

Toshkent temir yo'l muhandislari instituti

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Aniq integral

Barcha bakalavriat ta'lim yo'nalishlarida taxsil oluvchi talabalar uchun
mustaqil ishlarni bajarish bo'yicha uslubiy ko'rsatma va topshiriqlar.

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Ushbu uslubiy ko'rsatmada oliy matematika fanining "Aniq integral bo'limi bo'yicha" mustaqil ish topshiriqlarini bajarish uchun kerakli nazariy tushunchalar keltirilgan va namuna sifatida misollar yechib ko'rsatilgan. Har bir mavzuga doir saralangan mustaqil ish topshiriqlari berilgan.

Mazkur ko'rsatma barcha bakalavriat ta'lim yo'nalishlaridagi talabalar uchun mo'ljallangan.

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Muxarrir

Bosishga ruxsat etildi. Xajmi buyurtma № qog'oz bichimi nusxa.

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KIRISH

Oliy matematika fanining «Aniq integral» bo'limini turli texnika masalalarini echishga tadbiqi beqiyosdir. Talabalardan yechilishi talab etilayotgan masalalarning matematik modelini yarata bilishi, ilmiy adabiyotlardan mustaqil foydalana olishi, olgan bilimlarini amaliyotga tadbiq qilishi talab etiladi. Aniq integral bo'yicha mustaqil ish topshiriqlarini bajarish talabalarda etarli bilim va ko'nikmalar hosil qiladi.

Mazkur uslubiy ko'rsatma o'z ichiga quyidagi tushunchalarni qamrab olgan.

1. Aniq integral va uning xossalari
2. Aniq integralni hisoblash
3. xosmas integrallar
4. Tekis figuralarning yuzalarini hisoblash
5. Tekis chiziq yoy uzunligini hisoblash
6. Jismlarning xajmini hisoblash
7. Aylanma jism sirtlarni hisoblash
8. Statik momentlar. Inersiya momenti
9. Og'irlik markazi

1. Integral yig'indi.

1. $[a; b]$ kesmada aniqlangan $f(x)$ funksiya qaralayotgan bo'lsin. $[a; b]$ kesmani 4 ta ushbu $a = x_0 < x_1 < x_2 < \dots < x_{4-1} < x_4 = b$ ko'rinishdagi bo'lakchalarga bo'lib, uzunligi $\Delta x_i = x_i - x_{i-1}$ bo'lgan xar bir bo'lakchada ixtiyoriy ξ nuqta tanlaymiz.

$$\text{Ushbu } G = \sum_{i=1}^4 f(\xi) \Delta x_i$$

ko'rinishdagi yig'indiga $f(x)$ funksiyaning $[a; b]$ kesmadagi integral yig'indisi deyiladi.

2. Agar

$$\lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^4 f(\xi) \Delta x_i = I$$

chekki ilmiy mavjud bo'lsa, u xolda I soniga $f(x)$ funksiyadan $[a; b]$ kesmada olingan aniq integral deyiladi va quyidagicha yoziladi:

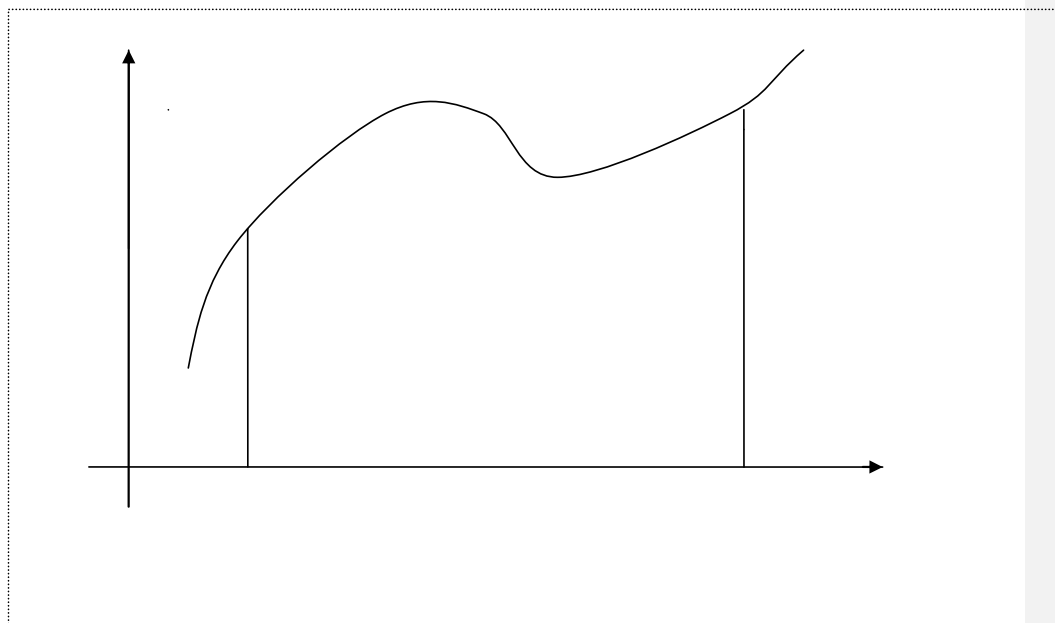
$$I = \int_a^b f(x) dx$$

a – integrallashning quyi chegarasi

b – integrallashni yuqori chegarasi.

3. Agar $f(x)$ funksiya $[a; b]$ temada uzluksiz bo'lsa, u xolda, I chekli limit mavjud bo'ladi. Bu limit $[a; b]$ kesmani bo'lakchalarga bo'lish usulidan va ξ nuqtani tanlashdan bog'liq bo'lmaydi.

4. Agar $[a; b]$ kesmada $f(x)$ funksiya manfiy bo'lmasa, u xolda aniq integral ushbu $\{y = f(x); x = 0; x = b\}$ chiziqlar bilan chegaralangan egri chiziqli trapesiyaning yuzini ifodalaydi (1-shakl.)



1-shakl

Misol. $\int_0^1 x^2 dx$ integralni integral yig'indidan foydalanib hisoblang.

Yechish. Bu erda $f(x) = x^2$, $a = 0$ $b = 1$. $[0/1]$ kesmani n ta teng bo'lakchalarga bo'lamiz. U xolda $\Delta x_i = \frac{(b-a)}{n} = \frac{1}{n}$ bo'ladi. ξ sifatida x_i ni tanlaymiz. Natijada:

$$x_0 = 0; \quad x_1 = \frac{1}{n}; \quad x_2 = \frac{2}{n}; \dots; \quad x_{n-1} = \frac{n-1}{n}; \quad x_n = 1$$

$$f(\xi_1) = \left(\frac{1}{n}\right)^2 = \frac{1}{n^2} \qquad f(\xi_2) = \left(\frac{2}{n}\right)^2 = \frac{2^2}{n^2}$$

$$f(\xi) = \left(\frac{3}{n}\right)^2 = \frac{3^2}{n^2} \qquad f(\xi) = \left(\frac{3}{n}\right)^2 = \frac{3^2}{n^2}$$

$$f(\xi_i) = \left(\frac{i}{n}\right)^2 = \frac{i^2}{n^2} \qquad f(\xi_n) = \left(\frac{n}{n}\right)^2 = \frac{n^2}{n^2}$$

Hosil bo'ladi va $\max \Delta x_i \rightarrow 0$ bo'lishi uchun $n \rightarrow \infty$ intilishi

Ta'rifga ko'ra:

$$\int_0^1 x^2 dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(\xi) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$$

Ma'lumki, $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n \cdot (n+1)(2n+1)}{6}$

Shunga ko'ra:

$$\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6} \cdot \lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{n})(2 + \frac{1}{n})}{1} = \frac{1}{6} \cdot 2 = \frac{1}{3}$$

Natijada $\int_0^1 x^2 dx = \frac{1}{3}$ hosil bo'ladi.

Javob: $\frac{1}{3}$.

1- topshiriq. Integrallarni integral yig'indiga ko'ra hisoblang.

