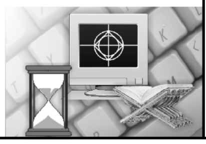


IV BOB

FUNKSIYANING LIMITI VA UZLUKSIZLIGI

1-§. Funksiyaning limiti

1. Funksiyaning nuqtadagi bir tomonlama limiti.

$$y = f(x) = \begin{cases} (x-1)^2 + 0,5, \text{ agar } x \le 1 \text{ bo‘lsa,} \\ 3 - x, \text{ agar } x > 1 \text{ bo‘lsa,} \end{cases}$$ funksiya berilgan bo‘lsin. Bu funksiyaning qiymatlar jadvalini tuzamiz va uning grafigini (IV.1-rasm) yasaymiz:

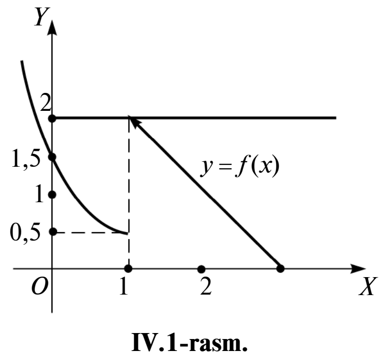
x	0	0,5	0,6	0,7	0,8	0,9	1	1,1	1,2	1,3	1,5	2	3
y	1,5	0,75	0,66	0,59	0,54	0,51	0,5	1,9	1,8	1,7	1,5	1	0

Jadval va grafikni kuzatib, x argument 1 soniga chapdan yaqinlashganda funksiyaning qiymatlari 0,5 sonidan, o‘ng tomondan yaqinlashganda esa 2 sonidan istalgancha kam farq qiladi deb tasdiqlash mumkin.

0,5 soni berilgan $y = f(x)$ funksiyaning $x = 1$ nuqtadagi chap limiti, 2 soni esa berilgan $y = f(x)$ funksiyaning $x = 1$ nuqtadagi o‘ng limiti deyiladi.

Funksiya chap va o‘ng limitlarining qat’iy matematik ta’rifini beramiz. Dastlab, chap limit ta’rifini keltiraylik.

$y = f(x)$ funksiya va $x = a$ nuqta berilgan bo‘lsin. Agar ixtiyoriy $\varepsilon > 0$ son uchun a dan kichik bo‘lgan shunday bir N haqiqiy son topilib, N va a sonlar orasida yotuvchi barcha x lar uchun $(N < x < a)$ $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, $b \in R$ son $y = f(x)$ funksiyaning $x = a$ nuqtadagi (yoki $x \to a$ dagi) *chap limiti* deyiladi.



IV.1-rasm.

Funksiyaning $x \to a$ dagi chap limiti $\lim\limits_{x \to a-0} f(x) = b$ ko‘rinishda belgilanadi. $x \to a - 0$ belgisi x ning a ga chapdan intilishini, ya'ni x argument a ga a dan kichik bo'lib intilishini bildiradi.

Yuqorida keltirilgan misoldan, $\lim\limits_{x \to 1-0} f(x) = 0,5$ ekanligi kelib chiqadi.

Funksiyaning $x \to a$ dagi o'ng limiti tushunchasi ham $x \to a$ dagi chap limiti tushunchasi kabi ta'riflanadi.

Agar $\varepsilon > 0$ son uchun a dan katta bo'lgan shunday M haqiqiy son topilib, a va M sonlar orasida yotuvchi barcha x lar ($a < x < M$) uchun $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, b son $y = f(x)$ funksiyaning $x = a$ nuqtadagi (yoki $x \to a$ dagi) *o'ng limiti* deyiladi va $\lim\limits_{x \to a+0} f(x) = b$ ko'rinishda belgilanadi.

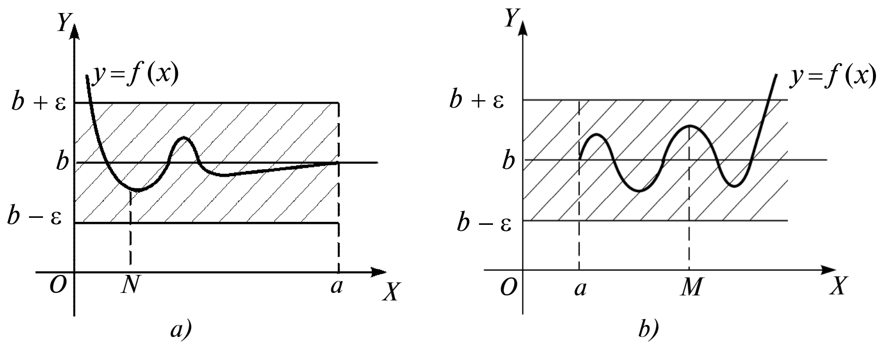
Yuqorida qaralgan misoldan $\lim\limits_{x \to 1+0} f(x) = 2$ ga ega bo'lamiz.

Funksiyaning $x \to a$ dagi chap limitining geometrik ma'nosi quyidagicha:

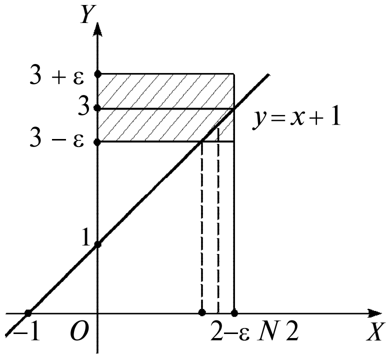
har qanday $\varepsilon > 0$ son uchun a dan kichik shunday N son topiladiki, N va a sonlari orasida yotuvchi barcha x lar uchun funksiyaning grafigi $y = b - \varepsilon$ va $y = b + \varepsilon$ to'g'ri chiziqlar bilan chegaralangan *yo'lakda* yotadi (IV.2-*a* rasm).

Agar $f(x)$ funksiyaning $x \to a$ dagi o'ng limiti b songa teng bo'lsa, u holda uning geometrik ma'nosi quyidagicha bo'ladi:

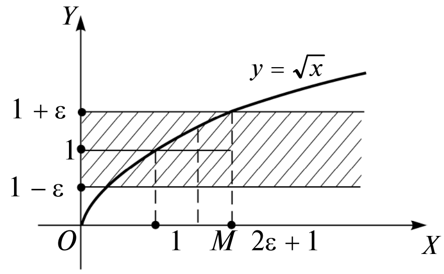
har qanday $\varepsilon > 0$ son uchun a dan katta shunday M son topiladiki, a va M sonlar orasida joylashgan barcha x lar uchun funksiyaning grafigi $y = b - \varepsilon$ va $y = b + \varepsilon$ to'g'ri chiziqlar bilan chegaralangan *yo'lak*da yotadi (IV.2-*b* rasm).



IV.2-rasm.



IV.3-rasm.



IV.4-rasm.

Funksiyaning $x = a$ nuqtadagi chap va o'ng limitlari uning shu nuqtadagi *bir tomonlama limitlari* deyiladi.

1 - m i s o l. $\lim\limits_{x \to 2-0} (x+1) = 3$ ekanligini isbotlang.

I s b o t. ε ixtiyoriy musbat son va $x < 2$ bo'lsin. U holda $|f(x) - 3| = |x + 1 - 3| = |x - 2| = 2 - x < \varepsilon$ bo'lishi uchun $2 - \varepsilon < x < 2$ bo'lishi yetarlidir. Demak, ta'rifdagi N son sifatida $2 - \varepsilon$ sonni yoki $2 - \varepsilon$ dan katta, lekin 2 dan kichik bo'lgan har qanday sonni olish mumkin. Bu esa $\lim\limits_{x \to 2-0} (x+1) = 3$ ekanligini ko'rsatadi (IV.3-rasm).

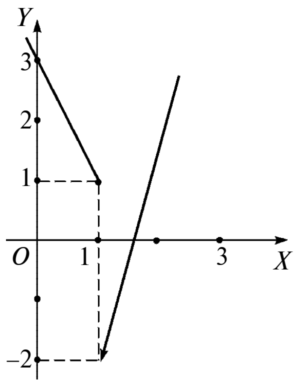
2 - m i s o l. $\lim\limits_{x \to 1+0} \sqrt{x} = 1$ ekanligini isbotlaymiz.

I s b o t. ε ixtiyoriy musbat son va $x > 1$ bo'lsin. U holda

$$|f(x) - 1| = |\sqrt{x} - 1| = \left|\frac{x-1}{\sqrt{x}+1}\right| = \frac{x-1}{\sqrt{x}+1} < \frac{x-1}{2}$$

tengsizlik bajariladi. Bu yerdan ko'rinadiki, $|f(x) - 1| < \varepsilon$ bo'lishi uchun $\frac{x-1}{2} < \varepsilon$ va $x > 1$ bo'lishi, ya'ni $1 < x < 2\varepsilon + 1$ bo'lishi yetarlidir. Demak, ta'rifdagi M son sifatida $(1;\ 2\varepsilon + 1)$ oraliqdagi har qanday sonni olish mumkin. Bu esa $\lim\limits_{x \to 1+0} \sqrt{x} = 1$ ekanligini ko'rsatadi (IV.4-rasm).

3 - m i s o l. $f(x) = \begin{cases} -2x + 3, & \text{agar } x \le 1 \text{ bo'lsa,} \\ 3x - 5, & \text{agar } x > 1 \text{ bo'lsa} \end{cases}$ funksiyaning $x \to 1$ dagi bir tomonlama limitlarini topamiz.



IV.5-rasm.

Y e c h i s h . $x \leq 1$ bo‘lsin. U holda, $f(x) = -2x + 3$. Demak, $\lim\limits_{x \to 1-0} f(x) =$

$$= \lim_{x \to 1-0} (-2x + 3) = -2 \cdot 1 + 3 = 1 \cdot$$

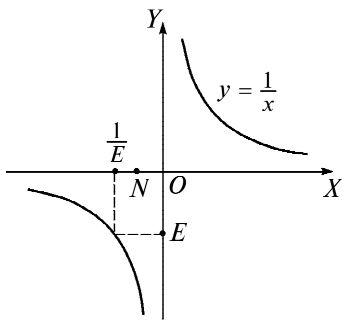
Agar $x > 1$ bo‘lsa, $f(x) = 3x - 5$ bo‘lib, $\lim\limits_{x \to 1+0} f(x) = \lim\limits_{x \to 1+0} (3x - 5) = 3 \cdot 1 - 5 = -2$ (IV.5-rasm).

IV.6- rasmda $y = \frac{1}{x}$ funksiyaning grafigi tasvirlangan. Grafikni kuzatib, x argument 0 soniga chapdan (o‘ngdan) yaqinlashganda funksiya qiymatlari $-\infty$ ga (mos ravishda $+\infty$ ga) yaqinlashadi deb tasdiqlash mumkin: $\lim\limits_{x \to 0-0} \frac{1}{x} = -\infty$, $\lim\limits_{x \to 0-0} \frac{1}{x} = +\infty$.

Cheksiz chap limit va cheksiz o‘ng limit tushunchalarining qat’iy matematik ta’rifini keltiramiz.

Agar ixtiyoriy $E < 0$ ($E > 0$) haqiqiy son uchun shunday $N < a$ haqiqiy son topilib, barcha $x \in (N;\ a)$ lar uchun $f(x) < E$ (mos ravishda, $f(x) > E$) tengsizlik bajarilsa, $f(x)$ funksiyaning a nuqtadagi chap limiti $-\infty$ (mos ravishda $+\infty$) ga teng deyiladi va $\lim\limits_{x \to a-0} f(x) = -\infty$ (mos ravishda $\lim\limits_{x \to a-0} f(x) = +\infty$) ko‘rinishda belgilanadi.

Agar ixtiyoriy $E < 0$ ($E > 0$) haqiqiy son uchun, shunday $M < a$ haqiqiy son topilib, barcha $x \in (a;\ M)$ lar uchun $f(x) < E$ (mos ravishda $f(x) > E$) tengsizlik bajarilsa, $f(x)$ funksiya ning a nuqtadagi o‘ng limiti $-\infty$ (mos ravishda $+\infty$) ga teng deyiladi va $\lim\limits_{x \to a-0} f(x) = -\infty$ (mos ravishda, $\lim\limits_{x \to a+0} f(x) = +\infty$) ko‘rinishda belgilanadi.



IV.6-rasm.

4 - m i s o l. $\lim\limits_{x \to 0-0} \frac{1}{x} = -\infty$ tenglikni isbotlang.

