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Qo'lyozma huquqida

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«KASR TARTIBLI INTEGRODIFFERENTSIAL OPERATORLARNING XOSSALARI VA TADBIQLARI»

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MAGISTRLIK DISSERTATSIYASI

Ilmiy rahbar: fizika-matematika fanlari nomzodi, dotsent H.N.Qosimov

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MUNDARIJA

KIRISH

O'zbekiston Respublikasi Prezidenti Islom Karimov o'zining "O'zbekiston mustaqillikka erishish ostonasida" asarida quyidagi fikrlarni aytib o'tgan.

O'zbekiston fani salmoqli kuch-quvvatga ega. Hozir respublikamizda 185 ta ilmiy tadqiqot muassasalari, konstruktorlik byurolari va ilmiy ishlab chiqarish birlashmalari bor. Moddiy ishlab chiqarish tarmoqlarining o'zidagina 500 dan ko'proq zavod konstruktorlik byurolari, laboratoriyalari, mexanizatsiyalash va avtomatlashtirish bo'limlari va shu kabilar ishlamoqda.

Shu katta imkoniyatdan to'g'ri foydalanish, kuchlarni samarali tarzda joy-joyiga qo'yish, fundamental tadqiqotlar bilan amaliy tadqiqotlar o'rtasida eng maqbul mutanosibini topish uchun nimalar qilish kerak? Bu eng avvalo respublika ilm-fanining qarorgohi-akdemiyamizga bog'liqdir.

Olimlarimiz an'anaviy yo'nalishlar bilan bir qatorda so'nggi davrda radiatsiya va geliomaterialshunosligi, fizika va kvant elektronikasi, lazer texnikasi, nurlanish kasalligiga yo'liqqanlarni davolashda qo'llaniladigan dori-darmonlarni yaratish, inshootlarning zilzilabardoshligi, o'simlik moddalari kimyosi sohalari va boshqa sohalarda katta obro' qozondilar.

Matematika sohasidagi va fanning boshqa tarmoqlaridagi tadqiqotlar va yechimlar hammaga ma'lum.

Biohimiya instituti kollektivi tadqiqotlarini omilkorlik bilan tashkil etish, ishga ijodiy yondoshishning namunasi bo'la oladi. Institutda so'nggi yillarda erishilgan natijalar butun jahonda tan olindi. Bu yerda nufuzli yo'nalishlar tez sur'atlar bilan rivojlanmoqda. Institutda ishlab chiqarilgan preparatlar mamlakatimizda ham, chet ellarda ham sotilmoqda. Kollektivda yuzaga kelgan sog'lom muhit kadrlarning ijodiy o'sishiga yordam berayapdi.

Endilikda ilm-fan yangi yo'llar ochishi, sifat jihatidan yangi texnalogiyalarni jadallik bilan yaratishi, jamiyatning yangi holatga o'tishini tahminlashi lozim.

Fan samaradorligini sifat jihatidan oshirishga quruq da'vatlar bilangina erishish mumkin emas. Ilmiy kadrlarga munosabatlarni ham tubdan o'zgartirish, ularning ijtimoiy maqomini qat'iyan oshirish, chuqur struktura o'zgarishlari qilish zarur.

Fanni malakali kadrlar bilan ta'minlash, hodimlarning professional bilimdonligi darajasini oshirish, ularning qobiliyatlarini ro'yobga chiqarish uchun barcha sharoitlarni yaratish ilmiy jarayonni jadallashtirishning asosiy omilidir.

Men anchagina masalalar va muammolar xususida gapirdim. Bizning nazarimizda bularning hal etilishi, O'zbekiston fanining yangi marralarga chiqib olishiga, yaqin o'n yilliklarda respublika uchun belgilanayotgan ijtimoiy-iqtisodiy o'zgarishlarda tayanch bo'lishiga imkon beradi. Aytilgan fikr-mulohaza va takliflar bizga hayotimizning barcha jabhalarida fanning samaradorligini oshirish, xalq farovonligini yaxshilash uchun tadbirlar belgilab olishimizda yordam beradi deb umid qilaman.[1]

O'zbekiston Respublikasi prezidenti Islom Karimovning O'zbekistonda Ta'lim sohasini isloh qilish , yuksak bilimli va intellektual rivojlangan avlodni tarbiyalash bo'yicha amalga oshirilayotgan dasturning asosiy maqsad va vazifalari, mazmun –mohiyati qisqacha bayon etilgan nutqi xalqaro konferentsiya ishida muhim ahamiyat kasb etdi.

Bundan 15 yil oldin (1997 yilda) qabul qilingan, Kadrlar tayyorlash milliy dasturi deb nom olgan Ta'lim sohasini isloh qilish dasturi mamlakatda yangi jamiyat qurishning bosqichma-bosqich va tadrijiy rivojlanish printsipiga asoslangan iqtisodiy va siyosiy islohotlarning O'zbekiston tanlagan "o'zbek modeli"ning ajralmas tarkibiy qismi ekani alohida ta'kidlandi.

Bugun yuksak ma'naviy va ahloqiy fazilatlarga ega, bilimli, samarali mehnat qilishga qodir fuqarolar jamiyatning eng muhim boyligi va asosiy kapitali, uni xarakatga keltiruvchi kuchiga aylanib bormoqda. Aynan shu bois XXI asr – intellektual bilimlar asrida inson kapitaliga investitsiyalar yo'naltirishni ustivor

[1] I.A.Karimov, "O'zbekiston mustaqillikka erishish ostonasida" Toshkent "O'zbekiston", 2011 yil.

vazifa sifatida tanlangan mamlakatlargina yuksak taraqqiyotga erishishi mumkin. Faqat tom ma'nodagi bilimli jamiyatgina zamonaviy tahdid va muammolarni yengib o'tishga qodir bo'ladi. Har qanday davlat mamlakatning intellektual va ma'naviy salohiyatini manfaatdor bo'lmog'i kerak. Globallashuv davrida bilimlilik mamlakatning iqtisodiy rivojlantirish va uning milliy boyligini ko'paytirishning muhim tarkibiy qismiga aylanmoqda, aholining yuksak ma'naviy darajasi esa odamlarda madaniyatni, erkin va demokratik huquqiy davlatda yashash hamda mehnat qilish qobilyatini shakllantirish, o'z huquq va erkinliklarini anglash, ulardan shaxs, davlat hamda jamiyat manfaatlari yo'lida foydalanish imkonini beradi.

Jamiyatda bilimli, ma'lumotli va yuksak intelektga ega odamlarning mavqeyini oshirish, ta'lim va kasb – hunar dasturlarning jahondagi ta'lim ilm-fan,texnika va texnalogiyalar, iqtisodiyot hamda madaniyat sohasidagi zamonaviy yutuqlarga muvofiqligi; ta'limni yoshlarning kelajagidagi amaliy mexnat faoliyati bilan uzviyligini ta'minlash, ta'lim olayotgan o'quvchi – yoshlar ongida umuminsoniy qadriyatlar, yuksak ma'naviyat, madaniyat va ijodiy fikrlash ustivorligini shakillantirish, ta'limning milliy tarix, xalq an'analarini va urf – odatlari bilan mushtarakligi, boshqa xalqlar tarixi va madaniyatini hurmat qilish ta'lim tizimini isloh etishning asosiy printsiplari bo'lishi lozim. [2]

Ta'lim sohasida, ham ushbu sohada mehnat qilayotganlar, ham zamonaviy kadrlar talab qilinadigan tarmoqlar vakillari o'rtasida xalqaro hamkorlikni muntazam rivojlantirish zarur.

Biz ta'lim tizimida o'quvchilarning nafaqat keng bilim va proffessional ko'nikmalarni egallashi, ayni paytda chet mamlakatlardagi tengdoshlari bilan faol muloqot qilish, bugungi dunyoda ro'y berayotgan barcha voqea-hodisalar, yangilik va o'zgarishlardan atroflicha xabardor bo'lish, jahondagi ulkan intellektual boylikni egallashning eng muhim sharti hisoblangan xorijiy tillarni ham chuqur o'rganishlari uchun katta ahamiyat bermoqdamiz

[2]I.A.Karimov, "Yuksak ma'naviyat-yengilmas kuch" Toshkent "Ma'naviyat", 2008 yil.

O’zbekiston Respublikasi prezidenti I.A.Karimov Oliy Majlisning XIV sessiyasida so’zlagan nutqida kadrlar tayyorlashning ahamiyatiga izoh berib shunday degan edi:

«Biz oldimizga qanday vazifa qo’ymaylik, qanday muammoni yechish zaruriyati tug’ilmasin, gap oxir oqibat, baribir kadrlarga borib qadalaveradi. Mubolag’asiz aytish mumkinki, bizning kelajagimiz, mamlakatimiz kalajagi, o’rnimizga kim kelishiga yoki boshqacharoq qilib aytganda, qanday kadrlar tayyorlashimizga bog’liq.

…Mamlakatimiz kelajagi uchun Oliy Majlisning IX sessiyasida qabul qilingan «Kadrlar tayyorlash bo’yicha milliy dasturi» ning amalga oshirishi juda ham muhim ahamiyatga ega.

…Yuqori malakali pedagog kadrlar tayyorlash va qayta tayyorlashga alohida e’tibor berish lozim. Kadrlar tayyorlashning sifati, erkin fikrlovchi shaxs-fuqaroni kamol toptirishiga ertaga sinf xonalar va auditoriyalarda kimlar dars va saboq berishiga bog’liq».

Darhaqiqat, barkamol inson shaxsning shakllanishi bevosita uzluksiz ta’lim jarayonida amalga oshadi. Shunday ekan, har jabhada muvaffaqiyatga erishish, jumladan yuqori malakali kadrlar tayyorlashda milliy dasturni o’rni va ahamiyati beqiyosdir.

