O’ZBEKISTON RESPUBLIKASI OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

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DIFFERENSIAL TENGLAMALAR KAFEDRASI

5460100 -“MATEMATIKA” YO’NALISHI BO’YICHA BAKALAVR AKADEMIK DARAJASINI OLISH UCHUN

MALAKAVIY BITIRUV ISHI

MAVZU: IKKI KARRALI INTEGRALLAR VA ULARNING TADBIQLARI.

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Himoyaga tavsiya etildi

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Mundarija

Kirish

1. **Masalaning qo'yilishi.** Bitiruv malakaviy ishida ikki karrali integral to'g'risida umumiy tushunchalar, uning ta'rifi, xossalari, ikki karrali integralni hisoblash, ikki karrali integralda o'zgaruvchilarni almashtirish, Ostragradskiy determinantidan foydalanish, tekis soha yuzasi, jismning hajmini, sirt yuzasini, jismning massasini, inertsiya moment, shuningdek jismning og'irlik markazining koordinatalarini topish masalalarini hal etish ko'zda tutiladi.
2. **Mavzuning dolzarbligi.** Ikki karrali integral matematik analizning asosiy masalalaridan biri bo'lib, faqatgina geometriya, fizika masalalarini hal etishda emas, balki texnika fanlarining ko'pgina sohalaridagi masalalarni hal qilishda ishlatiladi. Shuning uchun bitiruv ishining mavzusi va o'zi dolzarb.
3. **Bitiruv malakaviy ishning maqsadi.** Ikki karrali integralning barcha xossalari, hisoblash qoidalarini o'rgangan holda, o'zgaruvchilarni almashtirish usullarini qo'llab, geometriya, fizikaga oid masalalarni yechishdir.
4. **Ilmiy tadqiqot usullari.** Bitiruv malakaviy ishda qo'yilgan masala referativ xarakterda bo'lishiga qaramay, ikki karrali integralning yassi sohaning yuzini, jismlarning hajmini, jismning massasini, inertsiya momentini, og'irlik markazining koordinatalarini kabi amaliy masalalarni hal qilishga bag'ishlanadi.
5. **Ishning ilmiy ahamiyati.** Malakaviy bitiruv ishi – matematikada qo'llaniladigan barcha, geometriya, mexanika, fizikaga oid amaliyot masalalarini yechishga tadbiq etish mumkin.
6. **Ishning amaliy ahamiyati.** Malakaviy bitiruv ishida hal qilingan ba'zi bir masalalari hayotda uchrashi lozim bo'lib, ular yechimlari ikki karrali integralning barcha xossa, hisoblash usullari asosida bajarilgan.
7. **Ishning tuzilishi.** Malakaviy bitiruv ishi 2 bob va paragraflardan iborat.

I bob ikki karrali integral to'g'risidagi asosiy tushunchalardan tuzilgan bo'lib, 8 ta paragrafni o'z tarkibida saqlaydi:

1-§. Ikki o'lchovli integral

2-§. Ikki o'lchovli integralni hisoblash

3-§. Ikki o'lchovli integrallar yordamida yuzlar va hajmlarni hisoblash

4-§. Qutb koordinatalaridagi ikki o'lchovli integral. O'zgaruvchilarni almashtirish

5-§. Sirt yuzini hisoblash

6-§. Modda taqsimlanishining zichligi va ikki o'lchovli integral

7-§. Yassi (tekis) shakl yuzining inertsiya moment

8-§. Yassi (tekis) shakl yuzi og'irlik markazining koordinatalari.

II bob ikki karrali integralning tadbiqlaridan iborat bo'lib, 5 ta paragrafni o'z tarkibida saqlaydi:

1-§. Ikki karrali integralni hisoblash

2-§. Ikki karrali integral yordamida yuzlarni hisoblash

3-§. Ikki karrali integral yordamida jismlar hajmlarini hisoblash

4-§. Ikki karrali integral yordamida sirt yuzasini aniqlash

5-§. Ikki karrali integralning fizikadagi ba'zi tadbiqlari

I BOB. IKKI KARRALI INTEGRAL ASOSIY TUSHUNCHALARI

1-§. Ikki o'lchovli integral

Oxy tekislikda L chizik bilan chegaralangan, D yopiq sohani qaraymiz.

D sohada uzluksiz funksiya,

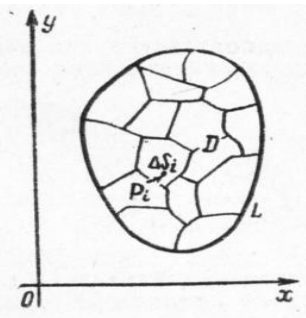
$$z = f(x, y)$$

berilgan bo'lsin.

D sohani ixtiyoriy chiziqlar bilan n ta bo'lakka bo'lamiz:

$$\Delta s_1, \Delta s_2, \Delta s_3, ..., \Delta s_n$$

(1-rasm), ularni yuzchalar deb ataymiz. Yangi simvollar kiritmaslik maqsadida $\Delta s_1, ..., \Delta s_n$ orqali bularning nomlarinigina emas, yuzlarini ham belgilaymiz. Δs_i yuzlarning har birida P_i, nuqta olamiz (bu nuqta yuzning ichida yoki chegarasida yotishining farqi yo'q), u holda n ta nuqta hosil bo'ladi: $P_1, P_2, ..., P_n$. Funksiyaning tanlangan nuqtalardagi qiymatlarini $f(P_1), f(P_2), ..., f(P_n)$ bilan belgilaymiz va $f(P_i)\Delta s_i$ ko'rinishdagi ko'paytmalarning yig'indisini tuzamiz



1-rasm

$$V_n = f(P_1)\Delta s_1 + f(P_2)\Delta s_2 + ... + f(P_n)\Delta s_n = \sum_{i=1}^{n} f(P_i)\Delta s_i. \qquad (1)$$

Bu yig'indi D sohada $f(x, y)$ funksiya uchun integral yig'andi deb ataladi.

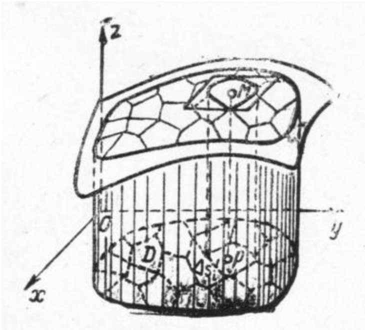
Agar D sohada $f \geq 0$ bo'lsa, u holda har bir $f(P_i)\Delta s_i$ qo'shiluvchini, geometrik jihatdan asosi Δs_i ga, balandligi esa $f(P_i)$ ga teng bo'lgan silindrchaning hajmi deb qarash mumkin.

V_n yig'indi, ko'rsatilgan elementar silindrchalar hajmlarining yig'indisi, ya'ni qandaydir „pog'onali* jismning hajmi bo'ladi (2-rasm).

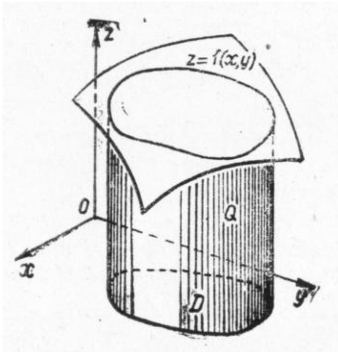
Berilgan D soha uchun $f(x, y)$ funksiya yordami bilan tuzilgan integral yig'indilarning D sohani Δs_i bo'laklarga turli usullar bilan bo'lishdan hosil qilingan ixtiyoriy

$$V_{n_1}, V_{n_2}, ..., V_{n_k}, ... \quad (2)$$

ketma-ketlikni qaraymiz. $n_k \to \infty$ da Δs_i yuzlarning maksimal diametri nolga intiladi deb faraz etamiz. Bu holda biz quyida isbotsiz keltiradigan teorema to'g'ri bo'ladi.



2- rasm.



3- rasm.

1-teorema. Agar $f(x, y)$ funksiya yopiq D sohada uzluksiz va $n \to \infty$ da Δs_i yuzning maksimal diametri nolga intilsa, (1) integral yig'indilardan hosil bo'lgan (2) ketma-ketlikning limiti mavjud bo'ladi. Bu limit (2) shakldaga har qanday ketma-ketlik uchun bir xildir, ya'ni u D sohani Δs_i yuzga bo'lish usuliga va bu Δs_i yuz ichida P_i nuqtani tanlab olish usuliga bog'liq emas.

Bu limitni $f(x, y)$ funksiyaning D soha bo'yicha olingan ikki o'lchovli integrali deyiladi va quyidagicha belgilanadi:

$$\iint_D f(P)ds \quad \text{yoki} \quad \iint_D f(x, y)dxdy,$$

ya'ni:

$$\lim_{diam\,\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i = \iint_D f(x, y)dxdy.$$

Bu yerda D soha integrallash sohasi deyiladi.

Agar $f(x, y) \geq 0$ bo'lsa, $f(x, y)$ funksiyaning D soha bo'yicha olingan ikki o'lchovli integrali $z = f(x, y)$ sirt, $z = 0$ tekislik va yasovchisi Oz o'qiga parallel, yo'naltiruvchisi esa D sohaning chegarasidan iborat bo'lgan silindrik sirt bilan chegaralangan jismning hajmi Q ga tengdir (3- rasm).

Endi ikki o'lchovli integral haqidagi quyidagi teoremalarni ko'rib chiqamiz.

2-**teorema.** Ikki funksiya yig'indisi $\varphi(x,y)+\psi(x,y)$ ning D soha bo'yicha olingan ikki o'lchovli integrala ularning har biridan shu D soha bo'yicha ayrim olingan ikki o'lchovli integrallar yig'indisiga teng:

$$\iint_D [\varphi(x,y)+\psi(x,y)]ds = \iint_D \varphi(x,y)ds + \iint_D \psi(x,y)ds.$$

3-teorema. O'zgarmas ko'paytuvchini ikki o'lchovli integral belgisining oldiga chiqarish mumkin:

agar $a = const$ bo'lsa, u holda:

$$\iint_D a\varphi(x,y)ds = a\iint_D \varphi(x,y)ds.$$

Bu ikkala teoremaning isboti, aniq integralga tegishli moc teoremalarning isbotiga o'xshashdir (I t. XI bobining 3-paragrafiga qarang).

4-teorema. Agar D soha ikki umumiy nuqtalarga ega bo'lmagan ikkita D_1 va D_2 sohaga bo'lingan bo'lsa va $f(x,y)$ funksiya D sohaning hamma nuqtalarida uzluksiz bo'lsa, u holda:

$$\iint_D f(x,y)dxdy. = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy. \qquad (3)$$

Isbot. D soha bo'yicha olingan integral yig'indini quyidagicha tasvirlash mumkin (4-rasm):

$$\sum_D f(P_i)\Delta s_i = \sum_{D_1} f(P_i)\Delta s_i + \sum_{D_2} f(P_i)\Delta s_i, \qquad (4)$$

bunda birinchi yig'indi D_1 soha yuzlariga tegishli qo'shiluvchilardan, ikkinchi yig'indi esa D_2 soha yuzlariga tegishli qo'shiluvchilardan iborat. Haqiqatan, ikki o'lchovli integral sohaning bo'linish usuliga bog'liq bo'lmagani uchun biz D sohani D_1 va D_2 sohalarning umumiy chegarasi yuzlarning ham umumiy chegarasi bo'la oladigan qilib bo'lamiz. $\Delta s_i \to 0$ deb (4) tenglikdan limit olsak, (3) tenglik hosil bo'ladi. Qo'shiluvchilar soni har qancha bo'lganda ham bu teoremaning to'g'ri ekanligi o'z-o'zidan ma'lum.

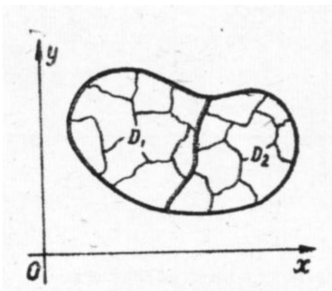
2-§. Ikki o'lchovli integralni hisoblash

Oxy tekislikda yotuvchi D sohani koordinata o'qlaridan biriga, masalan, Oy o'qqa parallel bo'lgan va sohaning ichki nuqtasidan o'tadigan har qanday to'g'ri chiziq uning chegarasini ikki N_1 va N_2 nuqtada kesib o'tadigan qilib olamiz (5-rasm).

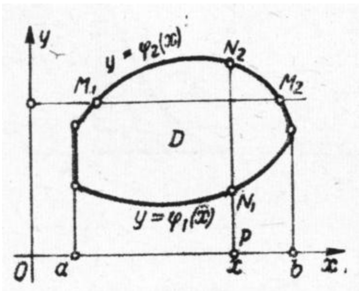
Qaralayotgan holda D soha $y=\varphi_1(x),\ \ y=\varphi_2(x),\ \ x=a,\ \ x=b$ chiziqlar bilan chegaralangan, bunda

$$\varphi_1(x)\le\varphi_2(x),\quad a<b$$

bo'lib, $\varphi_1(x),\ \varphi_2(x)$ funksiyalar $[a,b]$ kesmada uzluksiz deb faraz etamiz. Bunday sohani biz Oy o'q yo'nalishida to'g'ri bo'lgan soha deb ataymiz. Ox o'q yo'nalishida to'g'ri bo'lgan soha ham shuning singari aniqlanadi.



4- rasm.



5- rasm.

Ham Ox, ham Oy o'qlar yo'nalishida to'g'ri bo'lgan sohani qisqacha to'g'ri soha deymiz. 5- rasmda Ox va Oy o'qlar bo'yicha to'g'ri bo'lgan soha D ko'rsatilgan.

$f(x,y)$ funksiya D sohada uzluksiz bo'lsin. Quyidagi ifodani qaraymiz:

$$I_D=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)dy\right)dx.$$

Bu ifodani $f(x,y)$ funksiyaning D soha bo'yicha olingan ikki karrali integrali deb ataymiz. Uni hisoblash uchun x ni o'zgarmas deb qarab, kavs ichidagi ifodani avval y bo'yicha integrallaymiz. Integrallash natijasida x ning uzluksiz funksiyasi hosil bo'ladi:

$$\Phi(x)=\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy.$$

Bu funksiyani x bo'yicha a dan b gacha chegarada integrallaymiz

$$I_D=\int\limits_a^b \Phi(x)dx.$$

Natijada biror o'zgarmas son chiqadi.

Misol. Ushbu ikki karrali integral hisoblansin:

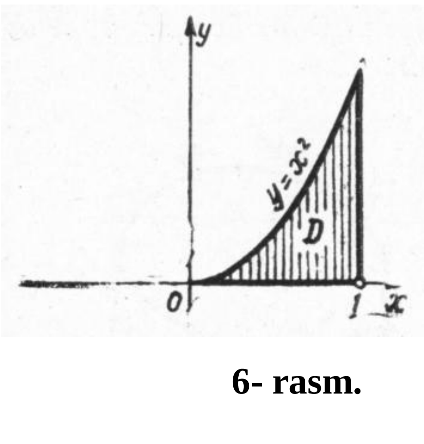
$$I_D=\int\limits_0^1\left(\int\limits_0^{x^2}(x^2+y^2)dy\right)dx.$$

Yechish. Avval ichki integralni (qavs ichidagini) hisoblaymiz;

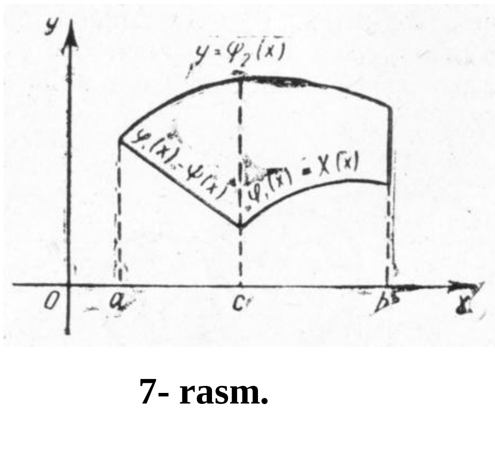
$$\Phi(x)=\int\limits_0^{x^2}(x^2+y^2)dy=(x^2y+\frac{y^3}{3})\Bigg|_0^{x^2}=x^2x^2+\frac{(x^2)^3}{3}=x^4+\frac{x^6}{3}.$$

Hosil bo'lgan funksiyani 0 dan 1 gacha chegarada integrallaymiz

$$\int\limits_0^1\left(x^4+\frac{x^6}{3}\right)dx=\left(\frac{x^5}{5}+\frac{x^7}{3\cdot 7}\right)\Bigg|_0^1=\frac{1}{5}+\frac{1}{27}=\frac{26}{105}.$$



6- rasm.



7- rasm.

D sohani aniqlaymiz. Bu holda D deb $y=0$, $y=x^2$, $x=0$, $x=1$ chiziqlar bilan chegaralangan soha qaraladi (6-rasm).

D soha x o'zgaradigan ($x = a$ dan $x = b$ gacha) oraliqning hammasida $y = \varphi_1(x),\ \ y = \varphi_2(x)$ funksiyalardan birini analitik ifodalab bo'lmaydigan soha bo'lib qolishi mumkin.

Masalan, $a < c < b$ bo'lib, $[a, c]$ kesmada $\varphi_1(x) = \psi(x),\ \ [c, b]$ kesmada $\varphi_1(x) = \chi(x),$ bo'lsin, bunda $\psi(x)$ va $\chi(x)$ funksiyalar analitik usulda berilgan (7-rasm). U vaqtda ikki karrali integral quyidagicha yoziladi:

$$\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx = \int_a^c \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx + \int_c^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx =$$

$$= \int_a^c \left(\int_{\psi(x)}^{\varphi_2(x)} f(x,y)dy \right) dx + \int_c^b \left(\int_{\chi(x)}^{\varphi_2(x)} f(x,y)dy \right) dx.$$

Bu tengliklardan birinchisi aniq integralning ma'lum xossasiga, ikkinchisi esa $[a, c]$ uchastkada $\varphi_1(x) \equiv \psi(x),\ \ [c, b]$ uchastkada $\varphi_1(x) \equiv \chi(x),$ ekanligiga asosan yozilgan.

