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# Local Asymptotic Normality of Family of Distributions from Incomplete Observations

**Abdurahim A. Abdushukurov***
**Nargiza S. Nurmuhamedova**
National University of Uzbekistan,
VUZgorodok, Tashkent, 100174,
Uzbekistan

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*In this paper we prove the property of local asymptotic normality of the likelihood ratio statistics in the competing risks model under random censoring by non-observation intervals.*

*Keywords: competing risks, random censoring, likelihood ratio, local asymptotic normality.*

## Introduction

The likelihood ratio statistics (LRS) plays an important role in decision theory. For example, while testing a simple hypothesis $H_0$ against a complicated alternative $H_1$ with an undefined law of distribution the criterions based on the LRS, according to the Neyman-Pearson lemma, are uniformly more powerful for any size $n$ of observations (see [1,2]). Here appear some interesting examples when the alternative $H_1$ depends on $n$ and is close to $H_0$, i.e. $H_1 = H_{1n} \to H_0$ as $n \to \infty$. In such cases asymptotic properties of the LRS become transparent, which are useful for estimation theory and hypothesis testing. Among them there is the local asymptotic normality (LAN) of LRS. There is a number of papers devoted to investigations of the LAN for LRS and its applications in statistics. The most remarkable works are [2–5], which show that the LAN allows the development of asymptotic theory for most maximum likelihood and Bayesian type estimators and prove the contiguality properties of the family of probability distributions. In the papers [6–11] the properties of the LAN for LRS in the competing risks model (CRM) under random censoring of observations on the right and both sides were established. This paper includes investigations of the LAN for LRS in the CRM under random censoring by non-observation intervals.

## 1. Competing risks model under random censoring by non-observation intervals

In the CRM it is interesting to investigate a random variable (r.v.) $X$ with values from a measurable space $(\mathcal{X}, \mathcal{B})$ and events $(A^{(1)}, \ldots, A^{(k)})$ forming a complete group, where $k$ is fixed. In practice, a r.v. $X$ means, obviously, the survival or reliability time of some object (individual, physical system) exposed to $k$ competing risks and failing in case one of the events $\{A^{(i)},\ i = 1, ..., k\}$. The pairs $\{(X, A^{(i)}),\ \ i = 1, ..., k\}$ denote the time and reason the object fails (see more about the CRM in [6,12,13]). During the experiment under homogenous conditions an ensemble $(X, A^{(1)}, ..., A^{(k)})$ is observed, and we obtain a sequence $\{(X_j, A_j^{(1)}, ..., A_j^{(k)}),\ \ j \geqslant 1\}$. Let $\delta_j^{(i)} = I(A_j^{(i)})$ be the indicator of the event $A_j^{(i)}$. Every vector $\zeta_j = (X_j, \delta_j^{(1)}, ..., \delta_j^{(k)})$ induces

*a_abdushukurov@rambler.ru

a statistical model with sample space $\mathcal{Y} = \mathcal{X} \times \{0,1\}^{(k)} = \mathcal{X} \times \{0,1\} \times ... \times \{0,1\}$ and a $\sigma$-algebra $\mathcal{C}$ of sets of the form $B \times D_1 \times ... \times D_k$, where $B \in \mathcal{B}$ and $D_i \subset \{0,1\}$, $i = 1, ..., k$. We suppose that the distribution of the vector $\zeta_j$ on $(\mathcal{Y}, \mathcal{C})$ depends on an unknown parameter $\theta = (\theta_1, ..., \theta_s) \in \Theta$:

$$Q^*_\theta(B \times D_1 \times ... \times D_k) = P_\theta(X_1 \in B, \delta_1^{(1)} \in D_1, ..., \delta_1^{(k)} \in D_k), \tag{1}$$

where $\Theta$ is an open set in $R^s$. Let the distribution (1) be absolutely continuous with respect to the $\sigma$-finite measure $\nu(x) = \mu(x) \times \varepsilon_1 \times ... \times \varepsilon_k$, where $\mu$ is the Lebesque measure on $R$ and $\varepsilon_i$ are counting measures concentrated at the points $y^{(i)} \in \{0,1\}$, $i = \overline{1,k}$. In what follows we consider a statistical scheme where the sample $(X_j, A_j^{(1)}, ..., A_j^{(k)})$ is nonobservable if the r.v. $X_j$ falls in the interval $[Y_{1j}, Y_{2j}]$, where $\{(Y_{1j}, Y_{2j}),\ j \geqslant 1\}$ is the sequence of independent and identically distributed (i.i.d) random vectors with an unknown distribution $G(u,v)$, $(u,v) \in R^2$ (possibly implicitly depending on $\theta$). Here the samples $(X_j, A_j^{(1)}, ..., A_j^{(k)})$ and the pairs $(Y_{1j}, Y_{2j})$ are assumed to be independent and $P_\theta(Y_{1j} \leqslant Y_{2j}) = 1$ for every $j \geqslant 1$. This scheme models the experiments where the observation of object $j$ with life time $X_j$ might be stopped at a random moment $Y_{1j}$ and renewed at a random moment $Y_{2j}$. We call such a statistical model the CRM under random censoring by non-observation intervals. In this case instead of events $(A_j^{(1)}, ..., A_j^{(k)})$ we observe the events $(D_j^{(0)}, D_j^{(1)}, ..., D_j^{(k)})$, where $D_j^{(0)} = \{\omega : Y_{1j}(\omega) \leqslant X_j(\omega) \leqslant Y_{2j}(\omega)\}$ and $D_j^{(i)} = A_j^{(i)} \cap (\{\omega : X_j(\omega) < Y_{1j}(\omega)\} \cup \{\omega : X_j(\omega) > Y_{2j}(\omega)\})$, $i = 1, ..., k$. Let $\Delta_j^{(i)} = I(D_j^{(i)})$, $i = 0, 1, ..., k$ and $w_j = \varepsilon_{1j} + \varepsilon_{2j}$, where $\varepsilon_{1j} = I(X_j < Y_{1j})$ and $\varepsilon_{2j} = I(X_j > Y_{2j})$. It is obvious that $\Delta_j^{(0)} = 1 - w_j$ and $\Delta_j^{(i)} = w_j \delta_j^{(i)}$. In the CRM we are interested in the properties of pairs $\{(X_j, A_j^{(i)}),\ i = \overline{1,k}\}$, therefore we consider the subdistributions

$$Q_{i\theta}(B) = Q^*_\theta(B \times \{0\} \times ... \times \{0\} \times \{1\} \times \{0\} \times ... \times \{0\}),\ i = 1, ..., k, \tag{2}$$

produced from (1) when $D_i = \{1\}$ and $D_l = \{0\}$, $i \neq l$, $l = 1, ..., k$. Let $Q_\theta(B) = \sum\limits_{i=1}^k Q_{i\theta}(B)$. By $h^{(i)}$ and $h$ we denote the densities of subdistributions $Q_{i\theta}$ and $Q_\theta$:

$$Q_{i\theta}(B) = \int_B h^{(i)}(x;\theta)\mu(dx),\ i = 1, ..., k, \quad Q_\theta(B) = \int_B h(x;\theta)\mu(dx), \tag{3}$$

where $h = h^{(1)} + ... + h^{(k)}$. For $B = (-\infty; x]$ we put $Q_{i\theta}((-\infty; x]) = H^{(i)}(x;\theta)$, $i = \overline{1,k}$ and $Q_\theta((-\infty; x]) = H(x;\theta)$. Now we define the cumulative hazard functions (c.h.f.) of the pairs $(X, A^{(i)})$:

$$\begin{aligned} \Lambda^{(i)}(x;\theta) &= \int_{(-\infty;x]} \lim_{\Delta \downarrow 0} P_\theta(t < X \leqslant t + \Delta,\ A^{(i)}/X > t)\mu(dt) = \\ &= \int_{(-\infty;x]} \frac{dH^{(i)}(t;\theta)}{1 - H(t;\theta)},\ \ i = 1, ..., k,\ x \in R^1. \end{aligned} \tag{4}$$