1. $\int_0^1 x dx$

2. $\int_1^{10} (1+x) dx$

3. $\int_a^b x dx$

4. $\int_2^1 x^2 dx$

5. $\int_0^{10} 2^x dx$

6. $\int_1^2 x^3 dx$

7. $\int_1^2 \frac{dx}{x}$

8. $\int_1^3 \frac{dx}{x}$

$$\begin{array}{lll}
9. \int_0^2 x^2 dx & 10. \int_0^1 x^3 dx & 11. \int_0^1 e^x dx \quad 12. \int_0^1 3x dx \\
13. \int_0^1 \sqrt{x} dx & 14. \int_0^4 \sqrt{x} dx & 15. \int_0^1 \varphi nx dx \quad 16. \int_0^1 \cos x dx \\
17. \int_0^5 4x dx & 18. \int_0^3 x^2 dx & 19. \int_0^2 e^x dx \quad 20. \int_0^{\pi/2} \sin x dx \\
21. \int_0^{\pi/2} \cos x dx & 22. \int_0^{\pi} \sin x dx & 23. \int_0^{\pi} \cos x dx \quad 24. \int_1^3 \frac{dx}{x} \\
25. \int_0^2 x^3 dx & 26. \int_0^9 \sqrt{x} dx & 27. \int_1^2 x^2 dx \quad 28. \int_0^5 e^x dx \\
29. \int_0^{\pi/4} \sin x dx & 30. \int_0^{\pi/4} \cos x dx &
\end{array}$$

2. Aniq integralni hisoblash

1. Agar $f(x)$ funksiya $[a; b]$ kesmada uzluksiz bo'lsa, u holda

$$F(x) = \int_a^x f(t) dt$$

funksiya $f(x)$ funksiyaning $[a; b]$ kesmadagi boshlang'ich funksiyasi bo'ladi.

2. Agar $F(x)$ funksiya $f(x)$ funksiyaning $[a; b]$ kesmadagi boshlang'ich funksiyani bo'lsa, u holda

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

O'rinli bo'ladi. Bu ifodaga N'yuton - Leybnis formulasi deb yuritiladi.

3. Aniq integral quyidagi asosiy hossalarga ega.

$$1) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$2) \int_a^a f(x) dx = 0$$

$$3) \int_a^b (f(x) \pm \varphi(x)) dx = \int_a^b f(x) dx \pm \int_a^b \varphi(x) dx$$

$$4) \int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx, c \text{ o'zgarmas son}$$

$$5) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

6) agar $[a; b]$ kesmada $f(x) \leq \varphi(x)$ shart bajarilsa u holda

$$\int_a^b f(x) dx \leq \int_a^b \varphi(x) dx$$

o'rinli bo'ladi.

7) agar m va M sonlari mos ravishda $f(x)$ funksiyaning $[a; b]$ kesmadagi eng kichik va eng katta qiymatlari bo'lsa, u holda

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

o'rinli bo'ladi.

4. Agar $f(x)$ funksiya $[a; b]$ kesmada uzluksiz; $x = \varphi(t)$ funksiya birinchi tartibli hosilasi $\varphi'(t)$ bilan $[\alpha; \beta]$ kesmada uzluksiz; $a = \varphi(\alpha)$, $b = \varphi(\beta)$ bo'lib; $f(\varphi(t))$ funksiya $[\alpha; \beta]$ kesmada aniqlangan va uzluksiz bo'lsa, u holda

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(\varphi(t)) \varphi'(t) dt$$

o'rinli bo'ladi. Bunga aniq igtegralda o'zgaruvchini almashtirish deyiladi.

5. Agar $u(x)$ va $v(x)$ funksiyalar $[a; b]$ kesmada uzluksiz differensionlanuvchi bo'lsa, u holda

$$\int_a^b u \cdot v' dx = \int_a^b u' v dx + \int_a^b u v dx - \int_a^b v dx$$

o'rinli bo'ladi. Bunga aniq integralni bo'laklab integrallash deyiladi.

6. Agar $f(x)$ funksiya juft bo'lsa, u holda

$$\int_{-a}^a f(x) dx = 0,$$

Agar $f(x)$ funksiya toq bo'lsa, u holda

$$\int_{-a}^a f(x) dx = 2 \cdot \int_0^a f(x) dx$$

o'rinli bo'ladi.

1-misol. $\int_0^{\pi/2} \cos x dx$ ni hisoblang.

$$\text{Yechish. } \int_0^{\pi/2} \cos x dx = \sin x \Big|_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1$$

Javob: 1

2-misol. $\int_0^1 x^2 \sqrt{1-x^2} dx$ ni toping.

Yechish.

$$\int_0^1 x^2 \sqrt{1-x^2} dx$$

$$dx \Rightarrow \left| \begin{array}{l} x = \sin t \\ dx = \cos t dt \\ x = 0 \Rightarrow t = 0 \\ x = 1 \Rightarrow t = \frac{\pi}{2} \end{array} \right| \Rightarrow \int_0^{\pi/2} \sin^2 t \sqrt{1-\sin^2 t} \cdot \cos t dt = \int_0^{\pi/2} \sin^2 t \cos^2 t dt =$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2 2t dt = \frac{1}{8} \int_0^{\pi/2} (1 - \cos 4t) dt = \frac{1}{8} \left(t - \frac{1}{4} \sin 4t \right) \Big|_0^{\pi/2} = \frac{\pi}{16}$$

Javob: $\frac{\pi}{16}$.

3-misol. $\int_0^1 x \cdot e^x dx$ ni integrallang.

Yechish. $\int_0^1 x e^x dx \Rightarrow \left| \begin{array}{l} u = x; du = dx \\ e^x dx = dv; v = e^x \end{array} \right| \Rightarrow x \cdot e^x \Big|_0^1 - \int_0^1 e^x dx = x \cdot e^x \Big|_0^1 - e^x \Big|_0^1 =$

$$= e - e + 1 = 1$$

Javob: 1.

2-topshiriq. Integralni o'zgaruvchini almashtirish usuli bilan hisoblang.

1. $\int_{e+1}^{e^2+1} \frac{\ln(x-1) dx}{x-1}$
2. $\int_0^1 \frac{(x^2+1) dx}{(x^3+3x+3)^2}$
3. $\int_0^1 \frac{\arctan x}{1+x^2} dx$
4. $\int_0^2 \frac{x^3 dx}{x^2+9}$
5. $\int_{\pi}^{2\pi} \frac{3x+3 \cos x}{x^2+\sin x} dx$
6. $\int_0^{\pi/4} \frac{2 \sin x + 3 \cos x}{(2 \cos x - 3 \sin x)^3} dx$
7. $\int_0^{1/3} \frac{\arctg 3x}{1+9x^2} dx$
8. $\int_0^1 \frac{x dx}{x^4+1}$
9. $\int_0^{\sin^2} \frac{(\arcsin x)^2}{\sqrt{1-x^2}} dx$
10. $\int_1^3 \frac{dx}{\sqrt{x}(x+1)}$
11. $\int_1^e \frac{1+\ln x}{x} dx$
12. $\int_0^{\pi/2} \frac{\sin x - \cos x}{(\cos x + \sin x)^3} dx$
13. $\int_0^1 \frac{(\arccos x)^3}{\sqrt{1-x^2}} dx$
14. $\int_0^1 \frac{(\arcsin x)^3}{\sqrt{1-x^2}} dx$
15. $\int_0^1 \frac{\lg(x+1)}{\cos^2(x+1)} dx$
16. $\int_1^4 \frac{dx}{2\sqrt{x}(\sqrt{x}+x)}$
17. $\int_0^{\pi/4} \lg x \ln \cos x dx$
18. $\int_0^4 \sqrt{4-x^2} dx$
19. $\int_0^{\pi/4} \frac{x + \sin x}{1 + \cos x} dx$
20. $\int_1^3 \sqrt{x+1} dx$
21. $\int_0^4 \frac{dx}{1+\sqrt{x}}$
22. $\int_0^{\pi} \frac{dx}{3+2 \cos x}$
23. $\int_0^{\ln 2} \sqrt{e^x-1} dx$
24. $\int_0^{\pi/2} \frac{dx}{1+\sin^2 x}$

$$25. \int_0^9 \sqrt{9 - x^2} dx$$

$$26. \int_1^3 \sqrt{x^2 - 1} dx$$

$$27. \int_0^1 \frac{x^3}{(x^2 + 1)^2} dx$$

$$28. \int_{\pi/4}^{\pi/2} \frac{x \cos x + \sin x}{(x \sin x)^2} dx$$

$$29. \int_0^1 \frac{x - (\arctg x)^3}{1 + x} dx$$

$$30. \int_0^1 \frac{1 + (\arcsin x)^3}{\sqrt{1 - x^2}} dx$$

3-topshiriq. Integrallarni bo'laklab integrallash usuli bilan hisoblang.

