II BOB
NOSTANDART TENGLAMALAR, TENGSIZLIKLAR VA ULARNING SISTEMALARI

1-§. Nostandart tenglamalar

Tashqi ko‘rinishi odatdagi tenglamalardan keskin farq qiladigan tenglamalar (masalan, $2^{|x|} = \cos x$, $x^2 + 4x\cos(xy) + 4 = 0$), shuningdek, tashqi ko‘rinishi odatdagi tenglamalarga o‘xshaydigan, lekin odatdagi usullar bilan yechish mumkin bo‘lmaydigan tenglamalar (masalan, $\sin 7x + \cos 2x = -2$, $\sin^4 x - \cos^7 x = 1$ va hokazo) ham uchraydi. Bunday tenglamalarni *nostandart tenglamalar* deb ataymiz.

Nostandart tenglamalarni yechishning umumiy usuli mavjud emas. Shu sababli bunday tenglamalarni yechishda funksiyalarning grafiklaridan, turli xossalaridan, tengsizliklardan va hokazolardan foydalanishga to‘g‘ri keladi. Buni misollarda qarab chiqamiz.

1 - m i s o l. $2^{|x|} = \cos x$ tenglamani yechamiz.

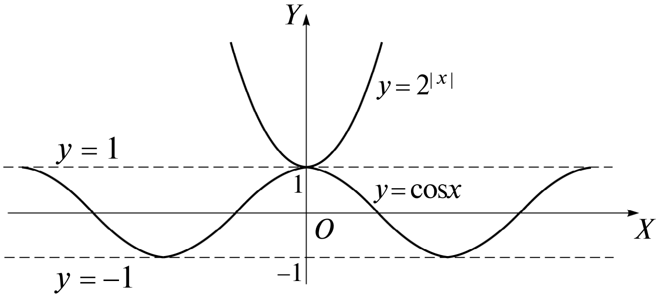
Y e c h i s h. $x = 0$ son tenglamaning yechimi ekanini ko‘ramiz (II.1-rasm).

Barcha $x \neq 0$ sonlar uchun $2^{|x|} > 1 \geq \cos x$ bo‘lgani uchun berilgan tenglama $x = 0$ dan boshqa yechimlarga ega emas.

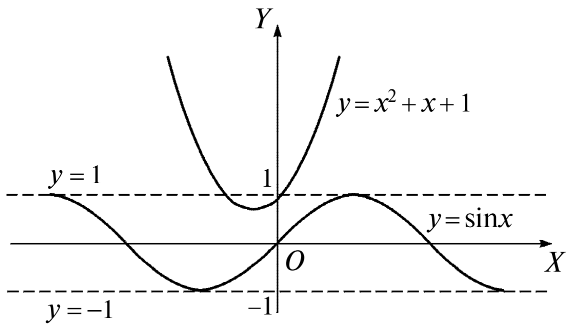
2 - m i s o l. $\sin x = x^2 + x + 1$ tenglamani yechamiz.

Y e c h i s h. II.2-rasmda $y = \sin x$ va $y = x^2 + x + 1$ funksiyalarning grafiklari tasvirlangan.

$x \in [-1; 0]$ bo‘lsa, $\sin x \leq 0$, $x^2 + x + 1 > 0$ bo‘lgani uchun tenglamaning $[-1; 0]$ oraliqqa tegishli yechimi mavjud emas.



II.1-rasm.



II.2-rasm.

$x \notin [-1;\ 0]$ lar uchun $\sin x \leq 1$, $x^2 + x + 1 > 0$ bo‘lgani sababli tenglama $[-1;\ 0]$ dan tashqarida ham yechimga ega emas. Demak, berilgan tenglama yechimga ega emas.

Eslatma. 1-misoldagi tenglamaning (shuningdek, 2-misoldagi tenglamaning ham) yechilishini bayon etishda grafiklarni chizish shart emas edi. Grafiklarni chizish esa tenglamani yechish usulini topish imkonini berdi. Quyida keltiriladigan misollarda u yoki bu funksiyaning grafigini chizish katta qiyinchiliklar tug‘diradi. Shu sababli, bu misollarning yechilishini grafik chizish bilan bog‘lash maqsadga muvofiq emas.