Kadrlar tayyorlash milliy dasturida oliy ta’limning asosiy maqsadi bozor iqtisodiyoti sharoitida mustaqil ishlashga qodir, raqobatbardosh, yuqori malakali mutaxassislar tayyorlashdan iborat. Bu maqsadga erishish uchun, shuningdek Respublikamiz prezidenti aytgani kabi «mamlakatimizning boy ilmiy – texnikaviy salohiyatidan keng foydalangan holda, yuksak texnologiya va fan yutuqlariga asoslangan ishlab chiqarish sohalari - avtomobilsozlik, samolyotsozlik, mikrobiologiya, elektrotexnika va elektronika sanoatlarini telekommunikatsiya va zamonaviy axborot texnologiya vositalarini tez sur’atlarda rivojlantirish» uchun saboq olayotgan har bir shaxs o’zi o’rgangan ta’lim mazmunini chuqur anglashi, qayerda va qanday tadbiq qilishni bilishi, hayotda esa o'zi amaliyotga tadbiq qila olishi kerak.

O'zbekiston Rekspublikasi prezidenti I.Karimovning Jaxon moliyaviy – iqtisodiy inqrozi, O'zbekiston sharoitida uni bartaraf etishning yo'llari va choralari nomli asarida jahon moliyaviy-iqtisodiy inqirozining kelib chiqish sabablari, oqibatlari va uning O'zbekiston iqtisodiyotiga ta'sirini kamaytirish yo'llari chuqur va atroflicha taxlil qilingan. Inqirozning mamlakatimiz iqtisodiyotiga ta'sirini yumshatishga qaratilgan. Inqirozga qarshi choralar dasturini ishlab chiqishga yo'naltirilgan amaliy tavsiyalar berilgan.[3]

Mavzuning dolzarbligi. Matematik analizning ixtiyoriy tartibli hosila va integrallarni o'rganish va qo'llashga bag'ishlangan sohasi funksiyalar nazariyasi, integral va differentsial tenglamalar bilan bog'liq bo'lgan ko'p yillik tarixga ega.

Hozirgi fan taraqqiyotida amaliy jarayonlarni matematik modelini tuzishda va hususiy hosilali giperbolik va aralash tipdagi tenglamalar uchun qo'yilgan chegaraviy masalalarni yechishda kasr tartibli integrodifferentsial operatorlar muhim ro'l o'ynaydi.

Kasr tartibli differentsiallash va integrallash tushunchasi odatda Liuvill nomi bilan bog'lanadi. Liuvill, Abel, Riman, Letnikov, Veyl, Adamar va boshqa mashxur matematiklar xozirgi vaqtda matematik analizni butun bir yo'nalishiga aylangan kasr tartibli integrodifferentsial operatorlarning rivojlanishiga katta xissa qo'shganlar.

A.M. Naxushevning kasr tartibli integrallar va hosilalar hamda ular yordamida siljishli masalalar yechishga bag'ishlangan ilmiy ishlari e'lon qilingandan keyin bu yo'nalish jadal sur'atlar bilan rivojlanib bormoqda.

Kasr tartibli integrodifferentsial operatorlar va ular yordamida siljishli masalalar yechish sohasida A.M. Naxushev, M.S.Saloxitdinov, A.Q.O'rinov va ularning shogirdlari samarali mehnat qilmoqdalar.

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[3]I.A.Karimov, "Jahon moliyaviy-iqtisodiy inqirozi, O'zbekiston sharoitida uni bartaraf etishning yo'llari va choralari" Toshkent "O'zbekiston", 2009 yil.

Ishning maqsadi. Giperbolik va aralash tipdagi tenglamalar uchun chegaraviy masalalarni o'rganishda kasr tartibli integrodifferentsial operatorlarning xossalarini o'rganish masalasi kelib chiqadi.

Ushbu magistrlik dissertatsiyasi kasr tartibli integrodifferentsial operatorlarning ba'zi xossalarini o'rganishga hamda bu xossalarni tor tebranish tenglamasi uchun siljishli masalalar yechish uchun tadbiq qilishga bag'ishlangan.

Tekshirish usullari. Qo'yilgan masalalar yechimining mavjudligi va yagonaligini isbotlashda integral tenglamalar nazariyasidan foydalanilgan.

Ilmiy yangiligi, nazariy va amaliy ahamiyati. Tor tebranish tenglamasi uchun chegaraviy shartida integrodifferentsial operatorlar qatnashgan siljishli masalalar qo'yilgan va ular yechimining mavjudligi hamda yagonaligi yoki yagona emasligi isbotlangan.

Dissertatsiyada olingan natijalar yangi bo'lib, nazariy harakterga ega. Ular giperbolik va aralash tipdagi tenglamalar nazariyasini rivojlantirishda qo'llanilishi mumkin.

Ishning muhokamasi. Dissertatsiyada olingan ilmiy natijalar Farg'ona davlat universiteti differentsial tenglamalar kafedrasida har xaftaning payshanba kuni A.Q.O'rinov rahbarligidagi "Differentsial tenglamalar va turdosh matematik sohalarning dolzarb muammolari" nomli seminarda, 2011 yil FarDU da o'tkazilgan "Ilm-zakovatimiz – senga, ona-Vatan!" mavzusidagi ilmiy-amaliy anjumanida, 2011 yil ADU da o'tkazilgan "Matematika fani va uni o'qitishning dolzarb muammolari" nomli Respublika ilmiy-amaliy anjumanida, 2012 yil FarDU da o'tkazilgan "Ilm-zakovatimiz – senga, ona-Vatan!" mavzusidagi Respublika ilmiy-amaliy anjumanida, FarDU magistrantlarining ilmiy amaliy anjumanlarida ma'ruza qilinib, muhokamadan o'tkazilgan.

Chop etilgan ishlar. Dissertatsiyada olingan asosiy natijalar
[17,18, 19] ilmiy maqolalar sifatida chop qilingan.

Dissertatsiyaning tuzilishi va hajmi. Magistrlik dissertatsiyasi kirish qismi, uchta bob, xulosa, foydalanilgan adabiyotlar ro'yxatidan iborat bo'lib, 72 betni tashkil qiladi.

Dissertatsiyaning mazmuni.

Birinchi bob to'rtta paragrafdan iborat bo'lib, kasr tartibli integrodifferentsial operatorlar va ularning xossalarini o'rganishga bag'ishlangan. Birinchi paragrafda birinchi va ikkinchi tur Eyler integrallari va ularning xossalari o'rganilgan.

Birinchi tur Eyler integrali (betta-funksiya).

Betta-funksiya ushbu

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx \tag{1.1}$$

tenglik bilan aniqlanadi. Bu tenglikning o'ng tomonidagi integral Eylerning birinchi tur integrali deyiladi.

Ikkinchi tur Eyler integrali (gamma-funksiya).

Gamma-funksiya ushbu

$$\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx \tag{1.8}$$

integral bilan aniqlanadi va bu integral ikkinchi tur Eyler integrali deb ataladi.

Ikkinchi paragrafda Abel integral tenglamasi o'rganilgan.

Ushbu

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}}=f(x),\quad 0<\alpha<1 \tag{1.15}$$

ko'rinishdagi integral tenglama *Abel integral tenglamasi* deyiladi.

Uchinchi va to'rttinchi paragraflarda kasr tartibli integral va differentsialllar o'rganilgan.

Kasr tartibli integrallarni quyidagicha aniqlaymiz.

Ta'rif. $\varphi(x) \in L_1(a,b)$ $(a < b < \infty)$ *bo'lsin. Ushbu*

$$D_{ax}^{-\alpha}\varphi(x) = \frac{1}{\Gamma(a)}\int_a^x (x-t)^{\alpha-1}\varphi(t)\,dt,\ \alpha > 0, \tag{1.27}$$

$$D_{bx}^{-\alpha}\varphi(x) = \frac{1}{\Gamma(a)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt,\ \alpha > 0 \tag{1.28}$$

ko'rinishdagi ifodalar Riman-Liuvill ma'nosidagi α $(0 < \alpha < \infty)$ kasr tartibli integrallar deyiladi va mos ravishda $D_{ax}^{-\alpha}\varphi(x)$ hamda $D_{xb}^{-\alpha}\varphi(x)$ kabi belgilanadi.

Ta'rif. $\varphi(x)$ funksiya [a,b] kesmada aniqlangan bo'lsin.

$$D_{ax}^{\alpha}\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{\varphi(t)dt}{(x-t)^{\alpha}}, \qquad 0 < \alpha < 1, \tag{2.1}$$

$$D_{xb}^{\alpha}\varphi(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_b^x \frac{\varphi(t)dt}{(x-t)^{\alpha}}, \qquad 0 < \alpha < 1 \tag{2.2}$$

ko'rinishdagi ifodalar Liuvill ma'nosidagi α (kasr) tartibli hosilalar deyiladi va mos ravishda $D_{ax}^{\alpha}\ \varphi(x)$ hamda $D_{xb}^{\alpha}\ \varphi(x)$ kabi belgilanadi.

Ikkinchi bob uchta paragrafdan iborat bo'lib, birinchi paragrafda tor tebranish tenglamasi uchun umumiy qo'yilgan Koshi masalasini o'rganishga bag'ishlangan.

D orqali $x, \ y$ o'zgaruvchilar tekisligida chegara bo'laklari silliq S Jordan chizig'idan iborat bo'lgan sohani belgilaymiz.

Faraz qilaylik, $u(x, y)$ funksiya D sohada

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.1}$$

tenglamaning regulyar yechimi bo'lib, $D \cup S$ da birinchi tartibli uzluksiz hosilalarga ega bo'lsin.