$$1. \int_0^1 (x^2 + 3x + 2) \sin x dx$$

$$2. \int_0^1 (x^2 + 4x + 3) \cos x dx$$

$$3. \int_0^1 (x^2 + 4x + 5) \sin 2x dx$$

$$4. \int_0^1 (x^2 + 3x + 6) \cos 2x dx$$

$$5. \int_0^1 (x^2 + 5x + 3) \sin 3x dx$$

$$6. \int_0^1 (x^2 + 5x + 7) \cos 3x dx$$

$$7. \int_0^1 (x^2 + 3x + 1) e^x dx$$

$$8. \int_0^1 (x^2 + 4x + 7) e^{2x} dx$$

$$9. \int_0^1 (x^2 + 4x + 5) e^{3x} dx$$

$$10. \int_0^1 (x^2 + 5x + 8) e^{4x} dx$$

$$11. \int_0^1 (x + 1) \ln(x + 1) dx$$

$$12. \int_0^1 (x + 1) \ln^2(x + 1) dx$$

$$13. \int_0^1 (x + 2) \ln(x + 2) dx$$

$$14. \int_0^1 (x + 2) \ln^2(x + 2) dx$$

$$15. \int_0^1 (x + 3) \ln^2(x + 3) dx$$

$$16. \int_0^1 (x + 4) \ln(x + 4) dx$$

$$17. \int_0^1 (x^2 + 3x + 5) e^{-x} dx$$

$$18. \int_0^1 (x^2 + 2x + 3) e^{2x} dx$$

$$19. \int_0^1 (x^2 + 6x + 3) \cos 5x dx$$

$$20. \int_0^1 (x^2 + 3x + 8) \sin 4x dx$$

$$21. \int_0^1 (x^2 + 5x + 7) e^{-3x} dx$$

$$22. \int_0^1 (x^2 + 4x + 5) e^{-4x} dx$$

$$23. \int_0^{\pi/2} e^x \cos x dx$$

$$24. \int_0^{\pi/2} e^x \sin x dx$$

$$25. \int_0^{\pi/2} e^{2x} \cos 2x dx$$

$$26. \int_0^{\pi/2} e^{2x} \sin 2x dx$$

$$27. \int_0^1 x \arctg x dx$$

$$28. \int_0^1 x \arcsin x dx$$

$$29. \int_0^{\pi} e^x \sin x dx$$

$$30. \int_0^1 e^x \cos x dx$$

3. Xosmas integrallar.

1. Integrallash chegarasi cheksiz bo'lgan integrallarga yosh chegaralanmagan funksiyalarning integrallariga xosmas integrallar deyiladi.

2. $f(x)$ funksiyaning $(a; \infty)$ oraliqdagi xosmas integrali quyidagicha aniqlanadi.

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

Agar chekli limit mavjud bo'lsa, u holda xosmas integralni yaqinlashuvchi deyiladi. Agar limit mavjud bo'lmasa yoki cheksizligiga intilsa, u holda integralni uzoqlashuvchi deyiladi.

Mos ravishda,

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$
$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^{\infty} f(x) dx$$

3. $[a; b]$ kesmaning biror $x = c$ nuqtasida chegaralanmagan, $[a; c)$ va $(c; b]$ oraliqlarda uzluksiz bo'lgan $f(x)$ funksiyaning $[a; b]$ kesmadagi xosmas integrali quyidagicha aniqlanadi:

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{c-\varepsilon} f(x) dx + \lim_{\varepsilon \rightarrow 0} \int_{c+\varepsilon}^b f(x) dx$$

Agar ikkala limit ham mavjud bo'lsa, u holda xosmas integral yaqinlashuvchi deyiladi. Agar limitlardan birortasi mavjud bo'lmasa, u holda integral uzoqlashuvchi deyiladi.

1-misol. $\int_{-\infty}^{-1} \frac{dx}{x^2}$ xosmas integralni hisoblang.

Echish. $\int_{-\infty}^{-1} \frac{dx}{x^2} = \lim_{a \rightarrow -\infty} \int_a^{-1} \frac{dx}{x^2} = \lim_{a \rightarrow -\infty} \left(-\frac{1}{x} \right) = \lim_{a \rightarrow -\infty} \left(1 + \frac{1}{a} \right) = 1$

Javob: 1; xosmas integral yaqinlashuvchi

2-misol. $\int_{-1}^1 \frac{dx}{x^2}$ ni hisoblang.

Echish. Integrallanayotgan $f(x) = \frac{1}{x^2}$ funksiya $x = 0$ nuqtada chegaralanmagan. Shuning uchun:

$$\int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon \rightarrow 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{dx}{x^2} = \lim_{\varepsilon \rightarrow 0} \left(-\frac{1}{x} \right) + \lim_{\varepsilon \rightarrow 0} \left(-\frac{1}{x} \right) = \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon} - 1 \right) + \lim_{\varepsilon \rightarrow 0} \left(-1 + \frac{1}{\varepsilon} \right) = \infty$$

Javob: xosmas integral uzoqlashuvchi.

4-topshiriq. Xosmas integrallarni hisoblang.

1. $\int_0^{\infty} \frac{\arctg x}{1+x^2} dx$
2. $\int_{-\infty}^0 \frac{dx}{4+x^2}$
3. $\int_{-\infty}^{\infty} \frac{dx}{9+x^2}$
4. $\int_{-\infty}^0 \frac{dx}{16+x}$
5. $\int_0^1 \frac{dx}{x}$
6. $\int_0^1 x \ln^2 x$
7. $\int_{-1}^0 \frac{dx}{x^3}$
8. $\int_0^1 \frac{dx}{x^3}$
9. $\int_2^{\infty} \frac{dx}{x \sqrt{x^2-1}}$
10. $\int_1^{\infty} \frac{\arctg x}{x^2} dx$
11. $\int_3^5 \frac{dx}{\sqrt[3]{(4-x)^2}}$
12. $\int_1^3 \frac{dx}{(x-2)^2}$
13. $\int_0^{\infty} x e^{-x^2} dx$
14. $\int_0^{\infty} x^2 e^{-x^3} dx$
15. $\int_1^{\infty} \frac{\ln x}{x^2} dx$
16. $\int_1^e \frac{dx}{x \ln x}$
17. $\int_0^{\infty} \frac{dx}{\sqrt{(1+x)^3}}$
18. $\int_1^{\infty} \frac{dx}{x^2+x^4}$
19. $\int_0^1 \frac{dx}{x^4}$
20. $\int_0^{\frac{1}{3}} \frac{dx}{x \ln^2 x}$
21. $\int_0^{\pi/2} \ctg x dx$
22. $\int_2^{\infty} \frac{dx}{(x^2-1)^2}$
23. $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$
24. $\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5}$
25. $\int_0^{\infty} e^{-3x} dx$
26. $\int_2^{\infty} \frac{dx}{x \ln^2 x}$
27. $\int_1^{\infty} \frac{dx}{(x+1) \ln(x+1)}$
28. $\int_3^{\infty} \frac{dx}{(x-1) \ln(x-1)}$
29. $\int_{-3}^{-1} \frac{dx}{(x+2)^2}$
30. $\int_4^6 \frac{dx}{(x-5)^2}$

4. Xosmas integrallarni baholash.

1. Agar ushbu $0 \leq f(x) \leq \varphi(x)$ shart barcha $x \geq a$ da o'rinli bo'lib,

$\int_a^{\infty} \varphi(x) dx$ xosmas integral yaqinlashuvi bo'lsa, u holda $\int_a^{\infty} \varphi(x) dx$ integral

ham yaqinlashuvchi va $\int_a^{\infty} \varphi(x) dx \leq \int_a^{\infty} \varphi(x) dx$ o'rinli bo'ladi.

2. Agar ushbu $0 \leq \varphi(x) \leq f(x)$ shart barcha $x \geq a$ da o'rinli bo'lib, $\int_a^{\infty} \varphi(x) dx$

integral uzoqlashuvchi bo'lsa, u holda $\int_a^{\infty} f(x) dx$ integral ham uzoqlashuvchi bo'ladi.

3. Agar $\int_a^{\infty} |f(x)| dx$ yaqinlashuvchi bo'lsa, u holda $\int_a^{\infty} f(x) dx$ ham yaqinlashuvchi bo'ladi.

4. Agar $[a; b]$ kesmaning biror $x = c$ nuqtasida $f(x)$ va $\varphi(x)$ funksiyalar chegaralanmagan bo'lib, kesmaning barcha nuqtalarida $0 \leq f(x) \leq \varphi(x)$ shart bajarilsa, u holda $\int_a^b \varphi(x) dx$ ning yaqinlashuvchiligidan $\int_a^b f(x) dx$ ning ham yaqinlashuvchiligi kelib chiqadi.

5. Agar $[a; b]$ kesmaning biror $x = c$ nuqtasida $f(x)$ va $\varphi(x)$ funksiyalar chegaralanmagan bo'lib, kesmaning barcha nuqtalarida $0 \leq \varphi(x) \leq f(x)$ shart bajarilsa, u holda $\int_a^b \varphi(x) dx$ ning uzoqlashuvchiligidan $\int_a^b f(x) dx$ ning ham uzoqlashuvchiligi kelib chiqadi.