Agar $\varphi_2(x)$ funksiya $[a, b]$ kesmaning turli uchastkalarida turlicha analitik ifodalar bilan berilganida ham ikki karrali integralni yuqoridagi kabi yozish o'rinli bo'lar edi.

Ikki karrali integralning ba'zi bir xossalarini aniqlab olaylik:

1-xossa. Agar Oy o'q yo'nalishida to'g'ri bo'lgan D sohani Oy yoki Ox o'qqa parallel to'g'ri chiziq bilan ikki D_1, D_2 sohaga bo'linsa, D soha bo'yicha olingan ikki karrali I_D integral D_1 va D_2 sohalar bo'yicha olingan ikki karrali integrallarning yig'indisiga teng bo'ladi, ya'ni:

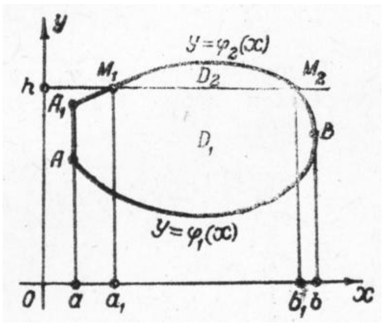
$$I_D = I_{D_1} + I_{D_2}. \qquad (1)$$

Isbot. a) $x = c \ \ a < c < b$ to'g'ri chiziq D sohani Oy o'q yo'nalishida to'g'ri bo'lgan ikki D_1 va D_2 sohaga bo'lsin. U holda:

$$I_D = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx = \int_a^b \Phi(x)dx = \int_a^c \Phi(x)dx + \int_c^b \Phi(x)dx =$$

$$= \int_a^c \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx + \int_c^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx = I_{D_1} + I_{D_2}.$$

b) Faraz qilaylik, $y = h$ to'g'ri chiziq D sohani 8-rasmda ko'rsatilganidek Oy o'q yo'nalishida to'g'ri bo'lgan ikki D_1 va D_2 sohaga bo'lsin. $y = h$ to'g'ri chiziq bilan D soha L chegarasining kesishish nuqtalarini M_1 va M_2 bilan, bu nuqtalarning abssissalarini esa a_1 va b_1 bilan belgilaymiz.



8-rasm.

D_1 soha quyidagi uzluksiz chiziqlar bilan chegaralangan:

1) $y = \varphi_1(x)$;

2) $A_1M_1M_2B$ egri chiziq; bu egri chiziqning tenglamasini shartli ravishda

$$y = \varphi_1^*(x)$$

shaklda yozamiz va bunda $a \le x \le a_1$ hamda $b_1 \le x \le b$ bo'lganda

$$\varphi_1^*(x) = \varphi_2(x)$$

va $a_1 \le x \le b_1$ bo'lganda

$$\varphi_1^*(x) = h$$

ekanligini nazarda tutamiz;

3) $x = a,\ x = b$ to'g'ri chiziqlar.

D_2 soha ushbu chiziqlar bilan chegaralangan:

$$y = \varphi_1^*(x),\quad y = \varphi_2(x) \text{ bunda } a_1 \le x \le b_1.$$

Ichki integralga integrallash oralig'ini bo'lish haqidagi teoremani tatbiq etib, ushbu ayniyatni yozamiz:

$$I_D = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x,y)dy + \int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx =$$

$$= \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x,y)dy \right) dx + \int_a^b \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx.$$

Tashqi integralga integrallash oralig'ini bo'lish haqidagi teoremani tatbiq etib, oxirgi integralni uchta integralga ajratamiz:

$$\int_a^b \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx = \int_a^{a_1} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx +$$

$$+ \int_{a_1}^{b_1} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx + \int_{b_1}^{b} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx;$$

$[a, a_1]$ va $[b_1, b]$ kesmalarda $\varphi_1^*(x) = \varphi_2(x)$ bo'lganligidan, birinchi va uchinchi integrallar aynan nolga teng. Shuning uchun

$$I_D = \int_a^b \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx = \int_{a_1}^{b_1} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,y)dy \right) dx.$$

Bu yerdagi birinchi integral D_1 soha bo'yicha, ikkinchisi D_2 soha bo'yicha olingan ikki karrali integraldir. Demak,

$$I_D = I_{D_1} + I_{D_2}.$$

$M_1 M_2$ kesuvchi to'g'ri chiziqning vaziyati har qanday bo'lganda ham isbot yuqoridagi singari qilinadi. Agar $M_1 M_2$ to'g'ri chiziq D sohani uchta yoki undan ortiq sohaga bo'lsa, o'ng tomonida mos sondagi qo'shiluvchilar turgan (1) munosabatga o'xshash tenglik hosil bo'ladi.

2-xossa (ikki karrali integralni baholash). $f(x,y)$ funksiyaning D sohadagi eng kichik va eng katta qiymatlari mos holda m va M bo'lsin, D sohaning yuzini S bilan belgilasak, u holda quyidagi munosabat o'rinla bo'ladi:

$$mS \le \int_a^b \left(\int_{\varphi_1}^{\varphi_2} f(x,y)dy \right) dx \le MS. \qquad (3)$$

Isbot. Ichki integralni $\Phi(x)$ bilan belgilab, uni baholaymiz:

$$\Phi(x)=\int_{\varphi_1}^{\varphi_2} f(x,y)dy \le \int_{\varphi_1}^{\varphi_2} M\,dy = M[\varphi_2(x)-\varphi_1(x)],$$

u holda:

$$I_D=\int_a^b\left(\int_{\varphi_1}^{\varphi_2} f(x,y)dy\right)dx \le \int_a^b M[\varphi_2(x)-\varphi_1(x)]dx = MS,$$

ya’ni:

$$I_D \le MS. \qquad (3')$$

Shunga o’xshash,

$$\Phi(x)=\int_{\varphi_1}^{\varphi_2} f(x,y)dy \ge \int_{\varphi_1}^{\varphi_2} mdy = m[\varphi_2(x)-\varphi_1(x)],$$

$$I_D=\int_a^b \Phi(x)dx \ge \int_a^b m[\varphi_2(x)-\varphi_1(x)]dx = mS,$$

ya’ni

$$I_D \le mS. \qquad (3'')$$

(3') va (3'') tengsizliklardan ham (3) munosabat kelib chi-qadi:

$$mS \le I_D \le MS.$$

Bu teoremaning geometrik ma’nosini keyingi paragrafda tushuntiramiz.

3-xossa (o’rta qiymat haqidagi teorema). $f(x,y)$ uzluksiz funksiyaning *D* soha bo’yicha olingan I_D ikki karrali integrali, *D* sohaning *S* yuzini funksiyaning *D* sohada olingan biror R nuqtadagi siymatiga ko’payti-rilganiga teng, ya’ni

$$\int_a^b\left(\int_{\varphi_1}^{\varphi_2} f(x,y)dy\right)dx = f(P)S. \qquad (4)$$

Isbot. (3) munosabatdan

$$m \le \frac{1}{S} I_D \le M$$

tengsizlikni hosil qilamiz.

$\frac{1}{S}I_D$ son $f(x,y)$ funksiyaning D sohadagi eng katta va eng kichik qiymatlari orasida joylashgan. $f(x,y)$ funksiyaning uzluksizligiga asosan ikki karrali integral D sohaning biror P nuqtasida $\frac{1}{S}I_D$ songa teng qiymat qabul qiladi, ya’ni:

$$\frac{1}{S}I_D = f(P),$$

bundan

$$I_D = f(P)S. \tag{5}$$

Ikki o’lchovli integralmi hisoblash (davomi)

Teorema. $f(x,y)$ uzluksiz funksiyaning D to’g’ri soha bo’yicha olingan ikki o’lchovli integrali funksiyaning o’sha D soha bo’yicha olingan ikki karrali integraliga teng, ya’ni

$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_{\varphi_1}^{\varphi_2} f(x,y)dy \right) dx.$$

Isbot. D sohani koordinata o’qlariga parallel to’g’ri chiziqlar bilan n ta to’g’ri (to’g’ri to’rt burchakli)

$$\Delta s_1, \Delta s_2, \ldots, \Delta s_n$$

sohalarga bo’lamiz.

Oldingi paragrafdagi 1-xossaning (2) formulasiga asosan:

$$I_D = I_{\Delta s_1} + I_{\Delta s_2} + \ldots + I_{\Delta s_n} = \sum_{i=1}^{n} I_{\Delta s_i}. \tag{1}$$

O’ng tomondagi qo’shiluvchilarning har birini ikki karrali integralning o’rta qiymati haqidagi teoremaga asosan almashtiramiz:

$$I_{\Delta s_i} = f(P_i)\Delta s_i.$$

U holda (1) tenglik quyidagi ko’rinishda bo’ladi:

$$I_D = f(P_1)\Delta s_1 + f(P_2)\Delta s_2 + ... + f(P_n)\Delta s_n = \sum_{i=1}^{n} f(P_i)\Delta s_i$$

bunda P_i nuqta Δs_i sohanig biror nuqtasi. O'ng tomonda $f(x,y)$ funksiya uchun D soha bo'yicha olingan integral yig'indi turibdi. Ikki o'lchovli integralning mavjudligi haqidagi teoremaga asosan $n \to \infty$ da va Δs_i yuzning eng katta diametri nolga intilganda bu yig'indining limiti mavjud va u $f(x,y)$ funksiyaning D soha bo'yicha olingan ikki o'lchovli integraliga teng ekanligi kelib chiqadi. (2) tenglikning chap tomonida turgan ikki karrali I_D iltegralning qiymati n ga bog'liq emas. Shunday qilib, (2) tenglikda limitga o'tib, quyidagini hosil qilamiz:

$$I_D = \lim_{diam\,\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i = \iint_D f(x,y)dxdy$$

yoki

$$\iint_D f(x,y)dxdy = I_D. \tag{3}$$

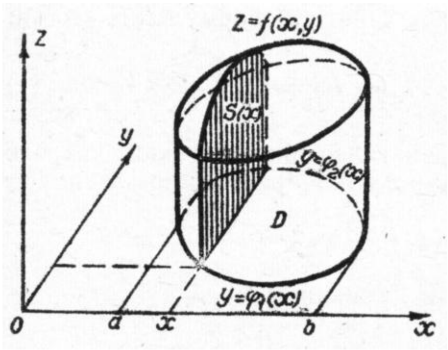
Endi I_D ikki karrali integralning aniq ifodasini yozib olib, oxirgi natijani chiqaramiz:

$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx. \tag{4}$$

1-izoh. $f(x,y) > 0$ bo'lganda, (4) formulaning geometrik ma'nosi yaqqol ko'rinadi. $z = f(x,y)$ sirt, $z = 0$ tekislik va yasovchisi Oz o'qqa parallel, yo'naltiruvchisi esa D sohaning chegarasidan iborat bo'lgan silindrik sirt bilan chegaralangan jismni qaraymiz (9-rasm). Bu jismning V hajmini hisoblaymiz. Bu jismning hajmi $f(x,y)$ funksiyaning D soha bo'yicha olingan ikki o'lchovli integraliga teng ekanligi yuqorida ko'rsatilgan edi:

$$V = \iint_D f(x,y)dxdy. \tag{5}$$

Endi bu jismning hajmini I t. XII bobining 4- paragrafidagi parallel kesimlarning yuzlari bo'yicha jismning hajmini hisoblashga doir natijadan foydalanib hisoblaymiz. Qaralayotgan jismni kesuvchi $x = const$ $a < x < b$ tekislik o’tkazamiz. $x = const$ kesimda hosil bo’ladigan figuraning $S(x)$ yuzini hisoblaymiz.



9-rasm.

Bu figura $z = f(x, y)$ $(x = const)$, $z = 0$, $y = \varphi_1(x)$, $y = \varphi_2(x)$ chiziqlar bilan chegaralangan egri chiziqli trapesiyadir. Demak, bu yuz:

$$S(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \tag{6}$$

integral bilan ifodalanadi. Parallel kesimlarning yuzlarini bilgan holda, jismning hajmini topish oson:

$$V = \int_a^b S(x)dx$$

yoki $S(x)$ yuz o’rniga uning (6) ifodadagi qiymatini qo’ysak, quyidagi tenglik hosil bo’ladi:

$$V = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx. \tag{7}$$

(5) va (7) formulalarning chap tomonlari teng, shuning uchun o’ng tomonlari ham tengdir:

$$\iint_D f(x, y)dxdy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx.$$

Endi ikki karrali integralni baholash haqidagi teoremaning geometrik ma’nosini tushuntirish qiyin emas (oldingi paragrafdagi 2-xossa): $z = f(x, y)$ sirt, $z = 0$ tekislik va yo’naltiruvchisi D sohaning chegarasidan iborat bo’lgan silindrik sirt bilan chegaralangan jismning V hajmi, asosi S balandligi m bo'lgan silindrning hajmidan katta, lekin asosi S, balandligi M bo'lgan silindrning hajmidan kichik [bunda m va M lar $z = f(x, y)$ funksiyaning D sohadagi eng kichik va eng katta qiymatlaridir (10-rasm). Bu esa I_D ikki karrali integral o'sha jismning V hajmiga teng ekanligidan kelib chiqadi.

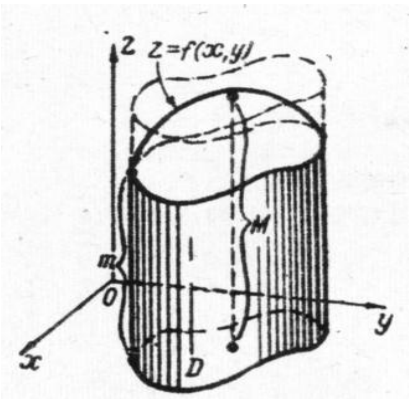
1-misol. Agar D soha $x = 0,\ x = 1,\ y = 0,\ y = \frac{3}{2}$ to'g'ri chiziqlar bilan chegaralangan bo'lsa, $\iint\limits_D (4 - x^2 - y^2)dxdy$ ikki o'lchovli integral hisoblansin.

Yechish. Bu integral (7) formulaga asosan hisoblanadi:

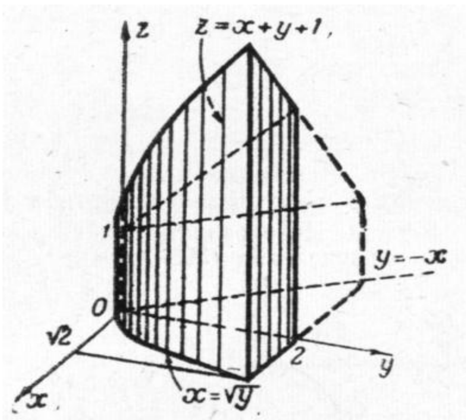
$$V = \int_0^{\frac{3}{2}} \left[\int_0^1 (4 - x^2 - y^2)dx \right] dy = \int_0^{\frac{3}{2}} \left[(4x - \frac{x^3}{3} - y^2 x) \right]_0^1 dy =$$

$$= \int_0^{\frac{3}{2}} (4 - \frac{1}{3} - y^2)dy = \left(4y - \frac{1}{3}y - \frac{y^3}{3} \right)\Bigg|_0^{\frac{3}{2}} = \frac{35}{8}.$$

2-misol. $f(x, y) = 1 + x + y$ funksiyaning $y = -x,\ x = \sqrt{y},\ y = 2,\ z = 0$ (11- rasm) chiziqlar bilan chegaralangan soha bo'yicha olingan ikki o'lchovli integrali hisoblansin.



10- rasm.



11- rasm.

Yechish.

$$V=\int_0^2\left[\int_{-y}^{\sqrt{y}}(1+x+y)dx\right]dy=\int_0^2\left[x+\frac{x^2}{2}+\frac{y^2}{2}\right]_{-y}^{\sqrt{y}}dy=$$

$$=\int_0^2\left[(\sqrt{y}+\frac{y}{2}+y\sqrt{y})-(-y+\frac{y^2}{2}-y^2)\right]dy=$$

$$=\int_0^2\left[\sqrt{y}+\frac{3y}{2}+y\sqrt{y})+\frac{y^2}{2}\right]dy=\left[\frac{2y^{\frac{3}{2}}}{3}+\frac{3y^2}{4}+\frac{2y^{\frac{5}{2}}}{5}+\frac{y^3}{6}\right]_0^2=\frac{44}{12}\sqrt{2}+\frac{13}{3}.$$

2-izoh. Ox o'q yo'nalishida to'g'ri bo'lgan D soha

$$x=\psi_1(y),\quad x=\psi_2(y),\quad y=c,\quad y=d$$

chiziqlar bilan chegaralangan va bunda $\psi_1(y)\le\psi_2(y)$ bo'lsin (12- rasm).

Ma'lumki, bu holda

$$\iint_D f(x,y)dxdy=d\int_c^b\left(\int_{\psi_1(x)}^{\psi_2(x)}f(x,y)dx\right)dy. \qquad (8)$$

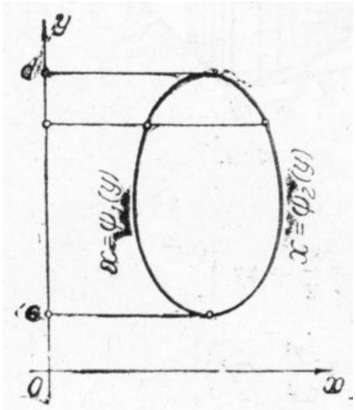
Ikki o'lchovli integralni hisoblash uchun uni ikki karrali integral shaklida tasvirlab olish kerak. Buni yuqorida ko'rib o'tganimiz kabi ikki xil usul bilan: yo (4) formula bo'yicha, yoki (8) formula bo'yicha bajarish mumkin. Har qaysi konkret hol uchun D sohaning yoki integral ostidagi funksiyaning shakliga qarab tegishli formulani tanlab olamiz.

