



New regular solutions with axial symmetry in Einstein–Yang–Mills theory

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Abstract

We construct new regular solutions in Einstein–Yang–Mills theory. They are static, axially symmetric and asymptotically flat. They are characterized by a pair of integers (k, n) , where k is related to the polar angle and n to the azimuthal angle. The known spherically and axially symmetric EYM solutions have $k = 1$. For $k > 1$ new solutions arise, which form two branches. They exist above a minimal value of n , that increases with k . The solutions on the lower mass branch are related to certain solutions of Einstein–Yang–Mills–Higgs theory, where the nodes of the Higgs field form rings.

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1. Introduction

The well-known regular Bartnik–McKinnon (BM) solutions [1] and the corresponding non-Abelian black hole solutions [2], are asymptotically flat, static spherically symmetric solutions of SU(2) Einstein–Yang–Mills (EYM) theory. They are unstable solutions, sphalerons [3], and are characterized by the number of nodes of the gauge field. Besides the BM solutions there are also asymptotically flat, static, only axially symmetric regular and black hole solutions [4]. These

are characterized by two integers, the node number of their gauge field function(s), and the winding number with respect to the azimuthal angle, denoted n . The spherically symmetric solutions have winding number $n = 1$, while the axially symmetric solutions have winding number $n > 1$.

In SU(2) Einstein–Yang–Mills–Higgs (EYMH) theory, with a triplet Higgs field, gravitating monopole solutions arise [5]. While gravitating monopoles with unit magnetic charge are spherically symmetric, the known gravitating multimono­poles possess axial symmetry [6]. As in EYM theory, these EYMH solutions are characterized by two integers, the node number of the gauge field and the azimuthal winding num-

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ber n , which corresponds to the topological charge of the monopoles.

But EYM theory allows for further static axially symmetric solutions, representing gravitating monopole–antimonopole pair, chain and vortex solutions [7–9]. These solutions can be characterized by the azimuthal winding number n , and by a second integer m , related to the polar angle. For the monopole–antimonopole chains, which in flat space arise for $n = 1$ and 2 , the integer m corresponds to the number of nodes of the Higgs field (and thus the number of poles on the symmetry axis) [8]. In vortex solutions, on the other hand, which in flat space arise for winding number $n > 2$, the Higgs field vanishes (for even m) on $m/2$ rings centered around the symmetry axis [8].

The existence of monopole–antimonopole pair, chain and vortex solutions in EYM theory [7–9] immediately leads to the question, whether there might be analogous solutions in EYM theory, even though there would be no Higgs field participating in the subtle interplay of attraction and repulsion. In this Letter we report the existence of one such new type of solution, related to vortex solutions in EYM theory.

In Section 2 we present the EYM action, the axially symmetric ansatz and the boundary conditions. In Section 3 we discuss the properties of the new axially symmetric solutions, and we present our conclusions in Section 4.

2. Action and ansatz

We consider the SU(2) EYM action

$$S = \int \left(\frac{R}{16\pi G} - \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \right) \sqrt{-g} d^4x \quad (1)$$

with Ricci scalar R , field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu], \quad (2)$$

gauge potential $A_\mu = \tau^a A_\mu^a/2$, and gravitational and Yang–Mills coupling constants G and e , respectively. Variation of the action (1) with respect to the met-

ric $g^{\mu\nu}$ leads to the Einstein equations, variation with respect to the gauge potential A_μ to the gauge field equations.

In isotropic coordinates the static axially symmetric metric reads [4]

$$ds^2 = -f dt^2 + \frac{m}{f} dr^2 + \frac{mr^2}{f} d\theta^2 + \frac{lr^2 \sin^2 \theta}{f} d\varphi^2, \quad (3)$$

where the metric functions f , m and l are functions of the coordinates r and θ , only. The z -axis ($\theta = 0, \pi$) represents the symmetry axis. Regularity on the z -axis requires $m = l$ there.