I s b o t . E ixtiyoriy manfiy son va $x < 0$ bo‘lsin. U holda $f(x) = \frac{1}{x} < E$ tengsizlik $\frac{1}{E} < x < 0$ tengsizlikka teng kuchlidir. Bu yerdan ko‘rinadiki, ta’rifda so‘z borgan N son sifatida $\left(\frac{1}{E};\ 0\right)$ oraliqdagi har qanday sonni olish mumkin.

5 - m i s o l. $\lim\limits_{x\to 0+0} \frac{1}{x} = +\infty$ tenglikni isbotlang.

I s b o t . E ixtiyoriy musbat son va $x > 0$ bo‘lsin. U holda $f(x) = \frac{1}{x} > E$ tengsizlik $0 < x < \frac{1}{E}$ tengsizlikka teng kuchlidir. Bu yerdan ko‘rinadiki, ta’rifda so‘z borgan M son sifatida $\left(0;\ \frac{1}{E}\right)$ oraliqdagi har qanday sonni olish mumkin.

Haqiqatan ham, $M \in \left(0;\ \frac{1}{E}\right)$ bo‘lsin. U holda, barcha $x \in (0;\ M)$ lar uchun $f(x) = \frac{1}{x} > \frac{1}{M} > \frac{1}{\frac{1}{E}} = E$ tengsizlik bajariladi. Demak, $\lim\limits_{x\to 0+0} \frac{1}{x} = +\infty$.

M a s h q l a r

4.1. $\lim\limits_{x\to a+0} f(x) = b$ ekanligini isbotlang, bunda:

1) $f(x) = 4x - 2,\ a = 1,\ b = 2$;   2) $f(x) = \sqrt{x^2},\ a = 0,\ b = 0$;
3) $f(x) = \sqrt{x},\ a = 9,\ b = 3$;   4) $f(x) = x^2 - 1,\ a = 1,\ b = 0$.

4.2. $\lim\limits_{x\to a-0} f(x) = b$ ekanligini isbotlang, bunda:

1) $f(x) = \begin{cases} x^2 - 1, & \text{agar } x > a \text{ bo‘lsa,} \\ x + 1, & \text{agar } x < a \text{ bo‘lsa,} \end{cases}$ $a = 2,\ b = 3$;

2) $f(x) = \begin{cases} x^2 - 1, & \text{agar } x \le a \text{ bo‘lsa,} \\ x + 1, & \text{agar } x > a \text{ bo‘lsa,} \end{cases}$ $a = 2,\ b = 1$.

4.3. $f(x)$ funksiyaning $x = a$ nuqtadagi bir tomonlama limitlarini toping:

1) $f(x) = \begin{cases} \sqrt{x}, & \text{agar } x > a \text{ bo‘lsa,} \\ x + 4, & \text{agar } x \le a \text{ bo‘lsa,} \end{cases}$ $a = 4$;

2) $f(x)=\begin{cases}\cos x, & \text{agar } x>a \text{ bo‘lsa,}\\ \sin x, & \text{agar } x\le a \text{ bo‘lsa,}\end{cases}\quad a=\frac{\pi}{2}.$

2. Funksiyaning nuqtadagi limiti. $f(x)=x-2$ funksiyaning $x=2$ nuqtadagi bir tomonlama limitini hisoblaymiz:

$$\lim_{x\to 2-0} f(x)=\lim_{x\to 2-0}(x-2)=2-2=0;$$

$$\lim_{x\to 2+0} f(x)=\lim_{x\to 2+0}(x-2)=2-2=0.$$

Bu yerda $\lim\limits_{x\to 2-0} f(x)=\lim\limits_{x\to 2+0} f(x)$ ekanini ko‘ramiz.

Agar $\lim\limits_{x\to a-0} f(x)=\lim\limits_{x\to a+0} f(x)=b$ bo‘lsa, b son $f(x)$ funksiyaning *$x\to a$ dagi limiti* deyiladi va $\lim\limits_{x\to a} f(x)=b$ ko‘rinishda belgilanadi.

Shunday qilib, agar ixtiyoriy $\varepsilon>0$ son uchun shunday M va N sonlar topilib (bunda $N<a<M$), $(N;\ M)$ oraliqda yotuvchi barcha x lar uchun (a nuqta bundan mustasno bo‘lishi mumkin) $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, $b\in R$ son $y=f(x)$ funksiyaning *$x\to a$ dagi limiti* deyiladi.

1 - m i s o l. $\lim\limits_{x\to 0}(x^2+2)=2$ ekanini isbotlang.

I s b o t. $\lim\limits_{x\to 0-0}(x^2+2)=0^2+2=2$ va $\lim\limits_{x\to 0+0}(x^2+2)=0^2+2=2$

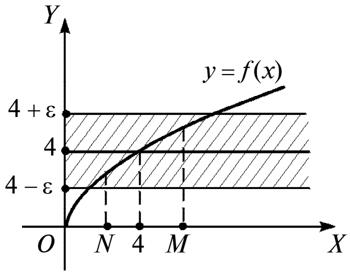
bo‘lgani uchun $\lim\limits_{x\to 0}(x^2+2)=2$.

2 - m i s o l. $\lim\limits_{x\to 4}\sqrt{x}=2$ ekanligini isbotlang.

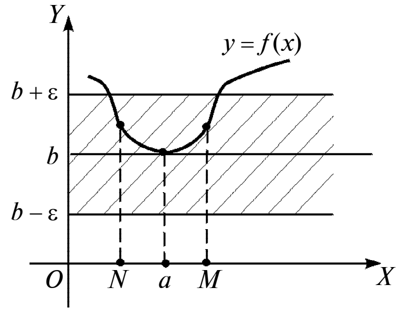
I s b o t. ε ixtiyoriy musbat son bo‘lsin. $x\ge 0$ bo‘lgani uchun,

$$|f(x)-2|=|\sqrt{x}-2|=\left|\frac{x-4}{\sqrt{x}+2}\right|\le\frac{|x-4|}{\sqrt{x}+2}\le\frac{|x-4|}{2}<|x-4|$$

tengsizlik o‘rinli. Demak, $|f(x)-2|<\varepsilon$ bo‘lishi uchun $|x-4|<\varepsilon$ va $x\ge 0$ bo‘lishi, ya’ni $4-\varepsilon<x<4+\varepsilon$, $x\ge 0$ bo‘lishi yetarli. Bu yerdan ko‘rinadiki, ta’rifdagi N son sifatida $(4-\varepsilon;\ 4)$ oraliqdagi har qanday musbat sonni, M son sifatida esa $(4;\ 4+\varepsilon)$ oraliq-



IV.7- rasm.



IV.8- rasm.

dagi har qanday sonni olish mumkin. Bu esa $\lim\limits_{x\to 4}\sqrt{x}=2$ ekanligini bildiradi (IV.7-rasm).

a nuqtani o‘z ichiga olgan har qanday ochiq oraliqni uning *atrofi* deb ataymiz. $(a-\delta;\ a+\delta)$ oraliq (bu yerda $\delta>0$) *a* nuqtaning δ - *atrofi,* $\delta>0$ son esa *atrofning radiusi* deb ataladi.

Agar *b* son $y=f(x)$ funksiyaning $x\to a$ dagi limiti bo‘lsa, u holda $|f(x)-b|<\varepsilon$ tengsizlik *a* nuqta biror atrofining barcha nuqtalari uchun (*a* nuqta bundan mustasno bo‘lishi mumkin) bajarilishini ko‘rish qiyin emas. $\lim\limits_{x\to a}f(x)=b$ ning geometrik ma’nosi IV.8-rasmdan ko‘rinib turibdi:

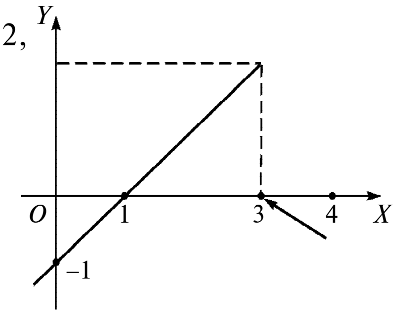
3 - m i s o l. [0; 4] kesmada quyidagicha aniqlangan $y=f(x)$ funksiyani qaraymiz:

$$f(x)=\begin{cases}x-1, & \text{agar } 0\le x\le 3 \text{ bo‘lsa,}\\ 3-x, & \text{agar } 3<x\le 4 \text{ bo‘lsa.}\end{cases}$$

Bu funksiyaning grafigi IV.9-rasmda tasvirlangan.

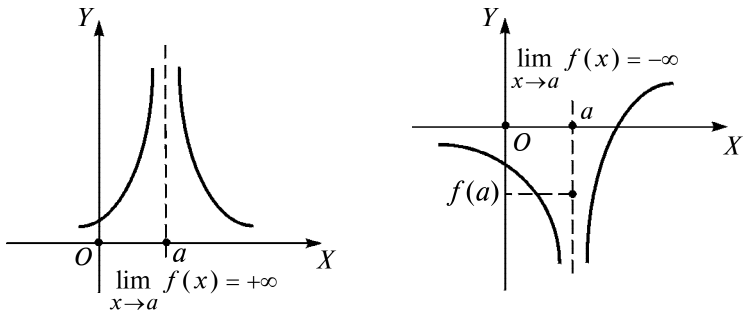
$$\lim_{x\to 3-0}f(x)=\lim_{x\to 3-0}(x-1)=3-1=2,$$
$$\lim_{x\to 3+0}(3-x)=3-0=3$$

tengliklardan ko‘rinadiki, $\lim\limits_{x\to 3}f(x)$ limit mavjud emas.



IV.9- rasm.

Endi funksiyaning nuqtadagi cheksiz limitini qaraymiz. Agar $y=f(x)$ funksiyaning $x=a$ nuq-



a) b)

IV.10- rasm.

tadagi chap limiti ham, o'ng limiti ham $+\infty$ $(-\infty)$ ga teng bo'lsa, $f(x)$ *funksiyaning* $x = a$ *nuqtadagi limiti* $+\infty$ (mos ravishda $-\infty$) *ga teng* deyiladi va $\lim\limits_{x\to a} f(x) = +\infty$ (mos ravishda $\lim\limits_{x\to a} f(x) = -\infty$) ko'rinishda belgilanadi (IV.10-rasm).

Agar $\lim\limits_{x\to a} f(x) = +\infty$ bo'lsa, ixtiyoriy $E > 0$ son uchun shunday $(N;\ a)$ va $(a;\ M)$ intervallar topiladiki, bu intervallardagi barcha x lar uchun $f(x) > E$ tengsizlik bajariladi.

Agar $\lim\limits_{x\to a} f(x) = -\infty$ bo'lsa, ixtiyoriy $E < 0$ son uchun shunday $(N;\ a)$ va $(a;\ M)$ intervallar topiladiki, bu intervaldagi barcha x lar uchun $f(x) < E$ tengsizlik bajariladi.

4 - m i s o l. $\lim\limits_{x\to 0} \frac{1}{|x|} = +\infty$ ekanligini isbotlang.

I s b o t. E ixtiyoriy musbat son va $x \neq 0$ bo'lsin. U holda $f(x) = \frac{1}{|x|} > E$ tengsizlik $\begin{cases} -\frac{1}{E} < x < 0, \\ 0 < x < \frac{1}{E} \end{cases}$ tengsizliklar sistemasiga teng kuchlidir. Bu yerdan ko'rinadiki, ta'rifda so'z borgan N son sifatida $\left(-\frac{1}{E};\ 0\right)$ oraliqdagi, M son sifatida esa $\left(0;\ \frac{1}{E}\right)$ oraliqdagi har qanday sonni olish mumkin. U holda $(N;\ M)$ oraliqdagi barcha $x \neq 0$ sonlar uchun $f(x) > E$ tengsizlik bajariladi. Demak, $\lim\limits_{x\to 0} \frac{1}{|x|} = +\infty$.

Agar $\lim\limits_{x\to a} f(x) = +\infty$ yoki $\lim\limits_{x\to a} f(x) = -\infty$ bo'lsa, $f(x)$ funksiya $x = a$ nuqtada ($x\to a$ da) *aniq ishorali cheksiz limitga ega* deyiladi.

Agar $f(x)$ funksiyaning $x = a$ nuqtadagi bir tomonlama limitlarining biri $+\infty$ ga, ikkinchisi esa $-\infty$ ga teng bo'lsa, $f(x)$ funksiya $x = a$ nuqtada *aniqmas ishorali cheksiz limitga ega* deyiladi.