6. Agar $f(x)$ funksiya $[a; b]$ kesmaning biror $x = c$ nuqtasida chegaralanmagan bo'lib, $\int_a^b |f(x)| dx$ xosmas integral yaqinlashuvchi bo'lsa, u holda $\int_a^b f(x) dx$ integral ham yaqinlashuvchi bo'ladi.

1- misol. $\int_1^{\infty} \frac{dx}{1+x^9}$ integralni yaqinlashishga tekshiring.

Echish. Integrallayotgan $f(x) = \frac{1}{1+x^9}$ funksiya integrallash oralig'ida ushbu $\varphi(x) = \frac{1}{x^9}$ funksiya kichik. Ma'lumki, $\int_1^{\infty} \frac{dx}{x^9}$ integral yaqinlashuvchi. Mos ravishda, qaralayotgan integral ham yaqinlashuvchi
Javob: yaqinlashuvchi

2- misol. $\int_0^1 \frac{dx}{\sqrt[3]{x} + 3x^2}$ integralni yaqinlashga tekshiring.

Echish. Integrallanayotgan $f(x) = \frac{1}{\sqrt[3]{x} + 3x^2}$ funksiya $[0;1]$ kesmaning $x = 0$ nuqtasida chegaralanmagan. SHuningdek $\varphi(x) = \frac{1}{\sqrt[3]{x}}$ funksiya ham $[0;1]$ kesmaning $x = 0$ nuqtasida chegaralanmagan. $[0;1]$ oraliqning barcha nuqtalarida $\frac{1}{\sqrt[3]{x} + 3x^2} < \frac{1}{\sqrt[3]{x}}$ shart bajariladi. Ma'lumki, $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ xosmas integral yaqinlashuvchi. Mos ravishda, qaralayotgan integral ham yaqinlashuvchi.

Javob: yaqinlashuvchi.

5-topshiriq. Xosmas integrallarni solishtirish belgilaridan foydalanib yaqinlashishga tekshiring.

$$1. \int_1^{\infty} \frac{dx}{x^3(1+e^x)}$$

$$2. \int_1^{\infty} \frac{dx}{x^4(1+e^x)}$$

$$3. \int_1^{\infty} \frac{dx}{1+x^{10}}$$

$$4. \int_1^{\infty} \frac{dx}{1+x^{11}}$$

$$5. \int_1^{\infty} \frac{1+x^2}{x^3}$$

$$6. \int_1^{\infty} \frac{e^{-x^2}}{x^2}$$

$$7. \int_1^{\infty} \frac{\ln(1+x)}{x} dx$$

$$8. \int_0^1 \frac{\cos^2 x}{\sqrt{1-x^2}}$$

$$9. \int_1^{\infty} \frac{\sin x}{x^4} dx$$

$$10. \int_1^{\infty} \frac{\cos x}{x^5} dx$$

$$11. \int_0^1 \frac{1}{\sqrt[4]{x} + 4x^3} dx$$

$$12. \int_0^1 \frac{dx}{\sqrt[5]{x} + 5x^2}$$

$$13. \int_{\pi}^{\infty} \frac{\sin x}{x^2} dx$$

$$14. \int_{\pi}^{\infty} \frac{\cos x}{x^4} dx$$

$$15. \int_0^{\infty} \frac{xdx}{\sqrt{x^3+1}}$$

$$16. \int_0^{\infty} \frac{xdx}{\sqrt{x^7+1}}$$

$$17. \int_3^{\infty} \frac{dx}{\sqrt[3]{x^3-2}}$$

$$18. \int_4^{\infty} \frac{dx}{\sqrt[3]{x^3-3}}$$

$$19. \int_1^{\infty} \frac{e^{-x^3}}{x^3} dx$$

$$20. \int_2^{\infty} \frac{xdx}{\sqrt{x^4+1}}$$

$$21. \int_1^{\infty} \frac{e^{-x^4}}{x^4} dx$$

$$22. \int_1^{\infty} \frac{xdx}{\sqrt{x^6+1}}$$

$$23. \int_1^{\infty} \frac{\sin 2x}{x^3} dx$$

$$24. \int_1^{\infty} \frac{\cos 2x}{x^5} dx$$

$$25. \int_0^1 \frac{dx}{\sqrt[3]{x} + 5x^4}$$

$$26. \int_0^1 \frac{dx}{\sqrt[5]{x} + 3x^3}$$

$$27. \int_0^{\infty} \frac{x \arctg x}{\sqrt{1+x^3}}$$

$$28. \int_1^{\infty} \frac{dx}{1+x^{12}}$$

$$29. \int_1^{\infty} \frac{dx}{x^5(1+e^{3x})}$$

$$30. \int_1^{\infty} \frac{\sin 3x}{x^4} dx$$

5§. Aniq integral yordamida tekis figuralarning yuzalarini hisoblash.

1⁰. Ushbu $\{y = f(x) (f(x) \geq 0); y = 0; x = a; x = b\}$ chiziqlar bilan chegaralangan egri chiziqli trapesiyaning yuzi (1-shakl)

$$S = \int_a^b f(x) dx$$

formula bilan topishdi.

2-shakl.

2⁰. Ushbu $\{y = f(x), y = \varphi(x); (\varphi(x) \leq f(x)), x = a, x = b\}$ chiziqlar bilan chegaralangan tekis figuraning yuzi (2-shakl.)

$$S = \int_a^b (f(x) - \varphi(x)) dx$$

formula bilan topiladi.

3⁰. Ushbu $y = \varphi(t), x = \varphi(t), t \in [\alpha; \beta]$ parametrik tenglama bilan berilgan egri chiziq orqali chegaralanmagan tekis figuraning yuzi

$$S = \int_a^b \varphi(t) \cdot \varphi'(t) dt$$

formula bilan topiladi.

4⁰. Qutb koordinata sistemasida ushbu $\rho = \rho(\theta), \theta \in [\theta_1; \theta_2]$ tenglama bilan berilgan chiziq orqali chegaralangan egri chiziqli sektorning yuzi

$$S = \frac{1}{2} \int_{\theta_1}^{\theta_2} \rho^2(\theta) d\theta$$

Formula bilan topiladi.

1-misol. Ushbu $y = x^3$ va $y = \sqrt{x}$ chiziqlar bilan chegaralangan tekis figuraning yuzini hisoblang.

Echish. Ko'rinib turibdiki, qaralayotgan chiziqlar $(0;0)$ va $(1;1)$ nuqtalarda kesishadi (3-shakl.)

3-shakl.

Natijada $a = 0; b = 1; \varphi(x) = x^3; f(x) = \sqrt{x}$ ga ega bo'lamiz. Mos formuladan foydalansak, tekis figuraning yuzi quyidagicha topiladi:

$$S = \int_0^1 (\sqrt{x} - x^3) dx = \left(\frac{2\sqrt{x^3}}{3} - \frac{x^4}{4} \right) = \frac{5}{12} \text{ yu.b.}$$

Javob: $\frac{5}{12}$ yuza birlik. (yu.b.)

2-misol. Ushbu $x = 2(t - \sin t), y = 2(1 - \cos t)$ sikloidaning bitta arkasi va ox o'qi bilan chegaralangan tekis figuraning yuzini toping.
Echish. SHakliga va formulaga mos asosan:

$$\alpha = 0$$

$$\beta = 2\pi$$

$$\psi(t) = 2(1 - \cos t)$$

$$\varphi(t) = 2(t - \sin t)$$

$$\varphi'(t) = 2(1 - \cos t)$$

4-shakl.

Natijada, qaralayotgan tekis figuraning yuzi quyidagicha topiladi:

$$S = \int_0^{2\pi} (2(1 - \cos t))^2 dt = 4 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) dt = 4 \int_0^{2\pi} \left(\frac{3}{2} - 2 \cos t + \frac{1}{2} \cos 2t \right) dt =$$

$$4 \left(\frac{3}{2} t - 2 \sin t + \frac{1}{4} \sin 2t \right) = 12\pi$$

yu.b.

Javob: 12π yu.b.

3-misol. Bernulli lemniskatasi $\rho^2 = 4^{\theta} \cos 2\theta$ bilan chegaralangan tekis figuraning yuzini hisoblang.

Echish: Simmetriyaga ko'ra, qidirilayotgan yuzaning to'rttdan bir qismini topamiz.

