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HAYDAROV FARHOD HALIMJANOVICH

GIBBS MEASURES FOR MODELS WITH UNCOUNTABLE SET OF SPIN VALUES ON A CAYLEY TREE

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scientific advisor

prof. Yu.Kh. Eshkobilov

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Contents

PREFECE

The Gibbs measure is a probablity measure, which has been an important object in many problems of problems of probability theory and statistical mechanics. It is the measure associated with the Hamiltonian of a physical system (a model) and generalizes the notion of a canonical ensemble. More importantly, when the Hamiltonian can be written as a sum of parts, the Gibbs measure has the Markov property (a certain kind of statistical independence), thus leading to its widespread appearance in many problems outside of physics, such as biology, Hopfield networks, Markov networks, and Markov logic networks. Moreover, the Gibbs measure is the unique measure that maximizes the entropy for a given expected energy.

One of the central problems in the theory of Gibbs measures is to describe infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the ground-breaking work of Dobrushin (see, e.g. [23]). However, a complete analysis of the set of limiting Gibbs measures for a specific Hamiltonian is often a difficult problem.

The structure of the lattice (graph) plays an important role in investigations of spin systems. For example, in order to study the phase transition problem for a system on Z^d and on Cayley tree there are two different methods: Pirogov-Sinai theory on Z^d, Markov random field theory and recurrent equations of this theory on Cayley tree. In [1]- [11] [16], [17]- [25], [29] for several models on Cayley trees, using the Markov random field theory Gibbs measures are described.

These papers are devoted to models with a *finite* set of spin values. Mainly were shown that these models have finitely many translation-invariant and uncountable numbers of the non-translation-invariant extreme Gibbs measures. Also for several models (see, for example, [8], [10], [20]) it were proved that there exist three periodic Gibbs measures (which are invariant with respect to normal subgroups of finite index of the group representation of the Cayley tree) and there are uncountable number of non-periodic Gibbs measures.

In [9] the Potts model with a countable set of spin values on a Cayley tree is considered. It was showed that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independently on parameters of the Potts model with countable set of spin values. This is a crucial difference from the models with a finite set of spin values, since the last ones may have more than one translation-invariant Gibbs measures.

In [21] It is considered models with nearest-neighbor interactions and with the set $[0,1]$ of spin values, on Cayley tree of order $k=1$ and reduced the problem of describing the "splitting Gibbs measures" of the model to the description of the solutions of some non-linear integral equation. Also For it is showed that the integral equation has a unique solution and for the Potts model with uncountable set of spin values it is proven that there is unique splitting Gibbs measure.

In this dissertation we consider models with nearest-neighbor interactions and with the set $[0,1]$ of spin values, on Cayley tree of order $k\geq 2$. Moreover we'll prove that each periodic Gibbs measures for given model is either translation-invariant or two-periodic. In case $k\geq 2$ some models (with the set $[0,1]$ of spin values) which have a unique splitting Gibbs measures are constructed. For arbitrary $k\geq 2$ we find a sufficient condition under which the integral equation has unique solution; hence under this condition the corresponding model has unique splitting Gibbs measure. Finally, We construct several models with the set $[0,1]$ of spin values and show that each of the constructed model has at least two (at least n Gibbs measures for $k \geq k_0$) periodic Gibbs measures.

1 BASIC CONCEPTS AND AUXILIARY FACTS

1.1 GIBBS MEASURE CORRESPONDING TO THE HAMILTONIAN ON A CAYLEY TREE.

Consider models where the spin takes values in the set $[0,1]$, and is assigned to the vertexes of the tree. For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to [0,1]$. Denote $\Omega_A = [0,1]^A$ the set of all configurations on A. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in [0,1]$; the set of all configurations is $[0,1]^V$. The (formal) Hamiltonian of the model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x),\sigma(y)}, \tag{1.1}$$

where $J \in R \setminus \{0\}$ and $\xi : (u,v) \in [0,1]^2 \to \xi_{u,v} \in R$ is a given bounded, measurable function. As usually, $\langle x, y\rangle$ stands for nearest neighbor vertices, i.e., $d(x,y) = 1$.

Let λ be the Lebesgue measure on $[0,1]$. On the set of all configurations on A the a priori measure λ_A is introduced as the $|A|$ fold product of the measure λ. Here and further on $|A|$ denotes the cardinality of A. We consider a standard sigma-algebra $\mathcal{B}$ of subsets of $\Omega = [0,1]^V$ generated by the measurable cylinder subsets. A probability measure μ on $(\Omega, \mathcal{B})$ is called a Gibbs measure (with Hamiltonian H) if it satisfies the DLR equation, namely for any $n = 1, 2, \ldots$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu\left(\left\{\sigma \in \Omega : \sigma\big|_{V_n} = \sigma_n\right\}\right) = \int_\Omega \mu(\mathrm{d}\omega)\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n),$$

where $\nu^{V_n}_{\omega|_{W_{n+1}}}$ is the conditional Gibbs density

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = \frac{1}{Z_n\left(\omega\big|_{W_{n+1}}\right)} \exp\left(-\beta H\left(\sigma_n \,||\, \omega\big|_{W_{n+1}}\right)\right),$$

and $\beta = \frac{1}{T}$, $T > 0$ is temperature. Here and below, Ω_{V_n} is the set of configurations in V_n (and Ω_{W_n} that in W_n; see below). Furthermore, $\sigma\big|_{V_n}$ and $\omega\big|_{W_{n+1}}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to V_n and W_{n+1}, respectively. Next, $\sigma_n: \ x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and $H\left(\sigma_n \,||\, \omega\big|_{W_{n+1}}\right)$ is defined as the sum $H(\sigma_n) + U\left(\sigma_n, \omega\big|_{W_{n+1}}\right)$ where

$$H(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \xi_{\sigma_n(x),\sigma_n(y)},$$

$$U\left(\sigma_n, \omega\big|_{W_{n+1}}\right) = -J \sum_{\langle x,y\rangle:\, x\in V_n, y\in W_{n+1}} \xi_{\sigma_n(x),\omega(y)}.$$

Finally, $Z_n\left(\omega\big|_{W_{n+1}}\right)$ stands for the partition function in V_n, with the boundary condition $\omega\big|_{W_{n+1}}$:

$$Z_n\left(\omega\big|_{W_{n+1}}\right) = \int_{\Omega_{V_n}} \exp\left(-\beta H\left(\widetilde{\sigma}_n \,||\, \omega\big|_{W_{n+1}}\right)\right) \lambda_{V_n}(d\widetilde{\sigma}_n).$$

Due to the nearest-neighbor character of the interaction, the Gibbs measure possesses a natural Markov property: for a given configuration ω_n on W_n, random configurations in V_{n-1} (i.e., 'inside' W_n) and in $V \setminus V_{n+1}$ (i.e., 'outside' W_n) are conditionally independent.

Let $h: \ x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in R^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right). \tag{1.2}$$

Here, as before, $\sigma_n: x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x\in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda_{V_n}(\widetilde{\sigma}_n). \tag{1.3}$$

The probability distributions $\mu^{(n)}$ are compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{1.4}$$

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. In this case there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$.

Definition 1. *The measure μ is called splitting Gibbs measure corresponding to Hamiltonian (1.1) and function $x \mapsto h_x$, $x \neq x^0$.*

The following statement describes conditions on h_x guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

Proposition 1. *The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (1.2) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{t,u}) f(u, y) du}{\int_0^1 \exp(J\beta\xi_{0,u}) f(u, y) du} \tag{1.5}$$

Here, and below $f(t, x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0, 1]$ and $du = \lambda(du)$ is the Lebesgue measure.

Proof. Necessity. Suppose that (1.4) holds; we want to prove (1.5). Substituting (1.2) in (1.4), obtain that for any configurations σ_{n-1}: $x \in V_{n-1} \mapsto \sigma_{n-1}(x) \in [0, 1]$:

$$\frac{Z_{n-1}}{Z_n} \int_{\Omega_{W_n}} \exp\left(\sum_{x \in W_{n-1}} \sum_{y \in S(x)} \left(J\beta d_{\sigma_{n-1}(x)\omega_n(y)} + h_{\omega_n(y),y} \right) \right) \lambda_{W_n}(d\omega_n) =$$

$$\exp\left(\sum_{x\in W_{n-1}} h_{\sigma_{n-1}(x),x}\right), \tag{1.6}$$

where ω_n: $x \in W_n \mapsto \omega_n(x)$.

From (1.6) we get:

$$\frac{Z_{n-1}}{Z_n}\int_{\Omega_{W_n}} \prod_{x\in W_{n-1}}\prod_{y\in S(x)} \exp\left(J\beta d_{\sigma_{n-1}(x)\omega_n(y)} + h_{\omega_n(y),y}\right)d(\omega_n(y)) =$$

$$\prod_{x\in W_{n-1}} \exp\left(h_{\sigma_{n-1}(x),x}\right).$$

Consequently, for any $t \in [0,1]$,

$$\prod_{y\in S(x)} \frac{\int_0^1 \exp\left(J\beta d_{tu} + h_{u,y}\right)du}{\int_0^1 \exp\left(J\beta d_{0u} + h_{u,y}\right)du} = \exp\left(h_{t,x} - h_{0,x}\right),$$

which implies (1.5).

Sufficiency. Suppose that (1.5) holds. It is equivalent to the representations

$$\prod_{y\in S(x)} \int_0^1 \exp\left(J\beta d_{tu} + h_{u,y}\right)du = a(x)\exp\left(h_{t,x}\right), t \in [0,1] \tag{1.7}$$

for some function $a(x) > 0, x \in V$. We have

$$\text{LHS of } (1.4) = \frac{1}{Z_n}\exp(-\beta H(\sigma_{n-1}))\lambda_{V_{n-1}}(d(\sigma_n))\times$$

$$\prod_{x\in W_{n-1}}\prod_{y\in S(x)} \int_0^1 \exp\left(J\beta d_{\sigma_{n-1}(x)u} + h_{u,y}\right)du. \tag{1.8}$$

Substituting (1.7) into (1.8) and denoting $A_n(x) = \prod_{x\in W_{n-1}} a(x)$, we get

$$\text{RHS of } (1.8) = \frac{A_{n-1}}{Z_n}\exp(-\beta H(\sigma_{n-1}))\lambda_{V_{n-1}}(d\sigma)\prod_{x\in W_{n-1}} h_{\sigma_{n-1}(x),x}. \tag{1.9}$$

Since $\mu^{(n)}$, $n \geq 1$ is a probability, we should have

$$\int_{\Omega_{V_{n-1}}} \lambda_{V_{n-1}}(d\sigma_{n-1}) \int_{\Omega_{W_n}} \lambda_{W_n}(d\omega_n)\mu^{(n)}(\sigma_{n-1}, \omega_n) = 1$$

Hence from (1.9) we get $Z_{n-1}A_{n-1} = Z_n$, and (1.4) holds.

From Proposition 1 it follows that for any $h = \{h_x \in R^{[0,1]}, \;\; x \in V\}$ satisfying (1.5) there exists a unique Gibbs measure μ and vice versa. However, the analysis of solutions to (1.5) is not easy. This difficulty depends on the given function ξ. In the next sections we will consider several examples of such functions and give some solutions of corresponding integral equations.

□

1.2 GROUP REPRESENTATION ON A CAYLEY TREE AND PERIODIC GIBBS MEASURE.

Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ...a_{k+1}$, respectively.

It is known that there exists a one to one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

To give this correspondence we fix an arbitrary element $x_0 \in V$ and let it correspond to the unit element e of the group G_k. Using $a_1, ..., a_{k+1}$ we numerate the nearest-neighbors of element e, moving by positive direction. Now we'll give numeration of the nearest-neighbors of each $a_i, i = 1, ..., k+1$ by $a_ia_j, j = 1, ..., k+1$. Since all a_i have the common neighbor e we give to it $a_ia_i = a_i^2 = e$. Other neighbor are numerated starting from a_ia_i by the positive direction. We numerate the set of all the nearest-neighbors of each a_ia_j by words $a_ia_ja_q, q = 1, ..., k+1$, starting from $a_ia_ja_j = a_i$ by the positive direction. Iterating this argument one gets a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

Since the set of vertices V has the group representation G_k. Without lost of generality we identify V with G_k, i.e., we sometimes replace V with G_k.

In this section we study periodic solutions of (1.5).

Definition 2. *Let K be a subgroup of $G_k, k \geq 1$. We say that a functions $h_x, x \in G_k$ is K-periodic if $h_{yx} = h_x$ for all $x \in G_k$, $y \in K$. A G_k- periodic function h is called translation-invariant.*

Definition 3. *A Gibbs measure is called K- periodic if it corresponds to K-periodic function h.*

Denote by $S_1(x) = \{y \in G_k :< x, y >\}$ the set of all nearest of the word $x \in G_k$. Let K- be a normal subgroup of index r in G_k, and let $G_k/K = \{K_0, K_1, ..., K_{r-1}\}$ be a quotient group, with the coset $K_0 = K$. In addition, let $q_i(x) = |S_1(x) \bigcap K_i|, i = 0, 1, ..., r-1$, and $Q(x) = (q_0(x), q_1(x), ..., q_{r-1}(x))$ where $x \in G_k$, $q_i(H_0) = q_i(e) = |\{j \in: H_i\}|$, $Q(H_0) = (q_0(H_0), ..., q_{n-1}(H_0))$.

Let $H_0 \subset G_k$ be an arbitrary normal subgroups of index n of the group G_k. Obviously, each normal subgroups of the group G_k is the kernel of some homomorphism φ of the group G_k into some group G^*. Introduce the following equivalence relation on the set $G_k : x \sim y$ if $xy^{-1} \in H_0$.

Proposition 2. *(i) $xy \sim xz$ if and only if $y \sim z$ and $x, y, z \in G_k$;*

(ii) $yx \sim zx$ if and only if $y \sim z$ and $x, y, z \in G_k$.

Proof. Necessity. Let $xy \sim xz$, i.e., $xy(xy)^{-1} = xyz^{-1}x^{-1} \in H_0$. Hence, $\varphi(xyz^{-1}x^{-1}) = e \in G^*$ (e is the unit element of the group G^*). Since φ is a homomorphism, we obtain

$$\varphi(x)\varphi(yz^{-1})\varphi(x^{-1}) = e. \tag{1.10}$$

or $\varphi(xz^{-1}) = [\varphi(x)]^{-1}[\varphi(x^{-1})]^{-1} = \varphi(x^{-1})[\varphi(x^{-1})]^{-1} = e$, i.e., $y \sim z$.

Sufficiency. Let $y \sim z$, i.e.,

$$\varphi(yz^{-1}) = e \tag{1.11}$$

Consider the element $\varphi(xyz^{-1}x^{-1})$. As φ is a homomorphism, we have, by virtue of 1.11, $\varphi(xyz^{-1}x^{-1}) = e$, i.e., $xy \sim xz$.

Statement (i) is proved. Statement (ii) can be proved analogously. □

Corollary 1. *(i) $xa_i \sim xa_j$ if and only if $a_i \sim a_j$, where*

$a_i, a_j \in \{e, a_1, ..., a_{k+1}\}$ and $x \in G_k$;

(ii) $xa_i \sim ya_i$ if and only if $x \sim y$, where $a_i \in \{e, a_1, ..., a_{k+1}\}$ and $x, y \in G_k$.

Denote by $S_1 = \{y \in G_k : \langle x, y\rangle\}$ the set of all nearest neighbors of the word $x \in G_k$. Let $G_k/H_0 = \{H_0, H_1, ..., H_{n-1}\}$ be the factor-group w.r.t H_0. In addition, let $q_i(x) = |S_1(x) \cap H_i|, i = 0, ..., n-1$, and

$Q(x) = (q_0(x), q_1(x), ..., q_{n-1}(x))$, $x \in G_k$.

