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NAVOIY DAVLAT PEDAGOGIKA INSTITUTI

FIZIKA –MATEMATIKA FAKULTETI

MATEMATIKA-INFORMATIKA TA’LIM YO’NALISHI

“UMUMIY MATEMATIKA”

KAFEDRASI

BITIRUV MALAKAVIY ISHI

Mavzu: Aniq integralni taqribiy hisoblash va uning tadbiqlari

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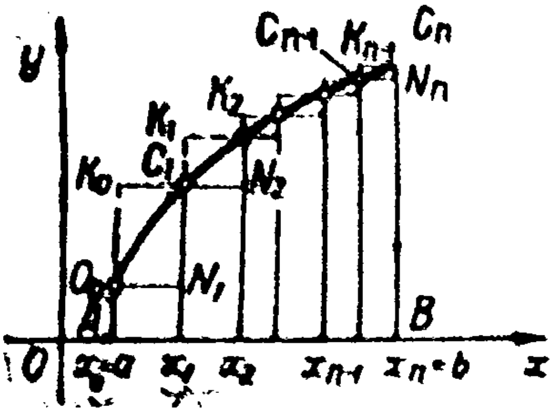
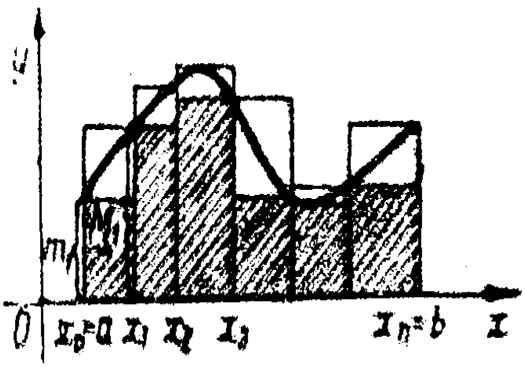
Mundarija

Kirish

Qadim zamonlardan buyon odamlar ekin maydoni yuzalarini o`lchash uchun ekin maydonini kichik to`rtburchaklarga ajratib, so`ngra ularning yuzalarini qo`shib maydon yuzi kattaligini taqribiy topishgan. Xuddi shu usulni Arximed geometric figuralarni yuzasi va hajmini topishda qo`llagan. Nyuton barcha fizikaviy hodisalar differensiallash va integrallash amallarining ketma-ket takrorlanish natijasida ro`y berishini kuzatadi. Shu prinsipni qo`llab ko`pgina natijalarga erishadi. Shu sababli ham integral va differensial tushunchalari nyuton nomi bilan bo`g`liq.

Integral tushunchasi matematik analizning asosiy tushunchalaridan biri bo'lib matematika, fizika, mexanika va boshqa fanlarning eng kuchli quroli hisoblanadi. Egri chiziqlar bilan chegaralangan yuzlarni, egri chiziq yoylari va uzunliklarini, hajmlarni, ishlarni, tezliklarni, yo'llarni, inersiya momentlarini va hokazolarni hisoblashga ishlarining hammasi integral hisoblashga keltiriladi.





[a,b] kesmada $y = f(y)$ uzlusiz funksiya berilgan bo'lsin uning bu kemadagi eng kichik va eng kata qiymatlarini m va M bilan belgilaymiz [a,b] kesmani

$$a = x_0, x_1, x_2, \dots\ x_{n-1}, x_n = b$$

nuqtalar bilan n ta qismga ajratamiz . $x_0 < x_1 < x_2 < \ \dots\ x_n$

deb hisoblaymiz va

$x_1 - x_0 = \Delta x_1, \qquad x_2 - x_1 = \Delta x_2, \qquad \ldots, x_n - x_{n-1} = \Delta x_n$

Deb faraz qilamiz. So`ngra $f(x)$ Funksiyaning eng kata va eng kichik qiymatlarini

$[x_0,\ x_1]$ kesmada m_1 va M_1 bilan,

$[x_1,\ x_2]$ kesmada m_2 va M_2 bilan,

………………………………………

$[x_{n-1},\ x_n]$ kesmada m_n va Mn bilan,

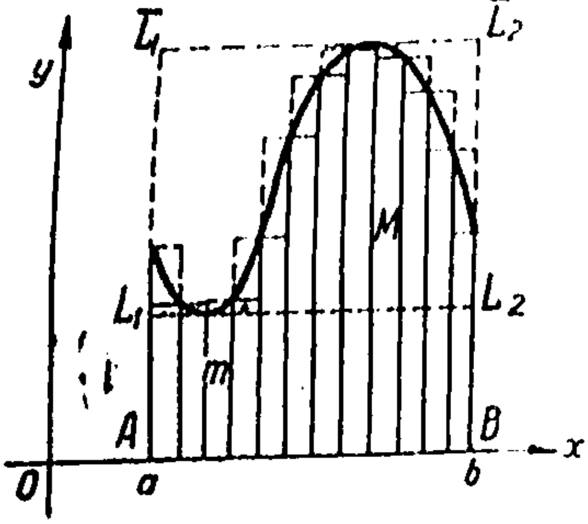
belgilaymiz,endi

$$s_n = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n = \sum_{i=1}^{n} m_i\Delta x_i \qquad (1)$$

$$s_n = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n = \sum_{i=1}^{n} M_i\Delta x_i \qquad (2)$$

yig’indilarni tuzamiz. Sn quyi integral yig’indi deb Sn esa yuqori integral yig’ndi deb ataladi.

Agar $f(x) \geq 0$ bo’lsa u holda quyi integral yig’indi son qiymati ichki chizilgan zinasimon shaklning $AC_0N_1C_1N_2{\ldots}C_{n-1}\ N_nBA$ siniq chiziq bilan chegarlangan yuzaga teng bo’lib yuqori integral yig’indining son qiymati esa Tashqi chizilgan zinasimon



$AK_0C_1K_1 \ldots C_{n-1}K_{n-1}C_nBA$

Saklning siniq chiziq bilan chegaralangan yuziga teng

Yuqori va quyi integral yig'indining ba'zi xossalarini ko'rsatib o'tamiz.

Har qanday $i(i = 1,2, \ldots, n)$ uchun $m \leq Mi$ bo'lgani sababli, (1) va (2) formulaga muvofiq

$\underline{s_n} \leq \overline{s_n}$

$F(x) = const$ bo'lgan holdagina tenglik belgisi bo'ladi

$m_1 \geq m, \quad m_2 \geq m, \ldots, m_n \geq m$ bo'lgani uchun m(f(x)) funksiyaning [a,b] kesmada eng kichik qiymati)

$\underline{s_n} = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n \geq m_1\Delta x_1 + m_2\Delta x_2 + \cdots +$

$+m_n\Delta x_n = m(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = m(b-a).$

Shunday qilib

$$\underline{s_n} \geq m(b-a)$$

$M_1 \leq M_1, M_2 \leq M, \ldots M_n \leq M$

bo'lgani sababli (m(f(x) funksiyaning [a,b] kesmada eng katta qiymati)

$\overline{s_n} = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n \geq M_1\Delta x_1 + M_2\Delta x_2 + \cdots +$

$+M_n\Delta x_n = M(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = M(b-a).$

Shunday qilib

$\overline{s_n} \leq M(b-a)$

Hosil qilingan tengsizlikni birgalikda yozamiz:

$$m(b-a) \leq \underline{s_n} \leq \overline{s_n} \leq M(b-a)$$

Agar $f(x) \geq 0$ bo'lsa bu tengsizliklar soda geometric ma'noga ega chunki

$m(b-a)$ va $M(b-a)$ qiymatlariga mos tartibda ichki chizilgan AL_1L_2B to’g’ri to’rtburchak yuziga tashqi chizilgan $A\overline{L_1, L_2\ B}$ to’g’ri to’rtburchak yuziga teng

1. Aniq integrallarni hisoblash

**1⁰. Aniq integrallarni ta'rifga ko`ra hisoblash.** Aytaylik, $f(x)\in R([a,b])$ bo`lsin. Unda integral ta'rifiga ko`ra

$$\lim_{\lambda_P\to 0}\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k=\int_a^b f(x)dx$$

bo`ladi.

1-Misol. Ushbu

$$\int_a^b \sin x\,dx$$

integral hisoblansin.

◀ Ravshanki, $f(x)=\sin x\in C[a,b]$. Demak, $f(x)\in R([a,b])$. $[a,b]$ oraliqni ushbu

$$a,a+\alpha_n,a+2\alpha_n,...,a+k\alpha_n,...,a+n\alpha_n=b$$

nuqtalar yordamida, bunda $\alpha_n=\dfrac{b-a}{n}$, n ta teng bo`lakka bo`lib, har bir

$$[a+k\alpha_n,a+(k+1)\alpha_n]\ \ (k=0,1,2,...,n-1)$$

bo`lakda ξ_k nuqtani quyidagicha

$$\xi_k=a+(k+1)\alpha_n\quad (k=0,1,2,...,n-1)$$

tanlaymiz. U holda $f(x)=\sin x$ funksiyaning integral yig`indisi quyidagicha

$$\sigma=\sum_{k=0}^{n-1}\sin(a+(k+1)\alpha_n)\cdot\alpha_n=\alpha_n\sum_{k=0}^{n-1}\sin(a+(k+1)\alpha_n)$$

ko`rinishga ega bo`ladi.

Ma'lumki,

$$\sin(a+(k+1)\alpha_n)=\frac{1}{2\sin\frac{\alpha_n}{2}}2\sin\frac{\alpha_n}{2}\sin(a+(k+1)\alpha_n)=$$

$$=\frac{1}{2\sin\frac{\alpha_n}{2}}\left[\cos(a+(k+\frac{1}{2})\alpha_n)-\cos(a+(k+\frac{3}{2})\alpha_n)\right]$$

bo`ladi.

Natijada integral yig`indi uchun ushbu

$$\sigma=\frac{\alpha_n}{2\sin\frac{\alpha_n}{2}}\sum_{k=0}^{n-1}\left[\cos(a+(k+\frac{1}{2})\alpha_n)-\cos(a+(k+\frac{3}{2})\alpha_n)\right]=$$

$$=\frac{\frac{\alpha_n}{2}}{\sin\frac{\alpha_n}{2}}(\cos(a+\frac{1}{2}\alpha_n)-\cos(b+\frac{1}{2}\alpha_n))$$

tenglikka kelamiz.

Keyingi tenglikda $\lambda_P=\Delta x_k=\alpha_n\to 0$ da limitga o`tib topamiz:

$$\int_a^b \sin x dx=\cos a-\cos b. \blacktriangleright$$

2⁰. Nyuton-Leybnits formulasi. Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan va shu segmentda uzluksiz bo`lsin. U holda $f(x)$ boshlang`ich funksiya

$$F(x)=\int_a^x f(t)dt$$

ga ega bo`ladi.

Ravshanki, $\Phi(x)$ funksiya $f(x)$ ning ixtiyoriy boshlang`ich funksiyasi bo`lsa, u holda

$$\Phi(x)=F(x)+C \qquad (C=const)$$

bo`ladi.