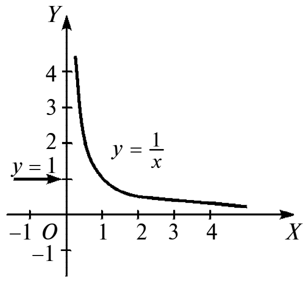
$y = \frac{1}{|x|}$ funksiya $x = 0$ nuqtada aniq ishorali cheksiz limitga (4-misol) ega, $y = \frac{1}{x}$ funksiya esa $x = 0$ nuqtada aniqmas ishorali cheksiz limitga ega (1-band, 4—5-misollar).

$x = a$ nuqtada aniq ishorali yoki aniqmas ishorali cheksiz limitga ega bo'lgan $f(x)$ funksiya shu nuqtada *cheksiz katta funksiya* deyiladi va $\lim\limits_{x\to a} f(x) = \infty$ ko'rinishda belgilanadi.

$y = \frac{1}{|x|}$, $y = \frac{1}{x}$ funksiyalarning har biri $x = 0$ nuqtada cheksiz katta funksiyalardir $\left(\lim\limits_{x\to a}\frac{1}{|x|} = +\infty,\ \lim\limits_{x\to a}\frac{1}{x} = \infty\right)$.

5-misol. $f(x) = \begin{cases} 1, & \text{agar } x \le 0 \text{ bo'lsa,} \\ \frac{1}{x}, & \text{agar } x > 0 \text{ bo'lsa} \end{cases}$ funksiyaning grafigi IV.11-rasmda tasvirlangan. $\lim\limits_{x\to 0-0} f(x) = \lim\limits_{x\to 0-0} 1 = 1$ va $\lim\limits_{x\to 0+0} f(x) = \lim\limits_{x\to 0+0}\frac{1}{x} = +\infty$ tengliklarga egamiz. Bu yerdan ko'rinadiki, $f(x)$ funksiya $x = 0$ nuqtada cheksiz katta funksiya ham emas, shuningdek chekli limitga ham ega emas.

Agar $\lim\limits_{x\to a} f(x) = 0$ bo'lsa, $f(x)$ funksiya $x = a$ nuqtada *cheksiz kichik funksiya* deyiladi. Masalan, $y = \sqrt{x-4}$, $y = \sqrt{x} - 2$ funksiyalarning har biri $x = 4$ nuqtada cheksiz kichikdir.



IV.11-rasm.

Cheksiz kichik va cheksiz katta funksiyalar orasidagi munosabatni ifodalovchi teoremani isbotsiz keltiramiz.

T e o r e m a . *Agar $f(x)$ funksiya $x = a$ nuqtada cheksiz katta (cheksiz kichik) funksiya bo'lsa, $\frac{1}{f(x)}$ funksiya $x = a$ nuqtada cheksiz kichik (cheksiz katta) funksiya bo'ladi.*

1 - e s l a t m a . Funksiyaning $x \to a$ dagi (yoki $x \to a - 0$, yoki $x \to a + 0$ dagi) limitining ta'rifida $x \neq a$ qiymatlar qaraldi, a nuqtaning o'zida funksiya aniqlanmagan bo'lishi ham mumkin.

2 - e s l a t m a . Funksiyaning $x \to a$ dagi (yoki $x \to a - 0$ dagi, yoki $x \to a + 0$ dagi) limitining ta'rifida ta'kidlanayotgan M va N sonlar ε va a ga bog'liqdir.

M a s h q l a r

4.4. Tenglikni isbotlang:

1) $\lim\limits_{x \to 0}(3x+5)=5$; 2) $\lim\limits_{x \to 8}\sqrt[3]{x}=2$; 3) $\lim\limits_{x \to \frac{1}{2}}\frac{x^2-0{,}25}{x-0{,}5}=1$;

4) $\lim\limits_{x \to 4}\frac{x-4}{\sqrt{x}-2}=4$; 5) $\lim\limits_{x \to 2}\frac{x-2}{x^2-5x+6}=-1$; 6) $\lim\limits_{x \to 1}\frac{1}{x}=1$.

4.5. Limitlarni hisoblang:

1) $\lim\limits_{x \to 2}(4x-5)$; 2) $\lim\limits_{x \to 3}\sqrt{x^2+7}$; 3) $\lim\limits_{x \to 8}\sqrt{x^2+36}$;

4) $\lim\limits_{x \to 9}(x^3-5)$; 5) $\lim\limits_{x \to 3}\frac{x^2-9}{x-3}$; 6) $\lim\limits_{x \to 0}\frac{1}{2x+1}$.

3. Funksiyaning nuqtadagi limiti haqidagi asosiy teoremalar. Oldingi bandlarda funksiyaning nuqtadagi limiti tushunchasini qaradik. Bu bandda funksiyaning nuqtadagi limiti haqidagi asosiy teoremalarni keltiramiz va limitni hisoblash masalasi bilan shug'ullanamiz.

1 - t e o r e m a . *$f(x)$ funksiya $x \to a$ da ko'pi bilan bitta limitga ega bo'lishi mumkin.*

2 - t e o r e m a . *Agar $\lim\limits_{x \to a} f(x)=b$ $(b \in R)$ bo'lsa, $x = a$ nuqtaning biror atrofida $f(x)$ funksiya chegaralangan bo'ladi.*

3 - t e o r e m a . *Agar $\lim\limits_{x \to a} f(x)=b$ bo'lib, $b \neq 0$* bo'lsa, *$x = a$ nuqtaning shunday bir atrofi topiladiki, bu atrofdagi barcha x lar uchun ($x = a$ bundan mustasno bo'lishi mumkin) $f(x)$ ning ishorasi b ning ishorasi bilan bir xil bo'ladi.*

4 - t e o r e m a . *Agar* $\lim\limits_{x\to a} f(x)=b$ *bo'lib,* $x=a$ *nuqtaning biror atrofidagi barcha* $x\neq a$ *lar uchun* $f(x)\geq 0$ $(f(x)\leq 0)$ *bo'lsa,* $b\geq 0$ (*mos ravishda,* $b\leq 0$) *bo'ladi.*

5 - t e o r e m a . *Agar* $x=a$ *nuqtaning biror atrofidagi barcha* $x\neq a$ *larda* $\varphi(x)\leq f(x)\leq g(x)$ *bo'lib,* $\lim\limits_{x\to a}\varphi(x)=\lim\limits_{x\to a} g(x)=b$ *bo'lsa,* $\lim\limits_{x\to a} f(x)=b$ *bo'ladi.*

6 - t e o r e m a . *O'zgarmasning limiti o'ziga teng:* $\lim\limits_{x\to a} c=c$.

7 - t e o r e m a . *O'zgarmas ko'paytuvchini limit belgisidan tashqariga chiqarish mumkin:* $\lim\limits_{x\to a}(k\cdot f(x))=k\cdot\lim\limits_{x\to a} f(x)$.

8 - t e o r e m a . *Agar* $f(x)$, $g(x)$ *funksiyalar* $x\to a$ *da chekli limitga ega bo'lsa,* $f(x)\pm g(x)$, $f(x)\cdot g(x)$ *funksiyalar ham* $x\to a$ *da chekli limitga ega va*

$$\lim_{x\to a}(f(x)\pm g(x))=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x),$$
$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

tengliklar o'rinlidir.

9 - t e o r e m a . *Agar* $f(x)$, $g(x)$ *funksiyalar* $x\to a$ *da chekli limitga ega va* $\lim\limits_{x\to a} g(x)\neq 0$ *bo'lsa,* $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ *tenglik o'rinli bo'ladi.*

Bu teoremalarning isboti oliy matematika kursida qaraladi.Biz shu teoremalarning tatbiqi yordamida limitlarni hisoblaymiz.

1 - m i s o l . $\lim\limits_{x\to 3}(x^2+4x-5)$ limitni hisoblaymiz.

Y e c h i s h . Yuqoridagi teoremalarga asosan

$$\lim_{x\to 3}(x^2+4x-5)=\lim_{x\to 3}x^2+\lim_{x\to 3}(4x)-\lim_{x\to 3}5=$$
$$=\lim_{x\to 3}x\cdot\lim_{x\to 3}x+4\lim_{x\to 3}x-5=3\cdot 3+4\cdot 3-5=16.$$

2 - m i s o l. $\lim\limits_{x\to 5}\frac{7x-5}{10+2x}$ limitni hisoblaymiz.

Y e c h i s h . $\lim_{x\to 5}(10+2x)=10+2\cdot 5=20\neq 0$ bo'lgani uchun 9-teoremani bevosita qo'llash mumkin:

$$\lim_{x\to 5}\frac{7x-5}{10+2x}=\frac{\lim\limits_{x\to 5}(7x-5)}{\lim\limits_{x\to 5}(10+2x)}=\frac{7\cdot 5-5}{10+2\cdot 5}=\frac{30}{20}=1{,}5\,.$$

3 - m i s o l . $\lim_{x\to 2}\frac{x^2-4}{x-2}$ limitni hisoblaymiz.

Y e c h i s h . $\lim_{x\to 2}(x-2)=0,\ \lim_{x\to 2}x^2-4=0$ bo'lgani uchun 9-teoremani bevosita qo'llash mumkin emas (bu holda $\frac{0}{0}$ ko'rinishdagi aniqmaslikka ega bo'lamiz). $x\to 2$ bo'lgani uchun $x\neq 2$ deb hisoblash mumkin. Shu sababli:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2}(x+2)=4\,.$$

4 - m i s o l . $\lim_{x\to 2}\frac{x^3+3x^2-9x+2}{x^3-x-6}$ limitni hisoblaymiz.

Y e c h i s h . Bevosita limitga o'tish natijasida $\frac{0}{0}$ ko'rinishdagi aniqmaslik hosil bo'ladi. $x\neq 2$ lar uchun

$$\frac{x^3+3x^2-9x+2}{x^3-x-6}=\frac{(x-2)(x^2+5x+1)}{(x-2)(x^2+2x+3)}=\frac{x^2+5x+1}{x^2+2x+3}$$

tenglik o'rinli bo'lgani sababli

$$\lim_{x\to 2}\frac{x^3+3x^2-9x+2}{x^3-x-6}=\lim_{x\to 2}\frac{x^2+5x+1}{x^2+2x+3}=\frac{2^2+5\cdot 2+1}{2^2+2\cdot 2+3}=1\frac{4}{11}\,.$$

5 - m i s o l . $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}$ limitni hisoblaymiz.

Y e c h i s h . Bevosita limitga o'tsak, $\frac{0}{0}$ ko'rinishdagi aniqmaslikka ega bo'lamiz. Aniqmaslikni ochish uchun kasrning surat va maxrajini $\sqrt{1+x}+1\neq 0$ ga (maxrajining qo'shmasiga) ko'paytirib olamiz:

$$\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}=\lim_{x\to 0}\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{x(\sqrt{1+x}+1)}{x}=$$

$$=\lim_{x\to 0}(\sqrt{1+x}+1)=2\,.$$

6 - m i s o l. $\lim\limits_{x\to 1}\frac{2x-2}{\sqrt[3]{26+x}-3}$ limitni hisoblaymiz.

Y e c h i s h. Bu yerda ham $\frac{0}{0}$ ko‘rinishdagi aniqmaslikni ochish kerak. $26+x=t^3$ deb olamiz (*o‘rniga qo‘yish usuli*). $x\to 1$ da $t=\sqrt[3]{26+x}\to 3$ bo‘lgani uchun

$$\lim_{x\to 1}\frac{2x-2}{\sqrt[3]{26+x}-3}=\lim_{t\to 3}\frac{2(t^3-26)-2}{t-3}=\lim_{t\to 3}\frac{2(t-3)(t^2+3t+9)}{t-3}=54.$$

1 0 - t e o r e m a. *Agar* $\lim\limits_{x\to a}g(x)=0$ *va* $\lim\limits_{x\to a}f(x)=b\neq 0$ $(b\in R)$ *bo‘lib, x = a nuqtaning biror atrofida (x = a nuqtaning o‘zi bundan mustasno bo‘lishi mumkin)* $g(x)\neq 0$ *bo‘lsa,* $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\infty$ *bo‘ladi.*

I s b o t. $\lim\limits_{x\to a}\frac{g(x)}{f(x)}=\frac{\lim\limits_{x\to a}g(x)}{\lim\limits_{x\to a}f(x)}=\frac{0}{b}=0$ bo‘lgani uchun 2-banddagi teoremaga ko‘ra, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\infty$ bo‘ladi.

7 - m i s o l. $\lim\limits_{x\to 2}\frac{3x+4}{\sqrt[3]{x-1}-1}$ limitni hisoblang.

