bir vaqt qatlamidan keyingi vaqt qatlamiga o’tishini ta’minlovchi koeffisiyent sifatida baholash mumkin. Buni esa (3.32) tenglik ham tasdiqlab turibdi:

$$\varepsilon_j^{n+1} / \varepsilon_j^n = G, \tag{3.34}$$

Shuni ham ta'kidlash mumkinki, G miqdor s va θ o'zgaruvchilarning funksiyasidir: $G(s,\theta)$.

Shuningdek, $G(s,\theta)$ to'r yacheykalari o'lchamlari Δx va Δt larga bog'liq bo'lib, u aynan Furye qatorining qaysi komponentasi qaralayotganligini ham anglatadi. Ushbu mulohazalar $s=\alpha\Delta t/\Delta x^2$ va $\theta=m\pi\Delta x$ ekanligidan o'z tasdig'ini topadi.

Agar G ning absolyut miqdori Furye qatorning ixtiyoriy garmonikasi G da birdan oshmasa hisoblash algoritmining (3.32) xatoliklari cheklangan bo'ladi.

Shunday qilib, turg'unlikning umumiy talabi quyidagi shartga keltirildi.

$$|G|\le 1 \text{ ixtiyoriy } \theta \text{ da .} \tag{3.34}$$

Demak, (3.33) ga asosan turg'unlik sharti (oshkor ayirmasi sxema uchun) ixtiyoriy θ da

$$-1\le 1-4s\cdot\sin^2(\theta/2)\le 1 \tag{3.35}$$

munosabatdan iborat ekan. Bu esa $s\le\frac{1}{2}$ bo'lsa o'rinli ekanligi ko'rinib turibdi. Ushbu natija matrisa usuli bilan olingan natija bilan bir xil.

§ 3.7. Ikki qatlamli oshkormas ayirmali sxemani turg'unligini tadqiq qilishda Neyman usulini qo'llash.

Endi Neyman usulini ikki qatlamli oshkormas ayirmali sxemaning turg'unligini tahlil qilishda qo'llaymiz.

Hisoblash algoritmi xatoligi ayirmali sxemani qanoatlantiradi:

$$\frac{\varepsilon_j^{n+1}-\varepsilon_j^n}{\Delta t}-\alpha\beta L_{xx}\varepsilon_j^{n+1}-\alpha(1-\beta)L_{xx}\varepsilon_j^n=0, \qquad (3.36)$$

bu yerda $L_{xx}\varepsilon_j=\frac{\varepsilon_{j-1}-2\varepsilon_j+\varepsilon_{j+1}}{\Delta x^2}$, α–koeffisiyent, β esa ayirmali sxemani oshkormaslik darajasini belgilovchi parametr.

Bu yerda ham turg'unlikni Neyman usuli bo'yicha tahlil qilishda (3.31) Furye qatoridan foydalanamiz. (3.36) tenglama chiziqli bo'lganligi bois ε_j^n ni ifodasida bitta Furye komponentasi kiritiladi, ya'ni

$$\varepsilon_j^n=(G)^n e^{i\theta j}, \qquad (3.37)$$

bu yerda $i=\sqrt{-1}$, j– to'rning tugun nomeri.

$$L_{xx}\varepsilon_j^n=(G)e^{i\theta(j-1)}-2(G)^n e^{i\theta j}+(G)^n e^{i\theta(j+1)}=(G)^n e^{i\theta j}\,\frac{2(\cos\theta-1)}{\Delta x^2}$$

almashtirishdan foydalansak, (3.36) tenglama quyidagi ko'rinishga keladi:

$$\frac{(G)^{n+1}e^{i\theta j}-(G)^n e^{i\theta j}}{\Delta t}-\alpha\beta(G)^{n+1}e^{i\theta j}\,\frac{2(\cos\theta-1)}{\Delta x^2}+$$

$$+\alpha(1-\beta)(G)^n e^{i\theta j}\,\frac{2(\cos\theta-1)}{\Delta x^2}=0.$$

Tenglikning ikki tomonini $(G)^n e^{i\theta j}$ ga bo'lamiz:

yoki

$$G=\frac{1-4s(1-\beta)\sin^2(\theta/2)}{1+4s\beta\sin^2(\theta/2)}$$

ni aniqlaymiz.

Agar $G \le 1$ ixtiyoriy θ ning qiymatida bajarilsa, sxema turg'un bo'ladi.

Agar $\sin(\theta/2)=0$ bo'lsa, $G=1$ bo'ladi, agar $\sin(\theta/2)=1$ bo'lsa, $G=\left(1-4s(1-\beta)\right)/\left(1+4s\beta\right)$ bo'ladi, agar $G \le 1$ bo'lsa, u holda $1-4s(1-\beta) \le 1+4s\beta$ yoki $s>0$ bo'ladi.

Agar $G>-1$ bo'lsa, u holda $1-4s(1-\beta) \le -1-4s\beta$ yoki $s \le 0,5/(1-2\beta)$ bo'ladi.

Ushbu mulohazalar matrisali usulda olingan natijalar bilan ustma-ust tushadi.

Misol. $\frac{\partial u}{\partial t}+c\frac{\partial v}{\partial x}=0, \quad \frac{\partial v}{\partial t}+c\frac{\partial u}{\partial x}=0, \quad c=const>0$ tenglamalar sistemasini Laks-Vendrof sxemasi bilan approksimatsiya qilamiz:

$$w_j^{n+1}-w_j^n+\frac{\tau}{2h}A\left(w_{j+1}^n-w_{j-1}^n\right)-\frac{\tau^2}{2h^2}A^2\left(w_{j+1}^n-2w_{j-1}^n+w_{j-1}^n\right)=0 \qquad (3.2.1)$$

bu yerda

$$w=\begin{pmatrix} u \\ v \end{pmatrix}, \quad A=\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \qquad (3.2.2)$$

Furye usulidan foydalanib (3.2.1) sxemaning turg'unligini isbot qiling. Bu sxemaning turg'unligi uchun yetarli bo'ladimi?

Yechish. Furye usuliga asosan (3.2.1) ifodaga $w_j^n=W_0\lambda^n e^{ijkh}$ yechimni qo'yamiz, bu yerda $i=\sqrt{-1}$, $k-$ haqiqiy son, W_0- o'zgarmas vektor. Tenglikning har ikki tomonini e^{ijkh} ga bo'lib, W_0 ni aniqlash uchun birjinsli $(G-\lambda I)W_0=0$ sistemani hosil qilamiz, bu yerda G (3.2.1) sxemaning o'tish matrisasi:

$$G=I-\left(\frac{\tau}{h}i\sin\xi\right)A+\frac{\tau^2}{h^2}(\cos\xi-1)A^2 \qquad (3.2.3)$$

Bu yerda I- birlik matrisa, $\xi=kh$. (3.2.3) ifodadan ko'rinib turibtiki G matrisa (3.2.2) tenglik bilan aniqlanadigan A matrisaning ko'phadi. Shuning uchun G matrisaning λ_1, λ_2 xos sonlari

$$\lambda_k=1-\frac{\tau}{h}i\sin\xi\mu_k+\frac{\tau^2}{h^2}(\cos\xi-1)\mu_k^2, \quad k=1,2 \qquad (3.2.4)$$

bu yerda μ_1,μ_2- A matrisaning xos sonlari. Ular $\det(A-\mu I)=0$ tenglamadan topiladi:

$$\begin{vmatrix} -\mu & c \\ c & -\mu \end{vmatrix}=\mu^2-c^2=0\Rightarrow \mu_1=c,\ \mu_2=-c.$$