Bu tenglikda, avval $x=a$ deb

$$\Phi(a)=C,$$

so`ngra $x=b$ deb

$$\Phi(b)=\int_a^b f(x)dx+C$$

bo`lishini topamiz. Demak,

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a). \qquad (1)$$

(1) formula Nyuton-Leybnits formulasi deyiladi.

Odatda, $\Phi(b) - \Phi(a)$ ayirma $\Phi(x)\Big|_a^b$ kabi yoziladi. Demak,

$$\int_a^b f(x)dx = \Phi(x)\Big|_a^b = \Phi(b) - \Phi(a).$$

Masalan,

$$\int_a^b \frac{1}{x}dx = \ln x\Big|_a^b = \ln b - \ln a = \ln\frac{b}{a}. \qquad (a > 0, b > 0)$$

**3^0. O`zgaruvchilarini almashtirish formulasi.** Faraz qilaylik, $f(x) \in C[a,b]$ bo`lsin. Ravshanki, bu holda

$$\int_a^b f(x)dx$$

integral mavjud bo`ladi.

Ayni paytda, bu funksiya $[a,b]$ da boshlang`ich $\Phi(x)$ funksiyaga ega bo`lib,

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a)$$

bo`ladi.

Aytaylik, aniq integralda x o`zgaruvchi ushbu

$$x = \varphi(t)$$

formula bilan almashtirilgan bo`lib, bunda $\varphi(t)$ funksiya quyidagi shartlarni bajarsin:

1) $\varphi(t) \in C[\alpha,\beta]$ bo`lib, $\varphi(t)$ funksiyaning barcha qiymat-lari $[a,b]$ ga tegishli;

2) $\varphi(\alpha) = a, \quad \varphi(\beta) = b$;

3) $\varphi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hosilaga ega bo`lsin.

U holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (2)$$

bo`ladi.

◄ Ravshanki, $\Phi(\varphi(t))$ murakkab funksiya $[\alpha,\beta]$ segmentda uzluksiz bo`lib,

$$(\Phi(\varphi(t)))' = \Phi'(\varphi(t))\cdot\varphi'(t)$$

bo`ladi.

Agar $\Phi'(x) = f(x)$ ekanini e'tiborga olsak, unda

$$(\Phi(\varphi(t)))' = f(\varphi(t))\cdot\varphi'(t)$$

bo`lishini topamiz. Bu esa $\Phi(\varphi(t))$ funksiya $[\alpha,\beta]$ da $f(\varphi(t))\cdot\varphi'(t)$ funksiyaning boshlang`ich funksiyasi ekanini bildiradi. Nyuton-Leybnits formulasiga ko`ra

$$\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt = \Phi(\varphi(\beta)) - \Phi(\varphi(\alpha)) = \Phi(b) - \Phi(a) \qquad (3\ )$$

bo`ladi.

(2) va (3) munosabatlardan

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (4)$$

bo`lishi kelib chiqadi. ►

(4) formula aniq integralda o`zgaruvchini almashtirish formulasi deyiladi.

2-misol. Ushbu

$$\int_0^1 \sqrt{1-x^2}\,dx$$

integral hisoblansin.

◄ Berilgan integralda $x = \sin t$ almashtirishni bajara-miz. Unda

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t\,dt = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt =$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)dt = \left(\frac{1}{2}t+\frac{1}{4}\sin 2t\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

bo`ladi. ►

4⁰. Bo`laklab integrallash formulasi. Aytaylik , $u(x)$ va $v(x)$ funksiyalarning har biri $[a,b]$ segmentda uzluksiz $u^{/}(x)$ va $v^{/}(x)$ hosilalarga ega bulsin. U holda

$$\int_a^b u(x)dv(x)=(u(x)\cdot v(x))\Big|_a^b-\int_a^b v(x)du(x) \qquad (5)$$

bo`ladi.

◄ Hosilani hisoblash qoidasiga ko`ra

$$(u(x)\cdot v(x))'=u^{/}(x)\cdot v(x)+u(x)\cdot v^{/}(x)$$

bo`ladi. Demak, $u(x)\cdot v(x)$ funksiya $[a,b]$ oraliqda $u^{/}(x)\cdot v(x)+u(x)\cdot v^{/}(x)$ funksiyaning boshlang`ich funksiyasi bo`ladi. Nyuton-Leybnits formulasidan foydalanib topamiz:

$$\int_a^b (u^{/}(x)\cdot v(x)+u(x)\cdot v^{/}(x))'dx=(u(x)\cdot v(x))\Big|_a^b .$$

Keyingi tenglikdan

$$\int_a^b u(x)dv(x)=(u(x)\cdot v(x))\Big|_a^b-\int_a^b v(x)du(x)$$

bo`lishi kelib chiqadi. ►

(5) formula aniq integrallarda bo`laklab integrallash formulasi deyiladi.

3-misol. Ushbu

$$\int_1^2 x\ln x dx$$

integral hisoblansin.

◄ Bu intervalda $u(x)=\ln x, dv(x)=x$ deb $du(x)=\frac{1}{x}dx, v(x)=\frac{x^2}{2}$

bo`lishini topamiz. Unda (5) formulaga ko`ra:

$$\int_1^2 x\ln x dx = \left(\frac{x^2}{2}\ln x\right)\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}dx = 2\ln 2 - \frac{1}{2}\int_1^2 x dx = 2\ln 2 - \frac{3}{4}$$ bo\`ladi. ►

4-misol. Ushbu

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x dx \qquad (n = 0,1,2,...)$$

integral hisoblansin.

◄ Ravshanki,

$$J_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \quad , \quad J_1 = \int_0^{\frac{\pi}{2}} \sin x dx = (-\cos x)\Big|_0^{\frac{\pi}{2}} = 1.$$

$n \ge 2$ bo\`lganda berilgan integralni

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x\, d(-\cos x)$$

ko\`rinishida yozib, unga bo\`laklab integrallash formulasini qo\`llaymiz. Natijada

$$J_n = (-\sin^{n-1} x \cdot \cos x)\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx =$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx =$$

$$= (n-1)J_{n-2} - (n-1)J_n$$

bo\`lib, undan ushbu

$$J_n = \frac{n-1}{n} J_{n-2}$$

rekurrent formula kelib chikadi.

Bu formula yordamida berilgan integralni $n = 1,2,3,........$ bo\`lganda ketma-ket hisoblash mumkin.

Aytaylik, $n = 2m$- juft son bo\`lsin. Unda

$$J_{2m} = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}......\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot J_0 = \frac{(2m-1)!!}{(2m)!!}\cdot\frac{\pi}{2}$$

bo`ladi.

Aytaylik, $n=2m+1$- toq son bo`lsin. Unda

$$J_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}......\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot J_1=\frac{(2m)!!}{(2m+1)!!}$$

bo`ladi. ($m!!$ simvol $m$ dan katta bo`lmagan va u bilan bir xil juftlikka ega bo`lgan natural sonlarning ko`paytmasini bildiradi.) ►

5⁰. Vallis formulasi. Ma'lumki, $0<x<\frac{\pi}{2}$ bo`lganda

$$\sin^{2n+1}x<\sin^{2n}x<\sin^{2n-1}x \qquad (n=1,2,3,...)$$

tengsizliklar o`rinli bo`ladi. Bu tengsizliklarni $[0,\frac{\pi}{2}]$ oraliq bo`yicha integrallab,

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1}xdx<\int_0^{\frac{\pi}{2}}\sin^{2n}xdx<\int_0^{\frac{\pi}{2}}\sin^{2n-1}xdx,$$

so`ngra 4^0 da keltirilgan formulalardan foydalanib topamiz:

$$\frac{(2n)!!}{(2n+1)!!}<\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}<\frac{(2n-2)!!}{(2n-1)!!}.$$

Bu tengsizliklardan

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\cdot\frac{1}{2n+1}<\frac{\pi}{2}<\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\cdot\frac{1}{2n}$$

bo`lishi kelib chiqadi.

Keyingi tengsizliklardan topamiz:

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{1}{2n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2. \qquad (6)$$

(6) formula Vallis formulasi deyiladi.

2. Aniq integralni taqribiy hisoblash

Odatda, aniq integrallar Nyuton-Leybnits formulasi yordamida hisoblanadi. Bu formula boshlang`ich funksiyaga asoslanadi. Ammo boshlang`ich funksiyani topish masalasi doim osongina hal bo`lavermaydi. Agar integral ostidagi funksiya murakkab bo`lsa, tegishli aniq integralni taqribiy hisoblashga to`g`ri keladi.

**1^0. To`g`ri to`rtburchaklar formulasi.** Faraz qilaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan va uzluksiz bo`lsin. Demak, $f(x)\in R([a,b])$.

Masala $\int_a^b f(x)dx$ integralni taqribiy hisoblashdan iborat.

$[a,b]$ oraliqni $a=x_0, x_1, x_2, ..., x_{n-1}, x_n=b$ nuqtalar $\left(x_0<x_1<x_2<...<x_n\right)$ yordamida n ta teng bo`lakka bo`lib, har bir $[x_k, x_{k+1}]$ $(k=0,1,2,...,n-1)$ bo`yicha integralni quyidagicha

$$\int_{x_k}^{x_{k+1}} f(x)dx \approx f\left(\frac{x_k+x_{k+1}}{2}\right)\cdot(x_{k+1}-x_k)=\frac{b-a}{n}f(x_{k+\frac{1}{2}})$$

taqribiy hisoblaymiz, bunda

$$x_k=a+k\frac{b-a}{n}\ ,\ x_{k+\frac{1}{2}}=\frac{x_k+x_{k+1}}{2}=a+(k+\frac{1}{2})\frac{b-a}{n},\quad x_{k+1}-x_k=\frac{b-a}{n}\quad (k=0,1,2,...,n-1).$$

Aniq integral xossasidan foydalanib topamiz:

$$\int_a^b f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_2} f(x)dx+...+\int_{x_k}^{x_{k+1}} f(x)dx+...$$

$$...+\int_{x_{n-1}}^{x_n} f(x)dx\approx\frac{b-a}{n}f(x_{\frac{1}{2}})+\frac{b-a}{n}f(x_{1+\frac{1}{2}})+\frac{b-a}{n}f(x_{2+\frac{1}{2}})+...$$

$$...+\frac{b-a}{n}f(x_{k+\frac{1}{2}})+...+\frac{b-a}{n}f(x_{n-\frac{1}{2}})=\frac{b-a}{n}[f(x_{\frac{1}{2}})+$$

$$+f(x_{1+\frac{1}{2}})+...+f(x_{k+\frac{1}{2}})+...+f(x_{n-\frac{1}{2}})].$$

Natijada

$$\int_a^b f(x)dx$$

integralni taqribiy hisoblash uchun quyidagi

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{k=1}^{n-1} f(x_{k+\frac{1}{2}}) \qquad (1)$$

formulaga kelamiz.

