O'ZBEKISTON RESPUBLIKASI

OLIY VA O'RTA-MAXSUS TA'LIM VAZIRLIGI

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R E F E R A T

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Masala

Faraz qilaylik, moddiy nuqta to'g'ri chiziq bo'yicha $[t_0,T]$ vaqt oralig'ida $V=V(t)$ tezlik bilan harakat qilsin. Uning shu vaqt oralig'ida bosib o'tgan yo'li S topilsin.

Ma'lumki, tezlik o'zgarmas, ya'ni $V(t)=V_0=const$ bo'lsa, u holda o'tilgan yo'l

$$S=V_0(T-t_0)$$

bo'ladi.

Agar tezlik t o'zgaruvchining $t\in[t_0,T]$ ixtiyoriy funksiyasi bo'lsa, unda masalani echishga quyidagicha kirishiladi:

1) vaqt oralig'i $[t_0,T]$ ni $t_0,t_1,t_2,\ldots,t_n$ $(t_n=T)$ nuqtalar yordamida n ta qismga ajratiladi, bunda

$$t_0<t_1<t_2<\cdots<t_k<\cdots t_{n-1}<t_n=T\ ;$$

2) har bir $[t_k,t_{k+1}]$ $(k=0,1,2,...,n-1)$ oraliqda ixtiyoriy ξ_k nuqtani olib, tezlikning shu nuqtadagi qiymati $V(\xi_k)$ topiladi:

3) $V(\xi_k)$ ni $[t_k,t_{k+1}]$ oraliqning uzunligi $(t_{k+1}-t_k)$ ga ko'paytiriladi.

$$V(\xi_k)\cdot(t_{k+1}-t_k) \qquad (k=0,1,2,...,n-1) \tag{1}$$

Bu miqdor, tezlik $V(t)$ ni $[t_k,t_{k+1}]$ oraliqda o'zgarmas va u $V(\xi_k)$ ga teng deb olinganda, nuqtaning $[t_k,t_{k+1}]$ oraliqda bosib o'tgan yo'lini (taqriban) ifodalaydi.

4) (1) ko'paytmani k ning $k=0,1,2,...,n-1$ qiymatlari uchun yozib, so'ng ularni yig'ib ushbu

$$V(\xi_0)\cdot(t_1-t_0)+V(\xi_1)(t_2-t_1)+\cdots+$$
$$+V(\xi_k)\cdot(t_{k+1}-t_k)+V(\xi_{n-1})(t_n-t_{n-1})$$

yig'indini hosil qilinadi. Bu yig'indi ushbu Σ orqali quyidagicha yoziladi:

$$\sum_{k=0}^{n-1}V(\xi_k)\cdot(t_{k+1}-t_k)\ . \tag{2}$$

Ravshanki, (2) yig'indi nuqtaning $[t_0,T]$ oraliqda bosib o'tgan yo'lini taqribiy ifodalaydi, chunki tezlik $V(t)$ vaqtning ixtiyoriy funksiyasi bo'lgan holda uni har bir $[t_k,t_{k+1}]$ da o'zgarmas $V(\xi_k)$ deb olindi. Demak,

$$S \approx \sum_{k=0}^{n-1} V(\xi_k)\cdot(t_{k+1}-t_k).$$

$t_{k+1}-t_k=\Delta t_k$ deb, bu Δt_k $(k=0,1,2,...,n-1)$ larning eng kattasini λ deylik.

Endi $[t_0,T]$ oraliqning bo'laklash sonini orttira borilsa (bunda har bir Δt_k nolga, ya'ni $\lambda \to 0$ intilsin), u holda

$$\sum_{k=0}^{n-1} V(\xi_k)\cdot\Delta t_k$$

miqdor izlanayotgan yo'lni tobora aniqroq ifodalay boradi. Binobarin,

$$S=\lim_{\lambda\to 0}\sum_{k=0}^{n-1} V(\xi_k)\cdot\Delta t_k$$

bo'ladi.