Shunday qilib, μ_1,μ_2-ildizlar haqiqiy sonlar ekan. Shuning uchun

$$|\lambda_k|^2=\left(1-\frac{\tau^2}{h^2}2\sin^2\frac{\xi}{2}\mu_k^2\right)^2+\left(\frac{\tau^2}{h^2}\sin^2\xi\right)\mu_k^2=1-4\frac{\tau^2}{h^2}\sin^2\frac{\xi}{2}\mu_k^2+4\frac{\tau^4}{h^4}\sin^4\frac{\xi}{2}\mu_k^4+$$

$$+4\frac{\tau^2}{h^2}\sin^2\frac{\xi}{2}\cos^2\frac{\xi}{2}\mu_k^2=1-4\frac{\tau^2}{h^2}\sin^4\frac{\xi}{2}\mu_k^2+4\frac{\tau^4}{h^4}\sin^4\frac{\xi}{2}\mu_k^4= \qquad (3.2.5)$$

$$=1-4\frac{\tau^2}{h^2}\left(\sin^4\frac{\xi}{2}\right)\mu_k^2\left(1-\frac{\tau^2}{h^2}\mu_k^2\right).$$

(3.2.5) tenglikdan tushunarliki fon Neyman zaruriy shartining bajarilishi uchun vaqt bo'yicha τ qadam

$$\frac{\tau}{h}c\le 1 \qquad (3.2.6)$$

shartni qanoatlantirishi kerak. Endi (3.2.6) shart Laks-Vendrof sxemasining turg'unligi uchun yetarli shart bo'lishi mumkin yoki mumkinmasligini tekshiramiz. Buning uchun G matrisaning normal matrisaligini, ya'ni G^*G va GG^* matrisalar ko'paytmasining tengligini tekshirishimiz lozim. Lekin bushbu jarayon ancha murakkab bo'lganligi sababli, biz matrisalar nazariyasida isbot qilingan quyidagi tasdiqdan foydalanamiz. Agar G matrisa haqiqiy yoki kompleks koeffitsiyentli simmetrik haqiqiy matrisaning matrisali ko'phadidan iborat bo'lsa, u holda G matrisa normal bo'ladi. Natijada (3.2.3) tenglikdagi G matrisa normal bo'ladi va (3.2.6) shart (3.2.1) Laks-Vendrof sxemasining turg'unligi uchun yetarli bo'ladi.

§ 3.8. Yechimning yaqinlashishi va aniqligi.

Berilgan xususiy hosilani differensial tenglamani approksimatsiya qiluvchi algebraik tenglamalar sistemasining yechimi yaqinlashuvchi deyiladi, agar taqribiy yechim xususiy hosilali differensial tenglamaning aniq yechimiga erkli o'zgaruvchining ixtiyoriy qiymatida yaqinlashsa, yoki hyech bo'lmaganda to'r yacheykalarining o'lchamlari nolga intilganda taqribiy yechim berilgan xususiy xosilali differensial tenglama yechimiga yaqinlashsa. Shunday qilib, taqribiy yechim T_i^n - xususiy hosilali differensal tenglamaning aniq yechimi $\overline{T}(x_i,t_n)$ ga yaqinlashadi deyiladi agarda $\Delta x \to 0$, $\Delta t \to 0$, da $T_i^n \to \overline{T}(x_i,t_n)$ bo'lsa.

Xususiy hosilali differensial tenglama aniq yechimi va algebraik tenglamalar sistemasining aniq yechimi o'rtasidagi ayirma yechimning xatoligi deyiladi va e_i^n bilan belgilanadi:

Algebraik tenglamalar sistemasining aniq yechimi xususiy hosilali differensial tenglama uchun taqribiy yechim hisoblaniladi. Algebraik tenglamalar sistemasining yechimi aniq deyiladi, agarda hisoblash jarayonida biron bir sonli xatolikka yo'l qo'yilmagan bo'lsa, masalan, yaxlitlash turidagi xatolikka. Odatda e_i^n xatolik miqdori (i,n) tugun nuqtada to'r yacheykasining o'lchamlari Δx va Δt ga hamda differensial tenglamani approksimatsiya qilishda tashlab yuboriladigan yuqori tartibli hosilalarni shu tugun nuqtadagi qiymatlariga bog'liq bo'ladi.

Algebraik tenglamalar sistemasining yechimi berilgan xususiy hosilali differensial tenglamaning aniq yechimiga yaqinlashishini hattoki oddiy masala uchun ham isbotlash ancha mushkul. Shu sababli ayrim masalalarni yaqinlashishini baholashda Laksning ekvivalentlik haqidagi teoremasidan foydalaniladi:

Teorema: Agar boshlang'ich shartli korrekt qo'yilgan chiziqli masala va uni approksimatsiyalovchi kelishilganlik shartini qanoatlantiruvchi chekli ayirmali masala mavjud bo'lsa, u holda turg'unlik yaqinlashishning zaruriy va yetarli shartidir.

Shunday qilib, yechimning yaqinlashishi va aniqligini o'rganish ayirmasi sxemaning approksimatsiya xatoligini va turg'unligini o'rganishga keltiriladi.

Xulosa sifatida shuni aytish mumkinki, agar ayirmali sxema berilgan differensial masalani approksimatsiya qilsa va turg'un bo'lsa ayirmali sxema yaqinlashadi deyiladi (odatda approksimatsiya va turg'unlikdan yaqinlashish kelib chiqadi deyiladi). Shuningdek aniqlik tartibi (yaqinlashish tezligi) ayirmali sxemaning approksimatsiya tartibi bilan aniqlanadi.

1-misol. Ayirmali sxemalardan foydalanib $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ parabolik tenglamaning $u(x,0)=3x(1-x)+0.12$ boshlang'ich shartni va $u(0,t)=2(t+0,06),\quad u(0,6;t)=0,84$ chegaraviy qiymatlarni qanoatlantiruvchi qiymatlari topilsin. Bunda $x\in[0;0,6]$, $t\in[0;0,01]$, $h=0,1$, $\sigma=1/6$. Nostasionar masalalarning yechimi funksiyaning $u(x_i,t_j)$ qiymatidan foydalanib, $u(x_i,t_{j+1})$ qiymatini topish orqali amalga oshiriladi, bunda $t_{j+1}=t_j+k,\ k=h^2/6=0,01/6=0,0017$.Hisoblashla $u_{i,j+1}=\frac{1}{6}(u_{i+1,j}+4u_{i,j}+u_{i-1,j})$ $(i=1,2,3,4,5,6;\quad j=1,2,3,4,5,6)$ formula bilan hisoblanadi. Barcha hisoblash natijalari quyidagi jadvalda keltirilgan:

j	i	0	1	2	3	4	5	6
	t_j \ x_i	0	0,1	0,2	0.3	0.4	0.5	0.6
0	0	0,12	0,39	0,60	0,75	0,84	0,87	0,84
1	0,0017	0,1233	0,3800	0,5900	0,7400	0,8300	0,8600	0,84
2	0,0033	0,1267	0,6372	0,5800	0,7300	0,8200	0,8500	0,84
3	0,0050	0,1300	0,3659	0,5704	0,7200	0,8103	0,8445	0,84
4	0,0067	0,1333	0,3607	0,5612	0,7101	0,8010	0,8380	0,84
5	0,0083	0,1367	0,3562	0,5526	0,7004	0,7920	0,8322	0,84
6	0,01	0,1400	0,3524	0,5445	0,6910	0,7834	0,8268	0,84

2-misol. Ayirmali sxemadan foydalanib $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ tor tebranish tenglamasining $u(x,0) = 2x(1-x^2),\ u_t(x,0) = (x+0.4)\cos(x+0,3)$ boshlang'ich va $u(0,t) = 0,5t^2,\ u(1,t) = 0$ chegaraviy shartlarni qanoatlantiruvchi sonli yechimini toping, bunda $h = 0.1,\quad 0 \le t \le 0,5$.