(1) formula to`g`ri to`rtburchaklar formulasi deyiladi.

Endi (1) taqribiy formulaning xatoligini aniqlaymiz.

(1) formulaning xatoligini

$$R_n = \int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}}) \qquad (2)$$

deylik.

Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda uzluksiz $f''(x)$ hosilaga ega bo`lsin.

Avvalo R_n ni quyidagicha yozib olamiz:

$$R_n = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f(x)dx - \frac{b-a}{n}\sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}}) = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f(x)dx -$$

$$-\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f(x_{k+\frac{1}{2}})dx = \sum_{k=0}^{n-1}\int [\,f(x) - f(x_{k+\frac{1}{2}})]dx.$$

Teylor formulasidan foydalanib topamiz:

$$f(x) - f(x_{k+\frac{1}{2}}) = f'(x_{k+\frac{1}{2}})\cdot(x - x_{k+\frac{1}{2}}) + \frac{1}{2} f''(\xi_k)\cdot(x - x_{k+\frac{1}{2}})^2$$

(bunda ξ_k son x va $x_{k+\frac{1}{2}}$ sonlar orasida). Natijada

$$R_n = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} (\,f'(x_{k+\frac{1}{2}})\cdot(x - x_{k+\frac{1}{2}}) + \frac{1}{2} f''(\xi_k)\cdot(x - x_{k+\frac{1}{2}})^2)dx =$$

$$= \sum_{k=0}^{n-1}(\,f'(x_{k+\frac{1}{2}})\int_{x_k}^{x_{k+1}}(x - x_{k+\frac{1}{2}})dx + \frac{1}{2}\int_{x_k}^{x_{k+1}} f''(\xi_k)\cdot(x - x_{k+\frac{1}{2}})^2)dx$$

bo`ladi.

Ravshanki, $\int\limits_{x_k}^{x_{k+1}}\left(x-x_{k+\frac{1}{2}}\right)dx=0$.

Demak, $$R_n=\frac{1}{2}\sum_{k=0}^{n-1}\int\limits_{x_k}^{x_{k+1}}f''(\xi_k)\cdot\left(x-x_{k+\frac{1}{2}}\right)dx.$$

O`rta qiymat haqidagi teoremaga binoan

$$\int\limits_{x_k}^{x_{k+1}}f''(\xi_k)\cdot(x-x_{k+\frac{1}{2}})^2dx=f''(\xi_k^*)\int\limits_{x_k}^{x_{k+1}}(x-x_{k+\frac{1}{2}})^2dx=$$
$$=\frac{(x_{k+1}-x_k)^2}{12}f''(\xi_k^*)=\frac{(b-a)^3}{12n^3}f''(\xi_k^*)\ \ (\xi_k^*\in[x_k,x_{k+1}])$$

bo`ladi.

Shunday qilib, R_n uchun ushbu

$$R_n=\frac{1}{2}\sum_{k=0}^{n-1}\frac{(b-a)^3}{12n^3}f''(\xi_k)=\frac{(b-a)^3}{24n^2}\cdot\frac{1}{n}\sum_{k=0}^{n-1}f''(\xi_k^*)$$

ifodaga kelamiz.

Ravshanki,

$$\frac{1}{n}\sum_{k=0}^{n-1}f''(\xi_k^*)=\frac{f''(\xi_0^*)+(\xi_1^*)+...+f''(\xi_{n-1}^*)}{n}$$

miqdor $(\xi_k^*\in[a,b],\ \ k=0,1,2,...,n-1)$ $f''(x)$ ning $[a,b]$ oraliqdagi eng kichik m'' hamda eng katta M'' qiymatlar orasida,

$$m''\le\frac{1}{n}\sum_{k=0}^{n-1}f''(\xi_k^*)\le M$$

bo`ladi.

Shartga ko`ra $f''(x)$ funksiya $[a,b]$ da uzluksiz. Uzluksiz funksiyaning xossasiga muvofiq (a,b) da shunday ζ nuqta topiladiki,

$$f''(\zeta)=\frac{1}{n}\sum_{k=0}^{n-1}f''(\xi_k^*)$$

bo`ladi.

Natijada R_n uchun quyidagi

$$R_n = \frac{(b-a)^3}{24n^2} f''(\zeta)$$

tenglikka kelamiz.

Demak,

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}}) + \frac{(b-a)^3}{24n^2} f''(\zeta)$$

bo`ladi.

Shunday qilib, $[a,b]$ oraliqda ikkinchi tartibli uzluksiz hosilaga ega bo`lgan $f(x)$ funksiyaning

$$\int_a^b f(x)dx$$

integralini (1) tug`ri to`rtburchaklar formulasi yordamida taqribiy hisoblansa, bu taqribiy hisoblash xatoligi quyidagi

$$R_n = \frac{(b-a)^3}{24n^2} f''(\zeta) \qquad (\zeta \in (a,b))$$

formula bilan ifodalanadi.

2⁰. Trapetsiyalar formulasi. $f(x)$ funksiyaning

$$\int_a^b f(x)dx$$

integralini taqribiy hisoblash uchun, avvalo $[a,b]$ segmentni

$$a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b$$

nuqtalar yordamida n ta teng bo`lakka bo`linadi. So`ng har bir $[x_k, x_{k+1}]$ $(k = 0,1,2,...,n-1)$ bo`yicha integralni quyidagicha

$$\int_{x_k}^{x_{k+1}} f(x)dx \approx \frac{f(x_k) + f(x_{k+1})}{2} \cdot (x_{k+1} - x_k) \qquad (k = 0,1,2,...,n-1)$$

taqribiy hisoblanadi. Natijada ushbu

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + ... + \int_{x_{n-1}}^{x_n} f(x)dx \approx$$

$$\approx \frac{f(x_0)+f(x_1)}{2}(x_1 - x_0) + \frac{f(x_1)+f(x_2)}{2}(x_2 - x_1) + ...$$

$$... + \frac{f(x_{n-1})+f(x_n)}{2}(x_n - x_{n-1}) = \frac{b-a}{2}\left[\ \frac{f(x_0)+f(x_n)}{2} + \right.$$

$$\left. + f(x_1) + f(x_2) + ... + f(x_{n-1})\ \right]$$

formulaga kelamiz. Demak,

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\left[\frac{f(x_0)+f(x_n)}{2} + \right.$$

$$\left. + f(x_1) + f(x_2) + ... + f(x_{n-1})\right]. \tag{3}$$

(3) formula trapetsiyalar formulasi deyiladi.

Bu taqribiy formulaning hatoligi R_n', $f(x)$ funksiya $[a,b]$ da uzluksiz $f''(x)$ hosilaga ega bo`lishi shartida ,

$$R_n' = -\frac{(b-a)^3}{12n^2} f''(\zeta) \qquad (\zeta \in (a,b))$$

bo`ladi.

Demak,

$$\int_a^b f(x)dx = \frac{b-a}{n}\left[\frac{f(x_0)+f(x_n)}{2} + f(x_1) + \right.$$

$$\left. + f(x_2) + ... + f(x_{n-1})\right] - \frac{(b-a)^3}{12n^2} f''(\zeta).$$

3^0. Simpson formulasi. Bu holda $f(x)$ funksiyaning

$$\int_a^b f(x)dx$$

integralini taqribiy hisoblash uchun $[a,b]$ segmentni $a = x_0,\ x_1, ..., x_{2k},\ x_{2k+1},$ $x_{2k+2}, ..., x_{2n-2}, x_{2n-1}, x_{2n} = b$ nuqtalar yordamida $2n$ ta teng bo`lakka bo`lib, har bir $[x_{2k}, x_{2k+2}]$ $(k = 0,1,2,...,n-1)$ bo`yicha integralni quyidagicha

$$\int_{x_{2k}}^{x_{2k+2}} f(x)dx \approx \frac{x_{2k+2}-x_{2k}}{6}\left[f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})\right]=$$

$$=\frac{b-a}{6n}\left[f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})\right] \qquad (k=0,1,\dots,n-1)$$

taqribiy hisoblanadi. Natijada

$$\int_a^b f(x)dx=\int_{x_0}^{x_2} f(x)dx+\int_{x_2}^{x_4} f(x)dx+\dots+\int_{x_{2n-2}}^{x_{2n}} f(x)dx \approx$$

$$\approx \frac{b-a}{6n}[(f(x_0)+4f(x_1)+f(x_2))+(f(x_2)+4f(x_3)+$$

$$+f(x_4))+\dots+(f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n}))]=$$

$$=\frac{b-a}{6n}[(f(x_0)+f(x_{2n}))+4(f(x_1)+f(x_3)+\dots$$

$$\dots+f(x_{2n-1}))+2(f(x_2)+f(x_4)+\dots+f(x_{2n-2}))].$$

hosil bo`ladi. Demak,

$$\int_a^b f(x)dx \approx \frac{b-a}{6n}[f(x_0)+f(x_{2n})+4(f(x_1)+f(x_3)+\dots$$

$$\dots+f(x_{2n-1}))+2(f(x_2)+f(x_4)+\dots+f(x_{2n-2}))]. \qquad (4)$$

(4) formula Simpson formulasi deyiladi.

Bu taqribiy formulaning hatoligi R''_n, $f(x)$ funksiya $[a,b]$ da uzluksiz $f^{(iv)}(x)$ hosilaga ega bo`lishi shartida,

$$R''_n=-\frac{(b-a)^5}{2880n^4} f^{(iv)}(\zeta) \qquad (\zeta \in (a,b))$$

bo`ladi. Demak,

$$\int_a^b f(x)dx=\frac{b-a}{6n}[f(x_0)+f(x_{2n})+4(f(x_1)+f(x_3)+\dots+f(x_{2n-1}))+$$

$$+2(f(x_2)+f(x_4)+\dots+f(x_{2n-2}))]-\frac{(b-a)^5}{2880n^4} f^{(iv)}(\zeta).$$

Misol. Ushbu

$$\int_0^1 e^{-x^2} dx$$

integral to`g`ri to`rtburchaklar, trapetsiyalar va Simpson formulalari yordamida taqribiy hisoblansin.