Shunday qilib nuqtaning tezligiga ko'ra o'tilgan yo'lni topish masalasi maxsus tuzilgan yig'indining limitini topishga kelar ekan.

Shunga o'xshash ko'pgina masalalar, jumladan sterjenning zichligiga ko'ra uning massasini topish, egri chiziqli trapesiyaning yuzini topish, o'zgaruvchi kuchning bajargan ishini topish masalalari ham yuqoridagiga o'xshash yig'indining limitini topishga keladi. Bunday yig'indining limiti oliy matematikada muhim bo'lgan aniq integral tushunchasiga olib keladi.

Aniq integral tushunchasi. Integralning mavjudligi

Aytaylik, $y=f(x)$ funksiya $[a,b]$ segmentda berilgan bo'lsin. Bu segmentni $x_0,x_1,x_2,...,x_{n-1},x_n$ $(x_0=a,\ x_n=b,\ x_0<x_1<...<x_n)$ nuqtalar yordamida n ta

$$[x_0,x_1],\ [x_1,x_2],...,\ [x_k,x_{k+1}],...,\ [x_{n-1},x_n]$$

bo'lakka ajratamiz. Bu bo'lakchalarning uzunliklarini mos ravishda quyidagicha belgilaymiz:

$$\begin{aligned}
&\Delta x_0 = x_1 - x_0 \qquad (x_0 = a),\\
&\Delta x_1 = x_2 - x_1,\\
&\dots\dots\dots\dots\dots\dots\\
&\Delta x_k = x_{k+1} - x_k,\\
&\dots\dots\dots\dots\dots\dots\\
&\Delta x_{n-1} = x_n - x_{n-1} \qquad (x_n = b)
\end{aligned}$$

Odatda $\{\Delta x_0, \Delta x_1, ..., \Delta x_{n-1}\}$ kesmalar sistemasi (to'plami) $[a,b]$ segmentni bo'laklash deyiladi va uni λ bilan belgilanadi:

$$\lambda = \{\Delta x_0, \Delta x_1, ..., \Delta x_{n-1}\}.$$

Bu $\Delta x_0, \Delta x_1, ..., \Delta x_{n-1}$ larning eng kattasini $|\lambda|$ deylik:

$$|\lambda| = \max\{\Delta x_0, \Delta x_1, ..., \Delta x_{n-1}\}.$$

Har bir tayin λ bo'laklash $[a,b]$ segmentining bitta bo'linishini aniqlaydi.

Har bir bo'lakchada ixtiyoriy ravishda bittadan

$$\xi_0, \xi_1, ..., \xi_k, ..., \xi_{n-1}$$

nuqtalarni olib, bu nuqtalardagi funksiyaning qiymatlari

$$f(\xi_0), f(\xi_1), ..., f(\xi_k), ..., f(\xi_{n-1})$$

ni mos ravishda bo'lakchalarning uzunliklariga ko'paytirib

$$f(\xi_0)\cdot\Delta x_0, f(\xi_1)\cdot\Delta x_1, ..., f(\xi_k)\cdot\Delta x_k, ..., f(\xi_{n-1})\cdot\Delta x_{n-1}$$

quyidagi

$$\begin{aligned}
\sigma &= f(\xi_0)\cdot\Delta x_0 + f(\xi_1)\cdot\Delta x_1 + ... + f(\xi_k)\cdot\Delta x_k + \\
&+ ... + f(\xi_{n-1})\cdot\Delta x_{n-1} = \sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k
\end{aligned}$$

yig'indini hosil qilamiz.

Odatda,

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k \tag{3}$$

yig'indi $f(x)$ funksiyaning integral yig'indisi deyiladi. Bu yig'indi $[a,b]$ segmentning bo'laklanishiga, hamda har bir bo'lakchada olingan ξ_k nuqtalarga bog'liq bo'ladi.

Endi $[a,b]$ segmentining shunday bo'laklashlar ketma-ketligi

$$\lambda_1, \lambda_2, ..., \lambda_n, ... \tag{4}$$

ni olaylik, ular uchun

$$\lim_{n\to\infty} |\lambda_n| = 0$$

bo'lsin.