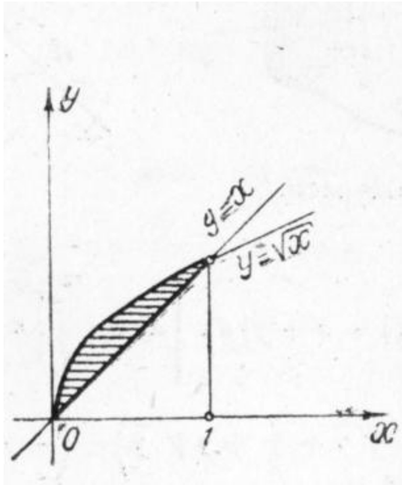
3- misol. Ushbu

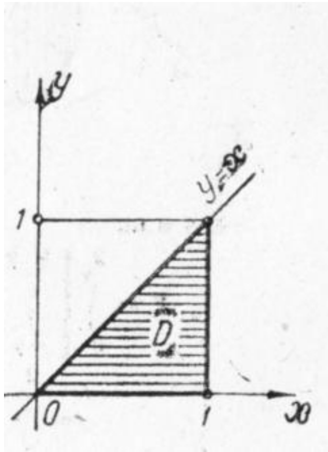
$$I=\int_0^1\left(\int_x^{\sqrt{x}}f(x,y)dy\right)dx$$

integralning integrallash tartibi o'zgartirilsin.

Yechish. Integrallash sohasi $y=x$ to'g'ri chiziq va $y=\sqrt{x}$ parabola bilan chegaralangan (13-rasm).







12- rasm. **13- rasm.** **14- rasm.**

Ox o'qqa parallel bo'lgan har qanday to'g'ri chiziq soha chegarasini ikkitadan ortiq nuqtada kesa olmaydi; demak, integralni

$$\psi_1(y) = y^2, \qquad \psi_2(y) = y, \qquad 0 \le x \le 1$$

deb olib, (8) formula bo'yicha hisoblash mumkin, u holda

$$I = \int_0^1 \left(\int_{y^2}^{y} f(x,y)dx \right) dy.$$

4-misol. Agar D soha $y = x,\ y = 0,\ x = 1$ to'g'ri chiziqlar bilan chegaralangan uchburchak shaklida (14- rasm) bo'lsa,

$$\iint_D e^{\frac{x}{y}} ds$$

integral hisoblansin.

Yechish. Berilgan ikki o'lchovli integralni ikki karrali integral bilan almashtiramiz. Bunda (4) formuladan foydalanamiz. (Agar (8) formulani tatbiq etsak, u holda $e^{\frac{x}{y}}$ funksiyani x bo'yicha integrallashga to'g'ri kelar edi, biroq bu integralni elementar funksiyalarda hisoblab bo'lmaydi),

$$\iint_D e^{\frac{x}{y}} ds = \int_0^1 \left(\int_0^x e^{\frac{x}{y}} dy \right) dx = \int_0^1 x e^{\frac{x}{y}} \Bigg|_0^x dx = \int_0^1 x(e-1)dx =$$

$$= (e-1)\frac{x^2}{2}\Big|_0^1 = \frac{e-1}{2} = 0.859...$$

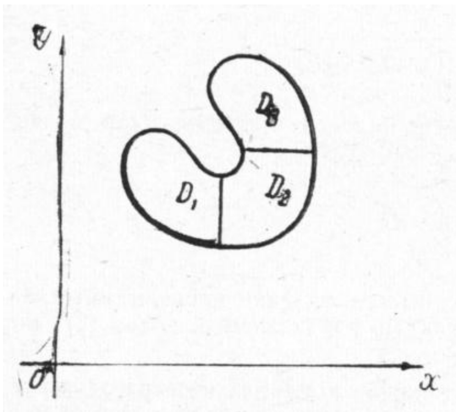
5-misol. Agar ichki kvadratning har bir tomoni 2 ga. tashqi kvadratning tomoni 4 ga teng bo’lsa, ushbu

$$\iint\limits_D e^{x+y}ds$$

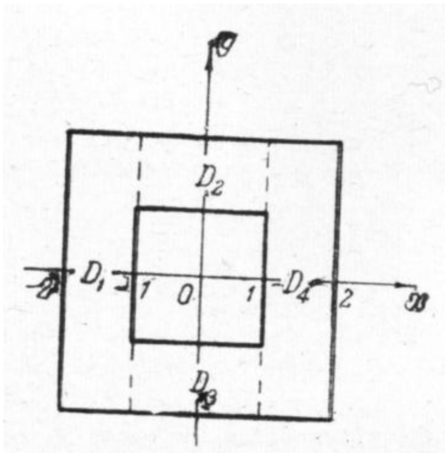
ikki ulchovli integral markazlari koordinatalar boshida bo’lib, tomonlari koordinata o’qlariga parallel holda joylashgan ikki kvadrat orasidagi D soha bo’yicha hisoblansin (16- rasm).

Yechish. D noto’g’ri soha. Biroq $x=-1$ va $x=1$ to’g’ri chiziqlar uni to’rtta $D_1, D_2, D_3,$ va D to’g’ri sohaga bo’ladi Shuning uchun

$$\iint\limits_D e^{x+y}ds = \iint\limits_{D_1} e^{x+y}ds + \iint\limits_{D_2} e^{x+y}ds + \iint\limits_{D_3} e^{x+y}ds + \iint\limits_{D_4} e^{x+y}ds.$$



15-rasm.



16-rasm.

Bu integrallarning har birini ikki karrali integral shakliga keltirib, quyidagini topamiz:

$$\iint\limits_D e^{x+y}ds = \int_{-2}^{-1}\left(\int_{-2}^{2} e^{x+y}dy\right)dx + \int_{-1}^{1}\left(\int_{1}^{2} e^{x+y}dy\right)dx +$$

$$+\int_{-1}^{1}\left(\int_{-2}^{-1} e^{x+y}dy\right)dx + \int_{1}^{2}\left(\int_{-2}^{2} e^{x+y}dy\right)dx =$$

$$=(e^2-e^{-2})(e^{-1}-e^{-2}) + (e^2-e)(e-e^{-1}) + (e^{-1}-e^{-2})(e-e^{-1}) +$$

$$+(e^2-e^{-2})(e^2-e) = (e^3-e^{-3})(e-e^{-1}) = 4shx3sh1.$$

3-§. Ikki o'lchovli integrallar yordami bilan yuzlar va hajmlarni hisoblash

1. Hajm. Biz $z = f(x, y)$ sirt, $z = 0$ tekislik va yo'naltiruvchi D sohaning chegarasidan iborat bo'lgan to'g'ri chiziq, yasovchisi esa Oz o'qqa parallel silindrik sirt bilan chegaralangan jismning V hajmi, D soha bo'yicha $f(x, y)$ funksiyadan (funksiya manfiy emas) olingan ikki o'lchovli integralga teng ekanligini 1-paragrafda ko'rgan edik:

1-misol. $x = 0,\ y = 0,\ x + y + z = 1,\ z = 0$ sirtlar bilan chegaralangan jismning hajmi hisoblansin (17- rasm).

Yechish.

$$V = \iint\limits_D (1 - x - y)dydx,$$

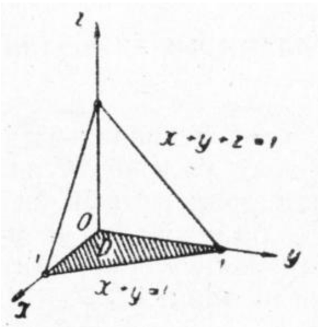
bundagi D soha 17-rasmda shtrixlab qo'yilgan va $x = 0,\ y = 0,\ x + y = 1$ to'g'ri chiziqlar bilan chegaralangan Oxy tekislikdagi uchburchak shakldagi sohadir. Chegaralarni ikki o'lchovli integralga qo'yib, hajmni hisoblaymiz:

$$V = \int\limits_0^1 \int\limits_0^{1-x} (1 - x - y)dydx = \int\limits_0^1 \left[((1 - x)y - \frac{y^2}{2}) \right]_0^{1-x} dx =$$

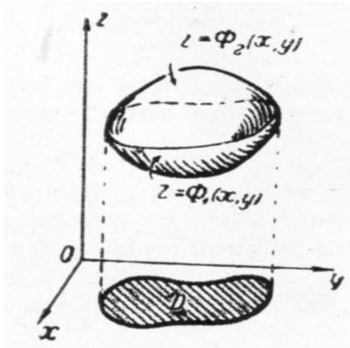
$$= \int\limits_0^1 \frac{1}{2}(1 - x)^2 dx = \frac{1}{6}.$$

Demak, $V = \dfrac{1}{6}$ kub birlik.

1-izoh. Agar hajmi izlanayotgan jism, yuqoridan $z = \Phi_2(x, y) \geq 0$ sirt, pastdan $z = \Phi_1(x, y) \geq 0$ sirt bilan chegaralangan va ikkala sirtning Oxy tekislikdagi proyeksiyasi D sohadan iborat bo'lsa, u holda bu jismning V hajmi ikkita „silindrik" jism hajmlarining ayirmasiga teng bo'ladi; bu silindrik jismlardan birinchisining pastki asosi D sohadan, ustki asosi $z = \Phi_2(x, y)$ sirtdan iboratdir, shuningdek, ikkinchi jismning pastki asosi D sohadan, ustki asosi $z=\Phi_1(x,y)$ sirtdan iboratdir (18- rasm).



17- rasm.



18- rasm.

Shuning uchun V hajm ikkita ikki o’lchovli integrallar-ning ayirmasiga teng:

$$V=\iint_D \Phi_2(x,y)ds-\iint_D \Phi_1(x,y)ds,$$

yoki

$$V=\iint_D \left[\Phi_2(x,y)-\Phi_1(x,y)\right]ds. \tag{1}$$

(1) formula faqat $\Phi_1(x,y)$ va $\Phi_2(x,y)$ funksiyalar manfiy bo’lmagandagina emas, balki bu funksiyalar $\Phi_2(x,y)\geq\Phi_1(x,y)$ munosabatni qanoatlantiruvchi har qanday uzluksiz funksiyalar bo’lganda ham to’g’ri ekanini osongina isbot qilish mumkin.

2-izoh. Agar D sohada $f(x,y)$ funksiya o’z ishorasini o’zgartirsa, sohani ikki qismga ajratamiz: 1) $f(x,y)\geq 0$ bo’lgan D_1 soha; 2) $f(x,y)\leq 0$ bo’lgan D_2 soha. D_1 va D_2 sohalarni, bu sohalar bo’yicha olingan ikki o’lchovli integrallar mavjud deb faraz etamiz. U vaqtda D soha bo’yicha olingan integral musbat bo’lib, Oxy tekislikdan yuqorida joylashgan jismning hajmiga teng. D_2 soha bo’yicha olingan integral manfiy va absolyut qiymat jihatdan Oxy tekislikdan pastda joylashgan jismning hajmiga tengdir. Demak, D bo’yicha olingan integral mos hajmlarning ayirmasini ifodalaydi.

2. Tekis sohaning yuzini hisoblash. Agar biz D soha bo'yicha $f(x,y)\equiv 1$ funksiya uchun integral yigindi tuzsak, u holda bu yigindi, bo'lish usuli har qanday bo'lganda ham shu sohaning S yuziga teng bo'ladi.

$$S=\sum_{i=1}^{n} 1\cdot \Delta s_i.$$

Tenglikning o'ng tomonida limitga o'tib, quyidagi integralni hosil qilamiz:

$$S=\iint_D dxdy.$$

Agar D soha to'g'ri bo'lsa (masalan, 296-rasmga qarang), u holda yuz ushbu

$$S=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} dy\right)dx$$

ikki karrali integral bilan ifodalanadi. Qavs ichidagi integralni hisoblab, ushbu tenglikni topamiz:

$$S=\int_a^b [\varphi_2(x)-\varphi_1(x)]dx$$

(I t. XII bobning 1-paragrafi bilan solishtiring).

2-misol. $y=2-x^2,\ y=x$ egri chiziqlar bilan chegaralangan sohaning yuzi hisoblansin.

Yechish. Berilgan egri chiziqlarning kesishish nuqtalarini topamiz (19-rasm). Kesishish nuqtada ordinatalar teng, ya'ni

$$x=2-x^2,$$

bundan

$$x^2+x-2=0,\quad x_1=-2,\ x_2=1.$$

Biz egri chiziqlar kesishgan ikki nuqtani topdik: $M_1(-2,-2),\ M_2(1,1).$ Demak, izlanayotgan yuz:

$$S=\int_{-2}^{1}\left(\int_{x}^{2-x^2} dy\right)dx=\int_{-2}^{1}(2-x^2-x)dx=\left[2x-\frac{x^3}{3}-\frac{x^2}{2}\right]_{-2}^{1}=\frac{9}{2}.$$

4-§. Qutb koordinatalaridagi ikki o'lchovli integral. O'zgaruvchilarni almashtirish

θ, ρ qutb koordinatalar sistemasida shunday D soha berilgan bo'lsinki, uning ichki nuqtasidan o'tuvchi har bir nur bu sohaning chegarasini ikkitadan ortiq nuqtada kesmasin. D soha $\rho = \Phi_1(\theta)$, $\rho = \Phi_2(\theta)$ egri chiziqlar hamda $\theta = \alpha$ va $\theta = \beta$ nurlar bilan chegaralangan, bunda $\Phi_1(\theta) \le \Phi_2(\theta)$ va $\alpha < \beta$ deb faraz qilamiz (20-rasm). Bunday sohani yana to'g'ri deb ataymiz.

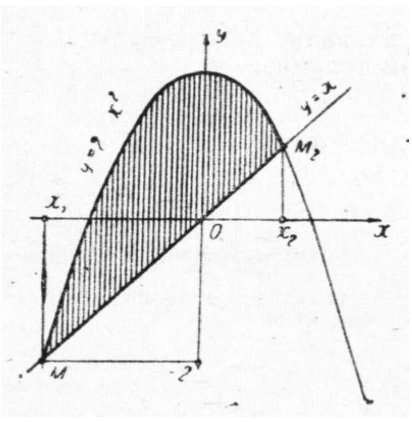
D sohada θ va ρ koordinatalarning ualuksiz funksiyasi berilgan bo'lsin: $z = F(\theta, \rho)$.

D sohani biror usul bilan $\Delta s_1, \Delta s_2, \Delta s_3, ..., \Delta s_n$ yuzlarga bo'lamiz. Integral yig'indini tuzamiz:

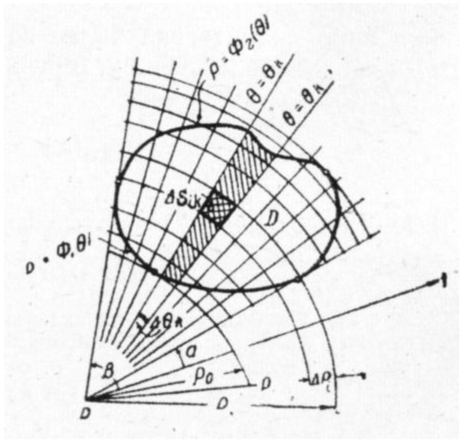
$$V_n = \sum_{k=1}^{n} F(P_k)\Delta s_k, \qquad (1)$$

bunda P_k nuqta Δs_k yuzning ixtiyoriy nuqtasi.

Ikki o'lchovli integralning mavjudligi haqidagi teoremadan Δs_k yuzlarning eng katta diametri nolga intilganda (1)



19-rasm.



20- rasm.

integral yig'indining limiti V mavjud ekanligi kelib chiqadi. Bu V limit, ta'rifga asosan, $F(\theta, \rho)$ funksiyadan D soha bo'yicha olingan ikki o'lchovli integralga teng:

$$V = \iint_D F(\theta, \rho) ds. \qquad (2)$$

1- misol. Sferik

$$x^2 + y^2 + z^2 = 4a^2$$

va silindrik

$$x^2 + y^2 - 2ay = 0$$

sirtlar bilan chegaralangan jismning v hajmi hisoblansin.

Yechish. Bu yerda integrallash sohasi uchun $x^2 + y^2 - 2ay = 0$ silindrning asosini, ya'ni markazi (o, a) nuqtada bo'lgan, a radiusli doirani olish va bu doiraning tenglamasini $x^2 + (y-a)^2 = a^2$ ko'rinishda yozish mumkin (20-rasm). Izlanayotgan V hajmining $\frac{1}{4}$ qismini, ya'ni uning birinchi oktantda joylashgan bo'lagining hajmini hisoblaymiz. U vaqtda integrallash sohasi sifatida chegarasi

$$x = \varphi_1(y) = 0,\ \ x = \varphi_2(y) = \sqrt{2ay - y^2},\ y = 0,\ \ y = 2a$$

tenglamalar bilan aniqlanuvchi yarim doirani olishga to'g'ri keladi. Integral ostidagi funksiya

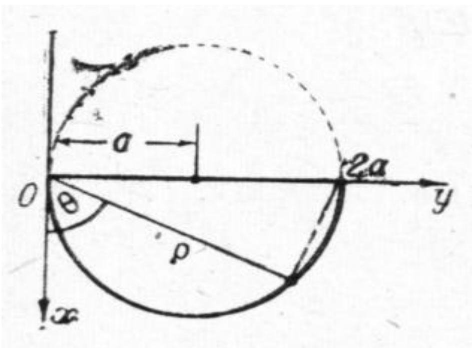
$$z = f(x, y) = \sqrt{4a^2 - x^2 - y^2}$$

ifodaga teng bo'ladi. Demak,

$$\frac{1}{4}V = \int_0^{2a}\left(\int_0^{\sqrt{2ay-y^2}} \sqrt{4a^2 - x^2 - y^2}\,dx\right)dy.$$

Hosil qilingan bu integralni $\theta,\ \rho$ qutb koordinatalarga almashtiramiz:

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta.$$



21-rasm.

