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MEXANIKA-MATEMATIKA FAKULTETI

GEOMETRIYA KAFEDRASI

DIFFERENSIALLANUVCHI FUNKSIYALAR GRADIYENT CHIZIQLARI

mavzusidagi

BITIRUV MALAKAVIY ISHI

Bajardi: "Matematika" ta'lim
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RO'ZIYEVA DILNOZA

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Bitiruv malakaviy ishi kafedradan dastlabki himoyadan o'tdi.

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KIRISH

“Xalqning boy intelektual merosi va umumbashariy qadriyatlar asosida zamonaviy madaniyat, iqtisodiyot, fan, texnika texnologiyalarning yutuqlari asosida kadrlar tayyorlashning mukammal tizimini shakllantirish O’zbekiston taraqqiyotining muhim shartidir”.

“Kadrlar tayyorlash milliy dasturi”dan.

Ushbu malakaviy bitiruv ishi referativ xarakterga bo’lib, unda differensiallanuvchi funksiyalar gradiyent chiziqlarining xossalari va ularning tadbiqi o’rganilgan.

Differensial geometriya va matematik analiz kursida differensiallanuvchi funksiyalar gradiyenti va gradiyent chiziqlarining xossalari muhim hisolanadi. Funksiya sath sirtlari va gradiyent chiziqlari orasidagi bog’lanish, shuningdek boshqa xossalri differensiallanuvchi funksiyalar sath sirtlari hosil qiluvchi geometrik ob’ektlarning xossalarini o’rganishda qo’llaniladi. Shu sababli bitiruv malakaviy ishi uchun shu mavzu tanlandi.

Malakaviy bitiruv ishining birinchi bobida bir va ko’p o’zgaruvchili funksiyalar differensiallanuvchiligi va sirtdagi silliq akslantirishlar keltirilgan Ikkinchi bob uchta paragrafdan iborat bo’lib, birinchi paragrafda funksiya gradiyenti tushunchasi kiritilgan va misollar keltirilgan. Ikkinchi paragrafda differensiallanuvchi funksiya gradiyenti xossalar o’rganilgan. Uchinchi paragraf asosiy qism bo’lib, unda differensiallanuvchi funksiyalar gradiyent chiziqlarining xossalari o’rganilgan. Xossalar tadbiqi bir nechta misollarda ko’rsatilgan.

I bob

Differensiallanuvchi funksiyalar

1-§. Bir o'zgaruvchili funksiya differensiali

$f(x)$ funksiya (a,b) intervalda aniqlangan, $x_0 \in (a,b), x_0 + \Delta x \in (a,b)$ bo'lsin. U holda $f(x)$ funksiya ham x_0 nuqtada $\Delta y = f(x_0 + \Delta x) - f(x_0)$ orttirmaga ega bo'ladi.

Ta'rif-1. Agar $f(x)$ funksiyaning $x_0 \in (a,b)$ nuqtadagi orttirmasi Δy ni $\Delta y = A\Delta x + \alpha\Delta x$ (1) ko'rinishda ifodalash mumkin bo'lsa, $f(x)$ funksiya x_0 nuqtada differensiallanuvchi deyiladi, bunda A – Δx ga bog'liq bo'lmagan o'zgarmas, α esa Δx ga bog'liq va $\Delta x \to 0$ da $\alpha = \alpha(\Delta x) \to 0$.

Agar $\Delta x \to 0$ da $\alpha \cdot \Delta x = \alpha(\Delta x)\Delta x = 0(\Delta x)$ ekanini e'tiborga olsak, u holda yuqoridagi (1) ifoda ushbu $\Delta y = A\Delta x + 0(\Delta x)$ ko'rinishni oladi.Funksiya orttirmasi uchun (1) formulada $A\Delta x$ ifoda orttirmaning bosh qismi deb yuritiladi.

Teorema-1. $f(x)$ funksiyaning $x \in (a,b)$ nuqtada differensiallanuvchi bo'lishi uchun uning shu nuqtada chekli hosilaga ega bo'lishi zarur va yetarli.

Isboti. Zarurligi. $f(x)$ funksiya $x \in (a,b)$ nuqtada differensiallanuvchi bo'lsin. Unda $\Delta y = A\Delta x + 0(\Delta x)$, $\dfrac{\Delta y}{\Delta x} = A + \dfrac{0(\Delta x)}{\Delta x}$ bo'lib, $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} (A + \dfrac{0(x)}{\Delta x}) = A$ bo'ladi.

Demak, $f'(x) = A$.

Yetarliligi. $f(x)$ funksiya $x \in (a,b)$ nuqtada chekli $f'(x)$ hosilaga ega bo'lsin:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Agar $\dfrac{\Delta y}{\Delta x} - f'(x) = \alpha$ deb olsak, undan $\Delta y = f'(x)\Delta x + \alpha\Delta x$ ekanini topamiz. Bu tenglikdagi α miqdor Δx ga bog'liq va $\Delta x \to 0$da $\alpha \to 0$. Demak, $f(x)$ funksiya $x \in (a,b)$ nuqtada differensiallanuvchi bo'lib, $A = f'(x)$ bo'ladi.

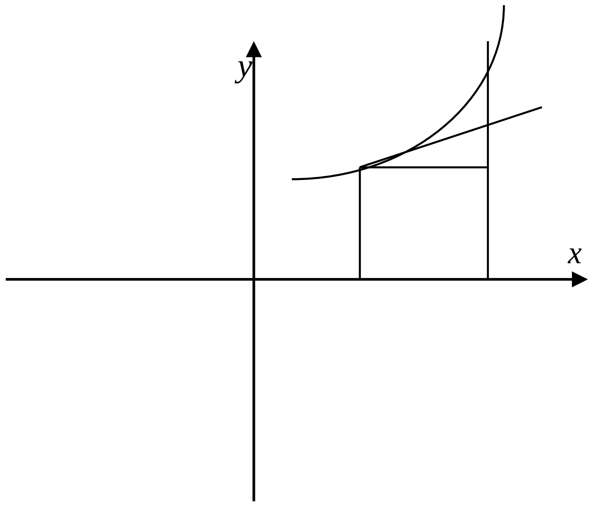
Isbot etilgan teorema $f(x)$ funksiyaning $x \in (a,b)$ nuqtada chekli $f'(x)$ hosilaga ega bo'lishi bilan uning shu nuqtada differensiallanuvchi bo'[lishi ekvivalent ekanini ko'rsatadi.

Funksiya differensiali va uning geometrik ma'nosi

$f(x)$ funksiya $x \in (a,b)$ nuqtada differensiallanuvchi bo'lsin. $\Delta y = A\Delta x + 0(\Delta x)$ bunda, $A = f'(x)$ bo'ladi. Bu tenglikda funksiya orttirmasi Δy ikki qo'shiluvchi, argument orttirmasi Δx ga nisbatan chiziqli $A\Delta x$ hamda Δx ga nisbatan yuqori tartibli ($\Delta x \to 0$) cheksiz kichik miqdorlar yig'indisidan iborat ekani ko'rinadi.

Ta'rif-2. $f(x)$ funksiya orttirmasi Δy ning Δx ga nisbatan chiziqli bosh qismi $A\Delta x = f'(x)\Delta x$ berilgan $f(x)$ funksiyaning x nuqtadagi differensiali deb ataladi.

Funksiyaning differensiali dy yoki $df(x)$ kabi belgilanadi: $dy = df(x) = A \cdot \Delta x = f'(x)\Delta x$. Endi $x \in (a,b)$ nuqtada differensiallanuvchi bo'lgan $f(x)$ funksiyaning grafigi chizmada ko'rsatilgan chiziqni ifodalasin deylik.



Bu chiziqning $(x, f(x))$, $(x+\Delta x, f(x+\Delta x))$ nuqtalari mos ravishda F va B bilan belgilaylik. Unda $FC = \Delta x$ $BC = \Delta y$ bo'ladi. $f(x)$ funksiya $x \in (a,b)$ nuqtada differensiallanuvchi bo'lgani uchun u x nuqtada chekli $f'(x)$ hosilaga ega. Demak, $f(x)$ funksiya grafigiga uning $F(x, f(x))$ nuqtasida o'tkazilgan FL urinma mavjud va bu urinmaning burchak koeffitsienti $tg\alpha = f'(x)$ shu FL urinmaning BC bilan kesishgan nuqtasini D bilan belgilaylik. Ravshanki, ΔFDC dan $\frac{DC}{FC} = tg\alpha$ va undan $DC = tg\alpha \cdot FC = f'(x) \cdot \Delta x$ ekani kelib chiqadi.