◀ [0,1] segmentni 5 ta teng bo`lakka bo`lamiz. Bunda bo`linish nuqtalari

$$x_0 = 0,\ x_1 = 0{,}2,\ x_2 = 0{,}4,\ x_3 = 0{,}6,\ x_4 = 0{,}8,\ x_5 = 1{,}0$$

bo`lib, bu nuqtalarda $f(x) = e^{-x^2}$ funksiyaning qiymatlari quyidagicha bo`ladi:

$$f(x_0) = 1{,}00000 \quad ,$$
$$f(x_1) = 0{,}96079 \quad ,$$
$$f(x_2) = 0{,}85214 \quad ,$$
$$f(x_3) = 0{,}69768 \quad ,$$
$$f(x_4) = 0{,}52729 \quad ,$$
$$f(x_5) = 0{,}36788 \quad .$$

Har bir bo`lakning o`rtasini ifodalovchi nuqtalar

$$x_{\frac{1}{2}} = 0{,}1\ ,\quad x_{\frac{3}{2}} = 0{,}3\ ,\quad x_{\frac{5}{2}} = 0{,}5\ ,\quad x_{\frac{7}{2}} = 0{,}7\ ,\quad x_{\frac{9}{2}} = 0{,}9$$

bo`lib, bu nuqtalardagi funksiyaning qiymatlari quyidagicha bo`ladi:

$$f(x_{\frac{1}{2}}) = 0{,}99005 \quad ,$$
$$f(x_{\frac{3}{2}}) = 0{,}91393 \quad ,$$
$$f(x_{\frac{5}{2}}) = 0{,}77680 \quad ,$$
$$f(x_{\frac{7}{5}}) = 0{,}61263 \quad ,$$
$$f(x_{\frac{9}{5}}) = 0{,}44486 \quad .$$

a) To`g`ri to`rtburchaklar formulasi bo`yicha

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{5}(0,99005+0,91393+0,77680+$$

$$+0,61263+0,44486)=\frac{1}{5}\cdot 3,74027\approx 0,74805$$

bo`lib,

$$|R_n|\le \frac{1}{12\cdot 25}=\frac{1}{300}\approx 0,003$$

bo`ladi.

b) Trapetsiyalar formulasi bo`yicha

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{5}(\frac{1,00000+0,36788}{2}+0,96079+0,85214+$$

$$+0,69768+0,52729)=\frac{1}{5}(0,68394+3,03790)=$$

$$=\frac{1}{5}\cdot 3,72184\approx 0,74437$$

bo`lib,

$$|R'_n|\le \frac{1}{6\cdot 25}=\frac{1}{150}\approx 0,006$$

bo`ladi.

v) Simpson formulasi bo`yicha

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{30}[(1,00000+0,36788)+4(0,99005+$$

$$+0,91393+0,77680+0,61263+0,44486)+2(0,96079+$$

$$+0,85214+0,69768+0,52729)]=\frac{1}{30}(1,36788+4\cdot 3,74027)+$$

$$+2\cdot 3,03790)=\frac{1}{30}(1,36788+6,07580+14,96108)\approx 0,74682$$

bo`lib,

$$|R''_n|\le \frac{12}{2880\cdot 5^4}=0,7\cdot 10^{-5}$$

bo`ladi.

3. Aniq integralning bazi tadbiqlari

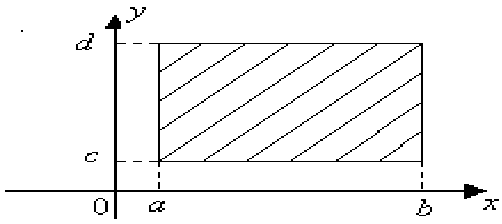
Tekis shaklning yuzi tushunchasi. Ma'lumki, (x, y) juftlik, $(x \in R, y \in R)$, tekislikda nuqtani ifodalaydi. Koordinatalari ushbu

$$a \le x \le b, \quad c \le y \le d \qquad (a \in R, b \in R, c \in R, d \in R)$$

tengsizliklarni qanoatlantiruvchi tekislik nuqtalaridan hosil bo`lgan $D_0$ to`plam :

$$D_0 = \{(x, y); x \in [a, b], y \in [c, d]\}$$

to`g`ri to`rtburchak deyiladi (8-chizma)



8-чизма

Bu to`g`ri to`rtburchakning tomonlari (chegaralari) mos ravishda koordinatalar o`qiga parallel bo`ladi.

D_0 to`g`ri to`rtburchakning yuzi deb (uning chegarasining, ya'ni

$$x = a \quad , \quad x = b \qquad\qquad (c \le y \le d),$$
$$y = c \quad , \quad y = d \qquad\qquad (a \le x \le b)$$

to`g`ri chiziq kesmalarining D_0 ga tegishli bo`lishi yoki tegishli bo`lmasligidan qat'iy nazar) ushbu

$$\mu(D_0) = (b - a) \cdot (d - c)$$

miqdorga aytiladi.

Aytaylik, tekislik nuqtalaridan iborat biror Q to`plam berilgan bo`lsin.

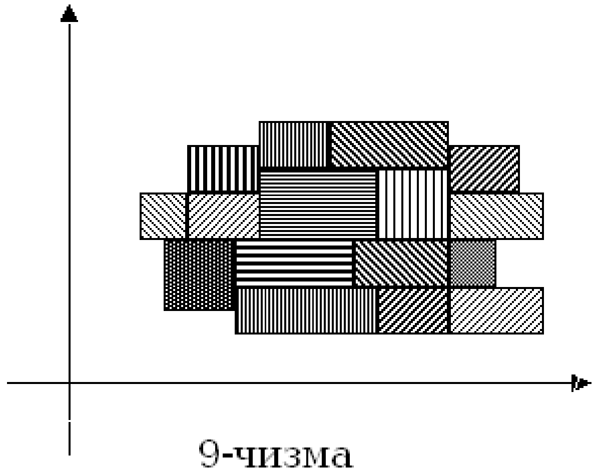
Agar shunday D_0 to`g`ri to`rtburchak topilsaki,

$$Q \subset D_0$$

bo`lsa, $Q$ chegaralangan to`plam deyiladi.

Har qanday chegaralangan tekislik nuqtalaridan iborat to`plam tekis shakl deyiladi.

Agar tekis shakl chekli sondagi kesishmaydigan to`g`ri to`rtburchaklarning birlashmasi sifatida ifodalansa, uni to`g`ri ko`pburchak deymiz.(9-chizma)



9-чизма

Bunday to`g`ri ko`pburchakning yuzi deb, uni tashkil etgan to`g`ri to`rtburchaklar yuzalari yig`indisiga aytiladi.

To`g`ri ko`pburchak yuzi quyidagi xossalarga ega:

1) To`g`ri ko`pburchak yuzi har doim manfiy bo`lmaydi: $\mu(D) \geq 0$;

2) Kesishmaydigan ikki D_1 va D_2 to`g`ri ko`pburchaklar-dan tashkil topgan to`g`ri ko`pburchak yuzi D_1 va D_2 larning yuzalari yig`indisiga teng:

$$\mu(D_1 \cup D_2) = \mu(D_1) + \mu(D_2) \;\; ;$$

3) Agar D_1 va D_2 to`g`ri ko`pburchaklar uchun

$$D_1 \subset D_2$$

bo`lsa, u holda

$$\mu(D_1) \leq \mu(D_2)$$

bo`ladi.

Tekislikda biror chegaralangan Q shakl berilgan bo`lsin. Bu shaklning ichiga $A$ to`g`ri ko`pburchak $(A \subset Q)$, so`ngra $Q$ shaklni o`z ichiga olgan B to`g`ri ko`pburchak $(Q \subset B)$ lar chizamiz. Ularning yuzlari mos ravishda $\mu(A)$ va $\mu(B)$ bo`lsin.

Ravshanki, bunday to`g`ri ko`pburchaklar ko`p bo`lib, ularning yuzalaridan iborat $\{\mu(A)\}$ va $\{\mu(B)\}$ to`plamlar hosil bo`ladi.

Ayni paytda, bu sonli to`plamlar chegaralangan bo`ladi. Binobarin, ularning aniq chegaralari

$$\sup\{\mu(A)\},\ \ \inf\{\mu(B)\}$$

lar mavjud.

1-ta'rif. Agar

$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo`lsa, Q shakl yuzaga ega deyiladi. Ularning umumiy qiymati Q shaklning yuzi deyiladi va $\mu(Q)$ kabi belgilanadi:

$$\mu(Q) = \sup\{\mu(A)\} = \inf\{\mu(B)\}$$

1-teorema. Tekis shakl Q yuzaga ega bo`lish uchun $\forall\varepsilon > 0$ son olinganda ham shunday $A$ $(A \subset Q)$ va $B$ $(Q \subset B)$ to`g`ri ko`pburchaklar topilib, ular uchun

$$\mu(B) - \mu(A) < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

◄ **Zarurligi.** Aytaylik, Q shakl yuzaga ega bo`lsin. Unda ta'rifga binoan

$$\sup\{\mu(A)\} = \inf\{\mu(B)\} = \mu(Q)$$

bo`ladi.

Modomiki,

$$\sup\{\mu(A)\} = \mu(Q),$$
$$\inf\{\mu(B)\} = \mu(Q)$$

ekan, unda $\forall\varepsilon > 0$ olinganda ham shunday to`g`ri ko`pburchak $A(A \subset Q)$ hamda shunday to`g`ri ko`pburchak $B(Q \subset B)$ topiladiki,

$$\mu(Q) - \mu(A) < \frac{\varepsilon}{2},$$

$$\mu(B) - \mu(Q) < \frac{\varepsilon}{2}$$

bo`ladi. Bu tengsizliklardan

$$\mu(B)-\mu(A)<\varepsilon$$

bo`lishi kelib chiqadi.

Yetarliligi. Aytaylik, A $(A\subset Q)$ va B $(Q\subset B)$ to`g`ri ko`pburchaklar uchun $\mu(B)-\mu(A)<\varepsilon$ tengsizligi bajarilsin.

Ravshanki,

$$\mu(A)\leq \sup\{\mu(A)\}\ ,$$
$$\mu(B)\geq \inf\{\mu(B)\ \}\ .$$

Bu munosabatlardan

$$\inf\{\mu(B)\}-\sup\{\mu(A)\}\leq \mu(B)-\mu(A)<\varepsilon$$

bo`lishini topamiz.

ε-ixtiyoriy musbat son bo`lganligidan

$$\sup\{\mu(A)\}=\inf\{\mu(B)\}$$

bo`lishi kelib chikadi. Demak, Q shakl yuzaga ega. ►

Shunga o`xshash quyidagi teorema isbotlanadi.

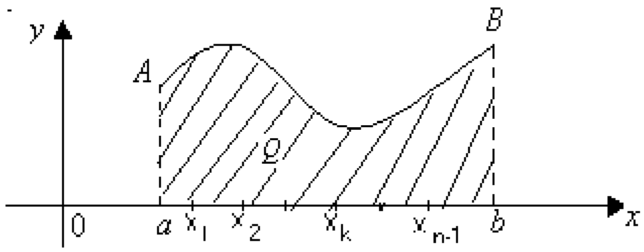
2-teorema. Tekis shakl Q yuzaga ega bo`lishi uchun $\forall\varepsilon>0$ son olinganda ham shunday yuzaga ega tekis shakllar P va S lar $(P\subset Q\ ,\ Q\subset S)$ topilib, ular uchun

$$\mu(S)-\mu(P)<\varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

2^0. Egri chizikli trapetsiyaning yuzini hisoblash. Faraz qilaylik, $f(x)\in C[a,b]$ bo`lib, $\forall x\in[a,b]$ da $f(x)\geq 0$ bo`lsin.