Yechish. $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1},\ i = 1,2\ldots,\ j = 1,2,3\ldots$ munosabatdan masalani yechish uchun foydalanamiz. Bunda $u_{i,0} = f_i$ bo'lib, $u_{i,1}$ ni masalan quyidagi usuldan foydalanib aniqlash mumkin:

$u_{i,1} = \frac{1}{2}(f_{i+1} + f_{i-1}) + h\Phi_i,\qquad x_i = 0 + ih\ (i = 0,1,2,\ldots,n),\ n = \frac{1-0}{h} = 10,$

$t_j = 0 + jh\ (i = 0,1,2,3,4,5),\qquad u_{0,j} = 0.5t_j^{\ 2};\ u_{n,j} = 0.$ Ushbu formulalar yordamida izlanayotgan funksiyaning to'r tugun nuqtalaridagi qiymatlarini topib, jadvalni quyidagi algoritm asosida to'ldiramiz:

- $x_i = 0,1i$ da $u_{i,0} = f(x_i) = 2x_i(1 - x_i^{\ 2})$ qiymatlarni topamiz va ularni jadvalning birinchi qatoriga yozamiz (ular $t_0 = 0$ ga mos keladi).
- $t_j = 0,1$ da $u_{0,j} = 0,5t_j^2$ qiymatlarni birinchi ustunga joylashtiramiz (ular $x_0 = 0$ ga mos keladi).
- $u_{10,j} = 0$ qiymatlarni oxirgi ustunga jaylashtiramiz (ular $x_{10} = 1$ ga mos keladi).
- $u_{i,1} = \frac{1}{2}(f_{i+1} + f_{i-1}) + h\Phi_i$ formula yordamida $u_{i,1}$ qiymatlarni hisoblaymiz, bu yerda f_{i+1} va f_{i-1} jadvalning birinchi qatoridan olinadi, $\Phi_i = (x_i + 0,4)\cos(x_i + 0,3);\quad x_i = 0,1i\ (i = 1,2,...,9);\ h = 0.1$ hisoblanib, ikkinchi qatorga yozamiz.

• Navbatdagi qatorlardagi $u_{i,j}$ qiymatlarni $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$ formuladan foydalanib topamiz, bu yerda $u_{i+1,j}, u_{i-1,j}, u_{i,j-1}$ qiymatlar jadvalning oldingi qatorlaridan olinadi.

x_i \ t_j	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
0	0	0,198	0,384	0,546	0,672	0,750	0,768	0,714	0,576	0,342	0
0,1	0,005	0,2381	0,4247	0,5858	0,7092	0,7677	0,7942	0,7315	0,5825	0,3354	0
0,2	0,02	0,2317	0,4399	0,5879	0,6815	0,7534	0,7312	0,6627	0,4909	0,2405	0
0,3	0,045	0,2218	0,3949	0,5356	0,6321	0,6450	0,6219	0,4906	0,3207	0,1555	0
0,4	0,08	0,2082	0,3175	0,4391	0,4991	0,5006	0,4044	0,2799	0,1552	0,0802	0
0,5	0,125	0,1757	0,2524	0,2810	0,3076	0,2585	0,1586	0,6090	0,0394	-0,0003	0

3- misol. $\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$ Laplas tenglamasiga uchlari A(0;0), B(0;1),C(1;1),D(1;0) nuqtalarda bo’lgan kvadratga $u|_{AB} = 45y(1-y)$, $u|_{BC} = 25x$, $u|_{CD} = 25$, $u|_{AD} = 25x\sin\frac{\pi x}{2}$ shartlar bilan qo’yilgan Dirixle ayirmali masalasini $h = 0{,}2$ qadam bilan to’r usuli yordamida taqribiy yeching.

Masalani yechish uchun Libman iterasiya usulidan foydalanamiz. Buning uchun $h = 0{,}2$ qadam bilan berilgan sohada to’r quramiz va chegaraviy nuqtalarda izlanayotgan funksiyaning qiymatlarini hisoblaymiz. AB kesmada funksiyaning qiymatini $u(x,y) = 45y(1-y)$ funksiya orqali hisoblaymiz, ya’ni

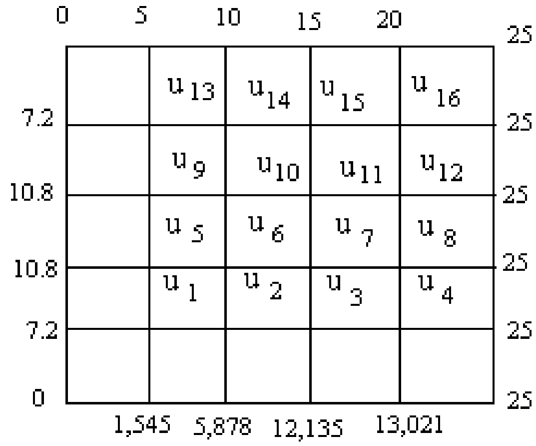
$$u(0{,}0) = 0; u(0;0.2) = 7.2; u(0;0{,}4) = 10.8; u(0;0{,}6) = 10.8; u(0;0{,}8) = 7{,}2; u(0;1) = 0,$$

BC tomonda: $u(x,y)=25x$ funksiya orqali

$u(0{,}2;1)=5,\ u(0{,}4;1)=10,\ u(0{,}6;1)=15,\ u(0.8;1)=20,\ u(1;1)=25$ qiymatlarni,

CD tomonda: $u(x,y)=25$ funksiya orqali

$u(1;0.8)=u(1;0.6)=u(1;0.4)=u(1;0.2)=u(1;0)=25$ qiymatlarni,



1-расм

AD tomonda $u(x,y)=25x\sin\frac{\pi x}{2}$ funksiya orqali

$u(0{,}2;0)=1.545;\ u(0{,}4;0)=5.878;\ u(0{,}6;0)=12.135;\ u(0.8;0)=19.021;\ u(1;1)=25$

qiymatlarni topamiz.