Demak, $f(x)$ funksiyaning x nuqtadagi differensiali $dy = f'(x)\Delta x$ funksiya grafigiga $F(x, f(x))$ nuqtada o'tkazilgan urinma orttirmasi DC ni$(DC = dy)$ ifodalaydi.

Ko'p o'zgaruvchili funksiya differensiali

$y = f(x)$ funksiya ochiq $M\left(M \subset R^m\right)$ to'plamda berilgan bo'lib, bu to'plamning x^0 nuqtasida differensillanuvchi bo'lsin. Ta'rifga ko'ra, u holda $f(x)$ funksiyaning x^0 nuqtadagi orttirmasi $\Delta f\left(x^0\right) = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m + 0(\rho)$ (2) bo'lib, bunda $A_i = \dfrac{\partial f\left(x^0\right)}{\partial x_i}$, $(i = 1, 2, ..., m)$ va $\Delta x_1 \to 0, \Delta x_2 \to 0, ..., \Delta x_m \to 0$ va $\rho \to 0$ bo'ladi. (2) tenglikning o'ng tomoni ikki qismdan

1). $\Delta x_1, \Delta x_2, ..., \Delta x_m$ orttirmalarga nisbatan chiziqli ifoda $A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m$ dan

2). $\Delta x_1 \to 0, \Delta x_2 \to 0, ..., \Delta x_m \to 0$ da, ya'ni $\rho \to 0$ da ρ ga nisbatan yuqori tartibli cheksiz kichik miqdor $0(\rho)$ dan iborat.

Shuningdek, (2) munosabatdan $\rho \to 0$ da $A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m$ - cheksiz kichik miqdor, $\Delta f\left(x^0\right)$ - cheksiz kichik miqdorning bosh qismi ekanligini payqaymiz.

Ta'rif-3. $f(x)$ funksiya orttirmasi $\Delta f\left(x^0\right)$ ning $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga nisbatan chiziqli bosh qismi

$$A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m = \frac{\partial f\left(x^0\right)}{\partial x_1}\Delta x_1 + \frac{\partial f\left(x^0\right)}{\partial x_2}\Delta x_2 + ... + \frac{\partial f\left(x^0\right)}{\partial x_m}\Delta x_m .$$

$f(x)$ funksiyaning x^0 nuqtadagi differensiali (to'liq differensiali) deb ataladi va $df\left(x^0\right)$ yoki $df\left(x_1^0, x_2^0, ..., x_m^0\right)$ kabi belgilanadi.

$$df(x^0) = df\left(x_1^0, x_2^0, ..., x_m^0\right) = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m = \frac{\partial f\left(x^0\right)}{\partial x_1}\Delta x_1 + \frac{\partial f\left(x^0\right)}{\partial x_2}\Delta x_2 + ... + \frac{\partial f\left(x^0\right)}{\partial x_m}\Delta x_m$$

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Agar $x_1, x_2, ..., x_m$ erkli o'zgaruvchilarning ixtiyoriy orttirmalari $\Delta x_1, \Delta x_2, ..., \Delta x_m$ lar mos ravishda bu o'zgaruvchilarning differensiallari $dx_1, dx_2, ..., dx_m$ ga teng ekanligini e'tiborga olsak, unda $f(x)$ funksiyaning differensiali quyidagi

$$df\left(x^0\right) = \frac{\partial f\left(x^0\right)}{\partial x_1} dx_1 + \frac{\partial f\left(x^0\right)}{\partial x_2} dx_2 + ... + \frac{\partial f\left(x^0\right)}{\partial x_m} dx_m \quad (2)$$

ko'rinishga keladi.

Odatda $\frac{\partial f}{\partial x_1} dx_1, \frac{\partial f}{\partial x_2} dx_2, ..., \frac{\partial f}{\partial x_m} dx_m$ lar $f(x)$ funksiyaning xususiy differensiallari deb ataladi va ular mos ravishda $dx_1 f, dx_2 f, ..., dx_m f$ kabi belgilanadi:

$$dx_1 f = \frac{\partial f}{\partial x_1} dx_1, dx_2 f = \frac{\partial f}{\partial x_2} dx_2, ..., dx_m f = \frac{\partial f}{\partial x_m} dx_m .$$

Demak, $f(x)$ funksiyaning x^0 nuqtadagi differensiali, uning shu nuqtadagi xususiy differensiallari yig'indisidan iborat.

Shuni ta'kidlash lozimki, $f(x_1, x_2, ..., x_m)$ funksiyaning differensiali $\left(x_1^0, x_2^0, ..., x_m^0\right)$ nuqtaga bog'liq bo'lishi bilan birga bu o'zgaruvchilarning orttirmalari $\Delta x_1 = dx_1, \Delta x_2 = dx_2, ..., \Delta x_m = dx_m$ larga ham bog'liqdir.

Funksiya differensiali soda geometrik ma'noga ega.Quyida uni keltiramiz. $y = f(x) = f(x_1, x_2, ..., x_m)$ funksiya ochiq M to'plamda berilgan bo'lib, $x^0 = \left(x_1^0, x_2^0, ..., x_m^0\right)$ nuqtada $\left(x^0 \in M\right)$ differensiallanuvchi bo'lsin. Demak, bu funksiyaning x^0 nuqtadagi orttirmasi

$$\Delta f\left(x^0\right) = f\left(x_1^0 + \Delta x, x_2^0 + \Delta x, ..., x_m^0 + \Delta x\right) - f\left(x_1^0, x_2^0, ..., x_m^0\right) \text{ uchun}$$

$$\Delta f\left(x^0\right) = f'_{x_1}\left(x^0\right)\left(x_1 - x_1^0\right) + f'_{x_2}\left(x_0\right)\left(x_2 - x_2^0\right) + ... + f'_{x_m}\left(x_0\right)\left(x_m - x_m^0\right) + 0(\rho)$$

bo'ladi.

Faraz qilaylik, $y = f(x)$ funksiyaning grafigi R^{m+1} fazodagi ushbu $(S) = \left\{(x_1, x_2, ..., x_m; y) : (x_1, x_2, ..., x_m) \in R^m, y \in R\right\}$ sirtdan iborat bo'lsin.Geometriyadan ma'lumki, bu sirtning $\left(x_1^0, x_2^0, ..., x_m^0; y_0\right)$ nuqtasidan $\left(y_0 = f\left(x_1^0, x_2^0, ..., x_m^0\right)\right)$ o'tuvchi hamda Oy o'qiga parallel bo'lmagan tekisliklarning umumiy tenglamasi

$$Y - y_0 = f'_{x_1}\left(x^0\right)\left(x_1 - x_1^0\right) + f'_{x_2}\left(x_0\right)\left(x_2 - x_2^0\right) + ... + f'_{x_m}\left(x_0\right)\left(x_m - x_m^0\right) \quad (4)$$

tekislik esa (S) sirtga $\left(x_1^0, x_2^0, ..., x_m^0; y_0\right)$ nuqtasida o'tkazilgan urinma tekislik deb ataladi.Agar

$$x_1 - x_1^0 = dx_1, x_2 - x_2^0 = dx_2, ..., x_m - x_m^0 = dx_m$$

deyilsa, unda (4) urinma tekislik

$$Y - y_0 = f'_{x_1}\left(x^0\right)dx_1 + f'_{x_2}\left(x_0\right)dx_2 + ... + f'_{x_m}\left(x_0\right)dx_m = df\left(x^0\right)$$

ko'rinishga keladi.

Natijada quyidagiga kelamiz: $y = f\left(x_1, x_2, ..., x_m\right)$ funksiya argumentlari $x_1, x_2, ..., x_m$ larning $x_1 = x_1^0, x_2 = x_2^0, ..., x_m = x_m^0$ qiymatlariga mos ravishda orttirma beraylik.