5-shakl

SHaklga va mos formulaga asosan, qaralayotgan tekis figuraning yuzi quyidagicha topiladi:

$$\frac{1}{4}S = \frac{1}{2} \int_0^{\pi/4} 4 \cos 2\theta d\theta = \sin 2\theta = 1 \quad \text{bundan } S = 4 \text{ tob.}$$

Javob: 4 yu.b.

6-topshiriq. Quyidagi chiziqlar bilan chegaralangan tekis figuraning yuzalarini hisoblang.

1. $y = x^3; y = 1; x = 2$
2. $y = \cos x; y = 0; x = -\frac{\pi}{4}; x = \frac{\pi}{4}$
3. $y = (x - 3)^3; y = 9x - 27$
4. $y = \sqrt{x}; y = 2; y = 9$
5. $y = x^4; y = x$
6. $y = 9 - x^2; y = x^2 - 3x$
7. $y = \frac{5}{x}; y = 6 - x$
8. $y = x\sqrt{4 - x^2}; y = 0; x = 0; x = 2$
9. $y = \sqrt{9 - x^2}; y = 0; x = 0; x = 2$
10. $y = \cos x \cdot \sin^2 x; y = 0; x = 0; x = \frac{\pi}{2}$
11. $y = \sin x \cos^2 x; y = 0; x = 0; x = \frac{\pi}{2}$
12. $y = x(x - 1)(x - 2); y = 0$
13. $y = \operatorname{tg} xy = 0; x = 0; x = \frac{\pi}{3}$
14. $y = \arccos x; y = 0; x = 0$
15. $y = (x + 1)^2; y^2 x + 1$
16. $y = 2x - x^2 + 3; y = x^2 - 4x + 3$
17. $y = x^2 \sqrt{8 - x^2}; y = 0; x = 0; x = 2\sqrt{2}$
18. $y = -x^2; x + y + z = 0$
19. $y^2 = 4x^3; y = 2x^2$
20. $y = \frac{16}{x^2}; y = 17 - x^2$ (I chorakda)
21. $xy = 20; x^2 + y^2 = 41$ (I chorakda)
22. $y = \sin x; y = \cos x; x = 0$
23. $y = 0,25x^2; y = 3x - 0,5x^2$
24. $y = x^3; y = 8; x = 0$
25. $y = 2x - x^2; y = -x$
26. $y = \frac{x^2}{2}; y = 2x$

$$27. y = \frac{x^2}{3}; y = 4 - \frac{2}{3}x^2$$

$$28. y = \sqrt{e^x - 1}; y = 0; x = \ln 2$$

$$29. y = x^2 \cos x; y = 0, x = 0, x = \frac{\pi}{2}$$

$$30. y = \arctg x; y = 0; x = \sqrt{3}$$

7- topshiriq. Parametrik tenglamalari bilan berilgan chiziqlar orqali chegaralangan tekis figuralarning yuzalarini toping.

$$1. \begin{cases} x = 3(t - \sin t) \\ y = 3(1 - \cos t) \end{cases} \\ y = 3(0 < x < 6\pi, y \geq 3)$$

$$2. \begin{cases} x = 5(t - \sin t) \\ y = 5(1 - \cos t) \end{cases} \\ y = 5(0 < x < 10\pi, y \geq 5)$$

$$3. \begin{cases} x = 8 \cos^3 t \\ y = 8 \sin^3 t \end{cases}$$

$$4. \begin{cases} x = 27 \cos^3 t \\ y = 27 \sin^3 t \end{cases}$$

$$5. \begin{cases} x = 2 \cos t \\ y = 6 \sin t \end{cases} \\ y = 3(y \geq 3)$$

$$6. \begin{cases} x = 6 \cos t \\ y = 2 \sin t \end{cases} \\ y = \sqrt{3}(y \geq \sqrt{3})$$

$$7. \begin{cases} x = 4(t - \sin t) \\ y = (1 - \cos t) \end{cases} \\ y = 4(0 < x < 8\pi, y \geq 4)$$

$$8. \begin{cases} x = 6(t - \sin t) \\ y = 6(1 - \cos t) \end{cases} \\ y = 6(0 < x < 12\pi, y \geq 6)$$

$$9. \begin{cases} x = 4\sqrt{2} \cos^3 t \\ y = 2\sqrt{2} \sin^3 t \end{cases} \\ x = 2(x \geq 2)$$

$$10. \begin{cases} x = 2\sqrt{2} \cos^3 t \\ y = \sqrt{2} \sin^3 t \end{cases} \\ x = 1(x \geq 1)$$

$$11. \begin{cases} x = 7(t - \sin t) \\ y = 7(1 - \cos t) \end{cases} \\ y = 7(0 < x < 14\pi, y \geq 7)$$

$$12. \begin{cases} x = 8(t - \sin t) \\ y = 8(1 - \cos t) \end{cases} \\ y = 8(0 < x < 16\pi : y \geq 8)$$

$$13. \begin{cases} x = 3 \cos t \\ y = 8 \sin t \end{cases}$$

$$x = 3\sqrt{3} (x \geq 3\sqrt{3})$$

$$14. \begin{cases} x = 32 \cos^3 t \\ y = \sin^3 t \end{cases}$$

$$x = 4 (x \geq 4)$$

$$15. \begin{cases} x = 3 \cos t \\ y = 8 \sin t \end{cases}$$

$$x = 3\sqrt{3} (x \geq 3\sqrt{3})$$

Qutb koordinata sistemasida berilgan chiziqlar bilan chegaralangan egri chiziqli sektorning yuzini toping.

$$16. \rho = \cos \theta; \rho = 2 \cos \theta$$

$$17. \rho = 4 \cos 2\theta$$

$$18. \rho^2 = 9 \sin 4\theta$$

$$19. \rho = 3 \sin 3\theta$$

$$20. \rho = 4 \sin 2\theta$$

$$21. \rho = 2 + \cos \theta$$

$$22. \rho = 7 \sin \theta; \rho = 5 \sin \theta$$

$$23. \rho = 2 \sin \theta; \rho = 4 \sin \theta$$

$$24. \rho = 3 \cos 6\theta$$

$$25. \rho = 16 \sin 3\theta$$

$$26. \rho = 9 \cos 3\theta$$

$$27. \rho = 5 \sin \theta; \rho = 3 \sin \theta$$

$$28. \rho = \sqrt{3} \cos \theta; \rho = \sin \theta \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$29. \rho = \cos \theta - \sin \theta$$

$$30. \rho = 25 \cos 4\theta$$

6. Aniq integral yordamida tekis egri chiziq yoyining uzunligini hisoblash.
 1^0 . $[a; b]$ kesmada birinchi tartibli hosilasi uzluksiz (silliqlik) bo'lgan $f(x)$ egri chiziqning shu kesmaga mos bo'lgan yoyining uzunligi

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

formula bilan topiladi.

2^0 . Ushbu $y = \psi(t)$, $x = \varphi(t)$, $\alpha \leq t \leq \beta$ parametrik tenglama bilan berilgan silliq egri chiziqning $[a; \beta]$ kesmaga mos yoyini uzunligi

$$L = \int_a^{\beta} \sqrt{\dot{\psi}^2 + \dot{\varphi}^2} dt$$

formula yordamida topiladi.

3⁰. Qutb koordinata sistemasida $\rho = \rho(\theta)$ $\theta_1 \leq \theta \leq \theta_2$ tenglama bilan berilgan silliq egri chiziqning $[\theta_1; \theta_2]$ kesmaga mos yoyini uzunligi

$$L = \int_{\theta_1}^{\theta_2} \sqrt{\rho^2 + (\rho')^2} d\theta$$

formula yordamida hisoblanadi.

1-misol. Ushbu $x^{2/3} + y^{2/3} = 8^{2/3}$ astroidaning uzunligini hisoblang.

Echish. Oshirmas funksiyaning hosilasiga ko'ra $y' = -\frac{y^{1/3}}{x^{1/3}}$ topiladi.

SHakldagi simmetriyaga ko'ra astroida uzunligining to'rdan bir qismini hisoblaymiz.

$$\frac{1}{4}L = \int_0^8 \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} dx = \int_0^8 \frac{2}{x^{1/3}} dx = 3\sqrt[3]{x^2} = 3 \cdot 4 = 12$$

u.b.

Javob: 48 uzunlik birlik (u.b.)

2-misol. Parametrik tenglamalari

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$$

bilan berilgan sikloida (4-shakl) ning bitta ariosini uzunligini toping.

$$x = 0; \beta = 2\pi$$

$$\varphi(t) = t - \sin t$$

Echish. SHartga ko'ra:

$$\psi(t) = 1 - \cos t$$

$$\varphi'(t) = 1 - \cos t$$

formulaga ko'ra:

$$L = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} dt = 2 \int_0^{2\pi} \sin \frac{t}{2} dt = -4 \cos \frac{t}{2} = -4(\cos \pi - \cos 0) = 8 \quad \text{u.b.}$$

Javob: 8 u.b.