Ixtiyoriy, yuqorida aytilgan (4) ketma-ketlikni olib, bu ketma-ketlikning har bir hadiga mos integral yig'indilarni tuzamiz. Ular

$$\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n, ... \tag{5}$$

ketma-ketlikni hosil qiladi, bunda

$$\sigma_n = \sum_{k=0}^{n-1} \cdot f(\xi_k) \cdot \Delta x_k$$

Ta'rif. *Agar har bir bo'lakchada olingan ixtiyoriy* ξ_k *nuqtalarda* σ_n *ketma-ketlik har doim bitta* J *songa intilsa, (uni* $\sigma_n = \sum_{k=0}^{n-1} \cdot f(\xi_k) \cdot \Delta x_k$ *ning limiti deyiladi),* $f(x)$ *funksiya* $[a,b]$ *segmentda integrallanuvchi,* J *son esa* $f(x)$ *funksiyaning* $[a,b]$ *segment bo'yicha aniq integrali deyiladi va u*

$$\int_a^b f(x)\,dx$$

kabi belgilanadi. Demak,

$$\lim_{n\to\infty} \sigma_n(x) = I = \int_a^b f(x)\,dx.$$

Bunda a son integralning quyi chegarasi, b son esa integralning yuqori chegarasi, $[a,b]$ segment integrallash oralig'i deyiladi.

1-§da keltirilgan masalaning echimi, o'tilgan S yo'l, tezlik $V(t)$ ning $[t_\circ, T]$ oraliq bo'yicha aniq integraldan iborat ekanligini bildiradi:

$$S = \int_{t_\circ}^{T} V(t)\,dt$$

Misol: *Agar* $[a,b]$ *da* $f(x) = c - const$ *bo'lsa, u holda*

$$\int_a^b c \cdot dx = c \cdot (b-a)$$

bo'lishi isbotlansin.

◄ $[a,b]$ segmentning ixtiyoriy bo'laklashi

$$[a,x_1], \quad [x_1,x_2], \quad [x_2,x_3], \ldots, \quad [x_k,x_{k+1}], \ldots \quad [x_{n-1},b]$$

ni olib, har bir bo'lakchada bittadan ixtiyoriy

$$\xi_0, \xi_1, \xi_2 \ldots, \xi_k, \ldots, \xi_{n-1}$$

nuqtalarni tanlaymiz. Ravshanki,

$$f(\xi_0) = c, f(\xi_1) = c, f(\xi_2) = c, \ldots, f(\xi_k) = c, \ldots, f(\xi_{n-1}) = c$$

bo'lib,

$$\sigma = c \cdot \Delta x_0 + c \cdot \Delta x_1 + c \cdot \Delta x_2 + \ldots + c \cdot \Delta x_k + \ldots + c \cdot \Delta x_{n-1} =$$
$$= c\,(\Delta x_0 + \Delta x_1 + \Delta x_2 + \ldots + \Delta x_k + \ldots + \Delta x_{n-1}) =$$
$$= c\,(x_1 - a + x_2 - x_1 + x_3 - x_2 + \ldots + x_{k+1} - x_k + \ldots + b - x_{n-1}) =$$
$$= c \cdot (b-a)$$

bo'ladi. Demak,

$$\int_a^b c \cdot dx = \lim_{n\to\infty} \sigma = \lim_{n\to\infty} c \cdot (b-a) = c \cdot (b-a). ►$$

Xususan, $f(x) \equiv 1$ bo'lsa,

$$\int_a^b 1 \cdot dx = \int_a^b dx = b-a$$

bo'ladi..

Yuqorida funksiyaning aniq integrali integral yig'indining limiti sifatida ta'riflandi. Albatta, yig'indining limiti integrallanadigan funksiyaga bog'liq bo'ladi.