Y e c h i s h. $\lim\limits_{x\to 2}(\sqrt[3]{x-1}-1)=0,\ \lim\limits_{x\to 2}(3x+4)=10$ bo‘lgani uchun 10- teoremaga ko‘ra, $\lim\limits_{x\to 2}\frac{3x+4}{\sqrt[3]{x-1}-1}=\infty$ bo‘ladi. Hisoblashni quyidagicha rasmiylashtirish mumkin:

$$\lim_{x\to 2}\frac{3x+4}{\sqrt[3]{x-1}-1}=\frac{3\cdot 2+4}{\sqrt[3]{2-1}-1}=\frac{10}{0}=\infty.$$

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4.6. Limitlarni hisoblang:

1) $\lim\limits_{x\to 2}(4x^2-3x+5)$; 2) $\lim\limits_{x\to 2}\frac{2x+3}{x-4}$; 3) $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$;

4) $\lim\limits_{x\to 2}\frac{x^2-5x+6}{x^2-4}$; 5) $\lim\limits_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$; 6) $\lim\limits_{x\to 3}\frac{\sqrt{x+3}-\sqrt{2x-3}}{\sqrt[3]{x+3}-2}$.

4.7. Tengliklar to'g'rimi:

1) $\lim\limits_{x\to 2}\frac{x^2-4}{x^3-14}=0$; 2) $\lim\limits_{x\to -2}\frac{x^4+5x^3+6x^2}{x^2-3x-10}=-\frac{4}{7}$;

3) $\lim\limits_{x\to 0}\frac{(1+x)(1+2x)(1+3x)}{x}=-5$; 4) $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{x^2-1}=\frac{1}{4}$;

5) $\lim\limits_{x\to 2}\frac{x-\sqrt{3x-2}}{x^2-4}=\frac{1}{8}$; 6) $\lim\limits_{x\to 4}\frac{\sqrt{1+2x}}{\sqrt{x}-2}=\frac{4}{3}$;

7) $\lim\limits_{x\to -8}\frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}=-2$; 8) $\lim\limits_{x\to 9}\frac{\sqrt[3]{x-1}-2}{x-9}=\frac{1}{12}$;

9) $\lim\limits_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}=\frac{1}{9}$; 10) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}=\frac{9}{2}$;

11) $\lim\limits_{x\to 1}\left[\left(\frac{4}{x^2-x^{-1}}-\frac{1-3x+x^2}{1-x^3}\right)^{-1}+3\cdot\frac{x^4-1}{x^3-x^{-1}}\right]=3$;

12) $\lim\limits_{x\to 2}\left[\left(\frac{x^3-4x}{x^3-8}\right)^{-1}-\left(\frac{x+\sqrt{2x}}{x-2}-\frac{\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right]=\frac{1}{2}$.

4. Funksiyaning cheksizlikdagi limiti. Biz natural argumentli funksiya (sonli ketma-ketlik)ning limiti tushunchasi bilan tanishmiz. Bu limitni natural argumentli funksiyaning $+\infty$ dagi limiti sifatida talqin etish mumkin.

Bu bandda uzluksiz argumentli $y=f(x)$ funksiyaning $+\infty$ dagi, $-\infty$ dagi va ∞ dagi limiti tushunchasi bilan tanishamiz.

Dastlab, funksiyaning cheksizlikdagi chekli limiti tushunchasini qaraylik.

$(T;+\infty)$ oraliqda aniqlangan (bu yerda T – biror haqiqiy son) $y=f(x)$ funksiya va $A\in R$ son berilgan bo'lsin. Agar $\forall\varepsilon>0$ son uchun, shunday $M\in(T;+\infty)$ son mavjud bo'lib, $\forall x\in(M;+\infty)$ sonlar uchun $|f(x)-A|<\varepsilon$ tengsizlik bajarilsa, *A son f (x) funksiyaning* $+\infty$ *dagi limiti* deyiladi va

$$\lim_{x \to +\infty} f(x) = A \qquad (1)$$

ko‘rinishda belgilanadi.

$|f(x) - A| < \varepsilon$ tengsizlik $A - \varepsilon < f(x) < A + \varepsilon$ tengsizlikka teng kuchli bo‘lgani uchun (1) tenglikning o‘rinli bo‘lishi geometrik jihatdan quyidagini anglatadi:

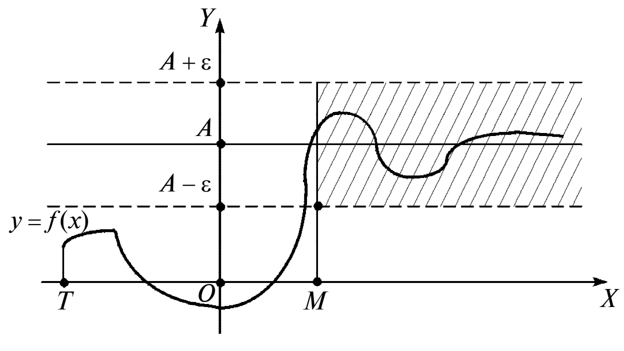
agar $\lim\limits_{x \to +\infty} f(x) = A$ bo‘lsa, $\forall\varepsilon > 0$ son uchun shunday M son topiladiki, $y = f(x)$ funksiyaning $(M; +\infty)$ oraliqdagi grafigi $y = A - \varepsilon$ va $y = A + \varepsilon$ to‘g‘ri chiziqlar bilan chegaralangan yo‘lakda yotadi (IV.12-rasm).

1 - m i s o l. $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$ tenglikni isbotlaymiz.

I s b o t. $y = \frac{1}{x}$ funksiya $(0; +\infty)$ oraliqda aniqlangan funksiyadir. Uning $+\infty$ dagi limiti 0 ga tengligini isbotlaymiz. ε ixtiyoriy musbat son bo‘lsin. $|f(x) - A| < \varepsilon$ tengsizlikni tuzamiz: $\left|\frac{1}{x} - 0\right| < \varepsilon$ yoki $\left|\frac{1}{x}\right| < \varepsilon$. $x > 0$ ekanligini e’tiborga olib, oxirgi tengsizlikdan $x > \frac{1}{\varepsilon}$ ni hosil qilamiz. Bundan, ta’rifdagi M son sifatida $\frac{1}{\varepsilon}$ dan katta bo‘lgan har qanday son olish mumkinligi ko‘rinadi. Demak, $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$ tenglik o‘rinlidir.

Endi $y = f(x)$ funksiya berilgan bo‘lib, $y = f(-x)$ funksiyaning $x \to +\infty$ dagi limiti haqida gapirish mumkin bo‘lsin.

Agar $\lim\limits_{x \to +\infty} f(-x) = A$ bo‘lsa (bu yerda $A \in R$), *A son* $f(x)$ *funksiyaning* $x \to -\infty$ *dagi limiti* deyiladi va $\lim\limits_{x \to -\infty} f(x) = A$ ko‘rinishda belgilanadi.



IV.12-rasm.

2 - m i s o l. $\lim\limits_{x\to-\infty}\left(-\frac{1}{x}\right)=0$ tenglikni isbotlaymiz.

I s b o t. $f(x)=-\frac{1}{x}$, $f(-x)=\frac{1}{x}$ funksiyalarni qaraymiz. $f(-x)$ funksiyaning $x\to+\infty$ dagi limiti mavjud va 0 soniga teng (1-misol) bo‘lgani uchun $\lim\limits_{x\to-\infty}f(x)=0$ tenglik o‘rinlidir.

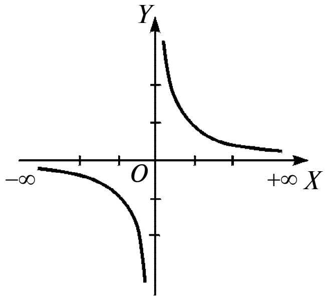
Agar $\lim\limits_{x\to-\infty}f(x)=A$ va $\lim\limits_{x\to+\infty}f(x)=A$ tengliklar bir vaqtda o‘rinli bo‘lsa (bu yerda $A\in R$), *A son y = f (x) funksiyaning $x\to\infty$ dagi limiti* deyiladi va $\lim\limits_{x\to\infty}f(x)=A$ ko‘rinishda belgilanadi.

3 - m i s o l. $\lim\limits_{x\to\infty}\frac{1}{x}=0$ ekanligini isbotlaymiz.

I s b o t. $\lim\limits_{x\to+\infty}\frac{1}{x}=0$, $\lim\limits_{x\to-\infty}\frac{1}{x}=0$ bo‘lgani uchun (1- va 2-misollar), $\lim\limits_{x\to\infty}\frac{1}{x}=0$ tenglik o‘rinlidir.

$\lim\limits_{x\to+\infty}\frac{1}{x}=0$ $\left(\lim\limits_{x\to-\infty}\frac{1}{x}=0\right)$ ekanligi, geometrik jihatdan, x ning yetarlicha katta musbat qiymatlarida (yetarlicha kichik manfiy qiymatlarida) $y=\frac{1}{x}$ funksiyaning grafigi $y=0$ to‘g‘ri chiziqqa yetarlicha yaqinlashib kelishini, $\lim\limits_{x\to\infty}\frac{1}{x}=0$ ekanligi esa x ning moduli yetarlicha katta bo‘lgan qiymatlarida $y=\frac{1}{x}$ funksiyaning grafigi $y=0$ to‘g‘ri chiziqqa yetarlicha yaqin bo‘lishini bildiradi (IV.13-rasm).

Agar $\lim\limits_{x\to+\infty}f(x)=0$ bo‘lsa, *f (x) funksiya $x\to+\infty$ da cheksiz kichik funksiya* deyiladi.



IV.13-rasm.

1-misolda $y=\frac{1}{x}$ funksiyaning $x\to+\infty$ da cheksiz kichik ekanligi isbotlandi.

Funksiyaning $x\to+\infty$ da, shuningdek, $x\to\infty$ da cheksiz kichik funksiya bo‘lishligi yuqoridagiga o‘xshash ta’riflanadi.

Funksiyaning nuqtadagi limitining xossalari bu bandda qaralgan limitlar uchun ham o‘z kuchini saqlaydi.

4-misol. $\lim\limits_{x\to+\infty}\frac{6x^2+x-1}{x^2-0{,}9}$ limitni hisoblaymiz.

Yechish. Bu limitni sonli ketma-ketlikning limitini hisoblashda tutilgan yo'ldan foydalanib hisoblaymiz:

$$\lim_{x\to+\infty}\frac{6x^2+x-1}{x^2-0{,}9}=\lim_{x\to+\infty}\frac{6+\frac{1}{x}-\frac{1}{x^2}}{1-\frac{0{,}9}{x^2}}=\frac{6+0-0}{1-0}=6.$$

Endi funksiyaning cheksizlikdagi cheksiz limiti tushunchasini qaraymiz.

$f(x)$ funksiya biror $(T;\ +\infty)$ oraliqda (bu yerda $T\in R$) aniqlangan funksiya va $\lim\limits_{x\to+\infty}\frac{1}{f(x)}=0$ bo'lsin. U holda, $f(x)$ funksiyaning $x\to+\infty$ dagi limiti ∞ ga teng deyiladi va $\lim\limits_{x\to+\infty}f(x)=\infty$ ko'rinishda belgilanadi.

5-misol. $\lim\limits_{x\to+\infty}(x^2+5)=\infty$ ekanligini isbotlaymiz.

Isbot. $\lim\limits_{x\to+\infty}\frac{1}{x^2+5}=\lim\limits_{x\to+\infty}\frac{\frac{1}{x^2}}{1+\frac{5}{x^2}}=\frac{0}{1+0}=0$ bo'lgani uchun yuqoridagi ta'rifga ko'ra $\lim\limits_{x\to+\infty}(x^2+5)=\infty$ bo'ladi.

Agar $\lim\limits_{x\to+\infty}f(x)=\infty$ bo'lib, $f(x)$ funksiya biror $(M;\ +\infty)$ oraliqdagi barcha x larda faqat musbat (faqat manfiy) qiymatlar qabul qilsa, $\lim\limits_{x\to+\infty}f(x)=+\infty$ (mos ravishda $\lim\limits_{x\to+\infty}f(x)=-\infty$) deyiladi.

6-misol. $\lim\limits_{x\to+\infty}(5-x^2)=-\infty$ ekanligini isbotlaymiz.

Isbot. $\lim\limits_{x\to+\infty}(5-x^2)=\infty$ ekanligi 5-misoldagidek ko'rsatiladi.

$(3;\ +\infty)$ oraliqdagi barcha x sonlar uchun, $5-x^2<0$ ekanligini ko'rish qiyin emas. Demak, $\lim\limits_{x\to+\infty}(5-x^2)=-\infty$.