Ushbu

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial y_1}\left(\frac{\partial u}{\partial y_1}\right) = 0$$

ayniyatni D soha bo'yicha integrallab, Gauss-Ostrogradskiy formulasini qo'llaymiz:

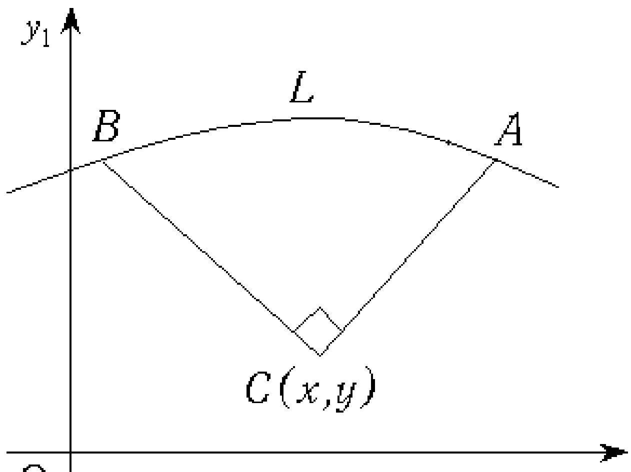
$$\int_D \left[\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial y_1}\left(\frac{\partial u}{\partial y_1}\right)\right] dx_1 dy_1 =$$

(2.2)

$$= \int_S \frac{\partial u}{\partial x_1} dy_1 + \frac{\partial u}{\partial y_1} dx_1 = 0.$$

Faraz qilaylik, L - yopiq bo'lmagan silliq Jordan chizig'i bo'lib, quyidagi ikki shartni qanoatlantirsin:

a) (2.1) tenglamaning $x + y = const, \qquad x - y = const$ xarakteristikalar oilasiga tegishli bo'lgan har bir to'g'ri chiziq L bilan bitta nuqtada kesishsin;



1-chizma

b) L egri chiziqqa o'tkazilgan urinmaning yo'nalishi hech bir nuqtada (2.1) tenglama xarakteristikalarining yo'nalishi bilan ustma-ust tushmasin.

Ixtiyoriy $C(x, y)$ nuqtadan chiqadigan $x_1 - x = y_1 - y$, $x_1 - x = y - y_1$ xarakteristikalar L egri chiziq bilan A va B nuqtalarda kesishsin (1-chizma).

(2.2) formulani AB egri chiziq, CA va CB xarakteristikalar bilan chegaralangan sohaga qo'llab, ushbu tenglikni hosil qilamiz:

$$\int\limits_{AB+BC+CA} \frac{\partial u}{\partial x_1} dy_1 + \frac{\partial u}{\partial y_1} dx_1 = 0.$$

CA va BC da, mos ravishda, $dx_1 = dy_1$ va $dx_1 = -dy_1$ bo'lgani uchun avvalgi tenglik quyidagi ko'rinishda yoziladi:

$$\int\limits_{AB} \frac{\partial u}{\partial x_1} dy_1 + \frac{\partial u}{\partial y_1} dx_1 - \int\limits_{BC} du + \int\limits_{CA} du = 0.$$

Bundan darhol

$$u(C) = \frac{1}{2} u(A) + \frac{1}{2} u(B) + \frac{1}{2} \int\limits_{AB} \frac{\partial u}{\partial x_1} dy_1 + \frac{\partial u}{\partial y_1} dx_1 \tag{2.3}$$

tenglikka ega bo'lamiz.

Agar (2.1) tenglamaning $u(x, y)$ yechimi

$$u\,|_L = \varphi, \quad \frac{\partial u}{\partial l}\,|_L = \psi \tag{2.4}$$

shartlarni qanoatlantirsa, bunda φ va ψ berilgan, mos ravishda ikki va bir marta uzluksiz differentsiallanuvchi funksiyalar, l esa L da berilgan yo'nalish bo'lib, L

ning urinmasi bilan ustma-ust tushmaydi, u holda $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}$ noma'lum funksiyalarni

$$\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial s}+\frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial s}=\frac{d\varphi}{ds},$$

$$\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial l}+\frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial l}=\psi$$

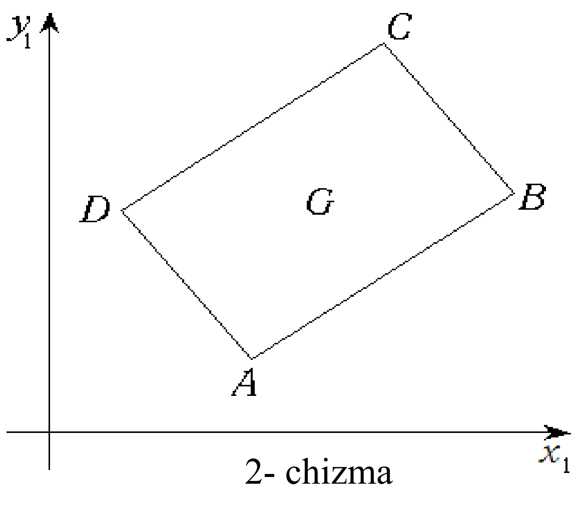
tengliklardan aniqlab olamiz, bu yerda s - L chiziq yoyining uzunligi.

Ma'lum u, $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}$ miqdorlarni (2.3) tenglikning o'ng tomoniga qo'yib, (2.1) tenglamaning (2.4) shartlarni qanoatlantiruvchi yechimini hosil qilamiz. Yuqorida yuritilgan mulohazalardan shu narsa kelib chiqadiki, Koshi masalasi birdan-bir turg'un yechimga egadir.

Odatda (2.1), (2.4) masala tor tebranish tenglamasi uchun umumiy qo'yilgan Koshi masalasi deyiladi.

Ikkinchi paragraf Gursa masalasini o'rganishga bag'ishlangan.

(2.1) tenglamaning tayin $A(x_0, y_0)$ nuqtadan chiqadigan $AB: x_1 - x_0 = y_1 - y_0$, $AD: x_1 - x_0 = y_0 - y_1$ xarakteristikalari va $C(x, y)$ nuqtadan chiqadigan $CB: x_1 - x = y - y_1$, $CD: x_1 - x = y_1 - y$ xarakteristikalaridan tashkil to'gan xarakteristik to'rtburchakni G orqali belgilab olamiz (2-chizma).



2- chizma

G soha uchun (2.2) formulani qo'llab,

$$\int\limits_{AB+BC+CD+DA} \frac{\partial u}{\partial x_1} dy_1 + \frac{\partial u}{\partial y_1} dx_1 = 0$$

tenglikka ega bo'lamiz.

AB va CD da $dx_1 = dy_1$, BC va DA da $dx_1 = -dy_1$ bo'lgani uchun avvalgi tenglikni quyidagi ko'rinishda yozib olamiz:

$$\int\limits_{AB} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 - \int\limits_{BC} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 + \int\limits_{CD} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 -$$

$$- \int\limits_{DA} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 = \int\limits_{AB} du - \int\limits_{BC} du + \int\limits_{CD} du - \int\limits_{DA} du =$$

$$= 2u(B) - 2u(A) - 2u(C) + 2u(D) = 0.$$

Bundan

$$u(C) = u(B) + u(D) - u(A) \tag{2.5}$$

yoki

$$u(A) + u(C) = u(B) + u(D).$$

Bu tenglik (2.1) tenglama uchun Asgeyrsson printsipi yoki o'rta qiymat to'g'risidagi teorema deb ataladi.

Bunga asosan (2.1) tenglama $u(x, y)$ yechimining xarakteristik to'rtburchak qarama-qarshi uchlaridagi qiymatlarining yig'indisi bir-biriga tengdir.

Asgeyrsson printsipi (2.1) tenglamaga masala qo'yishda va qo'yilgan tenglamalarni yechishda muhim rol o'ynaydi.

(2.1) tenglamaning $u(x, y)$ yechimi

$$u|_{AB} = \varphi(x_1), \qquad u|_{AD} = \psi(x_1), \quad \varphi(A) = \psi(A) \tag{2.6}$$

shartlarni qanoatlantirsin. U holda 2-chizmada B va D nuqtalarning koordinatalari mos ravishda $\left(\frac{x + x_0 + y - y_0}{2}, \frac{x - x_0 + y + y_0}{2}\right)$ va $\left(\frac{x + x_0 - y + y_0}{2}, \frac{-x + x_0 + y + y_0}{2}\right)$ lardan iboratligini e'tiborga olsak, (2.5) ga asosan, (2.1), (2.6) masalaning yechimi uchun

$$u(x, y) = \varphi\left(\frac{x + y + x_0 - y_0}{2}\right) + \psi\left(\frac{x - y + x_0 + y_0}{2}\right) - \varphi(x_0) \tag{2.7}$$

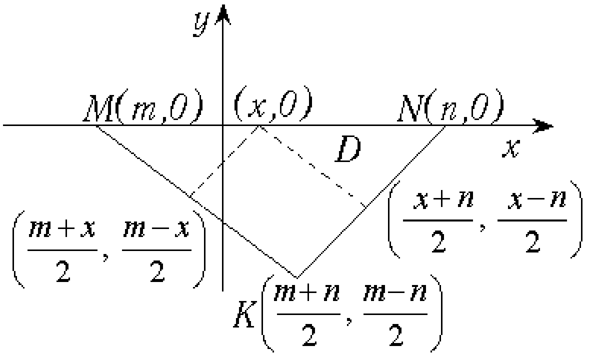
formula kelib chiqadi.

Agar $\varphi(x)$ va $\psi(x)$ funksiyalar ikki marta uzluksiz differentsiallanuvchi funksiyalar bo'lsa, (2.7) formula bilan aniqlangan $u(x, y)$ funksiya (2.1), (2.6) masalaning regulyar yechimidan iborat bo'ladi. Bu masala yechimining yagonaligi Asgeyrsson printsipidan yoki (2.7) formulani hosil qilish usulidan darhol kelib chiqadi.