3-misol. $2\cos\frac{x}{3} = 3^x + 3^{-x}$ tenglamani yechamiz.

Yechish. Barcha $x \in R$ lar uchun

$$2\cos\frac{x}{3} \leq 2, \quad 3^x + 3^{-x} \geq 2$$

tengsizliklarga egamiz. Shu sababli, berilgan tenglama $\begin{cases} 2\cos\frac{x}{3} = 2, \\ 3^x + 3^{-x} = 2 \end{cases}$ tenglamalar sistemasiga teng kuchlidir. Bu sistemaning ikkinchi tenglamasi $x = 0$ dan iborat yagona yechimga ega. $x = 0$ son sistemaning birinchi tenglamasini ham qanoatlantiradi. Shuning uchun, $x = 0$ sistemaning va berilgan tenglamaning ham yagona yechimi bo‘ladi.

4-misol. $2\sqrt{2}(\sin x + \cos x)\cos y = 3 + \cos 2y$ tenglamani yechamiz.

Y e c h i sh. $\sin x + \cos x = \sqrt{2}\sin\left(\frac{\pi}{4}+x\right)$ bo‘lgani uchun berilgan tenglamani $2\sin\left(\frac{\pi}{4}+x\right)\cos y = 1+\cos^2 y$ ko‘rinishda yozib olish mumkin. Oxirgi tenglama $\left(\cos y - \sin\left(\frac{\pi}{4}+x\right)\right)^2 + \left(1-\sin^2\left(\frac{\pi}{4}+x\right)\right) = 0$ yoki $\left(\cos y - \sin\left(\frac{\pi}{4}+x\right)\right)^2 + \cos^2\left(\frac{\pi}{4}+x\right) = 0$ tenglamaga teng kuchlidir. Hosil qilingan tenglama $\cos y = \sin\left(\frac{\pi}{4}+x\right)$, $\cos\left(\frac{\pi}{4}+x\right)=0$ tengliklar o‘rinli bo‘lgandagina to‘g‘ri tenglikka aylanadi, boshqacha qilib aytganda, u

$$\begin{cases}\cos y = \sin\left(\frac{\pi}{4}+x\right),\\ \cos\left(\frac{\pi}{4}+x\right)=0\end{cases}$$

tenglamalar sistemasiga teng kuchlidir.

Sistemaning ikkinchi tenglamasi $x_1 = \frac{\pi}{4}+2m\pi$ va $x_2 = \frac{5\pi}{4}+2n\pi$ (m, $n\in Z$) yechimlarga ega. Birinchi tenglamadan $y_1 = 2k\pi$, $y_2 = (2p+1)\pi$ larni topamiz (bu yerda k, $p\in Z$).

Shunday qilib, berilgan tenglama $x_1 = \frac{\pi}{4}+2m\pi, y_1 = 2k\pi$, ($m$, $k\in Z$) va $x_2 = \frac{5\pi}{4}+2n\pi$, $y_2 = (2p+1)\pi$, (n, $p\in Z$) yechimlarga ega.

5 - m i s o l . $\sin 7x + \cos 2x = -2$ tenglamani yechamiz.

Y e c h i s h . Barcha $x\in R$ lar uchun $\sin 7x \geq -1$, $\cos 2x \geq -1$ bo‘lgani sababli berilgan tenglama $\begin{cases}\sin 7x = -1,\\ \cos 2x = -1\end{cases}$ sistemaga teng kuchlidir. Birinchi tenglama $x = -\frac{\pi}{14}+\frac{2\pi k}{7}$, $k\in Z$ yechimlar guruhiga, ikkinchi tenglama esa $x = \frac{\pi}{2}+\pi n$, $n\in Z$ yechimlar guruhiga ega. Har ikki guruhga tegishli x largina sistemaning yechimi bo‘la oladi. Ularni aniqlaymiz:

$$-\frac{\pi}{14}+\frac{2\pi k}{7}=\frac{\pi}{2}+\pi n \ (k,\ n\in Z).$$

Bundan, $k = 2 + 3{,}5n$ ekanligini topib, $k\in Z$ ekanini e'tiborga olsak, $n = 2p$ ($p\in Z$) bo'lishi kelib chiqadi.

