PANJARADAGI LAPLAS OPERATORI UCHUN INVARIANT QISM FAZOLAR

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Ushbu maqolada bir o‘lchamli panjarada ikki zarrachali sistemaga mos Laplas operatorini invariant qism fazolar yordamida qaraymiz

Kirish. Hilbert fazolasida berilgan o‘z-o‘ziga qo‘shma operatorlarning spektrini o‘rganishda uchraydigan eng sodda masalalardan biri – bu invariant elementlarni, hech bo‘lmaganda bu operator ta’sirida o‘zining yo‘nalishini saqlab qoluvchi elementlarni topishdan, ya’ni

$$Hf = \lambda f \tag{1}$$

tenlamaning yechimlarini topishdan iborat. (1) tenglamaning noldan farqli har bir yechimi H operatorlarning xos vektori, λ - esa H operatorning xos qiymati deyiladi.

Panjaradagi ikki zarrachali sistema Hamiltonianining ba’zi spektral xossalari [1] ishda tahlil qilingan. [2] ishda panjaradagi ikki zarrachali Hamiltonian $H(k)$ ning invariant qism (xos qism) fazolari chekli bo‘lishining yetarli shartlari keltirilgan. [3] ishda esa ikki o‘lchamli panjarada berilgan ikki zarrachali sistema Hamiltonianining kvaziimpuls koordinatalaridan biri $k^{(1)} = \pi$ yoki $k^{(2)} = \pi$ bo‘lganda cheksiz ko‘p invariant qism fazolarga ega ekanligi isbotlangan. Ikki o‘lchamli panjarada berilgan ikki zarrachali Shryodinger operatorining invariant qism fazolari [4] ishda o‘rganilgan.

Bu ishda bir o‘lchamli panjarada ikki zarrachali sistemaga mos Laplas operatorini qaraymiz. Toq funksiyalardan tashkil topgan $L_2^-(T)$ va juft funksiyalardan iborat $L_2^+(T)$ qism fazolar bu operator uchun invariant qism fazolar bo‘lishligi isbotlangan.

Masalaning qo‘yilishi. Bir o‘lchamli panjara Z da ikki zarrachali sistemaga mos Laplas operatori $\hat{L}$ o‘z-o‘ziga qo‘shma chegaralangan operator bo‘lib, u kvadrati bilan jamlanuvchi funksiyalarning $\lambda_2(Z \times Z)$ Hilbert fazosida quyidagi formula yordamida beriladi

$$\hat{L} = -\frac{1}{2m_1}\Delta_1 - \frac{1}{2m_2}\Delta_2,$$

bu yerda m_1, m_2 zarrachalarning massalari bo‘lib, ularni biz birga teng deb hisoblaymiz, $\Delta_1 = \Delta \otimes I$ va $\Delta_2 = I \otimes \Delta$, Δ – ayirmali operator bo‘lib, u zarrachaning bir tugundan boshqa tugunga o‘tishini ifodalaydi, biz uni quyidagicha aniqlaymiz:

$$(\Delta\psi)(x) = \psi(x+2) + \psi(x-2), \quad \psi \in l_2(Z).$$

Bunday aniqlangan Laplasian $\lambda_2(Z \times Z)$ Hilbert fazosida o‘z-o‘ziga qo‘shma chegaralangan operator bo‘ladi. Ikki zarrachali sistema Laplasianining koordinat ko‘rinishidan uning impuls ko‘rinishiga o‘tish Furye almashtirishi deb ataluvchi akslantirish orqali amalga oshiriladi.

$$F : L_2(T^2) \to \lambda_2(Z^2), \quad (Ff)(n,m) = \frac{1}{2\pi}\int_{T^2} e^{int+ims} f(t,s)\,dt\,ds,$$

bu yerda $L_2(T^2) = L_2(T) \otimes L_2(T)$, $L_2(T)$ – sonlar o‘qida aniqlangan 2π davrli va $T = [-\pi, \pi]$ da kvadrati bilan Lebeg ma’nosida integrallanuvchi funksiyalarning Hilbert fazosi.

