O'ZGARMAS KOEFFITSIYENTLI CHIZIQLI BIR JINSLI DIFFERENSIAL TENGLAMALAR SISTEMASINI LAGRANJ VA KOSHI USULLARI BILAN YECHISH

Nurmamatov M.Q.
Mutaxassislik: Amaliy matematika va axborot texnologiyalari (2-bosqich)

Maqolada o'zgarmas koeffitsiyentli chiziqli birjinsli differensial tenglamalar sistemasini yechishda Lagraj va Koshi usullarining afzalli tomonlari yoritilib berilgan shu biln birga, bu usullar yordamida turdosh masalalarni shu tartibda yechish mumkinligi aytib o'tilgan.

O'zgarmas koeffitsiyentli chiziqli birjinsli differensial tenglamalar sistemasi deb quyidagi sistemaga aytiladi:

$$y_i' = \sum_{k=1}^{n} a_{ik} y_k + f_i(t), \qquad i = \overline{1,n} \tag{1}$$

buni vektor shaklida yozsak,

$$y' = Ay + f, \tag{2}$$

$$y = (y_1, ..., y_n)^T, \quad f = (f_1, ..., f_n)^T$$

bu yerda a_{ik}, $i,k = \overline{1,n}$ – o'zgarmaslar; f – $[0,T]$ intervalda uzluksiz funksiya.

Bizga ma'lumki, chiziqli birjinsli sistema $y' = Ay$ uchun yechimlarning fundamental sistemasini qurish mumkin (buning uchun xarakteristik tenglamaning ildizlarini topish lozim), u holda (2) sistemani integrallash uchun uning xususiy yechimini topish yetarli. Ana shu (2) sistemaning xususiy yechimini qurish o'zgarmaslarni variatsiyalash (yoki Lagranj) usuli va Koshi usuli deb ataladi. Koshi usulida o'zgarmas koeffitsiyentli (2) uchun Koshi matritsasi faqat $t–\tau$ dan bog'liq bo'ladi: $K(t,\tau) = K = (t-\tau)$. (2) sistemaning $t = t_0$ nuqtada boshlang'ich shartni qanoatlantiruvchi yechimi quyidagi formuladan topiladi [1]:

$$\tilde{y}(t) = \int_{t_0}^{t} K(t-\tau) f(\tau) d\tau . \tag{3}$$

Agar (2) sistemaning ixtiyoriy xususiy yechimini topish zarurati tug'ilsa, u holda (3) ni quyidagicha yozamiz (simvolik ravishda): $\tilde{y}(t) = \int^{t} K(t-\tau) f(\tau) d\tau$. Agar (3) sistemaga quyi chegarani $\tau = t_0$ deb qo'ysak, u holda birjinsli sistemaning biror yechimiga ega bo'lamiz. Agarda bir jinsli $y' = Ay$ sistema uchun Koshi matritsasini $K(t,\tau) = W(t)W^{-1}(\tau)$ kabi yozish mumkin bo'lsa, u holda (2) sistemaning xususiy yechimi ushbu $\tilde{y}(t) = W(t)\int_{t_0}^{t} W^{-1}(\tau) f(\tau) d\tau$ formuladan topiladi [1].

Misol. Birjinsli bo'lmagan ushbu

$$x' = y, \qquad y' = -x + \frac{1}{\sin t} \tag{4}$$

sistemaning umumiy yechimi topilsin.

Yechish. Berilgan sistemaning xarakteristik tenglamasini tuzamiz:

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0,$$

uning yechimlar $\lambda_{1,2} = \pm i$. (4) sistema birjinsli qismining umumiy yechimi quyidagicha:

$$y = \begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \tag{5}$$

(4) sistemaning $\tilde{y}$ - xususiy yechimini ikki xil usul: Lagraj va Koshi usullari bilan topamiz.

1-usul (*o'zgarmasni variatsiyalash yoki Lagranj usuli*). (4) ning umumiy yechim (5) ekanligini bilgan holda $\tilde{y}$ - xususiy yechimini ushbu

$$\tilde{y} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = C_1(t) \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2(t) \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \tag{6}$$

ko'rinishda izlaymiz. (6) ni (4) ga qo'yib, $C_1'(t)$ va $C_2'(t)$ lar uchun quyidagi algebraic sistemani hosil qilamiz:

$$C_1'(t)\cos t + C_1'(t)\sin t = 0,$$
$$-C_1'(t)\sin t + C_1'(t)\cos t = \frac{1}{\sin t},$$

bu yerdan $C_1'(t) = -1$, $C_2'(t) = \dfrac{\cos t}{\sin t}$ va $C_1(t) = -t + C_1$, $C_2(t) = \ln|\sin t| + C_2$. Buni (4) ga qo'ysak,

$$y = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} -t\cos t + \sin t \cdot \ln|\sin t| \\ t\sin t + \cos t \cdot \ln|\sin t| \end{pmatrix}. \tag{7}$$

Bu yerda oxirgi qo'shiluvchi (4) sistemaning biror xususiy yechimi.