Demak, $x=\frac{\pi}{2}+2\pi p$ ($p\in Z$) sonlargina sistemaning, binobarin, berilgan tenglamaning ham yechimlari bo'ladi.

6 - m i s o l. $\log_2^2 x+(x-1)\log_2 x-6+2x=0$ tenglamani yechamiz.

Y e c h i s h. Tenglamaning chap tomonini $t=\log_2 x$ ga nisbatan kvadrat uchhad sifatida qarab, odatdagi standart usulda ko'paytuvchilarga ajratamiz:

$$(\log_2 x+2)(\log_2 x+x-3)=0.$$

Bu tenglama $\log_2 x=-2$ va $\log_2 x=3-x$ tenglamalarga ajraladi. Ularning birinchisi $x=\frac{1}{4}$ dan iborat yagona yechimga ega. $\log_2 x=$ $=3-x$ tenglama esa $x=2$ dan iborat yagona yechimga ega.

Haqiqatan ham, $x>2$ bo'lganda $\log_2 x>\log_2 2=1>3-x$ tengsizlikka, $0<x<2$ bo'lganda esa $\log_2 x<\log_2 2=1<3-x$ tengsizlikka egamiz. $x\le 0$ da esa tenglama ma'noga ega emas.

Demak, berilgan tenglama $x=\frac{1}{4}$ va $x=2$ yechimlarga ega.

M a s h q l a r

Tenglamani yeching.

2.1. $2^{|x|}=\sin x$.

2.2. $2\cdot 3^{|x|}=2\cos\frac{x}{4}$.

2.3. $2\sin x=5x^2+2x+3$.

2.4. $4\cdot 2^{|x|-1}=\sin(\pi\sqrt{x})+4$.

2.5. $\cos x+\cos y-\cos(x+y)=1{,}5$.

2.6. $\left(\sin^2 x+\frac{1}{\sin^2 x}\right)^2+\left(\cos^2 x+\frac{1}{\cos^2 x}\right)^2=12+\frac{1}{2}\sin y$.

2.7. $\mathrm{tg}^4 x+\mathrm{tg}^4 y+2\,\mathrm{ctg}^2 x\cdot\mathrm{ctg}^2 y=3+\sin^2(x+y)$.

2.8. $8-x\cdot 2^x+2^{3-x}-x=0$.

2.9. $x\cdot 2^x=x(3-x)+2(2^x-1)$.

2.10. $\log_2\left(\cos^2 xy+\frac{1}{\cos^2 xy}\right)=\frac{1}{y^2-2y+2}$.

2.11. $\sqrt{\frac{x^2-x-2}{\sin x}}+\sqrt{\frac{\sin x}{x^2-x-2}}=1{,}5$.

2.12. $\sin^4 x+\cos^{15} x=1$.

2.13. $\sqrt{\sin^3 x}+\sqrt{\cos^3 x}=\sqrt{2}$.

2.14. $\sqrt{2+\cos^2 2x}=\sin 3x-\cos 3x$.

2.15. $\sqrt{5+\sin^2 3x}=\sin x+2\cos x$.

2-§. Nostandart tengsizliklar

Oldingi bandda nostandart tenglamalar qaraldi. Nostandart tenglamada tenglik belgisi tengsizlik belgisi bilan almashtirilsa, nostandart tengsizlik deb ataluvchi tengsizlik hosil bo‘ladi.

Nostandart tenglamalarni yechishning umumiy usuli mavjud bo‘lmagani kabi, nostandart tengsizliklarni yechishning ham umumiy usuli mavjud emas. Shu sababli, nostandart tengsizliklarni yechishda ham shu tengsizlikka xos bo‘lgan chuqur mantiqiy fikr yuritishga to‘g‘ri keladi.

Nostandart tengsizliklarni yechishga doir ayrim misollar bilan tanishaylik.

1 - m i s o l. $\cos x\ge y^2+\sqrt{y-x^2-1}$ tengsizlikni yechamiz.