Integrallash chegarasini aniqlaymiz. Buning uchun berilgan aylananing tenglamasini qutb koordinatalarda yozamiz:

$$x^2 - y^2 = \rho^2, \;\; y = \rho \sin\theta,$$

u vaqtda

$$\rho^2 - 2a\rho \sin\theta = 0$$

Demak, berilgan sohaning qutb koordinatalardagi (21-rasm) chegarasi

$$\rho = \Phi_1(\theta) = 0, \;\; \rho = \Phi_2(\theta) = 2a\sin\theta, \;\; \alpha = 0, \; \beta = \frac{\pi}{2}$$

tenglamalar bilan aniqlanadi. Integral ostidagi funksiya esa

$$F(\theta,\rho) = \sqrt{4a^2 - \rho^2}$$

ko'rinishda bo'ladi

Shunday qilib, quyidagini hosil qilamiz:

$$\frac{V}{4} = \int_0^{\pi/2} \left(\int_0^{2a\sin\theta} \sqrt{4a^2 - \rho^2}\,\rho d\rho \right) d\theta = \int_0^{\pi/2} \left[-\frac{(4a^2 - \rho^2)^{3/2}}{3} \right]_0^{2a\sin\theta} d\theta =$$

$$= \int_0^{\pi/2} \left[(4a^2 - 4a^2\sin\theta)^{\frac{3}{2}} - (4a^2)^{\frac{3}{2}} \right] d\theta = \frac{8a^3}{3} \int_0^{\pi/2} (1 - \cos 3\theta) d\theta = \frac{9}{4} a^3 (3\pi - 4).$$

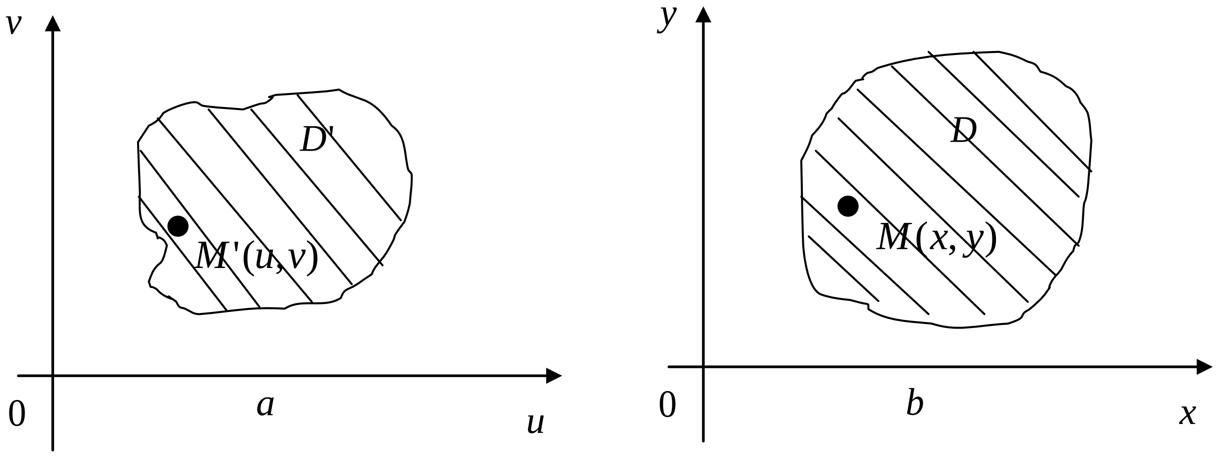
O'zgaruvchilarni almashtirish

Ba'zi hollarda ikki karrali integrallarni hisoblashni tezlashtirish uchun o'zgaruvchini almashtirish qulaylik tug'diradi. xOy tekislikda chegaralangan *(D)* sohada uzluksiz $f(x, y)$ funksiyadan olingan ikki karrali integralni hisoblash talab qilinsin.

Faraz qilaylik,

$$x = \varphi(u,v) \text{ va } y = \psi(u,v) \qquad (1)$$

funksiyalar xOy tekislikdagi $M(x, y) \in D$ nuqta bilan uOv tekislikning biror D' sohasidagi $M'(u,v)$ nuqtasi o'rtasida o'zaro bir qiymatli moslik o'rnatsin.



Demak, (1) formula tekislikning uOv sohasini xOy tekislikning D sohasiga almashtiradi:

$$u = \varphi_1(u,v), \quad v = \psi_1(u,v) \tag{2}$$

(2) formulalar aksincha. Bu yerda u, v lar M nuqtaning egri chiziqli koordinatalari deb yuritiladi. Umuman olganda, (1) formula va undan kelib chiquvchi (2) formulalarga koordinata almashtirish formulalari deyiladi. v M nuqtaning to'g'ri burchakli x, y koordinatalaridan, uning u, v egri chiziqli koordinatalariga va aksincha o'tishga imkon beradi. Ikki karrali integralda o'zgaruvchilarni almashtirish formulasini isbotsiz keltiramiz:

$$\iint_D f(x;y)dxdy = \iint_D \varphi(u,v);\psi(u,v)|I|dudv \tag{3}$$

Bu yerda

$$I = \begin{vmatrix} \varphi'_u(u;v) & \psi'_u(u;v) \\ \varphi'_v(u;v) & \psi'_v(u;v) \end{vmatrix}$$

bo'lib, bunda I Ostragradskiy determinant deyiladi. Ikki karrali integralda o'zgaruvchilarni almashtirishda ko'p ishlatiladigan hol bu to'g'ri burchakli x, y koordinatalarni ma'lum bo'lgan $x = r\cos\varphi, \quad y = r\sin\varphi$ formulalarga ko'ra r va φ qutb koordinatalariga o'tishdir. Bu holda

$$I(r,\varphi)=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r}\\ \dfrac{\partial x}{\partial \varphi} & \dfrac{\partial y}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & \sin\varphi\\ -r\sin\varphi & r\cos\varphi\end{vmatrix}=r,$$

u holda (3) tenglik $\{\alpha\le\varphi\le\beta,\ r_1(\varphi)\le r\le r_2(\varphi)\}$ (a-chizma)

$$\iint_D f(x;y)dxdy=\iint_D f(r\cos\varphi;r\sin\varphi)rdrd\varphi. \qquad (4)$$

(3), (4) formulalar

$$\iint_D f(r\cos\varphi;r\sin\varphi)rdrd\varphi=\int_\alpha^\beta d\varphi\int_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi;r\sin\varphi)rdr.$$

Misol. xOy tekislikda D soha bo'yicha olingan va xOy tekislikdagi $y=x+3,\quad y=x-5,\quad y=-\frac{2}{3}x+\frac{7}{3},\quad y=-\frac{2}{3}x+3$ to'g'ri chiziqlar bilan chegaralangan.

Yechish. Bu integralni to'g'ridan – to'g'ri hisoblash qiyinchilik tug'diradi. Ammo o'zgaruvchilarni oddiygina almashtirish sohani oddiy ko'rinishga keltiradi.

$$u=y-x,\ v=y+\frac{2}{3}x \qquad (6)$$

ko'rinishda yangi o'zgaruvchilarni kiritamiz. U holda $y=x+3$ va $y=x-5$ to'g'ri chiziqlar $u=3,\ u=-5$ to'g'ri chiziqlarga, $y=-\frac{2}{3}x+\frac{7}{3},\ \ y=-\frac{2}{3}x+3$ to'g'ri chiziqlar esa $v=\frac{7}{3},\ \ v=3$ to'g'ri chiziqlarga o'tadi (b-chizma). Ostragradskiy determinantini hisoblash uchun x va y larni u va v orqali ifodalab

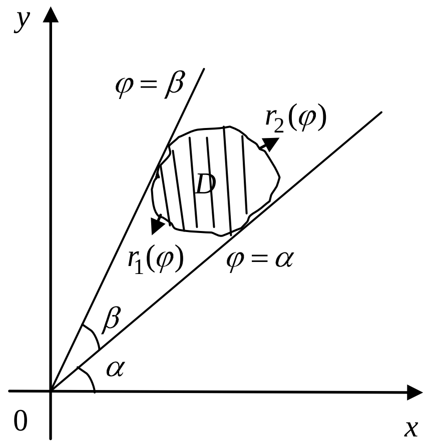
$x=-\frac{3}{5}u+\frac{3}{5}v;\ y=\frac{3}{5}u+\frac{3}{5}v$

$$I=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}-\dfrac{3}{5} & \dfrac{3}{5}\\ \dfrac{2}{5} & \dfrac{3}{5}\end{vmatrix}=-\frac{9}{25}-\frac{6}{25}=-\frac{15}{25}=-\frac{3}{5}.$$

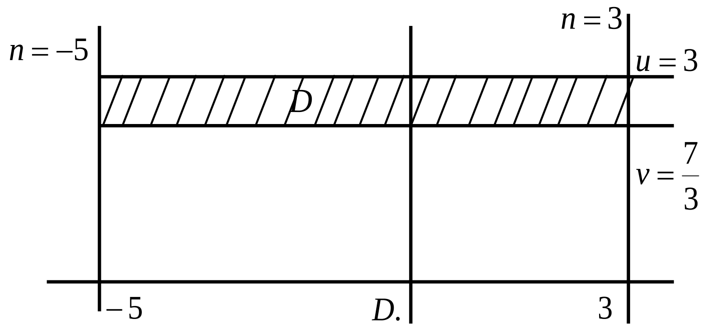
(6) formulaga asosan,

$$\iint_D f(x-y)dxdy = \iint_D \left[\left(\frac{2}{5}u+\frac{2}{5}v\right)-\left(\frac{2}{5}u+\frac{3}{5}v\right)\right]\frac{3}{5}dudv =$$

$$= \iint_D \frac{3}{5}ududv = \int_{\frac{7}{3}}^{3}\int_{-5}^{3}\frac{3}{5}ududv = \int_{\frac{7}{3}}^{3}\left(\int_{-5}^{3}\frac{3}{5}udu\right)dv = -3\frac{1}{5}.$$



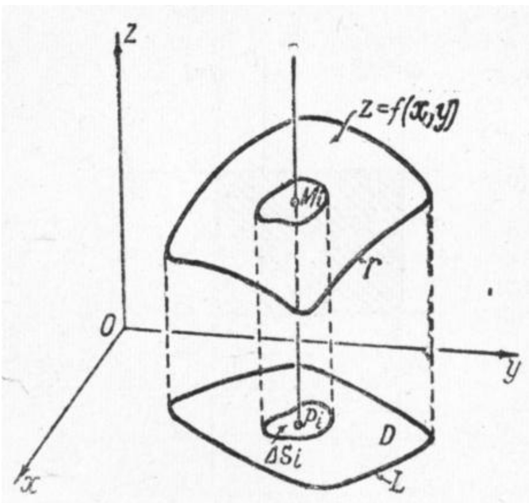
a-chizma



b-chizma

5-§. Sirtning yuzini hisoblash

Γ chiziq bilan chegaralangan sirtning yuzini hisoblash talab qilinsin (22-rasm); sirt o'zining $z = f(x,y)$ tenglamasi bilan berilgan, bundagi $f(x,y)$ funksiya uzluksiz va uzluksiz xususiy hosilalarga ega bo'lgan funksiyadir.



22-rasm.

Γ chiziqning Oxy tekislikdagi proyeksiyasini L bilan belgilaymiz. Oxy tekislikdagi L chiziq bilan chegaralangan sohani D bilan belgilaymiz.

D sohani ixtiyoriy yo’l bilan n ta elementar $\Delta s_1, \Delta s_2, ..., \Delta s_n$ yuzlarga bo’lamiz. Har qaysi Δs_i yuzning ichida $P_i(\xi_i, \eta_i)$ nuqta olamiz. P_i nuqtaga sirt ustida

$$M_i(\xi_i, \eta_i, f(\xi_i, \eta_i))$$

nuqta mos keladi. M_i nuqta orqali shu sirtga urinma tekislik o’tkazamiz. Uning tenglamasi quyidagi shakldа bo’ladi:

$$s - z_i = f_x'(\xi_i, \eta_i)(x - \xi_i) + f_y'(\xi_i, \eta_i)(y - \eta_i) \tag{1}$$

(I t. IX bobining 6-paragrafiga qarang.) Bu tekislikda, Oxy tekislikka Δs_i yuz shaklida proyeksiyalanadigan, $\Delta\sigma_i$ yuzni ajratamiz. Hamma $\Delta\sigma_i$ yuzlarning

$$\sum_{i=1}^{n} \Delta\sigma_i$$

yig’indisini qaraymiz.

$\Delta\sigma_i$ yuzlarning diametrlaridan eng kattasi nolga intilganda bu yig’indining σ limitini sirtning yuzi deb ataymiz, ya’ni ta’rifga ko’ra:

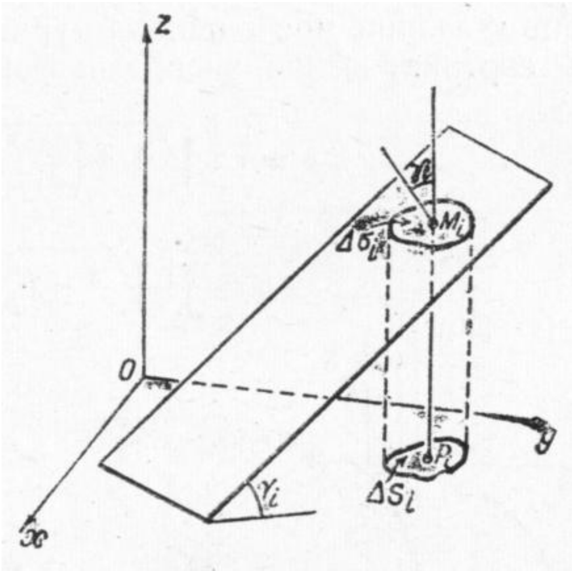
$$\sigma = \lim_{diam\Delta\sigma_i \to 0} \sum_{i=1}^{n} \Delta\sigma_i. \tag{2}$$

Endi sirtning yuzini hisoblashga kirishamiz. Urinma tekislik bilan Oxy tekislik orasidagi burchakni γ_i bilan belgilaymiz. Sirtning yuzini analitik geometriyaning ma’lum formulasiga asosan quyidagi ko’rinishda yozish mumkin (23- rasm):

$$\Delta s_i = \Delta\sigma_i \cos\gamma_i$$

yoki

$$\Delta\sigma_i = \frac{\Delta s_i}{\cos\gamma_i}. \tag{3}$$



23-rasm.

γ_i burchak bir vaqtda Oz o'q bilan (1) tekislikka tushirilgan perpendikulyar orasidagi burchak hamdir. Shuning uchun (1) tenglamaga va analitik geometriyaning formulasiga asosan

$$\cos\gamma_i = \sqrt{\frac{1}{1+f'^2_x(\xi_i,\eta_i)+f'^2_y(\xi_i,\eta_i)}}.$$

Demak,

$$\sigma_i = \sqrt{1+f'^2_x(\xi_i,\eta_i)+f'^2_y(\xi_i,\eta_i)}\Delta s_i.$$

Bu ifodani (2) formulaga qo'ysak, quyidagi tenglik hosil bo'ladi:

$$\sigma = \lim_{diam\Delta s_i \to 0}\sum_{i=1}^{n}\sqrt{1+f'^2_x(\xi_i,\eta_i)+f'^2_y(\xi_i,\eta_i)}\Delta s_i.$$

Ta'rifga ko'ra oxirgi tenglikning o'ng tomonida turgan integral yig'indining limiti

$$\iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy$$

ikki o'lchovli integral ko'rinishda bo'lganligi sababli oxirgi natija quyidagicha bo'ladi:

$$\sigma = \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy. \qquad (4)$$

Bu esa $z = f(x,y)$ sirtning yuzini hisoblash uchun qo'llaniladigan formuladir.

Agar sirtning tenglamasi

$$x = \mu(y, z) \text{ yoki } y = \chi(y, z)$$

ko’rinishda berilgan bo’lsa, u holda sirtning yuzini hisoblash formulalari mos ravishda

$$\sigma = \iint_D \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} dydz \tag{3'}$$

$$\sigma = \iint_D \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dxdz \tag{3''}$$

ko’rinishda bo’ladi, bundagi D' va D'' sohalar berilgan sirt proyeksiyalanadigan Oyz va Oxz tekislikda yotuvchi sohalardir.

1 - misol. Ushbu

$$x^2 + y^2 + z^2 = R^2$$

sferaning σ sirti hisoblansin.

Yechish. Sfera ustki yarmining sir- tini hisoblaymiz:

$$z = \sqrt{R^2 - x^2 - y^2}$$

(24- rasm). Bu holda

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}; \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}.$$

Demak,

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{R^2}{R^2 - x^2 - y^2}} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}.$$

Integrallash sohasi ushbu

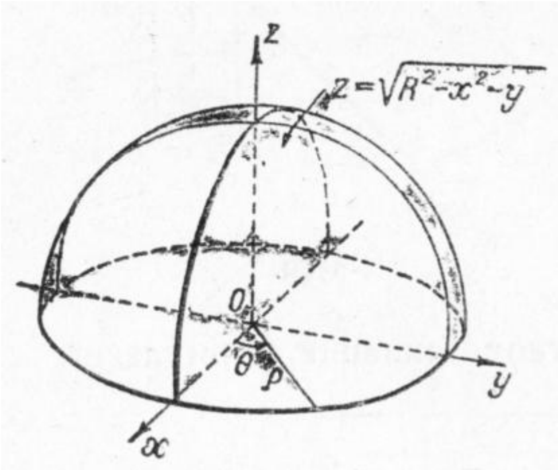
$$x^2 + y^2 \le R^2$$

shart bilan aniqlanadi. Shunday qilib, (4) formulaga asosan:

$$\frac{1}{2}\sigma = \int_{-R}^{R} \left(\int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dy \right) dx.$$

Hosil qilingan ikki o'lchovli integralni hisoblash uchun qutb koordinatalarga o'tamiz. Qutb koordinatalarda integrallash sohasining chegarasi $\rho = R -$ tenglama bilan aniqlanadi. Demak,

$$\sigma = 2\int_0^{2\pi}\left(\int_0^R \frac{R}{\sqrt{R^2-\rho^2}}\rho d\rho\right)d\theta = 2R\int_0^{2\pi}\left[-\sqrt{R^2-\rho^2}\right]_0^R d\theta = 2R\int_0^{2\pi}Rd\theta = 4\pi R^2.$$



24- rasm.

2- misol. Ushbu

$$x^2 + y^2 = a^2$$

silindrning

$$x^2 + z^2 = a^2$$

silindr bilan kesishganidan hosil bo'lgan qismining sirti hisoblansin.