Yuqoridan $f(x)$ funksiya grafigi, yon tomonlardan $x=a$, $x=b$ vertikal chiziqlar hamda pastdan absissa o`qi bilan chegaralangan Q shaklni qaraylik. (10-chizma)



10-chizma

Odatda, bu shakl egri chiziqli trapetsiya deyiladi. $[a,b]$ segmentni ixtiyoriy

$$P=\{x_0,x_1,x_2,...,x_n\} \quad (a=x_0<x_1<x_2<...<x_n=b)$$

bo`laklashni olamiz. Bu bo`laklashning har bir $[x_k,x_{k+1}]$ oralig`ida

$$\inf\{f(x)\}=m_k,\ \ \sup\{f(x)\}=M_k \quad (k=0,1,2,...,n-1)$$

mavjud bo`ladi.

Endi asosi $\Delta x_k=x_{k+1}-x_k$, balandligi m_k bo`lgan $(k=0,1,2,...,n-1)$ to`g`ri to`rtburchaklarning birlashmasidan tash-kil topgan to`g`ri ko`pburchakni A deylik.

Shuningdek, asosi $\Delta x_k=x_{k+1}-x_k$, balandligi M_k bo`lgan $(k=0,1,2,...,n-1)$ to`g`ri to`rtburchaklarning birlashmasidan tashkil topgan to`g`ri ko`pburchakni B deylik. Ravshanki,

$$A\subset Q\ ,\ \ Q\subset B$$

bo`lib, ularning yuzalari

$$\mu(A)=\sum_{k=0}^{n-1}m_k\cdot\Delta x_k,\ \ \mu(B)=\sum_{k=0}^{n-1}M_k\cdot\Delta x_k$$

bo`ladi.

Bu yig`indilarni $f(x)$ funksiyaning $[a,b]$ segmentining $P$ bo`laklashiga nisbatan Darbuning quyi hamda yuqori yig`indilari ekanini payqash qiyin emas:

$$\mu(A)=s(f;P) \quad , \quad \mu(B)=S(f;P).$$

$f(x)\in C[a,b]$ bo`lgani uchun $f(x)$ funksiya $[a,b]$ da integralla-nuvchi bo`ladi. Unda integrallanuvchilik mezoniga ko`ra, $\forall\varepsilon>0$ olinganda ham $[a,b]$ segmentning shunday $P$ bo`laklashi topiladiki,

$$S(f;P)-s(f;P)<\varepsilon$$

bo`ladi. Birobarin, ushbu

$$\mu(B)-\mu(A)<\varepsilon$$

tengsizlik bajariladi. Bu esa, 1-teoremaga muvofiq, qaralayotgan egri chiziqli trapetsiyaning yuzaga ega bo`lishini bildiradi. Unda ta’rifga ko`ra

$$\sup\{\mu(A)\}=\inf\{\mu(B)\}$$

bo`ladi.

Ayni paytda,

$$\sup\{\mu(A)\} = \int_{\bar{a}}^{b} f(x)dx,$$

$$\inf\{\mu(B)\} = \int_{a}^{\bar{b}} f(x)dx$$

bo`lganligi sababli Q egri chiziqli trapetsiyaning yuzi

$$\mu(Q) = \int_{a}^{b} f(x)dx \tag{1}$$

ga teng bo`ladi.

1-misol. Tekislikda ushbu

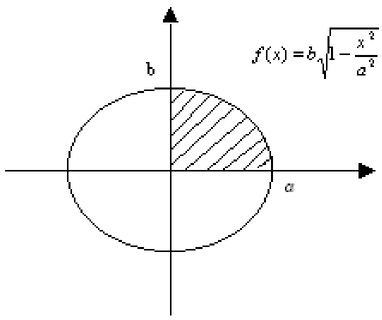
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ellips bilan chegaralangan Q shaklning yuzi topilsin.

◀Ellips bilan chegaralangan Q shaklning yuzi OX va OY koordinata o`qlari hamda

$$f(x) = b \cdot \sqrt{1 - \frac{x^2}{a^2}} \qquad , \qquad 0 \le x \le a$$

chiziqlar bilan chegaralangan egri chiziqli trapetsiya yuzi-ning 4 tasiga teng bo`ladi. (11-chizma).



11-чизма

Unda (1) formuladan foydalanib topamiz:

$$\mu(Q) = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx =$$

$$= \left| \begin{array}{l} x = a\sin t\ , \\ \qquad\qquad 0 \le t \le \dfrac{\pi}{2} \\ dx = a\cos t dt\ , \end{array} \right| =$$

$$= \frac{4b}{a}\cdot a^2 \int_0^{\frac{\pi}{2}} \cos^2 t dt = 4ab\cdot\frac{\pi}{4} = ab\pi. \blacktriangleright$$

Aytaylik, $f_1(x) \in C[a,b]$, $f_2(x) \in C[a,b]$ bo`lib, $\forall x \in [a,b]$ da

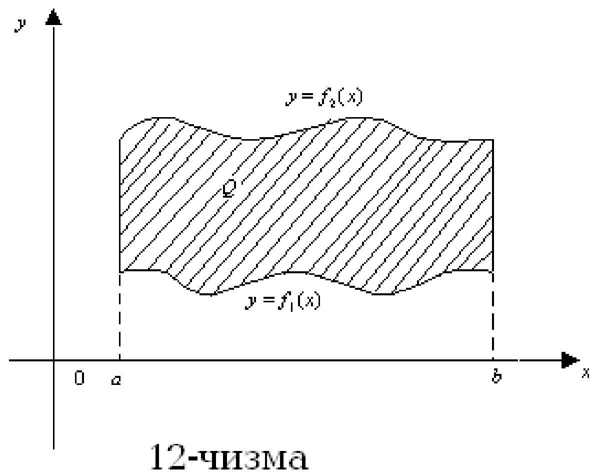
$$0 \le f_1(x) \le f_2(x)$$

bo`lsin.

Tekislikdagi Q shakl quyidagi

$$y = f_1(x) \quad , \quad y = f_2(x) \quad , \quad x = a \quad , \quad x = b$$

chiziqlar bilan chegaralangan shaklni ifodalasin (12-chizma)



12-чизма

Bu shaklning yuzi

$$\mu(Q) = \int_a^b f_2(x)dx - \int_a^b f_1(x)dx = \int_a^b [f_2(x) - f_1(x)]dx \qquad (2)$$

bo`ladi.

2-misol. Tekislikda ushbu

$$y = 4 - x^2 \quad , \qquad y = x^2 - 2x$$

chiziqlar (parabolalar) bilan chegaralangan Q shaklning yuzi topilsin.

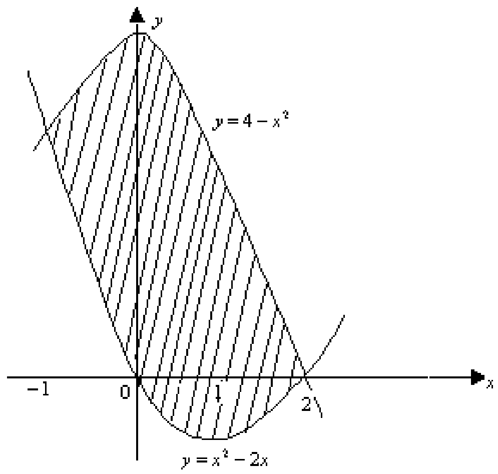
◄ Parabolalarning tenglamalari

$$y = 4 - x^2 \quad ,$$
$$y = x^2 - 2x$$

ni birgalikda yechib, ularning kesishish nuqtalarini topamiz:

$$4 - x^2 = x^2 - 2x \ ,$$
$$x_1 = -1 \ , \ x_2 = 2 \ ; \ y_1 = 3 \ , \ y_2 = 0 : \ A(-1;3) \ , \ B(2;0).$$

(13 -chizma).



13-чизма

Bu shaklning yuzini (2) formuladan foydalanib hisob-laymiz:

$$\mu(Q) = \int_{-1}^{2} \left[(4 - x^2) - (x^2 - 2x)\right] dx = \int_{-1}^{2} \left(4 + 2x - 2x^2\right) dx = \left.(4x + x^2 - \frac{2}{3}x^3)\right|_{-1}^{2} = 9. \ ►$$

Eslatma. Agar $f(x) \in C[a,b]$ funksiya $[a,b]$ da ishora saqlamasa, (1) integral egri chiziqli trapetsiyalar yuzalari-ning yig`indisidan iborat bo`ladi. Bunda OX o`qining yuqori-sidagi yuza musbat ishora bilan, $OX$ o`qining pastdagi yuza manfiy ishora bilan olinadi.

Masalan, OX o`qi hamda $f(x) = \sin x$, $0 \le x \le 2\pi$ funksiya grafigi bilan chegaralangan shaklning yuzi

$$\mu(Q)=\int_0^{\pi}\sin x+(-\int_{\pi}^{2\pi}\sin xdx)=(-\cos x)\Big|_0^{\pi}-(-\cos x)\Big|_{\pi}^{2\pi}=4$$

bo`ladi.

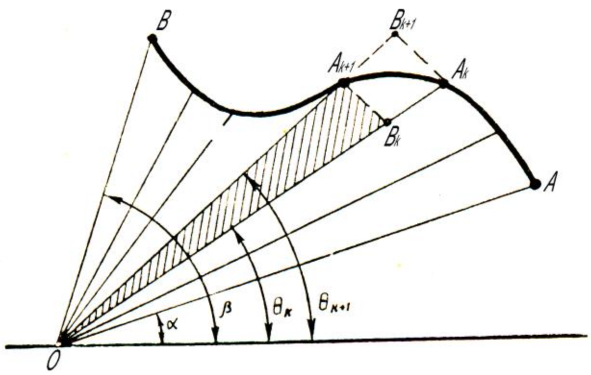
3⁰. Egri chiziqli sektorning yuzini hisoblash. Aytaylik, $\breve{AB}$ egri chiziq qutb koordinatalar sistemasida ushbu

$$\rho=\rho(\theta)\quad ,\quad \alpha\le\theta\le\beta\quad (\alpha\in R,\beta\in R)$$

tenglama bilan berilgan bo`lsin. Bunda

$$\rho(\theta)\in C[\alpha,\beta]\ ,\qquad \forall\theta\in[\alpha,\beta]\quad \text{да}\qquad \rho(\theta)\ge 0\ .$$

Tekislikda $\breve{AB}$ egri chiziq hamda OA va OB radius-vektorlar bilan chegaralangan Q shaklni qaraymiz. (14 -chizma).



14- chizma

$[\alpha,\beta]$ segmentni ixtiyoriy

$$P=\{\theta_0,\theta_1,...,\theta_n\}\qquad (\alpha=\theta_0<\theta_1<...<\theta_n=\beta)$$

bo`laklashini olamiz. $O$ nuqtadan har bir qutb burchagi $\theta_k$ ga mos $OA_k$ radius-vektor o`tkazamiz. Natijada OAB -egri chiziq-li sektor

$$OA_kA_{k+1}\ \left(k=0,1,2,...,n-1\quad ;\ A_0=A\ ,\ A_n=B\right)$$

egri chiziqli sektorchalarga ajraladi.