Then the c.h.f. corresponding to the r.v. $X$ is $\Lambda(x;\theta) = \sum\limits_{i=1}^k \Lambda^{(i)}(x;\theta)$. In the CRM the exponential hazard functionals $F^{(i)}(x;\theta) = 1 - \exp\{-\Lambda^{(i)}(x;\theta)\}$, $i = \overline{1,k}$ describes the distribution of the pairs $(X, A^{(i)})$ in terms of the $i$-th risk. In view of the equality $\Lambda(x;\theta) = -\log(1 - H(x;\theta))$, we have

$$1 - H(x;\theta) = P_\theta(X > x) = \prod_{i=1}^k (1 - F^{(i)}(x;\theta)). \tag{5}$$

Define the density $f^{(i)}(x;\theta) = \dfrac{\partial}{\partial x} F^{(i)}(x;\theta)$, $i = \overline{1,k}$. Then the density of intensity for $i$-th risk is $f^{(i)}/(1 - F^{(i)})$. On the other hand, by formulas (3–5) for every $(x;\theta) \in R^1 \times \Theta$ and $i = 1, ..., k$, we have

$$\frac{f^{(i)}(x;\theta)}{1-F^{(i)}(x;\theta)} = \frac{h^{(i)}(x;\theta)}{1-H(x;\theta)},$$

i.e.

$$h^{(i)}(x;\theta) = f^{(i)}(x;\theta)\prod_{\substack{j=1\\ j\neq i}}^{k}(1-F^{(j)}(x;\theta)). \tag{6}$$

Assume that on the $n$-th stage of experiments the sample $\mathbb{Z}^{(n)} = (Z_1,...,Z_n)$, where $Z_j = w_jX_j + (1-w_j)[Y_{1j}, Y_{2j}]$, is available for observation; this means that every observable $Z_j$ is a r.v. $X_j$ (when $w_j = 1$) or an interval $[Y_{1j}, Y_{2j}]$ (when $w_j = 0$). Denote by $p(z;\theta)$ the density of one observable without multipliers depending on the unknown nuisance distribution $G$. Then, according to (6), we have the following "truncated" likelihood function of the sample $\mathbb{Z}^{(n)}$:

$$\begin{aligned} p_n(\mathbb{Z}^{(n)};\theta) = \prod_{m=1}^{n} p(Z_m;\theta) = \prod_{m=1}^{n}\Bigg\{\Bigg[\prod_{i=1}^{k}\Bigg[f^{(i)}(X_m;\theta)\prod_{\substack{j=1\\ j\neq i}}^{k}(1-F^{(j)}(X_m;\theta))\Bigg]^{\delta_m^{(i)}}\Bigg]^{w_m} \times \\ \times[H(Y_{2m};\theta)-H(Y_{1m};\theta)]^{1-w_m}\Bigg\} = \\ = \prod_{m=1}^{n}\left\{\left[\prod_{i=1}^{k}\left[h^{(i)}(X_m;\theta)\right]^{\delta_m^{(i)}}\right]^{w_m}[H(Y_{2m};\theta)-H(Y_{1m};\theta)]^{1-w_m}\right\}. \end{aligned} \tag{7}$$

Let for every $u \in R^s$, $\theta + n^{-1/2}u = \Psi_n(u;\theta) \in \Theta$ and $\widetilde{Q}_\theta^{(n)}$ be the distribution induced by the sample $\mathbb{Z}^{(n)}$. Then we have the LRS of the model

$$\begin{aligned} L_{n,\theta}(u) = d\widetilde{Q}^{(n)}_{\Psi_n(u;\theta)}(\mathbb{Z}^{(n)})/d\widetilde{Q}^{(n)}_\theta(\mathbb{Z}^{(n)}) = \frac{p_n(\mathbb{Z}^{(n)};\Psi_n(u;\theta))}{p_n(\mathbb{Z}^{(n)};\theta)} = \\ = \prod_{m=1}^{n}\left\{\left[\prod_{i=1}^{k}\left[\frac{h^{(i)}(X_m;\Psi_n(u;\theta))}{h^{(i)}(X_m;\theta)}\right]^{\delta_m^{(i)}}\right]^{w_m}\left[\frac{H(Y_{2m};\Psi_n(u;\theta))-H(Y_{1m};\Psi_n(u;\theta)}{H(Y_{2m};\theta)-H(Y_{1m};\theta)}\right]^{1-w_m}\right\}. \end{aligned} \tag{8}$$

Put $\chi_{n,\theta}(u) = \log L_{n,\theta}(u)$, we shall now study the properties of the random function $\chi_{n,\theta}(u)$.

## 2. Local asymptotic normality

Let $N^{(i)} = \{x : h^{(i)}(x;\theta) > 0\}$ and $N = \bigcap_{i=1}^{k} N^{(i)}$. We need some regularity conditions:

(C1) The supports $\{N^{(i)},\ i=\overline{1,k}\}$ are independent of $\theta$ and $N \neq \varnothing$;

(C2) There exist the derivatives $\dfrac{\partial^m h^{(i)}(x;\theta)}{\partial\theta_j^m}$, $m = 1,2;\ i = 1,...,k;\ j = 1,...,s$, for all $\theta \in \Theta$;

(C3) $\displaystyle\int_{-\infty}^{\infty}\left|\frac{\partial^m h^{(i)}(x;\theta)}{\partial\theta_j^m}\right|\mu(dx) < \infty,\ m=1,2;\ i=1,...,k;\ j=1,...,s$ for all $\theta \in \Theta$;

(C4) There are finite integrals $I_{lj}^{(i)}(\theta) = M_\theta\left[\dfrac{\partial}{\partial\theta_l}\log h^{(i)}(X;\theta)\dfrac{\partial}{\partial\theta_j}\log h^{(i)}(X;\theta)\right]$ for all $l,j = 1,...,s$ and $\theta \in \Theta$;

(C5) The matrix $I_X(\theta) = \left\| I_{lj}^X(\theta) \right\|_{l,j=\overline{1,s}} = \left\| \sum\limits_{i=1}^{k} I_{lj}^{(i)}(\theta) \right\|_{l,j=\overline{1,s}} = \sum\limits_{i=1}^{k} I^{(i)}(\theta)$ is positively defined for all $\theta \in \Theta$.