For the gauge field we employ the ansatz [4,8,9]

$$A_\mu dx^\mu = \frac{1}{2er} [\tau_\varphi^n (H_1 dr + (1 - H_2)r d\theta) - n(\tau_r^{n,k} H_3 + \tau_\theta^{n,k} H_4)r \sin \theta d\phi]. \quad (4)$$

Here the symbols $\tau_r^{n,k}$, $\tau_\theta^{n,k}$ and τ_φ^n denote the dot products of the Cartesian vector of Pauli matrices, $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$, with the spatial unit vectors

$$\begin{aligned} \vec{e}_r^{n,k} &= (\sin k\theta \cos n\varphi, \sin k\theta \sin n\varphi, \cos k\theta), \\ \vec{e}_\theta^{n,k} &= (\cos k\theta \cos n\varphi, \cos k\theta \sin n\varphi, -\sin k\theta), \\ \vec{e}_\varphi^n &= (-\sin n\varphi, \cos n\varphi, 0), \end{aligned} \quad (5)$$

respectively. The gauge field functions H_i , $i = 1, \dots, 4$, depend on the coordinates r and θ , only. For $k = n = 1$ and $H_1 = H_3 = 0$, $H_2 = 1 - H_4 = w(r)$ the BM solutions [1] are recovered, while for $k = 1$, $n > 1$, one obtains the axially symmetric solutions of [4]. The new solutions reported here are obtained for $k > 1$. They are related to EYM solutions with $m = 2k$ in the limit of vanishing Higgs field [9].

The ansatz is form-invariant under the Abelian gauge transformation [4]

$$U = \exp\left(\frac{i}{2} \tau_\phi^n \Gamma(r, \theta)\right). \quad (6)$$

We fix the gauge by choosing the gauge condition [4,8,9]

$$r \partial_r H_1 - \partial_\theta H_2 = 0. \quad (7)$$

To obtain asymptotically flat solutions which are globally regular and possess the proper symmetries, we need to impose appropriate boundary conditions [4,8,9]. At the origin we impose the boundary conditions

$$\begin{aligned} \partial_r f = \partial_r m = \partial_r l = 0, \\ H_1 = H_3 = H_4 = 0, \quad H_2 = 1, \end{aligned} \quad (8)$$

at infinity we impose

$$\begin{aligned} f = m = l = 1, \\ H_1 = H_3 = 0, \quad H_2 = 1 - 2k, \\ H_4 = 2 \sin(k\theta) / \sin \theta, \end{aligned} \quad (9)$$

and on the z -axis we impose

$$\begin{aligned} \partial_\theta f = \partial_\theta m = \partial_\theta l = 0, \\ H_1 = H_3 = 0, \quad \partial_\theta H_2 = \partial_\theta H_4 = 0. \end{aligned} \quad (10)$$

We further introduce the dimensionless coordinate x , and the dimensionless mass μ ,

$$x = \frac{e}{\sqrt{4\pi G}} r, \quad \mu = \frac{eG}{\sqrt{4\pi G}} M. \quad (11)$$

3. Numerical results

Subject to the above boundary conditions, we solve the system of seven coupled nonlinear partial differential equations numerically. To map spatial infinity to the finite value $\bar{x} = 1$, we employ the radial coordinate [4,8,9]

$$\bar{x} = \frac{x}{1+x}. \quad (12)$$

The numerical calculations are based on the Newton–Raphson method, and are performed with help of the program FIDISOL [10]. The equations are discretized on a nonequidistant grid in \bar{x} and θ . Typical grids used have sizes 70×30 , covering the integration region $0 \leq \bar{x} \leq 1$ and $0 \leq \theta \leq \pi/2$. For the method, it is

essential to have a good first guess, to start the iteration procedure. For the $k = 1$ EYM solutions, the $n = 1$ BM solutions serve as a first guess, and then the ‘parameter’ n is varied (via unphysical noninteger values) to obtain solutions with higher winding number n [4]. For the $k = 2$ solutions we employ the $m = 4$, $n = 4$ EYMH vortex solution with (almost) vanishing Higgs field as a first guess [9], and then again vary n . Similarly, for the $k = 3$ solutions we start from the $m = 6$, $n = 6$ EYMH vortex solution.