Proposition 3. *If $x \sim y$, then $q_i(x) = q_i(y)$ for $i = 0, ..., n-1$.*

Proof. Let $x \sim y$. Then, by virtue of Corollary 1, $xa_i = ya_j$ for any $i = 0, ..., n-1$. Therefore, if $S_1(x) \cap H_i = \{xa_{i_1}, xa_{i_2}, ..., xa_{i_{q_i(x)}}\}$, then $S_1(y) \cap H_i = \{ya_{i_1}, ya_{i_2}, ..., ya_{i_{q_i(x)}}\}$ for any $i = 0, ..., n-1$, i.e., $|S_1(x) \cap H_i| = |S_1(y) \cap H_i|$. □

Corollary 2. *If $x \sim y$, then $Q(x) = Q(y)$.*

Introduce the following notations:

$$q_i(H_0) = q_i(e) = |\{j : a_j \in H_i\}|,$$

$$Q(H_0) = (q_0(H_0), ..., q_{n-1}(H_0)), N(H_0) = \{j : q_j(H_0) \neq 0\}.$$

Theorem 1. *For any $x \in G_k$, there exists a permutation π_x of the coordinates of the vector $Q(H_0)$ such that*

$$\pi_x(Q(H_0)) = Q(x). \tag{1.12}$$

Proof. Obviously, $S_1(x) = xS_1(e) = x\{a_1, a_2, ..., a_{k+1}\} = \{xa_1, ..., xa_{k+1}\}$. By virtue of Corollary 1, for any $i = 0, ..., n-1$, there exists an index $j(i) \in \{0, 1, ..., n-1\}$ such that

$$q_i(H_0) = |\{j : a_j \in H_i\}| = |\{xa_m : xa_m \in H_{j(i)}\}| = q_{j(i)}(x).$$

Set $\pi_x(i) = j(i)$. □

Corollary 3. *For any $x \in G_k$ we have $N(x) = N(H_0)$.*

Let $G_k^{(2)} = \{x \in G_k :$ the length of word x is even.$\}$

Theorem 2. *Let K be a normal subgroup of finite index in G_k. Then each K- periodic Gibbs measure for the model is either translation-invariant or $G_k^{(2)}$ − periodic.*

Proof. By Proposition 1

$$h_x(t) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{t,u})h_y(u)du}{\int_0^1 \exp(J\beta\xi_{0,u})h_y(u)du}.$$

Let $y_0 \in S_1(x) \setminus S(x)$ and $\widetilde{y} \in S(x)$. By Theorem 1

$$\prod_{y \in S_1(x)\setminus\{\widetilde{y}\}} \frac{\int_0^1 \exp(J\beta\xi_{t,u})h_y(u)du}{\int_0^1 \exp(J\beta\xi_{0,u})h_y(u)du} = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{t,u})h_y(u)du}{\int_0^1 \exp(J\beta\xi_{0,u})h_y(u)du}.$$

Then

$$\frac{\int_0^1 \exp(J\beta\xi_{t,u})h_{y_0}(u)du}{\int_0^1 \exp(J\beta\xi_{0,u})h_{y_0}(u)du} = \frac{\int_0^1 \exp(J\beta\xi_{t,u})h_{\widetilde{y}}(u)du}{\int_0^1 \exp(J\beta\xi_{0,u})h_{\widetilde{y}}(u)du}.$$

Hence

$$h_x(t) = \left(\frac{\int_0^1 \exp(J\beta\xi_{t,u}) h_{y_0}(u) du}{\int_0^1 \exp(J\beta\xi_{0,u}) h_{y_0}(u) du} \right)^k$$

Similarly we get

$$h_{\widetilde{y}}(t) = \left(\frac{\int_0^1 \exp(J\beta\xi_{t,u}) h_x(u) du}{\int_0^1 \exp(J\beta\xi_{0,u}) h_x(u) du} \right)^k$$

Thus $h_{y_1}(t) = h_{y_2}(t)$, $\forall y_1, y_2 \in S_1(x)$. Therefore

$$h_{\widetilde{x}} = h_{\widetilde{y}} = h, \quad \widetilde{x}, \widetilde{y} \in S_1(z), \quad z \in G_k^{(2)},$$

$$h_{\widetilde{x}} = h_{\widetilde{y}} = l, \quad \widetilde{x}, \widetilde{y} \in S_1(z), \quad z \in G_k \setminus G_k^{(2)}.$$

Thus measures are *translation-invariant* (if $h = l$) or $G_k^{(2)} -$ *periodic* (if $h \neq l$). This completes the proof. □

2 EXISTENCE AND UNIQUENESS OF GIBBS MEASURES ON A CAYLEY TREE

2.1 EXISTENCE OF GIBBS MEASURE

In this section we consider ξ_{tu} as a continuous function and we are going to solve equation (1.5) in the class of periodic functions $f(t,x)$. By Theorem 2 we have only two cases. One of them *translation-invariant*, i.e., $G_k -$ *periodic* and the second is $G_k^{(2)} -$ *periodic*.

For *translation-invariant* functions equation (1.5) can be written as

$$f(t) = \left(\frac{\int_0^1 K(t,u) f(u) du}{\int_0^1 K(0,u) f(u) du} \right)^k, \tag{2.1}$$

where $K(t,u) = \exp(J\beta\xi_{tu}) > 0, f(t) > 0, t, u \in [0,1]$.

We shall find positive continuous solutions to (2.1) i.e. such that

$f \in C^+[0,1] = \{f \in C[0,1] : f(x) > 0\}$.

Note that equation (2.1) is not linear for any $k \geq 1$.

Define the linear operator $W : C[0,1] \to C[0,1]$ by

$$(Wf)(t) = \int_0^1 K(t,u)f(u)du \tag{2.2}$$

and defined the linear functional $\omega : C[0,1] \to R$ by

$$\omega(f) \equiv (Wf)(0) = \int_0^1 K(0,u)f(u)du.$$

Then equation (2.1) can be written as

$$f(t) = (A_k f)(t) = \left(\frac{(Wf)(t)}{(Wf)(0)}\right)^k, \ f \in C^+[0,1], \ k \geq 1. \tag{2.3}$$

Similarly, $G_k^{(2)}$ $-$ *periodic* solutions to (1.5), i.e., $f(t,x) = f(t)$ if x - even and $f(t,x) = g(t)$ if x - odd. For such functions equation (1.5) can be written as

$$f(t) = \left(\frac{\int_0^1 K(t,u)g(u)du}{\int_0^1 K(0,u)g(u)du}\right)^k, \ g(t) = \left(\frac{\int_0^1 K(t,u)f(u)du}{\int_0^1 K(0,u)f(u)du}\right)^k, \tag{2.4}$$

where $K(t,u) = \exp(J\beta\xi_{tu}), \ f(t), g(t) > 0, \ t,u \in [0,1]$.

We are interested to positive continuous solutions to (1.5), i.e. such that

$$f, g \in C^+[0,1] = \{\varphi \in C[0,1] : \varphi(x) > 0\}.$$

Then (1.5) can be written as

$$A_k f = g, \ A_k g = f, \quad (f,g) \in \left(C_0^+[0,1]\right)^2. \tag{2.5}$$

Now we'll prove that there exist *translation−invariant* solution of (1.5). At first we check the case $k = 1$.

Proposition 4. *If $f \in C^+[0,1]$ is a solution to (2.1) then*

$$f(t) \geq \frac{\kappa^{\min}}{\kappa_0^{\max}}, \quad \text{for any} \ \ t \in [0,1],$$

where $\kappa^{\min} = \inf_{t,u\in[0,1]} K(t,u), \quad \kappa_0^{\max} = \sup_{u\in[0,1]} K(0,u).$

Proof. Straightforward. □

Denote

$$C_0^+ = \left\{ h \in C^+[0,1] : h(t) \geq \frac{\kappa^{\min}}{\kappa_0^{\max}} \right\}$$

The following Lemma is also obvious

Lemma 1. *(i) The set C_0^+ is a closed and convex subset of the space $C[0,1]$.*
(ii) The set C_0^+ is invariant w.r.t. operator A_1 i.e. $A_1(C_0^+) \subset C_0^+$.

Lemma 2. *Operator A_1 is continuous on C_0^+.*

Proof. Let $f \in C_0^+$ is an arbitrary element and $\{f_n\} \subset C_0^+$ such that $\lim_{n\to\infty} f_n = f$. We shall prove that $\|A_1 f_n - A_1 f\| \to 0$ as $n \to \infty$. We have

$$|A_1 f_n - A_1 f| \leq \frac{W f_n |\omega(f_n) - \omega(f)| + \omega(f_n)|W f_n - W f|}{\omega(f)\omega(f_n)}. \tag{2.6}$$

Since the functional $\omega(\cdot)$ and the operator $W(\cdot)$ are continuous on $C[0,1]$, for any small $\varepsilon > 0$ there exits $n_0 = n_0(\varepsilon) \in N$ such that

$$|\omega(f_n) - \omega(f)| < \varepsilon, \quad \|W f_n - W f\| < \varepsilon, \quad \forall n > n_0.$$

Consequently

$$\|A_1 f_n - A_1 f\| < \frac{\|W f_n\| + \omega(f_n)}{(\omega(f) - \varepsilon)\omega(f)} \cdot \varepsilon. \tag{2.7}$$

There are $M_i, i = 0, 1, 2$ such that $\omega(f) \geq M_0$, for all $f \in C_0^+$ and

$$\|W(f_n)\| \leq M_1, \quad \omega(f_n) \leq M_2, \; n \in N.$$

Thus from (2.7) we get

$$\|A_1 f_n - A_1 f\| < \frac{M_1 + M_2}{(M_0 - \varepsilon)M_0} \cdot \varepsilon, \quad n > n_0.$$

This completes the proof. □

Lemma 3. *The set $A_1(C_0^+)$ is relatively compact in $C[0,1]$.*

Proof. By Arzelá-Askoli's theorem (see [29], ch.III,§3) it suffices to prove that all functions of $A_1(C_0^+)$ are uniformly continuous and there exists $M > 0$ such that

$$|h(t)| \leq M, \quad \forall t \in [0,1] \quad \text{and} \quad \forall h \in A_1(C_0^+).$$

Let $h \in A_1(C_0^+)$ be an arbitrary function, then for a function $f \in C_0^+$ we have $h = A_1 f$. Consequently

$$|h(t)| \leq \frac{\kappa^{\max}}{\kappa_0^{\min}}, \ \forall t \in [0,1].$$

Now we shall prove that any $h \in A_1(C_0^+)$ is uniformly continuous. For arbitrary $t, t' \in [0,1]$ we have ($h = A_1 f$)

$$|h(t) - h(t')| \leq \frac{1}{\omega(f)} \int_0^1 |K(t,u) - K(t',u)| f(u) du. \tag{2.8}$$

Since the kernel $K(t,u)$ is uniformly continuous on $[0,1]^2$ we conclude that h also is a uniformly continuous function. This completes the proof. □

By Lemma 1-3 and Schauder's theorem (see [28], p.20) one obtains

Proposition 5. *[21] The equation $A_1 f = f$ has at least one solution in $C^+[0,1]$.*

2.2 THE HAMMERSTEIN'S NONLINEAR EQUATION

In this section we consider the existence of *periodic* solution of (1.5) for case $k \geq 2$ and connection of The Hammerstein's nonlinear equation. Denote

$$\mathcal{F}_k = \left\{ f \in C^+[0,1] : f(t) \geq \left(\frac{m}{M_0} \right)^k \right\}, k \in \mathbb{N},$$

where

$$m = \min_{t,u \in [0,1]} K(t,u), \quad M_0 = \max_{u \in [0,1]} K(0,u).$$

It is easy to see that $\mathcal{F}_k$ is a closed and convex subset of $C[0,1]$. Moreover this set is invariant with respect to operator A_k, i.e. $A_k(\mathcal{F}_k) \subset \mathcal{F}_k$.

Proposition 6. *The operator A_k is continuous on $\mathcal{F}_k$ for any $k \geq 2$.*

Proof. For arbitrary $C > 0$ we denote

$$\mathcal{F}_0 = \left\{ f \in C^+[0,1] : f(t) \geq C, \, \forall t \in [0,1] \right\}.$$

By Lemma 2 the operator A_1 is continuous on the set $\mathcal{F}_0$.

Let $f \in \mathcal{F}_k$ be an arbitrary element and $\{f_n\} \subset \mathcal{F}_k$ such that $\lim_{n\to\infty} f_n = f$. Since the operator A_1 is continuous we have $\lim_{n\to\infty} A_1 f_n = A_1 f$. Consequently, there exists $C_1 > 0$ such that $\|A_1 f_n\| \leq C_1$ for $n \in \mathbb{N}$. Moreover we have

$$(A_1 f)(t) \leq C_2 = \frac{M}{m_0}, \, t \in [0,1],$$

where

$$M = \max_{t,u \in [0,1]} K(t,u), \, m_0 = \min_{u \in [0,1]} K(0,u).$$

We have

$$A_k f_n - A_k f = (B f_n)^k - (B f)^k = q_{k,n}(t)(A_1 f_n - A_1 f), \tag{2.9}$$

where

$$q_{k,n}(t) = \sum_{j=0}^{k-1} (A_1 f_n)^{k-j-1}(t)(A_1 f)^j(t) > 0, \, t \in [0,1].$$

Consequently,

$$q_{k,n}(t) \leq C = \sum_{j=0}^{k-1} (C_1)^{k-j-1}(C_2)^j, \, t \in [0,1].$$

Hence

$$\|A_k f_n - A_k f\| \leq C \|A_1 f_n - A_1 f\|, \, n \in \mathbb{N}.$$

Since A_1 is a continuous from the last inequality it follows that A_k is continuous on $\mathcal{F}_k$. $\square$

Denote

$$\mathcal{F}_k^0 = \left\{ f \in C^+[0,1] : \left(\frac{m}{M_0}\right)^k \le f(t) \le \left(\frac{M}{m_0}\right)^k \right\}.$$

Proposition 7. *Let $k \ge 2$. If $f \in C_0^+[0,1]$ is a solution of the equation $A_k f = f$, then $f \in \mathcal{F}_k^0$.*

Proof. Straightforward. □

Proposition 8. *Let $k \ge 2$. The set $A_k(\mathcal{F}_k^0)$ is relatively compact in $C[0,1]$.*

Proof. By Arzelá-Askoli's theorem it suffices to prove that the set of functions $A_k(\mathcal{F}_k^0)$ is equi-continuous and there exists $\gamma > 0$ such that

$$h(t) \le \gamma, \quad \forall t \in [0,1] \text{ and } \forall h \in A_k(\mathcal{F}_k^0).$$

Let $h \in A_k(\mathcal{F}_k^0)$ be an arbitrary function, we have

$$0 < h(t) \le \left(\frac{M}{m_0}\right)^k$$

and there exists a function $f \in \mathcal{F}_k^0$ such that $h = A_k f$.

Now we shall prove that $A_k(\mathcal{F}_k^0)$ is equi-continuous. For arbitrary $t, t' \in [0,1]$ we have $(h = A_k f)$

$$|h(t) - h(t')| = |(A_1 f)^k(t) - (A_1 f)^k(t')| =$$

$$\sum_{j=0}^{k-1} (A_1 f)^{k-j-1}(t)(A_1 f)^j(t')|(A_1 f)(t) - (A_1 f)(t')| \le$$

$$k\left(\frac{M}{m_0}\right)^{k-1} \frac{1}{\omega(f)} \int_0^1 |K(t,u) - K(t',u)| f(u) du \le$$

$$k\left(\frac{M}{m_0}\right)^{2k-1} \frac{1}{\omega(f)} \int_0^1 |K(t,u) - K(t',u)| du.$$

We have

$$\omega(f) \ge m_0 \cdot \left(\frac{m}{M_0}\right)^k, \ f \in \mathcal{F}_k^0.$$

Consequently,

$$|h(t)-h(t')| \le \frac{k}{m_0}\left(\frac{M_0}{m}\right)^k \left(\frac{M}{m_0}\right)^{2k-1} \int_0^1 |K(t,u)-K(t',u)|du.$$

Since the kernel $K(t,u)$ is uniformly continuous on $[0,1]^2$, we conclude that $A_k(\mathcal{F}_k^0)$ also is equi-continuous. □

By Propositions 6-8 and Schauder's theorem one gets the following

Theorem 3. *The equation $A_k f = f$ has at least one solution in $C_0^+[0,1]$ and the set of all solutions of the equation is a subset in $\mathcal{F}_k^0$.*

For every $k \in \mathbb{N}$ we consider an integral operator H_k acting in $C^+[0,1]$ as follows:

$$(H_k f)(t) = \int_0^1 K(t,u) f^k(u) du.$$

If $k \ge 2$ then the operator H_k is a nonlinear operator which is called Hammerstein's operator of order k. Moreover the linear operator equation $H_1 f = f$ has a unique positive solution f in $C[0,1]$ (see [13], p.80).

For a nonlinear homogeneous operator A it is known that if there is one positive eigenfunction of the operator A then the number of the positive eigenfunctions is continuum (see [13], p.186).

Denote

$$\mathcal{M}_0 = \left\{ f \in C^+[0,1] : f(0) = 1 \right\}.$$

Lemma 4. *The equation*

$$A_k f = f, \ k \ge 2 \tag{2.10}$$

has a strongly positive solution iff the equation

$$H_k f = \lambda f, \ k \ge 2 \tag{2.11}$$

has a strongly positive solution in $\mathcal{M}_0$.

Proof. Necessariness. Let $f_0 \in C_0^+[0,1]$ be a solution of the equation (2.10). We have

$$(Wf_0)(t) = \omega(f_0)\sqrt[k]{f_0(t)}.$$

From this equality we get

$$(H_k h)(t) = \lambda_0 h(t),$$

where $h(t) = \sqrt[k]{f_0(t)}$ and $\lambda_0 = \omega(f_0) > 0$.