Y e c h i s h. $x=u,\ y=v$ sonlaridan tuzilgan (u; v) juftlik berilgan tengsizlikning yechimi bo‘lsin. U holda quyidagi to‘g‘ri sonli tengsizlikka ega bo‘lamiz:

$$\cos u\ge v^2+\sqrt{v-u^2-1} \tag{1}$$

(1) tengsizlikda ildiz ostidagi ifoda nomanfiy sondir, ya’ni $v-u^2-1\ge 0$ dir. Shu sababli,

$$v\ge u^2+1, \tag{2}$$

$$v\ge 1, \tag{3}$$

tengsizliklar to‘g‘ridir.

$\sqrt{v-u^2-1} \geq 0$ ekanligini e'tiborga olib, (1) va (3) tengsizliklardan

$$\cos u \geq v^2 + \sqrt{v-u^2-1} \geq v^2 \geq 1$$

tengsizlikni hosil qilamiz. Biroq $\cos u \leq 1$. Shu sababli quyidagi munosabatlar o'rinlidir:

$$1 \geq \cos u \geq v^2 + \sqrt{v-u^2-1} \geq v^2 \geq 1. \tag{4}$$

Bu esa $1 = \cos u = v^2 + \sqrt{v-u^2-1} = v^2 = 1$ ekanligini ko'rsatadi.

Oxirgi tenglikdan, (3) tengsizlikni e'tiborga olsak, $v = 1$, $u = 0$ ekanligi kelib chiqadi.

Yuqoridagi mulohazalar, (0; 1) juftlikdan boshqa juftliklar berilgan tengsizlikning yechimi bo'la olmasligini va (0; 1) juftlikgina berilgan *tengsizlikning yechimi bo'lishi mumkinligi*ni ko'rsatadi.

(0; 1) juftlik, haqiqatan ham, berilgan tengsizlikning yechimi bo'lishligini ko'rish qiyin emas. Demak, berilgan tengsizlik yagona yechimga ega: $x = 0$, $y = 1$.

2 - m i s o l. $\sin 2^{\frac{x}{\pi}} < 2^{|\sin x|}$ tengsizlikni yechamiz.

Y e c h i s h. Tengsizlikning aniqlanish sohasi R dan iborat va har qanday $x \in R$ son uchun quyidagi tengsizliklar o'rinlidir:

$$\sin 2^{\frac{x}{\pi}} \leq 1, \tag{5}$$

$$2^{|\sin x|} \geq 1. \tag{6}$$

Bu tengsizliklardan, barcha $x \in R$ uchun $\sin 2^{\frac{x}{\pi}} \leq 2^{|\sin x|}$ tengsizlik bajarilishi kelib chiqadi. Oxirgi tengsizlikning barcha yechimlari to'plami R dan $\sin 2^{\frac{x}{\pi}} = 2^{|\sin x|}$ tenglamaning barcha yechimlari chiqarib tashlansa, berilgan tengsizlikning yechimlari to'plami hosil bo'ladi.

(5) va (6) munosabatlardan ko'rinadiki, $\sin 2^{\frac{x}{\pi}} = 2^{|\sin x|}$ tenglama quyidagi sistemaga teng kuchli:

$$\begin{cases} \sin 2^{\frac{x}{\pi}} = 1, \\ 2^{|\sin x|} = 1. \end{cases} \tag{7}$$

(7) sistemaning birinchi tenglamasi $x = \pi \log_2 \left(\frac{\pi}{2} + 2\pi k\right)$, $k = 0,\ 1,\ 2,\ \ldots$ yechimlarga, ikkinchi tenglamasi esa $x = \pi n$, $n = 0$, ± 1, ± 2, ... yechimlarga ega. (7) sistemaning yechimlarini topish uchun

$$\pi \log_2 \left(\frac{\pi}{2} + 2\pi k\right) = \pi n,\ \ (k = 0,\ 1,\ 2, \ldots;\ n = 0;\ \pm 1;\ \pm 2;\ \ldots)$$

yoki

$$\frac{\pi}{2} + 2\pi k = 2^n,\ \ (k = 0,\ 1,\ 2, \ldots;\ n = 0;\ \pm 1;\ \pm 2;\ \ldots)$$

tenglamani qaraymiz. Oxirgi tenglikning chap tomoni irratsional son, o‘ng tomoni esa ratsional sondir. Shuning uchun bu tenglama va (7) sistema yechimga ega emas. Demak, $\sin 2^{\frac{x}{\pi}} = 2^{|\sin x|}$ tenglama yechimga ega emas. Bu yerdan, berilgan tengsizlik barcha $x \in R$ sonlarida bajarilishi kelib chiqadi.