Yechish. 25- rasmda izlangan sirtning 1/8 qismi tasvirlangan. Sirtning tenglamasi quyidagi shakldа bo'ladi:

$$y = \sqrt{a^2 - x^2}$$

shuning uchun

$$\frac{\partial y}{\partial x} = -\frac{x}{\sqrt{a^2-x^2}}, \quad \frac{\partial y}{\partial z} = 0;$$

$$\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2} = \sqrt{1+\frac{x^2}{a^2-x^2}} = \frac{a}{\sqrt{a^2-x^2}}.$$

Integrallash sohasi doira choragi ko'rinishida bo'ladi, ya'ni:

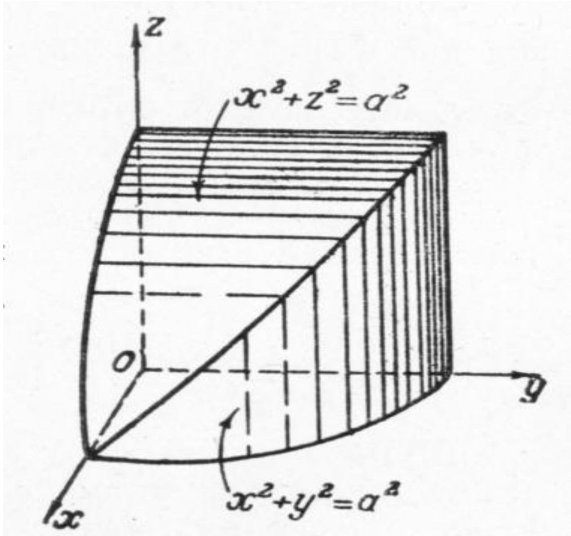
$$x^2+z^2\le a^2, \qquad x\ge 0; \qquad z\ge 0.$$

shartlar bilan aniqlanadi.

Demak,

$$\frac{1}{8}\sigma=\int_0^a\left(\int_0^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}dz\right)dx=a\int_0^a\frac{z}{\sqrt{a^2-x^2}}\Bigg|_0^{\sqrt{a^2-x^2}}dx=a\int_0^a dx=a^2,$$

$$\sigma=8a^2.$$



25- rasm.

6-§. Modda taqsimlanishining zichligi va ikki o'lchovli integral

D sohada biror modda taqsimlangan va *D* sohaning har qaysi birligi yuziga bu moddaning aniq miqdori to'g'ri keladigan bo'lsin. Garchi muhokamamiz elektr zaryadlari, issiqlik, miqdori va hokazolarning taqsimoti haqida so'z borganda ham to'g'ri bo'lsada, bundan buyon biz massa taqsimoti haqida so'zlaymiz.

D sohaning ixtiyoriy Δs yuzini qaraymiz. Shu yuzga to'g'ri keladigan moddaning massasi Δm bo'lsin. U holda $\dfrac{\Delta m}{\Delta s}$ nisbat moddaning Δs sohadagi o'rtacha sirt zichligi deb ataladi. Endi Δs yuz kichraya borib, $P(x,y)$ nuqtaga aylanib qoladi deb faraz qilamiz va ushbu $\lim\limits_{\Delta s\to 0}\dfrac{\Delta m}{\Delta s}$ limitni qaraymiz. Agar bu limit mavjud bo'lsa, umuman aytganda, u P nuqtaning vaziyatiga, ya'ni bu nuqtaning x va y koordinatalariga bog'liq bo'ladi hamda P nuqtaning qandaydir $f(P)$ funksiyasidan iborat bo'ladi. Biz bu limitni moddaning P nuqtadagi sirt zichligi deb ataymiz:

$$\lim_{\Delta s \to 0} \frac{\Delta m}{\Delta s} = f(P) = f(x, y) \qquad (1)$$

Shunday qilib, sirt zichligi soha nuqtasi koordinatalarining $f(x, y)$ funksiyasidan iborat.

Endi, aksincha, D sohada biror moddaning sirt zichligi qandaydir $f(P) = f(x, y)$ uzluksiz funksiya bilan berilgan bo'lsin va shu moddaning D sohadagi umumiy miqdorini aniqlash talab qilinsin.

D sohani $\Delta s_i \ (i = 1, 2, ..., n)$ yuzlarga bo'lamiz va har qaysi yuzda P_i nuqtani olamiz; u holda $f(P_i)$ funksiya P_i nuqtadagi sirt zichligi bo'ladi.

$f(P_i) \ \Delta s_i$ ko'paytma bizga yuqori tartibli cheksiz kichik miqdorgacha aniqlik bilan Δs_i yuzdagi moddaning miqdorini beradi,

$$\sum_{i=1}^{n} f(P_i) \Delta s_i$$

yig'indi esa D sohaga yoyilgan moddaning taqribiy umumiy miqdorini ifodalaydi. Biroq, bu D sohadagi $f(P)$ funksiyaning integral yig'indisidir. Uning aniq qiymatini esa biz $\Delta s_i \to 0$ da limitga o'tish bilan topamiz.

Shunday qilib,

$$M = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i) \Delta s_i = \iint_D f(P) ds = \iint_D f(x, y) dx dy, \qquad (2)$$

ya'ni D sohadagi moddaning umumiy miqdori shu moddaning $f(P) = f(x, y)$ zichligidan D soha bo'yicha olingan ikki o'lchovli integraliga tengdir.

Misol. R radiusli doiraviy plastinkaning har bir $P(x, y)$ nuqtasidagi plastinka materialining $f(x, y)$ sirt zichligi doira markazidan (x, y) nuqtagacha bo'lgan masofaga proporsional, ya'ni

$$f(x, y) = k\sqrt{x^2 + y^2}$$

bo'lsa, bu plastinkaning massasi aniqlansin.

Yechish. (2) formula bo'yicha:

$$M = \iint_D k\sqrt{x^2 + y^2}\,dxdy$$

bo'ladi, bunda integrallash sohasi D $x^2 + y^2 \le R^2$ doiradir. Qutb koordinatalariga o'tib, mana buni hosil qilamiz:

$$M = \int_0^{2\pi}\left(\int_0^R \rho\rho d\rho\right)d\theta = k2\pi \left.\frac{\rho^3}{3}\right|_0^R = \frac{2}{3}k\pi R^2.$$

7-§. Yassi (tekis) shakl yuzining inersiya momenti

Massasi m bo'lgan moddiy M nuqtaning biror O nuqtaga nisbatan I inersiya momenti deb, m massaning M dan O nuqtagacha bo'lgan r masofaning kvadrati bilan ko'paytmasiga aytiladi:

$$I = mr^2.$$

$m_1, m_2, m_3, ..., m_n$ moddiy nuqtalar sistemasining O nuqtaga nisbatan inersiya momenti shu sistema nuqtalari inersiya momentlarining yig'indisidan iborat bo'ladi:

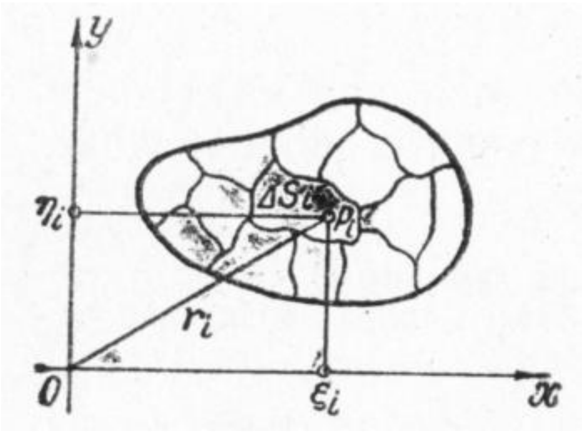
$$I = \sum_{i=1}^{m} m_i r_i^2.$$

Endi D moddiy yassi shaklning inersiya momentini aniqlaymiz. D shakl Oxy koordinata tekisligiga joylashgan bo'lsin. Bu shaklning sirt zichligi hamma yerida birga teng deb faraz qilib, uning inersiya momentini koordinatalar boshiga nisbatan aniqlaymiz. D sohani Δs_i $(i = 1,2,...,n)$ elementar yuzlarga ajratamiz (26-rasm). Har bir yuzda koordinatalari ξ_i, η_i bo'lgan P_i nuqta olamiz. Δs_i yuzning elementar ΔI_i inersiya momenti deb, Δs_i yuzning massasi bilan masofa kvadrati $r_i^2 = \xi_i^2 + \eta_i^2$ ning ko'paytmasiga aytamiz:

$$\Delta I_i = (\xi_i^2 + \eta_i^2)\Delta S_i$$

va shunday inersiya momentlarining yig'indisini tuzamiz:

$$\sum_{i=1}^{n}\left(\xi_i^2+\eta_i^2\right)\Delta S_i .$$



26-rasm.

Bu esa $f(x,y)=x^2+y^2$ funksiyaning D soha bo'yicha olingan integral yig'indisidir. D shaklning inersiya momentini har bir ΔS_i elementar yuzning diametri nolga intilgandagi shu integral yig'indining limiti sifatida aniqlaymiz:

$$I_0=\lim_{diam\Delta S_i\to 0}\sum_{i=1}^{n}\left(\xi_i^2+\eta_i^2\right)\Delta S_i .$$

Ammo bu yig'indining limita $\iint\limits_D\left(x^2+y^2\right)$ ikki o'lchovli integraldan iborat. Demak, D shaklning koordinatalar boshiga nisbatan inersiya momenti

$$I_0=\iint\limits_D\left(x^2+y^2\right)dxdy, \qquad (1)$$

integralga teng, bunda D soha berilgan tekis shakl bilan ustma-ust tushadi. Ushbu integrallar

$$I_{xx}=\iint\limits_D y^2dxdy, \qquad (2)$$

$$I_{yy}=\iint\limits_D x^2dxdy \qquad (3)$$

mos ravishda D shaklning Ox va Oy o'qlarga nisbatan inersiya momentlari deb ataladi.

1-misol. Radiusi R bo'lgan D doira yuzining O markazga nisbatan inersiya momenti hisoblansin.

Yechish (1) formulaga asosan:

$$I_0 = \iint_D \left(x^2 + y^2\right) dxdy.$$

Bu integralni hisoblash uchun $\theta,\ \rho$ qutb koordinatalarga o'tamiz. Aylananing qutb koordinatalaridagi tenglamasi

$$\rho = R.$$

Shuning uchun

$$I_0 = \int_0^{2\pi} \left(\int_0^R \rho^2 \rho d\rho \right) d\theta = \frac{\pi R^4}{2}.$$

Izoh. Agar sirt zichligi γ birga teng bo'lmay, x va y ning biror funksiyasi, ya'ni $\gamma = \gamma(x, y)$ bo'lsa, ΔS_i yuzning massasi yuqori tartibli cheksiz kichik miqdorgacha aniqlik bilan $\gamma(\xi_i, \eta_i)\Delta S_i$ ga teng bo'ladi, shuning uchun yassi shaklning koordinatalar boshiga nisbatan inersiya momenti quyidagicha bo'ladi:

$$I_0 = \iint_D \gamma(x, y) \left(x^2 + y^2\right) dxdy. \tag{1'}$$

2- misol. Agar

$$y^2 = 1 - x; \qquad x = 0, \qquad y = 0$$

chiziqlar bilan chegaralangan D moddiy yassi shaklning har bir nuqtasidagi sirt zichligi y ga teng bo'lsa, uning Oy o'qqa nisbatan inersiya momenti hisoblansin (27- rasm).

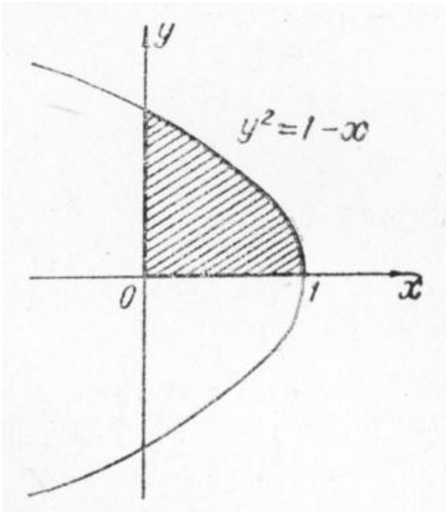
Yechish.

$$I_{yy} = \int_0^1 \left(\int_0^{\sqrt{1-x}} yx^2 dy \right) dx = \int_0^1 \frac{x^2 y^2}{2} \Bigg|_0^{\sqrt{1-x}} dx = \frac{1}{2} \int_0^1 x^2 (1 - x) dx = \frac{1}{24}.$$

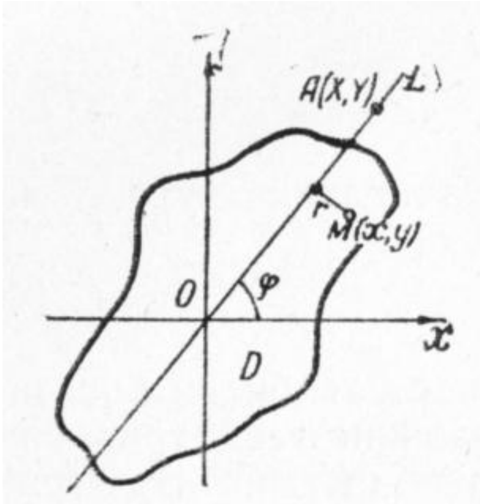
Inersiya ellipsi. Koordinatalar boshi deb olingan O nuqtadan o'tuvchi biror OL o'qqa nisbatan D yassi shakl yuzining inersiya momentini topaylik. OL to'g'ri chiziq bilan Ox o’qning musbat yo’nalishi orasidagi burchakni φ bilan belgilaymiz (28- rasm).

OL to’gri chiziqning normal tenglamasi

$$x\sin\varphi - y\cos\varphi = 0.$$



27-rasm.



28-rasm.

Biror $M(x,y)$ nuqtaning bu to’g’ri chiziqdan masofasi quyidagiga teng:

$$r = |x\sin\varphi - y\cos\varphi|.$$

D soha yuzining OL to’g’ri chiziqqa nisbatan I inersiya momenti ta’rifga muvofiq ushbu integral bilan ifodalanadi:

$$I = \iint\limits_D r^2 dxdy = \iint\limits_D (x\sin\varphi - y\cos\varphi)^2 dxdy = \sin^2\varphi \iint\limits_D x^2 dxdy -$$
$$-2\sin\varphi\cos\varphi \iint\limits_D xydxdy + \cos^2\varphi \iint\limits_D y^2 dxdy.$$

Demak,

$$I = I_{yy}\sin^2\varphi - 2I_{xy}\sin\varphi\cos\varphi + I_{xx}\cos^2\varphi, \tag{4}$$

bunda $I_{yy} = \iint\limits_D x^2 dxdy$ shaklning y o’qqa nisbatan inersiya momenti, $I_{xx} = \iint\limits_D y^2 dxdy - x$ o’ssa nisbatan inersiya momenti va $I_{xy} = \iint\limits_D xydxdy$. Oxirgi tenglikning hamma hadlarini I ga bo’lsak:

$$I = I_{xx}\left(\frac{\cos\varphi}{\sqrt{I}}\right)^2 - 2I_{xy}\left(\frac{\sin\varphi}{\sqrt{I}}\right)\left(\frac{\cos\varphi}{\sqrt{I}}\right) + I_{yy}\left(\frac{\sin\varphi}{\sqrt{I}}\right)^2.$$

OL to'g'ri chiziqda $A(X,Y)$ nuqtani $OA = \frac{1}{\sqrt{I}}$ bo'ladigan qilib olamiz. OL o'qning turlicha yo'nalishlariga, ya'ni φ burchakning har xil qiymatlariga A ning turli qiymatlari va turlicha A nuqtalar mos keladi. A nuqtalarning geometrik o'rinlarini topamiz. Ravshanki,

$$X = \frac{1}{\sqrt{I}}\cos\varphi, \qquad Y = \frac{1}{\sqrt{I}}\sin\varphi. \tag{5}$$

(5) tenglikka ko'ra X va Y miqdorlar o'zaro

$$I = I_{xx}X^2 - 2I_{xy}XY + I_{yy}Y^2 \tag{6}$$

munosabat bilan bog'langan. Shunday qilib, $A(X,Y)$ nuqtalarning geometrik o'rni ikkinchi tartibli (6) egri chizivdan iboratdir. Bu egri chiziqning ellips ekanligini isbot qilamiz.

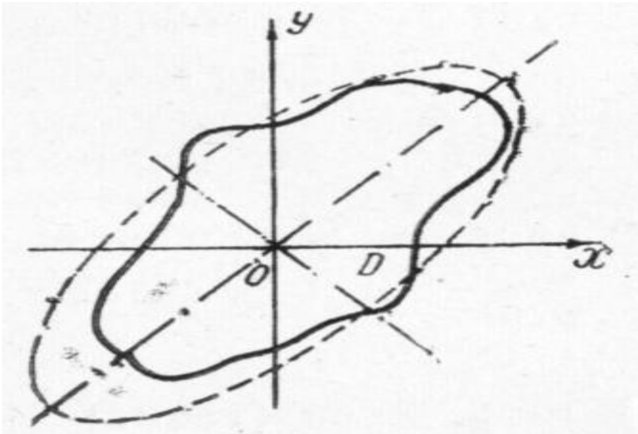
Rus matematigi V.Ya. Bunyakovskiy tomonidan topilgan ushbu tengsizlik to'g'ridir:

$$\left(\iint_D xydxdy\right)^2 < \left(\iint_D x^2dxdy\right)\left(\iint_D y^2dxdy\right)$$

yoki

$$I_{xx}I_{yy} - I_{xy}^2 > 0.$$

Shunday qilib, (6) egri chiziqning diskriminanti musbat ekan, demak, bu egri chiziq ellipsdir (29-rasm). Bu ellips inersiya ellipsi deb ataladi. Inersiya ellipsi tushunchasi mexanikada katta ahamiyatga ega.



29-rasm.

Inersiya ellipsi o'qlarining uzunliklari va tekislikdagi vaziyati berilgan yassi shaklning formasiga bog'liqdir.

Ellipsning ixtiyoriy A nuqtasidan koordinatalar boshigacha bo'lgan masofa $\frac{1}{\sqrt{I}}$ ga teng bo'lgani uchun (bunda *I* shaklning *OA* o'qqa nisbatan olingan inersiya momentidir) ellipsni yasab, koordinatalar boshidan o'tuvchi biror to'g'ri chiziqqa nisbatan *D* shaklning inersiya momentini osongina hisoblab chiqarishimiz mumkin. Jumladan, shaklning inersiya momenti inersiya ellipsining katta o'qiga nisbatan eng kichik va shu ellipsning kichik o'qiga nisbatan eng katta ekanligini ko'rish oson.