Odatda (2.1), (2.6) masala tor tebranish tenglamasi uchun umumiy qo'yilgan Gursa masalasi deyiladi.

Uchinchi paragrafda tor tebranish tenglamasi uchun siljishli masalalar o'rganishga bag'ishlangan.

Masalalarni bayon qilishda chegaraviy shartlar D soha chegarasining bir qismidagina berilib, qolgan qismi esa chegaraviy shartlardan ozod qolmoqda. Ikkinchi tomondan, Gursa va Darbu masalalarining qo'yilishidan (Asgeyrsson printsipidan ham) kelib chiqadiki, D sohada (2.1) tenglama uchun Dirixle masalasi, ya'ni D da (2.1) tenglamaning $u|_{\overline{MN}\cup\overline{MK}\cup\overline{NK}}=\psi(x,y)$ shartni qanoatlantiruvchi yechimini topish haqidagi masala qaralsa, u ortiqcha shartli masala bo'lar ekan. Bular asosida 1960 yillarga kelib quyidagi savol paydo bo'ldi: giperbolik tipdagi tenglamalar uchun xarakteristik uchburchak chegarasining barcha qismida chegaraviy shartlar berilgan korrekt masala mavjudmi? Bu savolga 1960 yillarning oxiriga kelib A.M. Naxushev tomonidan ijobiy javob berildi [8]. Quyida biz ana shu masalalarning ba'zilarini bayon qilamiz.



3- chizma

Shunday qilib (2.1) tenglamani 3-chizmadagi $D\equiv MNK$ xarakteristik uchburchakda qaraymiz.

1-masala. (2.1) tenglamaning quyidagi shartlarni qanoatlantiruvchi $u(x,y)\in C(\overline{D})\cap C^2(D)$ yechimi topilsin:

$$u(x,0)=\tau(x),\ m\le x\le n; \tag{2.8}$$

$$\alpha(x)u\left(\frac{m+x}{2},\frac{m-x}{2}\right)+\beta(x)u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=\gamma(x),\qquad m\le x\le n, \tag{2.14}$$

bu yerda $\tau(x),\ \alpha(x),\ \beta(x),\ \gamma(x)\in C[m,n]\cap C^2(m,n)$-berilgan funksiyalar bo'lib,

$$\alpha(x) \neq \beta(x), \quad m \leq x \leq n, \quad \alpha(n)\beta(m) \neq 0,$$

$$\alpha(n)[\gamma(m) - \alpha(m)\tau(m)] = \beta(m)[\gamma(n) - \beta(n)\tau(n)] \tag{2.15}$$

shartlarni qanoatlantiradi.

Uchinchi bob ikkita paragrafdan iborat bo'lib, birinchi paragrafda tor tebranish tenglamasi uchun chegaraviy masalalar qo'yish va o'rganishga bag'ishlangan.

(x,y)-o'zgaruvchilar tekisligida

$$U_{xx} - U_{yy} = 0 \tag{3.1}$$

tenglamaning $AC: x + y = 0, \quad BC: x - y = 1$ xarakteristikalari va $y = 0$ to'g'ri chiziqning $\overline{J} = [0,1]$ kesmasi bilan chegaralangan D sohani ko'raylik.

Masala. Quyidagi shartlarni qanoatlantiruvchi $U(x, y)$ funksiya topilsin.

1) $U(x, y) \in C(\overline{D}) \cap C^2(D)$,
2) D sohada (3.1) tenglamani qanoatlantiradi.
3) D soha chegarasida

$$U(x,0) = \tau(x), \qquad x \in \overline{J} \tag{3.2}$$

$$a(x)D_{0x}^{\alpha}U\left(\frac{x}{2}, -\frac{x}{2}\right) + b(x)D_{x1}^{\beta}U\left(\frac{x+1}{2}, \frac{x-1}{2}\right) =$$

$$= p(x)U_y(x,0) + q(x), \qquad x \in J \tag{3.3}$$

shartlarni qanoatlantiradi. Bu yerda $\alpha, \beta -$ haqiqiy sonlar; $\tau(x)$, $a(x)$, $b(x)$, $p(x)$, $q(x)$-berilgan funksiyalar, $D_{0x}^{\alpha}, D_{x1}^{\beta}$ -kasr tartibli integrodifferentsial operatorlar [8].

1-teorema. Agar $U(x,y)$ funksiya (3.1)-(3.3) masalaning regulyar yechimi bo'lsa, u holda $U_y(x,0)=\nu(x)$ funksiya

$$2p(x)\nu(x)+a(x)D_{0x}^{\alpha-1}\nu(x)+b(x)D_{x1}^{\beta-1}\nu(x)=\gamma(x), \tag{3.4}$$

tenglik bilan topiladi.

Bu yerda

$$\gamma(x)=a(x)D_{0x}^{\alpha}\left[\tau(0)+\tau(x)\right]+b(x)D_{x1}^{\beta}\left[\tau(x)+\tau(1)\right]-2q(x) \tag{3.5}$$

2-teorema. Agar $\tau(x)\in C^1(\overline{AB})\cap C^3(AB)$, $a(x)$, $b(x)$, $p(x)$, $q(x)\in C(\overline{AB})$, $2p(x)+b(x)\neq 0$ bo'lsa. U holda (3.1) – (3.3) masalaning yechimi mavjud va yagona bo'ladi.

3-teorema. Agar $\tau(x)\in C^1(\overline{AB})\cap C^3(AB)$, $a(x)$, $b(x)$, $p(x)$, $q(x)\in C(\overline{AB})$, $2p(x)+a(x)\neq 0$ bo'lsa. U holda (3.1) – (3.3) masalaning yechimi mavjud va yagona bo'ladi.

4-teorema. Agar quyidagi shartlar bajarilgan bo'lsa,

1) $\nu(x)=x^{\alpha-2}\nu_1(x),\quad \nu_1(x)\in C^1(\overline{J}),\quad \nu_1(0)\neq 0.$

2) $\tau(x)=x^{\sigma}\tau_1(x),\quad \tau_1(x)\in C^1(\overline{J})\cap C^3(J)$, $\alpha-1<\sigma<\alpha$

3) $a(x)$, $b(x)$, $p(x)$, $q(x)\in C^1(\overline{J})$, $2p(x)+b(x)\neq 0$

u holda (3.1)-(3.3) masala cheksiz ko'p yechimga ega bo'ladi.

Ikkinchi paragrafda yuqori ko'rsatkichli integrodifferentsial operatorlar yordamida siljishli masalalar yechish ko'rib chiqilgan.

(x,y)-o'zgaruvchilar tekisligida

$$U_{xx}-U_{yy}=0 \tag{3.31}$$

tenglamaning $AC: x+y=0,\quad BC: x-y=1$ xarakteristikalari va $y=0$ to'g'ri chiziqning $\overline{J}=[0,1]$ kesmasi bilan chegaralangan D sohani ko'raylik.

Masala. Quyidagi shartlarni qanoatlantiruvchi $U(x,y)$ funksiya topilsin.

1) $U(x,y)\in C(\overline{D})\cap C^2(D),$

2) D sohada (3.31) tenglamani qanoatlantiradi.

3) D soha chegarasida

$$U(x,0)=\tau(x),\qquad x\in\overline{J} \tag{3.32}$$

$$a(x)D_{0x^k}^{\alpha}U\left(\frac{x}{2},-\frac{x}{2}\right)+b(x)D_{x^k1}^{\beta}U\left(\frac{x+1}{2},\frac{x-1}{2}\right)=$$
$$=p(x)U_y(x,0)+q(x),\qquad x\in J \tag{3.33}$$

shartlarni qanoatlantiradi. Bu yerda $\alpha,\beta-$haqiqiy sonlar; $\tau(x)$, $a(x)$, $b(x)$, $p(x)$, $q(x)$-berilgan funksiyalar, $D_{0x^k}^{\alpha}$, $D_{x^k1}^{\beta}$ -kasr tartibli integrodifferentsial operatorlar [8].

Biz $k=2$ bo'lgan holni ko'rib chiqamiz.

1-teorema. Agar $U(x,y)$ funksiya (3.31)-(3.33) masalaning regulyar yechimi bo'lsa, u holda $U_y(x,0)=\nu(x)$ funksiya

$$4p(x)\nu(x)+a(x)D_{0x^2}^{\alpha-1}\frac{\nu(x)}{x}+b(x)D_{x^21}^{\beta-1}\frac{\nu(x)}{x}=\gamma(x), \tag{3.34}$$

tenglik bilan topiladi.

Bu yerda

$$\gamma(x)=2a(x)D_{0x^2}^{\alpha}\left[\tau(0)+\tau(x)\right]+2b(x)D_{x^21}^{\beta}\left[\tau(x)+\tau(1)\right]-4q(x) \tag{3.35}$$

2-teorema. Agar $\tau_1(x)\in C^1(\overline{J})\cap C^3(J)$, $a_1(x)$, $b_1(x)$, $p_1(x)$, $q_1(x)\in C(\overline{J})\cap C^2(J)$, $4p_1(x)+b_1(x)\neq 0$ bo'lsa. U holda (3.31) – (3.33) masalaning yechimi mavjud va yagona bo'ladi.

3-teorema. Agar $\tau_1(x) \in C^1(\overline{J}) \cap C^3(J)$, $a_1(x)$ $, b_1(x)$ $, p_1(x)$, $q_1(x) \in C(\overline{J}) \cap C^2(J)$, $4p_1(x) + a_1(x) \neq 0$ bo'lsa. U holda (3.31) – (3.33) masalaning yechimi mavjud va yagona bo'ladi.