U holda funksiyaning mos orttirmasi

$$\Delta f\left(x^0\right) = f\left(x_1^0 + \Delta x, x_2^0 + \Delta x, ..., x_m^0 + \Delta x\right) - f\left(x_1^0, x_2^0, ..., x_m^0\right) = y - y_0 \; (S) \text{ sirt}$$

$$\left(x_1^0, x_2^0, ..., x_m^0; y_0\right) \text{va} \left(x_1^0 + \Delta x_1, x_2^0 + \Delta x_2, ..., x_m^0 + \Delta x_m; Y\right)$$

nuqtalarning oxirgi, y koordinatasi olgan orttirmani bildiradi.

Funksiyaning shu nuqtadagi differensiali esa $df\left(x^0\right) = Y - y_0$ ga teng.

2-§. Sirtdagi silliq akslantirishlar.

Φ- regulyar sirt va $G : \Phi \to R^m$ akslantirish berilgan. p-sirtning biror nuqtasi bo'lsin.

Ta'rif-4. Φ sirtning p nuqta atrofida ixtiyoriy silliq (f, G) parametrlash usuli uchun $g \cdot f : G \to R^m$ silliq akslantirish bo'lsa, g -akslantirish p nuqtada silliq akslantirish deyiladi. [3].

Agar (f, G) parametrlash usuli $\vec{r} = \vec{r}(u, v)$ tenglama bilan berilgan bo'lsa, $g \cdot f$ akslantirish g akslantirishning egri chiziqli (u, v) koordinatalardagi ifodasi deyiladi.

Izoh.Ta'rifga ko'ra g silliq akslantirish bo'lishi uchun sirtning p nuqta atrofidagi ixtiyoriy (f, G) parametrlash usuli uchun, $g \cdot f : G \to R^m$ akslantirish differensiallanuvchi bo'lishi kerak. Bu yerda G -(u, v) tekislikdagi elementar soha bo'lganligi uchun $g \cdot f$ akslantirish m ta

$$y_1 = g_1(u,v)$$
$$y_2 = g_2(u,v)$$
$$\ldots$$
$$y_m = g_m(u,v)$$

funksiyalar differensiallanuvchi bo'lishi kerak, lekin quyidagi teorema ko'rsatadiki, g silliq akslantirish bo'lishi uchun birorta regulyar (f,G) parametrlash usuli uchun $g\cdot f$ ning differensiallanuvchi bo'lishi yetarlidir.

Teorema-2. Berilgan $g:\Phi\to R^m$ akslantirish p nuqtada silliq akslantirish bo'lishi uchun Φ sirtning p nuqta atrofidagi birorta regulyar (f_1,G) parametrlash usuli uchun $g\cdot f_1:G_1\to R^m$ akslantirishning differensiallanuvchi bo'lishi zarur va yetarlidir.[3]

Isbot.Tabiiyki, bu yerda faqat yetarlilik qismini isbotlash lozimdir.Demak, biz ixtiyoriy silliq parametrlash usuli (f,G) -uchun $g\cdot f:G\to R^m$ akslantirishning differensiallanuvchi ekanligini ko'rsatishimiz kerak. p nuqtaning (f_1,G_1) parametrlash usulidagi koordinatalari (w_0,s_0), (f,G) -parametrlash usulidagi koordinatalari (u_0,v_0) va $W=f(G)\cap f_1(G_1)$ bo'lsin. Shunda $U=f^{-1}(W)$ to'plam (u_0,v_0) nuqtaning atrofi bo'ladi va bu atrofda $g\cdot f=(g\cdot f_1)\cdot\left(f_1^{-1}\cdot f\right)$ tenglik o'rinli. Teorema shartiga ko'ra, $g\cdot f_1$ differensiallanuvchi akslantirishdir.Shuning uchun, biz $f_1^{-1}\cdot f:U\to G_1$ akslantirishning differensiallanuvchi ekanligini ko'rsatishimiz kerak. Regulyar parametrlash usuli (f_1,G_1) differensiallanuvchi

$$\begin{cases} x = x(w,s) \\ y = y(w,s) \\ z = z(w,s) \end{cases}$$

Funksiyalar yordamida beriladi va $rank\begin{pmatrix} x_w & y_w & z_w \\ x_s & y_s & z_s \end{pmatrix}=2$ tenglik o'rinlidir.

Faraz qilaylik, $\begin{vmatrix} x_w & y_w \\ x_s & y_s \end{vmatrix}\neq 0$ bo'lsin. Teskari funksiya haqidagi teoremani

$$\begin{cases} x = x(w,s) \quad x_0 = x(w_0,s_0) \\ y = y(w,s) \quad y_0 = y(w_0,s_0) \end{cases}$$

sistemaga qo'llaymiz.Shunda silliq $w=w(x,y), s=s(x,y)$ funksiyalar mavjud bo'lib,

$$x=x\big(w(x,y),s(x,y)\big) \quad w_0=w(x_0,y_0)$$
$$y=y\big(w(x,y),s(x,y)\big) \quad s_0=s(x_0,y_0)$$

tengliklar o'rinli bo'ladi. Uchinchi koordinatamiz $z=z\big(w(x,y),s(x,y)\big)=\gamma(x,y)$ x,y larning funksiyasi bo'ladi.

Demak, p nuqta atrofida (x,y)lar ichki koordinatalar bo'lib, sirt $z=\varphi(x,y)$ funksiya grafigidan iborat bo'ladi. Shunda $\pi:(x,y,z)\to(x,y)$ proyeksiya va $w=w(x,y), s=s(x,y)$ funksiyalar yordamida berilgan $\tilde{f}:(x,y)\to(w,s)$ akslantirish differensiallanuvchi bo'lganligi uchun $f_1^{-1}(x,y,z)=\tilde{f}\pi(x,y,z)$ akslantirish differensiallanuvchidir. Demak, $f_1^{-1}\cdot f$ ham differensiallanuvchidir.∆

Misol.

(U,r)parametrlash usuli S sirtda berilgan bo'lsin. Shunda $r^{-1}:r(U)\to R^2$ silliq bo'ladi.

Agar S_1,S_2-R^3 da 2ta sirt berilgan bo'lsin.$F:S_1\to R^3$akslantirish orqali $F:S_1\to S_2$akslantirish qandayligini tushunish mumkin, chunki $S_2\subset R^3$. Aniqki, $i\cdot F$ ko'rinishdagi akslantirish mavjud bo'ladi, qachonki $i:S_2\to R^3$akslantirish ochiq bo'lsa.

Ta'rif-5. $F:S_1\to S_2$akslantirish silliq deyiladi, agar $F:S_1\to R^3$ silliq akslantirish unga tegishli bo'lsa. R^3daS_1 sirt va $G:R^3\to R^3$ -diffeomorfizm berilgan bo'lsin. Shunda $S_2=G(S_1)$ -sirtda ham $G|_{S_1}:S_1\to S_2$ akslantirish silliq.Ko'rinib turibdiki, har bir silliq akslantirish uzluksiz ham bo'ladi.[3]

Ta'rif-6. $F:S_1\to S_2$akslantirish diffeomorfizm deyiladi, agar F -biyektiv va F va F^{-1} akslantirishlar silliq bo'lsa.[3]

Bizga Φ_1,Φ_2 sirtlar va$g:\Phi_1\to R^3$ akslantirish berilib, $g(\Phi_1)=\Phi_2$ bo'lsa, $g:\Phi_1\to\Phi_2$ akslantirish berilgan deyiladi.

Tabiiyki, $g:\Phi_1 \to R^3$ differensiallanuvchi bo'lsa, $g:\Phi_1 \to \Phi_2$ differensiallanuvchi deyiladi.Agar g differensiallanuvchi bo'lsa, Φ_1 sirtdagi silliq egri chiziqning obrazi Φ_2 sirtda silliq egri chiziq bo'ladi.