$I^{(i)}(\theta)$ is, obviously, the Fisher information matrix for the pair $(X, \delta^{(i)})$, and so is $I_X(\theta)$ for the r.v. $X$. Let

$$S_n(\mathbb{Z}^{(n)};\theta) = \frac{\partial \log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta} = \sum_{j=1}^{n} l_\theta(X_j, Y_{1j}, Y_{2j}, w_j),$$

where

$$l_\theta(x, y_1, y_2, w) = w\sum_{i=1}^{k}\delta^{(i)}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta} + (1-w)\frac{\partial \log(H(y_2;\theta) - H(y_1;\theta))}{\partial\theta}.$$

We note that $\mathbb{J}(\theta) = \mathbb{J}_1(\theta) + \mathbb{J}_2(\theta)$, where

$$\mathbb{J}_1(\theta) = \sum_{i=1}^{k}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left[\int_{-\infty}^{y_1}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta}\left(\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta}\right)^T dH^{(i)}(x;\theta) + \right.$$
$$\left. + \int_{y_2}^{\infty}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta}\left(\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta}\right)^T dH^{(i)}(x;\theta)\right] dG(y_1, y_2),$$

$$\mathbb{J}_2(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{y_1}\frac{\partial \log(H(y_2;\theta) - H(y_1;\theta))}{\partial\theta}\left(\frac{\partial \log(H(y_2;\theta) - H(y_1;\theta))}{\partial\theta}\right)^T \times$$
$$\times (H(y_2;\theta) - H(y_1;\theta))dG(y_1, y_2).$$

Let $(u; v)$ be the scalar product of vectors $u, v \in R^s$. The following theorem asserts the LAN for the LRS

**Theorem 2.1.** *Let the regularity conditions (C1)–(C5) hold and* $\det\{\mathbb{J}(\theta)\} \neq 0$. *Then for the LRS* $L_{n,\theta}(u)$ *we have the representation*

$$L_{n,\theta}(u) = \exp\left\{ n^{-1/2}\sum_{j=1}^{n}(l_\theta(X_j, Y_{1j}, Y_{2j}, w_j); u) - \frac{1}{2}\left(\mathbb{J}(\theta)u^T; u\right) + R_n(u;\theta)\right\}, \tag{9}$$

*where for all* $u \in R^s$

$$R_n(u;\theta) \overset{\widetilde{Q}_\theta^{(n)}}{\to} 0 \tag{10}$$

*as* $n \to \infty$, *and*

$$\mathcal{L}\left(n^{-1/2}\sum_{j=1}^{n} l_\theta(X_j, Y_{1j}, Y_{2j}, w_j)/\widetilde{Q}_\theta^{(n)}\right) \to \mathbb{N}_s(0; \mathbb{J}(\theta)). \tag{11}$$

It follows from (9) that the LRS $L_{n,\theta}(u)$ is approximated by the exponential density, and $\chi_{n,\theta}(u)$ has asymptotically $s$-dimensional normal distribution. For the proof of Theorem 3.1 we need the following lemmas.

Let $\{l_i(x) = (l_{i1}(x), ..., l_{is}(x)),\ i = 1, ..., k\}$ and $l_0(y_1, y_2) = (l_{01}(y_1, y_2), ..., l_{0s}(y_1, y_2))$ be vector-valued functions, possibly depending on $\theta$, and let

$$l(x, y_1, y_2, w) = w\sum_{i=1}^{k}\delta^{(i)} l_i(x) + (1-w) l_0(y_1, y_2).$$

**Lemma 3.1.** *Suppose the following conditions hold:*
*(A)* $E_\theta\left[\delta^{(i)}\left|l_{ij}(X)\right|\right]<\infty$ *for all* $i=1,...,k;\ j=1,...,s$ *and* $\theta\in\Theta$*;*
*(B)* $E_\theta\left[(1-w)\left|l_{0j}(Y_1,Y_2)\right|\right]<\infty$ *for all* $j=1,...,s$ *and* $\theta\in\Theta$*;*
*Then for any* $\theta\in\Theta$

$$\begin{aligned}E_\theta l(X,Y_1,Y_2,w)=&\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int_{-\infty}^{y_1}l_i(x)dH^{(i)}(x;\theta)+\int_{y_2}^{\infty}l_i(x)dH^{(i)}(x;\theta)\right)dG(y_1,y_2)+\\&+\int_{-\infty}^{\infty}\int_{y_1}^{\infty}l_0(y_1,y_2)(H(y_2;\theta)-H(y_1;\theta))\,dG(y_1,y_2).\end{aligned}\tag{12}$$

**Lemma 3.2.** *Let the regularity conditions (C1)–(C4) hold. Then*

$$E_\theta\left[\frac{\partial\log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta}\right]=0\ \text{ for all }\theta\in\Theta.\tag{13}$$

**Lemma 3.3.** *With the conditions (C1)–(C4), for all* $\theta\in\Theta$

$$E_\theta\left[\frac{\partial\log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta}\left(\frac{\partial\log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta}\right)^T\right]=-\left\|E_\theta\left(\frac{\partial^2 p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta_j\partial\theta_l}\right)\right\|_{j,l=\overline{1,s}}.\tag{14}$$

Note $g^{(i)}(x;\theta)=\sqrt{h^{(i)}(x;\theta)}$, $\xi_{ni}(x;u)=\dfrac{g^{(i)}(x;\theta+u)}{g^{(i)}(x;\theta)}-1$, $q(y_1,y_2;\theta)=\sqrt{H(y_2;\theta)-H(y_1;\theta)}$ and $\eta_n(y_1,y_2;u)=\dfrac{q(y_1,y_2;\theta+u)}{q(y_1,y_2;\theta)}-1$.

**Lemma 3.4.** *Let the regularity conditions (C1)–(C5) hold. Then for* $|u|\to 0$ *we have:*

$$E_\theta\left[w\sum_{i=1}^{k}\delta^{(i)}\xi_{ni}^2(X;u)\right]-\frac{1}{4}\left(\mathbb{J}_1(\theta)u;u\right)=o\left(|u|^2\right),\tag{15}$$

$$E_\theta\left[(1-w)\eta_n^2(Y_1,Y_2;u)\right]-\frac{1}{4}\left(\mathbb{J}_2(\theta)u;u\right)=o\left(|u|^2\right),\tag{16}$$

$$E_\theta\left|w\sum_{i=1}^{k}\delta^{(i)}\left[\xi_{ni}^2(X;u)-\left(u;\ \frac{\partial g^{(i)}(X;\theta)}{\partial\theta}\right)^2\right]\right|=o\left(|u|^2\right),\tag{17}$$

$$E_\theta\left|(1-w)\left[\eta_n^2(Y_1,Y_2;u)-\left(u;\ \frac{\partial q(Y_1,Y_2;\theta)}{\partial\theta}\right)^2\right]\right|=o\left(|u|^2\right),\tag{18}$$

$$P_\theta\left(\left|w\sum_{i=1}^{k}\delta^{(i)}\xi_{ni}(X;u)\right|>\varepsilon\right)=o\left(|u|^2\right),\tag{19}$$

$$P_\theta\left(|(1-w)\eta_n(Y_1,Y_2;u)|>\varepsilon\right)=o\left(|u|^2\right),\tag{20}$$

$$E_\theta\left[w\sum_{i=1}^{k}\delta^{(i)}\xi_{ni}(X;u)\right]+\frac{1}{8}\left(\mathbb{J}_1(\theta)u;u\right)=o\left(|u|^2\right),\tag{21}$$

$$E_\theta\left[(1-w)\eta_n(Y_1,Y_2;u)\right]+\frac{1}{8}\left(\mathbb{J}_2(\theta)u;u\right)=o\left(|u|^2\right).\tag{22}$$

