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OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

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№ 1. Tenglamaning umumiy yechimini toping.

$$u_{xy} - \frac{1}{x-y}(u_x - u_y) = 1 \quad (1), \text{ agar } x+y<0,\ x>2 \text{ bo'lsa.}$$

Yechimi : Birinchi bo'lib, $u(x,y) = \dfrac{1}{x-y} \cdot v(x,y)$ almashtirish kiritamiz.

(1) dagi mos xususiy hosilalarni olamiz :

$$u_x = -\frac{1}{(x-y)^2}v + \frac{1}{x-y}v_x \ , \ u_y = \frac{1}{(x-y)^2}v + \frac{1}{x-y}v_y \ ,$$

$$u_{xy} = -\frac{2}{(x-y)^3}v - \frac{1}{(x-y)^2}v_y + \frac{1}{(x-y)^2}v_x + \frac{1}{x-y}v_{xy}.$$

Hosil bo'lgan tengliklarni (1) ga olib borib qo'yamiz :

$$-\frac{2}{(x-y)^3}v - \frac{1}{(x-y)^2}v_y + \frac{1}{(x-y)^2}v_x + \frac{1}{x-y}v_{xy} - \frac{1}{x-y}(-\frac{1}{(x-y)^2}v + \frac{1}{x-y}v_x) +$$

$$+\frac{1}{x-y}(\frac{1}{(x-y)^2}v + \frac{1}{x-y}v_y) = 1$$

Mos hadlarni soddalashtirib,

$$v_{xy} = x - y \Rightarrow \int v_{xy}dx = \int (x-y)dx. \qquad v_y = \frac{x^2}{2} - xy + g_1(y).$$

$$v = \frac{x^2}{2}y - \frac{y^2}{2}x + g\ \ y\ \ + f(x) \Rightarrow u(x,y) = \frac{xy}{2} + \frac{1}{x-y}(g(y) + f(x)).$$

Javob : $u(x,y) = \dfrac{xy}{2} + \dfrac{1}{x-y}(g(y) + f(x)).$

№ 2. Tor tebranish tenglamasi uchun qo'yilgan quyidagi Koshi masalasi yechimi topilsin.

$$u_{tt} = a^2u_{xx} + \sin\omega x \quad (2)\ ,\ u(x,0) = 0,\ u_t(x,0) = 0.$$

Yechimi: Agarda $a=0$ bo'lsa, $u(x,t)=\frac{t^2}{2}\sin\omega x+t\cdot F(x)+g(x)\Rightarrow u(x,0)=g(x)=0, u_t(x,0)=F(x)=0.$

$u(x,t)=\frac{t^2}{2}\sin\omega x$ bo'ladi. Endi $a\neq 0$ bo'lsin :

Birinchi bo'lib, (2) ning bir jinsli qismini yechib olamiz : $u_{tt}-a^2u_{xx}=0$ (3).

Uning xarakteristik tenglamasini yozib olamiz :

$$(dx-adt)(dx+adt)=0\Rightarrow\begin{cases}\xi=x-at\\ \eta=x+at\end{cases}$$

Mos hosilalarni hisoblaymiz : $u_{xx}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$, $u_{tt}=a^2u_{\xi\xi}-2a^2u_{\xi\eta}+a^2u_{\eta\eta}$.

Hosil bo'lgan tengliklarni (3) ga olib borib qo'yamiz, hamda $a\neq 0$ ekanligini hisobga oladigan bo'lsak, $u_{\xi\eta}=0$ ekanligini hosil qilamiz.

Demak, (3) ning umumiy yechimi

$$u(\xi,\eta)=F(\xi)+g(\eta)\Rightarrow u(x,t)=F(x-at)+g(x+at).$$

Endi (2) chiziqli bir jinsli bo'lmagan tenglamaning bir xususiy yechimi $u_0(x,t)=\frac{1}{a^2\omega^2}\sin\omega x$ ekanligini tekshirib ko'rish qiyin emasligini hisobga oladigan bo'lsak, (2) ning umumiy yechimini hosil qilamiz :

$$u(x,t)=F(x-at)+g(x+at)+\frac{1}{a^2\omega^2}\sin\omega x .$$

Endi Koshi shartlari orqali noma'lum funksiyalarni topamiz :

$$u(x,0)=F(x)+g(x)+\frac{1}{a^2\omega^2}\sin\omega x=0 ,\ \ u_t(x,0)=-a\cdot F'(x)+a\cdot g'(x)=0.$$