To'rning ichki nuqtalarida funksiyaning qiymatlarini topish uchun Laplas tenglamasini quyidagi ayirmali nisbat bilan almashtiramiz: $u_{i,j}=u(x_i,y_j)=\frac{1}{4}\left(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}\right)$. Ushbu formuladan va berilgan chegaraviy qiymatlardan foydalanib to'rning ichki har bir nuqtasi uchun funksiyani qiymatini hisoblash formulalarini yozamiz:

$$u_1=\frac{1}{4}\left(7.2+1.545+u_2+u_5\right);\quad u_2=\frac{1}{4}\left(5.878+u_1+u_3+u_6\right);$$

$$u_3=\frac{1}{4}\left(12.135+u_2+u_4+u_7\right);\quad u_4=\frac{1}{4}\left(19.021+25+u_3+u_8\right);$$

$$u_5=\frac{1}{4}(0.8+u_1+u_6+u_9);\ u_6=\frac{1}{4}(u_2+u_5+u_7+u_{10});\ u_7=\frac{1}{4}(u_3+u_6+u_8+u_{11});$$

$$u_8=\frac{1}{4}(25+u_4+u_7+u_{12});\ u_9=\frac{1}{4}(0.8+u_5+u_{10}+u_{13});\ u_{10}=\frac{1}{4}(u_6+u_9+u_{11}+u_{14});$$

$$u_{11}=\frac{1}{4}(u_7+u_{10}+u_{12}+u_{15});\ u_{12}=\frac{1}{4}(25+u_8+u_{11}+u_{16});\ u_{13}=\frac{1}{4}(2.2+5+u_9+u_{14});$$

$$u_{14}=\frac{1}{4}(0+u_{10}+u_{13}+u_{15});\ u_{15}=\frac{1}{4}(5+u_{11}+u_{14}+u_{16});\ u_{16}=\frac{1}{4}(20+25+u_{12}+u_{15}).$$

Endi bu sistema yechimini Zeydel iterasion usuli bilan topamiz. Har bir qiymat $u_i^{(0)},u_i^{(1)},u_i^{(2)},\ldots,u_i^{(k)},\ldots$ ketma-ketlik hadlari uchun formulalarni beramiz:

$$u_1^k=\frac{1}{4}(3.745+u_2^{(k-1)}+u_5^{(k-1)});\ u_2^{(k)}=\frac{1}{4}(6.878+u_1^{(k)}+u_3^{(k-1)}+u_6^{(k-1)});$$

$$u_3^{(k)}=\frac{1}{4}(2.135+u_2^{(k)}+u_4^{(k-1)}+u_7^{(k-1)});\ u_4^{(k)}=\frac{1}{4}(44.021+u_3^{(k)}+u_8^{(k-1)});$$

$$u_5^{(k)}=\frac{1}{4}(0.8+u_1^{(k)}+u_6^{(k-1)}+u_9^{(k-1)});\ u_6^{(k)}=\frac{1}{4}(u_2^{(k)}+u_5^{(k)}+u_7^{(k-1)}+u_{10}^{(k-1)});$$

$$u_7^{(k)}=\frac{1}{4}(u_3^{(k)}+u_6^{(k)}+u_8^{(k-1)}+u_{11}^{(k-1)});\ u_8^{(k)}=\frac{1}{4}(25+u_4^{(k)}+u_7^{(k)}+u_{12}^{(k-1)});$$

$$u_9^{(k)}=\frac{1}{4}(0.8+u_5^{(k)}+u_{10}^{(k-1)}+u_{13}^{(k-1)});\ u_{10}^{(k)}=\frac{1}{4}(u_6^{(k)}+u_9^{(k)}+u_{11}^{(k-1)}+u_{14}^{(k-1)});$$

$$u_{11}^{(k)}=\frac{1}{4}(u_7^{(k)}+u_{10}^{(k)}+u_{12}^{(k-1)}+u_{15}^{(k-1)});\ u_{12}^{(k)}=\frac{1}{4}(25+u_8^{(k)}+u_{11}^{(k)}+u_{16}^{(k-1)});$$

$$u_{13}^{(k)}=\frac{1}{4}(2.2+u_9^{(k)}+u_{14}^{(k-1)});\ u_{14}^{(k)}=\frac{1}{4}(0+u_{10}^{(k)}+u_{13}^{(k)}+u_{15}^{(k-1)});$$

$$u_{15}^{(k)}=\frac{1}{4}(5+u_{11}^{(k)}+u_{14}^{(k)}+u_{16}^{(k-1)});\ u_{16}^{(k)}=\frac{1}{4}(45+u_{12}^{(k)}+u_{15}^{(k-1)}).$$

Ushbu formulalardan foydalanib noma’lum qiymatlarni topish uchun boshlang’ich yaqinlashish $u_i^{(0)}$ lar berilishi lozim. Bu qiymatlar biror usul bilan topiladi. Masalan, izlanayotgan $u(x,y)$ funksiya qaralayotgan sohada gorizotal bo'yicha tekis taqsimlangan deb faraz qilib topamiz. Bunda (0;0,2) va (1;0,2) chegarani qarab, ichki $u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_4^{()}$ nuqtalardagi qiymatlarni hisoblaymiz. Kesma 5 ta qismga bo'lingani uchun funksiyaning o'zgarish qadami $K_1 = (25 - 7.2)/5 = 3{,}56$ ekanligini topib, quyidagilarni hisoblaymiz:

$$u_1^{(0)} = 7.2 + K_1 = 7.2 + 3.56 = 10.76; \quad u_2^{(0)} = u_1^{(0)} + K_1 = 10.76 + 3.56 = 14.32;$$

$$u_3^{(0)} = u_2^{(0)} + K_1 = 14.32 + 3.56 = 17.88; \quad u_4^{(0)} = u_3^{(0)} + K_1 = 17.88 + 3.56 = 21.44.$$

Xuddi shu kabi qolgan bo'yicha qiymatlar topiladi. Natijada boshlang'ich qiymatlar uchun quyidagi jadvaldagi qiymatlar topiladi:

y_i \ x_i	0	0,2	0,4	0,6	0,8	1
0	0	1,545	5,878	12,135	19,021	25
0,2	7,2	10,76	14,32	17,88	21,44	25
0,4	10,8	13,64	16,48	19,32	22,16	25
0,6	10,8	13,64	16,48	19,32	22,16	25
0,8	7,2	10,76	14,32	17,88	21,44	25
1	0	5	10	15	20	25

Bu qiymatlarni boshlang'ich yaqinlashish deb qarab, navbatdagi iterasiyaning birinchi qadamida faqat funksiyaning ichki tugun nuqtadagi qiymatlarini jadvallarda hisoblaymiz. $k = 1$ da

9,790	13,258	17,027	20,904
12,641	15,363	18,411	21,589
12,524	15,170	18,241	21,506
9,176	12,354	16,312	20,623

va hokazo. $k = 14$ da

8,635	11,772	15,803	20,303
10,565	12,646	16,138	20,407

10,170	12,101	15,693	20,185
7,202	9,883	14,339	19,637

$k = 2$ da

9,346	12,708	16,561	20,679
11,927	14,460	17,630	21,153
11,754	14,243	17,443	21,079
8,406	11,442	15,610	20,384

$k = 15$ da

8,634	11,770	15,802	20,302
10,562	12,642	16,135	20,405
10,167	12,096	15,689	20,183
7,200	9,879	14,336	19,636

Ko'rinib turibdiki 14 va 15 iterasiya qadamidagi qiymatlar bir-biridan 0,01 (so'ralgan aniqlik) ga farq qiladi. Shuning uchun hisoblashlarni to'xtatamiz va natija

y_i \ x_i	0	0,2	0,4	0,6	0,8	1
0	0	1,54	5,88	12,14	19,02	25
0,2	7,2	7,20	9,88	14,34	19,64	25
0,4	10,8	10,17	12,10	15,69	20,18	25
0,6	10,8	10,56	12,64	16,14	20,40	25
0,8	7,2	8,63	11,77	15,80	20,30	25
1	0	5	10	15	20	25

ko'rinishda bo'ladi.

Mustaqil ishlash uchun misollar

1. $$\theta\frac{\varphi_{i+1}-\varphi_i}{h}+(1-\theta)\frac{\varphi_i-\varphi_{i-1}}{h}=f_i,\ i=\overline{1,n-1}$$

$$\varphi_0 = a,\ \varphi_1 = b$$

ayirmali sxemaning turg'unligini $\theta \in [0,1]$ da tekshiring.