Ikki zarrachali Laplasianning impuls tasviri o‘z-o‘ziga qo‘shma chegaralangan operator sifatida $L_2(T^2)$ Hilbert fazosida quyidagicha aniqlanadi

$$L = F^{-1}\hat{L}F, \quad (Lf)(k_1,k_2) = (\varepsilon(k_1) + \varepsilon(k_2)) f(k_1,k_2),$$

$\varepsilon(q) = \cos 2q$ - alohida olingan zarrachaning $q \in T$ momentdagi impulsi.

Ikki zarrachali Laplasian L unitar operatorlar gruppasi

$$(U_s f)(k_1,k_2) = e^{-is(k_1+k_2)} f(k_1,k_2), \qquad f \in L_2(T^2)$$

bilan o‘rin almashinuvchi bo‘lganligi uchun u $\{L(k),\, k \in T\}$ operatorlar oilasining to‘g‘ri integraliga yoyiladi [3-4] , ya’ni $L = \int_T \oplus L(k)dk$, bu yerda

$$L(k) : L_2(T) \to L_2(T), \tag{2}$$

$$(L(k)f)(p) = \varepsilon_k(p) f(p), \qquad \varepsilon_k(p) = 2\cos k \cos 2p, \tag{3}$$

Shuni aytib o‘tamizki, $L(k)$ operator har bir $k \in T$ da $L_2(T)$ fazoda chegaralangan va o‘z-o‘ziga qo‘shma operatordir.

$L(k)$ **operatorning invariant qism fazolari.** $L_2(T)$ fazoni toq funksiyalardan va juft funksiyalardan iborat

$$L_2^-(T)=\{f\in L_2(T):\, f(-p)=-f(p)\} \quad \text{va} \quad L_2^+(T)=\{f\in L_2(T):\, f(-p)=f(p)\}$$

qism fazolarning to‘g‘ri yig‘indisiga yoyish mumkin, ya’ni $L_2(T)=L_2^-(T)\oplus L_2^+(T)$.

1-teorema. $L_2^-(T)$ va $L_2^+(T)$ qism fazolar $L(k)$ operatorning invariant qism fazolari bo‘ladi.

Isbot. $L_2^-(T)$ fazodan ixtiyoriy f^- element olamiz. Bu elementga $L(k)$ operator ta’sirini qaraymiz: $(L(k)f^-)(p)=2\cos k\cos 2p\, f^-(p)$. f^- toq funksiya bo‘lganligi uchun uning juft funksiya $2\cos k\cos 2p$ ga ko‘paytmasi toq funksiya bo‘ladi. Demak, ixtiyoriy $f^-\in L_2^-(T)$ uchun $L(k)f^-\in L_2^-(T)$ ekan. Bu esa $L_2^-(T)$ qism fazoning $L(k)$ operatorga nisbatan invariant qism fazo ekanligini bildiradi. Juft funksiyalar ko‘paytmasi juft ekanligidan, ixtiyoriy $f^+\in L_2^+(T)$ uchun

$$(L(k)f^+)(p)=2\cos k\cos 2p\, f^+(p)$$

ning juft funksiya ekanligini olamiz, yani $L(k)f^+\in L_2^+(T)$. 1-teorema isbot bo’ldi.

Navbatdagi invariant qism fazolarni bayon qilish uchun biz $L_2^-(T)$ va $L_2^+(T)$ fazodagi ortonormal bazis $\left\{\varphi_n^-(t)=\frac{1}{\sqrt{\pi}}\sin nt\right\}_{n=1}^{\infty}$ va $\left\{\varphi_0^+(t)=\frac{1}{\sqrt{2\pi}},\ \varphi_n^+(t)=\frac{1}{\sqrt{\pi}}\cos nt\right\}_{n=1}^{\infty}$ lardan foydalanamiz. $L_2^-(T)$ va $L_2^+(T)$ azolarning har birini quyidagicha to‘g‘ri yig‘indiga yoyamiz:

$$L_2^-(T)=L_2^{-t}(T)\oplus L_2^{-j}(T) \quad \text{va} \quad L_2^+(T)=L_2^{+t}(T)\oplus L_2^{+j}(T).$$

Bu yerda $L_2^{-t}(T)$ bilan $\{\varphi_1^-,\varphi_3^-,\ldots,\varphi_{2n-1}^-,\ldots\}$ sistemadan hosil bo‘lgan qism fazo, $L_2^{-j}(T)$ bilan $\{\varphi_2^-,\varphi_4^-,\ldots,\varphi_{2n}^-,\ldots\}$ sistemadan hosil bo‘lgan qism fazo belgilangan.