3 - m i s o l. $\arcsin \frac{2}{x} + \sqrt{x-1} > 1$ tengsizlikni yechamiz.

Y e c h i s h. $\begin{cases} \left|\frac{2}{x}\right| \le 1, \\ x - 1 \ge 0 \end{cases}$ sistemani yechib, tengsizlikning aniqlanish sohasi $[2;\ +\infty)$ dan iborat ekanligini ko‘ramiz.

Barcha $x \ge 2$ larda $\sqrt{x-1} \ge 1$ va $\arcsin \frac{2}{x} > 0$ tengsizliklar to‘g‘ri bo‘lishini ko‘rish qiyin emas. Bu tengsizliklardan ko‘rinadiki, berilgan tengsizlik o‘zining aniqlanish sohasidagi barcha x lar uchun, ya’ni barcha $x \in [2;\ +\infty)$ lar uchun o‘rinli bo‘ladi.

4 - m i s o l. $\min\{2x - x^2,\ x - 1\} > -\frac{1}{2}$ tengsizlikni yechamiz.

Y e c h i s h. $\min\{2x - x^2,\ x - 1\}$ ni topib olamiz. Buning uchun, $2x - x^2 \ge x - 1$ tengsizlikning yechimlari to‘plami $\left[\frac{1-\sqrt{5}}{2};\ \frac{1+\sqrt{5}}{2}\right]$ oraliqdan iborat ekanligidan foydalanamiz.

$x \in \left[\frac{1-\sqrt{5}}{2};\ \frac{1+\sqrt{5}}{2}\right]$ larda $2x - x^2 \ge x - 1$ bo‘lgani uchun

$$\min_{x\in\left[\frac{1-\sqrt{5}}{2};\ \frac{1+\sqrt{5}}{2}\right]}\{2x-x^2,\ x-1\}=x-1,$$

$$\min_{x\in\left(-\infty;\ \frac{1-\sqrt{5}}{2}\right]}\{2x-x^2,\ x-1\}=2x-x^2 \quad \text{va}$$

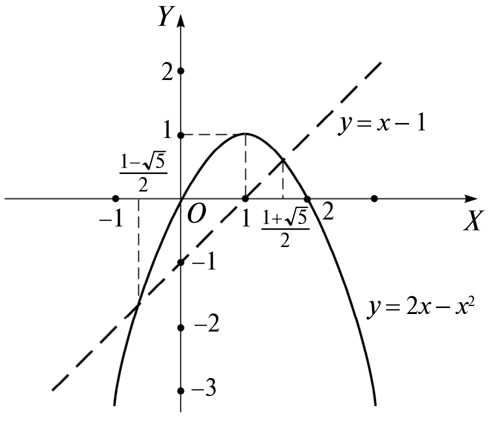
$$\min_{x\in\left[\frac{1+\sqrt{5}}{2};\ +\infty\right)}\{2x-x^2,\ x-1\}=2x-x^2$$

munosabatlar o'rinlidir (II. 3-rasmga qarang).

$$\min_{x\in R}\{2x-x^2,\ x-1\}=\begin{cases}2x-x^2,\ x\le\frac{1-\sqrt{5}}{2}\ \text{da},\\ x-1,\ \ \frac{1-\sqrt{5}}{2}\le x\le\frac{1+\sqrt{5}}{2}\ \text{da},\\ 2x-x^2,\ x\ge\frac{1+\sqrt{5}}{2}\ \text{da}\end{cases}$$

munosabatdan foydalansak, berilgan tengsizlik quyidagi majmuaga teng kuchli ekanligini ko'ramiz:

$$\left[\begin{array}{l}\begin{cases}2x-x^2>-\frac{1}{2},\\ x\le\frac{1-\sqrt{5}}{2};\end{cases}\\ \begin{cases}x-1>-\frac{1}{2},\\ \frac{1-\sqrt{5}}{2}\le x\le\frac{1+\sqrt{5}}{2};\end{cases}\\ \begin{cases}2x-x^2>-\frac{1}{2},\\ x\ge\frac{1+\sqrt{5}}{2}.\end{cases}\end{array}\right.$$



II.3-rasm.