Ta'rif-7. Φ_1 sirtdagi ixtiyoriy γ egri chiziqning p nuqtadagi urinma vektorini $g(\gamma)$ egri chiziqning $g(p)$ nuqtadagi urinma vektoriga o'tkazuvchi $T_p\Phi_1 \to T_{g(p)}\Phi_2$ akslantirish g akslantirishning p nuqtadagi differensial deb ataladi va $dg(p)$ ko'rinishda belgilanadi. [3]

Bizga $g:R^3 \to R^3$ differensiallanuvchi akslantirish berilgan va $\Phi_2 = g(\Phi_1)$ bo'lsa, $dg(p)$ - g akslantirishning p nuqtadagi Yakobi matritsasi bilan ustma-ust tushishini ko'rsataylik.

Φ_1 sirt p nuqta atrofida $\vec{r} = \vec{r}(u,v)$ (3) tenglama bilan berilgan va Φ_2 sirt $g(p)$ nuqta atrofida $\vec{\rho} = \vec{\rho}(u,v)$ (4) tenglama bilan berilgan bo'lsa, $\vec{\rho}(u,v)$ vektor $g\left(x(u,v), y(u,v), z(u,v)\right)$ -nuqtaning radius vektoridir. Endi $\rho(u_0,v_0)$ -nuqtadan o'tuvchi γ egri chiziq ichki koordinatalarda $\begin{cases} u = u(t) \\ v = v(t) \end{cases}$ (5) tenglamalar bilan berilgan va $u_0 = u(t_0), v_0 = v(t_0)$ bo'lsa, γ chiziqning p nuqtadagi urinma vektori $\vec{a} = \vec{r}_u u'(t_0) + \vec{r}_v v'(t_0)$ vektordir.

Φ_2 sirtda $g(\gamma)$ egri chiziqning $g(p)$ nuqtada urinma vektori $\vec{b} = \vec{\rho}_u u'(t_0) + \vec{\rho}_v v'(t_0)$ (6) vektordir.

Agar

$$g(x,y,z) = \left\{g^1(x,y,z), g^2(x,y,z), g^3(x,y,z)\right\} \text{ va } x_0 = x(u_0,v_0), y_0 = y(u_0,v_0), z_0 = z(u_0,v_0)$$

bo'lsa, $\vec{b} = I(g)(x_0,y_0,z_0)\cdot\vec{a}$ tenglik o'rinlidir. Bu yerda

$$I(g)(x_0,y_0,z_0) = \begin{pmatrix} g_x^1(x_0,y_0,z_0) & g_x^2(x_0,y_0,z_0) & g_x^3(x_0,y_0,z_0) \\ g_y^1(x_0,y_0,z_0) & g_y^2(x_0,y_0,z_0) & g_y^3(x_0,y_0,z_0) \\ g_z^1(x_0,y_0,z_0) & g_z^2(x_0,y_0,z_0) & g_z^3(x_0,y_0,z_0) \end{pmatrix}$$

g akslantirishning p nuqtadagi Yakobi matritsasidir.

Teorema-3. $dg(p)$ chiziqli akslantirishdir.

Isboti. (6) formuladan ko'rinib turibdiki, agar $\vec{a}$ vektor $T_p\Phi_1$ fazoda a_1, a_2 koordinatalarga ega bo'lsa, $\vec{b}$ vektor ham $T_{g(p)}\Phi_2$ fazoda xuddi shu koordinatalarga ega bo'ladi. Koordinatalarning chiziqliligidan $dg(p)$ akslantirishning chiziqli ekanligi kelib chiqadi.Δ

Silliq akslantirishga misollar.

1). Doimiy $f: S \to R^k : a \to A_0, A_0 \in R^k$ akslantirish silliq bo'ladi, egri chiziqli koordinata koeffitsienti $f_r(u,v) = A_0$ shaklida ifodalansa.

2). Ochiq akslantirish $i: S \to R^3 : a \to a$ silliq, agarda $f_r = i \circ r = r$ bo'lsa.

Sirtda differensial silliq akslantirishlar.

Akslantirish hosilasi uchun geometrik interperetatsiya quramiz.

$G: B \to R^3$ silliq akslantirish berilgan, $B - R^3$ da ochiq qism to'plam va

$$G(x,y,z) = \left(g_1(x,y,z), g_2(x,y,z), g_3(x,y,z)\right).$$

Shunda har bir $a = (x_0, y_0, z_0) \in B$ nuqta uchun $G'(a): \vec{R}_a^3 \to \vec{R}_{G(a)}^3$ akslantirish hosilasi chiziqli akslantirish bo'ladi, Yakobi matritsasi mavjud bo'ladi:

$$\left.\frac{D(g_1, g_2, g_3)}{D(x,y,z)}\right|_{(x_0,y_0,z_0)} = (\alpha_{ij}), 1 \le i, j \le 3.$$

Belgilash kiritamiz:

$$\partial_1 g = \frac{\partial g}{\partial x}, \partial_2 g = \frac{\partial g}{\partial y}, \partial_3 g = \frac{\partial g}{\partial z}$$

shunda $\alpha_{ij} = \partial_j g_i(x_0, y_0, z_0)$ bo'ladi. Har bir $h \in \vec{R}_\alpha^3$ vektor uchun $\{h_1, h_2, h_3\}$ koordinatalar bo'ladi, $G'(a)(h)$ vektorning koordinatalari $\left\{\sum_{j=1}^{3} \alpha_{1j} h_j, \sum_{j=1}^{3} \alpha_{2j} h_j, \sum_{j=1}^{3} \alpha_{3j} h_j\right\}$ t_0 nuqtada $\rho(t) = (x(t), y(t), z(t))$ yo'llar urinma vektori -h vektor berilgan, $\rho'(t_0) = h$. t_0 nuqtada$(G \circ \rho)(t)$ yo'lining urinma vektori $G'(a)(h)$ vektor bo'lishini ko'rsatamiz.

Buning uchun quyidagi ayniyatni differensiallaymiz:

$$(G \circ \rho)(t) = \left(g_1(x(t), y(t), z(t)), g_2(x(t), y(t), z(t)), g_3(x(t), y(t), z(t))\right)$$

$$\{h_1, h_2, h_3\} = \{x'(t_0), y'(t_0), z'(t_0)\}$$

bo'lganda,

$$\overrightarrow{(G \circ \rho)}'(t_0) = \left\{ \sum_{k=1}^{3} \partial_k g_1(x_0, y_0, z_0) h_k, \sum_{k=1}^{3} \partial_k g_2(x_0, y_0, z_0) h_k, \sum_{k=1}^{3} \partial_k g_3(x_0, y_0, z_0) h_k \right\} =$$

$$= \left\{ \sum_{k=1}^{3} \alpha_{1k} h_k, \sum_{k=1}^{3} \alpha_{2k} h_k, \sum_{k=1}^{3} \alpha_{3k} h_k \right\} \text{ bo'ladi.}$$

Keyingi kiritmaga kelamiz: $G'(a)$ akslantirish hosilasi $\rho(t)$ yo'l urinma vektorini $(G \circ \rho)t$ yo'l urinma vektoriga ko'chiradi.

Izoh. Qurilgan interpretatsiya ixtiyoriy o'lchamdagi fazolarni silliq akslantirish uchun ham o'rinli bo'ladi.

$F: S_1 \to S_2$ sirtda silliq akslantirish va $a \in S_1$ berilgan bo'lsin. Shunda S_1 sirtlar har qanday (I, ρ) silliq yo'llarga muvofiq S_2 sirtlarda $(I, F \circ \rho)$ silliq yo'l, agar $\rho(t)$ yo'l a nuqta orqali $t = t_0$ yoniga o'tadi.

Ta'rif-8. $T_a S_1 \to T_{F(a)} S_2$ akslantirish S_1 sirtda yotuvchi $\rho(t)$ yo'lning $\rho'(t_0)$ urinma vektoriga ($\rho(t_0) = a$) $F \circ \rho$ yo'lning t_0 nuqtadagi urinma vektori $\overrightarrow{(F \circ \rho)}'(t_0)$ ni mos qo'ysa, $F: S_1 \to S_2$ akslantirishning a nuqtadagi differensiali deyiladi va $dF(a)$ bilan belgilanadi.