*The proof of Theorem 3.1.* From (8) we have

$$L_{n,\theta}(u) = \exp\left\{\chi^{(1)}_{n,\theta}(u) + \chi^{(2)}_{n,\theta}(u)\right\}, \tag{23}$$

where

$$\begin{aligned}\chi^{(1)}_{n,\theta}(u) &= \sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \log\left[\frac{h^{(i)}(X_j;\, \theta + n^{-1/2}u)}{h^{(i)}(X_j;\theta)}\right] = \\ &= 2\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \log\Big(1 + \xi_{ni}(X_j; n^{-1/2}u)\Big),\end{aligned} \tag{24}$$

$$\begin{aligned}\chi^{(2)}_{n,\theta}(u) &= \sum_{j=1}^{n} (1-w_j) \log\left[\frac{H(Y_{2j};\, \theta + n^{-1/2}u) - H(Y_{1j};\, \theta + n^{-1/2}u)}{H(Y_{2j};\theta) - H(Y_{1j};\theta)}\right] = \\ &= 2\sum_{j=1}^{n} (1-w_j) \log\Big(1 + \eta_n(Y_{1j}, Y_{2j}; n^{-1/2}u)\Big),\end{aligned} \tag{25}$$

and

$$\chi^{(1)}_{n,\theta}(u) + \chi^{(2)}_{n,\theta}(u) = \chi_{n,\theta}(n^{-1/2}u). \tag{26}$$

Define $A_n = \left\{\max\limits_{1\leqslant j\leqslant n}\max\limits_{1\leqslant i\leqslant k}\left|\xi_{ni}(X_j; n^{-1/2}u)\right| < \varepsilon\right\}$, $B_n = \left\{\max\limits_{1\leqslant j\leqslant n}\left|\eta_n(Y_{1j}, Y_{2j}; n^{-1/2}u)\right| < \varepsilon\right\}$. Due to the fairness of those events and Taylor's formulas for some $\left|\alpha^{(i)}_{jn}\right| < 1$ and $\left|\beta_{jn}\right| < 1$ we have

$$\begin{aligned}\chi^{(1)}_{n,\theta}(u) = 2\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) - \sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi^2_{ni}(X_j; n^{-1/2}u) + \\ + \sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \alpha^{(i)}_{jn} \Big|\xi_{ni}(X_j; n^{-1/2}u)\Big|^3,\end{aligned} \tag{27}$$

and

$$\begin{aligned}\chi^{(2)}_{n,\theta}(u) = 2\sum_{j=1}^{n} (1-w_j)\eta_n(Y_{1j}, Y_{2j}; n^{-1/2}u) - \sum_{j=1}^{n} (1-w_j)\eta^2_n(Y_{1j}, Y_{2j}; n^{-1/2}u) + \\ + \sum_{j=1}^{n} (1-w_j)\beta_{jn}\Big|\eta_n(Y_{1j}, Y_{2j}; n^{-1/2}u)\Big|^3.\end{aligned} \tag{28}$$

To prove the theorem it is enough to show the following as $n \to \infty$:

$$P_\theta(\overline{A}_n) \to 0, \quad P_\theta(\overline{B}_n) \to 0, \tag{29}$$

$$\begin{aligned}P_\theta\Bigg(\Bigg|2\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) - n^{-1/2}\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left(\frac{\partial \log h^{(i)}(X_j;\theta)}{\partial\theta}; u\right) + \\ + \frac{1}{4}\left(\mathbb{J}_1(\theta)u; u\right)\Bigg| > \varepsilon\Bigg) \to 0,\end{aligned} \tag{30}$$

$$\begin{aligned}P_\theta\Bigg(2\sum_{j=1}^{n} (1-w_j)\eta_n(Y_{1j}, Y_{2j}; n^{-1/2}u) - \\ - n^{-1/2}\sum_{j=1}^{n} (1-w_j)\left(\frac{\partial \log\left(H(Y_{2j};\,\theta) - H(Y_{1j};\,\theta)\right)}{\partial\theta}; u\right) + \frac{1}{4}\left(\mathbb{J}_2(\theta)u; u\right)\Bigg| > \varepsilon\Bigg) \to 0,\end{aligned} \tag{31}$$

$$P_\theta\left(\left|\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}^2(X_j; n^{-1/2}u) - \frac{1}{4}\left(\mathbb{J}_1(\theta)u; u\right)\right| > \varepsilon\right) \to 0, \tag{32}$$

$$P_\theta\left(\left|\sum_{j=1}^{n} (1-w_j)\eta_n^2(Y_{1j}, Y_{2j}; n^{-1/2}u) - \frac{1}{4}\left(\mathbb{J}_2(\theta)u; u\right)\right| > \varepsilon\right) \to 0, \tag{33}$$

$$P_\theta\left(\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left|\xi_{ni}(X_j; n^{-1/2}u)\right|^3 > \varepsilon\right) \to 0, \tag{34}$$

$$P_\theta\left(\sum_{j=1}^{n} (1-w_j)\left|\eta_n(Y_{1j}, Y_{2j}; n^{-1/2}u)\right|^3 > \varepsilon\right) \to 0. \tag{35}$$

From (19), if $n \to \infty$

$$P_\theta(\overline{A}_n) \leqslant \sum_{j=1}^{n} P_\theta\left(\left|w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u)\right| > \varepsilon\right) = nP_\theta\left(\left|w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u)\right| > \varepsilon\right) = o(1).$$

The second convergence in (29) can be proved in the same way using (20). According to (17) and Markov's inequality, for $n \to \infty$ we have

$$\begin{aligned} &P_\theta\left(\left|\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}^2(X_j; n^{-1/2}u) - \frac{1}{n}\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left(u; \frac{\partial g^{(i)}(X_j;\theta)}{\partial\theta}\right)^2\right| > \varepsilon\right) \leqslant \\ &\leqslant \frac{1}{\varepsilon} E_\theta\left\{\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left[\xi_{ni}(X_j; n^{-1/2}u) - \frac{1}{n}\left(u; \frac{\partial g^{(i)}(X_j;\theta)}{\partial\theta}\right)^2\right]\right\} \leqslant \\ &\leqslant \frac{n}{\varepsilon} E_\theta\left|w_j \sum_{i=1}^{k} \delta_j^{(i)} \left[\xi_{ni}(X_j; n^{-1/2}u) - \left(n^{-1/2}u; \frac{\partial g^{(i)}(X_j;\theta)}{\partial\theta}\right)^2\right]\right| = o(1). \end{aligned} \tag{36}$$