$a\neq 0\Rightarrow F(x)=g(x)+c\Rightarrow g(x)=-\frac{1}{2a^2\omega^2}\sin\omega x-\frac{c}{2}$ va

$F(x)=-\frac{1}{2a^2\omega^2}\sin\omega x+\frac{c}{2}$ ekanligini hisobga oladigan bo'lsak,

$$u(x,t)=-\frac{1}{2a^2\omega^2}\sin\omega(x-at)+\frac{c}{2}-\frac{1}{2a^2\omega^2}\sin\omega(x+at)-\frac{c}{2}+\frac{1}{a^2\omega^2}\sin\omega x$$

$$u(x,t)=-\frac{1}{2a^2\omega^2}\cdot 2\sin\frac{\omega x-\omega at+\omega x+\omega at}{2}\cos\frac{\omega x-\omega at-\omega x-\omega at}{2}+\frac{1}{a^2\omega^2}\sin\omega x$$

$$u(x,t)=\frac{\sin\omega x}{a^2\omega^2}(1-\cos a\omega t)$$

Javob : $u(x,t)=\dfrac{\sin\omega x}{a^2\omega^2}(1-\cos a\omega t).$

№ 3. Tenglamaning umumiy yechimini toping.

$u_{xyyy}=60y^2$

Yechimi : Yuqoridagi tenglamani ketma – ket 4 marotaba integrallab (birinchi 3 marotaba y bo'yicha, so'ng x bo'yicha) uning umumiy yechimini topamiz :

$$u_{xyy}=20y^3+f(x)\Rightarrow u_{xy}=5y^4+y\cdot f(x)+f_1(x)\Rightarrow u_x=y^5+\frac{y^2}{2}\cdot f(x)+y\cdot f_1(x)+f_2(x)$$

Va nihoyat $u(x,y)=xy^5+\dfrac{y^2}{2}\cdot F(x)+y\cdot F_1(x)+F_2(x)+g(y)$ ekanligini hosil qilamiz.

Javob : $u(x,y)=xy^5+\dfrac{y^2}{2}\cdot F(x)+y\cdot F_1(x)+F_2(x)+g(y).$

№ 4. $u_{xxxy}-u_{xyyy}=0.$

Yechimi : Birinchi bo'lib, $u_{xy}=v(x,y)$ almashtirish bajaramiz (bu yerda tenglamaning regulyar yechimini qidirayotganimiz tufayli tenglamadagi funksiya aralash hosilalari teng).

$v_{xx}-v_{yy}=0$ Uning xarakteristikalari $\begin{cases}\xi=x-y\\ \eta=x+y\end{cases}$ ekanligi hisobga olsak,

$$v=F(x-y)+g(x+y)\Rightarrow u_{xy}=F(x-y)+g(x+y)\Rightarrow u_x=-F_1(x-y)+g_1(x+y)+f(x)$$

$u(x,y)=-F_2(x-y)+g_2(x+y)+F(x)+\psi(y)$ yoki

$u(x,y)=F_3(x-y)+g_2(x+y)+F(x)+\psi(y).$

Javob : $u(x,y)=F_3(x-y)+g_2(x+y)+F(x)+\varphi(y).$

№ 5. $u(x,t)=v(x,t)+w(x,t)$ **almashtirish bajarib,** $w(x,t)$ **ni shunday topingki, natijada** $v_{xx}-v_{tt}=F(x,t)$ **tenglama va bir jinsli chegaraviy shartlar hamda o'zgargan boshlang'ich shartlarga kelsin.**

$u_{xx}=u_{tt}+f(x,t),\ u(0,t)=\mu(t),\ u_x(l,t)+hu(l,t)=\nu(t)\ ,\ h>0,$

$u(x,0)=0,\ u_t(x,0)=\psi(x)$

Yechish : Masala berilishidagi kabi $u(x,t)=v(x,t)+w(x,t)$ almashtirish bajarib chegaraviy shartlarni qanoatlantiramiz. Undan quyidagi munosabatlarni hosil qilamiz :

$$\begin{cases} v(0,t)+w(0,t)=\mu(t), \\ v_x(l,t)+w_x(l,t)+hv(l,t)+hw(l,t)=\nu(t) \end{cases}$$