4-teorema. Agar quyidagi shartlar bajarilgan bo'lsa,

1) $\widetilde{\nu}(x) = x^{\alpha-2}\nu_{11}(x), \quad \nu_{11}(x) \in C^1(\overline{J}), \quad \nu_{11}(0) \neq 0.$

2) $\tau_1(x) = x^{\sigma}\tau_{11}(x), \quad \tau_{11}(x) \in C^1(\overline{J}) \cap C^3(J)$, $2\alpha-1<\sigma<2\alpha$

3) $a_1(x)$ $, b_1(x)$ $, p_1(x)$, $q_1(x) \in C^1(\overline{J})$, $4p_1(x) + b_1(x) \neq 0$

u holda (3.31)-(3.33) masala cheksiz ko'p yechimga ega bo'ladi.

I BOB. KASR TARTIBLI INTEGRODIFFERENTSIAL OPERATORLAR VA ULARNING XOSSALARI.

1.1-§. Birinchi va ikkinchi tur Eyler integrallari

Birinchi tur Eyler integrali (betta-funksiya).

Betta-funksiya ushbu

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx \tag{1.1}$$

tenglik bilan aniqlanadi. Bu tenglikning o'ng tomonidagi integral Eylerning birinchi tur integrali deyiladi.

Ko'rsatish mumkinki, $a>0$ *ва* $b>0$ bo'lganda (1.1) integral yaqinlashuvchi, agar a va b parametrlarning bittasi nolga teng yoki noldan kichik bo'lsa ham uzoqlashuvchi bo'ladi. Agar (1.1) integralda $x=1-t$ almashtirish bajarsak

$$B(a,b)=\int_0^1 t^{b-1}(1-t)^{a-1}dt=B(b,a)$$

tenglikni hosil qilamiz.

Demak, betta-funksiya o'zining a va b argumentlariga nisbatan simmetrik funksiya ekan. Endi (1.1) integralni bo'laklab integrallaymiz. Bo'laklab integrallash amallarini

$$u=(1-x)^{b-1},\ du=-(b-1)(1-x)^{b-2}dx$$
$$d\vartheta=x^{a-1}dx,\ \vartheta=\frac{1}{a}x^a$$

bajarib

$$x^a=x^{a-1}-x^{a-1}(1-x)$$

ayniyatni e'tiborga olsak, $b>1$ bo'lganda

$$B(a,b)=\left[\frac{(1-x)^{b-1}x^a}{a}\right]_0^1+\int_0^1\frac{x^a}{a}(b-1)(1-x)^{b-2}dx=$$

$$=\frac{b-1}{a}\int_0^1 x^{a-1}(1-x)^{b-2}dx-\frac{b-1}{a}\int_0^1 x^{a-1}(1-x)^{b-1}dx=\frac{b-1}{a}B(a,b-1)-\frac{b-1}{a}B(a,b).$$

Bundan

$$B(a,b)=\frac{b-1}{a+b-1}B(a,b-1) \tag{1.2}$$

Betta-funksiya a va b ga nisbatan simmetrik bo'lgani uchun, $a>1$ bo'lganda

$$B(a,b)=\frac{a-1}{a+b-1}B(a-1,b) \tag{1.3}$$

(1.2) va (1.3) formulalarga asosan

$$(a-1)B(a-1,b)=(b-1)B(a,b-1).$$

Agar

$$a-1=p,\ b-1=q$$

desak, u holda

$$B(p,q+1)=\frac{q}{p}B(p+1,q) \tag{1.4}$$

Agar b parametr butun songa teng bo'lsa, ya'ni $b=n$ bo'lsa, $B(a,n)$ funksiyaga (1.2) formulaga ketma-ket qo'llash natijasida

$$B(a,n)=\frac{n-1}{a+n-1}\cdot\frac{n-2}{a+n-2}\cdot\frac{n-3}{a+n-3}\cdots\frac{1}{a+1}B(a,1)$$

tenglikka ega bo'lamiz. Ammo

$$B(a,1)=\int_0^1 x^{a-1}dx=\frac{1}{a}$$

bo'lgani uchun

$$B(a,n)=\frac{1\cdot2\cdot3\cdots(n-1)}{a(a+1)(a+2)\cdots(a+n-1)}$$

Agarda a parametr ham butun songa teng bo'lsa, ya'ni $a=m$ bo'lsa, (1.3) formulani ketma-ket qo'llash natijasida quyidagi tenglikni hosil qilamiz:

$$B(a,n)=\frac{1\cdot2\cdot3\cdots(n-1)}{(m+1)(m+2)\cdots(m+n-1)}\cdot B(m,1)=\frac{1\cdot2\cdot3\cdots(n-1)}{(m+1)(m+2)\cdots(m+n-1)}\cdot$$
$$\cdot\frac{m-1}{m}\frac{m-2}{m-2}\cdots\frac{1}{2}B(1,1)$$

bundan, $B(1,1)=1$ bo'lgani uchun

$$B(m,n)=B(n,m)=\frac{(n-1)!(m-1)!}{(m+n-1)!}. \qquad (1.5)$$

Endi (1.1) formulada $a=b$ desak,

$$B(a,a)=2\int_0^{\frac{1}{2}}\left[\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right]^{a-1}dx$$

Ohirgi integralda $\frac{1}{2}-x=\frac{\sqrt{t}}{2}$ almashtirish bajaramiz.

U holda

$$2^{\frac{1}{2a-1}}\int_0^1 t^{\frac{1}{2}}(1-t)^{a-1}dt$$

yoki

$$B(a,a)=2^{\frac{1}{2a-1}}B\left(\frac{1}{2},a\right),$$

(1.1) integralda

$$x=\frac{y}{1+y} \quad ёки \quad y=\frac{x}{1-x}$$

almashtirishni bajarsak, betta-funksiya quyidagi ko'rinishda yoziladi:

$$B(a,b)=\int_0^{\infty}\frac{y^{a-1}}{(1+y)^{a+b}}dy \qquad (1.6)$$

Bu formulada $0<a<1$ hisoblab, $b=1-a$ desak,

$$B(a,1-a)=\int_0^{\infty}\frac{y^{a-1}}{1+y}dy$$

bo'ladi.

Hosil qilingan integral matematik analizda Eyler ismi bilan bog'langan integral bo'lib, uning qiymati $\frac{\pi}{\sin\pi a}$ ga tengdir.

Shunday qilib,

$$B(a,1-a)=\frac{\pi}{\sin\pi a} \qquad (1.7)$$

Agar xususiy holda, $a=1-a=\frac{1}{2}$ desak,

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\pi$$

hosil bo'ladi.

Ikkinchi tur Eyler integrali (gamma-funksiya).

Gamma-funksiya ushbu

$$\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx \qquad (1.8)$$

integral bilan aniqlanadi va bu integral ikkinchi tur Eyler integrali deb ataladi.

Bu integral $a>0$ da yaqinlashuvchi, $a\le 0$ da esa uzoqlashuvchi bo'ladi. Bo'laklab integrallash natijasida, (1.8) dan

$$a\int_0^\infty x^{a-1}e^{-x}dx = x^a e^{-x}\Big|_0^\infty + \int_0^\infty x^a e^{-x}dx = \int_0^\infty x^a e^{-x}dx,$$

ya'ni

$$\Gamma(a+1)=a\Gamma(a) \qquad (1.9)$$

rekurent formulani hosil qilamiz.

Bu formulani ketma-ket qo'llab, quydagiga ega bo'lamiz:

$$\Gamma(a+n)=(a+n-1)(a+n-2)...(a+1)a\Gamma(a)$$

Agar bunda $a=1$ desak va

$$\Gamma(1)=\int_0^\infty e^{-x}dx=1 \qquad (1.10)$$

bo'lishini e'tiborga olsak, u holda

$$\Gamma(n+1)=n! \qquad (1.11)$$

kelib chiqadi. $n=0$ bo'lganda (1.11) formula

$$0!=\Gamma(1)=1$$

ko'rinishga ega bo'ladi. (1.11) dan $a\to 0$ da

$$\Gamma(a) = \frac{\Gamma(a+b)}{a} \to +\infty$$

ekanligi ravshan.

Betta va gamma funksiyalarning o'zaro bog'lanishlarini o'rnatish maqsadida (1.8) da $x = ty\,(t = const > 0)$ almashtirish bajaramiz, u holda

$$\frac{\Gamma(a)}{t^a} = \int_0^\infty y^{a-1} e^{-ty} dy \qquad (1.12)$$

Bu yerda a ni $a+b$ bilan va bu vaqtda t ni $1+t$ bilan almashtirib, quyidagini hosil qilamiz:

$$\frac{\tilde{A}(a+b)}{(1+t)^{a+b}} = \int_0^\infty y^{a+b-1} e^{-(1+t)y} dy$$

Oxirgi tenglikning har ikki tomonini t^{a-1} ga ko'paytiramiz va 0 dan ∞ gacha integrallaymiz:

$$\Gamma(a+b) \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt = \int_0^\infty y^{a+b-1} e^{-y} dy \int_0^\infty t^{a-1} e^{-ty} dt$$

Bundan (1.6), (1.8) va (1.12) ga asosan

$$\Gamma(a+b) B(a,b) = \int_0^\infty y^{a+b-1} e^{-y} \frac{\Gamma(a)}{y^a} dy = \Gamma(a) \int_0^\infty y^{b-1} e^{-y} dy = \Gamma(a)\Gamma(b) \ .$$

Demak,

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad (1.13)$$

Agar (1.13) formulada $b = 1-a$ desak, u holda (1.7) va (1.10) ga asosan

$$\Gamma(a) \cdot \Gamma(1-a) = \frac{\pi}{\sin a\pi} \qquad (1.14)$$

formulaga ega bo'lamiz. Bu formula to'ldirish formulasi deyiladi. Bundan $a = \frac{1}{2}$ bo'lganda $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ kelib chiqadi.