It is easy to see that $h \in \mathcal{M}_0$ and $h(t)$ is an eigenfunction of the Hammerstein's operator H_k, corresponding the positive eigenvalue λ_0.

Sufficiency. Let $k \geq 2$ and $h \in \mathcal{M}_0$ be an eigenfunction of the Hammerstein's operator. Then there is a number $\lambda_0 > 0$ such that $H_k h = \lambda_0 h$. From $h(0) = 1$ we get $\lambda_0 = (H_k h)(0) = \omega(h^k)$. Then

$$h(t) = \frac{H_k h}{\omega(h^k)}.$$

From this equality we get $A_k f_0 = f_0$ with $f_0 = h^k \in C_0^+[0,1]$. This completes the proof. □

Theorem 4. *If $k \geq 2$ then every number $\lambda > 0$ is an eigenvalue of the Hammerstein's operator H_k.*

Proof. By Theorem 3 and Lemma 4 there exist $\lambda_0 > 0$ and $f_0 \in \mathcal{M}_0$ such that

$$H_k f_0 = \lambda_0 f_0.$$

Take $\lambda \in (0, +\infty)$, $\lambda \neq \lambda_0$. Define function $h_0(t) \in C_0^+[0,1]$ by

$$h_0(t) = \sqrt[k-1]{\frac{\lambda}{\lambda_0}} f_0(t), \quad t \in [0,1].$$

Then

$$H_k h_0 = H_k\left(\sqrt[k-1]{\frac{\lambda}{\lambda_0}} f_0\right) = \lambda h_0.$$

This completes the proof. □

Denote

$$\mathcal{K} = \left\{ f \in C^+[0,1] : M \cdot \min_{t\in[0,1]} f(t) \geq m \cdot \max_{t\in[0,1]} f(t) \right\},$$

$$\mathcal{P}_k = \left\{ f \in C[0,1] : \frac{m}{M} \cdot \left(\frac{1}{M}\right)^{\frac{1}{k-1}} \leq f(t) \leq \frac{M}{m} \cdot \left(\frac{1}{m}\right)^{\frac{1}{k-1}} \right\},\ k \geq 2.$$

Proposition 9. *Let $k \geq 2$.*

a) The following holds

$$H_k(C^+[0,1]) \subset \mathcal{K}.$$

b) If a function $f_0 \in C_0^+[0,1]$ is a solution of the equation

$$H_k f = f \tag{2.12}$$

then $f_0 \in \mathcal{P}_k$.

Proof. a) Let $h \in H_k(C^+[0,1])$ be an arbitrary function. Then there exists a function $f \in C^+[0,1]$ such that $h = H_k f$. Since h is continuous on $[0,1]$, there are $t_1, t_2 \in [0,1]$ such that

$$h_{\min} = \min_{t\in[0,1]} h(t) = h(t_1) = (H_k f)(t_1),$$
$$h_{\max} = \max_{t\in[0,1]} h(t) = h(t_2) = (H_k f)(t_2).$$

Hence

$$h_{\min} \geq m \int_0^1 f^k(u)du \geq m \int_0^1 \frac{K(t_2,u)}{M} f^k(u)du = \frac{m}{M} h_{max},$$

i.e. $h \in \mathcal{K}$.

b) Let $f \in C_0^+[0,1]$ be a solution of the equation (2.12). Then we have $\|f\| \leq M\|f\|^k$. Consequently,

$$\|f\| \geq \left(\frac{1}{M}\right)^{\frac{1}{k-1}}.$$

By the property a) we have

$$f(t) \geq f_{\min} = \min_{t\in[0,1]} f(t) \geq \frac{m}{M}\|f\|.$$

Then we obtain

$$f(t) \geq \frac{m}{M}\left(\frac{1}{M}\right)^{\frac{1}{k-1}}.$$

Also we have

$$f(t) = (H_k f)(t) \geq m \int_0^1 f^k(u)du \geq m f_{\min}^k.$$

Then $f_{\min} \geq m f_{\min}^k$, i.e.

$$f_{\min} \leq \left(\frac{1}{m}\right)^{\frac{1}{k-1}}.$$

Hence be the property a) we get

$$f(t) \leq f_{\max} \leq \frac{M}{m} f_{\min} \leq \frac{M}{m}\left(\frac{1}{m}\right)^{\frac{1}{k-1}}.$$

Thus we have $f \in \mathcal{P}_k$. □

The case $G_k^{(2)}$ $-$*periodic* is similar. That's why we'll give results without proof.

Put

$$\mathcal{M}_0 = \left\{ f \in C^+[0,1] : f(0) = 1 \right\}.$$

Lemma 5. *The system of equations:*

$$(A_k f)(t) = g(t), \quad (A_k g)(t) = f(t), \qquad k \geq 2. \tag{2.13}$$

has a positive solution iff the system of equations:

$$(H_k f)(t) = \lambda_1 g(t), \quad (H_k g)(t) = \lambda_2 f(t), \quad k \geq 2 \tag{2.14}$$

has a positive solution in $(\mathcal{M}_0)^2$.

Lemma 6. *The system of equations (2.14) has a positive solution iff the system of equations:*

$$(H_k f)(t) = g(t), \quad (H_k g)(t) = f(t), \quad k \geq 2 \tag{2.15}$$

has a positive solution.

Denote

$$\mathcal{K} = \left\{ f \in C^+[0,1] : M \cdot \min_{t\in[0,1]} f(t) \geq m \cdot \max_{t\in[0,1]} f(t) \right\},$$

$$\mathcal{P}_k = \left\{ \varphi \in C[0,1] : \frac{m}{M} \cdot \left(\frac{1}{M}\right)^{\frac{1}{k-1}} \leq \varphi(t) \leq \frac{M}{m} \cdot \left(\frac{1}{m}\right)^{\frac{1}{k-1}} \right\},\ k \geq 2,$$

where

$$M = \max_{t,u\in[0,1]^2} K(t,u), \quad m = \min_{t,u\in[0,1]^2} K(t,u).$$

Proposition 10. *Let $k \geq 2$. Then*

a) $H_k(C^+[0,1]) \subset \mathcal{K}$.

b) If $(f_0, g_0) \in (C_0^+[0,1])^2$ is a solution of the system (2.15) then $(f_0, g_0) \in (\mathcal{P}_k)^2$.

2.3 UNIQUENESS OF GIBBS MEASURES

Now we shall prove that every periodic Gibbs measures be a unique *translation−invariant* Gibbs measure for $k = 1$.

Since the equation $A_1 f = f$ is equivalent to

$$(Wf)(t) = \omega(f) \cdot f(t), \quad f \in C^+[0,1], \tag{2.16}$$

we shall study eigenvalues of the operator Wf.

Lemma 7. *If $\varphi_0 \in C^+[0,1]$ is an eigenfunction of the operator W i.e.*

$W\varphi_0 = \lambda_0\varphi_0$, $\lambda_0 > 0$ then there are $a_1 > 0$ and $b_1 > 0$ such that

$$a_1\omega_1(f)\varphi_0(t) \le (Wf)(t) \le b_1\omega_1(f)\varphi_0(t), \quad \forall t \in [0,1], \quad \forall f \in C^+[0,1], \tag{2.17}$$

where $\omega_1(f) = \int_0^1 f(u)du$.

Proof. Note that

$$a\omega_1(f) \le Wf \le b\omega_1(f), \ f \in C[0,1] \tag{2.18}$$

where $a = \min_{t,u\in[0,1]} K(t,u)$ and $b = \max_{t,u\in[0,1]} K(t,u)$. We have

$$a\omega_1(\varphi_0) \le W\varphi_0 = \lambda_0\varphi_0 \le b\omega_1(\varphi_0).$$

Hence

$$\frac{\lambda_0\varphi_0(t)}{b\omega_1(\varphi_0)} \le 1 \le \frac{\lambda_0\varphi_0(t)}{a\omega_1(\varphi_0)}, \quad \forall t \in [0,1]. \tag{2.19}$$

Using (2.18) and (2.19) we get (2.17) with

$$a_1 = \frac{a\lambda_0}{b\omega_1(\varphi_0)} > 0, \quad b_1 = \frac{b\lambda_0}{a\omega_1(\varphi_0)} > 0.$$

Theorem 5. *If $\lambda_0 > 0$ is an eigenvalue of W then $Wf = \lambda_0 f$ has a unique solution $f \in C^+[0,1]$.*

Proof. Assume that there are two solutions $f_0 \in C^+[0,1]$ and $f_1 \in C^+[0,1]$ i.e $Wf_i = \lambda_0 f_i$, $i = 0,1$. Denote

$$\delta_0 = \sup\{\delta \in [0,\infty) : f_0(t) - \delta f_1(t) \in C^+[0,1]\}.$$

We have

$$W(f_0 - \delta_0 f_1) = W(f_0) - \delta_0 W(f_1) = \lambda_0(f_0 - \delta_0 f_1) > 0.$$

By Lemma 7 we get

$$W(f_0 - \delta_0 f_1) \ge a_2 f_0(t) > a_2\delta_0 f_1(t) \ \text{ with some } \ a_2 > 0,$$

where we used $a_2(f_0(t) - \delta_0 f_1(t)) > 0$.

Consequently

$$\lambda_0(f_0 - \delta_0 f_1) > a_2\delta_0 f_1(t)$$

i.e.

$$f_0(t) - \delta_0(1 + \frac{a_2}{\lambda_0})f_1(t) > 0 \quad \text{for any} \quad t \in [0,1].$$

This contradicts the maximality of δ_0.

$$(Wf)(t) = w(f)g(t), \quad (Wg)(t) = w(g)f(t), \qquad f, g \in C^+[0,1] \tag{2.20}$$

Remark 1. *Let (f,g) satisfies 2.20 with $f \neq g$, $\delta_0 = sup\{\delta \in (0,\infty) : f - \delta g > 0\}$.*
Then $W(f - \delta_0 g) > 0$.

Proof We have $f - \delta_0 g \geq 0 \;\Rightarrow W(f - \delta_0 g) \geq 0$. Suppose $W(f - \delta_0 g) = 0$ then

$$f - \delta_0 g \equiv 0 \quad \Rightarrow \frac{f(t)}{g(t)} = \delta_0, \quad t \in [0,1].$$

For $\quad t = 0$

$$g(0) = \frac{(Wf)(0)}{w(f)} = 1 = \frac{(Wg)(0)}{w(g)} = f(0).$$

Then $\delta_0 = 1$. This contradicts to $f \neq g$. Thus we have proved $W(f - \delta_0 g) > 0$.

Theorem 6. *If $k = 1$ then every periodic solution of (1.5) is unique and this solution is a translation-invariant.*

Proof. At first we'll show the equation $A_1 f = f$ has a unique solution $f \in C^+[0,1]$. By Proposition 5 the equation has at least one solution. We shall prove its uniqueness. Assume that $A_1 f = f$ has two solutions f_0 and f_1, then there are $\lambda_0 = \lambda_0(f_0)$ and $\lambda_1 = \lambda_1(f_1)$ such that $Wf_i = \lambda_i f_i,\ i = 0, 1$. By Theorem 5 we have $\lambda_0 \neq \lambda_1$. Assume $\lambda_0 < \lambda_1$ (the case $\lambda_0 > \lambda_1$ is similar). Consider

$$h_\delta(t) = f_0(t) - \delta f_1(t),\ \ \delta \in [0, \infty)$$

and

$$\delta_0 = \sup\{\delta \in [0, \infty) : h_\delta(t) \in C^+[0, 1]\}$$

We have

$$W(h_{\delta_0})(t) = \lambda_0(f_0(t) - \delta_0 \frac{\lambda_1}{\lambda_0} f_1(t)) > 0,\ \ \forall t \in [0, 1].$$

Since δ_0 is maximal we get $\frac{\lambda_1}{\lambda_0} \leq 1$ i.e. $\lambda_0 \geq \lambda_1$, this contradicts our assumption $\lambda_0 < \lambda_1$.

Now we'll show the system of equations $(A_1 f)(t) = g(t)$, $(A_1 g)(t) = f(t)$ has not any solution $(f, g) \in (C^+[0, 1])^2$ with $f \neq g$. Let $(f_1(x), g_1(x))$ be a solution of the system of equations:

$$(A_1 f)(t) = g(t),\ \ \ (A_1 g)(t) = f(t).$$

Then

$$(Wf_1)(t) = w(f_1)g_1(t),\ \ (Wg_1)(t) = w(g_1)f_1(t).$$

Put

$$\lambda_1 = w(f_1),\ \ \ \lambda_2 = w(g_1)\ \ \ \ \Rightarrow\ \ \lambda_i > 0,\ \ i \in \{1, 2\}.$$

Denote

$$\delta_1 = sup\{\delta \in (0, \infty) : f - \delta g > 0\},\ \ \ \delta_2 = sup\{\delta \in (0, \infty) : g - \delta f > 0\}.$$

By Remark 1

$$\lambda_1 g(t) - \lambda_2 \delta_1 f(t) = W(f - \delta_1 g) > 0,$$

$$\lambda_2 f(t) - \lambda_1 \delta_2 g(t) = W(g - \delta_2 f) > 0.$$

Hence

$$\frac{\lambda_2}{\lambda_1}\delta_1 < \frac{g(t)}{f(t)}, \quad \frac{\lambda_1}{\lambda_2}\delta_2 < \frac{f(t)}{g(t)}, \quad t \in [0,1].$$

There exists $t_0, t_1 \in [0,1]$ such that $\delta_2 = \frac{g(t_0)}{f(t_0)}$ and $\delta_1 = \frac{f(t_1)}{g(t_1)}$. Then

$$\frac{g(t)}{f(t)} \geq \frac{g(t_0)}{f(t_0)} = \delta_2 > \frac{\lambda_2}{\lambda_1}\delta_1, \quad \frac{f(t)}{g(t)} \geq \frac{f(t_1)}{g(t_1)} = \delta_1 > \frac{\lambda_1}{\lambda_2}\delta_2.$$

Thus we have $\frac{\lambda_2}{\lambda_1}\delta_1 < \delta_2$ and $\frac{\lambda_1}{\lambda_2}\delta_2 < \delta_1$ this is a contradiction. This completes the proof. □

Example 1. If $K(t,u) = \alpha(t) + \alpha(u)$ where α is a given function, then one can easily check that

$$f(t) = \frac{\alpha(t) + \sqrt{\int_0^1 \alpha^2(u)du}}{\alpha(0) + \sqrt{\int_0^1 \alpha^2(u)du}}$$

is the unique solution of the equation $A_1 f = f$.

Theorem 7. *For model (1.1) with an arbitrary continuous function ξ_{tu} on $[0,1]^2$, $\forall J \in R$ and for any $\beta > 0$ on the Cayley tree of order 1 there exists a unique splitting Gibbs measure.*

Example 2. For any $k \geq 1$ we shall consider one simple case: let $\xi_{tu} = a(t) + b(u)$, where $a(t)$ and $b(u)$ are arbitrary given functions. Very simple calculations show that equation (1.5) has unique solution $f(t,x) = f(t) = \exp(kJ\beta(a(t) - a(0)))$. Thus for the model (1.1) with $\xi_{tu} = a(t) + b(u)$ there is unique splitting Gibbs measure.

Example 3. Consider $K(t,u) = \alpha(t) + \alpha(u)$ i.e. $\xi_{tu} = \frac{1}{J\beta}\ln(\alpha(t) + \alpha(u))$, where α is a given positive function on $[0,1]$. Then the unknown function f can be written as

$$f(t) = \left(\frac{\alpha(t)X + Y}{\alpha(0)X + Y}\right)^k = \left(\frac{\alpha(t)x + 1}{\alpha(0)x + 1}\right)^k,$$

where $X = \int_0^1 f(u)du$ and $Y = \int_0^1 \alpha(u)f(u)du$, $x = \frac{X}{Y}$. It is easy to see that x satisfies the equation

$$x = \frac{\sum_{j=0}^{k} \frac{a_j}{j!(k-j)!} x^j}{\sum_{j=0}^{k} \frac{a_{j+1}}{j!(k-j)!} x^j}, \ \ x > 0 \tag{2.21}$$

where $a_i = \int_0^1 \alpha^i(t)dt$, $i = 0, 1, ..., k+1$.

From (2.21) we get

$$\gamma(x) = a_{k+1}x^{k+1} + b_k x^k + b_{k-1}x^{k-1} + ... + b_1 x - 1 = 0, \tag{2.22}$$

where

$$b_j = \frac{k!(2j-k-1)}{j!(k-j+1)!} a_j, \quad j = 1, ..., k.$$

It is well known (see [18], p.28) that the number of positive roots of the polynomial (2.22) does not exceed the number of sign changes of the sequence:

$$a_{k+1}, b_k, b_{k-1}, ..., b_1, -1. \tag{2.23}$$

It is obvious that $a_{k+1} > 0, b_j > 0$ if $j > \frac{k+1}{2}$ and $b_j < 0$ if $j < \frac{k+1}{2}$. Thus the number of positive roots of the polynomial (2.22) is at most one. Since $\gamma(0) = -1$ and $\gamma(+\infty) = +\infty$ we get that (2.22) has a unique positive root.