In the other hand, according to Lemma 3.1 and the law of large numbers, for $n \to \infty$

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left(u; \frac{\partial g^{(i)}(X_j;\theta)}{\partial\theta}\right)^2 = \\ &= u^T \frac{1}{n}\sum_{j=1}^{n} w_j \sum_{i=1}^{k} \delta_j^{(i)} \frac{\partial g^{(i)}(X_j;\theta)}{\partial\theta}\left(\frac{\partial g^{(i)}(X_j;\theta)}{\partial\theta}\right)^T u \overset{\tilde{Q}_\theta^{(n)}}{\to} \frac{1}{4}\left(\mathbb{J}_1(\theta)u; u\right). \end{aligned} \tag{37}$$

From (36) and (37) we have (32). In the same way (33) is proved. From (29) and (32) we obtain (34). In the same way we prove (35). Let us now prove (30) and (31). According to (21), when $n \to \infty$

$$E_\theta\left(w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u)\right) = -\frac{1}{8}\left(\mathbb{J}_1(\theta)u; u\right) + o\left(\frac{1}{n}\right). \tag{38}$$

Consequently, when $n \to \infty$ the expression (30) is equivalent to

$$\begin{aligned} \lambda(u) = P_\theta\Bigg(2\Bigg|\sum_{j=1}^{n}\Bigg\{ &w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) - E_\theta\left[w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u)\right] - \\ &- \frac{n^{-1/2}}{2} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left(\frac{\partial \log h^{(i)}(X_j;\theta)}{\partial\theta}; u\right)\Bigg\}\Bigg| > \varepsilon\Bigg) \to 0. \end{aligned}$$

Now, as the summands are independent, from Markov's inequality we have

$$\lambda(u) \leqslant \frac{4}{\varepsilon^2} E_\theta \left\{ \sum_{j=1}^{n} w_j \left[ \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) - E_\theta \left[ w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) \right] - \right.\right.$$
$$\left.\left. - \frac{n^{-1/2}}{2} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left( \frac{\partial \log h^{(i)}(X_j;\theta)}{\partial \theta}; u \right) \right] \right\}^2 =$$
$$= \frac{4n}{\varepsilon^2} \left\{ E_\theta \left[ w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) - \frac{1}{2} w_j \sum_{i=1}^{k} \delta_j^{(i)} \left( \frac{\partial \log h^{(i)}(X_j;\theta)}{\partial \theta}; n^{-1/2}u \right) \right]^2 - \right. \tag{39}$$
$$\left. - \left[ E_\theta \left[ w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) \right] \right]^2 \right\} - \frac{4n^{-1/2}}{\varepsilon^2} E_\theta \left[ w_j \sum_{i=1}^{k} \delta_j^{(i)} \left( \frac{\partial \log h^{(i)}(X_j;\theta)}{\partial \theta}; u \right) \right] \times$$
$$\times E_\theta \left[ w_j \sum_{i=1}^{k} \delta_j^{(i)} \xi_{ni}(X_j; n^{-1/2}u) \right].$$

The first summand tends to zero when $n \to \infty$ because of (51) from the proof of Lemma 3.4 (see Section 4) and (38), and the second summand tends to zero too since (38). So, (30) is fulfilled. In the same way we prove (31). Now, (10) follows from (29)–(35), and to prove (11) we use the central limit theorem. The theorem is proved.

# 3. The proofs of Lemmas 3.1–3.4

*Proof of Lemma 3.1.* It is easy to see that for all $\theta \in \Theta$ and $m = 1, ..., s$ under the condition of Lemma

$$E_\theta \left[ |l_m(X, Y_1, Y_2, w)| \right] \leqslant k \max_{1 \leqslant i \leqslant k} \left\{ E_\theta \left[ \delta^{(i)} |l_{im}(X)| \right] \right\} + E_\theta \left[ (1-w) |l_{0m}(Y_1, Y_2)| \right] < \infty.$$

Compute the expectation for the events $\{w = 1\}$ and $\{w = 0\}$. We have

$$E_\theta \left[ w l(X, Y_1, Y_2, w) \right] = E_\theta \left\{ \sum_{i=1}^{k} \left[ E_\theta \left[ \delta^{(i)} l_i(X) I(X < Y_1)/Y_1 \right] I(Y_1 \leqslant Y_2) \right] + \right.$$
$$\left. + E_\theta \left[ \delta^{(i)} l_i(X) I(X > Y_2)/Y_2 \right] I(Y_1 \leqslant Y_2) \right\} =$$
$$= \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \sum_{i=1}^{k} \left( \int_{-\infty}^{y_1} l_i(x) dH^{(i)}(x;\theta) + \int_{y_2}^{\infty} l_i(x) dH^{(i)}(x;\theta) \right) dG(y_1, y_2)$$

also

$$E_\theta \left[ (1-w) l(X, Y_1, Y_2, w) \right] = E_\theta \left\{ E_\theta \left[ l_0(Y_1, Y_2) I(Y_1 \leqslant X \leqslant Y_2)/(Y_1, Y_2) \right] I(Y_1 \leq Y_2) \right] \right\} =$$
$$= E_\theta \left\{ l_0(Y_1, Y_2) E_\theta \left[ I(Y_1 \leqslant X \leqslant Y_2)/(Y_1, Y_2) \right] I(Y_1 \leqslant Y_2) \right] \right\} =$$
$$= E_\theta \left\{ l_0(Y_1, Y_2) \left( H(Y_2;\theta) - H(Y_1;\theta) \right) I(Y_1 \leqslant Y_2) \right\} =$$
$$= \int_{-\infty}^{\infty} \int_{y_1}^{\infty} l_0(Y_1, Y_2) \left( H(Y_2;\theta) - H(Y_1;\theta) \right) dG(y_1, y_2).$$

Now adding these formulas we obtain (11). □

*Proof of Lemma 3.2.* We have

$$\frac{\partial \log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta} = \sum_{j=1}^{n} l_\theta\left(X_j, Y_{1j}, Y_{2j}, w_j\right),$$

where $l_\theta\left(X_j, Y_{1j}, Y_{2j}, w_j\right) = w_j \sum\limits_{i=1}^{k} \delta_j^{(i)} l_{i\theta}(X_j) + (1-w_j) l_{0\theta}\left(Y_{1j}, Y_{2j}\right)$ is a vector-function from Lemma 3.1, where

$$l_{i\theta}(x) = \frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta}, \quad l_{0\theta}(y_1,y_2) = \frac{\partial \log(H(y_2;\theta) - H(y_1;\theta))}{\partial\theta}.$$