In EYMH theory, the monopole–antimonopole pair (MAP) solution is obtained, when $m = 2k = 2$, $n = 1$. Here a monopole and an antimonopole are located symmetrically on the z -axis. Similarly, one might try to find a sphaleron–antisphaleron pair (SAP) solution in EYM theory. The boundary conditions of such a SAP solution, however, do not differ from those of a BM solution. Consequently, when the EYMH coupling constant $\alpha = \sqrt{4\pi G} v$ (where v is the Higgs field expectation value) is varied, the gravitating MAP solution starts from the flat space solution, reaches a critical solution at a maximal value of α , and then with decreasing α smoothly reaches the BM solution in the limit $\alpha \rightarrow 0$ (after rescaling of the radial coordinate) [7]. Consequently, we do not find a SAP solution in EYM theory, and similarly, we do not find new solutions, when $m = 2k = 2$, $n > 1$, since their boundary conditions agree with those of the axially symmetric solutions of [4].

When $m = 2k = 4$, however, the boundary conditions Eq. (9) for the gauge field differ from those of the known solutions [1,4]. We therefore may expect the existence of new EYM solutions, subject to these boundary conditions. Indeed, for $m = 4$, $n = 4$, for instance, when α is varied again, the gravitating EYMH vortex solution starts from the flat space solution [8], reaches a critical solution at a maximal value of α , and then with decreasing α smoothly reaches a new EYM solution in the limit $\alpha \rightarrow 0$ (after rescaling of the radial coordinate) [9].

Let us in the following refer to the solutions characterized by the integers k and n as (k, n) solutions. In Fig. 1(a) we exhibit the energy density of the new $(2, 4)$ EYM solution. Its energy density has a pronounced maximum on the ρ -axis, thus the energy density is torus-like. The energy density of this $(2, 4)$ solution is thus similar to the energy density of the (lowest mass) known $(1, 4)$ EYM solution [4], exhibited in

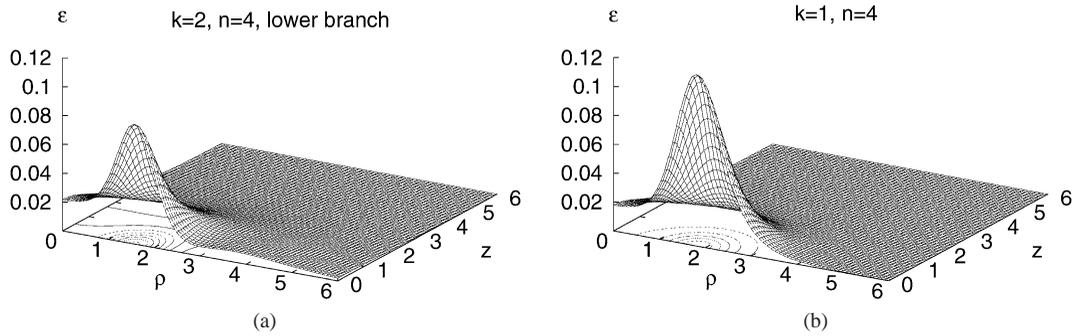


Fig. 1. The energy density of the (2, 4) EYM solution on the lower mass branch (a) and of the (1, 4) EYM solution (b).

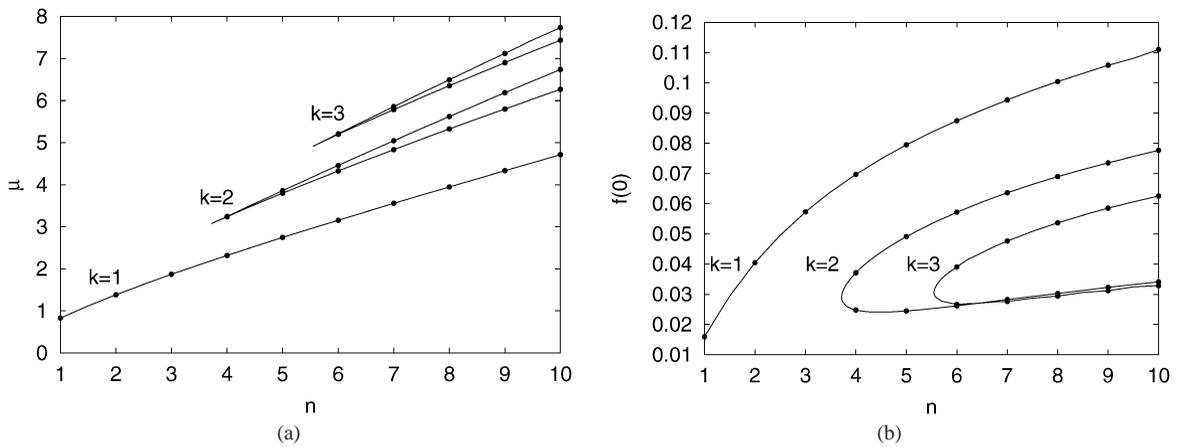


Fig. 2. The mass (a) and the value of the metric function f at the origin (b) of the (1, n), (2, n) and (3, n) EYM solutions is shown as a function of n .