Masala shartiga ko'ra $v_{xx}-v_{tt}=F(x,t)$ tenglama va bir jinsli chegaraviy shartlar bo'lishi quyidagi tengliklarga ekvivalentdir :

$$\begin{cases} w(0,t)=\mu(t), \\ w_x(l,t)+hw(l,t)=\nu(t) \end{cases} \qquad (1)$$

Umumiy holda $w(x,t)$ funksiyani $w(x,t)=(Ax^2+Bx+C)\mu(t)+(Dx^2+Ex+F)\nu(t)$ (2) ko'rinishda qidiramiz.

(2) ni (1) ga olib borib quyidagi tengliklarga ega bo'lamiz :

$$\begin{cases} C\mu(t)+F\nu(t)=\mu(t) \\ (2Al+B)\mu(t)+(2Dl+E)\nu(t)+h(Al^2+Bl+C)\mu(t)+h(Dl^2+El+F)\nu(t)=\nu(t) \end{cases}$$

Oxirgi tenglamalar sistemasidan $C=1, F=0$ ekanligi hamda

$2Al+B+h(Al^2+Bl+C)=0,\quad 2Dl+E+h(Dl^2+El+F)=1$ munosabatlarga ega bo'lamiz. Iloji boricha yuqori koeffitsiyentlarni yo'qotamiz. Buning uchun $A=0$ deb olsak, $B=-\frac{h}{1+hl}$ ekanligi kelib chiqadi. Shu kabi $D=0$ desak, $E=\frac{1}{1+hl}$ chiqadi. Demak,

$$w(x,t)=(1-\frac{hx}{1+hl})\mu(t)+\frac{x}{1+hl}\nu(t).$$

Oxirgi topilgan funksiya (1) shartlarni qanoatlantirishini ko'rish qiyin emas.

Javob : $w(x,t)=(1-\frac{hx}{1+hl})\mu(t)+\frac{x}{1+hl}\nu(t).$

№ 6. Quyidagi aralash masalani yeching.

$$u_{tt}=a^2u_{xx}\,,\; u_x(0,t)=0, u_x(l,t)=Ae^{-t}$$

$$u(x,0)=\frac{Aa\,ch\frac{x}{a}}{sh\frac{l}{a}}\;,\; u_t(x,0)=-\frac{Aa\,ch\frac{x}{a}}{sh\frac{l}{a}},\quad 0<x<l,\; t>0.$$

Yechish : Misolning berilishidagi 2- chegaraviy shartdan ko'rinib turibdiki, e^{-t} funksiya qatnashgan. Shuning uchun ham misolni yechish uchun yechimni $u(x,t)=v(x,t)+e^{-t}f(x)$ kabi ko'rinishda qidiramiz. Bu yerda $f(x)$ ixtiyoriy funksiya bo'lib, u faqatgina x o'zgaruvchiga bog'liqdir. Endi 2ta chegaraviy shartlarni qanoatlantiramiz :

$$u_x(0,t)=v_x(0,t)+e^{-t}f'(0)=0\,, u_x(l,t)=v_x(l,t)+e^{-t}f'(l)=Ae^{-t}.$$

Endi boshlang'ich shartlarni qanoatlantiramiz :

$u(x,0)=v(x,0)+f(x)=\frac{Aa\,ch\frac{x}{a}}{sh\frac{l}{a}}.$ Oxirgi tenglikda $v(x,0)=0$ desak, $f(x)=\frac{Aa\,ch\frac{x}{a}}{sh\frac{l}{a}}$ ekanligi kelib chiqadi. Endi ikkinchi boshlang'ich shartga qo'yamiz :

$u_t(x,0) = v_t(x,0) - e^{-0} f(x) = -\frac{Aach\frac{x}{a}}{sh\frac{l}{a}}$. $v_t(x,0) = 0$ desak, yuqoridagi topilgan $f(x)$ funksiyamiz ikkinchi boshlang'ich shartni ham qanoatlantirishini ko'rishimiz mumkin.