The fact that conditions (A), (B) of Lemma 3.1 hold follows from

$$E_\theta\left[\delta^{(i)}\left|l_{\theta ij}(X)\right|\right] = \int_{-\infty}^{\infty}\left|\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_j}\right|\mu(dx) < \infty \quad \text{for all} \quad \theta\in\Theta, \tag{40}$$

also

$$\begin{aligned}
E_\theta\left[(1-w)\left|l_{\theta 0j}(Y_1,Y_2)\right|\right] &= \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left|\frac{\partial}{\partial\theta_j}\left(H(y_2;\theta) - H(y_1;\theta)\right)\right| dG(y_1,y_2) = \\
&= \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left|\int_{y_1}^{y_2}\left[\frac{\partial \log h(x;\theta)}{\partial\theta_j}\sqrt{h(x;\theta)}\right]\sqrt{h(x;\theta)}\mu(dx)\right| dG(y_1,y_2) \leqslant \\
&\leqslant \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left|\int_{y_1}^{y_2}\left[\left(\frac{\partial \log h(x;\theta)}{\partial\theta_j}\right)^2 dH(x;\theta)\right]^{1/2}\left[H(y_2;\theta) - H(y_1;\theta)\right]^{1/2}\right| dG(y_1,y_2) \\
&\leqslant \left(I_{jj}^X(\theta)\right)^{1/2}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left[H(y_2;\theta) - H(y_1;\theta)\right]^{1/2} dG(y_1,y_2) \leqslant \left(I_{jj}^X(\theta)\right)^{1/2},
\end{aligned}\tag{41}$$

here we use the regularity conditions (C1)–(C4), formula (40), and also the Cauchy-Bunyakovsky-Schwarz inequality. Thus, by (40) and (41) the expectation in (13) exists, and

$$E_\theta\left[\frac{\partial \log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta}\right] = nE_\theta l_\theta\left(X, Y_1, Y_2, w\right). \tag{42}$$

By lemma 3.1, for any $\theta\in\Theta$, we have

$$\begin{aligned}
E_\theta l_\theta\left(X, Y_1, Y_2, w\right) &= \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int_{-\infty}^{y_1}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta} dH^{(i)}(x;\theta) + \right. \\
&\quad \left. + \int_{y_2}^{\infty}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta} dH^{(i)}(x;\theta)\right) dG(y_1,y_2) + \\
&\quad + \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\frac{\partial \log(H(y_2;\theta) - H(y_1;\theta))}{\partial\theta}(H(y_2;\theta) - H(y_1;\theta)) dG(y_1,y_2) = \\
&= \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int_{-\infty}^{y_1}\frac{\partial h^{(i)}(x;\theta)}{\partial\theta}\mu(dx) + \int_{y_2}^{\infty}\frac{\partial h^{(i)}(x;\theta)}{\partial\theta}\mu(dx)\right) + \\
&\quad + \frac{\partial}{\partial\theta}(H(y_2;\theta) - H(y_1;\theta)) dG(y_1,y_2) = \\
&= \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left[-\frac{\partial}{\partial\theta}(H(y_2;\theta) - H(y_1;\theta)) + \frac{\partial}{\partial\theta}(H(y_2;\theta) - H(y_1;\theta))\right] dG(y_1,y_2) = 0.
\end{aligned}\tag{43}$$

Now, (13) follows from (42) and (43). □

*Proof of lemma 3.3.* Since $l_\theta \cdot {l_\theta}^T = \|l_{\theta j} \cdot l_{\theta l}\|_{j,l=\overline{1,s}}$, where

$$l_{\theta j}(X,Y_1,Y_2,w)\,l_{\theta l}(X,Y_1,Y_2,w) = w\sum_{i=1}^{k}\delta^{(i)}\frac{\partial \log h^{(i)}(X;\theta)}{\partial\theta_j}\frac{\partial \log h^{(i)}(X;\theta)}{\partial\theta_l}+$$
$$+(1-w)\frac{\partial \log(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j}\frac{\partial \log(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l},$$

then by Lemmas 3.1 and 3.2

$$E_\theta\left[\frac{\partial \log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta}\left(\frac{\partial \log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta}\right)^T\right] = n\mathbb{J}(\theta),$$

where $\mathbb{J}(\theta)$ is the matrix with elements

$$\begin{aligned} J_{jl}(\theta) = \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\Bigg(\int_{-\infty}^{y_1}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_j}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_l}dH^{(i)}(x;\theta)+\\ +\int_{y_1}^{\infty}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_j}\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_l}dH^{(i)}(x;\theta)\Bigg)\,dG(y_1,y_2)+\\ +\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\frac{\partial \log(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j}\frac{\partial \log(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l}dG(y_1,y_2) =\\ = J_{jl}^{(1)}(\theta)+J_{jl}^{(2)}(\theta). \end{aligned} \tag{44}$$

It is easy to see that the first summand in (44) is estimated for all $\theta\in\Theta$ as

$$\begin{aligned} \left|J_{jl}^{(1)}(\theta)\right| \leqslant 2\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left|\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_j}\right|\left|\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_l}\right|dH^{(i)}(x;\theta) =\\ = 2\sum_{i=1}^{k}\left[\int_{-\infty}^{\infty}\left(\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_j}\right)^2 dH^{(i)}(x;\theta)\right]^{1/2}\times\\ \times\left[\int_{-\infty}^{\infty}\left(\frac{\partial \log h^{(i)}(x;\theta)}{\partial\theta_l}\right)^2 dH^{(i)}(x;\theta)\right]^{1/2} = 2\sum_{i=1}^{k}\left[I_{jj}^{(i)}(\theta)I_{ll}^{(i)}(\theta)\right]^{1/2}<\infty, \end{aligned} \tag{45}$$

where we use the condition (C4) and the Cauchy-Bunyakovsky-Schwarz inequality. Similarly, we estimate $J_{jl}^{(2)}(\theta)$ for all $\theta\in\Theta$ and $j,l=1,...,s$:

$$\begin{aligned} \left|J_{jl}^{(2)}(\theta)\right| \leqslant \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left|\int_{y_1}^{y_2}\frac{\partial h(x;\theta)}{\partial\theta_j}\mu(dx)\right|\left|\int_{y_1}^{y_2}\frac{\partial h(x;\theta)}{\partial\theta_l}\mu(dx)\right|\frac{dG(y_1,y_2)}{(H(y_2;\theta)-H(y_1;\theta))} =\\ = \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left[\int_{y_1}^{y_2}\left(\frac{\partial \log h(x;\theta)}{\partial\theta_j}\sqrt{h(x;\theta)}\right)\mu(dx)\right]\times\\ \times\left[\int_{y_1}^{y_2}\left(\frac{\partial \log h(x;\theta)}{\partial\theta_l}\sqrt{h(x;\theta)}\right)\sqrt{h(x;\theta)}\mu(dx)\right]\frac{dG(y_1,y_2)}{(H(y_2;\theta)-H(y_1;\theta))}\\ \leqslant \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left[\int_{y_1}^{y_2}\left(\frac{\partial \log h(x;\theta)}{\partial\theta_j}\right)^2 dH(x;\theta)\right]^{1/2}\int_{y_1}^{y_2}h(x;\theta)\mu(dx)\times\\ \times\left[\int_{y_1}^{y_2}\left(\frac{\partial \log h(x;\theta)}{\partial\theta_l}\right)^2 dH(x;\theta)\right]\frac{dG(y_1,y_2)}{(H(y_2;\theta)-H(y_1;\theta))} \leqslant \left[I_{jj}^X(\theta)I_{ll}^X(\theta)\right]^{1/2}<\infty. \end{aligned} \tag{46}$$