1.2-§. Abel integral tenglamasi

Ushbu

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}} = f(x), \quad 0<\alpha<1 \qquad (1.15)$$

ko'rinishdagi integral tenglama *Abel integral tenglamasi* deyiladi. Bu yerda $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-\alpha}dx$ - gamma funksiya. (1.15) tenglama quyidagi usulda yechiladi. Bu tenglamada x ni t bilan, t ni s bilan almashtirib yozamiz

$$\frac{1}{\Gamma(\alpha)}\int_a^t \frac{\varphi(s)}{(t-s)^{1-\alpha}}ds = f(t)$$

so'ngra tenglamaning har ikki tomonini $(x-t)^{-\alpha}$ ifodaga ko'paytiramiz va t bo'yicha a dan x gacha integrallaymiz:

$$\int_a^x \frac{dt}{(x-t)^\alpha}\int_a^t \frac{\varphi(s)\,ds}{(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)\,dt}{(x-t)^\alpha}.$$

Dirixle formulasiga ko'ra integrallash tartibini almashtirib

$$\int_a^x \varphi(s)\,ds\int_s^x \frac{dt}{(x-t)^\alpha(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)\,dt}{(x-t)^\alpha} \qquad (1.16)$$

tenglikni hosil qilamiz. Tenglikning chap tomonidagi ichki integralda $t=s+\tau(x-s)$ almashtirish bajarsak,

$$\int_s^x (x-t)^{-\alpha}\,(t-s)^{\alpha-1}dt = \int_0^1 \tau^{\ \alpha-1}(1-\tau)^{-\alpha}\,d\tau =$$

$$= B(\alpha,1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$$

tenglik kelib chiqadi. U holda, (1.16) ga asosan

$$\int_a^x \varphi(s)\,ds = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha}}. \tag{1.17}$$

tenglikni olamiz.

Bu tenglikni integrallab,

$$\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha}}. \tag{1.18}$$

Abel integral tenglamasining yechimini hosil qilamiz.

Shunday qilib, agar (1.15) tenglamaning yechimi mavjud bo'lsa, u (1.18) ko'rinishda ifodalanar ekan. Bu formulani hosil qilish jarayonidan kelib chiqadiki, agar yechim mavjud bo'lsa, u yagona.

Shu usulda ko'rsatish mumkinki, ushbu

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)\,dt}{(t-x)^{1-\alpha}} = f(x),\quad 0<\alpha<1 \tag{1.19}$$

ko'rinishdagi integral tenglamaning yechimi.

$$\varphi(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)\,dt}{(t-x)^\alpha} \tag{1.20}$$

formula bilan aniqlanadi.

Isbot. (1.19) tenglamada x ni t bilan t ni s bilan almashtiramiz.

$$\int_t^b \frac{dt}{(t-x)^\alpha}\int_t^b \frac{\varphi(s)}{(s-t)^{1-\alpha}}ds = \int_x^b \frac{f(t)}{(t-x)^\alpha}dt \tag{1.21}$$

Bu tenglamada ham yuqoridagi kabi Dirixle formulasini qo'llab, integrallash tartibini almashtiramiz

$$\int_x^b \varphi(s)ds\int_x^s \frac{dt}{(t-x)^\alpha(s-t)^{1-\alpha}} = \int_x^b \frac{f(t)}{(t-x)^\alpha}dt \tag{1.22}$$

hosil bo'lgan tenglikning ichki integralida $t = x + \tau(s-x)$ almashtirish bajarib integralning quyi chegarasi $\tau = 0$ yuqori chegarasi $\tau = 1$ $dt = (s-x)d\tau$

$$\int_0^1 \frac{(s-x)d\tau}{\tau^\alpha(s-x)^\alpha(1-\tau)^{1-\alpha}(s-x)^{1-\alpha}} = \int_0^1 \frac{d\tau}{(1-\tau)^\alpha\tau^{1-\alpha}} =$$
$$= B(1-\alpha,\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$$

olingan natijani (1.22) ga qo'ysak,

$$\int_x^b \varphi(s)ds = \frac{1}{\Gamma(1-\alpha)}\int_x^b \frac{f(t)}{(t-x)^\alpha}dt$$

hosil bo'ladi. Tenglikning har ikki tomonini differentsiallab,

$$\varphi(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^\alpha}dt$$

tenglikni olamiz.

Teorema. Abel integral tenglamasi $L_1(a,b)$ funksiyalar sinfida yechimga ega bo'lishi shart va uning uchun

$$f_{1-\alpha}(x) \in AC([a,b]) \quad \text{va} \quad f_{1-a}(a) = 0 \tag{1.23}$$

bo'lishi zarur va yetarli.

Bu yerda $AC([a,b])$ – $[a,b]$ da absolyut uzluksiz funksiyalar sinfi:

$$f_{1-\alpha}(x)=\frac{1}{\Gamma(1-\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{\alpha}}.$$

Isbot. Zarurligi. Faraz qilaylik $L_1(a,b)$ da (1) Abel integral tenglamasining yechimi mavjud bo'lsin. U holda, (1.16) va (1.17) tengliklar o'rinli bo'ladi. Bundan esa (1.23) shartlarning bajarilishi kelib chiqadi.

Yetarliligi. Teoremaning shartlari bajarilgan bo'lsin. $f_{1-\alpha}(x)\in AC([a,b])$ bo'lishidan $f'_{1-\alpha}(x)=\frac{d}{dx}f_{1-\alpha}(x)\in L_1(a,b)$ bo'lishi kelib chiqadi. Shuning uchun ham (1.4) ko'rinishda ifodalangan funksiya (a,b) oraliqning deyarli barcha nuqtalarida aniqlangan bo'lib, $L_1(a,b)$ funksiyalar sinfiga tegishli bo'ladi.

Bu funksiya haqiqatdan ham (1.15) tenglamaning yechimi ekanligini ko'rsataylik. Buning uchun (1.18) formula bilan aniqlangan funksiyani (1.15) tenglamaga qo'yamiz

$$\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f'_{1-\alpha}(t)}{(x-t)^{1-\alpha}}dt=f(x)$$

tenglamaning o'ng tomonini $g(x)$ bilan belgilaymiz:

$$\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f'_{1-\alpha}(t)}{(x-t)^{1-\alpha}}dt=g(x). \tag{1.24}$$

Endi $g(x)=f(x)$ tenglik (a,b) oraliqning deyarli barcha nuqtalarida bajarilishini ko'rsatamiz.

(1.24) tenglik $f'_{1-\alpha}(x)$ funksiyaga nisbatan Abel integral tenglamasi bo'ladi. Shuning uchun (1.18) formulaga asosan,

$$f'_{1-\alpha}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{g(t)dt}{(x-t)^{\alpha}},$$

ya'ni

$$f'_{1-\alpha}(x) = g'_{1-\alpha}(x). \qquad (1.25)$$

$f_{1-\alpha}(x)$ teorema shartiga asosan, $g_{1-\alpha}(x)$ funksiya esa (1.17) tenglikka asosan absolyut uzluksiz funksiyalardir. Shuning uchun ham, (1.25) tenglikka asosan, $f_{1-\alpha}(x) - g_{1-\alpha}(x) = C$. Lekin $f_{1-\alpha}(a) = 0$. (1.24) integral tenglamaning yechimga ega ekanligidan $g_{1-\alpha}(a) = 0$ tenglik kelib chiqadi. Shuning uchun ham *S*=0. U holda, $\int_a^x \frac{f(t) - g(t)}{(x-t)^{\alpha}} dt = 0$.

Bu - Abel tipidagi integral tenglama bo'lib, uning yechimi mavjud va yagona:

$f(x) - g(x) \equiv 0$, ya'ni $f(x) \equiv g(x)$. Teorema isbotlandi.

1.3-§. Kasr tartibli integral operatorlar.

Matematik analiz kursidan ma'lumki, n - karrali integral uchun quyidagi formula o'rinli:

$$\int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{n-1}} \varphi(t)dt = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt, \ n \in N. \qquad (1.26)$$

$(n-1)! = \Gamma(n)$ ekanligini e'tiborga olib, (1.26) tenglikning o'ng tomonini n ning kasr qiymatlari uchun ham aniqlash mumkin.

(1.26) tenglikka mos ravishda kasr tartibli integrallarni quyidagi tartibda aniqlaymiz.

Ta'rif. $\varphi(x) \in L_1(a,b)$ ($a < b < \infty$) *bo'lsin. Ushbu*

$$D_{ax}^{-\alpha}\varphi(x) = \frac{1}{\Gamma(a)}\int_a^x (x-t)^{\alpha-1}\varphi(t)\,dt, \ \alpha > 0, \qquad (1.27)$$

$$D_{bx}^{-\alpha}\varphi(x) = \frac{1}{\Gamma(a)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \ \alpha > 0 \qquad (1.28)$$

ko'rinishdagi ifodalar Riman-Liuvill ma'nosidagi α $(0 < \alpha < \infty)$ kasr tartibli integrallar deyiladi va mos ravishda $D_{ax}^{-\alpha}\varphi(x)$ hamda $D_{xb}^{-\alpha}\varphi(x)$ kabi belgilanadi.

$D_{ax}^{-\alpha}\varphi(x)$ va $D_{xb}^{-\alpha}\varphi(x)$ funksiyalar (a,b) oraliqning deyarli barcha nuqtalarida aniqlangan bo'lib, $L_1(a,b)$ sinfga tegishli bo'ladi.