Integral yig'indi limitining mavjudligini ko'rsatish (ya'ni funksiyaning integrallanuvchi bo'lishini isbotlash) ancha murakkab bo'lib, ular maxsus adabiyotlarda ma'lum sinf funksiyalari uchun isbotlanadi. Biz quyida bunday teoremalardan birini isbotsiz keltiramiz.

Teorema. *Agar* $f(x)$ *funksiya* $[a,b]$ *segmentda uzluksiz bo'lsa, u shu oraliqda integrallanuvchi bo'ladi.*

Eslatma.

1) Agar $f(x)$ *funksiya* $[a,b]$ *da integrallanuvchi bo'lsa, u* $[a,b]$ *da chegaralangan bo'ladi.*

2) *Agar* $f(x)$ *funksiya* $[a,b]$ *da chegaralangan bo'lib, u* $[a,b]$ *ning chekli sondagi nuqtalarida uzilishga ega va qolgan barcha nuqtalarida uzluksiz bo'lsa,* $f(x)$ *funksiya* $[a,b]$ *da integrallanuvchi bo'ladi.*

Aniq integralning xossalari

Funksiyaning aniq integrali qator xossalarga ega. Bu xossalardan aniq integralni hisoblashdi va uning turli sohalarga tatbiqlarida foydalaniladi. Ko'p hollarda xossalarning isboti aniq integral ta'rifi va funksiya limiti xossalaridan kelib chiqadi. Biz xossalarni keltirish bilan kifoyalanamiz:

1) Aniq integral

$$\int_a^b f(x)\,dx$$

da x ning o'rniga ixtiyoriy harf ishlatilishi mumkin:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(z)\,dz \text{ va h.k.}$$

2)Ushbu

$$\int_a^a f(x) = 0,$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

tengliklar o’rinli.

3) Agar $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo’lsa, u holda $c\cdot f(x)$ funksiya $(c = const)$ ham $[a,b]$ da integrallanuvchi va

$$\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$$

bo’ladi.

4) Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da integrallanuvchi bo’lsa, u holda $f(x)+g(x)$ funksiya ham $[a,b]$ da integrallanuvchi va

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

bo’ladi.

5) Agar $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo’lib, ixtiyoriy $x\in[a,b]$ da $f(x)\geq 0$ bo’lsa, u holda

$$\int_a^b f(x)\,dx \geq 0$$

bo’ladi.

6) Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da integrallanuvchi bo’lib, ixtiyoriy $x\in[a,b]$ da $f(x)\leq g(x)$ bo’lsa, u holda

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

bo’ladi.

7) Agar $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lsa, u holda bu funksiya $[a,b]$ ning istalgan $[\alpha,\beta]$ qismida $([\alpha,\beta]\subset[a,b])$ integrallanuvchi bo'ladi.

8) Agar $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lib, $a<c<b$ bo'lsa, u holda funksiya $[a,c]$ va $[c,b]$ da integrallanuvchi va

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$

bo'ladi.

Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo'lib, u shu segmentda integrallanuvchi bo'lsin.

Ushbu

$$\mathrm{M}(f)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

miqdor $f(x)$ funksiyaning $[a,b]$ dagi o'rta qiymati deyiladi.

9) Agar $f(x)$ funksiya $[a,b]$ da uzluksiz bo'lsa, u holda shunday c nuqta ($a<c<b$) topiladiki,

$$\int_a^b f(x)\,dx=f(c)\cdot(b-a)$$

bo'ladi. Bu xossa o'rta qiymat haqidagi teorema deb ham yuritiladi.

Aniq integralni hisoblash usullari

1⁰. Aniq integrallarni Nyuton-Leybnits formulasi yordamida hisoblash. Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo'lib, $F(x)$ funksiya esa uning $[a,b]$ segmentdagi boshlang'ich funksiyasi bo'lsin:

$$F'(x)=f(x).$$

Ravshanki, $f(x)$ funksiya $[a,b]$ da integrallanuvchi, ya'ni

$$\int_a^b f(x)\,dx$$

mavjud. Bu integral uchun

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad (1)$$

bo'lishini isbotlaymiz.