Consider the Hamiltonian

$$H(\sigma) = -\frac{1}{\beta} \sum_{\langle x,y \rangle \in L} \ln \left(\alpha(\sigma(x)) + \alpha(\sigma(y)) \right), \tag{2.24}$$

where α is a given positive, integrable function.

The Potts model.

Note that if $\xi_{tu} = \delta_{tu}$ where δ is the Kronecker's symbol then model (1.1) becomes the Potts model with uncountable set of spin values. It is easy to see that

$$\int_0^1 \exp\left(J\beta\delta_{tu}\right) f(u,y)du = \int_0^1 \exp\left(J\beta\delta_{0u}\right) f(u,y)du$$

for any $t \in [0,1], y \in V$. Consequently the equation (1.5) has the unique solution $f(t,x) = 1, t \in [0,1], x \in V$ for any $k \geq 1$, $J \in R$, and any $\beta > 0$. Thus we have

Theorem 8. *it The Potts model with uncountable set of spin values on Cayley tree of order $k \geq 1$ has unique splitting Gibbs measure for any $J \in R$ and $\beta > 0$.*

The following remarks give a comparison of the result with known results about ordinary Potts model.

Remarks. 1. It is known (see, for example ([15]) that the Potts model with $q \geq 2$ spin values on $Z^d, d \geq 2$ undergoes a first-order phase transition at a certain transition temperature $T_{\rm cr} = T_{\rm cr}(q)$, provided q is large enough. Namely, the model (on Z^d) has q different Gibbs measures for temperatures $T < T_{\rm cr}$, $q+1$ measures at $T = T_{\rm cr}$ and one measure for $T > T_{\rm cr}$.

2. Note that (see [7], [11], [16]) for the ferromagnetic Potts model with q spin values on Cayley tree *for any* $q \geq 2$ (even for $q = 2$ i.e. for the Ising model (see [3], [2]) there are $q+1$ distinct translation-invariant Gibbs measures. Namely, there are two critical temperatures $0 < T'_{\rm c} < T_{\rm c}$ such that (i) for $T \in (0, T'_{\rm c}]$ there are $q+1$ Gibbs measures. Among them only one, say μ_0, (with $\mu_0(\sigma(x) = i) = \frac{1}{q}, i = 1,..,q)$ is not extreme and called unordered Gibbs measure; (ii) for $T \in (T'_{\rm c}, T_{\rm c}]$ the $q+1$ Gibbs measures still exist and all of them are extreme. (iii) for $T > T_{\rm c}$ there is one Gibbs measure.

3. In [9] it was proven that the Gibbs measure is unique if $q \to \infty$ i.e.

when the set of spin values is a countable set. Theorem 1 shows that the uniqueness is also true for an uncountable set of spin values.

Now for arbitrary $k \geq 2$ we find a sufficient condition under which the integral equation has unique solution; hence under this condition the corresponding model has unique splitting Gibbs measure. We shall prove that $A_k f = f$ and $H_k f = f$ have a unique solution in $C_0^+[0,1]$.

Lemma 8. *Assume function $f \in C[0,1]$ changes its sign on $[0,1]$. Then for every $a \in \mathbb{R}$ the following inequality holds*

$$\|f_a\| \geq \frac{1}{n+1}\|f\|, \quad n \in \mathbb{N},$$

where $f_a = f_a(t) = f(t) - a$, $t \in [0,1]$.

Proof. By conditions of lemma there are $t_1, t_2 \in [0,1]$ such that

$$f_{\min} = f(t_1) < 0, \quad f_{\max} = f(t_2) > 0.$$

In case $a = 0$ the proof is obvious. We assume $a > 0$

a) Let $|f_{\min}| \geq f_{\max}$. Then $\|f\| = |f_{\min}| = |f(t_1)|$. Hence

$$\|f_a\| = \max\{|f(t_1) - a|, |f(t_2) - a|\} > |f(t_1)| = \|f\| \geq \frac{1}{n+1}\|f\|, \, n \in \mathbb{N}.$$

b) Let $|f_{\min}| < f_{\max}$ and $\frac{1}{2}\|f\| \geq a$. Then $\|f\| = f_{\max} = f(t_2)$ and $\|f\| - a \geq a > 0$. Consequently,

$$\|f_a\| = \max\{|f(t_1)-a|, |f(t_2)-a|\} \geq |f(t_2)-a| = \|f\|-a \geq \frac{1}{n+1}\|f\|, \, n \in \mathbb{N}.$$

c) Let $|f_{\min}| < f_{\max}$ and $\frac{1}{2}\|f\| < a$. Then $\|f\| = f(t_2)$ and

$$\|f_a\| = \max\{|f(t_1) - a|, |f(t_2) - a|\} > \frac{1}{2}\|f\| \geq \frac{1}{n+1}\|f\|, \, n \in \mathbb{N}.$$

Thus for $a > 0$ the proof is completed. For $a < 0$ we put $g_a(t) = g(t) - a'$ with $g(t) = -f(t)$ and $a' = -a > 0$. Then

$$\|f_a\| = \|g_a\| \geq \frac{1}{n+1}\|g\| = \frac{1}{n+1}\|f\|, \, n \in \mathbb{N}.$$

This completes the proof. □

Theorem 9. *Let $k \geq 2$. If the kernel $K(t,u)$ satisfies the condition*

$$\left(\frac{M}{m}\right)^k - \left(\frac{m}{M}\right)^k < \frac{1}{k}, \tag{2.25}$$

then the operator H_k has a unique fixed point in $C_0^+[0,1]$.

Proof. By Theorem 4 the Hammerstein's equation $H_k f = f$ has at least one solution. Assume that there are two solutions $f_1 \in C_0^+[0,1]$ and $f_2 \in C_0^+[0,1]$, i.e $H_k f_i = f_i$, $i = 1,2$. Denote $f(t) = f_1(t) - f_2(t)$. Then by Theorem 46.6 of [14] the function $f(t)$ changes its sign on $[0,1]$. From Lemma 16 we get

$$\max_{t\in[0,1]} \left| f(t) - \frac{k}{2}(\gamma_1 + \gamma_2) \int_0^1 f(s)ds \right| \geq \frac{1}{2}\|f\|,$$

where

$$\gamma_1 = \left(\frac{m}{M}\right)^k, \quad \gamma_2 = \left(\frac{M}{m}\right)^k.$$

By a mean value Theorem we have

$$f(t) = \int_0^1 K(t,u)k\xi^{k-1}(u)f(u)du,$$

here $\xi \in C^+[0,1]$ and

$$\min\{f_1(t), f_2(t)\} \leq \xi(t) \leq \max\{f_1(t), f_2(t)\}, \ t \in [0,1].$$

By Proposition 9 we have $\xi \in \mathcal{P}_k$, i.e.

$$\frac{m}{M}\left(\frac{1}{M}\right)^{\frac{1}{k-1}} \leq \xi(t) \leq \frac{M}{m}\left(\frac{1}{m}\right)^{\frac{1}{k-1}}, \ t \in [0,1].$$

Hence

$$\gamma_1 \leq K(t,u)\xi^{k-1}(u) \leq \gamma_2, \ t,u \in [0,1].$$

Therefore

$$\left| k \cdot K(t,u)\xi^{k-1}(u) - \frac{\gamma_1 + \gamma_2}{2} \right| \leq \frac{\gamma_2 - \gamma_1}{2}.$$

Then

$$\left|f(t) - \frac{k}{2}(\gamma_1 + \gamma_2)\int_0^1 f(u)du\right| \le \frac{k}{2}(\gamma_2 - \gamma_1)\|f\|. \tag{2.26}$$

Assume the kernel $K(t,u)$ satisfies the condition (2.25). Then $k(\gamma_2 - \gamma_1) < 1$ and the inequality (2.26) contradicts to Lemma 16. This completes the proof. □

Theorem 10. *Let $k \ge 2$. If the kernel $K(t,u)$ satisfies the condition (2.25), then for every $\lambda > 0$ the Hammerstein's equation $H_k f = \lambda f$ has unique solution in $C_0^+[0,1]$.*

Proof. Clearly the equation $H_k f = \lambda f$ is equivalent to the following equation

$$\int_0^1 K_\lambda(t,u) f^k(u)du = f(t), \tag{2.27}$$

where $K_\lambda(t,u) = \frac{1}{\lambda}K(t,u)$. The kernel $K_\lambda(t,u)$ satisfies the condition (2.25) with $\tilde{m} = \frac{m}{\lambda}$ and $\tilde{M} = \frac{M}{\lambda}$. Consequently, by Theorem 9 it follows that the equation (2.27) has unique solution in $C_0^+[0,1]$. □

Theorem 11. *Let $k \ge 2$. If the kernel $K(t,u)$ satisfies the condition (2.25), then the equation $A_k f = f$ has unique solution in $C_0^+[0,1]$.*

Proof. Assume there are two solutions $f_1, f_2 \in C^+[0,1]$, $f_1 \ne f_2$, i.e. $A_k f_i = f_i$, $i = 1,2$. By Lemma 4 the functions $h_i(t) = \sqrt[k]{f_i(t)}$, $t \in [0,1]$ are solutions of the Hammerstein's equation, i.e.

$$H_k h_i = \lambda_i h_i, \quad i = 1,2,$$

where $\lambda_i = \omega(f_i) > 0$ and $h_i \in \mathcal{M}_0$. On the other hand Theorem 10 implies that $\lambda_1 \ne \lambda_2$. Let $h_0(t) \in C^+[0,1]$ be a fixed point of the Hammerstein's operator H_k. Then by Theorems 4 and 10 we get

$$h_i = \sqrt[k-1]{\lambda_i}\, h_0(t),\ i = 1,2.$$

Consequently,

$$\frac{f_1(t)}{f_2(t)} = \gamma^k, \quad \text{with} \quad \gamma = \sqrt[k-1]{\frac{\lambda_1}{\lambda_2}}.$$

Using this equality we obtain

$$f_1(t) = (A_k f_1)(t) = A_k(\gamma^k f_2) = A_k f_2(t) = f_2(t).$$

This completes the proof. □

Consider the following Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x)\sigma(y)} = -\sum_{\langle x,y\rangle \in L} \ln K(\sigma(x), \sigma(y)), \tag{2.28}$$

where $J \in R \setminus \{0\}$ and $K(t,u)$ satisfies the condition (2.25). Then as a corollary of Proposition 1 and Theorem 11 we get the following

Theorem 12. *Let $k \geq 2$. If the function $K(t,u)$ of the Hamiltonian (2.28) satisfies the condition (2.25), then the model (2.28) has unique translational invariant Gibbs measure.*

Example 4. It is easy to see that the condition (2.25) is satisfied iff

$$\frac{M}{m} \leq \eta_k = \sqrt[k]{\frac{1+\sqrt{4k^2+1}}{2k}}, \; k \geq 2.$$

Consider the following function

$$K(t,u) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} t^i u^j + a, \; c_{ij} \geq 0, \; a > 0. \tag{2.29}$$

For this function we have $m = a$, $M = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} + a$. The following is obvious

a) If $\frac{1}{a} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \leq \eta_k - 1$ then for function (2.29) the condition (2.25) is satisfied.

b) If $\frac{1}{a} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} > \eta_k - 1$ then for function (2.29) the condition (2.25) is not satisfied.

Analogously we get

Proposition 11. *Let $k \geq 2$. If the kernel $K(t,u)$ satisfies the condition (2.25) then the system (2.15) has not solution (f,g) in $(C_0^+[0,1])^2$ with $f \neq g$.*

Corollary 4. *Let $k \geq 2$. Let the kernel $K(t,u)$ satisfies the condition (2.25). For every $\lambda_1 > 0$, $\lambda_2 > 0$ the Hammerstein's system of equations*

$$H_k f = \lambda_1 g, \quad H_k g = \lambda_2 f \tag{2.30}$$

has not solution $(f,g) \in (C_0^+[0,1])^2$, $f \neq g$.

Proposition 12. *Let $k \geq 2$. If the kernel $K(t,u)$ satisfies the condition (2.25), then the system of equations (2.15) has not solution in $(C_0^+[0,1])^2$, $f \neq g$.*

Theorem 13. *Let $k \geq 2$. If the kernel $K(t,u)$ satisfies the condition (2.25) then the model (1.1) has no periodic (non-translation-invariant) Gibbs measure.*

3 NON-UNIQUENESS OF PERIODIC GIBBS MEASURE ON A CAYLEY TREE.

3.1 EXISTENCE OF TWO GIBBS MEASURES FOR THE MODEL (1.1): CASE $k \geq 2$

Case: $k = 2$**.** Consider the case $k = 2$ in the model (1.1) and

$$\xi_{t,u} = \frac{1}{\beta J} \ln \left(1 + \frac{14}{15} \cdot \sqrt[5]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}\right), \quad t, u \in [0,1].$$

Then, for the kernel $K(t,u)$ of the Hammerstein's integral operator H_2 we have

$$K(t,u) = 1 + \frac{14}{15} \cdot \sqrt[5]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}.$$

Proposition 13. *The Hammerstein's operator H_2 :*

$$(H_2 f)(t) = \int_0^1 K(t,u) f^2(u) du$$

in the space $C[0,1]$ has at least two strictly positive fixed points.

Proof. a) Let $f_1(t) \equiv 1$. Then we have

$$(H_2 f_1)(t) = 1 + \frac{14}{15} \cdot \sqrt[5]{4\left(t - \frac{1}{2}\right)} \cdot \int_0^1 \left(u - \frac{1}{2}\right)^{\frac{1}{5}} du = 1 = f_1(t), \ \ t \in [0,1].$$

b) Denote

$$f_2(t) = \frac{3}{4} + \sqrt{\frac{21}{5}} \cdot \frac{\sqrt[5]{2}}{4} \cdot \left(t - \frac{1}{2}\right)^{\frac{1}{5}}, \ \ t \in [0,1].$$

Then $f_2 \in C[0,1]$ and the function $f_2(t)$ is strictly positive. Put

$$a = \frac{14}{15} \cdot \sqrt[5]{4}, \ \ b = \sqrt{\frac{21}{5}} \cdot \frac{\sqrt[5]{2}}{4}.$$

We have

$$H_2 f_2 = h_1(t) + h_2(t) + h_3(t) + \gamma,$$

where

$$h_1(t) = ab^2 \cdot \sqrt[5]{t - \frac{1}{2}} \cdot \int_0^1 \sqrt[5]{\left(u - \frac{1}{2}\right)^3} du,$$

$$h_2(t) = \frac{3ab}{2} \cdot \sqrt[5]{t - \frac{1}{2}} \cdot \int_0^1 \sqrt[5]{\left(u - \frac{1}{2}\right)^2} du,$$

$$h_3(t) = \frac{9a}{16} \cdot \sqrt[5]{t - \frac{1}{2}} \cdot \int_0^1 \sqrt[5]{u - \frac{1}{2}} du,$$

$$\gamma = \int_0^1 f_2^2(u) du.$$

It is clear that

$$h_1(t) = h_3(t) \equiv 0.$$

For the function $h_2(t)$ we obtain

$$h_2(t) = \frac{3ab}{2} \cdot \sqrt[5]{t - \frac{1}{2}} \cdot \int_{-1/2}^{1/2} u^{\frac{2}{5}} du = \frac{15ab}{14\sqrt[5]{4}} \cdot \sqrt[5]{t - \frac{1}{2}} \ .$$

Observe that

$$\gamma = \frac{5b^2}{7\sqrt[5]{4}} + \frac{9}{16}.$$

Consequently we have

$$(H_2 f_2)(t) = h_2(t) + \gamma = f_2(t).$$

□

Denote by μ_1 and μ_2 the translation-invariant Gibbs measures which by Proposition 1 correspond to solutions $f_1(t) = 1$ and $f_2(t) = \frac{3}{4} + \sqrt{\frac{21}{5}} \cdot \frac{\sqrt[5]{2}}{4} \cdot \left(t - \frac{1}{2}\right)^{\frac{1}{5}}$.

Thus we have proved the following

Theorem 14. *The model*

$$H(\sigma) = -\frac{1}{\beta}\sum_{\substack{<x,y>\\ x,y\in V}} \ln\left(1+\frac{14}{15}\cdot\sqrt[5]{4\left(\sigma(x)-\frac{1}{2}\right)\left(\sigma(y)-\frac{1}{2}\right)}\right), \quad \sigma\in\Omega_V$$

on the Cayley tree Γ^2 *has at least two translation-invariant Gibbs measures* μ_1, μ_2.