Fig. 1(b). But the maximum of the (2, 4) solution is smaller than the maximum of the (1, 4) solution, and it is shifted slightly inwards. In addition, the (2, 4) solution has a small saddle point on the ρ -axis, located further outwards. On the other hand comparison with the corresponding EYMH solution [8] shows, that the energy density of the EYM solution differs significantly from the energy density of the flat space (2, 4) EYMH solution, whose energy density exhibits two tori, located symmetrically with respect to the xy -plane [8].

The existence of a $k = 2$ EYM solution with winding number $n = 4$, immediately suggests that analogous solutions with winding number $n \neq 4$ should exist as well. Indeed, by increasing n , a whole set of such new solutions is obtained numerically, one for each integer n . On the other hand, a decrease of n does not lead to new solutions. Instead a bifurcation at the noninteger and thus unphysical value of $n = 3.7175$

is obtained, where a second branch of solutions with higher mass appears. Consequently there is a second solution for $n = 4$, and also for each higher integer n , but there are no such solutions for integer $n \leq 3$. This is illustrated in Fig. 2(a), where we exhibit the mass of these (2, n) EYM solutions. Only the integer values of n in the figure correspond to physical solutions. In Fig. 2(b) we illustrate the value of the metric function f at the origin. We note, that for (1, n) solutions (with a single node) we find only one branch, which is also exhibited in Fig. 2.

To illustrate the dependence of the solutions on n , we exhibit the energy density of the (2, n) solutions with $n = 4, 6, 8$ on the lower (mass) branch in Fig. 3(a) and on the upper branch in Fig. 3(b). With increasing n the maximum of the energy density decreases and moves outwards for the solutions on both branches. For the same n , the maximum of the upper mass so-

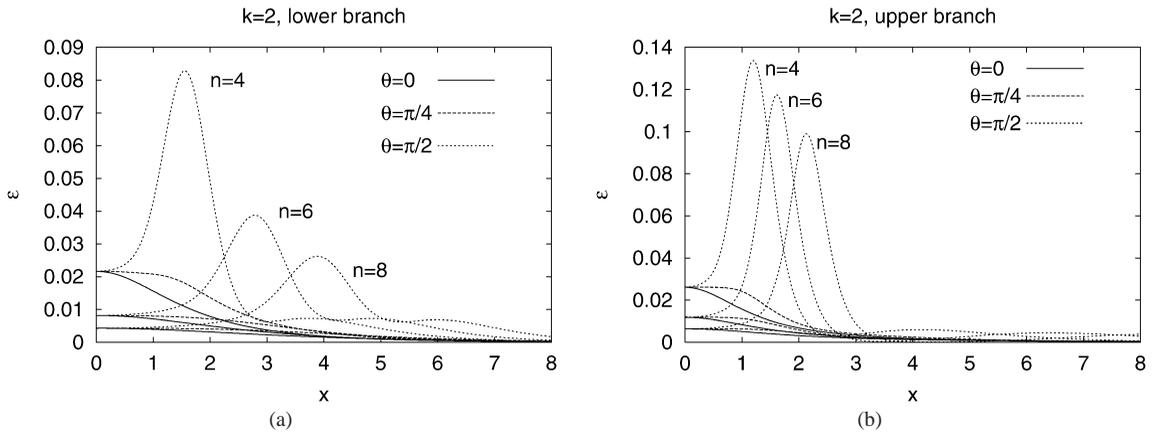


Fig. 3. The energy density of the $(2, n)$ EYM solution on the lower mass branch (a) and on the upper mass branch (b) for $n = 4, 6, 8$.