$[a,b]$ segmentni

$$x_0, x_1, ..., x_{n-1}, x_n \qquad (a = x_0 < x_1 < ... < x_{n-1} < x_n = b)$$

nuqtalar yordamida n ta

$$[a, x_1], \quad [x_1, x_2], ..., \quad [x_k, x_{k+1}], ..., \quad [x_{n-1}, b]$$

bo'lakchalarga ajratamiz. Har bir bo'lakchada $F(x)$ funksiyaga Lagranj teoremasini qo'llab topamiz:

$$F(x_1) - F(a) = F'(\xi_0)\cdot(x_1 - a) = f(\xi_0)\cdot\Delta x_0, \quad a < \xi_0 < x_1$$
$$F(x_2) - F(x_1) = F'(\xi_1)\cdot(x_2 - x_1) = f(\xi_1)\cdot\Delta x_1, \quad x_1 < \xi_1 < x_2$$
....
$$F(b) - F(x_{n-1}) = F'(\xi_{n-1})\cdot(b - x_{n-1}) = f(\xi_{n-1})\cdot\Delta x_{n-1}, \quad x_{n-1} < \xi_{n-1} < b$$

Bu tengliklarni hadlab qo'shish natijasida

$$F(b) - F(a) = \sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k \qquad (2)$$

hosil bo'ladi. $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lgani uchun

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k = \int_a^b f(x)\,dx$$

bo'ladi. (2) tenglikda limitga o'tsak, unda

$$\int_a^b f(x)\cdot dx = F(b) - F(a)$$

bo'lishi kelib chiqadi. Shuni isbotlash kerak edi.

Odatda (1) formula Nyuton-Leybnits formulasi deb yuritiladi. Bu formula yordamida ko'pgina aniq integrallar hisoblanadi.

(1) tenglikning o'ng tomonidagi $F(b) - F(a)$ (yozuvni qisqa qilish maqsadida) $F(x)\Big|_a^b$ kabi yoziladi:

$$F(b)-F(a)=F(x)\Big|_a^b,$$

$$\int_a^b f(x)dx=F(x)\Big|_a^b$$

Shunday qilib,

$$\int_a^b f(x)dx$$

integralni Nyuton-Leybnits formulasidan foydalanib hisoblash uchun avvalo $f\ \ x$ funksiyaning aniqmas integrali hisoblanadi:

$$\int f(x)dx=F(x)+C$$

So’ng

$$F\ \ x\ \ +C\ \Big|_a^b=F\ \ b\ \ +C-\ \ F\ \ a\ \ +C\ \ =F\ \ b\ \ -F\ \ a$$

topiladi.

Misollar.

1. *Ushbu*

$$\int_0^{\frac{\pi}{2}}\sin xdx$$

integral hisoblansin.

◄Ma’lumki,

$$\int\sin xdx=-\cos x+c\,.$$

Unda

$$\int_0^{\frac{\pi}{2}}\sin xdx=\ \ -\cos x+c\ \left|\begin{matrix}\frac{\pi}{2}\\0\end{matrix}\right.=-\cos\frac{\pi}{2}-\ \ -\cos 0\ \ =0+1=1$$

bo’ladi. ►

2. *Ushbu*

$$\int_0^1 x^n dx \qquad\qquad n\neq-1$$

integral hisoblansin.

◀ $f(x)=x^n$ funksiyaning aniqmas integrali

$$\int x^n dx=\frac{x^{n+1}}{n+1}+c$$

bo'lgani uchun

$$\int_0^1 x^n dx=\left(\frac{x^{n+1}}{n+1}+c\right)\Bigg|_0^1=\frac{1}{n+1}-\frac{0}{n+1}=\frac{1}{n+1}$$

bo'ladi. ▶

***3.** Ushbu*

$$\int_0^a \frac{x^2}{a^3+x^3}dx \qquad\qquad a>0$$

integral hisoblansin.