Dastlab akslantirish to'g'riligini tekshiramiz. Buning uchun quyidagini ko'rsatishimiz lozim, agar $\rho(t)$ va $\rho_1(t)$ 2ta yo'llarki, S_1 sirtda yotadilar, $\overrightarrow{\rho_1(t_0)} = \overrightarrow{\rho_1(s_0)}$ umumiy urinma vektor bo'ladi, qachonki, $\rho(t_0) = \rho_1(s_0) = a_1$, $F \circ \rho$ va $F \circ \rho_1$ yo'llar t_0 va s_0 nuqta tegishli bo'lgan umumiy urinma vektor bo'ladi.

Haqiqatdan ham $(u, r = r(u, v))$ parametrlash usuli S_1 sirtda a nuqtaning atrofi, $u = u(t)$, $v = v(t)$ va $u = u_1(s)$, $v = v_1(s)$ ichki berilish ρ va ρ_1 yo'lga tegishli. U holda quyidagi lemmaga ko'ra,

Lemma. $\rho = \rho(t)$ yo'l berilgan, S_1 sirtda yotadi, (U, r) parametrlash usulining ichki berilish tenglamalari $u = u(t)$, $v = v(t)$ bo'lsin. U holda

$$\vec{\rho}(t) = u'(t) \partial_u r + v'(t) \partial_v r .$$

Tabiiy reper $\partial_u r(u_0, v_0)$, $\partial_v r(u_0, v_0)$ koordinatalari $u'(t_o), v'(t_0)$ va $u_1'(s_0), v_1'(s_0)$. $\rho'(t_0)$ va $\overrightarrow{\rho_1'}(s_0)$ urinma vektorlarga tegishli bo'ladi.[3]

$F \circ \rho$ va $F \circ \rho_1$ yo'llar urinma vektori topiladi, u quyidagi munosabat differensialiga tegishli

$$(F \circ \rho)(t) = F_r\left(u(t), v(t)\right) \text{va } (F \circ \rho_1)(s) = F_r\left(u(s), v(s)\right)$$

qachonki $F_r = F \circ r$ F uchun koordinat ifoda.

Murakkab funksiyani differensiallash qoidasiga ko'ra:

$$\left(\overrightarrow{F \circ \rho}\right)'(t_0) = \overrightarrow{\partial_u F_r}(u_0, v_0)u'(t_0) + \overrightarrow{\partial_v F_r}(u_0, v_0)v'(t_0)$$

$$\left(\overrightarrow{F \circ \rho_1}\right)'(s_0) = \overrightarrow{\partial_u F_r}(u_0, v_0)u'(s_0) + \overrightarrow{\partial_v F_r}(u_0, v_0)v'(s_0).$$

Bu formuladan kuchli tenglik kelib chiqadi:

$$u'(t_0) = u_1'(s_0),\ v'(t_0) = v_1'(s_0) \quad \left(\overrightarrow{F \circ \rho}\right)'(t_0) = \left(\overrightarrow{F \circ \rho_1}\right)'(s_0).$$

Teorema-4. $dF : T_a S_1 \to T_{F(a)} S_2$ akslantirish chiziqli akslantirish bo'ladi.

Aytaylik, M $-n$ o'lchamli silliq ko'pxillik, (U, φ) -M dagi lokal karta, $f : M \to R$ dagi funksiya bo'lsin.

Ta'rif-9. f funksiyaning (U, φ) lokal kartaga nisbatan koordinatalik qiymati deb, $f \circ \varphi^{-1} : \varphi(U) \to R$ funksiyaga aytiladi. [3]

Ta'rif-10. $f : M \to R$ funksiya silliq deyiladi, agar M dagi har qanday (U, φ) lokal karta uchun $f \circ \varphi^{-1}$ koordinatalik qiymat silliiq funksiya bo'lsa, ya'ni barcha tartibli uzluksiz xususiy hosilalarga ega.

Xuddi shu kabi M dan olingan ochiq to'plamostida aniqlangan funksiyaning silliqligi aniqlanadi.

Teorema-5. $f : M \to R$ funksiya silliq bo'ladi, faqat va faqat shundaki, qachonki $x \in U$ va $f \circ \varphi^{-1}$ munosabat $\varphi(x)$ nuqtada silliq bo'ladigan har qanday $x \in M$ nuqta uchun M da (U, φ) lokal karta mavjud bo'lsa.

Isboti. Teoremaning zarurligi aniq.

Yetarliligi. (V,ψ) M da ixtiyoriy karta bo'lsin. $f \circ \psi^{-1} : \psi(V) \to R$ funksiyaning silliqligini isbot qilishimiz kerak. Buning uchun uning ixtiyoriy $\psi(x) \in \psi(V)$, $x \in V$ nuqtada silliqligini isbot qilish yetarli.

M da shunday (U,φ) karta topamizki, u teoremada keltirilgan. U holda $\psi(x)$ nuqta atrofida $f \circ \psi^{-1}$ funksiyani $f \circ \psi^{-1} = \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \psi^{-1}\right)$ ko'rinishda ifodalash mumkin. Shartga ko'ra, $f \circ \varphi^{-1}$ funksiya $\varphi(x)$ nuqtada silliq . $\varphi \circ \psi^{-1}$ akslantirish $\psi(x)$ nuqtada kartalarning silliq bog'langanliligiga asosan silliq bo'ladi. Demak, $f \circ \psi^{-1}$ funksiya silliq akslantirishlar kompozitsiyasi sifatida silliq bo'ladi.

Har qanday silliq f funksiya M da uzluksiz, chunki uni uzluksiz akslantirishlar kompozitsiyasi $f|_U = \left(f \circ \varphi^{-1}\right) \circ \varphi$ sifatida lokal ifodalash mumkin.

Natija. Funksiyaning sillioqlik sharti lokaldir, ya'ni f funksiyaning silliqligi uchun har qanday x nuqta uchun uning shunday U ochiq atrofi mavjud bo'lsaki, $f|_U$ silliq bo'ladi.

Misol. Aytaylik, M da (U,φ) lokal karta berilgan bo'lsin. $x^i : U \to R, 1 \le i \le n$ funksiyani qaraymiz, u $a \in U$ nuqtada $\varphi(a) = u^1\left(\varphi(a)\right),...,u^n\left(\varphi(a)\right)$ nuqtaning $i-$ koordinatasini mos qo'yadi, ya'ni $x^{i(a)} = u^i\left(\varphi(a)\right)$. Ko'rilgan $x^i : U \to R$ funksiyalar koordinatlik deyiladi. Biz koordinatlik funksiyalarni silliq deb ta'kidlaymiz. Buning uchun teoremaga asosan,

$$\left(x^i \circ \varphi^{-1}\right)(b) = x^i\left(\varphi^{-1}(b)\right) = u^i\left(\varphi\left(\varphi^{-1}(b)\right)\right) = u^i(b) \quad \forall b \in \varphi(U)$$

funksiyalar $\varphi(U) : u^i\left(b^1,...,b^n\right) = b^i$ dagi oddiy koordinatlik funksiyalar bilan ustma-ust tushishini ko'rsatish yetarli.

II bob

Funksiya gradiyenti va uning xossalari

1-§. Gradiyent ta'rifi va misollar

$$\overrightarrow{\nabla F}\left\{\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\} \text{ yoki } \overrightarrow{\nabla F}=\frac{\partial F}{\partial x}i+\frac{\partial F}{\partial y}j+\frac{\partial F}{\partial z}k$$

bu vektor $F(x,y,z)$ skalyar funksiyaning gradiyent vektori deyilib, "nable ef" deb o'qiladi. Shartga ko'ra $\overrightarrow{\nabla F}\neq 0$ bo'lib, sirtning har bir oddiy nuqtasida tayin gradiyent vektor bordir.

Ta'rif-11. $u=F(x,y,z)$ maydon funksiyasi yordamida aniqlangan skalyar maydonning $P(x,y,z)$ nuqtasida aniqlangan ushbu

$$F'_x(x,y,z)\cdot\vec{i}+F'_y(x,y,z)\cdot\vec{j}+F'_z(x,y,z)\cdot\vec{k}$$

vektorga skalyar maydon gradiyenti deyiladi va $grad\,F(x,y,z)$ kabi yoki qisqacha

$$grad\,u:=\frac{\partial u}{\partial x}\cdot\vec{i}+\frac{\partial u}{\partial y}\cdot\vec{j}+\frac{\partial u}{\partial z}\cdot\vec{k}$$

kabi belgilanadi.[6]

Gradiyent tushunchasini skalyar maydonga bevosita aloqador ekanini ko'rsataylik.Buning uchun bir nuqtadan ikkinchi nuqtaga o'tganida $U(x,y,z)$ ning qanday o'zgarishini tekshirib ko'ramiz.