Bu majmuani yechib, berilgan tengsizlikning barcha yechimlarini topamiz: $\frac{1}{2}<x<1+\sqrt{\frac{3}{2}}$.

Mashqlar

Tengsizlikni yeching:

2.16. $-|y|+x \geq \sqrt{x^2+y^2-1}+1$.

2.17. $-x-y^2-\sqrt{x-y^2-1} \geq -1$.

2.18. $\sqrt{\sin x}+\sqrt{\cos x}>1$.

2.19. $|x|+\sqrt{x^2-1} \geq 1$.

2.20. $\lg x+\sqrt{x^2-1} \geq 0$.

2.21. $\log_2(x^2+3)+\sqrt{x^2-1} \geq 2$.

2.22. $x+2^x \leq \sqrt{1-x}+3$.

2.23. $x+2^x+\sqrt{x-1} \geq 2+\sqrt{x}$.

2.24. $2^{\sqrt{1-x}} \geq x$.

2.25. $2^{\sqrt{1-x}}-x\lg x>0$.

2.26. $\sqrt{x-1}+\sqrt{4-x^2}+\operatorname{tg}\frac{\pi x}{4}>1$.

2.27. $\sqrt{\lg \sin x}<x-13\pi$.

2.28. $\frac{\lg(|x|+1)}{\lg(|x|-1)}+\sqrt{x^2-4}>1$.

2.29. $\cos^2(x+1)\cdot\lg(9-2x-x^2) \geq 1$.

2.30. $(4x-x^2-3)\log_2(\cos^2\pi x+1) \geq 1$.

2.31. $\cos(x+3\operatorname{tg}x)+(\operatorname{tg}x-\operatorname{tg}^2 x)^2 \leq -1$.

2.32. $\sin(\sin x) \leq \frac{\sqrt{3}}{2}$.

2.33. $[\sin x+\cos x] \leq 2^{|\cos x|}$.

2.34. $\frac{4}{3}\sin^2\{5x\} \leq 2^{1-|\sin x|}$.

2.35. $\sin\{x\}<\sin(\{x\}+1)$.

2.36. $\sqrt{x-1}+2^x+\log_2 x \geq 2$.

2.37. $\sqrt{x-1}+2^x+\log_2 x<6$.

2.38. $\sqrt{1-x}+3 \geq x+2^x$.

2.39. $\{x\}>\{2x\}$.

2.40. $\max\left\{x^2-2x,\ \frac{x}{2}-1\right\}<-\frac{1}{2}$.

3-§. Nostandart sistemalar

Nostandart sistema deyilganda qanday sistemalar nazarda tutilishi quyidagi misollardan oydinlashadi.

1 - m i s o l. $\begin{cases} 2^{x+1} = 4y^2 + 1, \\ 2^x \le 2y \end{cases}$ sistemani yechamiz.

Y e c h i s h . Sistemadagi tenglamadan $2^x = 2y^2 + \frac{1}{2}$ tenglamani hosil qilib, sistemadagi tengsizlikda 2^x ni $2y^2 + \frac{1}{2}$ bilan almashtiramiz:

$$\begin{cases} 2^x = 2y^2 + \frac{1}{2}, \\ 2y^2 + \frac{1}{2} \le 2y. \end{cases}$$

Bu sistemadagi tengsizlik $\left(y - \frac{1}{2}\right)^2 \le 0$ tengsizlikka teng kuchli bo‘lgani uchun $y = \frac{1}{2}$ dan iborat yagona yechimga egadir. $y = \frac{1}{2}$ qiymatni sistemadagi tenglamaga qo‘yib, $x = 0$ ni topamiz. Demak, berilgan sistema $\left(0;\ \frac{1}{2}\right)$ dan iborat yagona yechimga ega.

2 - m i s o l. $\begin{cases} x^2 - 2xy + 12 = 0, \\ x^2 + 4y^2 \le 60, \\ x \in Z \end{cases}$ sistemani yechamiz.