◀Bu integralni Nyuton-Leybnits formulasidan foydalanib hisoblaymiz:

$$\int_0^a \frac{x^2}{a^3+x^3}dx=\int_0^a \frac{\frac{1}{3}dx^3}{a^3+x^3}=\frac{1}{3}\int_0^a \frac{d(a^3+x^3)}{a^3+x^3}=\int_0^a \frac{x^2}{a^3+x^3}dx=\left(\frac{1}{3}\ln\left(a^3+x^3\right)\right)\Big|_0^a=.$$

$$=\frac{1}{3}\ln 2a^3-\frac{1}{3}\ln a^3=\frac{1}{3}\ln\frac{2a^3}{a^3}=\frac{1}{3}\ln 2 .▶$$

Eslatma. *Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo'lsin. U $[a,b]$ da integrallanuvchi bo'lib, integralning xossasiga ko'ra $[a,x]$ da $a\le x\le b$ da ham integrallanuvchi bo'ladi:*

$$\int_a^x f(t)\,dt .$$

Agar $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi bo'lsa, ya'ni

$$F'(x)=f(x)$$

bo'lsa, u holda Nyuton-Leybnits formulasiga ko'ra

$$\int_a^x f(t)\cdot dt=F(x)-F(a)$$

bo'lib, undan

$$\left(\int_a^x f(t)\cdot dt\right)' = \left(F(x)-F(a)\right)' = F'(x)$$

bo'lishi kelib chiqadi.

Demak,

$$\int_a^x f(t)\,dt$$

funksiya $f(x)$ ning boshlang'ich funksiyasi. Bu uzluksiz funksiya har doim boshlang'ich funksiyaga ega bo'lishini bildiradi.

2^0. Aniq integrallarni o'zgaruvchini almashtirish usuli bilan hisoblash. $[a,b]$ segmentda uzluksiz bo'lgan $f(x)$ funksiyaning aniq integrali

$$\int_a^b f(x)\,dx \tag{3}$$

ni hisoblash kerak bo'lsin. Ko'p hollarda bu integralda o'zgaruvchini almashtirish natijasida u soddaroq, hisoblash uchun qulayroq integralga keladi.

(3) integralda

$$x=\varphi(t)$$

deylik, bunda $\varphi(t)$ funksiya quyidagi shartlarni qanoatlantirsin:

1) $x=\varphi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hosilaga ega,

2) ixtiyoriy $t\in[\alpha,\beta]$ da $a\le\varphi(t)\le b$ va $\varphi(\alpha)=a,\quad \varphi(\beta)=b$.

U holda

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt$$

bo'ladi.

◄Faraz qilaylik, $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi bo'lsin:

$$F'(x)=f(x).$$

Ravshanki,

$$\left[F(\varphi(t))\right]'=F'(\varphi(t))\cdot\varphi'(t)=f(\varphi(t))\cdot\varphi'(t).$$

Demak, $F\ \ \varphi\ \ t$ funksiya α,β da $f\ \ \varphi\ \ t\ \ \cdot\varphi'\ \ t$ funksiyaning boshlang'ich funksiya bo'ladi. Unda Nyuton-Leybnits formulasiga ko'ra

$$\int_{\alpha}^{\beta} f\ \ \varphi\ \ t\ \ \cdot\varphi'\ \ t\ \ dt = F\ \ \varphi\ \ \beta\ \ - F\ \ \varphi\ \ \alpha\ \ = F\ \ b\ \ - F\ \ a$$

bo'ladi. Ikkinchi tomondan

$$\int_{a}^{b} f(x)dx = F(b) - F(a)$$

bo'lishi ma'lum. Keyin ikki tenglikdan

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)dt \qquad (4)$$

bo'lishi kelib chiqadi.

Misollar.

4. *Ushbu*

$$\int_{1}^{3} \sqrt{x+1}dx$$

integral hisoblansin.