Case: $k=3$. Now we shall consider the case $k=3$ and

$$\xi_{t,u} = \frac{1}{\beta J}\ln\left(1+\frac{1}{2}\cdot\sqrt[7]{4\left(t-\frac{1}{2}\right)\left(u-\frac{1}{2}\right)}\right), \quad t,u\in[0,1].$$

Then, for the kernel $K(t,u)$ of the operator H_3 we have

$$K(t,u) = 1+\frac{1}{2}\sqrt[7]{4\left(t-\frac{1}{2}\right)\left(u-\frac{1}{2}\right)}.$$

Proposition 14. *The operator* H_3 :

$$(H_3 f)(t) = \int_0^1 \left(1+\frac{1}{2}\cdot\sqrt[7]{4\left(t-\frac{1}{2}\right)\left(u-\frac{1}{2}\right)}\right) f^3(u)du$$

in the space $C[0,1]$ *has at least two strictly positive fixed points.*

Proof. a) Let $f_1(t)\equiv 1$. Then

$$(H_3 f_1)(t) = 1+\frac{1}{2}\cdot\sqrt[7]{4\left(t-\frac{1}{2}\right)}\cdot\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{\frac{1}{7}}du = 1 = f_1(t), \quad t\in[0,1].$$

b) We define the function f_2:

$$f_2(t) = \frac{1}{2}\left(\sqrt{\frac{57}{17}}+\sqrt{\frac{33}{119}}\cdot\sqrt[7]{2\left(t-\frac{1}{2}\right)}\right), \quad t\in[0,1].$$

Then $f_2\in C[0,1]$ and the function $f_2(t)$ is strictly positive. Put

$$a = \frac{1}{2}\sqrt{\frac{57}{17}}\ , \quad b = \frac{1}{2}\sqrt{\frac{33}{119}}\ .$$

We have

$$(H_3 f_2)(t) = h_1(t)+h_2(t)+h_3(t)+h_4(t)+\gamma\ ,$$

where

$$h_1(t) = \frac{a^3}{2}\varphi(t) \cdot \int_0^1 \sqrt[7]{u - \frac{1}{2}}du,$$

$$h_2(t) = \frac{3a^2b}{2} \cdot \sqrt[7]{2}\varphi(t) \cdot \int_0^1 \sqrt[7]{\left(u - \frac{1}{2}\right)^2}du,$$

$$h_3(t) = \frac{3ab^2}{2} \cdot \sqrt[7]{4}\varphi(t) \cdot \int_0^1 \sqrt[7]{\left(u - \frac{1}{2}\right)^3}du,$$

$$h_4(t) = \frac{b^3}{2} \cdot \sqrt[7]{8}\varphi(t) \cdot \int_0^1 \sqrt[7]{\left(u - \frac{1}{2}\right)^4}du,$$

$$\gamma = \int_0^1 f_2^3(u)du.$$

Here $\varphi(t) = \sqrt[7]{4\left(t - \frac{1}{2}\right)}$, $t \in [0, 1]$. It is clear that

$$h_1(t) = h_3(t) \equiv 0.$$

For the functions $h_2(t)$ and $h_4(t)$ we obtain, that

$$h_2(t) = \frac{3a^2b\sqrt[7]{2}}{2} \cdot \varphi(t) \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{\frac{2}{7}}du = \frac{7a^2b}{6\sqrt[7]{2}} \cdot \varphi(t),$$

$$h_4(t) = \frac{b^3\sqrt[7]{8}}{2} \cdot \varphi(t) \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{\frac{4}{7}}du = \frac{7b^3}{22\sqrt[7]{2}} \cdot \varphi(t).$$

Observe that

$$\gamma = a^3 + 3ab^2\sqrt[7]{4} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{\frac{2}{7}}du = a^3 + \frac{7ab^2}{3} = a.$$

Consequently, we have

$$H_3f_2 = h_2 + h_4 + a = a + \frac{7b}{2\sqrt[7]{2}}\left(\frac{a^2}{3} + \frac{b^2}{11}\right)\varphi(t) = a + \frac{b}{\sqrt[7]{2}}\varphi(t) = f_2(t).$$

From Proposition 14 and Proposition 1 we get

Theorem 15. *The model*

$$H(\sigma) = -\frac{1}{\beta}\sum_{\substack{<x,y>\\ x,y\in V}} \ln\left(1+\frac{1}{2}\sqrt[7]{4\left(\sigma(x)-\frac{1}{2}\right)\left(\sigma(y)-\frac{1}{2}\right)}\right) \quad , \quad \sigma\in\Omega_V$$

on the Cayley tree Γ^3 has at least two translation-invariant Gibbs measures.

Case: $k \geq 4$**.**

Let $k \in \mathbb{N}$ and $k \geq 2$. We consider sequences of continuous functions $P_n(x)$ $(n \in \mathbb{N})$ and $Q_m(x)$ $(m \in \mathbb{N},\ m > k)$ defined by

$$P_n(x) \equiv P_{n,k}(x) = \left(1+\frac{x^{n-1}}{2}\right)^{k+1} - \left(1-\frac{x^{n-1}}{2}\right)^{k+1}, \ \ x\in\mathbb{R},$$

$$Q_m(x) \equiv Q_{m,k}(x) = (k+1)x^{m-k}, \ \ m > k, \ \ x \in \mathbb{R}.$$

Proposition 15. *Let $k \geq 2$. Then*

$$P_n(1) > Q_n(1), \tag{3.1}$$

for any $n \in \mathbb{N},\ \ n > k$.

Proof. Let $k \geq 2$ and $n > k$. We have

$$P_n(1) = \mu_k = \frac{3^{k+1}-1}{2^{k+1}} \ \ , \ \ Q_n = \eta_k = k+1.$$

In the case $k = 2$ we obtain, that

$$P_n(1) = \frac{13}{4} > Q_n(1) = 3.$$

We now suppose, that the inequality (3.1) holds for $k = m > 2$. Then we show that the inequality (3.1) also is true for $k = m+1$.

Obviously, that

$$\mu_{m+1} = \frac{3^{(m+1)+1}-1}{2^{(m+1)+1}} > \frac{3^{(m+1)+1}-3}{2^{m+1}\cdot 2} =$$

$$= \frac{3^{m+1}-1}{2^{m+1}} \cdot \frac{3}{2} = \mu_m \cdot \frac{3}{2} > (m+1) \cdot \frac{3}{2} > m+2 = \eta_{m+1},$$

i.e. $\mu_{m+1} > \eta_{m+1}$. Thus we get

$$P_n(1) > Q_n(1)$$

for any $k \geq 2$ and $n > k$.

Proposition 16. *Let $k \geq 2$. The equation*

$$\left(1+\frac{x}{2}\right)^{k+1} - \left(1-\frac{x}{2}\right)^{k+1} - (k+1)x = 0, \quad x \geq 0 \tag{3.2}$$

has a unique solution $x = 0$.

Proof. Let $k \geq 2$. Define the continuous function $\varphi(x)$:

$$\varphi(x) = \left(1+\frac{x}{2}\right)^{k+1} - \left(1-\frac{x}{2}\right)^{k+1} - (k+1)x, \;\; x \in [0,\infty).$$

We have

$$\varphi'(x) = (k+1)\left(\frac{1}{2}\left(1+\frac{x}{2}\right)^{k} + \frac{1}{2}\left(1-\frac{x}{2}\right)^{k} - 1\right).$$

However,

$$\left(1+\frac{x}{2}\right)^{k} + \left(1-\frac{x}{2}\right)^{k} > 2, \;\; \text{for all} \;\; x \in (0,\infty).$$

Consequently, we have $\varphi'(x) > 0$ for all $x \in (0,\infty)$, i.e. the function $\varphi(x)$ is an increasing on $[0,\infty)$. So, the zero is a unique solution of the equation (3.2).

Proposition 17. *Let $k \geq 2$. Then for each $n \in \mathbb{N}$, $n > k$ the equation*

$$P_n(x) - Q_n(x) = 0 \tag{3.3}$$

has at least one solution $\xi = \xi(k;n)$ in (0,1).

Proof. Let $k \geq 2$ and $n > k$. We have

$$\lim_{x\to 0+} \frac{P_n(x)}{Q_n(x)} = \frac{1}{k+1} \lim_{x\to 0+} \frac{\left(1+\frac{x^{n-1}}{2}\right)^{k+1} - \left(1-\frac{x^{n-1}}{2}\right)^{k+1}}{x^{n-k}} =$$

$$= \frac{1}{k+1} \lim_{x\to 0+} \frac{\left(\left(1+\frac{x^{n-1}}{2}\right) - \left(1-\frac{x^{n-1}}{2}\right)\right) \sum_{j=0}^{k} \left(1+\frac{x^{n-1}}{2}\right)^{k-j} \left(1-\frac{x^{n-1}}{2}\right)^{j}}{x^{n-k}} =$$

$$= \frac{1}{k+1} \lim_{x\to 0+} x^{k-1} \cdot \sum_{j=o}^{k} \left(1+\frac{x^{n-1}}{2}\right)^{k-j} \left(1-\frac{x^{n-1}}{2}\right)^{j} = 0.$$

Since the functions $P_n(x)$ and $Q_n(x)$ are continuous, the exists a number $\delta > 0$ such that

$$P_n(x) < Q_n(x) \text{ for all } x \in (0, \delta).$$

However $P_n(0) = Q_n(0) = 0$ and by Proposition 15 we have $P_n(1) > Q_n(1)$. Consequently there exists a number $\xi = \xi(k; n) \in (0, 1)$ such that $P_n(\xi(k; n)) = Q_n(\xi(k; n)) = 0$.

Let $k \geq 2$ be a fixed number and suppose that $\{\xi(k; n)\}_{n>k} \subset (0, 1)$ – some set of solutions of the following system of equations:

$$P_n(x) - Q_n(x) = 0, \ \ n \in \mathbb{N}, \ \ n > k.$$

We have $0 < \xi(k; n) < 1$ for all $n \in \mathbb{N}, \ \ n > k$. Consequently $0 < \xi(k; n)^{n-1} < 1$ for all $n > k$. Then there exists a upper limit of the sequence $\xi(k; n)^{n-1}, \ \ n > k$, i.e. there exists a subsequence $\alpha_p = \xi(k; n_p)^{n_p-1}, \ \ p \in \mathbb{N}$ of the sequence $\xi(k; n)^{n-1}, \ \ n > k$ such that

$$\alpha = \limsup_{n\to\infty} \xi(k; n)^{n-1} = \lim_{p\to\infty} \xi(k; n_p)^{n_p-1} = \lim_{p\to\infty} \alpha_p.$$

Obviously, that $0 \leq \alpha \leq 1$. Define the sequence $\beta_p, \ \ p \in \mathbb{N}$ by

$$\beta_p = \xi(k; n_p), \ \ p \in \mathbb{N}.$$

Then

$$\alpha_p = \beta_p^{n_p - 1}, \;\; p \in \mathbb{N}.$$

Lemma 9. $\alpha = \lim\limits_{p\to\infty} \alpha_p = 0.$

Proof. a)Assume $\alpha = 1$. Put

$$\beta = \limsup_{p\to\infty} \xi(k; n_p) = \limsup_{p\to\infty} \beta_p.$$

Then, there exists a subsequence $\{\beta_{p_q}\}_{q\in\mathbb{N}} \subset \{\beta_p\}_{p\in\mathbb{N}}$ such that

$$\lim_{q\to\infty} \beta_{p_q} = \beta.$$

We have $0 \le \beta \le 1$. If $0 \le \beta < 1$, there exists $q_0 \in \mathbb{N}$ such that $\beta_{p_q} < \frac{1+\beta}{2}$ for all $q > q_0$. From that

$$0 \le \alpha_{p_q} \le \left(\frac{1+\beta}{2}\right)^{n_{p_q}-1}, \;\; q \in \mathbb{N}, \;\; q > q_0.$$

Therefore $\alpha = \lim\limits_{q\to\infty} \alpha_{p_q} = 0$. The last equality is a contradiction to the assumption $\alpha = 1$. However, we obtain that $\beta = 1$. Then from the equality

$$P_{n_{p_q}}\left(\xi(k; n_{p_q})\right) = Q_{n_{p_q}}\left(\xi(k; n_{p_q})\right), \;\; q \in \mathbb{N} \tag{3.4}$$

as $q \to \infty$ we observe that

$$\left(1 + \frac{1}{2}\right)^{k+1} - \left(1 - \frac{1}{2}\right)^{k+1} = k + 1,$$

i.e.

$$P_m(1) = Q_m(1), \;\; m > k.$$

The last equality is a contradiction to the assertion of Proposition 15. Thus, we have proved that $\alpha \neq 1$.

b) Assume that $0 < \alpha < 1$. In the case $0 \le \beta < 1$ we get $\alpha = 0$. So $\beta = 1$. Then from (3.4) as $q \to \infty$ we get

$$\left(1 + \frac{\alpha}{2}\right)^{k+1} - \left(1 - \frac{\alpha}{2}\right)^{k+1} = (k + 1)\alpha.$$

The last equality is contradict to the assertion of Proposition 16. Thus, we have proved that $\alpha \notin (0,1)$. Consequently, $\alpha = 0$. □

Corollary 5. $\lim\limits_{p\to\infty} \beta_p = 1.$

Proof. From the equality (3.4) we get

$$\beta_p = \xi(k; n_p) = \sqrt[k-1]{\frac{k+1}{\sum\limits_{j=0}^{k} \left(1+\frac{\alpha_p}{2}\right)^{k-j}\left(1-\frac{\alpha_p}{2}\right)^{j}}}\ , \ \ p \in \mathbb{N}.$$

Hence by Lemma 9 it follows that

$$\lim_{p\to\infty} \beta_p = 1.$$

□

Define the sequence $C_n, \ \ n > k \geq 2:$

$$C_n = C_n(k) = \frac{\xi(k;n)^{3n-k-2}}{\frac{1}{2+k}\cdot\left[\left(1+\frac{\xi(k;n)^{n-1}}{2}\right)^{k+2} - \left(1-\frac{\xi(k;n)^{n-1}}{2}\right)^{k+2}\right] - \xi(k;n)^{n-k}}, \tag{3.5}$$

where $\xi(k;n) \in (0,1)$ is an arbitrary solution to the equation (3.3).

Put

$$\gamma_p = \gamma_p(k) = C_{n_p}(k), \ \ p \in \mathbb{N}.$$

Lemma 10. *For every $k \in \mathbb{N}, \ \ k \geq 2$ the following equality holds*

$$\lim_{p\to\infty} \gamma_p(k) = \frac{12}{k}.$$

Proof. We have

$$\gamma_p = \frac{\alpha_p^3 \cdot \beta_p^{1-k}}{\frac{1}{k+2}\cdot\left(\left(1+\frac{\alpha_p}{2}\right)^{k+2} - \left(1-\frac{\alpha_p}{2}\right)^{k+2}\right) - \xi(k;n_p)^{n_p-k}} =$$

$$= \frac{\alpha_p^3 \cdot \beta_p^{1-k}}{\frac{1}{k+2}\cdot\left(\left(1+\frac{\alpha_p}{2}\right)^{k+2} - \left(1-\frac{\alpha_p}{2}\right)^{k+2}\right) - \frac{1}{k+1}\cdot\left(\left(1+\frac{\alpha_p}{2}\right)^{k+1} - \left(1-\frac{\alpha_p}{2}\right)^{k+1}\right)}\ .$$

However

$$\left(1+\frac{\alpha_p}{2}\right)^{k+2}-\left(1-\frac{\alpha_p}{2}\right)^{k+2}=\sum_{j=0}^{k+2}C_{k+2}^{j}\cdot\left(\frac{\alpha_p}{2}\right)^{j}-\sum_{j=0}^{k+2}C_{k+2}^{j}\cdot\left(-\frac{\alpha_p}{2}\right)^{j}=$$

$$=2C_{k+2}^{1}\cdot\frac{\alpha_p}{2}+2C_{k+2}^{3}\cdot\frac{\alpha_p^3}{2^3}+...+2C_{k+2}^{m_1}\cdot\frac{\alpha_p^{m_1}}{2^{m_1}}\ ,$$

where

$$m_1\equiv m_1(k)=\begin{cases}k+2, & \text{if } k \text{ is odd}\\ k+1, & \text{if } k \text{ is even.}\end{cases}$$

Analogously we have

$$\left(1+\frac{\alpha_p}{2}\right)^{k+1}-\left(1-\frac{\alpha_p}{2}\right)^{k+1}=2C_{k+1}^{1}\cdot\frac{\alpha_p}{2}+2C_{k+1}^{3}\cdot\frac{\alpha_p^3}{2^3}+...+2C_{k+1}^{m_2}\cdot\frac{\alpha_p^{m_2}}{2^{m_2}},$$

where

$$m_2\equiv m_2(k)=\begin{cases}k+1, & \text{if } k \text{ is even}\\ k, & \text{if } k \text{ is odd,}\end{cases}$$

i.e. $m_2=2m_0-1,\ \ m_0\in\mathbb{N}$.