2. Quyidagi ayirmali sxemaning turg'unligini isbotlang:

$$-\frac{2}{x_{i+1}-x_{i-1}}\left(\frac{\varphi_{i+1}-\varphi_i}{\int\limits_{x_i}^{x_{i+1}}\frac{dx}{p(x)}}-\frac{\varphi_i-\varphi_{i-1}}{\int\limits_{x_{i-1}}^{x_i}\frac{dx}{p(x)}}-\varphi_i\int\limits_{x_{i-1/2}}^{x_{i+1/2}}q(x)dx\right)=\frac{2}{x_{i+1}-x_{i-1}}\int\limits_{x_{i-1/2}}^{x_{i+1/2}}f(x)dx,$$

$$p(x)>0,\ q(x)>0,\ i=\overline{1,\,n-1},$$

$$\varphi_0=0,\ \varphi_n=0,\ x_0=0,\ x_n=1,\ x\in[0,1].$$

3. Quyidagi differensial va ayirmali masalaning yechimini toping.

$$\frac{du}{dx}+2u=0,\ u(0)=1,\ \ x\in[0,1];$$

$$\frac{\varphi_{i+1}-\varphi_{i-1}}{2h}+2\varphi_i=0,\ i=\overline{1,\,n-1},\ h=\frac{1}{n},$$

$$\varphi_0=1,\ \varphi_0=1-2h.$$

Topilgan yechimga va yaqinlashish teoremasiga asosan $\left\|(u)_h-\varphi^h\right\|_{\Phi_h}$ ni baholang.

4. 3-masalani quyidagi ayirmali sxema uchun yeching.

$$\frac{\varphi_{i+1}-\varphi_{i-1}}{2h}+2\varphi_i=0,\ i=\overline{1,\,n-1},$$

$$\varphi_0=1,\varphi_1=1.$$

5. Quyidagi differensial va ayirmali masalaning yechimini toping.

$$\frac{du}{dx}=0,\ u(0)=1,\ x\in\ 0,1\ ;$$

$$4\frac{\varphi_{i+1}-\varphi_{i-1}}{2h}-3\frac{\varphi_{i+1}-\varphi_i}{h}=0,\; i=\overline{1,n-1},\; h=\frac{1}{n},$$

$$\varphi_0=1,\; \varphi_1=1+h^2,$$

$\left\| (u)_h-\varphi^h \right\|_{\Phi_h}$ ni baholang.

6. $$\frac{\varphi_i-\varphi_{i-1}}{h}+l\psi_{i-1}=0,\; \varphi_0=a,\; h=\frac{1}{n}$$

$$\frac{\psi_i-\psi_{i-1}}{h}-l\varphi_{i-1}=0,\; \psi_0=b,\; i=\overline{1,n}$$

ayirmali sxemaning yechimini

$$\frac{du}{dx}+l\vartheta=0,\; u(0)=a,$$

$$\frac{du}{dx}-lu=0,\; \vartheta(0)=b,$$

$$x\in [0,1] \quad l=const,$$

differensial masala yechimi uchun turg'unligini ikkala masala yechimlaridan foydalanib tekshiring.

7. 6-masalani quyidagi ayirmali sxema uchun yeching.

$$\frac{\varphi_i-\varphi_{i-1}}{h}+l\psi_i=0,\; \varphi_0=a,\; h=\frac{1}{n},$$

$$\frac{\psi_i-\psi_{i-1}}{h}-l\varphi_i=0,\; \psi_0=b,\; i=\overline{1,n}.$$

8. 6-masalani quyidagi ayirmali sxema uchun yeching.

$$\frac{\varphi_i-\varphi_{i-1}}{h}+l\frac{\psi_i+\psi_{i-1}}{2}=0,\; \varphi_0=a,\; h=\frac{1}{n},$$

$$\frac{\psi_i - \psi_{i-1}}{h} - l\frac{\varphi_i + \varphi_{i-1}}{2} = 0,\ \psi_0 = b,\ i = \overline{1, n}.$$

9. 6-masalani quyidagi ayirmali sxema uchun yeching.

$$\frac{\varphi_{i+1} - \varphi_{i-1}}{2h} + l\psi_i = 0,\ \varphi_0 = a,\ \varphi_1 = a - lhb,$$

$$\frac{\psi_{i+1} - \psi_{i-1}}{2h} - l\varphi_i = 0,\ \psi_0 = b,\ \psi_1 = b + lha$$

$$i = \overline{1, n-1},\ h = \frac{1}{n}.$$

10. 6-masalani quyidagi ayirmali sxema uchun yeching.

$$4\frac{\varphi_{i+1} - \varphi_{i-1}}{2h} - 3\frac{\varphi_{i+1} - \varphi_i}{h} + l\psi_i = 0,\ h = \frac{1}{n},$$

$$4\frac{\psi_{i+1} - \psi_{i-1}}{2h} - 3\frac{\psi_{i+1} - \psi_i}{h} - l\varphi_i = 0,\ i = \overline{1, n-1},$$

$$\varphi_0 = a,\ \varphi_1 = a - lhb,$$

$$\psi_0 = b,\ \psi_1 = b + lha.$$

11. Yaqinlashish haqidagi teoremadan foydalanib, ayirmali sxema yechimining differensial masala yechimiga yaqinlashish tartibini baholang.

$$\frac{du}{dx} + 5u = \cos x + 5\sin x,\ u(\pi) = 0,\ x \in \ \pi, 2\pi\ ,$$

$$\frac{\varphi_{i+1} - \varphi_i}{h} + 5\varphi_i = \cos\ ih\ + 5\sin\ ih\ ,\ i = \overline{0, n-1},$$

$$\varphi_0 = 0,\ h = \pi / n.$$

12. 11-masalani quyidagi tenglamalar uchun yeching.

$$\frac{du}{dx}+2u=2\ \ x+x^2\ \ ,\ u(4)=16\,,\ x\in\ \ 4,7\ \ ,$$

$$\frac{\varphi_{i+1}-\varphi_i}{h}+\varphi_{i+1}+\varphi_i=2\ \ 4+ih+\ \ 4+ih\ \ ^2\ \ ,\ i=\overline{0,\,n-1},$$

$$\varphi_0=16\,,\ h=3/n\,.$$

13. 11-masalani quyidagi tenglamalar uchun yeching.

$$\frac{du}{dx}+2u=2\ \ x+x^2\ \ ,\ u(4)=16\,,\ x\in\ \ 4,7\ \ ,$$

$$\frac{\varphi_{i+1}-\varphi_i}{h}+\varphi_{i+1}+\varphi_i=2\ \ 4+ih+\ \ 4+ih\ \ ^2\ \ +h\ \ 1+2\ \ 4+ih\ \ \ \ ,\ i=\overline{0,\,n-1},$$

$$\varphi_0=16\,,\ h=3/n\,.$$

14. 11-masalani quyidagi tenglamalar uchun yeching.

$$\frac{du}{dx}+u\sin x=\frac{\sin\ \ 2x}{2}+\sin x\,,\ x\in\ \ 0,1\ \ ,$$

$$u(0)=1,$$

$$\frac{\varphi_{i+1}-\varphi_i}{h}+\varphi_i\sin\ \ ih\ \ =\frac{\sin\ \ 2ih}{2}+\sin\ \ ih\ \ ,\ i=\overline{0,\,n-1},$$

$$\varphi_0=1,\ h=1/n\,.$$

15. $$-\frac{\varphi_{i+1}-2\varphi_i+\varphi_{i-1}}{h^2}+(1+ih)\varphi_i=(ih)^3-ih-2\,,\ i=\overline{1,\,n-1},$$

$$\varphi_0=\varphi_n=0\,,\ h=1/n$$

ayirmali sxema yechimini

$$-\frac{d^2u}{dx^2}+(1+x)u=x^3-x-2\,,\ x\in[0,1],$$

$$u(0) = u(1) = 0$$

differensial masala yechimiga yaqinlashishini tekshiring.