Ravshanki, $\rho=\rho(\theta)\in C[\alpha,\beta]$

bo`lganligi uchun $[\theta_k\ ,\theta_{k+1}]$ da $(k=0,1,2,...,n-1)$

$$m_k=\inf\{\rho(\theta)\}\ \ ,\quad M_k=\sup\{\rho(\theta)\}$$

lar mavjud.

Endi har bir $[\theta_k, \theta_{k+1}]$ segment uchun radius-vektorlari mos ravishda m_k hamda M_k bo`lgan doiraviy sektorlarni hosil qilamiz. Bunday doiraviy sektorlar yuzaga ega bo`lib, ularning yuzi mos ravishda

$$\frac{1}{2}m_k^2 \cdot \Delta\theta_k \quad , \quad \frac{1}{2}M_k^{2\cdot} \cdot \Delta\theta_k \qquad (\Delta\theta_k = \theta_{k+1} - \theta_k)$$

bo`ladi.

Radius-vektorlari m_k $\quad (k = 0,1,2,...,n-1)$ bo`lgan barcha doiraviy sektorlar birlashmasidan hosil bo`lgan shaklni Q_1 desak, unda $Q_1 \subset Q$ bo`lib, uning yuzi

$$\mu(Q_1) = \frac{1}{2}\sum_{k=0}^{n-1} m_k^2 \cdot \Delta\theta_k \tag{3}$$

bo`ladi.

Shuningdek, radius-vektorlari M_k $\quad (k = 0,1,2,...,n-1)$ bo`l-gan barcha doiraviy sektorlar birlashmasidan hosil bo`lgan shaklni Q_2 desak, unda $Q \subset Q_2$ bo`lib, uning yuzi

$$\mu(Q_2) = \frac{1}{2}\sum_{k=0}^{n-1} M_k^2 \cdot \Delta\theta_k \tag{4}$$

bo`ladi.

(3) va (4) yig`indilar $\frac{1}{2}\rho^2(\theta)$ funksiyaning Darbu yig`indilari bo`ladi. Ayni paytda, $\frac{1}{2}\rho^2(\theta)$ funksiya $[\alpha, \beta]$ da uzluksiz bo`lgani uchun u integrallanuvchidir. Demak, $\forall \varepsilon > 0$ olinganda ham $[\alpha, \beta]$ segmentning shunday P bo`laklashi topiladiki,

$$S(\frac{1}{2}\rho^2(\theta); P) - s(\frac{1}{2}\rho^2(\theta); P) < \varepsilon$$

bo`ladi. Binobarin, ushbu

$$\mu(Q_2) - \mu(Q_1) < \varepsilon$$

tengsizlik bajariladi. Bu esa, 2-teoremaga muvofiq, qaralayotgan egri chiziqli sektorning yuzaga ega bo`lishini bildiradi. Unda ta’rifga ko`ra

$$\sup\{\mu(Q_1)\} = \inf\{\mu(Q_2)\}$$

bo`ladi.

Ayni paytda,

$$\sup\{\mu(Q_1)\} = \underline{\int_{\alpha}^{\beta}} \rho^2(\theta)d\theta,$$

$$\inf\{\mu(Q_2)\} = \overline{\int_{\alpha}^{\beta}} \rho^2(\theta)d\theta$$

bo`lgani sababli Q egri chiziqli sektorning yuzi

$$\mu(Q) = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\theta)d\theta$$

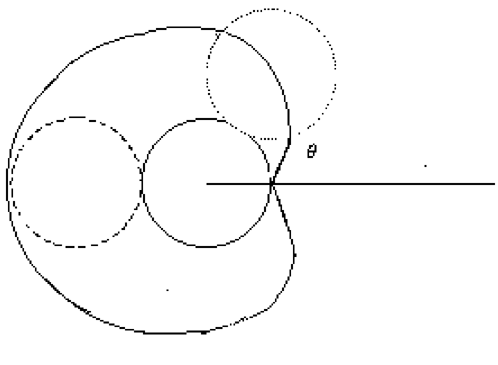
ga teng bo`ladi.

3-misol. Ushbu

$$\rho = \rho(\theta) = a(1-\cos\theta) \quad (a \in R,\ 0 \le \theta \le 2\pi)$$

funksiya grafigi bilan chegaralangan shaklning yuzi topilsin.

◄ Bu funksiya grafigi kardioidani ifodalaydi. Ma’lumki, kardioida radiusi r ga teng bo`lgan aylananing shu radiusli ikkinchi qo`zg`almas aylana bo`ylab xarakati (sirpanmasdan dumalashi) natijasida birinchi aylana ixtiyoriy nuqtasining chizgan chizig`idir. (15-chizma).



15-чизма

Kardioida qutb o`qiga nisbatan simmetrik bo`lganligi sababli yuqori yarim tekislikdagi shaklning yuzini topib, so`ngra uni 2 ga ko`paytirsak, izlanayotgan yuza kelib chiqadi.

θ o`zgaruvchi $[0,\pi]$ da o`zgarganda ρ radius-vektor kardioidaning yuqori yarim tekislikdagi qismini chizadi. Shuning uchun

$$\mu(Q) = 2\cdot\frac{1}{2}\int_0^{\pi}\rho^2(\theta)d\theta = \int_0^{\pi}a^2(1-\cos\theta)^2d\theta =$$

$$= a^2\int_0^{\pi}\left[\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right]d\theta =$$

$$= a^2\left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi} = \frac{3}{2}\pi a^2$$

bo`ladi. ►

4. Yoy uzunligi va uni hisoblash

1⁰. Yoy uzunligi tushunchasi. Ma'lumki, tekislikdagi ikki $A(x_1, y_1)$ va $B(x_2, y_2)$ nuqtalarni birlashtiruvchi to`g`ri chiziq kesmasi l_0 uzunlikka ega va uning uzunligi

$$\mu(l_0) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

ga teng bo`ladi.

Aytaylik, tekislikdagi l chiziq $A_0(x_0, y_0), A_1(x_1, y_1), ...,$ $A_n(x_n, y_n)$ nuqtalarni $(n \in N)$ birin-ketin to`g`ri chiziq kesmalari bilan birlashtirishidan hosil bo`lgan bo`lsin. Odatda, bunday chiziq siniq chiziq deyiladi.

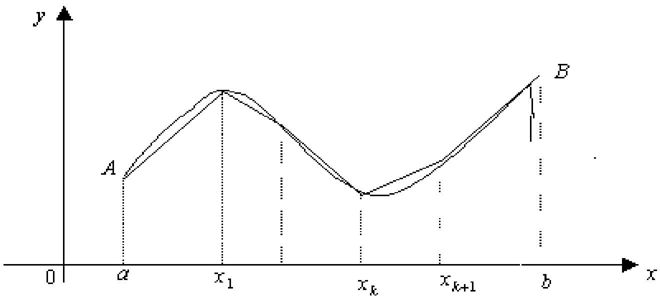
Siniq chiziq uzunligi (perimetri) deb, uni tashkil etgan to`g`ri chiziq kesmalari uzunliklarining yig`indisiga aytiladi:

$$\mu(l) = \sum_{k=0}^{n-1}\sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}.$$

Faraz qilaylik, tekislikdagi $\breve{AB}$ egri chizig`i (uni $\breve{AB}$ yoyi deb ham ataymiz) ushbu

$$y = f(x) \qquad (a \le x \le b)$$

tenglama bilan berilgan bo`lsin, bunda $f(x) \in C[a,b]$.



16-чизма

$[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \qquad (a = x_0 < x_1 < ... < x_n = b$$

bo`laklashni olib, bo`luvchi x_k $(k = 0,1,2,...,n)$ nuqtalar orqali OY o`qiga parallel to`g`ri chiziqlar o`tkazamiz. Bu to`g`ri chiziqlarning $\breve{AB}$ yoyi bilan kesishgan nuqtalari

$$A_k(x_k, f(x_k)) \qquad (k = 0,1,2,...,n \ ; \quad A_0 = A \ , \ A_n = B)$$

bo`ladi.

$\breve{AB}$ yoyidagi bu $A_k(x_k, f(x_k))$ nuqtalarni bir-biri bilan to`g`ri chiziq kesmalari yordamida birlashtirib, l siniq chiziqni hosil qilamiz. (16-chizma)

Odatda, l siniq chiziq $\breve{AB}$ yoyiga chizilgan siniq chiziq deyiladi. U uzunlikka ega bo`lib, uzunligini (perimetrini) $\mu(l)$ deylik.

Agar P_1 va P_2 lar $[a,b]$ segmentning ikkita bo`laklashi bo`lib, $P_1 \subset P_2$ bo`lsa, u holda bu bo`laklashlarga mos $\breve{AB}$ yoyiga chizilgan siniq chiziq l_1 , l_2 larning perimetrlari uchun

$$\mu(l_1) \leq \mu(l_2)$$

bo`ladi.

◀$[a,b]$ segmentning P_1 bo`laklashi quyidagi

$$P_1 = \{x_0, x_1, ..., x_k, x_{k+1}, ..., x_n\}$$
$$(a = x_0 < x_1 < ... < x_k < x_{k+1} < ... < x_n = b)$$

ko`rinishda bo`lib, P_2 bo`laklash esa $P_1$ bo`laklashning barcha bo`luvchi nuqtalari hamda qo`shimcha bitta $x^* \in [a,b]$ nuqtani qo`shish natijasida hosil bo`lgan bo`laklash bo`lsin. Bu x^* nuqta x_k hamda x_{k+1} nuqtalar orasida joylashsin: $x_k < x^* < x_{k+1}$. Demak,

$$P_2 = \{x_0, x_1, ..., x_k, x^*, x_{k+1}, ..., x_n \}$$

$$(a = x_0 < x_1 < ... < x_k < x* < x_{k+1} < ... < x_n = b \quad)$$

Ravshanki, $P_1 \subset P_2$ bo`ladi.

$A\breve{B}$ yoyiga chizilgan P_1 bo`laklashga mos siniq chiziq $l_1$, shu yoyga chizilgan $P_2$ bo`laklashga mos siniq chiziq l_2 dan faqatgina bitta bo`lagi bilangina farq qiladi: $l_1$ da $A_k A_{k+1}$ bo`lak bo`lgan holda $l_2$ da ikkita $A_k A^*$ hamda $A^* A_{k+1}$ bo`laklar bo`ladi.

Ammo $A_k A_{k+1}$ to`g`ri chiziq kesmasining uzunligi $\mu(A_k A_{k+1})$, $A_k A^*$ hamda $A^* A_{k+1}$ kesmalar uzunliklari $\mu(A_k A^*)$, $\mu(A^* A_{k+1})$ yig`indisidan har doim katta bo`lmaganligi, ya'ni

$$\mu(A_k A_{k+1}) \le \mu(A_k A^*) + \mu(A^* A_{k+1})$$

uchun

$$\mu(l_1) \le \mu(l_2)$$

bo`ladi. ►

Demak, P bo`laklashning bo`luvchi nuqtalari sonini orttira borilsa, $A\breve{B}$ yoyiga chizilgan ularga mos siniq chiziqlar perimetrlari ham ortib boradi.