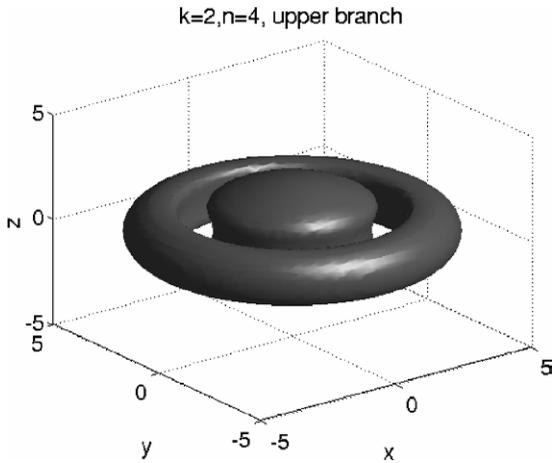


Fig. 4. A surface of constant energy density ($\varepsilon = 0.005$) of the $(2, 4)$ EYM solution on the upper mass branch.

lutions is higher and located further inwards than the maximum of the lower mass solutions. The small saddle point turns into a small maximum on the upper branch, thus the energy density of the solutions on the upper branch has a more complicated structure as seen in Fig. 4, where a surface of constant energy density is shown for the $(2, 4)$ solution. We exhibit the gauge field function H_2 for the same set of solutions in Fig. 5.

For $k = 3$ we obtain a similar pattern of solutions. But the new type of solutions appears only for $n > 5$. The mass and the value of the metric function f at the origin of these $(3, n)$ solutions are also shown in Fig. 2. The energy density for $(3, n)$ solutions with $n = 6, 8, 10$ and the gauge field function H_2 are shown

in Figs. 6 and 7, respectively. Based on these results, we conjecture the existence of (k, n) solutions with $k > 3$, but expect that they will appear only for still larger values of n .

4. Conclusions

We have constructed numerically new static regular solutions of EYM theory with axial symmetry. These solutions are characterized by the integers (k, n) , related to the azimuthal and polar angles, respectively. In particular, we have obtained solutions for $k = 2, n = 4, 6, 8, 10$, and $k = 3, n = 6, 8, 10$.

Like the previously known $(1, n)$ solutions [4], the $(2, n)$ and $(3, n)$ solutions on the lower branch have a torus-like shape. On the upper branch of the $(3, n)$ solutions double tori appear for larger n . The $(1, n)$ solutions (with one node) most likely exist for any integer $n \geq 1$ and form a single branch. The new $(2, n)$ and $(3, n)$ solutions have lower bounds on $n, n = 2k$, which imply the existence of two branches of solutions for both values of k . We expect, that the $(2, n)$ and $(3, n)$ solutions represent only the first sequences of new solutions, and conjecture the existence of (k, n) solutions also for higher values of k .

The $k = 1$ BM solutions are unstable [3], and there is all reason to believe, that the new (k, n) solutions are also unstable. Having constructed the regular axially symmetric solutions it appears straightforward to construct analogous black hole solutions, and, in particular, to look for rotating solutions [11].

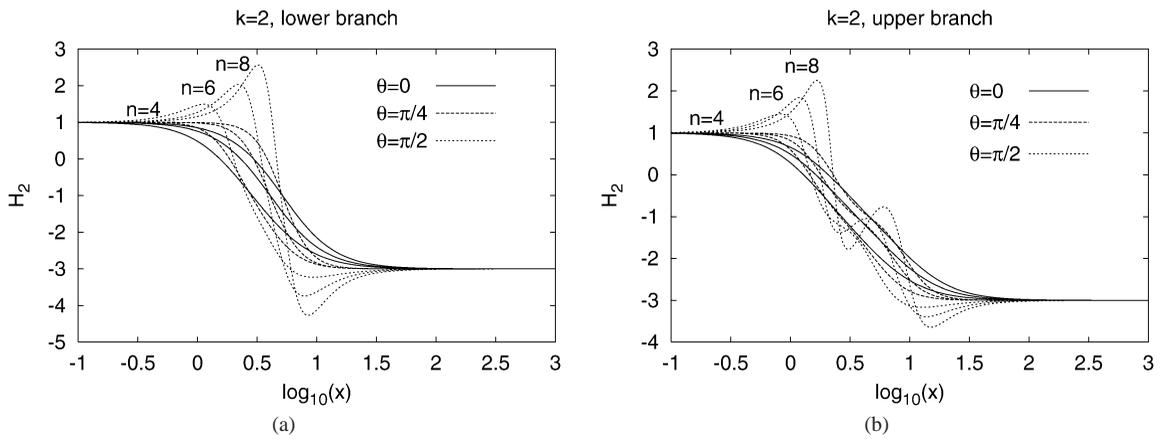


Fig. 5. The gauge field function H_2 of the $(2, n)$ EYM solutions on the lower mass branch (a) and on the upper mass branch (b) for $n = 4, 6, 8$.