8-§. Yassi (tekis) shakl yuzi og'irlik markazining koordinatalari

It. XII bobining 8-paragrafida massalari $m_1, m_2, m_3, ..., m_n$ bo'lgan $P_1, P_2, P_3, ..., P_n$ moddiy nuqtalar sistemasi ogirlik markazining koordinatalari

$$x_c = \frac{\sum x_i m_i}{\sum m_i}, \qquad y_c = \frac{\sum y_i m_i}{\sum m_i}, \tag{1}$$

formulalar bilan aniqlanishi ko'rsatilgan edi.

Endi *D* yassi shakl ogirlik markazining koordinatalarini topamiz. Bu shaklni juda kichik ΔS_i elementar yuzlarga bo'lib chiqamiz. Agar sirt zichligini birga teng deb qabul qilsak, yuzning massasi uning yuziga teng bo'ladi. Agar taqriban ΔS_i elementar yuzning barcha massasi uning biror $P_i(\xi_i, \eta_i)$ nuqtasiga to'plangan deb hisoblansa, D shaklni moddiy nuqtalarning sistemasi deb qarash mumkin. U vaqtda (1) formulaga muvofiq bu shakl og'irlik markazining koordinatalari taxminan

$$x_c \approx \frac{\sum_{i=1}^{n} \xi_i \Delta S_i}{\sum_{i=1}^{n} \Delta S_i}, \qquad y_c \approx \frac{\sum_{i=1}^{n} \eta_i \Delta S_i}{\sum_{i=1}^{n} \Delta S_i}, \tag{2}$$

tengliklar bilan aniqlanadi,

$\Delta S_i \to 0$ da limitga o'tsak, kasrlarning suratida va maxrajida turgan integral yig'indilar ikki o'lchovli integralga aylanadi va biz yassi shakl og'irlik markazining koordinatalarini hisoblash uchun aniq formulalar hosil qilamiz:

$$x_c = \frac{\iint\limits_D x\,dxdy}{\iint\limits_D dxdy}, \qquad y_c = \frac{\iint\limits_D y\,dxdy}{\iint\limits_D dxdy}. \tag{2}$$

Sirt zichligi 1 ga teng bo'lgan yassi shakl uchun chiqarilgan bu formulalar hamma nuqtalarida o'zgarmas γ zichlikka ega bo'lgan har qanday shakllar uchun ham o'z kuchida qolishi ravshan.

Agar sirt zichligi o'zgaruvchan bo'lsa,

$$\gamma = \gamma(x, y),$$

u holda bu formulalar quyidagi ko'rinishda bo'ladi:

$$x_c = \frac{\iint\limits_D \gamma(x, y)\,x\,dxdy}{\iint\limits_D \gamma(x, y)\,dxdy}, \qquad y_c = \frac{\iint\limits_D \gamma(x, y)\,y\,dxdy}{\iint\limits_D \gamma(x, y)\,dxdy}.$$

Mana bu

$$M_y = \iint\limits_D \gamma(x, y)\,x\,dxdy \quad \text{va} \quad M_x = \iint\limits_D \gamma(x, y)\,y\,dxdy$$

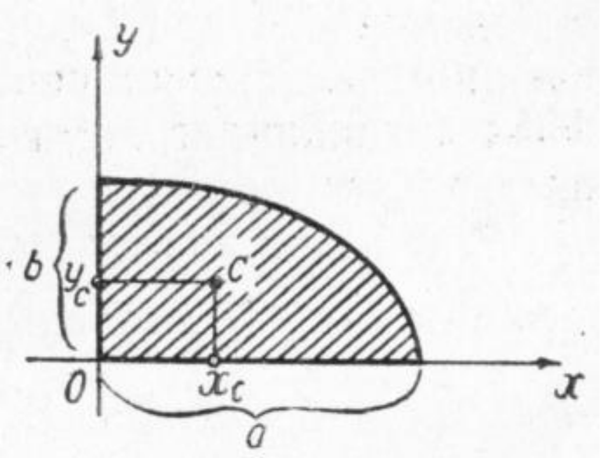
ifodalar Oy va Ox o'qlarga nisbatan D yassi shaklning statik momentlari deb ataladi.

$\iint \gamma(x, y)\,x\,dxdy$ integral qaralayotgan shaklning massasini ifodalaydi.

Misol. Sirt zichligi hamma nuqtalarida 1 ga teng deb olib,

$$\frac{x^2}{a^2}+\frac{x^2}{b^2}=1$$

ellips choragi og’irlik markazining koordinatalari topilsin (30- rasm).



30-rasm.

Yechish. (2) formulaga asosan:

$$x_c=\frac{\int\limits_0^a\left(\int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}}xdy\right)dx}{\int\limits_0^a\left(\int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}}dy\right)dx}=\frac{\int\limits_0^a\sqrt{a^2-x^2}xdx}{\frac{1}{4}\pi ab}=\frac{-\frac{b}{a}\frac{1}{3}(a^2-x^2)^{3/2}\Big|_0^a}{\frac{1}{4}\pi ab}=\frac{4a}{3\pi},$$

$$y_c=\frac{\int\limits_0^a\left(\int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}}ydy\right)dx}{\frac{1}{4}\pi ab}=\frac{4b}{3\pi}.$$

II BOB. IKKI KARRALI INTEGRALLARNING TADBIQLARI

1-§. Ikki karrali integralni hisoblash

1-misol. (P) sohada markazi koordinatalar boshoida bo’lgan R radiusli doirada ushbu

$$I = \iint_D y^2 \sqrt{R^2 - x^2}\, dp$$

ikki karrali integral hisoblansin.

Yechish. (P) soha konturining tenglamasi $x^2 + y^2 = R^2$ ko’rinishga ega, bundan $y = \pm\sqrt{R^2 - x^2}$ bu esa $y = +\sqrt{R^2 - x^2}$ yuqori yarim aylananing, $y = -\sqrt{R^2 - x^2}$ quyi aylananing tenglamasidir. $[R, -R]$ dan olingan x o’zmaganda, o’zgaruvchi $y = -\sqrt{R^2 - x^2}$ dan $y = +\sqrt{R^2 - x^2}$ ga qadar o’zgaradi (A-chizma).

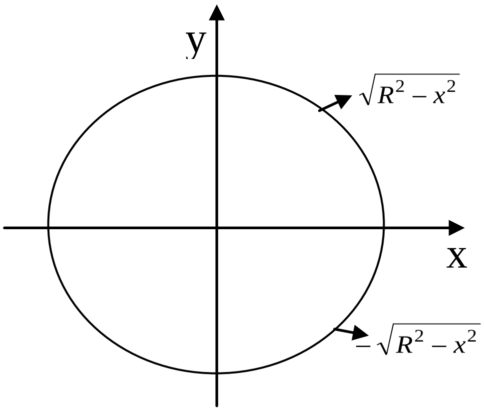
$$\iint_{(P)} f(x, y)dp = \int_a^b dx \int_{y_0(x)}^{y(x)} f(x, y)dy \qquad \text{(A)}$$

formulaga asosan

$$I = \int_{-R}^{R} dx \int_{-\sqrt{R^2 - x^2}}^{+\sqrt{R^2 - x^2}} y^2 \sqrt{R^2 - x^2}\, dy = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx \int_0^{\sqrt{R^2 - x^2}} y^2 dy =$$

$$= 2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx \left[\frac{1}{3}(R^2 - x^2)^{\frac{3}{2}}\right] = \frac{2}{3}\int_{-R}^{R} (R^2 - x^2)^2 dx =$$

$$= \frac{4}{3}\int_0^R (R^2 - x^2)^2 dx = \frac{32}{45} R^3.$$



A-chizma

2-misol. Ushbu

$$\iint_D \sqrt{4x^2 - y^2}\,dxdy$$

integralni $y = 0,\ x = 1,\ y = x$ to'g'ri chiziqlar bilan tashkil qilgan uchburchak bo'yicha hisoblansin.

Yechish. (A) formulaga asosan

$$I = \int_0^1 dx \int_0^x \sqrt{4x^2 - y^2}\,dy = \int_0^1 dx \left[\frac{y}{2}\sqrt{4x^2 - y^2} + 2x^2 \arcsin\frac{y}{2x} \right]_0^x =$$

$$= \int_0^1 dx \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) x^2 = \frac{1}{3}x^3 \Big|_0^1 \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) = \frac{1}{3}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right).$$

3-misol. Ushbu

$$I = \int_0^2 dx \int_{-1}^1 (x^2 + 2y^2)\,dy.$$

integralni hisoblang.

Yechish. Dastlab ichki integralni hisoblaymiz, so'ngra tashqi integralni hisoblaymiz.

$$\int_{-1}^1 (x^2 + 2y^2)\,dy = \left(x^2 y + \frac{2}{3}y^3 \right)_{-1}^1 = \left(x^2 + \frac{2}{3} \right) - \left(x^2 - \frac{2}{3} \right) = 2x^2 + \frac{4}{3},$$

$$\int_0^2 \left(2x^2 + \frac{4}{3} \right) dx = \left(2\frac{x^3}{3} + \frac{4}{3}x \right)_0^2 = \frac{16}{3} + \frac{8}{3} = \frac{24}{3} = 8.$$

Demak,

$$I=\int_0^2 dx\int_{-1}^{1}\left(x^2+2y^2\right)dy=8.$$

4-misol. Ushbu

$$I=\int_0^4 dx\int_0^1 \frac{dy}{(2x+y+1)^2}.$$

integralni hisoblang.

Yechish.

$$I=\int_0^4 dx\int_0^1 \frac{dy}{(2x+y+1)^2}=\int_0^4 dx\left[-\frac{1}{2x+y+1}\right]_0^1=$$

$$=\int_0^4\left(-\frac{1}{2(x+1)}+\frac{1}{2x+1}\right)dx=-\int_0^4\frac{dx}{2(x+1)}+\int_0^4\frac{dx}{2x+1}=$$

$$=-\frac{1}{2}\left[\ln(x+1)\right]_0^4+\frac{1}{2}\left[\ln(2x+1)\right]_0^4=-\frac{1}{2}\ln 9+\frac{1}{2}\ln 5=\frac{1}{2}\ln\frac{9}{5}.$$

3) $I=\iint\limits_D \frac{ydxdy}{(1+x^2+y^2)^{3/2}}=\left[(D)\ \text{soha}\ \ 0\le x\le 1,\ 0\le y\le 1\right]=$

$$=\int_0^1 dx\int_0^1 y\left(1+x^2+y^2\right)^{-\frac{3}{2}}dy.$$

Ichki integralni hisoblaymiz:

$$\int_0^1\left(1+x^2+y^2\right)^{-\frac{3}{2}}ydy=-\left[\frac{1}{\sqrt{1+x^2+y^2}}\right]_0^1=\frac{1}{\sqrt{1+x^2}}-\frac{1}{\sqrt{2+x^2}}.$$

Demak,

$$I=\int_0^1\frac{dx}{\sqrt{1+x^2}}-\int_0^1\frac{dx}{\sqrt{2+x^2}}=\ln(1+\sqrt{2})-\ln(1+\sqrt{3})+\ln\sqrt{2}=\ln\frac{2+\sqrt{2}}{1+\sqrt{3}}.$$

4) $I=\iint\limits_D\left(1-x^2-y^2\right)dxdy$ integral (D) soha $x=0,\ x=1,\ y=2,\ y=\frac{3}{2}$ lar

bo’yicha hisoblansin.

$$I=\int_0^{3/2}\left[\int_0^1 (4-x^2-y^2)dx\right]dy=\int_0^{3/2}\left[\left(4x-\frac{x^3}{3}-y^2x\right)\right]_0^1 dy=$$

$$=\int_0^{3/2}\left(4-\frac{1}{3}-y^2\right)dy=\left(4y-\frac{1}{3}y-\frac{y^3}{3}\right)_0^{3/2}=\frac{35}{8}.$$

5) $f(x,y)=1+x+y$ funksiyadan $y=-x,\ x=\sqrt{y},\ y=2,\ y=0$ sohalar bo’yicha olingan ikki karrali integral hisoblansin.

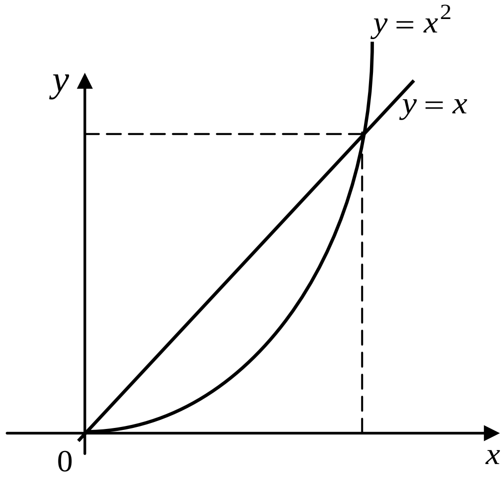
Yechish.

$$I=\int_0^2\left[\int_{-y}^{\sqrt{y}}(1+x+y)dx\right]dy=\int_0^2\left[\left(x+xy+\frac{x^2}{2}\right)\right]_{-y}^{\sqrt{y}}dy=$$

$$=\int_0^2\left[\left(\sqrt{y}+y\sqrt{y}+\frac{y}{2}\right)-\left(-y-y^2+\frac{y^2}{2}\right)\right]dy=$$

$$=\int_0^2\left(\sqrt{y}+y\sqrt{y}+\frac{3y}{2}+\frac{y^2}{2}\right)dy=\left(\frac{2}{3}\sqrt{y^3}+\frac{2}{5}\sqrt{y^5}+\frac{3y^2}{4}+\frac{y^3}{6}\right)_0^2=\frac{44\sqrt{2}}{15}+\frac{13}{3}.$$

6) $\iint\limits_D xydxdy$ integralni hisoblang. Bu yerda D soha yuqoridan $y=x$ to’g’ri chiziq bilan pastdan $y=x^2$ parabola bilan chegaralangan (B-chizma).



B - chizma

Yechish. To’g’ri chiziq va parabola tenglamalarini birgalikda yechamiz:

$$\begin{cases} y=x, \\ y=x^2. \end{cases}$$

$x^2=x,$

$x(x-1)=0,$

$x_1=0,\ x_2=1.$

$$\iint_D xydxdy = \int_0^1 dx\int_{x^2}^{x} xydy = \int_0^1 xdx\int_{x^2}^{x} ydy = \int_0^1 xdx\left[\frac{y^2}{2}\right]_{x^2}^{x} =$$

$$= \frac{1}{2}\int_0^1 x(x^2 - x^4)dx = \frac{1}{2}\left[\frac{x^4}{4}\Big|_0^1 - \frac{x^6}{6}\Big|_0^1\right] = \frac{1}{2}\left[\frac{1}{4}-\frac{1}{6}\right] = \frac{1}{24}.$$

7) $\iint_D (x^2+y^2)dxdy$ integralni hisoblang. Bu yerda D soha yuqoridan $y=0,\ y=-x+1,\ y=x+1\ (y-1\le x\le -y+1;\ 0\le y\le 1)$ to'g'ri chiziq bilan chegaralangan uchburchakdan iborat.

Yechish. $$\iint_D (x^2+y^2)dxdy = \int_0^1 dy\int_{y-1}^{-y+1}(x^2+y^2)dx = \int_0^1 dy\left[\left(\frac{x^3}{3}+y^2x\right)\right]_{y-1}^{-y+1} =$$

$$= \int_0^1 dy\left[\left(\frac{(-y+1)^3}{3} + y^2(-y+1) - \frac{(y-1)^3}{3} + y^2(y-1)\right)\right] =$$

$$= \int_0^1 dy\left(-\frac{2}{3}(y-1)^3 + 2y^3 + 2y^2\right) = -\frac{2}{3}\int_0^1 (y-1)^3dy - \int_0^1 y^3dy + 2\int_0^1 y^2dy =$$

$$= -\frac{1}{6}(y-1)^4\Big|_0^1 - \frac{1}{2}y^4\Big|_0^1 + \frac{2}{3}y^3\Big|_0^1 = \frac{1}{6} - \frac{1}{2} + \frac{2}{3} = \frac{1}{3}.$$

8) $\iint_D x\ln ydxdy$ integralni $\{D:\ 0\le x\le 4,\ 1\le y\le l\}$ soha bo'yicha hisoblang.

Yechish.

$$\iint_D x\ln ydxdy = \int_0^4 xdx\int_0^l \ln ydy =$$

$$= \frac{x^2}{2}\Big|_0^4 (y\ln y - y)\Big|_0^l = 8(l - l + 1) = 8.$$

9) $\iint_D (\cos^2 x + \sin^2 y)dxdy$ integralni $\left\{D: 0\le x\le \frac{\pi}{4}, 1\le y\le l\right\}$ soha bo'yicha hisoblang.

Yechish.

$$\iint_D (\cos^2 x + \sin^2 y)\,dxdy = \int_0^{\frac{\pi}{4}} dx \int_0^{\frac{\pi}{4}} (\cos^2 x + \sin^2 y)\,dy =$$

$$= \int_0^{\frac{\pi}{4}} \left[y\cos^2 x + \frac{4}{2} - \frac{1}{4}\sin 2y \right]_0^{\frac{\pi}{4}} dx = \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{4}\cos^2 x + \frac{\pi}{8} - \frac{1}{4} \right) dx =$$

$$= \left[\frac{\pi}{8}\left(x + \frac{1}{2}\sin 2x \right) + \left(\frac{\pi}{8} - \frac{1}{4} \right) x \right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}\left(\frac{\pi}{4} + \frac{1}{2} \right) + \left(\frac{\pi}{8} + \frac{1}{2} \right)\frac{\pi}{4} = \frac{\pi^2}{16}.$$

10)

$$I = \int_0^2 dx \int_x^{x^2} (2x - y)dy = \int_0^2 dx \left[2xy - \frac{1}{2}y^2 \right]_x^{x^2} =$$

$$= \int_0^2 \left(2x^3 - \frac{1}{2}x^4 - 2x^2 + \frac{1}{2}x^2 \right) dx = \left[\frac{1}{2}x^4 - \frac{1}{10}x^5 - \frac{1}{2}x^3 \right]_1^2 =$$

$$= \left(8 - \frac{16}{5} - 4 \right) - \left(\frac{1}{2} - \frac{1}{10} - \frac{1}{2} \right) = \frac{1}{2}.$$

11) $\iint_D (x - 2y)\,dxdy$ integralni hisoblang. Bu yerda D soha $y = x,$ $y = 2x,\ x = 2,\ x = 3$ to’g’ri chiziqlar bilan chegaralangan.