Endi $f'(x)$ ni hisoblaymiz : $f'(x) = \frac{Ash\frac{x}{a}}{sh\frac{l}{a}}$. Ko'rinib turibdiki, $f'(0) = 0, f'(l) = A$ tengliklar o'rinlidir. Endi yuqoridagi yozib olgan ikkita chegaraviy shartdan $v_x(0,t) = 0$,$v_x(l,t)$ ekanligini hosil qilamiz. Demak, $f(x)$ ni yuqoridagi belgilab olganimiz kabi yozib olsak, quyidagi masalaga kelamiz :

$v_{tt} = a^2 v_{xx}$, $v_x(0,t) = 0, v_x(l,t) = 0$. $v(x,0) = 0$, $v_t(x,0) = 0$.

Oxirgi masalani Furye usulida yechamiz : yechimni $v(x,t) = X(x)T(t)$ shu kabi ko'rinishda qidiramiz.

$X''(x) + л^2 X(x) = 0$ differensial tenglamani hamda ikkita $X'(0) = 0, X'(l) = 0$ shartlardan $л_n = \frac{pn}{l}, n \in N$ hamda $X_n(x) = \cos\frac{pnx}{l}$ ekanligini hosil qilamiz.

Shu kabi $T_n(t) = A_n \cos л_n t + B_n \sin л_n t$ ekanligi va yechim

$v(x,t) = \sum_{n=1}^{\infty}(A_n \cos л_n t + B_n \sin л_n t)\cos л_n x$ kabi ko'rinishda bo'lishini hosil qilamiz. Yuqoridagi bir jinsli boshlang'ich shartlarni qanoatlantirsak, $v(x,t) = 0$ ekanligi kelib chiqadi. U holda $u(x,t) = v(x,t) + e^{-t} f(x)$ ekanligidan misolimiz yechimini $u(x,t) = e^{-t}\frac{Aach\frac{x}{a}}{sh\frac{l}{a}}$ kabi ko'rinishda bo'lishini hosil qilamiz.

Javob : $u(x,t) = e^{-t}\frac{Aach\frac{x}{a}}{sh\frac{l}{a}}$.

№ 7. $0 < x < l, t > 0$ **yarim yo'lda quyidagi aralash masalani yeching :**

$u_t = a^2 u_{xx}$, $u_x(0,t) = u_x(l,t) = q$, $u(x,0) = Ax$.

Yechish : Masalada qo'yilgan chegaraviy shartlar bir jinsli bo'lmagani uchun, yechimni $u(x,t) = v(x,t) + w(x,t)$ ko'rinishda qidiramiz. Undan quyidagi munosabatlarni hosil qilamiz :

$$\begin{cases} u_x(0,t) = v_x(0,t) + w_x(0,t) = q \\ u_x(l,t) = v_x(l,t) + w_x(l,t) = q \end{cases} \Rightarrow \begin{cases} w_x(0,t) = q, v_x(0,t) = 0 \\ w_x(l,t) = q,\ v_x(l,t) = 0 \end{cases}$$

$w(x,t)$ funksiyani $w(x,t) = (Ax+B)q$ ko'rinishda qidiramiz. $w_x(x,t) = Aq$ va yuqoridagi chegaraviy shartlardan $A=1, B=0$ desak, $w(x,t) = qx$ funksiyaga ega bo'lamiz.

$v(x,t)$ funksiyaga nisbatan esa quyidagi tenglamalar sistemasini hosil qilamiz :

$$\begin{cases} v_t = a^2 v_{xx} \\ v_x(0,t) = v_x(l,t) = 0 \\ v(x,0) = x(A-q) \end{cases}$$

Endi yuqoridagi masalani Furye usulida yechamiz : $v(x,t) = X(x)T(t)$.