Skalyar maydondagi biror M nuqta orqali ma'lum bir l yo'nalishli (l) to'g'ri chiziq o'tkazaylik.

U funksiyaning shu nuqtadagi qiymati $U(M)$ bo'lib, (L) to'g'ri chiziqning M ga yaqin M' nuqtadagi qiymati $U(M')$ bo'lsin. Ushbu $\frac{U(M')-U(M)}{\overline{MM'}}$ nisbatning $M'\to M$ dagi limiti $U(M)$ funksiyaning l yo'nalish bo'ylab olingan hosilasi deyiladi va $\frac{\partial U(M)}{\partial l}$ simvol bilan belgilanadi ya'ni

$$\frac{\partial U(M)}{\partial l}=\lim_{M'\to M}\frac{U(M')-U(M)}{\overline{MM'}}.$$

Bu hosila $U(M)$ funksiyaning M nuqtada l yo'nalish bo'yicha o'zgarish suratini aniqlaydi. M nuqtadan xohlagancha to'g'ri chiziq o'tkazish mumkin. Shuning uchun U funksiya berilgan nuqtada cheksiz ko'p yo'nalishlar bo'yicha hosilalarga egadir. l vektorni bir-biriga tik uchta (i, j, k) yo'nalish bo'yicha yoyish mumkinligi sababli $\frac{\partial U}{\partial l}$ hosilani shu uch yo'nalish bo'yicha olingan hosilalar orqali ifodalash mumkin:

$$\frac{\partial U(M)}{\partial l} = \frac{\partial U}{\partial x}\cos(l, x) + \frac{\partial U}{\partial y}\cos(l, y) + \frac{\partial U}{\partial z}\cos(l, z).$$

Endi (l) to'g'ri chiziq o'rniga M nuqtadan o'tuvchi biror(L) chiziqni olib,

$$\lim_{M' \to M} \frac{U(M') - U(M)}{\overset{\smile}{MM'}}$$

ni qaraylik. Bu limit $U(x, y, z)$ funksiyadan (L) chiziqning S yoyi bo'yicha olingan hosilasi bo'lib, uni biz $\frac{\partial U}{\partial s}$ bilan belgilaymiz: $\frac{\partial U}{\partial s} = \lim\limits_{M' \to M} \frac{U(M') - U(M)}{\overset{\smile}{MM'}}$.

Bu hosila U ning (L) chiziq bo'yicha hosilasi deyiladi.

Murakkab funksiyani differensiallash qoidasiga muvofiq:

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial y}\frac{dy}{ds} + \frac{\partial U}{\partial z}\frac{dz}{ds} \qquad (7)$$

(L) chiziq uchun $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ hosilalar uning M nuqtadagi urinmasining birlik vektori τ ning koordinatalaridir (yo'naltiruvchi kosinuslari) $\tau\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$. Agar M nuqta ma'lum bo'lsa, $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}$ hosilalar tezda topiladi, demak, U funksiyaning chiziq bo'ylab olingan hosilasi, shu chiziqning M nuqtadagi urinmasi yo'nalishi bo'ylab olingan hosilasiga teng ekan.

$\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}$ hosilalar M nuqtadan o'tuvchi ekvipotensial $U = C$ sirtning shu nuqtadagi gradiyent vektorining koordinatalari ekani va bu gradiyentning shu sirt normali bo'ylab yo'nalganligi bizga ma'lum. Demak,

$$\left\{\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right\} = \frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k \quad (8)$$

ni eslasak, (7)ni skalyar ko'paytma shaklida yoza olamiz:

$$\frac{dU}{ds} = grad\,U\tau$$

(9) bu formula ikki vektorning skalyar ko'paytmasini ifodalaydi, ulardan biri (τ) birlik vektordir, demak, $\frac{dU}{ds}$ ni $gradU$ ning τ yo'nalishdagi proyeksiyasi deb qarash mumkin:

$$\frac{dU}{ds} = grad\,\tau U \quad (10).$$

Skalyar maydonning gradiyent vektor maydonni tashkil etadi.[4]

(10) formuladan muhim natijalar keltirib chiqarish mumkin:

1. Berilgan maydonning berilgan M nuqtasida U dan biror (L) yo'nalish bo'yicha olingan hosila faqat shu yo'nalishga bog'liq bo'lib, (L) chiziqning tanlanishiga bog'liq emas, chunki berilgan yo'nalishga M nuqtada urinuvchi chiziqlar ko'p;[4]

2. Har qanday vektorni o'z yo'nalishidagi to'g'ri chiziqqa proyeksiyalaganda, proyeksiya absolyut qiymat jihatidan eng kata qiymatga erishadi$\left(pr_l a = a\cos\left(\widehat{a l}\right)\right)$. shuning uchun $\frac{dU}{ds}$ hosila gradiyent yo'nalishida eng katta qiymatga erishadi, ya'ni U maydonning eng zo'r sur'atda o'zgarishi $grad\,U$ yo'nalishida bo'lib, bu eng katta "tezlik"$(grad\,U)$ ga tengdir; maydon qancha tez o'zgarsa, bu modul shuncha katta. Misol, uydagi pechkadan tarqalgan temperatura maydonini olsak, uning gradiyent pechka tomon yo'nalgan bo'ladi.

3. (8) dan $|grad\,U| = \sqrt{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2}$ demak, faqat

$$\frac{\partial U}{\partial x} = 0,\ \frac{\partial U}{\partial y} = 0,\ \frac{\partial U}{\partial z} = 0 \quad (11)$$

bo'lgandagina gradiyent nolga teng bo'ladi. Ammo, o'zgaruvchan bo'lsa-da, maydonning ayrim nuqtalarida (11) shart yuz berishi mumkin.Biz bunday nuqtalarni maydonning maxsus nuqtalari deb ataymiz. Ularda $grad\,U = 0$.[4]

4. Keltirilgan xossalardan gradiyent sof fizik miqdor bo'lib, u koordinatalar sistemasini tanlab olishga bog'liq emasligi kelib chiqadi.[4]

Misollar.

1). Quyidagi maydon gradientlarini toping.

$$grad\,u = \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k$$

a). $u = x^2 + y^2 - z^2$ javob: $grad\,u = i + j - k$

b). $u = x^2 + 2y^2 + 3z^2 - xz + yz - xy$

yechish: $\frac{\partial u}{\partial x} = 2x - z - y,\ \frac{\partial u}{\partial y} = 4y + z - x,\ \frac{\partial u}{\partial z} = 6z - x + y$

javob: $grad\,u = (2x - z - y)i + (4y + z - x)\,j + (6z - x + y)k$

2). $u = x^2 + 2y^2 + 3z^2 + xy + 3x - 2y - 6z$ skalyar maydonning $O(0,0,0)$ va $A(2,0,1)$ nuqtalardagi gradiyenti topilsin.

Yechish. $grad\,u = (2x + y + 3)i + (4y + x - 2)\,j + (6z - 6)k$.

Berilgan nuqtalardagi gradiyentlarini esa quyidagicha topamiz:

$$grad\,u(0,0,0) = 3i - 2j - 6k$$
$$grad\,u(2,0,1) = 7i$$

2-§. Differensiallanuvchi funksiyalar gradiyentining xossalari

Biz bu paragrafda evklid fazosida aniqlangan differensiallanuvchi funksiyalar gradiyentining ba'zi xossalarini o'rganamiz.

1. Ikkita skalyar funksiya yig'indisining gradiyenti $grad\,(u + v) = grad\,u + grad\,v$.

Bu xossa gradiyent ta'rifidan kelib chiqadi, chunki

$$grad\,U = \frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k\ .$$

2. $grad\,(u \cdot v) = u\,grad\,v + v\,grad\,u$.

Haqiqatdan,

$$grad_x(u\cdot v)=\frac{\partial(u,v)}{\partial x}=v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}$$

boshqa koordinatalar uchun ham shunga o'xshash tengliklarni yozsak, aytilgan xossa isbot bo'ladi.