Y e c h i s h . Agar $(x;\ y)$ juftlik sistemaning yechimi bo‘lsa, u holda $(-x;\ -y)$ juftlik ham sistemaning yechimi bo‘lishligini ko‘rish qiyin emas. Shu sababli, $x \ge 0$ holni qarash yetarli.

Agar $x = 0$ bo‘lsa, sistemadagi tenglama noto‘g‘ri tenglikka aylanadi. Demak, $x > 0$ bo‘lishi zarur ($x \ge 0$ bo‘lgan hol qaralmoqda!).

Sistemadagi tenglamadan y ni topaylik: $y = \frac{x^2 + 12}{2x} = \frac{x}{2} + \frac{6}{x}$. O‘rta arifmetik va o‘rta geometrik miqdorlar orasidagi munosabatni ifodalovchi $a+b\geq 2\sqrt{ab}$ $(a\geq 0,\ b\geq 0)$ tengsizlikdan foydalanib, $y\geq 2\sqrt{\frac{x}{2}\cdot\frac{6}{x}}=2\sqrt{3}$ yoki $y^2\geq 12$ ekanligini ko'ramiz. Bunga ko'ra, sistemadagi tengsizlikdan, $x^2\leq 60-4y^2\leq 60-4\cdot 12=12$ ni, ya'ni $x^2\leq 12$ ni olamiz. $x>0$, $x\in Z$ ekanligini e'tiborga olsak, $x=1$, $x=2$, $x=3$ bo'lishi zarurligini ko'ramiz.

$x=1$ bo'lsin. U holda, tenglamadan $y=12$ ekanligini topamiz. Lekin (1; 12) juftlik uchun $x^2+4y^2\leq 60$ shart bajarilmaydi. Demak, $x=1$ soni sistemaning yechimini aniqlamaydi.

$x=3$ holda sistema yechimga ega. Tenglamadan $y=\frac{7}{2}$ ekani topiladi. Yechim $\left(3;\frac{7}{2}\right)$ va $\left(-3;-\frac{7}{2}\right)$.

$x=2$ bo'lgan holni qaraymiz. Tenglamadan $y=9$ ekanligi topiladi. (2; 9) juftlik sistemadagi qolgan shartlarni ham qanoatlantiradi. Demak, (2; 9) – yechim.

Yuqoridagi eslatilganiga asosan, (–2; –9) juftlik ham sistemaning yechimi bo'ladi.

Shunday qilib, sistema (2; 9) va (–2; –9) dan iborat ikkita yechimga ega.

3-misol. $\begin{cases}\operatorname{tg}^2 x+\operatorname{ctg}^2 x=2\sin^2 y,\\ \sin^2 y+\cos^2 z=1\end{cases}$ sistemani yechamiz.

Yechish. $\operatorname{tg}^2 x+\operatorname{ctg}^2 x\geq 2,\ 2\sin^2 y\leq 2$ bo'lgani uchun sistemadagi birinchi tenglama $\begin{cases}\operatorname{tg}^2 x+\operatorname{ctg}^2 x=2,\\ \sin^2 y=1\end{cases}$ sistemaga teng kuchlidir. Shu sababli, berilgan sistema quyidagi sistemaga teng kuchli bo'ladi:

$$\begin{cases}\operatorname{tg}^2 x+\operatorname{ctg}^2 x=2,\\ \sin^2 y=1,\\ \sin^2 y+\cos^2 z=1.\end{cases}$$

Bu sistemani quyidagicha yozib olish mumkin:

$$\begin{cases} \mathrm{tg}^2 x = 1, \\ \sin^2 y = 1, \\ \cos^2 z = 0. \end{cases}$$

Oxirgi sistemadan $x = \frac{\pi}{4} + \frac{\pi}{2}k$, $y = \frac{\pi}{2} + \pi l$, $z = \frac{\pi}{2} + \pi m$ bo‘lishligini topamiz (bu yerda $k \in Z$, $l \in Z$, $m \in Z$).

4-misol.
$$\begin{cases} y^6 + y^3 + 2x^2 = \sqrt{xy - x^2 y^2}, \\ 4xy^3 + y^3 + \frac{1}{2} \geq 2x^2 + \sqrt{1 + (2x - y)^2} \end{cases} \qquad (*)$$

sistemani yechamiz.