1-ta'rif. Agar $\lambda_p \to 0$ da $A\breve{B}$ yoyiga chizilgan siniq chiziq perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + \left[f(x_{k+1}) - f(x_k)\right]^2}$$

chekli limitga ega bo`lsa, $A\breve{B}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p \to 0} \mu(l) = \mu(A\breve{B})$$

limit $A\breve{B}$ yoyining uzunligi deyiladi.

Masalan, agar

$$f(x) = kx + C \qquad (a \le x \le b)$$

bo`lsa, unda $\breve{AB}$ ning uzunligi

$$\mu(\breve{AB}) = \lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + k^2 (x_{r+1} - x_k)^2} =$$

$$= \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{1+k^2} \cdot (x_{k+1} - x_k) = \sqrt{1+k^2} \cdot (b-a)$$

bo`ladi.

Aytaylik, $\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo`lsin.

(Bu holda egri chiziq parametrik ko`rinishda berilgan deyiladi). Bunda:

1) $\varphi(t) \in C[\alpha, \beta]$, $\psi(t) \in C[\alpha, \beta]$;

2) $\forall t_1, t_2 \in [\alpha, \beta]$, $t_1 \ne t_2$ uchun (1)

$$A_1(x_1, y_1) = A_1(\varphi(t_1), \psi(t_1)) ,$$
$$A_2(x_2, y_2) = A_2(\varphi(t_2), \psi(t_2))$$

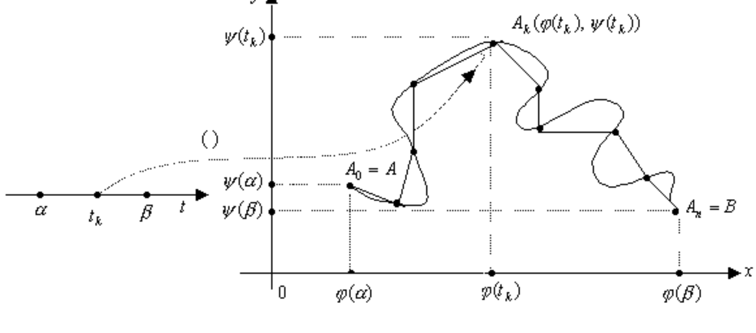
nuktalar turlicha ;

3) $t = \alpha$ ga A nuqta, $t = \beta$ ga B nuqta mos kelsin.

$[\alpha, \beta]$ segmentning ixtiyoriy

$$P = \{t_0, t_1, ..., t_n\} \qquad (\alpha = t_0 < t_1 < ... < t_n = \beta$$

bo`laklashni olib, bu bo`laklashning bo`luvchi $t_k$ $(k = 0,1,2,...,n)$ nuqtalariga mos kelgan $\breve{AB}$ yoydagi $A_k = A_k(x_k, y_k)$ $(x_k = \varphi(t_k)$ , $y_k = \psi(t_k)$ ; $k = 0,...,n)$ nuqtalarni bir-biri bilan to`g`ri chiziq kesmalari yordamida birlashtirib, $\breve{AB}$ yoyga chizilgan siniq chiziq l ni hosil qilamiz. (17-chizma).



17-чизма

Bu siniq chiziq perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{[\varphi(t_{k+1}) - \varphi(t_k)]^2 + [\psi(t_{k+1}) - \psi(t_k)]^2}$$

bo`ladi.

2-ta'rif. Agar $\lambda_p \to 0$ da $A\breve{B}$ yoyiga chizilgan siniq chiziq perimetri $\mu(l)$ chekli limitga ega bo`lsa, $A\breve{B}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p \to 0} \mu(l) = \mu(A\breve{B})$$

limit $A\breve{B}$ yoyining uzunligi deyiladi.

Yuqorida keltirilgan ta'riflardan yoy uzunligining (agar u mavjud bo`lsa ) musbat bo`lishi kelib chiqadi.

Endi yoy uzunligining ikkita xossasini isbotsiz keltiramiz:

1) Agar $A\breve{B}$ yoyi uzunlikka ega bo`lib, u $A\breve{B}$ yoydagi nuqtalar yordamida $n$ ta $A_k\breve{A}_{k+1}$ yoylarga $(k = 0,1,2,...,n\ ;\ A_0 = A, B = A_{n+1})$ ajralgan bo`lsa, u holda har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega va

$$\mu(A\breve{B}) = \sum_{k=0}^{n} \mu(A_k\breve{A}_{k+1})$$

bo`ladi.

2) Agar $A\breve{B}$ yoyi n ta $A_k\breve{A}_{k+1}$ yoylarga ajralgan bo`lib, har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega bo`lsa, u holda $A\breve{B}$ yoyi ham uzunlikka ega bo`ladi.

2^0. $y = f(x)$ tenglama bilan berilgan egri chiziq uzunligini hisoblash. Faraz qilaylik, $A\breve{B}$ egri chiziq ushbu

$$y = f(x) \quad , \quad a \le x \le b$$

tenglama bilan berilgan bo`lsin. Bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x)$ hosilaga ega.

$[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \quad (a = x_0 < x_1 < ... < x_n = b)$$

bo`laklashini olib, unga mos $\breve{AB}$ yoyiga chizilgan l siniq chiziqni hosil qilamiz. Bu siniq chiziqning perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

bo`ladi.

Har bir $[x_k, x_{k+1}]$ segmentda $f(x)$ funksiyaga Lagranj teoremasini qo`llab topamiz:

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f'(\tau_k) \cdot (x_{k+1} - x_k)]^2} =$$

$$= \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)} \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)}\, \Delta x_k \quad ,$$

bunda $\tau_k \in [x_k, x_{r+1}]$.

Bu tenglikdagi yig`indining $\sqrt{1 + f'^2(x)}$ funksiyaning integral yig`indisidan farqi shuki, integral yig`indida $\xi_k \in [x_k, x_{k+1}]$ nuqta ixtiyoriy bo`lgan holda yuqoridagi yig`indida esa $\tau_k$ nuqta Lagranj teoremasiga muvofiq olingan tayin nuqta bo`lishidadir. Ammo $\sqrt{1 + f'^2(x)}$ funksiya integrallanuvchi bo`lganligi sababli $\xi_k = \tau_k$ deb olinishi mumkin. Natijada

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k$$

bo`lib, undan

$$\lim_{\lambda_p \to 0} \mu(l) = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k = \int_a^b \sqrt{1 + f'^2(x)} dx$$

bo`lishi kelib chiqadi.

Demak, $\breve{AB}$ yoyining uzunligi

$$\mu(\breve{AB}) = \int_a^b \sqrt{1 + f'^2(x)} dx \qquad (2)$$

bo`ladi. Bu formula yordamida yoy uzunligi hisoblanadi.

1-misol. Ushbu

$$f(x)=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}}) \qquad (a>0\ \ ,\quad -a\le x\le a)$$

tenglama bilan berilgan $A\breve{B}$ egri chizig`ining uzunligi topilsin.

Bu tenglama bilan aniqlanadigan chiziq zanjir chizig`i deyiladi.

◀ Ravshanki,

$$f'(x)=\frac{1}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})\ \ ,$$

$$1+f'^2(x)=\frac{1}{4}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2\ \ ,$$

$$\sqrt{1+f'^2(x)}=\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$$

bo`ladi. (2) formuladan foydalanib, zanjir chizig`ining uzunligini topamiz:

$$\mu(A\breve{B})=\int_{-a}^{a}\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})dx=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})\Big|_{-a}^{a}=a(e-\frac{1}{e}).\ ▶$$

3^0. Parametrik ko`rinishda berilgan egri chiziq uzunligini hisoblash.

Faraz qilaylik, $A\breve{B}$ egri chiziq ushbu

$$\begin{cases} x=\varphi(t), \\ y=\psi(t) \end{cases} \qquad \left(\alpha\le t\le\beta\right)$$

tenglamalar sistemasi bilan berilgan bo`lib, (1) shartlar-ning bajarilishi bilan birga $\varphi(t),\psi(t)$ funksiyalari $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hamda $\psi'(t)$ hosilalarga ega bo`lsin.

$[\alpha,\beta]$ segmentning ixtiyoriy

$$P=\{t_0,t_1,...,t_n\} \qquad (\alpha=t_0<t_1<...<t_n=\beta)$$

bo`laklashini olib, ularga mos $A\breve{B}$ yoyiniig $A_k=A_k(x_k,y_k)$ $(x_k=\varphi(t_k)\ ,\ \ y_k=\varphi(t_k))$ nuqtalarini bir-biri bilan to`g`ri chiziq kesmasi yordamida birlashtirishdan hosil bo`lgan l siniq chiziq perimetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{[\varphi(t_{k+1})-\varphi(t_k)]^2+[\psi(t_{k+1})-\psi(t_k))]^2}$$

ni qaraymiz.

Lagranj teoremasidan foydalanib topamiz:

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\tau_k)\cdot(t_{k+1}-t_k)^2+\psi'^2(\theta_k)\cdot(t_{k+1}-t_k)^2}=$$

$$=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}\cdot\Delta t_k \qquad \left(\Delta t_k=t_{k+1}-t_k\right)$$

bunda

$$\tau_k\in[t_k,t_{k+1}]. \qquad \theta_k\in[t_k,t_{k+1}].$$

Keyingi tenglikni quyidagicha yozib olamiz:

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\cdot\Delta t_k+$$

$$+\sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\cdot\Delta t_k \qquad (*)$$

bunda,

$$\xi_k\in[t_k,t_{k+1}].$$

Modomiki,

$$\sqrt{\varphi'^2(t)+\psi'^2(t)}\in C[\alpha,\beta]$$

ekan unda

$$\sqrt{\varphi'^2(t)+\psi'^2(t)}\in R[\alpha,\beta]$$

bo`lib,

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\cdot\Delta t_k=\int_{\alpha}^{\beta}\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (3)$$

bo`ladi.

Ixtiyoriy a, b, c, d haqiqiy sonlar uchun ushbu

$$\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right|\le|a-c|+|b-d|$$

tengsizlik o`rinli bo`ladi.

◄ Haqiqatan ham,

$$\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right| = \left|\frac{(a^2-c^2)+(b^2-d^2)}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}\right| \le |a-c|\cdot$$

$$\cdot\frac{|a+c|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}+|b-d|\cdot\frac{|b+d|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}} \le$$

$$\le |a-c|+|b-d|. \ \blacktriangleright$$

Bu tengsizlikdan foydalanib topamiz:

$$\left|\sum_{k=0}^{n-1}\left[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\right]\Delta t_k\right| \le$$

$$\le \sum_{k=0}^{n-1}|\varphi'(\tau_k)-\varphi'(\xi_k)|\Delta t_k + \sum_{k=0}^{n-1}|\psi'(\theta_k)-\psi'(\xi_k)|\Delta t_k \le$$

$$\le \sum_{k=0}^{n-1}\omega_k(\varphi')\cdot\Delta t + \sum_{k=0}^{n-1}\omega_k(\psi')\cdot\Delta t.$$

$$\varphi'(t)\in R[\alpha,\beta], \qquad \psi'(t)\in R[\alpha,\beta]$$

bo`lganligi sababli

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\left[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)}-\right.$$
$$\left.-\sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}\right]\Delta t_k = 0 \qquad (4)$$

bo`ladi.