Therefore

$$\sum_{j=2}^{m_0}a_j\alpha_p^{2j-1}+a_{m_0+1}\alpha_p^{2m_0+1}=\alpha_p^3(a_2+a_3\alpha_p^2+a_4\alpha_p^4+\ldots+a_{m_0+1}\alpha_p^{2(m_0-1)}),$$

where

$$a_j=\frac{2}{2^{2j-1}}\cdot\left(\frac{C_{k+2}^{2j-1}}{k+2}-\frac{C_{k+1}^{2j-1}}{k+1}\right),\ \ j=2,3,...\ ,$$

$$a_{m_0+1}=\begin{cases}0 & \text{if } m_1=m_2,\\ \frac{1}{2^{2m_0}}\cdot\frac{C_{k+2}^{2m_0+1}}{k+2} & \text{if } m_2<m_1.\end{cases}$$

Obviously that

$$a_2=\frac{k}{12}.$$

Thus we get

$$\gamma_p=\frac{\beta_p^{1-k}}{\frac{k}{12}+a_3\alpha_p^2+a_4\alpha_p^4+...+a_{m_0+1}\alpha_p^{2(m_0-1)}},\ \ p\in\mathbb{N}.$$

Hence by Corollary 5 it follows that

$$\lim_{p\to\infty}\gamma_p = \frac{12}{k} \ .$$

Corollary 6. *If* $k \geq 4$ *then* $0 < \lim_{p\to\infty}\gamma_p \leq 3.$

For each $k \geq 4$ we define the set $\mathbb{N}_0(k)$:

$$\mathbb{N}_0(k) = \{p \in \mathbb{N} : |\gamma_p(k)| < 4\}.$$

Note that, the set $\mathbb{N}_0(k)$ is a countable subset in the set of all natural numbers. For each $p \in \mathbb{N}_0(k)$, $(k \geq 4)$ we define the continuous function $K_p(t, u; k)$ on $[0, 1]^2$ by

$$K_p(t, u; k) = 1 + \gamma_p(k)\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right), \quad t, u \in [0, 1].$$

By the inequality $|\gamma_p(k)| < 4$ it follows that, the function $K_p(t, u; k)$ is strictly positive.

Theorem 16. *Let* $k \geq 4$*. For each* $p \in \mathbb{N}_0(k)$ *the Hammerstein's equation*

$$\int_0^1 K_p(t, u; k) f^k(u)du = f(t) \tag{3.6}$$

in the $C[0, 1]$ *has at least two positive solutions.*

Proof. Obviously, that the function $f_0(t) \equiv 1$ is a solution of the equation (3.6). Define the strictly positive continuous function $f_1(t)$ on $[0, 1]$ by

$$f_1(t) = \xi(k; n_p) + \xi(k; n_p)^{n_p}\left(t - \frac{1}{2}\right), \quad t \in [0, 1].$$

We shall prove that the function $f_1(t)$ also is a solution of the Hammerstein's equation (3.6) :

$$\int_0^1 K_p(t, u; k) f_1^k(u)du = \int_0^1 \left(1 + \gamma_p(k)\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)\right) \times$$

$$\times\left(\xi(k;n_p)+\xi(k;n_p)^{n_p}\left(u-\frac{1}{2}\right)\right)^k du=$$

$$=\int_{-1/2}^{1/2}\left(\beta_p+\beta_p^{n_p}u\right)^k du+\gamma_p(k)\left(t-\frac{1}{2}\right)\int_{-1/2}^{1/2}u\left(\beta_p+\beta_p^{n_p}u\right)^k du=$$

$$=\frac{\beta_p^k}{\beta_p^{n_p-1}}\int_{-1/2}^{1/2}\left(1+\beta_p^{n_p-1}u\right)^k d\left(1+\beta_p^{n_p-1}u\right)+\gamma_p(k)\left(t-\frac{1}{2}\right)\times$$

$$\times\frac{\beta_p^k}{\beta_p^{n_p-1}}\int_{-1/2}^{1/2}u\left(1+\beta_p^{n_p-1}u\right)d\left(1+\beta_p^{n_p-1}u\right)=\frac{\beta_p^k}{\alpha_p}\cdot\frac{1}{k+1}\left(1+\alpha_p u\right)^{k+1}\Big|_{-1/2}^{1/2}+$$

$$+\frac{\gamma_p(k)\beta_p^k}{\alpha_p^2}\left(t-\frac{1}{2}\right)\int_{-1/2}^{1/2}\left((1+\alpha_p u)^{k+1}-(1+\alpha_p u)^k\right)d(1+\alpha_p u)=$$

$$=\beta_p+\beta_p^{n_p}\cdot\left(t-\frac{1}{2}\right)=\xi(k;n_p)+\xi(k;n_p)^{n_p}\left(t-\frac{1}{2}\right)=f_1(t).$$

$\square$

From Theorem 16, Proposition 1 we get the following theorem.

Theorem 17. *Let $k\geq 4$ and $p\in\mathbb{N}_0(k)$. The model*

$$H(\sigma)=-\frac{1}{\beta}\sum_{\substack{<x,y>\\ x,y\in V}}\ln K_p(\sigma(x),\sigma(y);k),\quad \sigma\in\Omega_V$$

on the Cayley tree Γ^k has at least two translations-invariant Gibbs measures.

3.2 EXISTENCE OF PERIODIC GIBBS MEASURES FOR MODEL (1.1)

In this section we construct a function $K(t,u)$ such that corresponding equation (1) has a solution (f,g) with $f\neq g$.

Case: $k=2$.

Put

$$K_n(t,u)=\frac{1-b_n c_n^3\sqrt[n]{u-\frac{1}{2}}\left(\sqrt[n]{(u-\frac{1}{2})^2}-4\right)^2\sqrt[n]{t-\frac{1}{2}}}{c_n^2\left(\sqrt[n]{u-\frac{1}{2}}+2\right)^2},\quad t,u\in[0,1]\quad(3.7)$$

where

$$b_n=\left(\frac{1}{\sqrt[n]{4}}\right)^{(n-1)}\left(1+\frac{2}{n}\right),\quad c_n^3=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{(2+\sqrt[n]{u})^2}du.$$

Lemma 11. *For all $t, u \in [0,1]$, the following holds:*

$$\lim_{n\to\infty} K_n(t,u) > 0.$$

Proof. It is easy to see

$$\lim_{n\to\infty} K_n(t,u) > 0 \quad \Leftrightarrow$$

$$\lim_{n\to\infty}\left(1 - b_n c_n^3 \sqrt[n]{u-\frac{1}{2}}\left(\sqrt[n]{(u-\frac{1}{2})^2 - 4}\right)^2 \sqrt[n]{t-\frac{1}{2}}\right) > 0.$$

We have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{4}}\right)^{(n-1)}\left(1+\frac{2}{n}\right) = \frac{1}{4},$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\sqrt[3]{\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{(2+\sqrt[n]{u})^2}du} \geq \sqrt[3]{\frac{1}{8}}.$$

Then

$$\lim_{n\to\infty} K_n(t,u) > 0 \quad \Leftrightarrow$$

$$\lim_{n\to\infty}\left(1 - b_n c_n^3 \sqrt[n]{u-\frac{1}{2}}\left(\sqrt[n]{\left(u-\frac{1}{2}\right)^2 - 4}\right)^2 \sqrt[n]{t-\frac{1}{2}}\right) > 0.$$

□

Corollary 7. *There exists n_0 such that for every $n \geq n_0$ the function $K_{n_0}(t,u)$ is a positive function.*

Proof. Straightforward. □

Theorem 18. *The system of Hammerstain's equation:*

$$\int_0^1 K_{n_0}(t,u) f^2(u)du = g(t), \quad \int_0^1 K_{n_0}(t,u) g^2(u)du = f(t) \tag{3.8}$$

in the space $(C[0,1])^2$ has at least two positive solutions with $f \neq g$.

Proof. Let

$$f_1^{(n_0)}(t) = c_{n_0}\left(\sqrt[n_0]{t-\frac{1}{2}}+2\right),\ \ g_1^{(n_0)}(t)=1,\quad t\in[0,1],$$

Then $(f_1^{(n_0)}, g_1^{(n_0)}) \in (C[0,1])^2$ and positive.

(a) Consider the first equation:

$$\int_0^1 K_{n_0}(t,u)f^2(u)du = g(t).$$

$$\int_0^1 K_{n_0}(t,u)\left(f_1^{(n_0)}(u)\right)^2 du =$$

$$= 1-\int_0^1 b_{n_0}\cdot c_{n_0}^3\ \sqrt[n_0]{u-\frac{1}{2}}\left(\sqrt[n_0]{\left(u-\frac{1}{2}\right)^2-4}\right)^2\ \sqrt[n_0]{t-\frac{1}{2}}du =$$

$$= 1-b_{n_0}\cdot c_{n_0}^3\cdot\ \sqrt[n_0]{t_1}\int_{-\frac{1}{2}}^{\frac{1}{2}}\ \sqrt[n_0]{u_1}\left(\ \sqrt[n_0]{u_1^2-4}\right)^2 du_1 = 1 = g_1^{(n_0)}(t).$$

where $u_1 = u-\frac{1}{2}$, $t_1 = t-\frac{1}{2}$.

(b) Now we consider the second equation:

$$\int_0^1 K_{n_0}(t,u)\left(g_1^{(n_0)}(u)\right)^2 du = f_1^{n_0}(t).$$

$$\int_0^1 K_{n_0}(t,u)\left(g_1^{n_0}(u)\right)^2 du =$$

$$= \int_0^1 \frac{1-b_{n_0}c_{n_0}^3\ \sqrt[n_0]{u-\frac{1}{2}}\left(\ \sqrt[n_0]{(u-\frac{1}{2})^2-4}\right)^2\ \sqrt[n_0]{t-\frac{1}{2}}}{c_{n_0}^2\left(\ \sqrt[n_0]{u-\frac{1}{2}}+2\right)^2}du =$$

Let $u_1 = u-\frac{1}{2}$, $t_1 = t-\frac{1}{2}$. Then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{c^2(\ \sqrt[n_0]{u_1}+2)^2}du_1 - b_{n_0}c_{n_0}\ \sqrt[n_0]{t_1}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\sqrt[n_0]{u_1}\left(\ \sqrt[n_0]{u_1}-4\right)^2}{(\ \sqrt[n_0]{u_1}+2)^2}du_1 =$$

$$\frac{1}{c_{n_0}^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{(\ \sqrt[n_0]{u_1}+2)^2}du_1 - b_{n_0}c_{n_0}\ \sqrt[n_0]{t_1}\int_{-\frac{1}{2}}^{\frac{1}{2}}\ \sqrt[n_0]{u_1}\left(\ \sqrt[n_0]{u_1}-2\right)^2 du_1 =$$

$$= 2c_{n_0} + 4b_{n_0}c_{n0}\sqrt[n_0]{t_1}\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt[n_0]{u_1^2}du_1 = c_{n_0}\left(\sqrt[n_0]{t-\frac{1}{2}}+2\right) = f_1^{n_0}(t).$$

By symmetry of (f,g) we have $(g_1^{n_0}(t), f_1^{n_0}(t))$ is also solution of (3.8).

This completes the proof. □

From this we get

Theorem 19. *The model:* $H(\sigma) =$

$$-\frac{1}{\beta}\sum_{<x,y>} ln\left(\frac{1-b_{n_0}c_{n_0}^3\sqrt[n_0]{\sigma(x)-\frac{1}{2}}\left(\sqrt[n_0]{(\sigma(x)-\frac{1}{2})^2}-4\right)^2\sqrt[n_0]{\sigma(y)-\frac{1}{2}}}{c_{n_0}^2\left(\sqrt[n_0]{\sigma(x)-\frac{1}{2}}+2\right)^2}\right)$$

on the Cayley tree Γ^2 *has at least two periodic Gibbs measures.*

Case: $k=3$.

Lemma 12. *Let* $a \in R$. *Then for every odd (even) function* $\varphi(x) \in C[0,1]$ *the following equation holds:*

$$\int_{-a}^{a}\frac{\varphi(x)}{(1+\sin x)^3}dx = -2\int_0^a\frac{\varphi(x)\sin x(3+\sin^2 x)}{\cos^6 x}dx.$$

$$\left(\int_{-a}^{a}\frac{\varphi(x)}{(1+\sin x)^3}dx = 2\int_0^a\frac{\varphi(x)(1+3\sin^2 x)}{\cos^6 x}dx\right).$$

Proof. Let $\varphi(x)$ be odd (the case even is similar) function

$$\int_{-a}^{a}\frac{\varphi(x)}{(1+\sin x)^3}dx = \int_0^a\frac{\varphi(x)}{(1+\sin x)^3}dx + \int_{-a}^{0}\frac{\varphi(x)}{(1+\sin x)^3}dx =$$

$$\int_0^a\frac{\varphi(x)}{(1+\sin x)^3}dx - \int_0^a\frac{\varphi(x)}{(1-\sin x)^3}dx = -2\int_0^a\frac{\varphi(x)\sin x(3+\sin^2 x)}{\cos^6 x}dx.$$

Put

$$K(t,u) = \frac{1-\frac{22}{17}\sin\frac{\pi(2t-1)}{3}\sin\frac{\pi(2u-1)}{3}}{a^3(1+\sin\frac{\pi(2u-1)}{3})^3}, \quad t,u \in [0,1], \tag{3.9}$$

where $a = \sqrt[4]{\frac{198\sqrt{3}}{5\pi}}$. It is easy to see that $K(t,u)$ is a positive and continuous function. □

Theorem 20. *The system of Hammerstain's equations*

$$\int_0^1 K(t,u)f^3(u)du = g(t), \quad \int_0^1 K(t,u)g^3(u)du = f(t), \tag{3.10}$$

in the space $(C[0,1])^2$ has at least two positive solutions with $f \neq g$.

Proof. (a) Denote

$$f_1(t) = a\left(1 + \sin\frac{\pi(2t-1)}{3}\right), g_1(t) = 1, \quad t \in [0,1],$$

where $a = \sqrt[4]{\frac{198\sqrt{3}}{5\pi}}$. Then $(f_1, g_1) \in (C[0,1])^2$ and the functions f_1 and g_1 are positive. Consider the first equation of (3.10)

$$\int_0^1 K(t,u)f_1^3(u)du = 1 - \frac{22}{17}\sin\frac{\pi(2t-1)}{3}\int_0^1 \sin\frac{\pi(2u-1)}{3}du = 1.$$

(b) Now we check the second equation.

$$\int_0^1 K(t,u)g_1^3(u)du = \int_0^1 \frac{1 - \frac{22}{17}\sin\frac{\pi(2t-1)}{3}\sin\frac{\pi(2u-1)}{3}}{a^3(1+\sin\frac{\pi(2u-1)}{3})^3}du.$$

Let $t_1 = \frac{\pi}{3}(2t-1)$, $u_1 = \frac{\pi}{3}(2u-1)$. Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(t_1,u_1)g_1^3(u_1)du_1 = \frac{3}{2a^3\pi}\left(\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1 - \frac{22}{17}\sin t_1 \sin u_1}{(1+\sin u_1)^3}\right)du_1 =$$

$$= \frac{3}{2a^3\pi}\left(\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{(1+\sin u_1)^3}du_1 - \frac{22}{17}\sin t_1 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin u_1}{(1+\sin u_1)^3}\right)du_1.$$

By Lemma 12 LHS of this equality is

$$\frac{3}{2a^3\pi}\left[\int_0^{\frac{\pi}{3}} \frac{1+3\sin^2 u_1}{\cos^6 u_1}du_1 + \frac{22}{17}\sin t_1 \int_0^{\frac{\pi}{3}} \frac{\sin^2 u_1(3+\sin^2 u_1)}{\cos^6 u_1}du_1\right] =$$

$$= \frac{3}{a^3\pi}\left[\int_0^{\sqrt{3}} (1+4y^2)(1+y^2)dy + \frac{22}{17}\sin t_1 \int_0^{\sqrt{3}} y^2(3+4y^2)dy\right] =$$

$$= \frac{198\sqrt{3}}{a^3\pi}(1+\sin t_1) = \frac{198\sqrt{3}}{a^3\pi}\left(1+\sin\frac{\pi(2t-1)}{3}\right) = f(t).$$

By symmetry of (f_1, g_1) we have $(g_1(t), f_1(t))$ is also solution to (3.8).

This completes the proof. □

Theorem 21. *The model:*

$$H(\sigma) = -\frac{1}{\beta}\sum_{<x,y>} ln\left(\frac{1-\frac{22}{17}\sin\frac{\pi(2\sigma(x)-1)}{3}\sin\frac{\pi(2\sigma(y)-1)}{3}}{a^3(1+\sin\frac{\pi(2\sigma(x)-1)}{3})^3}\right)$$

on the Cayley tree Γ^3 *has at least two periodic Gibbs measures.*

Case: $k \geq 4$.