Yechish. $f(z) = \sqrt{z - z^2}$ funksiyani qaraymiz. Bu funksiya $\frac{1}{2}$ ga teng eng katta qiymatga ega, ya’ni ixtiyoriy z haqiqiy son uchun $f(z) = \sqrt{z - z^2} \leq \frac{1}{2}$ munosabat o‘rinlidir.

(x*; y*) juftlik berilgan sistemaning yechimi bo‘lsin. U holda $(y^*)^6 + (y^*)^3 + 2(x^*)^2 = \sqrt{x^* y^* - (x^*)^2 (y^*)^2} \leq \frac{1}{2}$ ga ega bo‘lamiz.

Demak, berilgan sistemaning har qanday (x*; y*) yechimi quyidagi sistemaning ham yechimi bo‘ladi:

$$\begin{cases} (y^*)^6 + (y^*)^3 + 2(x^*)^2 \leq \frac{1}{2}, \\ 4x^* y^* + (y^*)^3 + \frac{1}{2} \geq 2(x^*)^2 + \sqrt{1 + (2x^* - y^*)^2}. \end{cases}$$

Bu sistemaning birinchi tengsizligidan ikkinchi tengsizligini ayirsak,

$$(y^*)^6 + 2(x^*)^2 - 4x^*(y^*)^3 - \frac{1}{2} \leq \frac{1}{2} - 2(x^*)^2 - \sqrt{1 + (2x^* - y^*)^2}$$

tengsizlik va shakl almashtirishlardan so‘ng

$$((y^*)^3 - 2x^*)^2 \leq 1 - \sqrt{1 + (2x^* - y^*)^2}$$

tengsizlikni hosil qilamiz. Bu tengsizlikning chap tomoni ≥ 0, o‘ng tomoni esa ≤ 0 ekanini ko‘rish qiyin emas. Demak, bu tengsizlik

$$\begin{cases} (y^*)^3 - 2x^* = 0, \\ \sqrt{1+(2x^* - y^*)^2} = 1 \end{cases}$$

sistemaga teng kuchlidir. Oxirgi sistema quyidagi yechimlarga ega:

$$x_1^* = 0,\ y_1^* = 0;\ x_2^* = -\tfrac{1}{2},\ y_2^* = -1;\ x_3^* = \tfrac{1}{2},\ y_3^* = 1.$$

Demak, (0; 0), $\left(-\frac{1}{2};\ -1\right)$, $\left(\frac{1}{2};\ 1\right)$ juftliklargina (*) sistemaning yechimi bo‘lishi mumkin. Bu juftliklarni (*) sistemaga qo‘yib ko‘rish bilan, $\left(-\frac{1}{2};\ -1\right)$ juftlikgina (*) sistemaning yechimi bo‘lishiga ishonch hosil qilamiz.

Mashqlar

Sistemani yeching:

2.41. $\begin{cases} 3^{x+4} = y^2 + 1, \\ 3^x \le \frac{2y}{81}. \end{cases}$ **2.42.** $\begin{cases} x + y + z = 2, \\ 2xy - z^2 = 4. \end{cases}$

2.43. $\begin{cases} x + y + z = 4, \\ 2xy - z^2 = 16. \end{cases}$ **2.44.** $\begin{cases} y^2 + xy - z^2 = 4, \\ x + 5y = 8. \end{cases}$

2.45. $\begin{cases} x^2 - 2yz = -1, \\ y + z - x = 1. \end{cases}$ **2.46.** $\begin{cases} \log_2(u+v) - \log_3(u-v) = 1, \\ u^2 - v^2 = 2. \end{cases}$

2.47. $\begin{cases} 3^{\lg x} = 4^{\lg y}, \\ (4x)^{\lg 4} = (3y)^{\lg 3}. \end{cases}$ **2.48.** $\begin{cases} \cos x - \cos y = \sin(x+y), \\ |x| + |y| = \frac{\pi}{4}. \end{cases}$

2.49. $\begin{cases} 2x^2 - xy + 10 = 0, \\ x^2 + y^2 \le 90, \\ x \in Z. \end{cases}$ **2.50.** $\begin{cases} 2x^2 - xy + 9 = 0, \\ 2x^2 + y^2 \le 81, \\ x \in Z. \end{cases}$