Agar $0<\alpha_1,\ \alpha_2<\infty$ bo'lsa, deyarli barcha $x\in(a,b)$ uchun

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1}f(x)=D_{ax}^{-\alpha_1}D_{ax}^{-\alpha_2}f(x)=D_{ax}^{-(\alpha_1+\alpha_2)}f(x) \qquad (1.29)$$

tenglik o'rinli bo'ladi. Haqiqatdan ham,

$$D_{ax}^{-\alpha_2}D_{ax}^{-\alpha_1}f(x)=\frac{1}{\Gamma(\alpha_1)}D_{ax}^{-\alpha_2}\int_a^x(x-t)^{\alpha_1-1}f(t)dt=$$

$$=\frac{1}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\int_a^x\left(\int_a^t(t-s)^{\alpha_1-1}f(s)ds\right)(x-t)^{\alpha_2-1}dt=$$

$$=\frac{1}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\int_a^x f(s)ds\int_s^x(x-t)^{\alpha_2-1}(t-s)^{\alpha_1-1}dt.$$

Oxirgi ichki integralda $t=(x-s)\tau+s$ almashtirish bajarish natijasida quyidagi tenglikni hosil qilamiz:

$$\int_s^x(x-t)^{\alpha_2-1}(t-s)^{\alpha_1-1}ds=(x-s)^{\alpha_1+\alpha_2-1}\int_0^1\tau^{\alpha_1-1}(1-\tau)^{\alpha_2-1}d\tau=$$

$$=\frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}(x-s)^{\alpha_1+\alpha_2-1}.$$

Bu esa (1.29) tenglikning to'g'ri ekanligini ko'rsatadi.

Ta'rifga asosan

$$D_{ax}^{0}f\left(x\right)=f\left(x\right),\;\; D_{xb}^{0}g\left(x\right)=g\left(x\right) \tag{1.30}$$

deb hisoblaymiz.

Isbot.

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt=\frac{1}{\Gamma(\alpha+1)}\int_a^x \frac{\alpha}{(x-t)^{\alpha-1}}\varphi t dt=$$

$$=\frac{1}{\Gamma(\alpha+1)}\left[-(x-t)^{\alpha}\varphi(t)\Big|_a^x+\int_a^x (x-t)^{\alpha}\varphi'(t)dt\right]=$$

$$\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\varphi(a)+\int_a^x (x-t)^{\alpha}\varphi'(t)dt\right]$$

ifoda kelib chiqadi.

Bundan $\alpha\to 0$ intilganda

$$\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\varphi(a)+\int_a^x (x-t)^{\alpha}\varphi'(t)dt\right]\Rightarrow \varphi(\alpha)+\varphi(t)\Big|_a^x=$$

$$=\varphi(\alpha)+\varphi(x)-\varphi(\alpha)=\varphi(x)$$

hisobimiz to'g'riligiga amin bo'lamiz.

1.4-§. Kasr tartibli differentsial operatorlar.

Ta'rif. $\varphi(x)$ funksiya [a,b] kesmada aniqlangan bo'lsin.

$$D_{ax}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}\frac{\varphi(t)dt}{(x-t)^{\alpha}},\qquad 0<\alpha<1, \qquad (1.31)$$

$$D_{xb}^{\alpha}\varphi(x)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{b}^{x}\frac{\varphi(t)dt}{(x-t)^{\alpha}},\qquad 0<\alpha<1 \qquad (1.32)$$

ko'rinishdagi ifodalar Liuvill ma'nosidagi α (kasr) tartibli hosilalar deyiladi va mos ravishda $D_{ax}^{\alpha}\ \varphi(x)$ hamda $D_{xb}^{\alpha}\ \varphi(x)$ kabi belgilanadi.

Eslatib o'tamizki, kasr tartibli integrallar ixtiyoriy $\alpha>0$ tartibgacha aniqlangan. Lekin (1.31), (1.32) kasr tartibli hosilalar faqatgina $0<\alpha<1$ bo'lganda aniqlangan.

Endi chegaralarni tekshiraylik.

1. $$\lim_{\alpha\to 0}D_{ax}^{\alpha}\ f(x)=\lim_{\alpha\to 0}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}\frac{f(t)dt}{(x-t)^{\alpha}}=$$

$$=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{x}\frac{\varphi(t)dt}{(x-t)^{0}}=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{x}f(t)dt=f(x)$$

2. $$\lim_{\alpha\to 1} D_{ax}^{\alpha} f(x) = \lim_{\alpha\to 1}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha}} = \left\langle \frac{\infty}{\infty} \ \text{ёки } 0\cdot\infty \right\rangle =$$

$$= \lim_{\alpha\to 1}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(x)-\left[f(x)-f(t)\right]}{(x-t)^{\alpha}}dt =$$

$$= \lim_{\alpha\to 1}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[f(x)\int_a^x (x-t)^{-\alpha}dt - \int_a^x \frac{f(x)-f(t)}{(x-t)^{\alpha}}dt \right] =$$

$$= \lim_{\alpha\to 1}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[f(x)\frac{(x-t)^{1-\alpha}}{1-\alpha} - \int_a^x K(x-t)^{\beta-\alpha}\,dt \right] =$$

$$= \left\langle Bu \ \ yerda \ Gyoldersharti \ \left|f(x)-f(t)\right| \le K\left|x-t\right|^{\beta} \ \ ga \right.$$

$$\left. k\dot{o}ra \ \ f(x)\in H^{\beta} \ va \beta > \alpha \ \ shartlarga \ \ muvofiq \right\rangle =$$

$$= \lim_{\alpha\to 1}\frac{1}{\Gamma(2-\alpha)}\frac{d}{dx} f(x)(x-a)^{1-\alpha} -$$

$$-\lim_{\alpha\to 1}\frac{K(\beta-\alpha)}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{\beta-\alpha-1}\,dt = f(x).$$

3. $$\lim_{\alpha\to 0} D_{xb}^{\alpha} g(x) = -\lim_{\alpha\to 0}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{g(t)\,dt}{(t-x)^{\alpha}} =$$

$$= -\frac{d}{dx}\int_x^b g(t)\,dt = g(x).$$

4. $$\lim_{\alpha\to1} D_{xb}^{\alpha}\, g(x) = -\lim_{\alpha\to1}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{\left[g(t)-g(x)+g(x)\right]dt}{(t-x)^{\alpha}} =$$

$$= -\lim_{\alpha\to1}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[\int_x^b \frac{g(t)-g(x)dt}{(t-x)^{\alpha}} + g(x)\int_x^b (t-x)^{-\alpha}\,dt\right] =$$

$$= \left\langle \begin{matrix} \left|g(t)-g(x)\right| \le K\left|t-x\right|^{\beta},\ \beta>\alpha \\ g(t)-g(x) = g'(\xi)(t-x),\ t<\xi<x \end{matrix} \right\rangle =$$

$$= -\lim_{\alpha\to1}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[\int_x^b K(t-x)^{\beta-\alpha}\,dt + \frac{g(x)}{1-\alpha}(b-x)^{1-\alpha}\right] =$$

$$= -\lim_{\alpha\to1}\left\{-\frac{1}{\Gamma(1-\alpha)}\int_x^b K(\beta-\alpha)(t-x)^{\beta-\alpha-1}\,dt + \frac{d}{dx}\frac{g(x)}{1-\alpha}(b-x)^{1-\alpha}\right\} =$$

$$= -\lim_{\alpha\to1}\left\{-\frac{g'(x)}{\Gamma(2-\alpha)}(b-x)^{1-\alpha} + \frac{g(x)}{1-\alpha}(b-x)^{-\alpha}\right\} = -g'(x)$$

$$D_{ax}^{1}\, f(x) = f'(x), \qquad D_{xb}^{1}\, g(x) = -g'(x)$$

Kasr tartibli hosilalarni $\alpha \ge 1$ bo'lganda aniqlashga o'tishdan oldin kasr tartibli hosilalar mavjudligining yetarli shartini keltiramiz.

Lemma. Agar $\varphi(x) \in AC([a,b])$ bo'lsa, $[a,b]$ kesmaning deyarli barcha nuqtalarida $\varphi(x)$ funksiyaning kasr tartibli hosilalari mavjud bo'lib, quyidagi formulalar o'rinli bo'ladi:

$$D_{ax}^{\alpha}\,\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\varphi(a)}{(x-a)^{\alpha}} + \int_a^x \frac{\varphi'(t)\,dt}{(x-t)^{\alpha}}\right],\ 0<\alpha<1,$$

$$D_{xb}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{\varphi(b)}{(b-x)^{\alpha}}-\int_{x}^{b}\frac{\varphi'(t)\,dt}{(t-x)^{\alpha}}\right],\ 0<\alpha<1.$$

Misol. $\varphi(x)=(x-a)^{\alpha-1}$ bo'lsin. U holda, (1.31) tenglikka asosan, $D_{ax}^{\alpha}\varphi(x)\equiv 0$ bo'ladi.

$$D_{ax}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}(x-t)^{-\alpha}(t-a)^{\alpha-1}dt$$

integralda $t=a+(x-a)z$ almashtirish bajarsak, integralning yuqori chegarasi x ni 1 bilan quyi chegarasi a ni 0 bilan almashtirsak $x-t=(x-a)(1-z)$ va $dt=(x-a)dz$ bo'ladi. Topilganlarni oxirgi ifodamizga qo'ysak

$$D_{ax}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{1}(x-a)^{\alpha-1}(x-a)^{-\alpha}(1-z)^{-\alpha}(x-a)dz=$$

$$=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{1}(1-z)^{-\alpha}dz=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}$$

$$B(1,1-\alpha)=\frac{1}{\Gamma(\alpha-1)}\cdot\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(2-\alpha)}\frac{d}{dx}\equiv 0$$

tenglik kelib chiqadi. Demak, $\varphi(x)=(x-a)^{\alpha-1}$ funksiya kasr tartibli hosila uchun o'zgarmas son vazifasini bajaradi.

Endi $\alpha\geq 1$ bo'lib, $[\alpha]$ - uning butun qismi, $\{\alpha\}$ - esa kasr qismi bo'lsin.