Agar $\lim\limits_{x\to+\infty}f(x)=\infty$ tenglik bajarilsa, $f(x)$ *funksiya* $x\to+\infty$ *da cheksiz katta funksiya* deyiladi. Masalan, $y=x^2+5$ va $y=5-x^2$

funksiyalarning har biri $x\to+\infty$ da cheksiz katta funksiyalardir (5- va 6-misollar).

$x\to-\infty$ va $x\to\infty$ dagi cheksiz limit va cheksiz katta funksiya tushunchalari yuqoridagiga o‘xshash aniqlanadi.

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4.8. Limitlarni hisoblang:

1) $\lim\limits_{x\to\infty}\frac{(2x-3)(3x+5)(4x-6)}{3x^2+x-1}$; 2) $\lim\limits_{x\to\infty}\frac{x}{\sqrt[3]{x^3+10}}$;

3) $\lim\limits_{x\to\infty}\frac{(x+1)^2}{x^2+1}$; 4) $\lim\limits_{x\to\infty}\frac{x^2-5x+1}{3x+7}$;

5) $\lim\limits_{x\to\infty}\frac{1000x}{x^2-1}$; 6) $\lim\limits_{x\to\infty}\frac{2x^2-x+3}{x^3-8x+5}$;

7) $\lim\limits_{x\to\infty}\frac{(2x+3)^3(3x-2)^2}{x^5+5}$; 8) $\lim\limits_{x\to\infty}\frac{x^2}{10+x\sqrt{x}}$;

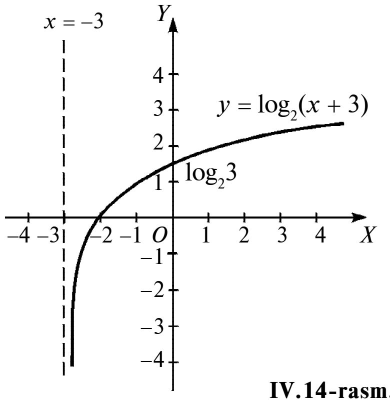
9) $\lim\limits_{x\to\infty}\frac{2x^2-3x-4}{\sqrt{x^4+1}}$; 10) $\lim\limits_{x\to\infty}\frac{\sqrt[3]{x^2+1}}{x+1}$;

11) $\lim\limits_{x\to\infty}\frac{2x+3}{x+\sqrt[3]{x}}$; 12) $\lim\limits_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

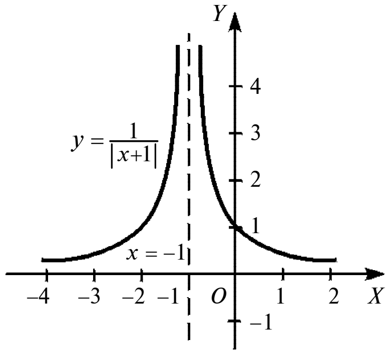
5. Funksiya grafigining asimptotasi. $y=f(x)$ funksiya va uning grafigi Γ berilgan bo‘lsin. Agar Γ chiziqning shunday bir $M(x_0; f(x_0))$ nuqtasi va shunday bir l to‘g‘ri chiziq mavjud bo‘lib, M nuqtadan boshlab hisoblanganda, Γ chiziqning va l to‘g‘ri chiziqning mos nuqtalari orasidagi masofa $x\to a$ ($x\to a+0$, $x\to a-0$, $x\to+\infty$, $x\to-\infty$) da cheksiz kichraysa, *l to‘g‘ri chiziq Γ grafikning asimptotasi* deyiladi.

Funksiya grafigining vertikal, gorizontal va og‘ma asimptotalari mavjud bo‘lishi mumkin. Ularni alohida-alohida qarab chiqamiz.

Agar $\lim\limits_{x\to a+0}f(x)$ yoki $\lim\limits_{x\to a-0}f(x)$ limitlarning aqalli birortasi $+\infty$ yoki $-\infty$ ga teng bo‘lsa, $x=a$ to‘g‘ri chiziq f funksiya grafigining *vertikal asimptotasi* deyiladi.



IV.14-rasm.



IV.15-rasm.

1 - m i s o l. $y = \log_2(x+3)$ funksiya $x = -3$ vertikal asimptotaga ega, chunki $\lim\limits_{x \to -3+0} \log_2(x+3) = -\infty$ (IV.14-rasm).

2 - m i s o l. $x = -1$ to‘g‘ri chiziq $y = \frac{1}{|x+1|}$ funksiya grafigining vertikal asimptotasidir, chunki $\lim\limits_{x \to -1} \frac{1}{|x+1|} = +\infty$ (IV. 15-rasm).

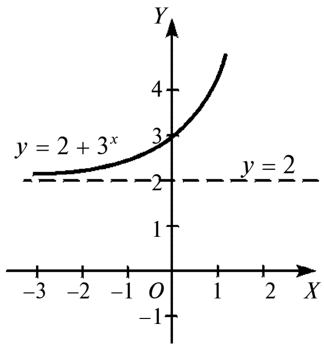
Agar $\lim\limits_{x \to -\infty} f(x) = b$, $\lim\limits_{x \to +\infty} f(x) = b$ shartlarning aqalli birortasi bajarilsa, $y = b$ *to‘g‘ri chiziq* $y = f(x)$ *funksiya grafigining gorizontal asimptota*si deyiladi.

3 - m i s o l. $\lim\limits_{x \to +\infty} \frac{1}{|x+1|} = 0$ bo‘lgani uchun $y = 0$ to‘g‘ri chiziq $y = \frac{1}{|x+1|}$ funksiya grafigining $x \to +\infty$ dagi gorizontal asimptotasi bo‘ladi. Bu to‘g‘ri chiziq shu funksiya grafigining $x \to -\infty$ dagi gorizontal asimptotasi hamdir (IV. 15-rasm).

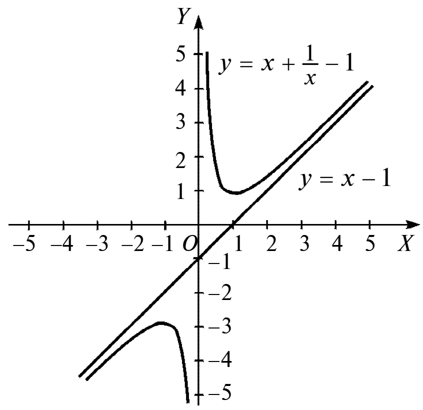
4 - m i s o l. $\lim\limits_{x \to -\infty} (2 + 3^x) = 2$ bo‘lgani uchun $x = 2$ to‘g‘ri chiziq $y = 2 + 3^x$ funksiya grafigining gorizontal asimptotasidir (IV.16-rasm).

A g a r $\lim\limits_{x \to +\infty} [f(x) - (kx+b)] = 0$, $\lim\limits_{x \to -\infty} [f(x) - (kx+b)] = 0$ tengliklarning aqalli birortasi bajarilsa, $y = kx + b$ *to‘g‘ri chiziq* $f(x)$ *funksiya grafigining og‘ma asimptotasi* deyiladi, bunda $k \neq 0$.

5 - m i s o l. $y = x + \frac{1}{x} - 1$ funksiya $y = x - 1$ og‘ma asimptotaga ega (IV.17-rasm).



IV.16-rasm.



IV.17-rasm.

Agar $\lim\limits_{x\to+\infty}\frac{f(x)}{x} = k,\ (k \neq 0)$ *va* $\lim\limits_{x\to+\infty}(f(x) - kx) = b$ *chekli limitlar mavjud bo'lsa,* $y = kx + b$ *to'gri chiziq* $f(x)$ *funksiya grafigining* $x\to+\infty$ *dagi og'ma asipmtotasi bo'lishligi, shuningdek,* $\lim\limits_{x\to-\infty}\frac{f(x)}{x} = k$ *va* $\lim\limits_{x\to-\infty}(f(x) - kx) = b$ *chekli limitlar mavjud bo'lsa,* $y = kx + b$ *to'g'ri chiziq* $f(x)$ *funksiya grafigining* $x\to-\infty$ *dagi og'ma asimptotasi bo'lishligi oliy matematika kursida isbotlanadi.*

Biz funksiyaning og'ma asimptotasini izlashda shu tasdiqlardan foydalanamiz.

6 - m i s o l. $f(x) = \frac{x^4 - 1}{x^3 + 6}$ funksiya grafigining og'ma asimptotasini topamiz.

Y e c h i s h.

$$k = \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{x^4 - 1}{x(x^3 + 6)} = 1, \qquad b = \lim_{x\to+\infty}(f(x) - 1\cdot x) =$$

$= \lim\limits_{x\to+\infty}\left(\frac{x^4 - 1}{x^3 + 6} - x\right) = 0$ bo'lgani uchun $y = x$ to'g'ri chiziq $f(x)$ funksiya grafigining $x\to+\infty$ dagi og'ma asimptotasi bo'ladi. $y = x$ to'g'ri chiziq shu funksiya grafigining $x\to-\infty$ dagi og'ma asimptotasi bo'lishligini ham ko'rsatish mumkin.

Kasr-ratsional funksiya grafigining asimptotasini topishning yana bir usulini keltiramiz.

$f(x)=\frac{x^4-1}{x^3+6}$ funksiya berilgan bo‘lsin. Suratdagi ko‘phadni maxrajdagi ko‘phadga bo‘lib, $\frac{x^4-1}{x^3+6}=x-\frac{6x+1}{x^3+6}$ ekanligini ko‘ramiz. $x\to\pm\infty$ da $\frac{6x+1}{x^3+6}\to 0$ bo‘lgani uchun $\frac{x^4-1}{x^3+6}-x\to 0$ bo‘ladi. Bu esa $y=x$ to‘g‘ri chiziq $y=\frac{x^4+1}{x^3+6}$ funksiyaning asimptotasi bo‘lishini bildiradi (6-misol bilan solishtiring).

M a s h q l a r

4.9. $f(x)=\frac{x^4-1}{x^4+1}$ funksiya grafigining gorizontal asimptotalarini toping.

4.10. $f(x)=\frac{x^4-1}{x^3+6}$ funksiya grafigining vertikal asimptotalarini toping.

4.11. $f(x)=x\cdot\cos\frac{\pi}{x}$ funksiya grafigining og‘ma asimptotalarini toping.

4.12. Funksiya grafigining asimtotalarini toping:

1) $y=\frac{1}{(x-2)^2}$; 2) $y=\frac{x^2}{x^2+9}$;

3) $y=\sqrt{x^2-4}$; 4) $y=\frac{x+1}{\sqrt{x^2+9}}$;

5) $y=\frac{x^2+1}{\sqrt{x^2-1}}$; 6) $y=\frac{\sin x}{x}$;

7) $y=x\mathrm{arctg}x$; 8) $y=x\left(2+\sin\frac{1}{x}\right)$;

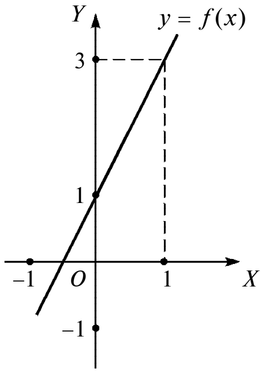
9) $y=\arcsin\frac{1}{x}$; 10) $y=\arccos\frac{1}{x}$.

2-§. Funksiyaning uzluksizligi

1. Funksiyaning nuqtada uzluksizligi va uzilishi. Funksiyaning nuqtadagi limiti tushunchasini o‘rganganimizda, funksiya qaralayotgan nuqtada aniqlanmagan bo‘lishi ham mumkinligi aytildi.

Endi $y=f(x)$ funksiya $x=a$ nuqtaning faqat o‘zidagina emas, balki uning biror atrofida ham aniqlangan bo‘lsin.

Agar $\lim\limits_{x\to a} f(x)=f(a)$ tenglik bajarilsa, $y=f(x)$ funksiya $x=a$ *nuqtada uzluksiz* deyiladi.



IV.18-rasm.

1 - m i s o l. $f(x) = 2x + 1$ funksiyani $x = 1$ nuqtada uzluksizlikka tekshiramiz.

Y e c h i s h . Barcha haqiqiy sonlar to‘plamida aniqlangan bu funksiyaning grafigi IV.18-rasmda tasvirlangan.

$f(1) = 2 \cdot 1 + 1 = 3$ va $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}(2x+1) = 2\cdot 1 + 1 = 3$ bo‘lgani uchun $\lim\limits_{x\to 1} f(x) = f(1)$ tenglik o‘rinlidir. Demak, berilgan funksiya $x = 1$ nuqtada uzluksizdir.