16. $$-\frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2} = -\frac{1}{(1+ih)^2}, \; i = \overline{1, n-1}, \; h = \frac{1}{n},$$

$$\frac{\varphi_1 - \varphi_0}{h} = 1, \; \varphi_n = \ln 2$$

ayirmali sxema yechimini

$$-\frac{d^2u}{dx^2} = -\frac{1}{(1+x)^2}, \quad x \in [0,1],$$

$$\left.\frac{du}{dx}\right|_{x=0} = 1, \; u(1) = \ln 2$$

differensial masala yechimiga yaqinlashishini tekshiring.

17. $$-\frac{d^2u}{dx^2} - 3\frac{du}{dx} - 4u = 0, \; x \in [0,1],$$

$$u(0) = 1, \; \left.\frac{du}{dx}\right|_{x=0} = -1,$$

differensial masala uchun

$$\frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2} - 3\frac{\varphi_{i+1} - \varphi_{i-1}}{2h} - 4\varphi_i = 0, \; i = \overline{1, n-1},$$

$$\varphi_0 = 1, \; \frac{\varphi_1 - \varphi_0}{h} = -1, \; h = \frac{1}{n}$$

ayirmali sxema tuzilgan.

a) Approksimatsiya tartibini aniqlang.

b) Ikkinchi tartibli approksimatsiyaga erishish uchun chegaraviy shartlarni approksimatsiyasini qanday o'zgartirish kerak?

18. $$\frac{-3\varphi_i + 4\varphi_{i+1} - \varphi_{i+2}}{2h} + a\varphi_i = f(x_i),\ i = 0, 2, 4, ..., 2n-2,$$

$$\frac{-\varphi_{i-1} + \varphi_{i+1}}{2h} + a\varphi_i = f(x_i),\ i = 1, 3, 5, ..., 2n-1,$$

$$\varphi_0 = b,\ h = \frac{1}{n},\ x_i = ih$$

ayirmali sxema yechimini

$$\frac{du}{dx} + au = f(x),\ x \in [0,1],$$

$u(0) = b$ differensial masala yechimiga yaqinlashishini tekshiring.

19. $$\frac{\varphi_{i+1} + \varphi_{i-1}}{2h} + 2\varphi_i = -\frac{5}{2}(1-ih)^{3/2} + 2(1-ih)^{5/2},\ i = \overline{1, n-1},$$

$$\varphi_0 = 1,\ \varphi_1 = (1-h)^{5/2},\ h = 1/n$$

ayirmali sxema yechimini

$$\frac{du}{dx} + 2u = -\frac{5}{2}(1-x)^{3/2} + 2(1-x)^{5/2},\ x \in [0,1],$$

$u(0) = 1$ differensial masala yechimiga yaqinlashishini tekshiring.

20. $$\frac{\varphi_{i+1} + 2\varphi_i + \varphi_{i-1}}{h^2} + 3\varphi_i = \frac{35}{4}(1-ih)^{3/2} - 3(1-ih)^{7/2},\ i = \overline{1, n-1},$$

$$\varphi_0 = 1,\ \frac{\varphi_1 - \varphi_0}{h} = -\frac{7}{2} + \frac{35}{8}h,\ h = \frac{1}{n}$$

ayirmali sxema yechimini differensial masala yechimiga yaqinlashishini tekshiring.

21. Quyidagi differensial va ayirmali masalalarning yechimlarini toping.

$$\frac{du}{dx}+u=0,\ x\in[0,1],$$

$$u(0)=1;$$

$$-\frac{\varphi_{i+1}-\varphi_{i-1}}{2h}+\varphi_i=0,\ i=\overline{1,\ n-1},\ h=\frac{1}{n},$$

$$\varphi_0=1,\ \varphi_1=e^{-h}$$

Agar $\psi_i=\frac{4}{3}\varphi_{2i}^{(2)}-\frac{1}{3}\varphi_i^{(1)},\ i=\overline{0,\ n_1}\ n_1=\frac{1}{h_1}$ bo'lsa, $\left\|(u)_h-\psi^h\right\|_\infty$ ni baholang.

Bu yerda $\varphi_i^{(1)}$ - $h=h_1$ qadamli, φ_{2i}^{2} - $h=h_1/2$ qadamli ayirmali sxemalarni yechimlari.

22. Quyidagi differensial va ayirmali masalalarning yechimlarini toping.

$$-\frac{d^2u}{dx^2}+u=0,\ x\in[0,1],$$

$$u(0)=1,\ \left.\frac{du}{dx}\right|_{x=0}=-1;$$

$$-\frac{\varphi_{i+1}+2\varphi_i+\varphi_{i-1}}{h^2}+\varphi_i=0,\ i=\overline{1,\ n-1},\ h=\frac{1}{n},$$

$$\varphi_0=1,\ \frac{\varphi_1-\varphi_0}{h}=-1+\frac{h}{2},$$

Agar $\psi_i = \frac{4}{3}\varphi_{2i}^{(2)} - \frac{1}{3}\varphi_i^{(1)},\ i = \overline{0,\ n_1},\ n_1 = \frac{1}{h_1}$ bo'lsa, $\left\|(u)_h - \psi^h\right\|_\infty \le Ch^4$

ekanligini ko'rsating. Bu yerda yerda $\varphi_i^{(1)}$ - $h = h_1$ qadamli, φ_{2i}^{2} - $h = h_1/2$ qadamli ayirmali sxemalarning yechimlari.

23. $$Lu = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}$$

differensial operatorni

$$(L_{h\tau}\varphi^{h\tau})_i^j = \frac{\varphi_i^{j+1} - \frac{\varphi_{i+1}^j + \varphi_{i-1}^j}{2}}{\tau} + a\frac{\varphi_{i+1}^j - \varphi_{i-1}^j}{2h}$$

ayirmali operator nechanchi tartib bilan lokal approksimatsiya qilishini aniqlang.

24. $$(L_{h\tau}\varphi^h)_{i,k} = \frac{\varphi_{i-1,k} + \varphi_{i,k-1} + \varphi_{i+1,k} + \varphi_{i,k+1} - 4\varphi_{i,k}}{h^2}$$

ayirmali operator qaysi differensial operatorni nechanchi tartib bilan lokal approksimatsiya qilishini aniqlang.

25. Quyidagi ayirmali operator uchun 24-masalani yeching.

$$(L_h\varphi^h)_{i,k} = \frac{\varphi_{i-1,k+1} + \varphi_{i-1,k-1} + \varphi_{i+1,k+1} + \varphi_{i+1,k-1} - 4\varphi_{i,k}}{2h^2}.$$

26. $$\frac{\varphi_i^{j+1} - \varphi_i^j}{\tau} = \frac{\varphi_{i-1}^j - 2\varphi_i^j + \varphi_{i+1}^j}{h^2}$$

ayirmali sxema

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

differensial operatorni $\tau / h^2 = 16$ bo'lganda τ bo'yicha ikkinchi tartib bilan va h bo'yicha to'rtinchi tartib bilan lokal approksimatsiya qilishini ko'rsating.