Yechish.

$$\int_2^3 dx \int_x^{2x} (x + 2y)dy = \int_2^3 dx \left(xy + y^2 \right)\Big|_x^{2x} = \int_2^3 (2x^2 + 4x^2 - x^2 - x^2)\,dx =$$

$$= 4\int_2^3 x^2 dx = \frac{4}{3}x^3 \Big|_2^3 = 25\frac{1}{3}.$$

12) Qutb koordinata sistemasiga o’tib, $\iint_D \sqrt{x^2 + y^2}\,dxdy$ integralni hisoblang. Bu yerda D soha $x^2 + y^2 = a^2$ doiraning birinchi choragi.

Yechish. $x = \rho\cos\varphi,\ y = \rho\sin\varphi$

$$\iint\limits_D \sqrt{x^2+y^2}\,dxdy = \iint\limits_D \sqrt{\rho^2\cos^2\varphi+\rho^2\sin^2\varphi}\,\rho\cos\varphi\sin\varphi d\varphi =$$

$$= \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^a \rho^2 d\rho = \frac{1}{3}\int\limits_0^{\frac{\pi}{2}} a^3 d\varphi = \frac{a^3}{3}\frac{\pi}{2} = \frac{a^3\pi}{6}.$$

13) $\iint\limits_D (x+y)^3(x-y)^2 dxdy$ integralni hisoblang. Bu yerda D soha $x+y=1,\quad x-y=1,\ x+y=3,\quad x-y=-1$ to'g'ri chiziqlardan tashkil topgan kvadrat.

Yechish. $x+y=u,\ x-y=v$ desak, $x=\frac{1}{2}(u+v),\ \ y=\frac{1}{2}(u-v).$ U holda Ostragradskiy determinant

$$I = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}-\frac{1}{4} = -\frac{1}{2};$$

$$|I| = \frac{1}{2}\int\limits_1^2 u^3 du \int\limits_{-1}^1 v^2 dv.$$

Demak, $\iint\limits_D (x+y)^3(x-y)^2 dxdy = \frac{1}{2}\int\limits_1^3 u^3 du \int\limits_{-1}^1 v^2 dv = \frac{1}{2}\int\limits_1^3 u^3\left[\frac{1}{3}v^3\right]du =$

$$= \frac{1}{6}\int\limits_1^3 u^3(1+1)du = \frac{1}{12}u^4\Big|_1^3 = \frac{1}{12}(81-1) = \frac{20}{3}.$$ (E, E' – chizmalar.)

2-§. Ikki karrali integral yordamida yuzlarni hisoblash

1) $y=2-x^2,\ \ y=x$ chiziqlar bilan chegaralangan yuzani hisoblash.

Yechish. Chiziqlarning kesishish nuqtalarini topamiz. Buning uchun ularning tenglamalarini birgalikda yechamiz:

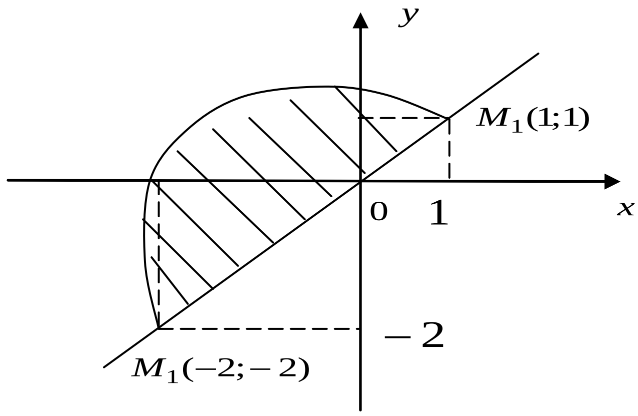
$$x^2+x-2=0$$

$$x_1=-2,\ \ x_2=1.$$

Biz ikkita $M_1(-2;-2),\ \ M_2(1;1)$ nuqtalarga ega bo'lamiz (a-chizma).

$$S = \int_{-2}^{1}\left(\int_{-2}^{2-x^2} dy\right)dx = \int_{-2}^{1}(2 - x^2 - x)\,dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1} =$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - \frac{4}{2}\right) = \frac{27}{6} \text{ kvadrat birlik.}$$



a-chizma.

2) $y = 6 - x^2,\ y = -x$ chiziqlar bilan chegaralangan soha yuzi topilsin.

Yechish. Yuzaning chegaralarini aniqlaymiz. Buning uchun berilgan chiziq tenglamalarini birgalikda yechamiz:

$$\begin{cases} y = 6 - x^2, \\ y = -x; \end{cases} \quad 6 - x^2 = -x,\ \ x^2 - x - 6 = 0,$$

bundan $x_1 = 3,\ \ x_2 = -2$. Demak, chiziqlar $M_1(3;\,3),\ \ M_2(-2;\,2)$ nuqtalarda kesishadi.

3) $x = 4y - y^2,\ x + y = 6$ chiziqlar bilan chegaralangan yuzani hisoblash.

Yechish. Berilgan chiziqlarning kesishish nuqtalarini topish uchun tenglamalarni birgalikda yechamiz:

$$\begin{cases} x = 4y - y^2 \\ x + y = 6 \end{cases}$$

$$4y - y^2 = 6 - y$$

$$y^2 - 5y + 6 = 0$$

$$y_1 = 3,\ \ y_2 = 2,\ \ x_1 = 3,\ \ x_2 = 4$$

$$S=\iint_D dxdy=\int_2^3 dy\int_{6-y}^{4y-y^2}dx=\int_2^3 x\Big|_{6-y}^{4y-y^2}dy=\int_2^3(5y-y^2-6)dy=$$

$$=\left(\frac{5}{2}y^2-\frac{1}{3}y^3-6y\right)\Bigg|_2^3=\left(\frac{45}{2}-9-18\right)-\left(10-\frac{8}{3}-12\right)=\frac{1}{6}$$

4) $\rho=1,\ \rho=\frac{2}{\sqrt{3}}\cos\varphi$ aylanalar bilan chegaralangan yuza hisoblansin (b-chizma).

Yechish. Ularning kesishish nuqtalarini topamiz:

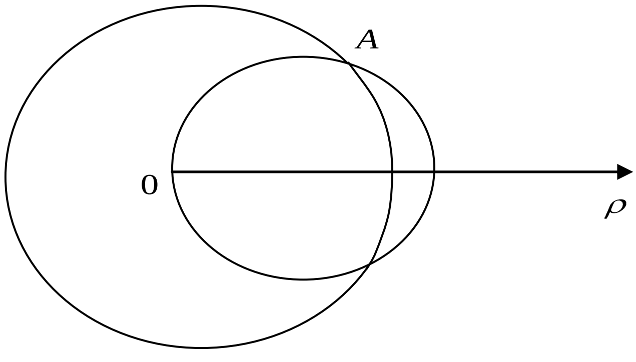
$$1=\frac{2}{\sqrt{3}}\cos\varphi,\quad \cos\varphi=\frac{\sqrt{3}}{2},\ \varphi=\frac{\pi}{6}.$$

Demak, $A\left(1;\ \frac{\pi}{6}\right)$. U holda

$$S=\iint_D\rho d\varphi=2\int_0^{\frac{\pi}{6}}\left[\frac{1}{2}\rho^2\right]_1^{\frac{2}{\sqrt{3}}\cos\varphi}d\varphi=\int_0^{\frac{\pi}{6}}\left(\frac{4}{3}\cos^2\varphi-1\right)d\varphi=$$

$$=\int_0^{\frac{\pi}{6}}\left(\frac{2}{3}+\frac{2}{3}\cos 2\varphi-1\right)d\varphi=\frac{1}{3}\int_0^{\frac{\pi}{6}}(2\cos 2\varphi-1)d\varphi=$$

$$=\frac{1}{3}(\sin 2\varphi-\varphi)\Big|_0^{\frac{\pi}{6}}=\frac{1}{3}\left[\sin\frac{\pi}{3}-\frac{\pi}{6}\right]=\frac{1}{18}(3\sqrt{3}-\pi)\ \text{ kvadrat birlik.}$$



b-chizma.

5) $x^2+y^2=2a^2xy$ lemniskata bilan chegaralangan yuzani toping.

Yechish. $x=\rho\cos\varphi,\ y=\rho\sin\varphi$ deb olib, egra chiziq tenglamasidan qutb koordinatalar sistemasiga o’tamiz. Natijada

$\rho^2 = 2a^2 \sin\varphi\cos\varphi = a^2 \sin 2\varphi$ tenglamaga ega bo'lamiz. Bu holda $0 \le \varphi \le \dfrac{\pi}{4}$ bo'ladi. Yuzaning to'rtdan bir qismini olamiz. Demak,

$$S = 4\iint\limits_D \rho d\rho d\varphi = 4\int\limits_0^{\frac{\pi}{4}} d\varphi \int\limits_0^{a\sqrt{\sin 2\varphi}} \rho d\rho = 2\int\limits_0^{\frac{\pi}{4}} \rho^2\Big|_0^{a\sqrt{\sin 2\varphi}} d\varphi =$$

$$= 2a^2 \int\limits_0^{\frac{\pi}{4}} \sin 2\varphi d\varphi = -\,2a^2\frac{1}{2}\cos 2\varphi\Big|_0^{\frac{\pi}{4}} = -a^2\cos\frac{\pi}{2} + a^2\cos 0 =$$

$$= 0 + a^2 = a^2 \text{ kvadrat birlik.}$$

6) $x^3 + y^3 = axy$ (d-chizma) egri chiziq bilan chegaralangan yuzani toping.

Yechish. Qutb koordinatalar sistemasiga o'tib

$$\rho^3(\sin^3\varphi + \cos^3\varphi) = a\rho^2\sin\varphi\cos\varphi$$

yoki

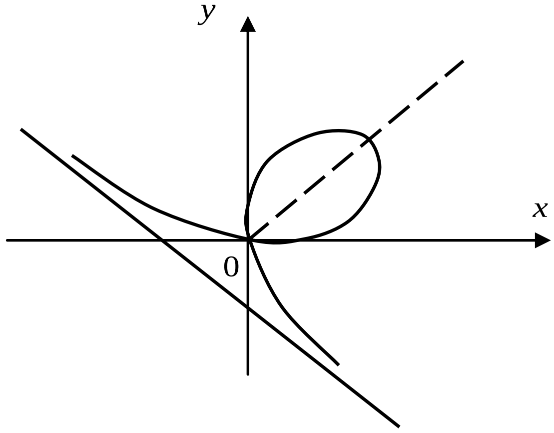
$$\rho = \frac{a\sin\varphi\cos\varphi}{\sin^3\varphi + \cos^3\varphi}$$

yuza simmetrik bo'lganligi uchun nur $\varphi = \dfrac{\pi}{4}$ bo'ladi. Shuning uchun

$$S = 2\iint\limits_D \rho d\rho d\varphi = 2\int\limits_0^{\frac{\pi}{4}} d\varphi \int\limits_0^{a\sin\varphi\cos\varphi/\sin^3\varphi+\cos^3\varphi} \rho d\rho = a^2\int\limits_0^{\frac{\pi}{4}} \frac{\sin^2\varphi\cos^2\varphi}{\left(\sin^3\varphi + \cos^3\varphi\right)^2} d\varphi =$$

$$= a^2\int\limits_0^{\frac{\pi}{4}} \frac{tg^2\varphi\cos^4\varphi}{\cos^6\varphi\left(1 + tg^3\varphi\right)^2} d\varphi = \frac{a^2}{3}\int\limits_0^{\frac{\pi}{4}} \frac{3tg^2\varphi\, d(tg\varphi)}{\left(1 + tg^3\varphi\right)^2} = \frac{a^2}{3}\int\limits_0^{\frac{\pi}{4}} \frac{d\left(1 + tg^3\varphi\right)}{\left(1 + tg^3\varphi\right)^2} =$$

$$\left[\frac{a^2}{3\left(1 + tg^3\varphi\right)}\right]_0^{\frac{\pi}{4}} = \frac{a^2}{6} \text{ kvadrat birlik.}$$



d-chizma.

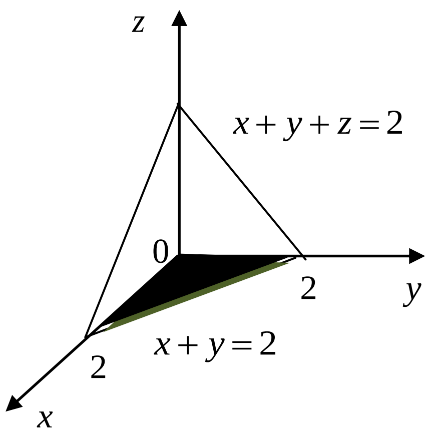
3-§. Ikki karrali integral yordamida jismlar hajmlarini hisoblash

1. $x=0,\quad y=0,\ x+y+z=2,\quad z=0$ sirtlar bilan chegaralangan jism hajmi hisoblansin (a -chizma).

Yechish. D soha chizmada bo'yalgan**.** U xOy tekislikda $x=0,$ $y=0,\ x+y=2$ to'g'ri chiziqlar bilan chegaralangan uchburchak shaklidagi sohalar. Chegaralarni ikki karrali integralga qo'yib hajmni hisoblaymiz:

$$V=\int_0^2\left(\int_0^{2-x}(2-x-y)dy\right)dx=\int_0^2\left[(2-x)y-\frac{y^2}{2}\right]_0^{2-x}dx=\frac{1}{2}\int_0^2(2-x)^2dx=\frac{4}{3}\ \text{kub}$$

birlik.



(a - chizma)

2. $x=0,\quad y=0,\ x+y+z=2,\quad z=0$ sirtlar bilan chegaralangan jism hajmi hisoblansin.

Yechish.

$$V = \iint_D (2-x-y)dxdy = \int_0^2 \left[(2-x)y - \frac{y^2}{2} \right]_0^{2-x} dx = \frac{1}{2}\int_0^2 (2-x)^2 dx = \frac{4}{3} \text{ kub}$$

birlik.

3. Oxy tekisligi bilan, yon tamonlaridan $x=0,\ x=a,\ y=0,\ y=b$ tekisliklar bilan, yuqoridan elliptik paraboloid $z=\frac{x^2}{2p}+\frac{y^2}{2q}$ bilan chegaralangan jism hajmi hisoblansin.

Yechish.

$$V = \iint_{[0,a;0,b]} \left(\frac{x^2}{2p}+\frac{y^2}{2q}\right)dp = \int_0^b dy \int_0^a \left(\frac{x^2}{2p}+\frac{y^2}{2q}\right)dx =$$

$$= \int_0^b \left(\frac{a^2}{2p}+\frac{ay^2}{2q}\right)dy = \frac{ab}{6}\left(\frac{a^2}{p}+\frac{b^2}{q}\right).$$

4. $y=1+x^2,\ \ z=3x,\ \ y=5,\ \ z=0$ sirtlar bilan chegaralangan jismning I oktantda joylashgan qismining hajmi hisoblansin.

Yechish. Jism yuqoridan $z=3x$ tekislik, yon yog'idan $y=1+x^2$ silindr va $y=5$ tekislik bilan chegaralangan, ya'ni u silindrik jism. D soha $y=1+x^2$ parabola va $y=5,\ x=0$ to'g'ri chiziqlar bilan chegaralangan. Shunday qilib,

$$V = \iint_D 3xdxdy = 3\int_0^2 xdx \int_{1+x^2}^5 dy = 3\int_0^2 x\,(y)\Big|_{1+x^2}^5 dx =$$

$$= 3\int_0^2 (4x - x^3)dx = 3\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 3(8-4) = 12 \text{ kub birlik.}$$

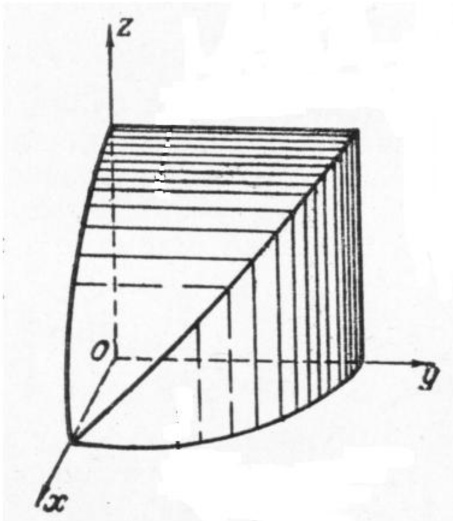
5. $x^2+y^2=a^2$ va $x^2+z^2=a^2$ sirtlar bilan chegaralangan jism hajmi hisoblansin.

Yechish. Jismning sakkizdan bir qismining hajmini hisoblaymiz (b-chizma).

$$\frac{1}{8}V=\iint_D\sqrt{a^2-x^2}\,dxdy=\int_0^a\sqrt{a^2-x^2}\,dx\int_0^{\sqrt{a^2-x^2}}dy=$$

$$=\int_0^a(a^2-x^2)dx=\left[a^2x-\frac{1}{3}x^3\right]_0^a=\frac{2}{3}a^3 \text{ kub birlik.}$$

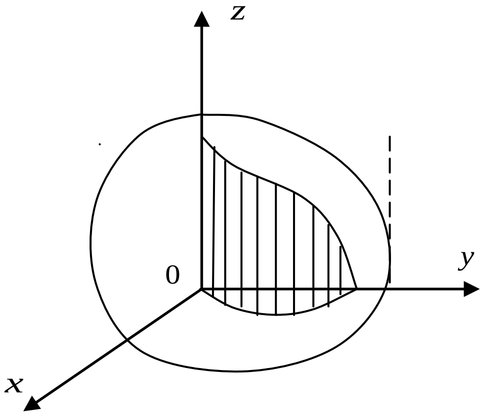
Demak, $V=\dfrac{16}{3}a^3$ kub birlik.



b-rasm.