2 - m i s o l. $\varphi(x) = \begin{cases} \dfrac{x^2-1}{x-1}, \text{ agar } x \neq 1 \text{ bo‘lsa,} \\ 4, \text{ agar } x = 1 \text{ bo‘lsa,} \end{cases}$ funksiyani $x = 1$ nuqtada uzluksizlikka tekshiramiz.

Y e c h i s h . $\varphi(1) = 4$ va $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1}(x+1) = 1+1 = 2$ munosabatlardan $\lim\limits_{x\to 1}\varphi(x) \neq \varphi(1)$ ekanligini ko‘ramiz. Demak, berilgan $\varphi(x)$ funksiya $x = 1$ nuqtada uzluksiz emas (IV.19-rasm).

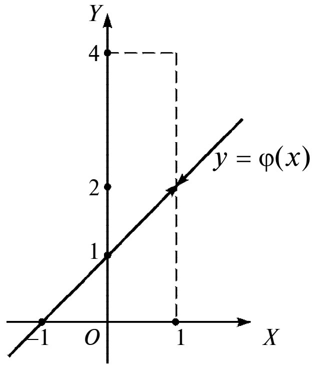
$x = a$ nuqtada uzluksiz bo‘lgan funksiyalarning uzluksizlik ta’rifidan va funksiyaning nuqtadagi limitining mos xossalaridan kelib chiqadigan asosiy xossalarini keltiramiz:

1°. *Agar $y = f(x)$ va $y = g(x)$ funksiyalar $x = a$ nuqtada uzluksiz bo‘lsa, $f(x) \pm g(x)$, $f(x) \cdot g(x)$ va $\dfrac{f(x)}{g(x)}$ $(g(a) \neq 0)$ funksiyalar ham $x = a$ nuqtada uzluksiz bo‘ladi.*

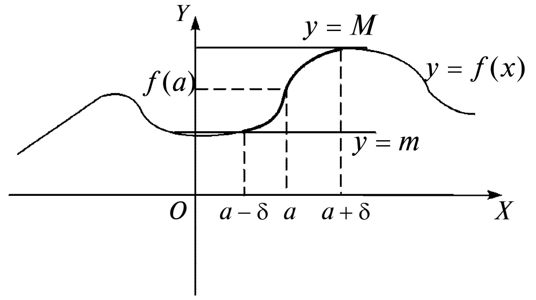
2°. *Agar $f(x)$ funksiya $x = a$ nuqtada uzluksiz bo‘lsa, u holda $x = a$ nuqtaning shunday bir δ-atrofi topiladiki, $f(x)$ funksiya bu atrofda chegaralangan bo‘ladi va agar $f(a) \neq 0$ bo‘lsa, bu atrofda $f(x)$ ning ishorasi $f(a)$ ishorasi bilan bir xil bo‘ladi* (IV.20-rasm).

Endi funksiyaning nuqtada chapdan va o‘ngdan uzluksizligi tushunchalarini ta’riflaymiz.

Agar $\lim\limits_{x\to a-0} f(x) = f(a)$ $\left(\lim\limits_{x\to a+0} f(x) = f(a)\right)$ tenglik bajarilsa, $y = f(x)$ funksiya $x = a$ nuqtada *chapdan* (*o‘ngdan*) *uzluksiz* deyiladi.



IV.19-rasm.



$m \le f(x) \le M,\ x \in (a-\delta;\ a+\delta)$

$f(x) > 0,\ x \in (a-\delta;\ a+\delta)$

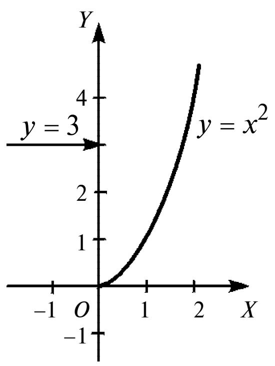
IV.20-rasm.

3-m i s o l. $y = \begin{cases} x^2, & \text{agar } x \ge 0 \text{ bo‘lsa,} \\ 3, & \text{agar } x < 0 \text{ bo‘lsa} \end{cases}$ funksiyani va $x = 0$ nuqtani qaraymiz. Bu funksiya $x = 0$ nuqtada aniqlangan va bu nuqtada $f(0) = 0^2 = 0$ ga teng qiymat qabul qiladi.

$$\lim_{x\to 0-0} f(x) = \lim_{x\to 0-0} 3 = 3 \quad \text{va} \quad \lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} x^2 = 0^2 = 0$$

tengliklardan ko‘rinadiki, $\lim\limits_{x\to 0-0} f(x) \ne f(0),\ \lim\limits_{x\to 0+0} f(x) = f(0)$ munosabatlar bajariladi. Demak, qaralayotgan funksiya $x = 0$ nuqtada o‘ngdan uzluksiz, lekin chapdan uzluksiz emas (IV.21-rasm).

$x = a$ nuqtada chapdan ham, o‘ngdan ham uzluksiz bo‘lgan funksiya shu nuqtada uzluksiz bo‘ladi va, aksincha, $x = a$ nuqtada uzluksiz bo‘lgan funksiya shu nuqtada chapdan ham, o‘ngdan ham uzluksiz bo‘ladi (isbotlang). Bu yerdan, 3-misoldagi $y(x)$ funksiyaning $x = 0$ nuqtada uzluksiz emasligi kelib chiqadi.



IV.21-rasm.

Agar $\lim\limits_{x\to a} f(x) = f(a)$ tenglik ma’noga ega bo‘lmasa yoki bajarilmasa, $f(x)$ funksiya *$x = a$ nuqtada uzilishga ega* (*uzluksiz emas*) deyiladi va $x = a$ nuqta *funksiyaning uzilish nuqtasi* deb ataladi.

2-misoldagi $\varphi(x)$ funksiya $x = 1$ nuqtada, 3-misoldagi $y(x)$ funksiya esa $x = 0$ nuqtada uzilishga egadir.

Agar $f(x)$ funksiya o'zining uzilish nuqtasi $x = a$ da chekli bir tomonli limitlarga ega bo'lsa, $x = a$ nuqta $f(x)$ funksiyaning *birinchi tur uzilish nuqtasi* deyiladi. Birinchi tur uzilish nuqtasi $x = a$ dagi chekli bir tomonli limitlar teng, ya'ni $\lim\limits_{x\to a+0} f(x) = \lim\limits_{x\to a-0} f(x)$ bo'lsa, $x = a$ nuqta *tuzatib (yo'qotib) bo'ladigan uzilish nuqtasi* deyiladi.

2-misoldagi $\varphi(x)$ funksiya $x = 1$ nuqtada tuzatib bo'ladigan uzilishga ega. Funksiyaning $x = 1$ nuqtadagi qiymati sifatida 4 ni emas, balki $\lim\limits_{x\to 1} \varphi(x) = \lim\limits_{x\to 1-0} \varphi(x) = \lim\limits_{x\to 1+0} \varphi(x) = 2$ ni olsak, $\varphi(x)$ funksiya $x = 1$ nuqtada uzluksiz bo'lib qoladi (IV.19-rasm).

Agar $x = a$ nuqtada $f(x)$ funksiya birinchi tur uzilishga ega bo'lib, $\lim\limits_{x\to a-0} f(x) \neq \lim\limits_{x\to a+0} f(x)$ munosabat bajarilsa, $f(x)$ funksiya $x = a$ nuqtada «sakrashga» ega deyiladi va $\lim\limits_{x\to a+0} f(x) - \lim\limits_{x\to a-0} f(x)$ ayirma funksiyaning *$x = a$ nuqtadagi sakrashi* deyiladi.

3-misoldagi $y(x)$ funksiya $x = 0$ nuqtada birinchi tur uzilishga ega va $\lim\limits_{x\to 0-0} y(x) = 3$, $\lim\limits_{x\to 0+0} y(x) = 0$ munosabatlar o'rinli. $\lim\limits_{x\to 0-0} y(x) \neq \lim\limits_{x\to 0+0} y(x)$ bo'lgani uchun $y(x)$ funksiya nuqtada $x = 0$ nuqtada sakrashga ega va bu sakrash $0 - 3 = -3$ ga teng (sakrash pastga qarab sodir bo'ldi!) (IV.21-rasm).

Agar $y = f(x)$ funksiya $x = a$ nuqtada uzilishga ega bo'lsa va $x = a$ nuqta funksiyaning birinchi tur uzilish nuqtasi bo'lmasa, $x = a$ nuqta funksiyaning *ikkinchi tur uzilish nuqtasi* deyiladi.

4-misol. $f(x) = \begin{cases} 1, & \text{agar } x \le 0 \text{ bo'lsa,} \\ \frac{1}{x}, & \text{agar } x > 0 \text{ bo'lsa} \end{cases}$ funksiya $x = 0$ nuqtada ikkinchi tur tuzilishga ega, chunki $\lim\limits_{x\to 0-0} f(x) = 1$, $\lim\limits_{x\to 0+0} f(x) = +\infty$ (1-§, 2- band, 5-misol) bo'lgani uchun berilgan funksiya $x = 0$ nuqtada uzilishga ega va bu uzilish birinchi tur uzilish emas (IV.11-rasm).

Mashqlar

4.13. Funksiyani $x = 0$ nuqtada uzluksizlikka tekshiring:

1) $y = x^3$;

2) $y = \begin{cases} 1 + x, & x > 0, \\ x^2, & x \le 0; \end{cases}$

3) $y = \begin{cases} -x^2 + 1, & x < 0, \\ x + 1, & x > 0, \\ 3, & x = 0; \end{cases}$

4) $y = \begin{cases} -x, & x \ge 0, \\ x + 2, & x < 0; \end{cases}$

5) $y = \begin{cases} \frac{1}{x}, & x \ne 0, \\ 2, & x = 0; \end{cases}$

6) $y = \begin{cases} x^2 - 2x, & x > 0, \\ 3 - x, & x \le 0. \end{cases}$

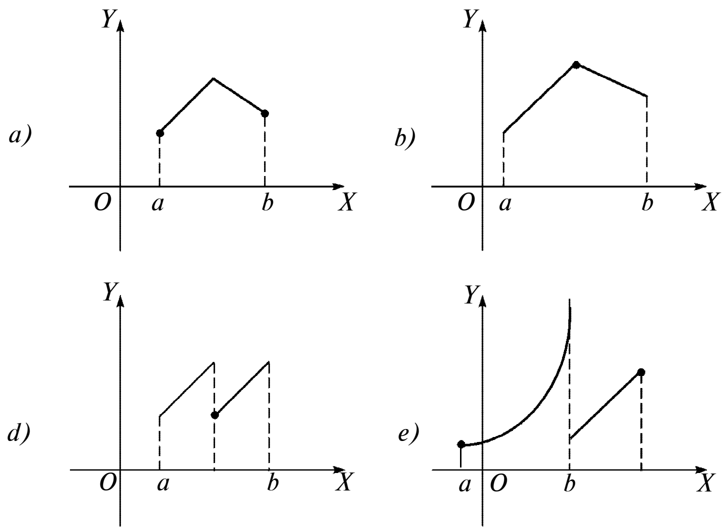
4.14. $y = f(x)$ funksiyalar berilgan:

1) $\frac{1}{4x+12}$; 2) $\frac{1}{x^2-9}$; 3) $\frac{x^2}{x^2-6x+9}$;

4) $\frac{x+3}{x^2+2x-8}$; 5) $\frac{x^2+2x-8}{x+3}$.

Funksiyalarning uzilish nuqtalarini toping.

4.15. IV.22-rasmda tasvirlangan grafik bo‘yicha, grafigi tasvirlangan funksiyaning $(a;\ b)$ oraliqqa tegishli bo‘lgan uzilish nuqtalarini toping.



IV.22- rasm.

4.16. Funksiyaning uzilish nuqtalarini ko‘rsating (isbotini keltirish shart emas):

1) $y=\frac{1}{x}$; 2) $y=\mathrm{tg}x$; 3) $y=\frac{1}{x^2-1}$; 4) $y=\frac{x-1}{x^2-1}$;

5) $y=[x]$; 6) $y=\mathrm{tg}x+\mathrm{ctg}x$; 7) $y=\{x\}$; 8) $y=\mathrm{tg}2x$.

4.17. Butun son to‘g‘ri chizig‘ida aniqlangan va

1) 0; 1 va 2; 2) –1; 0 va 3;

3) $\pi n,\ n\in Z$; 4) $\frac{\pi}{2}+2\pi n,\ n\in Z$

nuqtalardan farqli bo‘lgan barcha nuqtalarda uzluksiz bo‘lgan funksiya quring.