The expressions (45) and (46) imply the existence of $J_{jl}(\theta)$. To prove (14), note first that for all $\theta \in \Theta$ and $j, l = 1, ..., s$:

$$E_\theta\left[\frac{\partial^2 \log p_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta_j\partial\theta_l}\right] = nE_\theta\left[\frac{\partial^2 l_\theta\left(X, Y_1, Y_2, w\right)}{\partial\theta_j\partial\theta_l}\right], \tag{47}$$

where, by Lemma 3.1, we have the chain of equalities

$$\begin{gathered}
E_\theta\left[\frac{\partial^2 l_\theta\left(X, Y_1, Y_2, w\right)}{\partial\theta_j\partial\theta_l}\right] = \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int_{-\infty}^{y_1}\frac{\partial^2\log h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}dH^{(i)}(x;\theta)+\right.\\
\left.+\int_{y_2}^{\infty}\frac{\partial^2\log h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}dH^{(i)}(x;\theta)\right)dG(y_1,y_2)+\\
+\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\frac{\partial^2\log(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j\partial\theta_l}(H(y_2;\theta)-H(y_1;\theta))dG(y_1,y_2) =\\
=\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\left\{\int_{-\infty}^{y_1}\left[\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}h^{(i)}(x;\theta)+\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_l}\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_j}\right]\frac{\mu(dx)}{h^{(i)}(x;\theta)}+\right.\\
\left.+\int_{y_2}^{\infty}\left[\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}h^{(i)}(x;\theta)-\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_l}\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_j}\right]\frac{\mu(dx)}{h^{(i)}(x;\theta)}\right\}dG(y_1,y_2)+\\
+\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left[\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j\partial\theta_l}(H(y_2;\theta)-H(y_1;\theta))-\right.\\
\left.-\frac{\partial(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l}\frac{\partial(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j}\right]\frac{dG(y_1,y_2)}{(H(y_2;\theta)-H(y_1;\theta))} =\\
=\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int_{-\infty}^{y_1}\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_l\partial\theta_j}\mu(dx)+\int_{y_2}^{\infty}\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_l\partial\theta_j}\mu(dx)\right)dG(y_1,y_2)+\\
+\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l\partial\theta_j}dG(y_1,y_2)-J_{lj}(\theta) =\\
+\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left[-\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l\partial\theta_j}+\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l\partial\theta_j}\right]dG(y_1,y_2)-\\
-J_{lj}(\theta) = -J_{lj}(\theta).
\end{gathered} \tag{48}$$

The equality (14) follows from (47) and (48). □

*Proof of lemma 3.4.* Under the regularity conditions of the lemma

$$\begin{gathered}
\sum_{i=1}^{k}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left(\int_{\Gamma_1}h^{(i)}(x;\theta+u)\mu(dx)+\int_{\Gamma_2}h^{(i)}(x;\theta+u)\mu(dx)\right)dG(y_1,y_2) =\\
=\frac{1}{2}\left(\nabla_1^{(2)}u;u\right) = o\left(|u|^2\right),
\end{gathered} \tag{49}$$

where $\Gamma_1 = \overline{N}^{(i)} \cap (-\infty, y_1)$, $\Gamma_2 = \overline{N}^{(i)} \cap (y_2, \infty)$,

$$\nabla_1^{(2)} = \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left(\int_{\Gamma_1}\frac{\partial^2 h(x;\theta*)}{\partial\theta^2}\mu(dx)+\int_{\Gamma_2}\frac{\partial^2 h(x;\theta*)}{\partial\theta^2}\mu(dx)\right)dG(y_1,y_2),$$

$\theta*$ is between $\theta$ and $\theta + u$. It is easy to verify that for $|u| \to 0$

$$\frac{\sqrt{h^{(i)}(x;\theta+u)}-\sqrt{h^{(i)}(x;\theta)}}{\sqrt{h^{(i)}(x;\theta)}}-\frac{1}{2}\left(u;\frac{\partial\log h^{(i)}(x;\theta)}{\partial\theta}\right) = o\left(|u|\right). \tag{50}$$

Therefore, when $|u| \to 0$

$$E_\theta \left\{ w \sum_{i=1}^{k} \delta^{(i)} \left[ \xi_{ni}(X;u) - \frac{1}{2} \left( u; \frac{\partial \log h^{(i)}(X;\theta)}{\partial \theta} \right) \right] \right\}^2 = o\left(|u|^2\right). \tag{51}$$

Similarly, we have

$$\int_{-\infty}^{\infty} \int\limits_{\overline{N}^{(i)} \cap (y_1,\infty)} \left( H(y_2;\theta+u) - H(y_1;\theta+u) \right) dG(y_1,y_2) = \frac{1}{2} \left( \nabla_2^{(2)} u; u \right) = o\left(|u|^2\right), \tag{52}$$

where

$$\nabla_2^{(2)} = \int_{-\infty}^{\infty} \int\limits_{\overline{N}^{(i)} \cap (-\infty,y_1)} \left[ \frac{\partial^2}{\partial \theta^2} \left( H(y_2;\theta*) - H(y_1;\theta*) \right) \right] dG(y_1,y_2),$$

also

$$\begin{gathered} \frac{\sqrt{H(y_2;\theta+u) - H(y_1;\theta+u)} - \sqrt{H(y_2;\theta) - H(y_1;\theta)}}{\sqrt{H(y_2;\theta) - H(y_1;\theta)}} - \\ -\frac{1}{2} \left( u; \frac{\partial \log \left( H(y_2;\theta) - H(y_1;\theta) \right)}{\partial \theta} \right) = o\left(|u|\right), \end{gathered} \tag{53}$$

and hence

$$E_\theta \left\{ (1-w) \left[ \eta_n(Y_1,Y_2;u) - \frac{1}{2} \left( u; \frac{\partial \log \left( H(Y_2;\theta) - H(Y_1;\theta) \right)}{\partial \theta} \right) \right] \right\}^2 = o\left(|u|^2\right). \tag{54}$$