Denote

$$c_k = \frac{2\left(1-(\frac{1}{3})^{k-1}\right)}{\frac{k-1}{k-2}\left(1-(\frac{1}{3})^{k-2}\right)-2\left(1-(\frac{1}{3})^{k-1}\right)}. \tag{3.11}$$

Lemma 13. *For every* $k \in N,\ \ k \geq 4$ *the following inequality holds:* $|c_k| < 4$.

Proof For $k \geq 4$ we have

$$|c_k| = \frac{2\left(1-(\frac{1}{3})^{k-1}\right)}{\frac{k-3}{k-2}+(\frac{1}{3})^{k-1}(\frac{k+1}{k-2})} < \frac{2}{\frac{k-3}{k-2}+(\frac{1}{3})^{k-1}(\frac{k+1}{k-2})} < \frac{2(k-2)}{k-3},$$

Thus

$$|c_k| < \frac{2(k-2)}{k-3} = 2+\frac{2}{k-3} \leq 4. \tag{3.12}$$

Hence $|c_k| < 4$ for $k \geq 4$.

For each $k \geq 4$, $a > 0$ we define the continuous function

$$K(t,u,k) = \frac{1+c_k(t-\frac{1}{2})(u-\frac{1}{2})}{a^k(u+\frac{1}{2})^k}, \qquad t,u \in [0,1].$$

By the inequality (3.12) it follows that the function $K(t,u,k)$ is positive.

Theorem 22. *For each* $k \geq 4$ *the Hammerstein's system of equations:*

$$\int_0^1 K(t,u,k)f^k(u)du = g(t), \ \ \int_0^1 K(t,u,k)g^k(u)du = f(t) \tag{3.13}$$

in $(C[0,1])^2$ *have at least two positive solutions with* $f \neq g$.

Proof. Let $k \geq 4$. Define the positive continuous functions $f_1(t)$, $g_1(t)$ on $[0,1]$ by the equality

$$f_1(t) = a\left(t + \frac{1}{2}\right), \quad g_1(t) = 1$$

where

$$a = a(k) = \sqrt[k+1]{\frac{2^{k-1}}{k-1}\left(1 - \left(\frac{1}{3}\right)^{k-1}\right)}, \quad k \geq 4.$$

It is easy to see that $a > 0$. We shall show that (f_1, g_1) is a solution to the Hammerstein's system of equations (3.12). □

We shall check the first equation.

$$\int_0^1 K(t,u,k) f_1^k(u)du = \int_0^1 \frac{1 + c_k\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}{a^k\left(u + \frac{1}{2}\right)^k}\left(a\left(u + \frac{1}{2}\right)\right)^k du =$$

$$= \int_0^1 \left(1 + c_k\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)\right)du = 1 + c_k t_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} u_1 du_1 = 1.$$

Where $t_1 = t - \frac{1}{2}$ and $u_1 = u - \frac{1}{2}$. Hence

$$\int_0^1 K(t,u,k) f_1^k(u)du = g_1(t).$$

Now we shall check the second equation.

$$\int_0^1 K(t,u,k) g_1^k(u)du = \int_0^1 \left(\frac{1 + c_k(t - \frac{1}{2})(u - \frac{1}{2})}{a^k(u + \frac{1}{2})^k}\right) du =$$

$$\int_0^1 \frac{1}{a^k(u + \frac{1}{2})^k}du + \int_0^1 \frac{c_k(t - \frac{1}{2})(u - \frac{1}{2})}{a^k(u + \frac{1}{2})^k}du$$

$$= \frac{1}{a^k}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(u_1 + 1)^k}du_1 + c_k t_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u_1}{(u_1 + 1)^k}du_1\right) =$$

$$\left(where \;\; t_1 = t - \frac{1}{2}, u_1 = u - \frac{1}{2}\right) = \frac{1}{a^k}\frac{2^{k-1}}{k-1}\left(1 - \left(\frac{1}{3}\right)^{k-1}\right) +$$

$$+\frac{c_k t_1}{a^k}\left[\frac{2^{k-2}}{k-2}\left(1 - \left(\frac{1}{3}\right)^{k-2}\right) - \frac{2^{k-1}}{k-1}\left(1 - \left(\frac{1}{3}\right)^{k-1}\right)\right] =$$

$$= a + at_1 \frac{2\left(1-(\frac{1}{3})^{k-1}\right)}{\frac{k-1}{k-2}\left(1-(\frac{1}{3})^{k-2}\right) - 2\left(1-(\frac{1}{3})^{k-1}\right)} \times$$

$$\times \left(\frac{(k-1)\left(1-(\frac{1}{3})^{k-2}\right) - 2(k-2)\left(1-(\frac{1}{3})^{k-1}\right)}{2(k-2)\left(1-(\frac{1}{3})^{k-1}\right)} \right) = f_1(t).$$

Moreover, $(g_1(t), f_1(t))$ is also solution to (3.13).

Theorem 23. *Let $k \geq 4$. The model*

$$H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} ln \left(\frac{1 + c_k(\sigma(x) - \frac{1}{2})(\sigma(y) - \frac{1}{2})}{a^k(\sigma(x) + \frac{1}{2})^k} \right)$$

on the Cayley tree Γ^k has at least two periodic Gibbs measures.

Now we construct models which existence of four periodic Gibbs measures for model (1.1). Denote

$$c_{ij}(m) = \frac{1}{m + 2(i-1) + 2(j-1)}, \quad (n, m, p) \in N \times N \times N_0, \quad 1 \leq i, j \leq n,$$

$A_n^{(m,p)} = \left(\frac{c_{ij}(m)}{4^{p+j+i-2}} \right)_n$ be $n \times n$ square matrix.

If $n \in \{2, 3\}$ then it's easy to check $\det(A_n^{(m,p)}) \neq 0$.

Put

$$\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = \left(A_3^{(1,0)}\right)^{-1} \begin{pmatrix} 0 \\ \frac{1}{6} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = \left(A_3^{(3,1)}\right)^{-1} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{20} \end{pmatrix}$$

and

$$\begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} = \left(A_2^{(5,2)}\right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix} = \left(A_2^{(7,3)}\right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\left(A_n^{(m,p)}\right)^{-1}$ is inverse of $A_n^{(m,p)}$.

So we define following functions:

$$\psi_1(u) = a_{11} + a_{12}u^2 + a_{13}u^4, \quad \psi_2(u) = a_{21}u^2 + a_{22}u^4 + a_{23}u^6,$$

$$\psi_3(u) = b_{11}u + b_{12}u^3, \quad \psi_4(u) = b_{21}u^3 + b_{22}u^5.$$

Finally

$$K_1(t,u;k) = \psi_1(u)\left(\sqrt[k]{20t^4 + \frac{3}{4}} - 1\right) + \psi_2(u)\left(\sqrt[k]{6t^2 + \frac{1}{2}} - 1\right),$$

$$K_2(t,u;k) = \psi_3(u)\left(\sqrt[k]{t^3+1} - 1\right) + \psi_4(u)\left(\sqrt[k]{t^5+1} - 1\right),$$

$$\widetilde{K}(t,u;k) = 1 + K_1(t,u;k) + K_2(t,u;k).$$

Remark 2. *There exist $k_0 \in N$ such that for all $k \geq k_0$ the following inequality holds*

$$\widetilde{K}\left(t - \frac{1}{2}, u - \frac{1}{2}; k\right) > 0, \quad (t,u) \in [0,1]^2.$$

Proof. It is sufficient to show:

$$\lim_{k\to\infty} \widetilde{K}\left(t - \frac{1}{2}, u - \frac{1}{2}; k\right) > 0, \quad (t,u) \in [0,1]^2.$$

Let $\gamma : [0,1] \to [m, M]$ be a function, $m > 0$.

We have

$$0 = \lim_{k\to\infty}(\sqrt[k]{m} - 1) \leq \lim_{k\to\infty}(\sqrt[k]{\gamma(t)} - 1) \leq \lim_{k\to\infty}(\sqrt[k]{M} - 1) = 0.$$

Hence

$$\lim_{k\to\infty}(\sqrt[k]{\gamma(t)} - 1) = 0 \ \Rightarrow \lim_{k\to\infty} K_i\left(t - \frac{1}{2}, u - \frac{1}{2}; k\right) = 0, \quad i \in \{1,2\}$$

and

$$\lim_{k\to\infty} \widetilde{K}\left(t - \frac{1}{2}, u - \frac{1}{2}; k\right) = 1 > 0.$$

This completes the proof. □

Lemma 14. *If $k \in \{1,2\}$, then*

(i) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u)u^{2k}du = \frac{1}{12}\left(1 + (-1)^{k+1}\right),$

(*ii*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u)u^{2k}du = \frac{1}{12}\left(1+(-1)^k\right),$

(*iii*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u)u^{2k+1}du = \frac{1}{12}\left(1+(-1)^k\right),$

(*iv*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u)u^{2k+1}du = \frac{1}{12}\left(1+(-1)^{k+1}\right).$

Proof. For $k \in \{1,2\}$

(*i*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u)u^{2k}du = a_{11}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2k}du + a_{12}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+1)}du + a_{13}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du =$

$$= \frac{a_{11}}{4^k(2k+1)} + \frac{a_{12}}{4^{k+1}(2k+3)} + \frac{a_{13}}{4^{k+2}(2k+5)} =$$

$$= a_{11} \times c_{k+1,1}^{(0)}(1) + a_{12} \times c_{k+1,2}^{(0)}(1) + a_{13} \times c_{k+1,3}^{(0)}(1) = \frac{1}{12}\left(1+(-1)^{k+1}\right).$$

(*ii*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u)u^{2k}du = a_{21}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+1)}du + a_{22}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du +$

$$a_{23}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+3)}du = \frac{a_{21}}{4^k(2k+1)} + \frac{a_{22}}{4^{k+1}(2k+3)} + \frac{a_{23}}{4^{k+2}(2k+5)} =$$

$$= a_{21} \times c_{k+1,1}^{(1)}(3) + a_{22} \times c_{k+1,2}^{(1)}(3) + a_{23} \times c_{k+1,3}^{(1)}(3) = \frac{1}{12}\left(1+(-1)^k\right).$$

(*iii*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u)u^{2k+1}du = b_{11}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+1)}du + b_{12}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du =$

$$= b_{11} \times c_{k,1}^{(2)}(5) + b_{12} \times c_{k,2}^{(2)}(5) = \frac{1}{12}\left(1+(-1)^k\right) = \frac{1}{12}\left(1+(-1)^k\right).$$

(*iv*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u)u^{2k+1}du = b_{21}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du + b_{22}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+3)}du =$

$$= b_{21} \times c_{k,1}^{(3)}(7) + b_{22} \times c_{k,2}^{(3)}(7) = \frac{1}{12}\left(1+(-1)^{k+1}\right).$$

□

Lemma 15. *The function* $\varphi_0(u) = 1$ *is a fixed point of the operator* H_k :

$$(H_k f)(t) = \int_0^1 \widetilde{K}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) f^k(u)du, \quad k \geq 2. \tag{3.14}$$

Proof. Let $u_1 = u - \frac{1}{2}$, $v_1 = v - \frac{1}{2}$ then

$$(H_k\varphi_0)\left(t - \frac{1}{2}\right) = \int_0^1 \widetilde{K}\left(t - \frac{1}{2}, u - \frac{1}{2}; k\right) du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{K}(t_1, u_1; k) du_1 =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[1 + K_1(t_1, u_1; k) + K_2(t_1, u_1; k)\right] du_1$$

$$= 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) du_1.$$

Now we'll prove the following

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_i(t_1, u_1; k) du_1 = 0, \quad i \in \{1, 2\}. \tag{3.15}$$

Case: $i = 1$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) du_1$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\psi_1(u_1)\left(\sqrt[k]{20t_1^4 + \frac{3}{4}} - 1\right) du_1 + \psi_2(u_1)\left(\sqrt[k]{6t_1^2 + \frac{1}{2}} - 1\right)\right] du_1 =$$

$$= \left(\sqrt[k]{20t_1^4 + \frac{3}{4}} - 1\right)\left(a_{11} + a_{12}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^2 du + a_{13}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^4 du\right) +$$

$$+ \left(\sqrt[k]{6t_1^2 + \frac{1}{2}} - 1\right)\left(a_{21}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^2 du + a_{22}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^4 du + a_{23}\int_{-\frac{1}{2}}^{\frac{1}{2}} u^6 du\right) =$$

$$= \left(\sqrt[k]{20t_1^4 + \frac{3}{4}} - 1\right)\left(a_{11} + \frac{a_{12}}{3 \cdot 4} + \frac{a_{13}}{5 \cdot 4^2}\right) +$$

$$+ \left(\sqrt[k]{6t_1^2 + \frac{1}{2}} - 1\right)\left(\frac{a_{21}}{3 \cdot 4} + \frac{a_{22}}{5 \cdot 4^2} + \frac{a_{23}}{7 \cdot 4^{k+2}}\right) =$$

$$= \left(\sqrt[k]{20t_1^4 + \frac{3}{4}} - 1\right)\left(a_{11} \times c_{1,1}^{(0)}(1) + a_{12} \times c_{1,2}^{(0)}(1) + a_{13} \times c_{1,3}^{(0)}(1)\right) +$$

$$+ \left(\sqrt[k]{6t_1^2 + \frac{1}{2}} - 1\right)\left(a_{21} \times c_{2,1}^{(1)}(3) + a_{22} \times c_{2,2}^{(1)}(3) + a_{23} \times c_{2,3}^{(1)}(3)\right) = 0.$$

Case: $i = 2$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1)\left(\sqrt[k]{t^3 + 1}\right) du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u_1)\left(\sqrt[k]{t^5 + 1}\right) du_1$$

It's easy to check for $j \in \{3,4\}$ the functions $\psi_j(u_1)$ is odd, i.e:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_j(u_1)du_1 = 0 \;\Rightarrow\; \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1,u_1;k)du_1 = 0.$$

Thus we have proved

$$\left(H_k\varphi_0\right)(t) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1,u_1;k)du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1,u_1;k)du_1 = 1.$$

This completes the proof. □

Denote

$$f_1(u) = \sqrt[k]{6u^2 + \frac{1}{2}}, \quad f_2(u) = \sqrt[k]{20u^4 + \frac{3}{4}},$$
$$g_1(u) = \sqrt[k]{u^3+1}, \quad g_2(u) = \sqrt[k]{u^5+1}.$$

Theorem 24. *For all $k \geq k_0$ the Hammerstein's system of equations:*

$$\int_0^1 \widetilde{K}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) f^k(u)du = g(t),$$

$$\int_0^1 \widetilde{K}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) g^k(u)du = f(t) \tag{3.16}$$

in $\left(C[0;1]\right)^2$ have at least four positive solutions with $f \neq g$.

Proof. We'll show

$$\left(f_1\left(u-\frac{1}{2}\right),\; f_2\left(u-\frac{1}{2}\right)\right),\; \left(f_2\left(u-\frac{1}{2}\right),\; f_1\left(u-\frac{1}{2}\right)\right)$$

and

$$\left(g_1\left(u-\frac{1}{2}\right),\; g_2\left(u-\frac{1}{2}\right)\right),\; \left(g_2\left(u-\frac{1}{2}\right),\; g_1\left(u-\frac{1}{2}\right)\right)$$

are solutions to the system of equations (3.16).