◄Bu integralda

$$x = 2t - 1$$

almashtirish bajaramiz. Unda

$$x = 1 \text{ da } 1 = 2t - 1, \text{ ya'ni } t = 1,$$

$$x = 3 \text{ da } 3 = 2t - 1, \text{ ya'ni } t = 2$$

bo'lib,

$$dx = 2dt$$

bo'ladi. Natijada

$$\int_{1}^{3} \sqrt{x+1}dx = \int_{1}^{2} \sqrt{2t}\cdot 2dt = 2\sqrt{2}\int_{1}^{2} t^{\frac{1}{2}}dt$$

bo'lib,

$$\int_1^2 t^{\frac{1}{2}} dt = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_1^2 = \frac{2}{3}\left(2^{\frac{3}{2}} - 1\right) = \frac{2}{3}\left(\sqrt{8} - 1\right)$$

bo'lganligidan

$$\int_1^3 \sqrt{x+1}dx = 2\sqrt{2}\cdot\frac{2}{3}(\sqrt{8}-1) = \frac{4\sqrt{2}}{3}(\sqrt{8}-1)$$

bo'lishi kelib chiqadi. ►

5. *Ushbu*

$$J = \int_0^1 x\sqrt{1+x^2}dx$$

integral hisoblansin.

◄Bu integralda

$$\sqrt{1+x^2} = t \text{, ya'ni } x = \sqrt{t^2-1}$$

almashtirish bajaramiz.

Unda

$$x = 0 \text{ da } t = 1,$$
$$x = 1 \text{ da } t = \sqrt{2}$$

$$dx = \left(\sqrt{t^2}-1\right)' \cdot dt = \frac{t}{\sqrt{t^2-1}}dt$$

bo'lib,

$$J = \int_1^{\sqrt{2}} t^2 dt = \frac{t^3}{3}\Bigg|_1^{\sqrt{2}} = \frac{2\sqrt{2}-1}{3}$$

bo'ladi. Demak,

$$\int_0^1 x\sqrt{1+x^2}dx = \frac{2\sqrt{2}-1}{3}. ►$$

6. *Ushbu*

$$J = \int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}}$$

integral hisoblansin.

◄Bu integralda

$$e^x = t \ \text{ ya'ni } \ x = \ln t$$

almashtirish bajaramiz. Unda

$$x = \ln 2 \ \text{ da } \ t = 2,$$

$$x = \ln 3 \ \text{ da } \ t = 3,$$

$$dx = \frac{1}{t} dt$$

bo'lib,

$$J = \int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}} = \int_2^3 \frac{dt}{t\left(t - t^{-1}\right)} = \int_2^3 \frac{dt}{t^2 - 1}$$

bo'ladi. Ravshanki,

$$\int_2^3 \frac{dt}{t^2 - 1} = \frac{1}{2} \int_2^3 \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \frac{1}{2} \Big[\ln (t-1) - \ln (t+1) \Big] \Big|_2^3 =$$

$$= \frac{1}{2} \ln \frac{t-1}{t+1} \Big|_2^3 = \frac{1}{2} \left(\ln \frac{2}{4} - \ln \frac{1}{3} \right) = \frac{1}{2} \ln \frac{3}{2}.$$

Demak,

$$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}} = \frac{1}{2} \ln \frac{3}{2}. ►$$

3^0. Aniq integrallarni bo'laklab integrallash usuli yordamida hisoblash.

Aytaylik, $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da uzluksiz va uzluksiz $f'(x)$ va $g'(x)$ hosilalarga ega bo'lsin. Ravshanki,

$$\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Ayni paytda

$$\int_a^b \left(f(x) \cdot g(x)\right)' dx = f(x) \cdot g(x) \Big|_a^b$$

bo'ladi. Demak,

$$\int_a^b \left[f'(x) \cdot g(x) + f(x) \cdot g'(x) \right] dx = f(x) \cdot g(x) \Big|_a^b$$

bo’lib, undan

$$\int_a^b f'(x)\cdot g(x)\,dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f(x)\cdot g'(x)\,dx \quad (5)$$

bo’lishi kelib chiqadi. Bu formula yordamida aniq integrallar hisoblanadi.