3. $grad\, c=0,\ c=const$. Ta'rifdan foydalanamiz:

$$grad\, c=\frac{\partial c}{\partial x}i+\frac{\partial c}{\partial y}j+\frac{\partial c}{\partial z}k=0\cdot i+0\cdot j+0\cdot k \quad\Rightarrow\quad grad\, c=0$$

4. $grad(cu)=cgrad\, u,\ c=const$ ta'rifdan:

$$gradcu=\frac{\partial cu}{\partial x}i+\frac{\partial cu}{\partial y}j+\frac{\partial cu}{\partial z}k=c\frac{\partial u}{\partial x}i+c\frac{\partial u}{\partial y}j+c\frac{\partial u}{\partial z}k=c\left(\frac{\partial u}{\partial x}i+\frac{\partial u}{\partial y}j+\frac{\partial u}{\partial z}k\right)=cgradu\,.$$

5. $grad\left(\frac{u}{v}\right)=\frac{vgradu-ugradv}{v^2}$

$$grad\left(\frac{u}{v}\right)=\frac{\partial\left(\frac{u}{v}\right)}{\partial x}i+\frac{\partial\left(\frac{u}{v}\right)}{\partial y}j+\frac{\partial\left(\frac{u}{v}\right)}{\partial z}k=\frac{v\partial_x u-u\partial_x v}{v^2}i+\frac{v\partial_y u-u\partial_y v}{v^2}j+\frac{v\partial_z u-u\partial_z v}{v^2}k=$$

$$=\frac{1}{v^2}\Big((v\partial_x u-u\partial_x v)i+(v\partial_y u-u\partial_y v)j+(v\partial_z u-u\partial_z v)k\Big)=\frac{1}{v^2}\Big(v\left(\partial_x ui+\partial_y uj+\partial_z uk\right)-u\left(\partial_x vi+\partial_y vj+\partial_z vk\right)\Big)=$$

$$=\frac{1}{v^2}(vgradu-ugradv)=\frac{vgradu-ugradv}{v^2}\ .$$

6. $grad\ f(u)=f'(u)gradu$ quyidagicha isbotlaymiz:

$$grad\ f(u)=f'_x\frac{\partial u}{\partial x}i+f'_y\frac{\partial u}{\partial y}j+f'_z\frac{\partial u}{\partial z}k=f'(u)gradu\ .$$

7. $grad\, u^n=nu^{n-1}gradu$

$$grad\, u^n=\frac{\partial u}{\partial x}nu^{n-1}i+\frac{\partial u}{\partial y}nu^{n-1}j+\frac{\partial u}{\partial z}nu^{n-1}k=nu^{n-1}\left(\frac{\partial u}{\partial x}i+\frac{\partial u}{\partial y}j+\frac{\partial u}{\partial z}k\right)=nu^{n-1}gradu\ .$$

8. Skalyar maydonda berilgan $f(u,v,w)$ funksiya (egri chiziqli ortogonal koordinatada) gradiyentini topish formulasini keltirib chiqaring.[5]

Yechish. e_u, e_v, e_w - ko'chma chiziqli bog'lanmagan vektorlar berilgan. e_u, e_v - vektorlar tekislikda yotadi, $w=const$ sirt koordinatalariga urinma, e_w vektor bu tekislikka orthogonal va, demak, $w=const$ sirt koordinalariga ortogonal, w vektor skalyar gradientga kolleniar.

$$e_w = k_3 grad\, w \quad (14)$$

Bu yerda, k_3 -biror ko'paytuvchi.

$$u = u(s), v = v(s), w \neq w(s)$$

chiziqlarni qaraymiz.

$$r = r\left[u(s), v(s), w(s)\right]$$

differensial radius vektorni hosilasini olamiz:

$$\frac{dr}{ds} = r_u \frac{du}{ds} + r_v \frac{du}{ds} + r_w \frac{du}{ds} \; .$$

$grad\, w$ skalyar orqali ifodalaymiz.

$$\frac{dw}{ds} = \frac{dw}{ds} r_w \cdot grad\, w \; .$$

Agar $r_w \cdot grad\, w = 1$ bo'lsa, $r_w = |r_w| e_w$ va (14)dan

$|r_w| = k_3 \quad k_3 |grad\, w| = 1 \quad u = const \quad v = const$

$$grad\; f(u,v,w) = \frac{e_u}{k_1}\frac{\partial f}{\partial u} + \frac{e_v}{k_2}\frac{\partial f}{\partial v} + \frac{e_w}{k_3}\frac{\partial f}{\partial w} \quad k_1 = |r_u| = \frac{1}{|grad\, u|} \quad k_2 = |r_v| = \frac{1}{|grad\, v|} \quad k_3 = |r_w| = \frac{1}{|grad\, w|}$$

9. Silindrik koordinatalar sistemasida berilgan maydon uchun gradiyent topish formulasini keltirib chiqaring.[5]

Yechish. Silindrik koordinatalar sistemasi koordinatalari:

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases} .$$

8-misoldan foydalanamiz: $u = f(r, \varphi, z)$

$$k_1 = \frac{1}{|grad\, r|} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2}} = \frac{1}{\sqrt{\cos^2\varphi + \sin^2\varphi}} = 1$$

$$k_2 = \frac{1}{|grad\, \varphi|} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2}} = \frac{1}{\sqrt{\rho^2 \sin^2\varphi + \rho^2 \cos^2\varphi}} = \frac{1}{\rho}$$

$$k_3 = \frac{1}{|grad\, z|} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2}} = 1$$

javob:

$$grad\, u = \frac{\partial u}{\partial r} e_r + \frac{1}{r}\frac{\partial u}{\partial \varphi} e_\varphi + \frac{\partial u}{\partial z} e_z \; .$$

10. Sferik koordinatalar sistemasida berilgan maydon uchun gradiyent topish formulasini keltirib chiqaring.

Yechish. Sferik koordinatalar sistemasi koordinatalari:

$$\begin{cases} x = \rho \cos\varphi \cos\theta \\ y = \rho \sin\varphi \cos\theta \\ z = \rho \sin\theta \end{cases} .$$

8-misoldan foydalanamiz: $u = f(r, \theta, \varphi)$

$$k_1 = \frac{1}{|grad\, r|} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2}} = \frac{1}{\sqrt{\cos^2\varphi \cos^2\theta + \sin^2\varphi \cos^2\theta + \sin^2\theta}} = 1$$

$$k_2 = \frac{1}{|grad\, \theta|} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}} = \frac{1}{\sqrt{\rho^2\cos^2\varphi \sin^2\theta + \rho^2\sin^2\varphi \sin^2\theta + \rho^2\cos^2\theta}} = \frac{1}{\rho}$$

$$k_3 = \frac{1}{|grad\, \varphi|} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2}} = \frac{1}{\sqrt{\rho^2\sin^2\varphi \cos^2\theta + \rho^2\cos^2\varphi \cos^2\theta}} = \frac{1}{\rho\cos\theta}$$

javob: $grad\, u = \frac{\partial u}{\partial \rho} e_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \theta} e_\theta + \frac{1}{\rho\cos\theta}\frac{\partial u}{\partial \varphi} e_\varphi .$

3-§. Differensiallanuvchi funksiyalarning gradiyent chiziqlari

Bizga n-o'lchamli Evklid fazosida aniqlangan $y = f(x_1, x_2, ..., x_n)$ differensiallanuvchi funksiya berilgan bo'lsin.

Ta'rif-12. $\dot{x} = grad\; f(x)$ gradiyent sistema yechimlarining integral chiziqlari berilgan funksiyaning gradiyent chiziqlari deyiladi.

Berilgan maydonning $P_0(x_0, y_0, z_0)$ nuqtasidagi $grad\,U = grad\,F(x,y,z)$ vektori va P_0 nuqtadan o'tuvchi sath sirtlarini qanday joylashganligini aniqlashdan boshlaymiz.