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\text{л}^2 \Rightarrow \begin{cases} X(x) = c_1 \cos \text{л} x + c_2 \sin \text{л} x \\ X'(0) = X'(l) = 0 \end{cases} \Rightarrow \begin{cases} X'(0) = c_2 = 0 \\ \text{л}_n l = pn, n \in N \end{cases}$$

$$X_n(x) = \cos \text{л}_n x = \cos\frac{pn}{l}x, n \in N,\ \frac{T'(t)}{a^2 T(t)} = -\text{л}^2 \Rightarrow T_n(t) = A_n e^{-a^2 \text{л}^2 t}$$

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-a^2\text{л}^2 t} \cos\frac{pn}{l}x, v(x,0) = \sum_{n=1}^{\infty} A_n \cos\frac{pn}{l}x = x(A-q)$$

$$A_n = \frac{2}{l}\int_0^l x(A-q)\cos\frac{pn}{l}x\,dx = \frac{2l}{p^2n^2}(A-q)(\cos pn - 1)$$

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2l}{p^2n^2}(A-q)((-1)^n - 1)e^{-a^2\text{л}^2 t}\cos\frac{pn}{l}x$$

Ko'rinib turibdiki, $n = 2k, k = 0,1,2...$ da $v(x,t) = 0$ munosabat o'rinlidir. $n = 2k+1, k = 0,1,2...$ da esa

$$v(x,t)=\sum_{k=0}^{\infty}\frac{-4l}{p^2(2k+1)^2}(A-q)e^{-\frac{a^2p^2(2k+1)^2}{l^2}t}\cos\frac{p}{l}(2k+1)x$$

Demak, masalamiz yechimi quyidagi ko'rinishda ekan :

$$u(x,t)=v(x,t)+w(x,t)=q\,x-\frac{4l(A-q)}{p^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}e^{-\frac{(2k+1)^2a^2p^2}{l^2}t}\cos(2k+1)\frac{px}{l}$$

Javob : $u(x,t)=q\,x-\dfrac{4l(A-q)}{p^2}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}e^{-\frac{(2k+1)^2a^2p^2}{l^2}t}\cos(2k+1)\frac{px}{l}$

№ 8. ***Issiqlik o'tkazuvchanlik tenglamasi uchun Koshi masalasini yeching(n=1 holda):***

$$u_t=u_{xx}\,,\ u(x,0)=xe^{-x^2}.$$

Yechish : Tenglamaning yechimi quyidagi ko'rinishda bo'ladi :

$$u(x,t)=\frac{1}{2\sqrt{pt}}\int_{-\infty}^{\infty}o\,e^{-o^2}\cdot e^{\frac{-(x-o)^2}{4t}}do.$$

O'ng tomondagi integralni hisoblaymiz :

$$\int_{-\infty}^{\infty}o\,e^{-o^2}\cdot e^{\frac{-(x-o)^2}{4t}}do=-\frac{1}{2}\int_{-\infty}^{\infty}e^{\frac{-(x-o)^2}{4t}}de^{-o^2}=\frac{x}{4t}\int_{-\infty}^{\infty}e^{-o^2-\frac{(x-o)^2}{4t}}do-\frac{1}{4t}\int_{-\infty}^{\infty}oe^{-o^2-\frac{(x-o)^2}{4t}}do.$$

Bundan

$$I=\int_{-\infty}^{\infty}oe^{-o^2-\frac{(x-o)^2}{4t}}do=\frac{x}{4t+1}\int_{-\infty}^{\infty}e^{-o^2-\frac{(x-o)^2}{4t}}do.$$

Quyidagi

$$o^2+\frac{(x-o)^2}{4t}=\frac{4t+1}{4t}\left(o^2-\frac{2xo}{4t+1}\right)+\frac{x^2}{4t}=\frac{x^2}{4t+1}+\frac{4t+1}{4t}\left(o-\frac{x}{4t+1}\right)^2$$

tenglikka va $\displaystyle\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{p}$ Eyler - Puasson integral formulasiga ko'ra

$$I=\frac{x}{4t+1}\cdot e^{-\frac{x^2}{4t+1}}\int_{-\infty}^{\infty}e^{-\frac{4t+1}{4t}\left(o-\frac{x}{4t+1}\right)^2}do=\frac{2\sqrt{pt}}{\sqrt{(4t+1)^3}}xe^{-\frac{x^2}{4t+1}}.$$

Shunday qilib, berilgan Koshi masalasining yechimi quyidagi ko'rinishda ekan :

$$u(x,t)=\frac{x}{\sqrt{(4t+1)^3}}\cdot e^{-\frac{x^2}{4t+1}}.$$

Javob : $u(x,t)=\frac{x}{\sqrt{(4t+1)^3}}\cdot e^{-\frac{x^2}{4t+1}}.$