Yuqoridagi (5) formula aniq integrallarda bo’laklab integrallash formulasi deyilib, uni

$$\int_a^b f(x)\,dg(x) = f(x)\cdot g(x)\Big|_a^b - \int_a^b g(x)\,df(x)$$

kabi ham yozish mumkin.

Misollar.

7. *Ushbu*

$$\int_0^1 xe^x dx$$

integral hisoblansin.

◀Bu integralda

$$f(x) = x, \qquad dg(x) = e^x dx$$

deymiz. U holda

$$df(x) = f'(x)\,dx = (x)'\,dx = dx,$$
$$g(x) = \int e^x dx = e^x$$

bo’lib, (5) formulaga ko’ra

$$\int_0^1 xe^x dx = xe^x\Big|_0^1 - \int_0^1 e^x dx$$

bo’ladi. Ravshanki,

$$xe^x\Big|_0^1 = 1\cdot e^1 - 0\cdot e^0 = e, \qquad \int_0^1 e^x dx = e^x\Big|_0^1 = e^1 - e^0 = e - 1.$$

Demak,

$$\int_0^1 xe^x dx = e - (e-1) = 1.$$ ▶

8. *Ushbu*

$$\int_1^2 \ln x dx$$

integral hisoblansin.

◀Bu integralda

$$f(x) = \ln x, \qquad dg(x) = dx$$

deymiz. U holda

$$df(x) = f'(x)dx = (\ln x)'dx = \frac{1}{x}dx, \qquad g(x) = x$$

bo’lib, (5) formulaga ko’ra

$$\int_1^2 \ln x dx = x\ln x\Big|_1^2 - \int_1^2 x\cdot\frac{1}{x}dx = 2\ln 2 - 1\cdot\ln 1 -$$
$$-\int_1^2 dx = 2\ln 2 - (x)\Big|_1^2 = 2\ln 2 - (2-1) = 2\ln 2 - 1$$

bo’ladi.▶

9. *Ushbu*

$$\int_0^1 x^3 arctgx dx$$

integral hisoblansin.

◀Bu integralda

$$f(x) = arctgx, \qquad dg(x) = x^3dx$$

deymiz. U holda

$$df(x) = f'(x)dx = (arctgx)'dx = \frac{1}{1+x^2}dx,$$
$$g(x) = \int x^3dx = \frac{1}{4}x^4$$

bo’lib, (5) formulaga ko’ra

$$\int_0^1 x^3 arctgx dx = \frac{1}{4}x^4\cdot arctgx\Big|_0^1 - \frac{1}{4}\int_0^1 \frac{x^4}{1+x^2}dx$$

bo’ladi. Ravshanki,

$$\frac{1}{4}x^4 arctgx\Big|_0^1 = \frac{1}{4}arctg1 - \frac{1}{4}\cdot 0\cdot arctg0 = \frac{1}{4}\cdot\frac{\pi}{4} - 0 = \frac{\pi}{16}\ .$$

Keyingi integral quyidagicha hisoblanadi:

$$\int_0^1 \frac{x^4}{1+x^2}dx = \int_0^1 \frac{(x^4-1)+1}{1+x^2}dx = \int_0^1 \left[\frac{(x^2+1)(x^2-1)}{1+x^2} + \frac{1}{1+x^2}\right]dx =$$

$$= \int_0^1 (x^2-1)dx + \int_0^1 \frac{dx}{1+x^2} = (\frac{x^3}{3} - x)\Big|_0^1 + arctgx\Big|_0^1 = \frac{1}{3} - 1 + arctg1 = -\frac{2}{3} + \frac{\pi}{4}.$$

Demak,

$$\int_0^1 x^3 arctgxdx = \frac{\pi}{16} - \frac{1}{4}\left(-\frac{2}{3} + \frac{\pi}{4}\right) = \frac{\pi}{16} + \frac{1}{6} - \frac{\pi}{16} = \frac{1}{6}\ .\blacktriangleright$$