At first we'll prove $\left(f_1(u-\frac{1}{2}),\;\; f_2(u-\frac{1}{2})\right)$ is a solution to equation (3.16). Let $u-\frac{1}{2} = u_1,\; t-\frac{1}{2} = t_1.$ Then

$$\left(H_k f_i\right)(t) = \int_0^1 \widetilde{K}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) f_i^k\left(u-\frac{1}{2}\right) du$$

$$= \int_0^1 \widetilde{K}(t_1, u_1; k) f_i^k(u_1) du_1 =$$

$$= \int_0^1 \left[1 + K_1(t_1, u_1; k) + K_2(t_1, u_1; k)\right] f_i^k(u_1) du_1.$$

It's easy to see that

$$K_2(t_1, -u_1; k) = -K_2(t_1, u_1; k), \quad f_i(u_1) = f_i(-u_1), \quad i \in \{1, 2\}.$$

Hence

$$K_2(t_1, -u_1; k) f_i(u_1) = -K_2(t_1, u_1; k) f_i(u_1) \;\Rightarrow\; \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) f_i(u_1) du_1 = 0.$$

Thus

$$(H_k f_i)\left(t - \frac{1}{2}\right) = (H_k f_i)(t_1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[1 + K_1(t_1, u_1; k)\right] f_i^k(u_1) du_1.$$

Case: $i = 1$

$$(H_k f_1)\left(t - \frac{1}{2}\right) = (H_k f_1)(t_1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[1 + K_1(t_1, u_1; k)\right] \left(\sqrt[k]{6u_1^2 + \frac{1}{2}}\right)^k du_1$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(6u_1^2 + \frac{1}{2}\right) du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) \left(6u_1^2 + \frac{1}{2}\right) du_1$$

$$1 + 6 \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^2 du_1 + \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) du_1$$

By (3.15) we get

$$= 1 + 6 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\psi_1(u_1) \left(\sqrt[k]{20t_1^4 + \frac{3}{4}} - 1\right) + \psi_2(u_1) \left(\sqrt[k]{6t_1^2 + \frac{1}{2}} - 1\right)\right] u_1^2 du_1 =$$

$$= 1 + 6 \left(\sqrt[k]{20t_1^4 + \frac{3}{4}} - 1\right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1) u_1^2 du_1$$

$$+6 \left(\sqrt[k]{6t_1^2 + \frac{1}{2}} - 1\right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1) u_1^2 du_1.$$

By Lemma 14

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1) u_1^2 du_1 = \frac{1}{6}, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1) u_1^2 du_1 = 0. \tag{3.17}$$

By (3.15) and (3.17) we obtain

$$(H_k f_1)\left(t-\frac{1}{2}\right) = 1+\left(\sqrt[k]{20t_1^4+\frac{3}{4}}-1\right) = \sqrt[k]{20t_1^4+\frac{3}{4}} = f_2(t_1) = f_2\left(t-\frac{1}{2}\right).$$

Case: $i=2$

$$(H_k f_2)\left(t-\frac{1}{2}\right) = (H_k f_2)(t_1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} [1+K_1(t_1,u_1;k)]\, f_2^k(u_1)du_1 =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} [1+K_1(t_1,u_1;k)]\left(20u_1^4+\frac{1}{2}\right)du_1 =$$

$$= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1,u_1;k)du_1 + 20\int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1,u_1;k)u_1^2du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(20u_1^4+\frac{1}{2}\right)du_1 =$$

By (3.15)

$$= 1+20\int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1,u_1;k)u_1^4du_1 =$$

$$= 1+20\left(\sqrt[k]{20t_1^4+\frac{3}{4}}-1\right)\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1)u_1^4du_1 +$$

$$+20\left(\sqrt[k]{6t_1^2+\frac{1}{2}}-1\right)\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1)u_1^4du_1.$$

By Lemma 14

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1)u_1^4du_1 = 0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1)u_1^4du_1 = \frac{1}{20}.$$

Then

$$(H_k f_2)\left(t-\frac{1}{2}\right) = 1+\left(\sqrt[k]{6t_1^2+\frac{1}{2}}-1\right) = \sqrt[k]{6t_1^2+\frac{1}{2}} = f_1(t_1) = f_1\left(t-\frac{1}{2}\right).$$

By symmetry of (f_1, f_2) we have (f_2, f_1) is also solution to equation (3.16).

□

Now we'll prove $\left(g_1(u-\frac{1}{2}), g_2(u-\frac{1}{2})\right)$ is a solution to equation (3.16). For $i \in \{1,2\}$

$$(H_k g_i)\left(t-\frac{1}{2}\right) = \int_0^1 \widetilde{K}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) g_i^k\left(u-\frac{1}{2}\right)du =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{K}(t_1, u_1; k)(1 + u_1^{2i+1}) du_1,$$

where $u_1 = u - \frac{1}{2}$, $t_1 = t - \frac{1}{2}$. Then

$$(H_k g_i)\left(t - \frac{1}{2}\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{K}(t_1, u_1; k) du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{K}(t_1, u_1; k) u_1^{2i+1} du_1 =$$

By Lemma 15

$$= (H_k g_i)\left(t - \frac{1}{2}\right) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{K}(t_1, u_1; k) u_1^{2i+1} du_1 =$$

$$= 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k) + K_2(t_1, u_1; k)] u_1^{2i+1} du_1.$$

Hence

$$(H_k g_i)\left(t - \frac{1}{2}\right) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k)] u_1^{2i+1} du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) u_1^{2i+1} du_1 \tag{3.18}$$

One can easily check that

$$K_1(t_1, -u_1; k) = K_1(t_1, u_1; k) \quad \Rightarrow \quad K_1(t_1, -u_1; k)(-u_1^{2i+1}) = -K_1(t_1, u_1; k) u_1^{2i+1}$$

then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^{2i+1} du_1 = 0, \quad i \in \{1, 2\} \tag{3.19}$$

By (3.18) and (3.19) we obtain

$$(H_k g_1)\left(t - \frac{1}{2}\right) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} u_1^{2i+1} du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^{2i+1} du_1 +$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) u_1^{2i+1} du_1 = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) u_1^{2i+1} du_1 =$$

$$= 1 + \left(\sqrt[k]{t_1^3 + 1} - 1\right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1) u_1^{2i+1} du_1 + \left(\sqrt[k]{t_1^5 + 1} - 1\right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u_1) u_1^{2i+1} du_1.$$

Case: $i = 1$

$$(H_k g_1)\left(t - \frac{1}{2}\right) = 1 + \left(\sqrt[k]{t_1^3 + 1} - 1\right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1) u_1^3 du_1 +$$

$$+\left(\sqrt[k]{t_1^5+1}-1\right)\int_{-\frac{1}{2}}^{\frac{1}{2}}\psi_4(u_1)u_1^3du_1$$

By Lemma 14

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\psi_3(u_1)u_1^3du_1=0,\quad \int_{-\frac{1}{2}}^{\frac{1}{2}}\psi_4(u_1)u_1^3du_1=1.$$

Then

$$(H_kg_1)\left(t-\frac{1}{2}\right)=1+\left(\sqrt[k]{1+t_1^5}-1\right)=$$

$$=\sqrt[k]{1+t_1^5}=\sqrt[k]{1+\left(t-\frac{1}{2}\right)^5}=g_2\left(t-\frac{1}{2}\right).$$

Case: $i=2$

$$(H_kg_1)\left(t-\frac{1}{2}\right)=1+\left(\sqrt[k]{t_1^3+1}-1\right)\int_{-\frac{1}{2}}^{\frac{1}{2}}\psi_3(u_1)u_1^5du_1+$$

$$+\left(\sqrt[k]{t_1^5+1}-1\right)\int_{-\frac{1}{2}}^{\frac{1}{2}}\psi_4(u_1)u_1^5du_1.$$

By Lemma 14 we get

$$(H_kg_1)\left(t-\frac{1}{2}\right)=1+\left(\sqrt[k]{t_1^3+1}-1\right)=\sqrt[k]{t_1^3+1}=g_1\left(t-\frac{1}{2}\right).$$

Thus we have proved

$$(H_kg_1)\left(t-\frac{1}{2}\right)=g_2\left(t-\frac{1}{2}\right),\quad (H_kg_2)\left(t-\frac{1}{2}\right)=g_1\left(t-\frac{1}{2}\right).$$

Theorem 25. *Let $k\geq k_0$. The model*

$$H(\sigma)=-\frac{1}{\beta}\sum_{<x,y>}ln\widetilde{K}\left(\sigma(x)-\frac{1}{2},\sigma(y)-\frac{1}{2};k\right)$$

on the Cayley tree Γ^k has at least four periodic Gibbs measures.

3.3 EXISTENCE OF AT LEAST N GIBBS MEASURES ON A CAYLEY TREE

Put

$$\mathbf{A}_n^{(p)} = \left\{ \frac{1}{2(2p+i+j)-3} \left(\frac{1}{2}\right)^{2(2p+i+j-2)} \right\}_{i,j=\overline{1,n}}, \quad n,p \in N. \tag{3.20}$$

$$\mathbf{B}_n[a_1,...,a_n;b_1,...b_n] = \left(\frac{1}{a_i+b_j}\right)_{i,j=\overline{1,n}}, \quad a_i,b_j > 0. \tag{3.21}$$

$$\mathbf{C}_n^{(p)} = B_n[4p,4(p+1),...,2(p+n-1);1,5,...,4n-3]. \tag{3.22}$$

Lemma 16. *Let $n \geq 2$. Then*

$$\det \mathbf{B}_n[a_1,...,a_n;b_1,...,b_n] = \frac{\prod_{1\leq i<j\leq n}[(a_i-a_j)(b_i-b_j)]}{\prod_{i,j=1}^n (a_i+b_j)}$$

Corollary 8. $\det \mathbf{A}_n^{(p)} = \left(\frac{1}{2}\right)^{2n(2p+n-1)} \det \mathbf{C}_n^{(p)}$.

Proof. Let $i,j=\overline{1,n}$. We multiply by $2^{2(p+j-1)}$ the column j of the matrix $\mathbf{A}_n^{(p)}$ and then multiply by $2^{2(i-1)}$ the row i of the matrix obtained. As a result we get $\mathbf{C}_n^{(p)}$. □

Lemma 17. *Let* $\mathbf{B}^{-1}[a_1,a_2,...,a_n;b_1,b_2,...,b_n] = \{\beta_{ij}\}_{i,j=\overline{1,n}}$ *is an inverse matrix of*

$\mathbf{B}[a_1,a_2,...,a_n;b_1,b_2,...,b_n]$. *Then*

$$\beta_{ji} = (-1)^{i+j} \frac{\prod_{s=1}^n (a_s+b_j) \prod_{s=1,s\neq j}^n (a_i+b_s)}{\prod_{s=1,s\neq j}^n (b_i-b_s) \prod_{s=1,s\neq i}^n (a_i-a_s)}$$

Proof. Subtracting the ith column of $\mathbf{B}[a_1,a_2,...,a_n;b_1,b_2,...,b_n]$ from every other column we get a following equality

$$\det \mathbf{B}[a_1,a_2,...,a_n;b_1,b_2,...,b_n] =$$

$$= \frac{\prod_{s=1,s\neq j}^{n}(b_j-b_s)}{\prod_{s=1}^{n}(a_s+b_j)} \begin{pmatrix} \frac{1}{a_1+b_1} & \dots & \frac{1}{a_1+b_{j-1}} & 1 & \frac{1}{a_1+b_{j+1}} & \dots & \frac{1}{a_1+b_n} \\ \frac{1}{a_2+b_1} & \dots & \frac{1}{a_2+b_{j-1}} & 1 & \frac{1}{a_2+b_{j+1}} & \dots & \frac{1}{a_2+b_n} \\ & \dots & & \dots & & \dots & \\ \frac{1}{a_n+b_1} & \dots & \frac{1}{a_n+b_{j-1}} & 1 & \frac{1}{a_n+b_{j+1}} & \dots & \frac{1}{a_n+b_n} \end{pmatrix}.$$

Now we subtract from the jth row the ith row for every $j \in \{1, 2..., i-1, i+1, ...n\}$. Then

$$\det \mathbf{B}[a_1, ..., a_n; b_1, ..., b_n] =$$
$$= \frac{\prod_{s=1}^{n}(a_s+b_j)\prod_{s=1,s\neq j}^{n}(a_i+b_s)}{\prod_{s=1,s\neq j}^{n}(b_i-b_s)\prod_{s=1,s\neq i}^{n}(a_i-a_s)} \times \mathbf{B}^{(j,i)}[a_1, ..., a_n; b_1, ..., b_n]$$

where $\mathbf{B}^{(j,i)}[a_1, ..., a_n; b_1, ..., b_n]$ is the cofactor of the element $\frac{1}{a_i+a_j}$ in $\mathbf{B}[a_1, ..., a_n; b_1, ..., b_n]$. Since $\beta = \frac{\mathbf{B}^{(j,i)}[a_1,...,a_n;b_1,...,b_n]}{\det \mathbf{B}[a_1,...,a_n;b_1,...,b_n]}$ the lemma is proved. □

Denote

$$(\mathbf{A}_n^{(p)})^{-1} = \{\alpha_{ij}\}_{i,j\in\overline{1,n}}$$

$$\varphi_{s,n,q}(u) = \alpha_{s1}u^{2p-1} + ... + \alpha_{sn}u^{2(n+p)-3}, \quad (s, n, p) \in \mathbf{N}$$

$$K_{(n,p)}(t, u; k) = 1 + \sum_{s=1}^{n}\left(\sqrt[k]{1+t^{2(p+s)-1}} - 1\right)\varphi_{(s,n,p)}(u), \quad k \geq 2.$$

Remark 3. *For α_{ji} element of $(\mathbf{A}_n^{(p)})^{-1}$ following equality is holds*

$$\alpha_{ji} = (-1)^{i+j}4^{2p+i+j-4(n-1)}\frac{\prod_{s=1}^{n}(4p+2s+2j-3)\prod_{s=1,s\neq j}^{n}(4p+2s+2j-3)}{\prod_{s=1,s\neq j}^{n}(j-s)\prod_{s=1,s\neq i}^{n}(i-s)}$$

Proof. By Corollary 8 and Lemma 17 we get

$$\alpha_{ji} = (-1)^{i+j}4^{2p+i+j}\frac{\mathbf{B}^{(i,j)}[4p, 4p+2, ...4p+2(n-1); 1, 3, ...2n-1]}{\det \mathbf{B}[4p, 4p+2, ...4p+2(n-1); 1, 3, ...2n-1]} =$$
$$= (-1)^{i+j}4^{2p+i+j}\frac{\prod_{s=1}^{n}(4p+2s+2j-3)\prod_{s=1,s\neq j}^{n}(4p+2s+2j-3)}{\prod_{s=1,s\neq j}^{n}(2j-2s)\prod_{s=1,s\neq i}^{n}(2i-2s)}.$$

□

Remark 4. *There exist $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$ the following inequality holds*

$$K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) > 0, \quad (t,u) \in [0,1]^2.$$

Indeed, it's not difficult to check

$$\lim_{n\to\infty} K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) = 1 > 0.$$

Theorem 26. *Let $k \geq k_0$ for $k, n \in \mathbf{N}$. Then there exist positive kernel $K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right)$ such that Hammerstein's operator H_k has at least n positive fixed points.*

Proof. Let $f_m(u) = \sqrt[k]{1+u^{2(p+m)-1}}$, $m = \overline{1,n}$ and $u_1 = u - \frac{1}{2}$, $t_1 = t - \frac{1}{2}$. We show these functions are fixed points of operator H_k.

$$\int_0^1 K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) f_m^k\left(u-\frac{1}{2}\right) du =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} K_{(n,p)}\left(t_1, u_1; k\right) f_m^k\left(u_1\right) du_1 =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[1 + \sum_{s=1}^{n}\left(\sqrt[k]{1+t_1^{2(p+s)-1}} - 1\right)\varphi_{(s,n,p)}(u_1)\right]\left(1+u_1^{2(p+m)-1}\right) du_1 =$$

$$1 + \sum_{s=1}^{n}\left(\sqrt[k]{1+t_1^{2(p+s)-1}} - 1\right)$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\alpha_{s1}u_1^{4(p-1)+2s+2m} + ... + \alpha_{sn}u_1^{4p+2(s+m+n)-6}\right) du_1 =$$

$$= 1 + \sum_{s=1}^{n}\left(\sqrt[k]{1+t_1^{2(p+s)-1}} - 1\right) \times \left(\alpha_{sm}\beta_{s1} + ... + \alpha_{nm}\beta_{sn}\right) = \sqrt[k]{1+t_1^{2(p+m)-1}}.$$

Hence

$$\int_0^1 K_{(n,p)}\left(t-\frac{1}{2}, u-\frac{1}{2}; k\right) f_m^k\left(u-\frac{1}{2}\right) du = f_m\left(u-\frac{1}{2}\right).$$

□

Theorem 27. *Let $k \geq k_0$. The model*

$$H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} ln\left(K_{(n,p)}\left(\sigma(x) - \frac{1}{2}, \sigma(y) - \frac{1}{2}; k\right)\right)$$

on the Cayley tree Γ^k has at least n translation-invariant Gibbs measures.

INFERENCE

This dissertation is based on results of very recently written papers. The Gibbs measure is a probablity measure. It has been an important object in many problems of problems of probability theory and statistical mechanics. In the theory of Gibbs measures is one of the central problems is to describe limiting Gibbs measures corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the ground-breaking work of Dobrushin. However, a complete analysis of the set of limiting Gibbs measures for a specific Hamiltonian is often a difficult problem. These papers are devoted to models with a $finite$ set of spin values. Mainly were shown that these models have finitely many translation-invariant and uncountable numbers of the non-translation-invariant extreme Gibbs measures.

In this dissertation it is considered models with nearest-neighbor interactions and with the set $[0, 1]$ of spin values, on Cayley tree of order $k \geq 2$. Moreover it is proved that each periodic Gibbs measures for given model is either translation-invariant or two-periodic. In case $k \geq 2$ some models (with the set $[0, 1]$ of spin values) which have a unique splitting Gibbs measures are constructed. For arbitrary $k \geq 2$ it is found a sufficient condition under which the integral equation has unique solution; hence under this condition the corresponding model has unique splitting Gibbs measure. Finally, It is constructed several models with the set $[0, 1]$ of spin values and show that each of the constructed model has at least two (at least n Gibbs measures for $k \geq k_0$) periodic Gibbs measures.

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