3-misol. Qutb koordinata sistemasida berilgan $\rho(\theta) = \sin^3 \frac{\theta}{3}$ $0 \leq \theta \leq 3\pi$ chiziqning uzunligini hisoblang.

Yechish. $\rho'(\theta) = 3 \sin^2 \frac{\theta}{3} \cdot \cos \frac{\theta}{3} \cdot \frac{1}{3} = \sin^2 \frac{\theta}{3} \cdot \cos \frac{\theta}{3}$

formulaga asosan: $L = \int_0^{3\pi} \sqrt{\sin^6 \frac{\theta}{3} + \sin^4 \frac{\theta}{3} \cos^2 \frac{\theta}{3}} d\theta = \int_0^{3\pi} \sin^2 \frac{\theta}{3} d\theta =$

$$= \int_0^{3\pi} \frac{1 - \cos \frac{2\theta}{3}}{2} dt = \frac{1}{2} \left(\theta - \frac{3}{2} \sin \frac{2\theta}{3} \right) = \frac{3\pi}{2} \quad \text{u.b.}$$

Javob: $\frac{3\pi}{2}$ u.b.

8-topshiriq. Dekart koordinata sistemasida berilgan tekis silliq chiziqlarning mos yoyi uzunligini hisoblang.

1. $y = \ln x; \sqrt{5} \leq x \leq \sqrt{15}$

2. $y = \frac{x^2}{4} - \frac{\ln x}{2}; 1 \leq x \leq e$

3. $y = e^x; 0 \leq x \leq 1$

4. $y = \arcsin x; 0 \leq x \leq 1$

5. $y = e^x; 1 \leq x \leq e$

6. $y = \ln \cos x + 3; 0 \leq x \leq \frac{\pi}{4}$

7. $y = 1 - \ln(x^2 - 1); 2 \leq x \leq 3$

8. $y = 1 + \arcsin x - \sqrt{1 - x^2}; 0 \leq x \leq 3/4$

9. $y = \ln 5 - \ln x; \sqrt{3} \leq x \leq \sqrt{7}$

10. $y = \sqrt{1 - x^2} - \arccos x; 0 \leq x \leq 1$

11. $y = e^x + 5; 0 \leq x \leq 1$

12. $y = 1 - \ln \sin x; \pi/4 \leq x \leq \frac{\pi}{3}$

13. $y = e^x + 10; 0 \leq x \leq 1$

14. $y = \ln x; \sqrt{3} \leq x \leq \sqrt{8}$

15. $y = \arcsin(e^{-x}); 0 \leq x \leq 1$

16. $y = 2\sqrt{x}; 0 \leq x \leq 1$

17. $y = \ln \sin x; \frac{\pi}{3} \leq x \leq \frac{\pi}{2}$

18. $y = \frac{2}{5}x^4\sqrt{x} - \frac{2}{3}\sqrt[4]{x^3}; 0 \leq x \leq 1$

$$19. y = \frac{x^2}{2} \quad 0 \leq x \leq 1$$

$$20. y^2 = x^3 \quad 0 \leq x \leq 1 \quad (y \geq 0)$$

$$21. y = \ln(1 - x^2) \quad 0 \leq x \leq \frac{1}{2}$$

$$22. y = \ln \frac{e^x + 1}{e^x - 1} \quad 0 \leq x \leq 1$$

$$23. y = \arccos \sqrt{x} \quad 0 \leq x \leq \frac{1}{2}$$

$$24. y = \frac{1}{2}(1 - e^x - e^{-x}) \quad 0 \leq x \leq 3$$

$$25. y = \frac{1}{4}(e^{2x} + e^{-2x} + 3) \quad 0 \leq x \leq 2$$

$$26. y = \operatorname{ch} x + 3 \quad 0 \leq x \leq 1$$

$$27. y = 2 + \operatorname{cl} x \quad 0 \leq x \leq 1$$

$$28. y = \ln(1 - x^2) \quad 0 \leq x \leq \frac{1}{4}$$

$$29. y = e^x + 25 \quad \ln \sqrt{8} \leq x \leq \ln \sqrt{24}$$

$$30. y = \sqrt{x - x^2} - \arccos \sqrt{x} + 5 \quad 1/9 \leq x \leq 1$$

9- topshiriq. Parametrik tenglamalari bilan berilgan tekis silliq egri chiziqning mos yoyi uzunligi topilsin.

$$1. x = \frac{1}{3}t^3 - t, \quad y = t^2 + 2 \quad 0 \leq t \leq 3$$

$$2. x = e^t \cos t, \quad y = e^t \sin t \quad 0 \leq t \leq \ln \pi$$

$$3. x = 8 \sin t + 6 \cos t, \quad y = 6 \sin t - 8 \cos t \quad 0 \leq t \leq \frac{\pi}{2}$$

$$4. x = 3(t - \sin t), \quad y = 3(1 - \cos t) \quad 0 \leq t \leq \pi$$

$$5. x = 5(2 \cos t - \cos 2t), \quad y = 3(2 \sin t - \sin 2t) \quad 0 \leq t \leq 2\pi$$

$$6. x = 6(\cos t + t \sin t), \quad y = 6(\sin t - t \cos t) \quad 0 \leq t \leq 2$$

$$7. x = 9 \cos^3 t, \quad y = 9 \sin^3 t \quad 0 \leq t \leq \frac{\pi}{2}$$

$$8. x = e^t(\cos t + \sin t), \quad y = e^t(\cos t - \sin t)$$

$$9. x = 7(t - \sin t), \quad y = 7(1 - \cos t) \quad \pi \leq t \leq 2\pi$$

$$10. x = 8(\cos t + t \sin t), \quad y = 8(\sin t - t \cos t) \quad 0 \leq t \leq \frac{\pi}{3}$$

$$11. x = 5 \cos^3 t, \quad y = 5 \sin^3 t \quad 0 \leq t \leq \frac{\pi}{3}$$

$$12. x = 4(t - \sin t), \quad y = 4(1 - \cos t) \quad 0 \leq t \leq \frac{\pi}{2}$$

$$13. x = (t^2 - 2)\sin t + 2t \cos t, \quad y = (2 - t^2)\cos t + 2t \sin t \quad 0 \leq t \leq 2\pi$$

14. $x = e^t (\cos t + \sin t), \quad y = e^t (\cos t - \sin t) \quad 0 \leq t \leq \frac{\pi}{6}$
15. $x = 12 \cos^3 t, \quad y = 12 \sin^3 t \quad 0 \leq t \leq \frac{\pi}{3}$
16. $x = 13 (\cos t + t \sin t), \quad y = 13 (\sin t - t \cos t) \quad 0 \leq t \leq \frac{\pi}{4}$
17. $x = (t^2 - 2) \sin t + 2t \cos t, \quad y = (2 - t^2) \cos t + 2t \sin t \quad 0 \leq t \leq \pi$
18. $x = e^t (\cos t + \sin t), \quad y = e^t (\cos t - \sin t) \quad 0 \leq t \leq 2\pi$
19. $x = 13 \cos^3 t, \quad y = 13 \sin^3 t \quad 0 \leq t \leq \frac{\pi}{2}$
20. $x = 14 (\cos t + t \sin t), \quad y = 14 (\sin t - t \cos t), \quad 0 \leq t \leq \pi$
21. $x = (t^2 - 2) \sin t + 2t \cos t, \quad y = (2 - t^2) \cos t + 2t \sin t \quad \frac{\pi}{6} \leq t \leq \frac{\pi}{3}$
22. $x = 15 (t - \sin t), \quad y = 15 (1 - \cos t) \quad 0 \leq t \leq \frac{\pi}{2}$
23. $x = 16 (\cos t + t \sin t), \quad y = 16 (\sin t - t \cos t) \quad 0 \leq t \leq \frac{\pi}{2}$
24. $x = \frac{1}{2} \cos t - \frac{1}{4} \cos 2t, \quad y = \frac{1}{2} \sin t - \frac{1}{4} \sin 2t \quad \frac{\pi}{6} \leq t \leq \frac{\pi}{3}$
25. $x = 17 \cos^3 t, \quad y = 17 \sin^3 t \quad \pi \leq t \leq \frac{3\pi}{2}$
26. $x = (t^2 - 2) \sin t + 2t \cos t, \quad y = (2 - t^2) \cos t + 2t \sin t \quad \frac{\pi}{2} \leq t \leq \pi$
27. $x = \frac{1}{3} \cos t - \frac{1}{6} \cos 2t, \quad y = \frac{1}{3} \sin t - \frac{1}{6} \sin 2t \quad \frac{\pi}{3} \leq t \leq \frac{\pi}{2}$
28. $x = 18 (\cos t + t \sin t), \quad y = 18 (\sin t - t \cos t) \quad 0 \leq t \leq \pi$
29. $x = 19 \cos^3 t, \quad y = 19 \sin^3 t \quad \pi \leq t \leq 2\pi$
30. $x = 8 \sin t + 6 \cos t, \quad y = 6 \sin t - 8 \cos t \quad \frac{\pi}{2} \leq t \leq \pi$

10-topshiriq. Qutb koordinata sistemasida berilgan silliq tekis egri chiziqli yoyining uzunligini hisoblang.