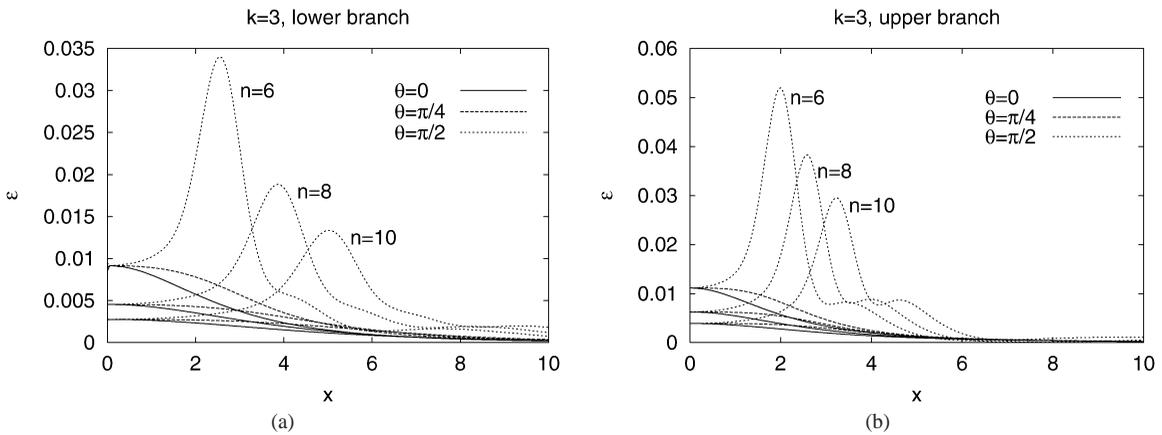


Fig. 6. The energy density of the $(3, n)$ EYM solution on the lower mass branch (a) and on the upper mass branch (b) for $n = 6, 8, 10$.

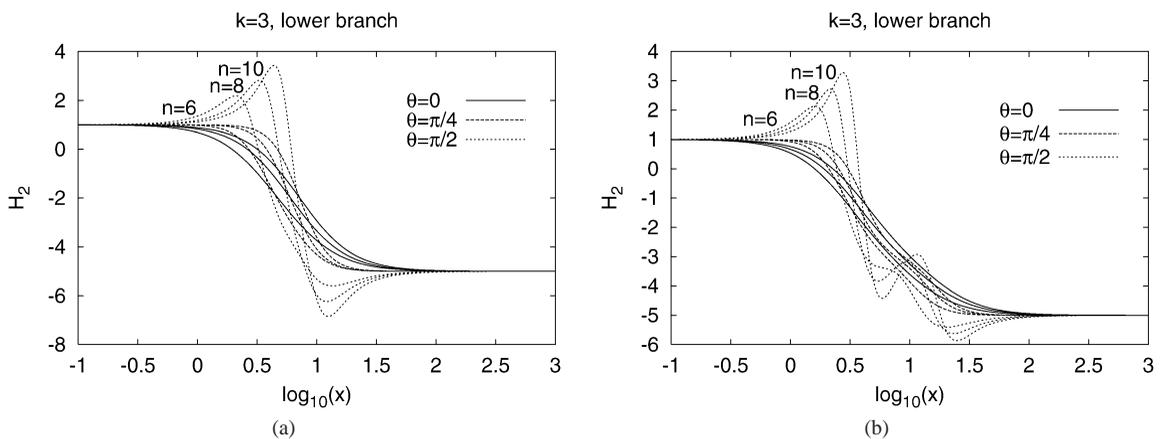


Fig. 7. The gauge field function H_2 of the $(3, n)$ EYM solutions on the lower mass branch (a) and on the upper mass branch (b) for $n = 6, 8, 10$.

Acknowledgements

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