(3) va (4) munosabatlarni e'tiborga olib, $\lambda_p \to 0$ da (*) tenglikda limitga o`tsak, u holda $A\breve{B}$ yoyining uzunligi uchun

$$\mu(A\breve{B}) = \int_{\alpha}^{\beta}\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (5)$$

bo`lishi kelib chiqadi. Bu formula yordamida yoy uzunligi hisoblanadi.

2-misol. Ushbu

$$x = a(t-\sin t)$$
$$y = a(1-\cos t) \qquad (0\le t\le \pi)$$

tenglamalar sistemasi bilan berilgan $A\breve{B}$ egri chiziqning (sikloidaning) uzunligi topilsin.

◄ Ravshanki,

$$x'(t) = a(1-\cos t) \quad , \qquad y'(t) = a\sin t \quad ,$$

$$x'^2(t) + y'^2(t) = a^2(1-\cos t)^2 + a^2\sin^2 t = a^2 2(1-\cos t) \ ,$$

$$\sqrt{x'^2(t) + y'^2(t)} = a\sqrt{2(1-\cos t)}$$

bo`ladi. (5) formulaga ko`ra izlanayotgan egri chiziqning uzunligi

$$\mu(A\breve{B}) = \int_0^{2\pi} a\sqrt{2(1-\cos t)}dt = 2a\int_0^{2\pi} \sin\frac{t}{2}dt = -4a\cdot(\cos\frac{t}{2})_0^{2\pi} = 8a$$

bo`ladi. ►

4⁰. Qutb koordinatalar sistemasida berilgan egri chiziqning uzunligini hisoblash.

Faraz qilaylik, $A\breve{B}$ egri chiziq qutb koordinatalar sistemasida quyidagi

$$r = \rho(\theta) \ , \quad (\alpha \le \theta \le \beta)$$

tenglama bilan berilgan bo`lsin. Bunda $\rho(\theta) \in C[\alpha,\beta]$ bo`lib, u uzluksiz $\rho'(\theta)$ hosilaga ega bo`lsin.

Қutb koordinatalari (ρ,θ) dan Dekart koordinatalari (x,y) ga o`tish formulasiga binoan

$$x = \rho(\theta)\cdot\cos\theta \ ,$$

$$y = \rho(\theta)\cdot\sin\theta \qquad (\alpha \le \theta \le \beta)$$

bo`ladi. Natijada $A\breve{B}$ parametrik ko`rinishda

$$\varphi(\theta) = \rho(\theta)\cdot\cos\theta \ ,$$

$$\psi(\theta) = \rho(\theta)\cdot\sin\theta \qquad (\alpha \le \theta \le \beta)$$

berilgan egri chiziq sifatida ifodalanadi, bunda $\varphi(\theta)$, $\psi(\theta)$ funksiyalari 3^0 da keltirilgan shartlarni bajaradigan funksiyalar bo`ladi.

(5) formuladan foydalanib $A\breve{B}$ egri chiziqning uzunli-gini topamiz:

$$\mu(A\breve{B}) = \int_\alpha^\beta \sqrt{(\rho(\theta)\cdot\cos\theta)'^2 + (\rho(\theta)\cdot\sin\theta)'^2}d\theta =$$

$$= \int_\alpha^\beta \sqrt{\rho'^2(\theta) + \rho^2(\theta)}d\theta.$$

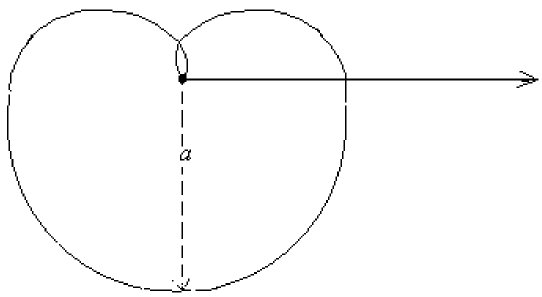
Bu formula yordamida egri chiziqning uzunligi hisoblanadi.

3-misol. Ushbu

$$\rho = a \cdot Sin^3 \frac{\theta}{3}$$

tenglama bilan berilgan egri chiziqning uzunligi topilsin.

◀ θ o`zgaruvchi 0 dan $3\pi$ gacha o`zgargandan (ρ, θ) nuqta 18-chizmada tasvirlangan l egri chiziqni chizib chiqadi:



18-чизма

(2) formuladan foydalanib l chiziqning uzunligini topamiz:

$$\mu(l) = \int_0^{3\pi} \sqrt{(a\sin^3\frac{\theta}{3})'^2 + (a\sin^3\frac{\theta}{3})'^2}\, d\theta =$$

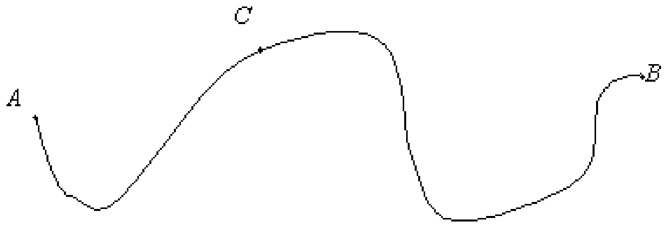
$$= \int_0^{3\pi} \sqrt{a^2 \cdot \sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3} + a^2\sin^6\frac{\theta}{3}}\, d\theta =$$

$$= a\int_0^{3\pi} \sin^2\frac{\theta}{3}\, d\theta = \frac{3\pi a}{2}. \blacktriangleright$$

5⁰. Yoy differensiali. Aytaylik, tekislikdagi $\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t) \ , \\ y = \psi(t) \end{cases} \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo`lib, bunda $\varphi(t)$ hamda $\psi(t)$ funksiyalari $[\alpha, \beta]$ da uzluksiz $\varphi'(t)$ hamda $\psi'(t)$ hosilalarga ega bo`lsin (19-chizma)



19-чизма

Ma’lumki, t o`zgaruvchining $t=\alpha$ qiymatiga $A\breve{B}$ egri chiziqda A nuqta mos keladi.

Endi ixtiyoriy $t\in[\alpha,\beta]$ ni olib, unga mos $A\breve{B}$ egri chiziqdagi nuqtani C bilan belgilaylik:

$$C(\varphi(t)\,,\psi(t))\quad ,\alpha\le t\le\beta.$$

Ravshanki, $A\breve{C}$ yoyining uzunligi C nuqtaning $A\breve{B}$ egri chiziqdagi holatiga qarab o`zgaradi va ayni paytda $t$ ning har bir tayin qiymatida yagona $A\breve{C}$ yoyining uzunligiga ega bo`lamiz. Binobarin $A\breve{C}$ yoyining uzunligi $\mu(A\breve{C})$ t o`zgaruv-chining funksiyasi bo`ladi:

$$\mu_t(A\breve{C})\qquad(\alpha\le t\le\beta)$$

(5) formuladan foydalanib topamiz:

$$\mu_t(A\breve{C})=\int_{\alpha}^{t}\sqrt{\varphi'^2(t)+\psi'^2(t)}dt.$$

Modomiki, $\sqrt{\varphi'^2(t)+\psi'^2(t)}\in C[\alpha,\beta]$ ekan, unda $\mu_t(A\breve{C})$ funksiya hosilaga ega bo`lib,

$$(\mu_t(A\breve{C}))'=\sqrt{\varphi'^2(t)+\psi'^2(t)}$$

bo`ladi.

Keyingi tenglikning kvadratini dt^2 ga ko`paytirib, ushbu

$$(\mu_t(A\breve{C}))'^2\cdot dt^2=\varphi'^2(t)dt^2+\psi'^2(t)dt^2\ ,$$

ya’ni

$$d(\mu_t(A\breve{C}))'^2=dx^2+dy^2$$

munosabatga kelamiz. Bu munosabat yoy differensialining kvadratini ifodalaydi. Demak, yoy differensiali $d\mu_t(\breve{AC})$ yuqoridagi $x=\varphi(t)$, $y=\psi(t)$ funksiyalarning differensial-lari dx hamda dy lar orqali ifodalanadi. Binobarin, (5) formula, uzluksiz hosilaga ega bo`lgan $x(t)$ , $y(t)$ funksiyalar yordamida egri chiziq yoyining turli usullarda parametrlash-tirishda o`z ko`rinishini saqlaydi.

Xulosa

$y = f(x)$ tenglama bilan berilgan egri chiziq uzunligini hisoblash. Faraz qilaylik, $\breve{AB}$ egri chiziq ushbu

$$y = f(x) \quad , \quad a \le x \le b$$

tenglama bilan berilgan bo`lsin. Bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x)$ hosilaga ega.

$[a,b]$ segmentning ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \quad (a = x_0 < x_1 < ... < x_n = b)$$

bo`laklashini olib, unga mos $\breve{AB}$ yoyiga chizilgan l siniq chiziqni hosil qilamiz. Bu siniq chiziqning perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f(x_{k+1}) - f(x_k)]^2}$$

bo`ladi.

Har bir $[x_k, x_{k+1}]$ segmentda $f(x)$ funksiyaga Lagranj teoremasini qo`llab topamiz:

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + [f'(\tau_k) \cdot (x_{k+1} - x_k)]^2} =$$

$$= \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)} \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)}\, \Delta x_k \quad ,$$

bunda $\tau_k \in [x_k, x_{r+1}]$.

Bu tenglikdagi yig`indining $\sqrt{1 + f'^2(x)}$ funksiyaning integral yig`indisidan farqi shuki, integral yig`indida $\xi_k \in [x_k, x_{k+1}]$ nuqta ixtiyoriy bo`lgan holda yuqoridagi yig`indida esa $\tau_k$ nuqta Lagranj teoremasiga muvofiq olingan tayin nuqta bo`lishidadir. Ammo $\sqrt{1 + f'^2(x)}$ funksiya integrallanuvchi bo`lganligi sababli $\xi_k = \tau_k$ deb olinishi mumkin. Natijada

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k$$

bo`lib, undan

$$\lim_{\lambda_p\to 0}\mu(l)=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k=\int_a^b\sqrt{1+f'^2(x)}dx$$

bo`lishi kelib chiqadi.

Demak, $A\breve{B}$ yoyining uzunligi

$$\mu(A\breve{B})=\int_a^b\sqrt{1+f'^2(x)}dx$$

bo`ladi. Bu formula yordamida yoy uzunligi hisoblanadi.

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