27. Quyidagi differensial tenglama berilgan.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\varphi_i^{j+1}-\varphi_i^j}{\tau}=\theta\frac{\varphi_{i+1}^{j+1}-2\varphi_i^{j+1}+\varphi_{i-1}^{j+1}}{h^2}+(1-\theta)\frac{\varphi_{i+1}^{j}-2\varphi_i^{j}+\varphi_{i-1}^{j}}{h^2}$$

ayirmali sxema θ ning qanday qiymatlarida τ bo'yicha ikkinchi tartib bilan va h bo'yicha to'rtinchi tartib bilan lokal approksimatsiya qilishini toping.

28. $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ t\in [0,T],\ x\in [0,1]$$

$$u(0,x)=\vartheta(x),\ u(t,0)=u(t,1)=0$$

differensial masalani

$$\frac{1}{12}\frac{\varphi_{i+1}^{j+1}-\varphi_{i+1}^{j}}{\tau}+\frac{5}{6}\frac{\varphi_{i}^{j+1}-\varphi_{i}^{j}}{\tau}+$$

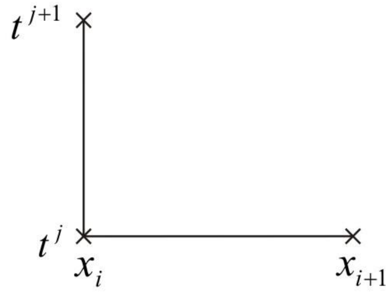
$$+\frac{1}{12}\frac{\varphi_{i-1}^{j+1}-\varphi_{i-1}^{j}}{\tau}=\frac{1}{2h^2}((\varphi_{i-1}^{j+1}-2\varphi_{i}^{j+1}+\varphi_{i+1}^{j+1})+(\varphi_{i-1}^{j}+\varphi_{i+1}^{j}-2\varphi_{i}^{j})),$$

$$j=\overline{0,\ m-1},\ i=\overline{1,\ n-1},$$

$$\varphi_i^0 = \vartheta(x_i), \quad i = \overline{0, n}, \quad h = 1/n, \quad x_i = ih,$$

$\varphi_i^0 = \varphi_n^j = 0$, $\quad j = \overline{0, m}$, $\tau = T/m$ ayirmali sxema nechanchi tartib bilan approksimatsiya qiladi?

29. 8-rasmda keltirilgan shablondan foydalanib,

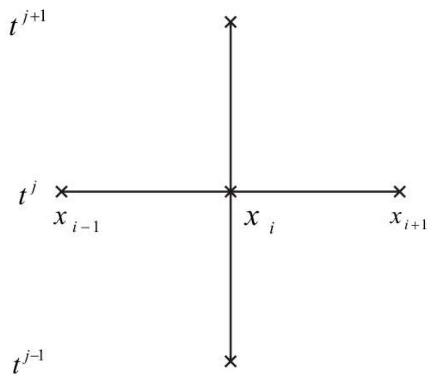


8-rasm.

$$Lu = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}$$

operator uchun (mumkin bo'lsa) $\tau = rh\ (= const)$ da 8-rasmda ko'rsatilgan shablonda birinchi va ikkinchi tartib bilan lokal approksimatsiya qiluvchi ayirmali operatorni quring.

30. 9-rasmda keltirilgan shablondan foydalanib,

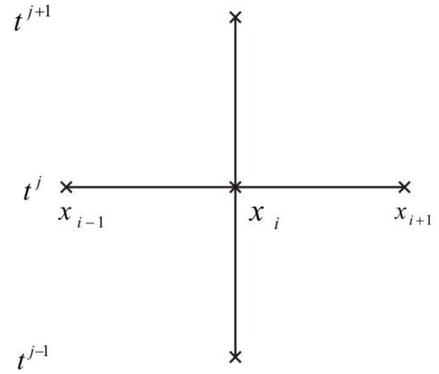


9-rasm.

$$Lu = \frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x}$$

differensial operatorni τ va h bo'yicha ikkinchi tartib bilan lokal approksimatsiya qiluvchi ayirmali operatorni quring.

31. 10-rasmda keltirilgan shablondan foydalanib,



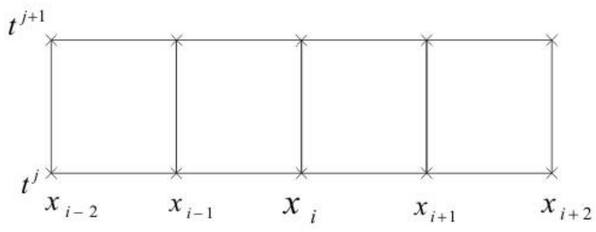
10-rasm.

(t^j, x_i) nuqtada

$$Lu = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}$$

differensial operatorni τ va h bo'yicha ikkinchi tartib bilan lokal approksimatsiya qiluvchi ayirmali operatorni quring.

32. 11-rasmda keltirilgan shablondan foydalanib,



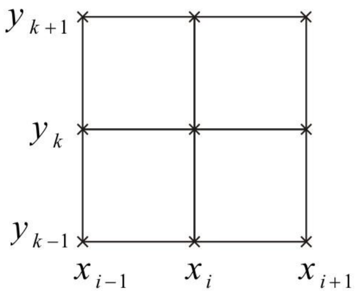
11-rasm.

$(t^{j+1/2}, x_i)$ nuqtada

$$Lu = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}$$

differensial operatorni τ bo'yicha ikkinchi tartib bilan va h bo'yicha to'rtinchi tartib bilan lokal approksimatsiya qiluvchi ayirmali operatorni quring.

33. 12-rasmda keltirilgan shablondan foydalanib,



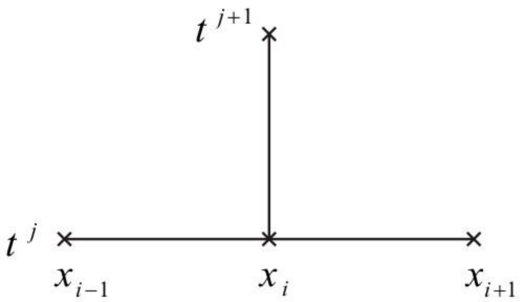
12-rasm.

$h = x_i - x_{i-1} = y_k - y_{k-1} = const$ - o'zgarmas qadam bilan noma'lum koeffitsiyentlar usuli bo'yicha

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f\left(x, y\right)$$

differensial tenglamani to'rtinchi tartib bilan lokal approksimatsiya qiluvchi ayirmali sxemani quring.

34. 13-rasmda keltirilgan shablondan foydalanib,



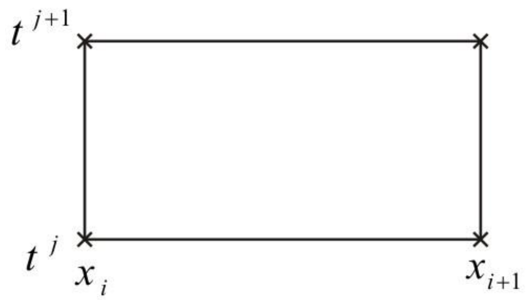
13-rasm.

noma’lum koeffitsiyentlar usuli bo’yicha

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=f\ (t,x\ ),\qquad t\in\ (0,T\ ],$$

$u(0,x)=g(x)$, $u\ (t,x\ )=u\ (t,x+1\ )$ differensial masalani τ va h bo’yicha ikkinchi tartib bilan approksimatsiya qiluvchi ayirmali sxemani quring.