2-teorema. $L_2^{-t}(T)$ va $L_2^{-j}(T)$ qism fazolar $L(k)$ operator uchun invariant qism fazolar bo‘ladi.

Isbot. $L_2^{-t}(T)$ va $L_2^{-j}(T)$ qism fazolarning $L(k)$ operator uchun invariant qism fazolar bo‘lishini ko‘rsatamiz. Buning uchun $L_2^{-t}(T)$ fazoning ixtiyoriy φ_{2n-1}^- bazis elementiga $L(k)$ operatorning ta’sirini qaraymiz:

$$(L(k)\varphi^{-}_{2n-1})(p)=\varepsilon_k(p)\varphi^{-}_{2n-1}(p)=\frac{2}{\sqrt{\pi}}\cos k\cos 2p\sin(2n-1)p\ .$$

Ko‘paytmani yig‘indiga keltirish

$$2\cos 2p\sin(2n-1)p=\sin[(2n-1)-2]p+\sin[(2n-1)+2]p=\sin(2n-2-1)p+\sin(2n+2-1)p$$

formulasidan foydalanib uni

$$(L(k)\varphi^{-}_{2n-1})(p)=\varepsilon_k(p)\varphi^{-}_{2n-1}(p)=\cos k[\varphi^{-}_{2n-2-1}(p)+\varphi^{-}_{2n+2-1}(p)] \qquad (5)$$

shaklda yozish mumkin, ya’ni $L(k)\varphi^{-}_{2n-1}\in L_2^{-t}(T)$ ekanligini olamiz. Bu yerda $n=1$ bo‘lgan holda

$$\varphi^{-}_{2n-2-1}(p)=\varphi^{-}_{-1}(p)=\frac{1}{\sqrt{\pi}}\sin(-p)=-\frac{1}{\sqrt{\pi}}\sin p=-\varphi^{-}_{1}(p)\in L_2^{-t}(T)$$

ekanligini hisobga olish kerak. (5) dan istalgan $f^{-}=\sum_{n=1}^{\infty}c_n\varphi^{-}_{2n-1}\in L_2^{-t}(T)$ uchun

$$(L(k)f^{-})(p)=\varepsilon_k(p)f^{-}(p)=\cos k\sum_{n=1}^{\infty}c_n[\varphi^{-}_{2n-2-1}(p)+\varphi^{-}_{2n+2-1}(p)]\in L_2^{-t}(T)$$

ekanligini olamiz. Shunday qilib $L_2^{-t}(T)$ qism fazo $L(k)$ operator uchun invariant qism fazo ekan. $L_2^{-j}(T)$ qism fazoning $L(k)$ operator uchun invariant qism fazo ekanligini shunga o’xshash isbotlanadi.

Foydalanilgan adabiyotlar

1. P P.A. Faria da Viega, L.Ioriatti and M.O‘Carrol. Energy-momentum spectrum of some two-particle lattice Schrodinger Hamiltonian. Phys. Rev. E(3) 66, 016130, 9 pp. (2002).

2. J.I. Abdullaev and I.A. Ikromov. Finitness of the number of eigenvalues of the two-particle Shrodinger operator on a lattice. Theoretical and Mathematical Physics, 152(3): 1299-1312 (2007).

3. Абдуллаев Ж.И., Собственные значения двухчастичного оператора Шредингера на двумерной решетке, Узбекский Математический Журнал, № 1, 2005. С.3-11.

4. Абдуллаев Ж.И., Файзиев М.Ш., Шотемиров Й.С. Инвариантные подпространства двухчастичного оператора Шредингера на решетке, Узбекский Математический Журнал, № 3, 2009. С.3-10.