4.18. Funksiyaning uzilish nuqtalarini va uzilish turlarini aniqlang:

1) $f(x)=\begin{cases} x^2-4, & \text{agar } x\le 2 \text{ bo‘lsa,}\\ 6-2x, & \text{agar } x>2 \text{ bo‘lsa;}\end{cases}$

2) $f(x)=\begin{cases} 9-x^2, & \text{agar } x\le 1 \text{ bo‘lsa,}\\ 2x+3, & \text{agar } x>1 \text{ bo‘lsa;}\end{cases}$

3) $f(x)=\begin{cases} x+4, & \text{agar } x<-1 \text{ bo‘lsa,}\\ x^2+2, & \text{agar } -1\le x<1 \text{ bo‘lsa,}\\ 2x, & \text{agar } x\ge 1 \text{ bo‘lsa;}\end{cases}$

4) $f(x)=\frac{2x}{x-1}$; 5) $f(x)=\frac{3x}{2x-3}$.

4.19. $y=f(x)$ funksiya berilgan. Funksiyaning uzilish nuqtalaridagi bir tomonli limitlarni, sakrashlarni va [–4; 4] kesmadagi grafigini yasang:

1) $f(x)=\begin{cases} 4x+5, & \text{agar } x\le -1 \text{ bo‘lsa,}\\ x^2-4x, & \text{agar } x>-1 \text{ bo‘lsa;}\end{cases}$

2) $f(x)=\begin{cases} x^2+2, & \text{agar } x\le 2 \text{ bo‘lsa,}\\ x+1, & \text{agar } x>2 \text{ bo‘lsa;}\end{cases}$

3) $f(x)=\begin{cases} -x, & \text{agar } x\le 0 \text{ bo‘lsa,}\\ -x(x-1)^2, & \text{agar } 0<x<2 \text{ bo‘lsa,}\\ x-3, & \text{agar } x\ge 2 \text{ bo‘lsa;}\end{cases}$

$$4)\quad f(x)=\begin{cases}-2x, & \text{agar } x\le 0 \text{ bo‘lsa,}\\ \sqrt{x}, & \text{agar } 0<x<4 \text{ bo‘lsa,}\\ 1, & \text{agar } x\ge 4 \text{ bo‘lsa.}\end{cases}$$

4.20. Agar:

1) $f(x)=(x^2+1)^5$, $x_0=0$; 2) $f(x)=\frac{x^2+4x-5}{x+5}$, $x_0=-5$;

3) $f(x)=\frac{3x-2}{9x-6}$, $x_0=\frac{2}{3}$; 4) $f(x)=\frac{x-1}{\sqrt{x}-1}$, $x_0=1$

bo‘lsa, A ning qanday qiymatida

$$F(x)=\begin{cases}f(x), & \text{agar } x\ne x_0 \text{ bo‘lsa,}\\ A, & \text{agar } x=x_0 \text{ bo‘lsa}\end{cases}$$

funksiya $x=x_0$ nuqtada uzluksiz bo‘ladi?

2. Funksiyaning oraliqda uzluksizligi. Agar $y=f(x)$ funksiya X oraliqdagi barcha nuqtalarda uzluksiz bo‘lsa, $f(x)$ funksiya shu *oraliqda uzluksiz* deyiladi.

1-misol. $y=x^2+x+1$ funksiya $X=(-\infty;+\infty)$ oraliqda uzluksizmi?

Yechish. $(-\infty;+\infty)$ oraliqdagi barcha $x=a$ nuqta uchun

$$\lim_{x\to a} y(x)=\lim_{x\to a}(x^2+x+1)=a^2+a+1=f(a)$$

tenglik o‘rinli bo‘lgani sababli, $y(x)$ funksiya $X=(-\infty;+\infty)$ oraliqda uzluksizdir.

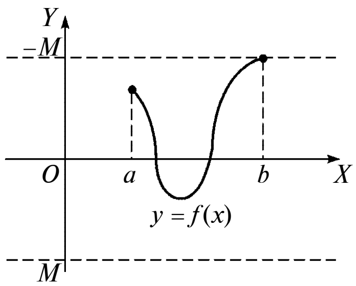
2-misol. $y=\frac{1}{x-3}$ funksiya $x=(0;5)$ oraliqda uzluksizmi?

Yechish. Berilgan funksiya $x=3\in(0;5)$ nuqtada aniqlanmagan. Shu sababli, u $x=3$ nuqtada uzilishga ega. Demak, berilgan funksiya $x=(0;5)$ oraliqda uzluksiz emas.

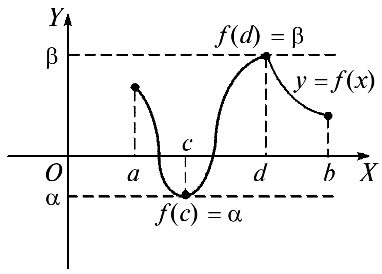
Kesmada uzluksiz bo‘lgan funksiya bir qator ajoyib xossalarga ega. Shu sababli $X=[a;b]$ bo‘lgan holga alohida to‘xtalamiz.

Agar $y=f(x)$ funksiya barcha $x\in(a;b)$ nuqtalarda uzluksiz bo‘lsa va $x=a$ nuqtada o‘ngdan, $x=b$ nuqtada esa chapdan uzluksiz bo‘lsa, $f(x)$ *funksiya* $[a;b]$ *kesmada uzluksiz* deyiladi.

Endi kesmada uzluksiz bo‘lgan funksiyaning xossalarini ifodalovchi teoremalarni va ularning geometrik talqinini keltiramiz.



IV.23-rasm.



IV.24-rasm.

1-teorema. *Agar* $f(x)$ *funksiya* $[a;\ b]$ *kesmada uzluksiz bo'lsa, bu funksiya* $[a;\ b]$ *oraliqda chegaralangan bo'ladi, ya'ni shunday o'zgarmas* $M>0$ *son topiladiki, barcha* $x\in[a;\ b]$ *lar uchun*

$$|f(x)|\le M$$

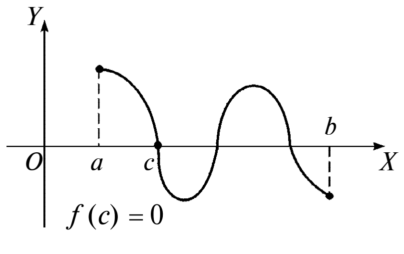
tengsizlik bajariladi (IV.23-rasm).

$y=f(x)$ funksiyaning $[a;\ b]$ kesmadagi grafigi $[-M;\ M]$ yo'lakda joylashgan.

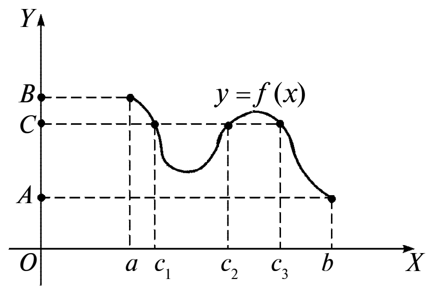
2-teorema. $[a;\ b]$ *kesmada uzluksiz bo'lgan* $f(x)$ *funksiya shu oraliqda o'zining eng katta va eng kichik qiymatlariga ega bo'ladi* (IV.24-rasmga qarang: $f(c)=\alpha$ — eng kichik qiymat, $f(d)=\beta$ — eng katta qiymat).

3-teorema. $[a;\ b]$ *kesmada uzluksiz* $f(x)$ *funksiya uchun* $f(a)\cdot f(b)<0$ *bo'lsa, funksiya nolga teng qiymat qabul qiladigan, ya'ni* $f(c)=0$ *bo'ladigan kamida bitta* $c\in(a;\ b)$ *nuqta mavjud bo'ladi* (IV.25-rasm).

4-teorema. $f(x)$ *funksiya* $[a;\ b]$ *kesmada uzluksiz va* min $\{f(a), f(b)\}=A$, max$\{f(a),\ f(b)\}=B$ *bo'lsa,* $f(x)$ *funksiya A va B orasidagi har qanday C qiymatni qabul qiladi* (IV.26-rasm).

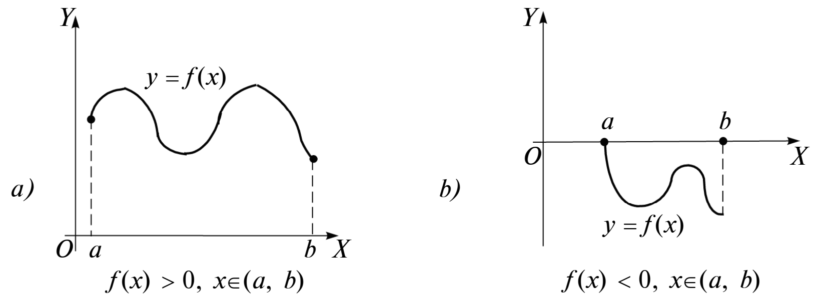


IV.25-rasm.



$f(c_1)=C,\ f(c_2)=C,\ f(c_3)=C$

IV.26-rasm.



IV.27-rasm.

5 - t e o r e m a . *Agar $f(x)$ funksiya $[a; b]$ kesmada uzluksiz va $(a; b)$ intervalda nolga aylanmasa, u shu oraliqning barcha ichki nuqtalarida bir xil ishorali bo'ladi* (IV.27-rasm).

$y = c$ ($c = \text{const}$), $y = x^n$, $y = a^x$ ($a > 0$, $a \neq 0$), $y = \log_a x$ ($a > 0$, $a \neq 0$), $y = \sin x$, $y = \cos x$, $y = \text{tg}x$, $y = \text{ctg}x$, $y = \arcsin x$, $y = \arccos x$, $y = \text{arctg}x$ va $y = \text{arcctg}x$ funksiyalar *eng sodda elementar funksiyalar*, ular ustida chekli marta arifmetik amallar, shuningdek funksiyadan funksiya olish (funksiyalar superpozitsiyasi) amallarini bajarishdan hosil bo'lgan funksiyalar esa *elementar funksiyalar* deb atalishini eslatib o'tamiz.

Masalan, $y = \sin(\ln^2 x + \cos x - x^2 + \text{arctg}(\cos x^2) + 1)$ funksiya elementar funksiya, 1-bandda keltirilgan 2-misoldagi funksiya esa elementar funksiya emas.

Oliy matematika kursida quyidagi teorema isbotlanadi.

T e o r e m a . *Barcha elementar funksiyalar o'zining aniqlanish sohasida uzluksizdir.*

3 - m i s o l . $y = \frac{1}{x^2 - 5x + 6}$ funksiya uzluksiz bo'ladigan barcha nuqtalar to'plamini topamiz.

Y e c h i s h . Berilgan funksiya elementar funksiya va uning aniqlanish sohasi $(-\infty; 2) \cup (2; 3) \cup (3; +\infty)$ to'plamdan iborat. Yuqorida keltirilgan teoremaga ko'ra bu to'plam izlangan to'plamdir.

4 - m i s o l . $y = (1+x)^{\frac{1}{x}}$ funksiya uzluksiz bo'ladigan barcha nuqtalar to'plamini toping.

Y e c h i s h . Bu funksiya darajali-ko'rsatkichli $y = u(x)^{v(x)}$ funksiyaning xususiy holi bo'lib, darajali-ko'rsatkichli funksiya ta'rifiga ko'ra $y = (1+x)^{\frac{1}{x}} = e^{\ln(1+x)^{\frac{1}{x}}} = e^{\frac{1}{x}\ln(1+x)}$ tenglik o'rinlidir. Bundan qaralayotgan funksiya elementar funksiya ekanligi va uning aniqlanish sohasi $(-1;\ 0)\cup(0;\ +\infty)$ to‘plamdan iboratligini ko‘ramiz.

Yuqorida keltirilgan teoremaga ko‘ra, $(-1;\ 0)\cup(0;\ +\infty)$ to‘plam izlangan to‘plamdir.