Aytaylik, P_0 nuqtadan o'tuvchi sath sirti

$$F(x,y,z) = c_0 \text{ yoki } F(x,y,z) - c_0 = 0 \quad (12)$$

bo'lsin. l egri chiziq esa (12) sirtda yotuvchi va P_0 nuqtadan o'tuvchi egri chiziq bo'lib, uning tenglamasi

$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ parametrik ko'rinishda berilgan bo'lsin. Bunda $x(t), y(t), z(t)$ – differensiallanuvchi funksiyalar va $x_0 = x(t_0), y_0 = y(t_0), z_0 = z(t_0)$. Bu chiziq sirtda yotgani uchun $F\left[x(t), y(t), z(t)\right] - c_0 = 0$ bo'ladi. Bu tenglikda t parametr bo'yicha differensiallab, quyidagini hosil qilamiz: $\frac{\partial F}{\partial x}\cdot x'(t) + \frac{\partial F}{\partial y} y'(t) + \frac{\partial F}{\partial z} z'(t) = 0$.

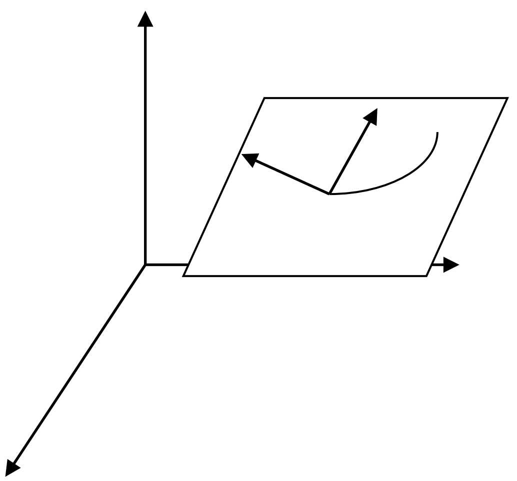
Agar $t = t_0$ desak, unda

$$F_x'(x_0, y_0, z_0)\cdot x'(t_0) + F_y'(x_0, y_0, z_0)\cdot y'(t_0) + F_z'(x_0, y_0, z_0)\cdot z'(t_0) = 0$$

bo'ladi. Bu tenglikni chap tomoni ikkita

$$grad\,F(P_0) = F_x'(x_0, y_0, z_0)\cdot \vec{i} + F_y'(x_0, y_0, z_0)\cdot \vec{j} + F_z'(x_0, y_0, z_0)\cdot \vec{k}$$

va $\vec{r}'(t_0) = x'(t_0)\cdot \vec{i} + y'(t_0)\cdot \vec{j} + z'(t_0)\cdot \vec{k}$ vektorlarning skalyar ko'paytmasiga teng. $\vec{r}'(t_0)$ vektor L egri chiziqning urinmasi bo'ylab yo'nalgan.



Demak, $grad\, u(P_0)\cdot \vec{r}'(t_0)=0$ (13). Agar $grad\, u(P_0)\neq 0$ bo'lsa, unda (13)-tenglikka ko'ra $grad\, u(P_0)\perp \vec{r}'(t_0)$ bo'ladi.Egri chiziq ixtiyoriy bo'lgani uchun $grad\, u(P_0)$ vektor sirtning sath sirti ustidagi P_0 nuqtasiga o'tkazilgan barcha urinmalariga perpendikulyar bo'ladi.[6]

Demak, biz quyidagi teoremani isbotladik.

Teorema-6. Tekislikda aniqlangan differensiallanuvchi funksiyalar gradiyent chiziqlari sath chiziqlariga perpendikulyar bo'ladi.

Ta'rif-13. Funksiyaning gradiyent vektorining uzunligi har bir sath sirtida o'zgarmas bo'lsa, bu funksiya metrik funksiya deyiladi.

Misol. Ushbu $f(x,y)=x^2+y^2$ qoida bilan aniqlangan $f:R^2\to R^1$ funksiyani qaraymiz. Funksiyaning sath sirti $x^2+y^2=c$ dan iborat. Bu funksiya metrik funksiya bo'ladi, chunki $grad\ f=\{2x,2y\}$. Bundan

$$|grad\ f|^2=\sqrt{4x^2+4y^2}=2\sqrt{x^2+y^2}=2\sqrt{c}$$

o'zgarmas bo'ladi.

Teorema-7. Aytaylik, $f:\Phi\to R^1$ metrik funksiya bo'lsin. U holda bu funksiyaning har bir gradiyent chizig'i Φ sirtda yotuvchi geodezik chiziq bo'ladi.

Teorema-8. Metrik funksiyalar har bir gradiyent chizig'ining egriligi nolga teng bo'ladi.

Natija. Agar f metrik funksiya kritik nuqtalarga ega bo'lmasa, u holda har bir gradiyent chiziq to'g'ri chiziq bo'ladi.

Endi biz differensiallanuvchi funksiyalarning gradiyent chiziqlarini topamiz va ularning grafiklarini chizamiz.

1-misol. $u(x,y)=x^2+y^2$ differensiallanuvchi funksiyaning gradiyent chiziqlarini toping.

Yechish. Bu funksiyaning gradiyentlarini topamiz va quyidagi sistemani tuzib olamiz: $\begin{cases} \dot{x} = 2x \\ \dot{y} = 2y \end{cases}$ sistemani yechimini topamiz: $\frac{dx}{dt} = 2x \Rightarrow \frac{dx}{x} = 2dt \Rightarrow \ln x = 2t + x_0$

$$\begin{cases} x = x_0 e^t \\ y = y_0 e^t \end{cases} \Rightarrow y = \frac{y_0}{x_0} x .$$

Demak, bu funksiyaning gradiyent chiziqlari $y = \frac{y_0}{x_0} x$ chiziqlar oilasidan iborat ekan. Agar $k = \frac{y_0}{x_0}$ deb belgilash kiritsak, berilgan funksiyaning gradiyent chiziqlari koordinatalar boshidan o'tuvchi $y = kx$ to'g'ri chiziqlarda yotuvchi to'g'ri chiziq qismlaridan iborat bo'ladi.

2-misol. Quyidagi differensiallanuvchi funksiyaning gradiyent chiziqlarini toping: $u(x, y) = x^2 - y^2$. Bu misolni ham yuqoridagi tartibda bajaramiz:

$$\begin{cases} \dot{x} = 2x \\ \dot{y} = -2y \end{cases} \Rightarrow \begin{cases} x = e^{2t} x_0 \\ y = e^{-2t} y_0 \end{cases} \Rightarrow y = \frac{x_0\, y_0}{x} .$$

Bu funksiya gradiyent chiziqlari $y = \frac{x_0\, y_0}{x}$ tenglama bilan aniqlanadigan chiziqlar oilasidan iborat.

3-misol. Quyidagi differensiallanuvchi funksiyaning gradiyent chiziqlarini toping:

$$u(x, y) = \frac{y}{x^2} .$$

Buning uchun gradiyent sistemani tuzib olamiz va uni yechamiz:

$$\begin{cases} \dot{x} = -\dfrac{2y}{x^3} \\ \dot{y} = \dfrac{1}{x^2} \end{cases}$$

$$\frac{dy}{dx} = \frac{1}{x^2} \cdot \frac{x^3}{(-2y)} = -\frac{x}{2y}$$

$$ydy = -\frac{1}{2} xdx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{4} + C \Rightarrow \frac{y^2}{2} + \frac{x^2}{4} = C$$

Javob: funksiya gradiyenti $y^2 + \frac{x^2}{2} = C$ tenglama bilan aniqlanadigan chiziqlar oilasidan iborat.

XULOSA

Differensial geometriya kursida chiziqlar va sirtlarning lokal xossalari differensiallanuvchi funksiyalar yordamida o'rganiladi. O'z navbatida funksiyalarning sath sirtlari va gradiyent chiziqlari sirtlar geometriyasini o'rganishda muhim rol o'ynaydi. Mazkur bitiruv malakaviy ishi differensiallanuvchi funksiyalarning gradiyent chiziqlarining xossalarini o'rganishga bag'ishlangan. Unda funksiya differensiallanuvchi funksiyalar sath sirtlari va gradiyent chiziqlari orasidagi bog'lanish, ixtiyoriy metrik funksiyaning gradiyent chiziqlarining egriligi nolga teng bo'lishi va boshqa xossalari o'rganilgan, hamda misollar orqali batafsil yoritilgan.

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