2-usul (*Koshi usuli*). Birjinsli bo'lmagan (4) sistemaning xususiy yechimini (3) dan topamiz. Koshining $K(t-\tau)$ matritsasini topish uchun ushbu

$$\frac{dK(t)}{dt} = AK(t), \qquad K(0) = E$$

matritsali Koshi masalasini yechishimiz lozim bo'ladi, yani $y = (x, y)^T$ vektor uchun ikkita vektorli masalani yechamiz:

$$\begin{matrix} x' = y, & x(0) = 1, \\ y' = -x, & y(0) = 0 \end{matrix} \quad \text{va} \quad \begin{matrix} x' = y, & x(0) = 0, \\ y' = -x, & y(0) = 1 \end{matrix}$$

va bu sitemaning yechimlarida t ni $t-\tau$ bilan almashtirib, $y_1(t-\tau)$ va $y_2(t-\tau)$ vektorlardan $K(t-\tau)$ matritsaning ustuni hosil qilinadi. Ushbu

$$W(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

fundamental matritsa Koshi matritsasining zaruriy xossalarini o'zida mujassamlashtirgan va $t = 0$ da u birlik matritsaga aylanadi, undan ushbu

$$K(t-\tau)=\begin{pmatrix}\cos(t-\tau) & \sin(t-\tau)\\ -\sin(t-\tau) & \cos(t-\tau)\end{pmatrix}$$

Koshi matritsasini tuzib olishimiz mumkin. (4) sistemaning xususiy yechimi:

$$\tilde{y}=\int_{t_0}^{t}\begin{pmatrix}\cos(t-\tau) & \sin(t-\tau)\\ -\sin(t-\tau) & \cos(t-\tau)\end{pmatrix}\begin{pmatrix}0\\ \dfrac{1}{\sin\tau}\end{pmatrix}d\lambda=\int_{t_0}^{t}\begin{pmatrix}\dfrac{\sin(t-\tau)}{\sin\tau}\\ \dfrac{\cos(t-\tau)}{\cos\tau}\end{pmatrix}dt=$$

$$=\begin{pmatrix}\sin t\displaystyle\int_{t_0}^{t}\frac{\cos\tau}{\sin\tau}d\tau-\cos t\int_{t_0}^{t}d\tau\\ \cos t\displaystyle\int_{t_0}^{t}\frac{\cos\tau}{\sin\tau}d\tau+\sin t\int_{t_0}^{t}d\tau\end{pmatrix}=\begin{pmatrix}\sin t\ln|\sin\tau|\Big|_{t_0}^{t}-(t-t_0)\cos t\\ \cos t\ln|\sin\tau|\Big|_{t_0}^{t}+(t-t_0)\sin t\end{pmatrix}.$$

Bu yechim $t=t_0$ da boshlang'ich shartni qanoatlantiradi, ammo u (7) ning uchinchi qo'shiluvchisi bilan mos tushmaydi. Integralni hisoblash qulay bo'lsin uchun Koshi matritsasini $K(t-\tau)=W(t)W^{-1}(\tau)$ kabi yozib olish qulay. (4) sistema uchun

$$K(t-\tau)=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}\cos\tau & -\sin\tau\\ -\sin\tau & \cos\tau\end{pmatrix}$$

va

$$\tilde{y}=\int_{t_0}^{t}K(t-\tau)f(\tau)d\tau=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\int_{t_0}^{t}\begin{pmatrix}\cos\tau & -\sin\tau\\ -\sin\tau & \cos\tau\end{pmatrix}\begin{pmatrix}0\\ \dfrac{1}{\sin\tau}\end{pmatrix}d\tau .$$

Ana shu holda integralning natijasiga ko'ra (4) dagi xususiy yechim kelib chiqadi.

Shunday qilib, o'zgarmas koeffitsiyentli chiziqli birjinsli differensial tenglamalar sistemasi Lagraj va Koshi usullari bilan yechildi. Har ikkala usul bilan olingan natijalar bir xil. Bu foydalanilgan usullarning ishonchli va samarali ekanligini bildiradi. Bu usullar yordamida turdosh masalalarni shu tartibda yechish mumkin.

Adabiyot

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