Mashqlar

4.21. Funksiyalarning uzluksizlik oraliqlarini toping:

1) $y=\frac{x}{(x-1)(x-3)}$; 2) $y=\frac{1}{x^2+2x-3}$;

3) $y=\frac{x^2+2x-8}{x^3-x^2-12x}$; 4) $y=\frac{1}{x+3}-\frac{1}{x-4}$;

5) $f(x)=\begin{cases}-x^2, & x<0,\\ x-1, & x>0;\end{cases}$ 6) $f(x)=\begin{cases}\frac{1}{x}, & x\le 2,\ x\ne 0,\\ 0, & x=0,\\ x, & x>2.\end{cases}$

4.22. $f(x)$ funksiyaning $x=k,\ l,\ m,\ n$ dagi qiymatlarini hisoblang, ishoralarining saqlanish va nollari mavjud bo‘lgan intervallarini aniqlang, $[a;\ b]$ oraliqdagi nolini ε gacha aniqlikda toping («kesmani teng ikkiga bo‘lish» va al-Koshiy usullarini tatbiq eting, mikrokalkulator yoki EHM dan foydalaning):

1) $f(x)=x^3-6x+5,\ k=-3,\ l=-2,\ m=1{,}5,\ n=2,\ a=l,$ $b=m,\ \varepsilon=0{,}001$;

2) $f(x)=x^3-4x-3,\ k=-1{,}5,\ l=-1{,}2,\ m=0,\ n=4,\ a=-1{,}5,$ $b=-1{,}2,\ \varepsilon=0{,}001$;

3) $f(x)=x^3-3x+2=0,\ k=-3,\ l=-1,\ m=2,\ n=3,\ a=-3,$ $b=0,\ \varepsilon=0{,}01$.

4.23. $x^4+x-\frac{2}{x}=0$ tenglama $[0{,}5;\ 1{,}5]$ kesmada ildizga ega ekanini isbot qiling va shu ildizni 0,01 gacha aniqlik bilan toping.

4.24. Tengsizlikni yeching:

1) $(x+5)(x-4)>0$; 2) $(x+4)(x-3)<0$;

3) $(x+2)(x+4)(x+5)\ge 0$; 4) $(x^2-1)(x+6)<0$.

4.25. Tengsizlikni yeching:

1) $(x-1)^7(x+2)(x+4)^{10} > 0$; 2) $x^3 - 4x^2 - x + 4 > 0$.

4.26. Tengsizlikni yeching:

1) $\frac{x^3+27}{(x^4-16)(x^2-25)} > 0$; 2) $\frac{x^2-9}{x^3-2x^2+4x} \geq 0$.

3. Ajoyib limitlar. Ko'pchilik hollarda limitlarni hisoblash masalasi

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \tag{1}$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \tag{2}$$

formulalar yordamida hal etilishi mumkin. Ulardan birini, masalan, (1) tenglikni isbotlash bilan cheklanamiz. $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$ tenglik barcha $x \neq 0$ sonlari uchun o'rinli bo'lgani sababli, (1) tenglikni $x \to 0+0$ bo'lgan hol uchun isbotlash yetarlidir.

$x \to 0+0$ bo'lsin. U holda $x \in \left(0;\ \frac{\pi}{2}\right)$ va $\sin x > 0$ deb hisoblash mumkin.

Birlik aylananing $2x$ radianli MN yoyini qaraymiz (IV.28-rasm). U holda $\breve{MN} = 2x$ va $MN = 2\sin x$, $2MK = 2\mathrm{tg}x$ tengliklar o'rinli bo'ladi, chunki M nuqtaning ordinatasi $\sin x$ ga, MK uzunlik esa $\mathrm{tg}x$ ga tengdir.

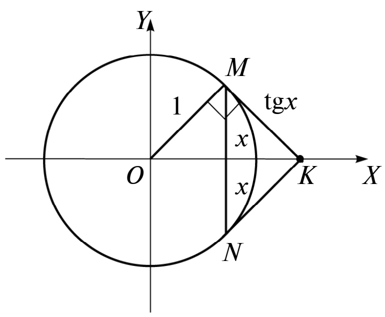
$MN < \breve{MN} < 2MK$ yoki $2\sin x < 2x < 2\mathrm{tg}x$, ya'ni $\sin x < x < \mathrm{tg}x$ ekanligini ko'ramiz. Bu tengsizlikning hamma hadlarini $\sin x > 0$ ga bo'lib, $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ va demak,

$$\cos x < \frac{\sin x}{x} < 1 \tag{3}$$

tengsizlikka ega bo'lamiz.

$y = \cos x$ funksiya uzluksiz bo'lgani uchun $\lim_{x\to 0}\cos x = \cos 0 = 1$ tenglik o'rinli bo'ladi.

1-§, 3-band, 5-teoremaga ko'ra, (3) tengsizlikdan $\lim_{x\to 0}\frac{\sin x}{x} = 1$ ekanligi kelib chiqadi.



IV.28-rasm.

1 - m i s o l. $\lim\limits_{x\to 0}\frac{\sin kx}{mx}$ limitni hisoblaymiz ($k\neq 0,\ m\neq 0$).

Y e c h i s h. $kx=t$ deb olamiz. $x\to 0$ da $t\to 0$ va, aksincha, $t\to 0$ da $x\to 0$ ekanini ko'rish qiyin emas. U holda (1) tenglikka ko'ra

$$\lim_{x\to 0}\frac{\sin kx}{mx}=\frac{1}{m}\lim_{x\to 0}\frac{k\sin kx}{kx}=\frac{k}{m}\lim_{x\to 0}\frac{\sin kx}{kx}=\frac{k}{m}\lim_{x\to 0}\frac{\sin t}{t}=\frac{k}{m}\cdot 1=\frac{k}{m}$$

bo'ladi.

2 - m i s o l. $\lim\limits_{x\to 0}\frac{\sin kx}{\sin mx}$ limitni hisoblaymiz ($k\neq 0,\ m\neq 0$).

Y e c h i s h. $\frac{\sin kx}{\sin mx}=\frac{m}{k}\cdot\frac{\sin kx}{mx}\cdot\frac{kx}{\sin mx}=\frac{m}{k}\cdot\frac{\sin kx}{mx}\cdot\frac{1}{\frac{\sin mx}{kx}}$ bo'lgani uchun 1-misolga ko'ra

$$\lim_{x\to 0}\frac{\sin kx}{\sin mx}=\lim_{x\to 0}\left(\frac{m}{k}\cdot\frac{\sin kx}{mx}\cdot\frac{1}{\frac{\sin mx}{kx}}\right)=\frac{m}{k}\cdot\lim_{x\to 0}\frac{\sin kx}{mx}\cdot\lim_{x\to 0}\frac{1}{\frac{\sin mx}{kx}}=$$

$$=\frac{m}{k}\cdot\frac{k}{m}\cdot\frac{1}{\frac{m}{k}}=\frac{k}{m}$$

bo'ladi.

3 - m i s o l. $\lim\limits_{x\to 0}\frac{\cos 5x-\cos 9x}{x^2}$ limitni hisoblaymiz.

Y e c h i s h. $\frac{\cos 5x-\cos 9x}{x^2}=\frac{2\sin 7x\sin 2x}{x^2}=2\cdot\frac{\sin 7x}{x}\cdot\frac{\sin 2x}{x}$

bo'lgani uchun $\lim\limits_{x\to 0}\frac{\cos 5x-\cos 9x}{x^2}=2\cdot\lim\limits_{x\to 0}\frac{\sin 7x}{x}\cdot\lim\limits_{x\to 0}\frac{\sin 2x}{x}=$
$=2\cdot\frac{7}{1}\cdot\frac{2}{1}=28$ tenglikka egamiz.

(1) tenglik *birinchi ajoyib limit*, (2) tenglik *ikkinchi ajoyib limit* deb yuritiladi.

4 - m i s o l. $\lim\limits_{x\to 0}\frac{a^x-1}{x}$, ($a>0,\ a\neq 1$) limitni hisoblaymiz.

Y e c h i s h. $y=a^x-1$ tenglik yordamida yangi o'zgaruvchi kiritamiz. U holda, $x=\log_a(1+y)$ bo'lgani uchun

$$\frac{a^x-1}{x}=\frac{y}{\log_a(1+y)}\quad\text{yoki}\quad\frac{a^x-1}{x}=\frac{y}{\frac{1}{y}\log_a(1+y)}=\frac{1}{\log_a(1+y)^{\frac{1}{y}}}$$

tenglik o'rinli bo'ladi.

$x \to 0$ da $y \to 0$ ekanini va, aksincha, $y \to 0$ da $x \to 0$ ekanini ko'rish qiyin emas.

$\log_a (1+y)^{\frac{1}{y}}$ uzluksiz funksiya bo'lgani uchun (2) tenglikka ko'ra $\lim\limits_{y \to 0} \log_a (1+y)^{\frac{1}{y}} = \log_a \left(\lim\limits_{y \to 0} (1+y)^{\frac{1}{y}} \right) = \log_a e = \frac{1}{\ln a}$ tenglikka egamiz. U holda

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{1}{\log_a (1+y)^{\frac{1}{y}}} = \frac{1}{\lim\limits_{y \to 0} \log_a (1+y)^{\frac{1}{y}}} = \ln a .$$

4-misolni yechish jarayonida $\lim\limits_{y \to 0} \frac{\log_a (1+y)}{y} = \frac{1}{\ln a}$ tenglik ham isbotlanganligini eslatib o'tamiz.

Mashqlar

4.27. Hisoblang:

1) $\lim\limits_{x \to 0} \frac{\sin 3x}{3x}$; 2) $\lim\limits_{x \to 0} \frac{\sin 6x}{4x}$;

3) $\lim\limits_{x \to 0} \frac{\operatorname{tg} 4x}{3x}$; 4) $\lim\limits_{x \to 0} \frac{\sin 5x}{\sin 6x}$;

5) $\lim\limits_{x \to 0} \frac{\operatorname{tg} 8x}{\operatorname{tg} 7x}$; 6) $\lim\limits_{x \to 0} x \operatorname{ctg} 6x$.

4.28. Limitlarni hisoblang:

1) $\lim\limits_{x \to 0} \frac{\sin 13x}{13x}$; 2) $\lim\limits_{x \to 0} \frac{\sin 3x}{\sin 6x}$;

3) $\lim\limits_{x \to 0} \frac{\sin 16x}{12x}$; 4) $\lim\limits_{x \to 0} \frac{\operatorname{tg} 5x}{\operatorname{tg} 6x}$;

5) $\lim\limits_{x \to 0} \frac{\operatorname{tg} 8x}{7x}$; 6) $\lim\limits_{x \to 0} x \operatorname{ctg} 3x$.

4.29. Limitlarni hisoblang:

1) $\lim\limits_{x \to 0} \frac{\operatorname{tg} 3x}{\sin 3x}$; 2) $\lim\limits_{x \to 0} \frac{\sin 4x}{2x \cos 3x}$;

3) $\lim\limits_{x\to 0}\frac{x\operatorname{tg}3x}{\sin^2 2x}$; 4) $\lim\limits_{x\to 0}\frac{\sin 5x\operatorname{tg}3x}{x^2}$;

5) $\lim\limits_{x\to 0}\frac{3x\cos 5x}{\sin 3x}$; 6) $\lim\limits_{x\to 0}\frac{2x\operatorname{tg}4x}{\sin^2 6x}$;

7) $\lim\limits_{x\to 0}\frac{1-\cos x}{5x^2}$; 8) $\lim\limits_{x\to 0}\frac{\arcsin 3x}{5x}$;

9) $\lim\limits_{x\to 0}\frac{\sqrt{1-\cos 2x}}{|x|}$; 10) $\lim\limits_{x\to 0}\frac{5x}{\operatorname{arctg}x}$;

11) $\lim\limits_{x\to 0}\frac{\cos x-\cos^3 x}{x^2}$; 12) $\lim\limits_{x\to 0}\frac{x^2\operatorname{ctg}x}{\sin 3x}$;

13) $\lim\limits_{x\to 0}\frac{1-\cos 6x}{1-\cos 2x}$; 14) $\lim\limits_{x\to 0}\frac{1-\cos 4x}{2x\cdot\operatorname{tg}2x}$;

15) $\lim\limits_{x\to 0}5x\cdot\operatorname{ctg}3x$; 16) $\lim\limits_{x\to 0}\frac{\sin 6x\operatorname{tg}2x}{x^2}$;

17) $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\frac{\pi x}{2}$; 18) $\lim\limits_{x\to\frac{\pi}{4}}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{3\pi}{4}+x\right)}$;

19) $\lim\limits_{x\to\pi}\sin 2x\operatorname{ctg}x$; 20) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)}{x-\frac{\pi}{2}}$.

4.30. Limitlarni hisoblang:

1) $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{x}}$; 2) $\lim\limits_{x\to 0}(1+7x)^{8x^{-1}}$;

3) $\lim\limits_{x\to 0}\frac{\ln(1+4x)}{x}$; 4) $\lim\limits_{x\to 0}\frac{\ln(1+9x)}{7x}$.