4-§. Ikki karrali integral yordamida sirt yuzasini aniqlash

1. $x^2+y^2+z^2=a^2$ sferaning $x^2+y^2=ay$ (1-chizma) silindr ichiga joylashgan qismining sirti topilsin.



1-chizma

Yechish. Sferaning (I oktant uchun) tenglamasidan:

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}; \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}.$$

Demak,

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy.$$

D soha sirtning $0xy$ tekislikdagi proyeksiyasi. Sirt tenglamasi $z = f(x, y)$ ko'rinishda berilgan holda yarim doira, D sohaning integrallash chegarasi bo'ladi. Sirt to'rtta oktantda joylashgan bo'lgani uchun, izlangan yuza

$$S = 4a\iint_D \frac{1}{\sqrt{a^2 - x^2 - y^2}}\, dxdy.$$

Qutb koordinatalar sistemasi $x = \rho\cos\varphi,\ y = \rho\sin\varphi$ ga o'tsak, aylana tenglamasi

$$\rho^2 \sin^2\varphi + \rho^2\cos^2\varphi = a\rho\sin\varphi$$

$$\rho^2 (\sin^2\varphi + \cos^2\varphi) = a\rho\sin\varphi$$

$$\rho^2 = a\sin\varphi$$

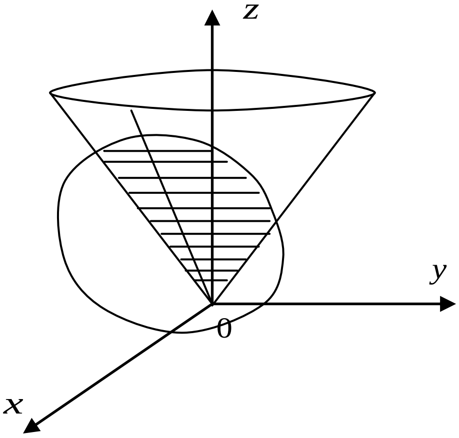
bo'ladi.

$$S = 4a\iint_D \frac{\rho d\rho d\varphi}{\sqrt{a^2 - \rho^2}} = 4a\int_0^{\frac{\pi}{2}} d\varphi \int_0^{a\sin\varphi} \frac{\rho d\rho}{\sqrt{a^2 - \rho^2}} =$$

$$= -4a\int_0^{\frac{\pi}{2}} \sqrt{a^2 - \rho^2}\Big|_0^{a\sin\varphi} d\varphi = -4a^2\int_0^{\frac{\pi}{2}} (\cos\varphi - 1)d\varphi =$$

$$= -4a^2 (\sin\varphi - \varphi)\Big|_0^{\frac{\pi}{2}} = 4a^2\left(-\frac{\pi}{2}\right) \text{ kvadrat birlik.}$$

2. $z = \sqrt{x^2 + y^2}$ konusning $x^2 + y^2 = 2x$ (2-chizma) silindr ichidagi joylashgan qismining sirti yuzi topilsin.



2-chizma.

Yechish. Konusning tenglamasidan

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}; \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

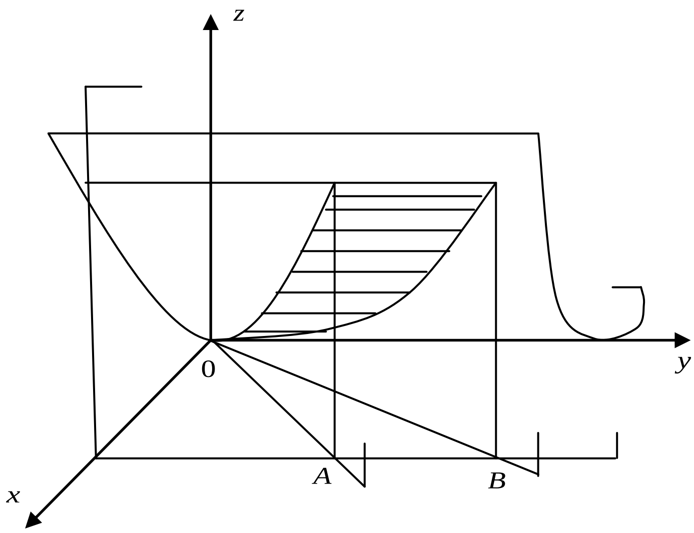
Integrallash sohasi doira bo'lib, $x^2 + y^2 = 2x$ aylanadan iborat yoki $\rho = a\cos\varphi$. U holda

$$\iint_D \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}}\,dxdy = \sqrt{2}\iint_D dxdy = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \rho d\rho =$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\rho^2 \Big|_0^{2\cos\varphi} d\varphi = 2\sqrt{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} 4\cos^2\varphi d\varphi = 2\sqrt{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\varphi) d\varphi =$$

$$= 2\sqrt{2}\left[\varphi + \frac{1}{2}\sin 2\varphi\right]_0^{\frac{\pi}{2}} = \pi\sqrt{2} \quad \text{kvadrat birlik.}$$

3. $x^2 = 2z$ silindrning $x - 2y = 0,\ y = 2x,\ x = 2\sqrt{2}$ chiziqlar bilan kesilgan qismining sirtini toping (3-chizma).



3-chizma

Yechish. Integrallash chegarasi OAB uchburchak. Silindr tenglamasidan $\frac{\partial z}{\partial x}=x,\ \frac{\partial z}{\partial y}=0$. U holda

$$S=\iint_D \sqrt{1+x^2}dxdy=\int_0^{2\sqrt{2}}\sqrt{1+x^2}dx\int_{\frac{x}{2}}^{2x}dy=\int_0^{2\sqrt{2}}\frac{3}{2}x\sqrt{1+x^2}dx=$$

$$=\frac{3}{4}\int_0^{2\sqrt{2}}(1+x^2)^{\frac{1}{2}}d(1+x^2)=\frac{3}{4}\cdot\frac{2}{3}(1+x^2)^{\frac{3}{2}}\Big|_0^{2\sqrt{2}}=13 \quad \text{kvadrat birlik.}$$

4. $x=1-y^2-z^2$ paraboloidning $y^2+z^2=1$ slindr bilan kesgan qismining sirti yuzi topilsin.

Yechish. Integrallash sohasi $y^2+z^2=1$ aylana . Paraboloid tenglamasidan

$$\frac{\partial x}{\partial y}=12y; \qquad \frac{\partial x}{\partial z}=-2z$$

ga ega bo'lamiz. U holda

$$S=\iint_D \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}dydz=\iint_D \sqrt{1+4(y^2+z^2}dydz$$

qutb koordinata sistemasiga o'tib,

$$S=\int_0^{2\pi}d\varphi\int_0^1 dy=\int_0^{2\sqrt{2}}\rho\sqrt{1+4\rho^2}\,d\rho=\int_0^{2\pi}\left[\frac{2}{3}\cdot\frac{1}{8}\left(1+4\rho^2\right)^{\frac{3}{2}}\right]_0^1 d\varphi=$$

$$=\int_0^{2\pi}\frac{1}{12}\left(5\sqrt{5}-1\right)d\varphi=\frac{5\sqrt{5}-1}{6}\pi \quad \text{kvadrat birlik.}$$

5-§. Ikki karrali integralning fizikadagi ba'zi tadbiqlari

1-misol. Sirt zichligi $\rho(x,y)=xy^2$ bo'lgan, Ox o'qi, $y=x^2$ parabola va $x+y=6$ to'g'ri chiziq bilan chegaralangan egri chiziqli uchburchakdan iborat D yupqa plastinkaning massasini hisoblang.

Yechish. Plastinkaning massasini hisoblash uchun dastlab D sohani aniqlaymiz:

$$D:\sqrt{y}\le x\le 6-y;\ 4\le y\le 9.$$

$$m=\int_4^9 dy\int_{\sqrt{y}}^{6-y}xy^2dx=\int_4^9\frac{x^2y^2}{2}\Bigg|_{\sqrt{y}}^{6-y}dy=\int_4^9\left(\frac{(6-y)^2y^2}{2}-\frac{y^3}{2}\right)dy=$$

$$=\int_4^9\left(18y^2-\frac{13y^3}{2}-\frac{y^4}{2}\right)dy=\left(18\frac{y^3}{3}-\frac{13y^4}{8}-\frac{y^5}{10}\right)\Bigg|_4^9=$$

$$=\left(6\cdot 9^3-\frac{13}{8}\cdot 9^4-\frac{9^5}{10}\right)-\left(6\cdot 4^3-\frac{13}{8}\cdot 4^4-\frac{4^5}{10}\right)=453\frac{1}{8}.$$

2-misol. Sirt zichligi $\rho(x,y)=x+y$ ga teng bo'lgan $y=4-x^2$ parabola va Ox o'qi bilan chegaralangan yuz og'irlik markazining koordinatalarini toping (a-chizma).

Yechish. D soha $-2\le x\le 2;\ 0\le y\le 4-x^2$ shakl Oy o'qqa nisbatan simmetrik bo'lgani uchun $x_0=0$ og'irlik markazining koordinatasini topish uchun M_x statik momentni hisoblaymiz.

$$M_x = \iint_D y\rho(x,y)dxdy = \int_{-2}^{2}dx\int_{0}^{4-x^2} y(x+y)dy = \int_{-2}^{2}dx\int_{0}^{4-x^2}(yx+y^2)dy =$$

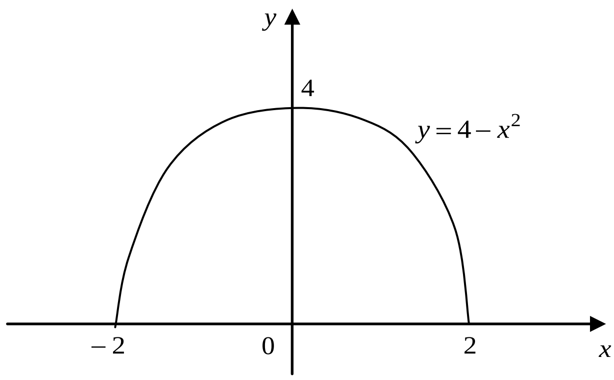
$$= \int_{-2}^{2}\left[\frac{x(4-x^2)^2}{2}+\frac{(4-x^2)^3}{3}\right]dx =$$

$$= \frac{1}{6}\int_{-2}^{2}\left(-2x^6+3x^5+24x^4-24x^3-96x^2+48x+128\right)dx =$$

$$= \frac{1}{6}\left[-2\frac{x^7}{7}+3\frac{x^6}{6}+24\frac{x^5}{5}-24\frac{x^4}{4}-96\frac{x^3}{3}+48\frac{x^2}{2}+128x\right]_{-2}^{2} = 40\frac{34}{105}.$$

Og'irlik markazining ordinatasi

$$y_0 = \frac{M_x}{m} = \frac{40\frac{34}{105}}{\iint_D \rho(x-y)dxdy} = \frac{40\frac{34}{105}}{17\frac{1}{15}} \approx 2.37.$$



a – chizma

3-misol. Bir jinsli ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

ning koordinata o'qlariga nisbatan inertsiya momentini toping.

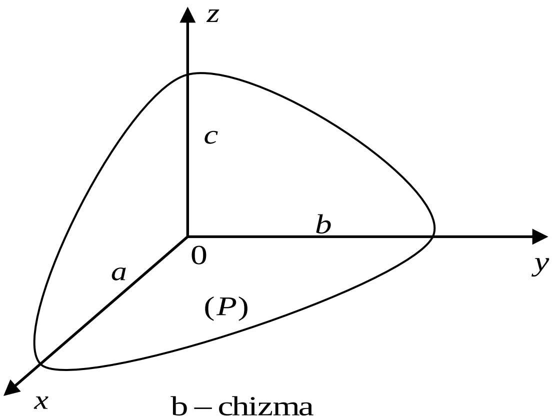
Yechish. Ellipsoidning birgina oktanti (b-chizma) bilan chegaralanib, natijasini 8 ga ko'paytirish mumkin. Bu holda P soha $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ ellipsning kvadranti bo'ladi.

$$I_{zx}=8\iint\limits_{(P)} y^2 z\,dp=\frac{8c}{a}\int_0^b y^2dy\int_0^{a\sqrt{1-\frac{y^2}{b^2}}}\sqrt{a^2\left(1-\frac{y^2}{b^2}\right)-x^2}\,dx=$$

$$=2\pi ac\int_0^b y^2\left(1-\frac{y^2}{b^2}\right)dy=\frac{4}{15}\pi ab^3c.$$

Xuddi shunga o'xshash

$$I_{yz}=\frac{4}{15}\pi a^3bc,\quad I_{xy}=\frac{4}{15}\pi abc^3.$$



b – chizma

4-misol. $y^2=4x+4,\quad y^2=-2x+4$ (d-chizma) chiziqlar bilan chegaralangan figuraning og'irlik markazining koordinatalari topilsin.

Yechish. Figura Ox o'qiga simmetrik bo'lgani uchun $\bar{y}=0$. $\bar{x}$ ni topish qoldi.

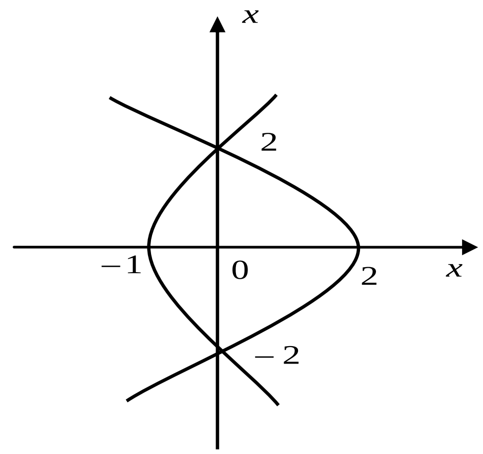
$$S=\iint\limits_D dxdy=2\int_0^2 dy\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}}dx=2\int_0^2\left(\frac{4-y^2}{2}-\frac{y^2-4}{4}\right)dy=2\int_0^2\left(3-\frac{3y^2}{4}\right)dy=$$

$$=6\left[y-\frac{y^3}{12}\right]_0^2=6\left(2-\frac{2}{3}\right)=8.$$

U holda

$$\bar{x}=\frac{1}{8}\iint\limits_D xdxdy=\frac{1}{8}\cdot 2\int_0^2 dy\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}}xdx=\frac{1}{8}\int_0^2\left(\frac{(4-y^2)^2}{4}-\frac{(y^2-4)^2}{16}\right)dy=$$

$$=\frac{1}{8}\int_0^2\left(3-\frac{3}{2}y^2+\frac{3}{16}y^4\right)dy=\frac{1}{8}\left(3y-\frac{1}{2}y^3+\frac{3}{80}y^5\right)\Bigg|_0^2=\frac{1}{8}\left(6-4+\frac{3\cdot 32}{80}\right)=\frac{2}{5}.$$



d – chizma

5-misol. $\frac{x^2}{25}+\frac{y^2}{9}=1$ ellips va uning $\frac{x}{5}+\frac{y}{3}=1$ yoyidan iborat qismining yoyidan tashkil topgan qismining og'irlik markazining koordinatalari topilsin.

Yechish. Sigmentning yuzini topamiz:

$$S=\iint\limits_D dxdy=\int_0^5 dx\int_{3\left(1-\frac{x}{5}\right)}^{\frac{3}{5}\sqrt{25-x^2}} dy=\int_0^5\left(\frac{3}{5}\sqrt{25-x^2}-3\left(1-\frac{x}{5}\right)\right)dx=\frac{15}{4}(\pi-2)$$ kvadrat

birlik.

U holda

$$\bar{x}=\frac{1}{S}\iint\limits_D xdxdy=\frac{4}{15(\pi-2)}\int_0^5 xdx\int_{3\left(1-\frac{x}{5}\right)}^{\frac{3}{5}\sqrt{25-x^2}} dy=$$

$$=\frac{4}{15(\pi-2)}\int_0^5\left[\frac{3}{5}x\sqrt{25-x^2}-3x\left(1-\frac{x}{5}\right)\right]dx=$$

$$=\frac{4}{15(\pi-2)}\left[-\frac{3}{5}\cdot\frac{1}{2}\cdot\frac{2}{3}(25-x^2)^{3/2}-\frac{3x^2}{2}+\frac{x^3}{5}\right]_0^5=$$

$$=\frac{4}{15(\pi-2)}\left(25-\frac{75}{2}+25\right)=\frac{10}{3(\pi-2)}.$$

$$\bar{y} = \frac{1}{S}\iint\limits_D ydxdy = \frac{4}{15(\pi-2)}\int_0^5 dx \int_{3\left(1-\frac{x}{5}\right)}^{\frac{3}{5}\sqrt{25-x^2}} ydy =$$

$$= \frac{4}{15(\pi-2)} \cdot \frac{1}{2}\int_0^5 \left[\frac{9}{25}\left(25-x^2\right) - 9\left(1-\frac{x}{5}\right)^2\right]dx =$$

$$= \frac{2\cdot 9\cdot 2}{15(\pi-2)\cdot 25}\int_0^5 \left(5x - x^2\right)dx = \frac{12}{125(\pi-2)}\left[\frac{5x^2}{2} - \frac{1}{3}x^3\right]_0^5 =$$

$$\frac{12}{125(\pi-2)}\left[\frac{125}{2} - \frac{125}{3}\right] = \frac{2}{\pi-2}.$$

Demak, og’irlik markazi koordinatalari:

$$\left(\frac{10}{3(\pi-2)};\ \ \frac{2}{\pi-2}\right).$$

Xulosa

Ikki karrali integral va uning ba’zi bir tadbiqlari to’g’risidagi ushbu bitiruv malakaviy ish talaba tomonidan quyidagi holda bajarilgan:

1. Ikki karrali integral va uni hisoblash to’g’risida to’liq ma’lumot keltirilgan.

2. Yuzalarni hisoblash misollar asosida hal etilgan.

3. Hajmlarni hisoblashda qutb koordinata sistemasiga o’tishlardan foydalanilgan.

4. Sirt yuzini topishda ikki o’zgaruvchili funksiya hosilalaridan unumli foydalanilgan.

5. Jismning og’irlik markazi koordinatalarini hisoblashda jism massasini hisoblash to’g’ri hal etilgan.

Umuman Abdullayev Jamshid bakalavr darajasini olishga loyiq va bitiruv malakaviy ishni hal eta olgan.

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