1. $\rho = \theta^2 \quad 0 \leq \theta \leq \pi$
2. $\rho = 5 \sin \theta \quad 0 \leq \theta \leq \frac{\pi}{2}$
3. $\rho = 3 \cos^3 \left(\frac{\theta}{3} \right), \quad 0 \leq \theta \leq \frac{\pi}{2}$
4. $\rho = 4(1 - \cos \theta), \quad 0 \leq \theta \leq \frac{\pi}{3}$
5. $\rho = 5(1 + \sin \theta), \quad 0 \leq \theta \leq \frac{\pi}{2}$

$$6. \rho = 7(1 - \cos \theta), \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$$

$$7. \rho = 5e^{\frac{3\theta}{4}}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$8. \rho = 3e^{\frac{4\theta}{3}}, \quad 0 \leq \theta \leq \frac{\pi}{3}$$

$$9. \rho = 7e^{\frac{4\theta}{5}}, \quad 0 \leq \theta \leq \frac{\pi}{6}$$

$$10. \rho = 8 \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{3}$$

$$11. \rho = 6 \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{6}$$

$$12. \rho = 13 \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{4}$$

$$13. \rho = 12 \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$14. \rho = 5\theta, \quad 0 \leq \theta \leq \frac{4}{5}$$

$$15. \rho = 7\theta, \quad 0 \leq \theta \leq \frac{5}{7}$$

$$16. \rho = 8(1 + \sin \theta), \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$17. \rho = 9(1 - \cos \theta), \quad -\frac{\pi}{6} \leq \theta \leq 0$$

$$18. \rho = 13e^{\frac{7\theta}{8}}, \quad 0 \leq \theta \leq \frac{\pi}{6}$$

$$19. \rho = 12e^{\frac{8\theta}{7}}, \quad 0 \leq \theta \leq \frac{\pi}{4}$$

$$20. \rho = 14 \cos \theta, \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{4}$$

$$21. \rho = 15 \sin \theta, \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

$$22. \rho = 13\theta, \quad 0 \leq \theta \leq \frac{9}{10}$$

$$23. \rho = 17\theta, \quad 0 \leq \theta \leq \frac{13}{14}$$

$$24. \rho = 19e^{\frac{3\theta}{8}}, \quad 0 \leq \theta \leq \frac{\pi}{4}$$

$$25. \rho = 18 \sin \theta, \quad \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$$

$$26. \rho = 1 - \cos \theta, \quad \frac{\pi}{2} \leq \theta \leq \pi$$

$$27. \rho = 1 + \sin \theta, \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

$$28. \rho = 22 \sin \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$$

$$29. \rho = 25 \cos \theta, \quad \frac{\pi}{3} \leq \theta \leq \frac{\pi}{4}$$

7§. Aniq integral yordamida hajm hisoblash.

1⁰. Agar jismni ox o'qiga perpendikulyar kesimining yuzi $Q(x)$ ma'lum bo'lsa, u holda jismning $x = a$ va $x = b$ tekisliklar orasidagi qismini hajmi

$$V = \int_a^b Q(x) dx$$

Formula bilan topiladi.

2⁰. Ushbu $\{y = f(x), y = 0, x = a, x = b\}$ chiziqlar bilan chegaralangan egri chiziqli trapesiyaning ox o'qi atrofida aylanishdan hosil bo'lgan jismning hajmi

$$V_{ox} = \pi \int_a^b (f(x))^2 dx$$

formula bilan hisoblaniladi.

3⁰. Ushbu $\{x = \varphi(y), x = 0, y = c, y = d\}$ chiziqlar bilan chegaralangan egri chiziqli trapesiyaning oy o'qi atrofida aylanishdan hosil bo'lgan jismning hajmi

$$V_{oy} = \pi \int_c^d (\varphi(y))^2 dy$$

formula bilan topiladi.

Misol. $y^2 = (x-1)^3$ va $x = 3$ chiziqlar bilan chegaralangan tekis figuraning ox o'qi atrofida aylanishdan hosil bo'lgan jismning hajmi topilsin. Echish.

SHaklga va mos formulaga asosan izlayotgan hajm quyidagicha topiladi.

$$V = \pi \int_1^3 (x-1)^2 dx = \frac{1}{4} \pi (x-1)^4 \Big|_1^3 = \frac{1}{4} \pi \cdot 16 = 4\pi \text{ hajm birlik.}$$

Javob. 4π h.b.

11- topshiriq. Berilgan chiziqlar bilan chegaralagan figuralarni ox o'qi atrofida aylanishdan hosil bo'lgan jismning hajmini toping.

$$1. y^2 = x, x^2 = y$$

$$2. y = \sqrt{x} e^x, x = 1, y = 0$$

$$3. y = \frac{1}{2}x^2, y = \frac{1}{8}x^3,$$

$$4. y = \frac{64}{x^2 + 16}, x^2 = 8y$$

$$5. y = \sin^2 x, y = 0, x = 0, x = \pi$$

$$6. y = -x^2 + 3x - 2, y = 0$$

$$7. y = 5 \sin x, y = \sin x, x = 0, x = \pi$$

$$8. y = xe^x, y = 0, x = 1$$

$$9. y = 4 - x^2, y = 0$$

$$10. y = 2x - x^2, y = 2 - x, x = 0$$

$$11. y = x^3, y = \sqrt{x}$$

$$12. y = -x^2 + 4x - 3, y = 0$$

$$13. y = -x^2 + 6x - 5, y = 0$$

$$14. y = 3 \cos x, y = \cos x, x = 0, x \geq 0$$

$$15. y = \sin x, y = 0, x = 0, x = \frac{\pi}{2}$$

Berilgan chiziqlar bilan chegaralangan figuralarni o'y o'qi atrofida aylanishdan hosil bo'lgan jismning hajmini toping.

$$16. y = x^2, y = \sqrt{x}$$

$$17. y = x^3, y = x^2$$

$$18. y = \arccos \left(\frac{x}{4} \right), y = \arccos x, y = 0$$

$$19. y = \arcsin \left(\frac{x}{3} \right), y = \arcsin x, y = \frac{\pi}{2}$$

$$20. y = x^2 - 4x + 4, x = 3, y = 0$$

$$21. y = x^3, y = x$$

$$22. y^2 = x - 2, y = 0, y = x^3, y = 1$$

$$23. y = \arccos \frac{x}{6}, y = \arccos \frac{x}{3}, y = 0$$

$$24. y = 3 \ln x, x = 2, y = 0$$

$$25. y = \arcsin x, y = \arccos x, y = 0$$

$$26. y = y = x^2 + 1, y = x, x = 0, x = 1$$

$$27. y = \arcsin \frac{x}{7}, x = \arcsin \frac{x}{2}, y = \frac{\pi}{2}$$

$$28. y = e^x, y = 0, x = 0$$

$$29. y^2 x^3, y = 0, x = 1$$

$$30. \frac{x^2}{4} + \frac{y^2}{16} = 1$$

8. Aniq integralni aylanma sirt yuzalarini, tekis figuralarning statik va inersiya momentlarini hamda og'irlik markazining koordinatalarini topishga tadbiqu.

1⁰. Silliqliq egri $y = f(x)$ chiziqning $[a; b]$ kesmaga mos yoyini ox o'qi atrofida aylanishdan hosil bo'lgan aylanma sirtning yuzi

$$S = 2\pi \int_a^b y \sqrt{1 + y'^2} dx$$

Formula bilan topiladi.

2⁰. Agar egri chiziq ushbu $x = \varphi(t); y = \psi(t) t \in [\alpha; \beta]$ parametrik tenglamalari bilan berilgan bo'lsa, u holda aylanma sirt yuzi.

$$S = 2\pi \int_{\alpha}^{\beta} \psi(t) \sqrt{\varphi'^2 + \psi'^2} dt$$

Formula bilan hisoblaniladi.