35. 14-rasmda keltirilgan shablondan foydalanib,



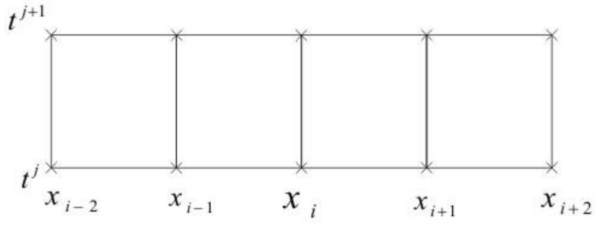
14-rasm.

integro-interpolyatsion usul bilan

$$\frac{\partial u}{\partial x}+a\frac{\partial u}{\partial x}=f(t,x),\ x\in\ (0,1\ ),\ t\in\ (0,T\ ],$$

$u(0,x)=g^{0}(x)$, $u(t,0)=g_{0}(t)$ differensial masalani τ va h bo’yicha ikkinchi tartib bilan approksimatsiya qiluvchi ayirmali sxemani quring.

36. 15-rasmda keltirilgan shablondan foydalanib,



15-rasm

integro-interpolyatsion usul bilan τ bo'yicha ikkinchi tartib bilan va h bo'yicha oltinchi tartib bilan approksimatsiya qiluvchi 5.13 masalani yeching.

37. Quyidagi diffensial masalani τ va h bo'yicha ikkinchi tartib bilan approksimatsiya qiluvchi ayirmali sxemani quring.

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=f(x,y)\ ,\ t\in\ 0,T\ \ ,$$

$$u(0,\ x,\ y)=g(x,y)\,,$$

$$u(t,\ x,\ y)=u(t,\ x+1,\ y)\,,\ u(\mathrm{t,\,x,\,y})=u(\mathrm{t,\,x,\,y}+1).$$

38. Quyidagi ayirmali sxemani turg'unligini tekshiring.

$$\frac{\varphi_i^{j+1}-\varphi_i^j}{\tau}+a\frac{\varphi_i^{j+1}-\varphi_{i-1}^{j+1}}{h}=0\,,\quad i=\overline{1,\,n}\ ,\ j=\overline{0,\ m-1}\ ,$$

$$\varphi_i^0=g_i\,,\ i=\overline{0,\ n}\,,\ \ nh=1\ ,$$

$$\varphi_0^j=g^j\,,\ j=\overline{0,\ m}\,,\ m\tau=T\ ,\ a>0\ .$$

39. Quyidagi ayirmali sxemani turg'unligini tekshiring.

$$\frac{\varphi_i^{j+1}-\varphi_i^j}{\tau}+a\frac{\varphi_{i+1}^{j}-\varphi_{i}^{j}}{h}=0\,,\ i=0,\pm1,\pm2,...\ \ j=\overline{0,\,m-1}$$

$$\varphi_i^0=g_i\,,$$

$$j=\overline{0,\ m}\ ,\ m\tau=T\ ,\ a>0\ .$$

40. Quyidagi ayirmali sxemani turg'unligini tekshiring.

$$\frac{\varphi_i^{j+1}-\varphi_i^j}{\tau}+a\frac{\varphi_{i+1}^{j}-\varphi_{i-1}^{j}}{2h}=f_i^{\,j}\,,$$

$$i=\overline{0,\ n-1}\ ,\ j=\overline{0,\ m-1}\ ,$$

$$\varphi_i^0=g_i\ ,\ i=\overline{0,\ n}\ ,nh=1\ ,$$

$$\varphi_0^j=\varphi_n^j\ ,\ \varphi_{-1}^j=\varphi_{n-1}^j\ ,\ j=\overline{0,\ m}\ ,\ m\tau=T\ ,a>0\ .$$

4.1 Quyidagi ayirmali sxemani turg'unligini tekshiring.

$$\frac{\varphi_i^{j+1}-\dfrac{\varphi_{i+1}^j+\varphi_{i-1}^j}{2}}{\tau}-\frac{\varphi_{i+1}^j-\varphi_{i-1}^j}{2h}=f_i^j\ ,$$

$$i=\overline{0,\ n-1}\ ,\quad j=\overline{0,\ \ m-1}\ ,$$

$$\varphi_i^0=g_i\ ,\ i=\overline{0,\ n}\ \ nh=1\ ,$$

$$\varphi_0^j=\varphi_n^j\ ,\ \varphi_{-1}^j=\varphi_{n-1}^j\ ,\ j=\overline{0,\ m}\ ,\ m\tau=T\ .$$

42. Quyidagi ayirmali sxemani turg'unligini tekshiring.

$$\frac{\varphi_i^{j+1}-\varphi_i^{j-1}}{2\tau}=\frac{\varphi_{i-1}^j-2\varphi_i^j+\varphi_{i+1}^j}{h^2}\ ,$$

$$i=\overline{1,\ n-1}\ ,\ j=\overline{1,\ m-1}\ ,$$

$$\varphi_i^0=g_i\quad \varphi_i^1=g_i\ ,\ i=\overline{0,\ n}\ ,\ nh=1\ ,$$

$$\varphi_i^0=\varphi_n^j=0\ ,\ j=\overline{0,\ m}\ ,\ m\tau=T\ .$$

43. Quyidagi ayirmali sxemani turg'unligini tekshiring.

$$\frac{\varphi_i^{j+1}-\varphi_i^j}{\tau}=\ 1-\theta\ \frac{\varphi_{i-1}^{j+1}-2\varphi_i^{j+1}+\varphi_{i+1}^{j+1}}{h^2}+\theta\frac{\varphi_{i-1}^j-2\varphi_i^j+\varphi_{i+1}^j}{h^2}\ ,$$

$$i=\overline{1,n-1}\ ,\ j=\overline{0,m-1}\ ,$$

$$\varphi_i^0=g_i,\ i=\overline{0,n}\ ,\ nh=1,$$

$$\varphi_0^j = \varphi_n^j = 0, \; j = \overline{0, m}, \; m\tau = T .$$

44. $\tau / h = 1$ da quyidagi ayirmali sxema turg'unmasligini isbotlang.

$$\frac{\varphi_i^{j+1} - 2\varphi_i^j + \varphi_i^{j-1}}{\tau^2} = \frac{\varphi_{i-1}^j - 2\varphi_i^j + \varphi_{i+1}^j}{h^2},$$

$$i = \overline{1, n-1}, \; j = \overline{1, m-1},$$

$$\varphi_i^0 = g_i^{(0)}, \; \frac{\varphi_i^1 - \varphi_i^0}{\tau} = g_i^1, \; i = \overline{0, n}, \; nh = 1,$$

$$\varphi_0^j = \varphi_n^j = 0, \; j = \overline{0, m}, \; m\tau = T$$

45. $$\frac{\varphi_i^{j+1} - \varphi_i^j}{\tau} - \frac{\varphi_{i+1}^j - \varphi_{i-1}^j}{2h} - \frac{\tau}{2h^2} \; \varphi_{i-1}^j - 2\varphi_i^j + \varphi_{i+1}^j \; = 0,$$

$$i = \overline{1, n}, \; j = \overline{0, m-1},$$

$$\varphi_i^0 = g_i, \; i = \overline{0, n}, \; nh = 1,$$

$\varphi_{i+n}^j = \varphi_i^j$, $j = \overline{0, m}$, $m\tau = T$ ayirmali sxema yechimining

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0, \; t \in \; 0, T \; ,$$

$u(0, x) = g(x)$, $u(t, x) = u(t, x+1)$ differensial masala yechimiga yaqinlashishini tekshiring.

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