By (49)

$$\begin{gathered} E_\theta \left[ w \sum_{i=1}^{k} \delta^{(i)} \left[ \xi_{ni}^2(X;u) \right] \right] = \sum_{i=1}^{k} \left[ \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left( \int_{\Gamma_1} \left( \sqrt{h^{(i)}(x;\theta+u)} - \sqrt{h^{(i)}(x;\theta)} \right)^2 \mu(dx) + \right.\right. \\ \left.\left. + \int_{\Gamma_2} \left( \sqrt{h^{(i)}(x;\theta+u)} - \sqrt{h^{(i)}(x;\theta)} \right)^2 \mu(dx) \right) dG(y_1,y_2) \right] = \\ = 2 \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left( \int_{-\infty}^{y_1} h(x;\theta)\mu(dx) + \int_{y_2}^{\infty} h(x;\theta)\mu(dx) \right) dG(y_1,y_2) + o\left(|u|^2\right) - \\ -2 \sum_{i=1}^{k} \left[ \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left( \int_{-\infty}^{y_1} \sqrt{h^{(i)}(x;\theta+u)} \sqrt{h^{(i)}(x;\theta)} \mu(dx) + \right.\right. \\ \left.\left. + \int_{y_2}^{\infty} \sqrt{h^{(i)}(x;\theta+u)} \sqrt{h^{(i)}(x;\theta)} \mu(dx) \right) \right] dG(y_1,y_2) = \\ = 2 \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left[ 1 - \left( H(y_2;\theta) - H(y_1;\theta) \right) \right] dG(y_1,y_2) + o\left(|u|^2\right) = \\ = 2 \sum_{i=1}^{k} \left[ \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left( \int_{-\infty}^{y_1} \left( \xi_{ni}(x;u) + 1 \right) dH^{(i)}(x;\theta) + \right.\right. \\ \left.\left. + \int_{y_2}^{\infty} \left( \xi_{ni}(x;u) + 1 \right) dH^{(i)}(x;\theta) \right) dG(y_1,y_2) \right] = \\ = 2 \sum_{i=1}^{k} \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left( \int_{-\infty}^{y_1} \xi_{ni}(x;u) dH^{(i)}(x;\theta) + \int_{y_2}^{\infty} \xi_{ni}(x;u) dH^{(i)}(x;\theta) \right) dG(y_1,y_2) + \\ + o\left(|u|^2\right) = 2 E_\theta \left[ w \sum_{i=1}^{k} \delta^{(i)} \xi_{ni}(X;u) \right] + o\left(|u|^2\right). \end{gathered} \tag{55}$$

Now, (15) and (17) follow from (51), also (16) and (18) follow from (15). From (15) and (55) we obtain (21). On the other hand, by (52)

$$
\begin{aligned}
&E_\theta\left[(1-w)\eta_n^2(Y_1,Y_2;u)\right] = \\
&= \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left(\sqrt{H(y_2;\theta+u)-H(y_1;\theta+u)}-\sqrt{H(y_2;\theta)-H(y_1;\theta)}\right)^2 dG(y_1,y_2) = \\
&= 2\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left(H(y_2;\theta)-H(y_1;\theta)\right)dG(y_1,y_2)+o\left(|u|^2\right)- \\
&-2\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left(\eta_n(y_1,y_2;u)+1\right)\left(H(y_2;\theta)-H(y_1;\theta)\right)dG(y_1,y_2) = \\
&= 2E_\theta\left[(1-w)\eta_n(Y_1,Y_2;u)\right]+o\left(|u|^2\right).
\end{aligned} \tag{56}
$$

Now (22) follows from (16) and (56). In order to establish (19) and (20), note that from (51) and (54), respectively, we have

$$
E_\theta\left\{w\sum_{i=1}^{k}\delta^{(i)}\left[\xi_{ni}(X;u)-\frac{1}{2}\left(u;\frac{\partial\log h^{(i)}(X;\theta)}{\partial\theta}\right)\right]\right\}^2 = o\left(|u|^2\right), \tag{57}
$$

$$
E_\theta\left\{(1-w)\left[\eta_n(Y_1,Y_2;u)-\frac{1}{2}\left(u;\frac{\partial\log\left(H(Y_2;\theta)-H(Y_1;\theta)\right)}{\partial\theta}\right)\right]\right\}^2 = o\left(|u|^2\right). \tag{58}
$$

By (57)

$$
\begin{aligned}
&P_\theta\left[\left|w\sum_{i=1}^{k}\delta^{(i)}\xi_{ni}(X;u)\right|>\varepsilon\right]\leqslant \\
&\leqslant P_\theta\left[\left|w\sum_{i=1}^{k}\delta^{(i)}\left[\xi_{ni}(X;u)-\frac{1}{2}\left(u;\frac{\partial\log h^{(i)}(X;\theta)}{\partial\theta}\right)\right]\right|>\frac{\varepsilon}{2}\right]\leqslant \\
&\leqslant P_\theta\left[\left|w\sum_{i=1}^{k}\delta^{(i)}\left(u;\frac{\partial\log h^{(i)}(X;\theta)}{\partial\theta}\right)^2\right|>\varepsilon^2\right]\leqslant \\
&\leqslant \frac{4}{\varepsilon^4}E_\theta\left\{w\sum_{i=1}^{k}\delta^{(i)}\left[\xi_{ni}(X;u)-\frac{1}{2}\left(u;\frac{\partial\log h^{(i)}(X;\theta)}{\partial\theta}\right)\right]\right\}^2+ \\
&+\frac{1}{\varepsilon^2}\int_{-\infty}^{\infty}\sum_{i=1}^{k}\int_{y_1}^{\infty}\left(\int_{\Omega_1}\left(u;\frac{\partial\log h^{(i)}(x;\theta)}{\partial\theta}\right)^2 dH^{(i)}(x;\theta)+\right. \\
&\left.+\int_{\Omega_2}\left(u;\frac{\partial\log h^{(i)}(x;\theta)}{\partial\theta}\right)^2 dH^{(i)}(x;\theta)\right)dG(y_1,y_2)=o\left(|u|^2\right),
\end{aligned} \tag{59}
$$

$$
\left(\Omega_1=(-\infty,y_1)\cap\left\{x:\left|\frac{\partial\log h^{(i)}(x;\theta)}{\partial\theta}\right|>\frac{\varepsilon}{|u|}\right\},\ \Omega_2=(y_2,\infty)\cap\left\{x:\left|\frac{\partial\log h^{(i)}(x;\theta)}{\partial\theta}\right|>\frac{\varepsilon}{|u|}\right\}\right)
$$

where the first summand on the right hand side of (59) is $o\left(|u|^2\right)$ by (51), and the second is also of the order $o\left(|u|^2\right)$ thanks to the convergence of the integral $\mathbb{J}_1(\theta)$. Quite similarly, using (58) we have

$$
\begin{aligned}
&P_\theta\left(|(1-w)\eta_n(Y_1,Y_2;u)|>\varepsilon\right)\leqslant \\
&\leqslant P_\theta\left(\left|(1-w)\eta_n(Y_1,Y_2;u)-\frac{1}{2}\left(u;\frac{\partial\log(H(Y_2;\theta)-H(Y_1;\theta))}{\partial\theta}\right)\right|>\frac{\varepsilon}{2}\right)+ \\
&\leqslant P_\theta\left((1-w)\eta_n^2(Y_1,Y_2;u)>\varepsilon^2\right)=o\left(|u|^2\right).
\end{aligned} \tag{60}
$$

Now, (19) and (20) follow from (59) and (60), respectively. □

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## Локальная асимптотическая нормальность семейств распределений по неполным наблюдениям


**Абдурахим А. Абдушукуров**
**Наргиза С. Нурмухамедова**



*В данной статье доказано свойство локальной асимптотической нормальности статистики отношения правдоподобия в модели конкурирующих рисков при случайном цензурировании интервалом ненаблюдения.*



*Ключевые слова: конкурирующие риски, случайное цензурирование, статистика отношения правдоподобия, локальная асимптотическая нормальность.*