3⁰. Agar xoy koordinata tekisligida $m_1, m_2, \dots, m_n$ massalariga ega bo'lgan $A_1(x_1; y_1), A_2(x_2; y_2), \dots, A_n(x_n; y_n)$ moddiy nuqtalar sistemasi berilgan bo'lsa, u holda

$$M_x = \sum_{k=1}^n m_k y_k, M_y = \sum_{k=1}^n m_k x_k$$

$$I_x = \sum_{k=1}^n m_k y_k^2, I_y = \sum_{k=1}^n m_k x_k^2$$

ifodalarga sistemaning ox va oy o'qlariga nisbatan statik va inersiya momentlari deyiladi.

Mos ravishda, $y = f(x)$ tekis egri chiziqning $[a; b]$ kesmadagi yoyini statik va inersiya momentlari quyidagicha topiladi

$$M_x = \int_a^b y dL \quad M_y = \int_a^b x dL$$

$$I_x = \int_a^b y^2 dL \quad I_y = \int_a^b x^2 dL$$

bu erda $dL = \sqrt{1 + y'^2} dx$ - egri chiziq yoyining differensial.

4⁰. Ushbu $\{y = f(x), y = 0; x = a; x = b\}$ chiziqlar bilan chegaralangan egri chiziqli trapesiyaning statik va inersiya momentlari quyidagicha hisoblaniladi:

$$M_x = \frac{1}{2} \int_a^b y^2 dx \quad M_y = \frac{1}{2} \int_a^b xy dx$$

$$I_x = \frac{1}{3} \int_a^b y^3 dx \quad I_y = \frac{1}{2} \int_a^b x^2 y dx$$

5⁰. $y = f(x)$ egri chiziqning $[a; b]$ kesmaga mos yoyini og'irlik markazining koordinatalari quyidagicha topiladi:

$$x_c = \frac{1}{L} \int_a^b x dL \quad y_c = \frac{1}{L} \int_a^b y dL$$

bu erda: $dL = \sqrt{1 + y'^2} dx$, L - yoy uzunligi

6⁰. Ushbu $\{y = f(x), y = 0; x = a; x = b\}$ chiziqlar bilan chegaralangan egri chiziqli trapesiyaning og'irlik markazini koordinatalari quyidagicha hisoblaniladi.

$$x_c = \frac{1}{S} \int_a^b xy dx \quad y_c = \frac{1}{2S} \int_a^b y^2 dx$$

bu erda S - egri chiziqli trapesiyaning yuzi

1- misol. Ushbu $y = \frac{1}{3} \sqrt{x(3-x)}$ egri chiziqning $[0; 3]$ kesmaga mos yoyning

o'qi atrofida aylanishdan hosil bo'lgan aylanma sirt yuzini toping.

Echish. YOyning differensialini aniqlaymiz:

$$dL = \sqrt{1 + y'^2} dx = \frac{x+1}{2\sqrt{x}} dx$$

Mos formulaga ko'ra:

$$S = 2\pi \int_0^3 \frac{1}{3} (3-x) \cdot \sqrt{x} \cdot \frac{x+1}{2\sqrt{x}} dx = \frac{\pi}{3} \int_0^3 (3-x)(x+1) dx = 3\pi$$

Javob: 3π yuza birlik.

2-misol. $x = 2 \cos t, y = 3 \sin t$ ellips yuzasining oy o'qiga nisbatan inersiya momentini toping.

Echish.

$$I_y = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 y dx = 4 \int_{-\frac{\pi}{2}}^0 4 \cos^2 t \cdot 3 \sin t \cdot (-2 \sin t) dt = -96 \int_{-\frac{\pi}{2}}^0 \sin^2 t \cos^2 t dt = -24 \int_{-\frac{\pi}{2}}^0 \sin^2 2t dt =$$

$$= -12 \int_{-\frac{\pi}{2}}^0 (1 - \cos 4t) dt = -12 \left(t - \frac{1}{4} \sin 4t \right) \Big|_{-\frac{\pi}{2}}^0 = 6\pi$$

Javob 6π

3-misol. $x^2 + y^2 = 4$ ($y \geq 0$) yarim aylana yoyining og'irlik markazi koordinatalarini toping.

Echish. Quyidagilarni topamiz:

$$y = \sqrt{4 - x^2} \quad y' = -\frac{x}{\sqrt{4 - x^2}}$$

$$dL = \sqrt{1 + y'^2} dx = \frac{2 dx}{\sqrt{4 - x^2}}$$

Natijada:

$$x_c = \frac{1}{2\pi} \int_{-2}^2 x dL = \frac{1}{2\pi} \int_{-2}^2 \frac{2x dx}{\sqrt{4 - x^2}} = -\frac{1}{2\pi} \sqrt{4 - x^2} \Big|_{-2}^2 = 0$$

$$y_c = \frac{1}{2\pi} \int_{-2}^2 y dL = \frac{1}{2\pi} \int_{-2}^2 \sqrt{4 - x^2} \cdot \frac{2 dx}{\sqrt{4 - x^2}} = \frac{1}{\pi} \int_{-2}^2 dx = \frac{1}{\pi} x \Big|_{-2}^2 = \frac{4}{\pi}$$

Javob: $x_c = 0$ $y_c = \frac{4}{\pi}$

12 -topshiriq. Berilgan chiziqlarning mos yo'ylarini ox o'qi atrofida aylanishidan hosil bo'lgan aylanma sirtlarning yuzini hisoblang.

1. $y = \operatorname{tg} x$ $x \in \left[0; \frac{\pi}{4} \right]$

2. $y = \sin x$ $x \in [0; \pi]$

3. $y = x^3$ $x \in \left[0; \frac{1}{2} \right]$

4. $y = x^3$ $x \in [0; \infty]$

5. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

6. $\frac{x^2}{16} + \frac{y^2}{4} = 1$

7. $\begin{cases} x = 2 \cos t - \cos 2t \\ y = 2 \sin t - \sin 2t \end{cases}$

8. $x^2 + y^2 = 4$

9. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

10. $y = \sin x$ $x \in \left[0; \frac{\pi}{2}\right]$

11. $y = \sqrt{9 - x^2}$ yarim aylananing koordinata o'qlariga nisbatan statik momentlarini toping.

12. $y = \sqrt{16 - x^2}$ yarim aylananing koordinata o'qlariga nisbatan inersiya momentlarini toping.

13. $x = 5 \cos t$; $y = 4 \sin t$ ellips yuzasining koordinata o'qlariga nisbatan inersiya momentlarini toping.

14. $y = \frac{1}{2}(e^x + e^{-x})$ egri chiziqning $[0;1]$ kesmaga mos yoyini ox o'qiga nisbatan statik momentini toping.

15. $y = 4 - x^2$ parabola va $y = 3$ chiziq bilan chegaralangan figuraning ox o'qiga nisbatan inersiya momentini toping.

16. Tomonlari 3 va 4 ga teng bo'lgan to'g'ri to'rtburchakning simmetriya o'qlariga nisbatan inersiya momentlarini toping.

17. $y = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$ egri chiziq yoyining $[0;2]$ kesmaga mos qismini ox o'qiga nisbatan inersiya momentini toping.

18. Tomonlari 6 va 8 bo'lgan to'g'ri to'rtburchakning simmetriya o'qlariga nisbatan statik momentlarini toping.

19. Radiusi 5 ga teng bo'lgan aylananing diametriga nisbatan inersiya momentini toping.

20. $\frac{x^2}{16} + \frac{y^2}{4} = 1$ ellips yuzasining uning bosh o'qlariga nisbatan inersiya momentlarini toping.

Berilgan chiziqlarning mos yoylarining og'irlik markazi koordinatalarini toping.

21. $y = 3cl \frac{x}{3}$ $x \in [-3;3]$

22. $x = a \cos^3 t$ $y = a \sin^3 t$ $t \in \left[0; \frac{\pi}{2}\right]$

23. $y = \sqrt{16 - x^2}$ $x \in [-4;4]$

24. $y = 5cl \frac{x}{5}$ $x \in [-5;5]$

25. $y = \sqrt{36 - x^2}$ $x \in [-6;6]$

Berilgan chiziqlar bilan chegaralangan figuralarning og'irlik markazi koordinatalarini toping.

26. $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{\pi}{2}$

27. $y = 9 - x^2$, $y = 0$

28. $y = 2x - x^2, y = 0$

29. $y = \sin x, y = 0, x = 0, x = \frac{\pi}{2}$

30. $y = x^2; y = \sqrt{x}$