1. Agar α - butun son bo'lsa, α tartibli hosilalar sifatida oddiy hosilalarni olamiz:

$$D_{ax}^{\alpha} f\left(x\right)=\frac{d^{\alpha}}{dx^{\alpha}} f\left(x\right), \quad D_{xb}^{\alpha} g\left(x\right)=\left(-1\right)^{\alpha}\frac{d^{\alpha}}{dx^{\alpha}} g\left(x\right), \quad \alpha=1,2,3,...$$

2. Agar α - kasr son, ya'ni $\alpha=\left[\alpha\right]+\left\{\alpha\right\}$ bo'lsa, α tartibli hosilalarni quyidagicha aniqlaymiz:

$$D_{ax}^{\alpha}\varphi(x)=\left(\frac{d}{dx}\right)^{[\alpha]} D_{ax}^{\{\alpha\}}\varphi(x)=\left(\frac{d}{dx}\right)^{[\alpha]+1} D_{ax}^{\{\alpha\}-1}\varphi(x),$$

$$D_{xb}^{\alpha}\varphi(x)=\left(-\frac{d}{dx}\right)^{[\alpha]} D_{xb}^{\{\alpha\}}\varphi(x)=\left(-\frac{d}{dx}\right)^{[\alpha]+1} D_{xb}^{\{\alpha\}-1}\varphi(x).$$

Demak, umumiy holda, $\alpha\geq 1$ bo'lganda

$$D_{ax}^{\alpha}\varphi(x)=\left(\frac{d}{dx}\right)^{n} D_{ax}^{\alpha-n}\varphi(x), \qquad n=[\alpha]+1, \tag{1.33}$$

$$D_{xb}^{\alpha}\varphi(x)=(-1)^{n}\left(\frac{d}{dx}\right)^{n} D_{xb}^{\alpha-n}\varphi(x), \quad n=[\alpha]+1. \tag{1.34}$$

(1.33) va (1.34) hosilalar mavjud bo'lishi uchun

$$\int_{a}^{x}\frac{\varphi(t)\,dt}{(x-t)^{\{\alpha\}}}\in AC^{([\alpha])}([a,b]), \quad \int_{x}^{b}\frac{\varphi(t)\,dt}{(t-x)^{\{\alpha\}}}\in AC^{([\alpha])}([a,b])$$

bo'lishi zarur. Buning uchun esa $\varphi\left(x\right)\in AC\left(\left[a,b\right]\right)$ bo'lishi yetarli. Bu yerda $AC^{[\alpha]}\left(\left[a,b\right]\right)-\left[a,b\right]$ kesmada $\left[\alpha-1\right]$ tartibgacha hosilalari mavjud va $\varphi(x)^{([\alpha]-1)}\in AC\left(\left[a,b\right]\right)$ bo'lgan funksiyalar sinfi.

Yuqorida biz $0<\alpha<1$ bo'lgan holni ko'rgan edik. Endi biz $\alpha\geq 1$ bo'lgan holni ko'rib chiqamiz.

Agar $\varphi(x)=(x-a)^{\alpha-k},\ k=1,2,...1+\left[\alpha\right]$ bo'lsa u holda $D_{ax}^{\alpha}\varphi(x)\equiv 0$ tenglik o'rinli bo'ladi.

α $(\alpha>0)$ kasr tartibli integrallar ko'rinishida ifodalanuvchi funksiyalar sinfini $D_{ax}^{-\alpha}(L_p)$ bilan belgilaylik, ya'ni

$$D_{ax}^{-\alpha}\left(L_p\right)=\left\{f\left(x\right):f(x)=D_{ax}^{-\alpha}\varphi(x)\ ,\ \ \varphi(x)\in L_p(a,b),\ 1\leq p<\infty\right\}.$$

U holda, quyidagi teoremalar o'rinli.

1-teorema. $f(x)$ funksiya $D_{ax}^{-\alpha}(L_1)$ sinfga tegishli bo'lishi uchun

$$f_{n-\alpha}(x)=D_{ax}^{\alpha-n}\varphi(x)\in AC^{(n)}([a,b]),$$

$$\varphi_{n-\alpha}^{(k)}(a)=0\quad,\qquad k=0,1,2,...,n-1$$

bo'lishi zarur va yetarli, bu yerda $n=\left[\alpha\right]+1$.

2-teorema. $\alpha>0$ bo'lsin. U holda

$$D_{ax}^{\alpha} D_{ax}^{-\alpha} \varphi(x) = \varphi(x) \tag{1.35}$$

tenglik barcha $\varphi(x) \in L_1(a,b)$ funksiyalar uchun,

$$D_{ax}^{-\alpha} D_{ax}^{\alpha} \varphi(x) = \varphi(x) \tag{1.36}$$

tenglik esa barcha

$$\varphi(x) \in D_{ax}^{-\alpha}(L_1) \tag{1.37}$$

funksiyalar uchun bajariladi.

Agar (1.37) o'rniga $\varphi(x) \in L_1(a,b)$ bo'lsa, (1.36) tenglik umuman olganda noto'g'ri bo'ladi va u quyidagi formula bilan almashtiriladi:

$$D_{ax}^{-\alpha} D_{ax}^{\alpha} \varphi(x) = \varphi(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \varphi_{n-\alpha}^{(n-k-1)}(a),$$

bu yerda $n = [\alpha] + 1$, $\varphi_{n-\alpha}(x) = D_{ax}^{\alpha-n} \varphi(x)$.

II BOB. TOR TEBRANISH TENGLAMASI UCHUN BA'ZI CHEGARAVIY MASALALAR

2.1-§ Koshi masalasi

D orqali $x,\ y$ o'zgaruvchilar tekisligida chegara bo'laklari silliq S Jordan chizig'idan iborat bo'lgan sohani belgilaymiz.

Faraz qilaylik, $u(x,y)$ funksiya *D* sohada

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0 \tag{2.1}$$

tenglamaning regulyar yechimi bo'lib, $D\cup S$ da birinchi tartibli uzluksiz hosilalarga ega bo'lsin.

Ushbu

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial y_1}\left(\frac{\partial u}{\partial y_1}\right)=0$$

ayniyatni D soha bo'yicha integrallab, Gauss-Ostrogradskiy formulasini qo'llaymiz:

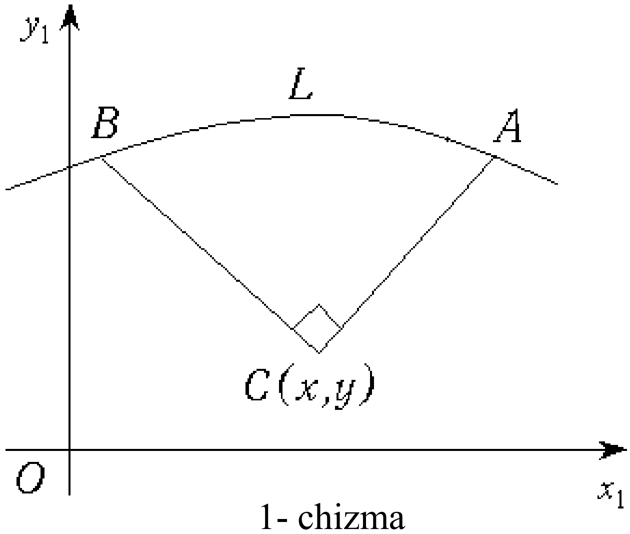
$$\int_D\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial y_1}\left(\frac{\partial u}{\partial y_1}\right)\right]dx_1dy_1=$$

$$=\int_S\frac{\partial u}{\partial x_1}dy_1+\frac{\partial u}{\partial y_1}dx_1=0. \tag{2.2}$$

Faraz qilaylik, L - yopiq bo'lmagan silliq Jordan chizig'i bo'lib, quyidagi ikki shartni qanoatlantirsin:

a) (2.1) tenglamaning $x+y=const,\quad x-y=const$ xarakteristikalar oilasiga tegishli bo'lgan har bir to'g'ri chiziq L bilan bitta nuqtada kesishsin;

b) L egri chiziqqa o'tkazilgan urinmaning yo'nalishi hech bir nuqtada (2.1) tenglama xarakteristikalarining yo'nalishi bilan ustma-ust tushmasin.



1- chizma

Ixtiyoriy $C(x,y)$ nuqtadan chiqadigan $x_1-x=y_1-y,$ $x_1-x=y-y_1$ xarakteristikalar L egri chiziq bilan A va B nuqtalarda kesishsin (1-chizma).

(2.2) formulani AB egri chiziq, CA va CB xarakteristikalar bilan chegaralangan sohaga qo'llab, ushbu tenglikni hosil qilamiz:

$$\int\limits_{AB+BC+CA} \frac{\partial u}{\partial x_1}\, dy_1 + \frac{\partial u}{\partial y_1}\, dx_1 = 0\,.$$

CA va BC da, mos ravishda, $dx_1 = dy_1$ va $dx_1 = -dy_1$ bo'lgani uchun avvalgi tenglik quyidagi ko'rinishda yoziladi:

$$\int\limits_{AB} \frac{\partial u}{\partial x_1} dy_1 + \frac{\partial u}{\partial y_1} dx_1 - \int\limits_{BC} du + \int\limits_{CA} du = 0\,.$$

Bundan darhol

$$u(C)=\frac{1}{2}u(A)+\frac{1}{2}u(B)+\frac{1}{2}\int_{AB}\frac{\partial u}{\partial x_1}dy_1+\frac{\partial u}{\partial y_1}dx_1 \tag{2.3}$$

tenglikka ega bo'lamiz.

Agar (2.1) tenglamaning $u(x,y)$ yechimi

$$u\,|_L=\varphi,\quad \frac{\partial u}{\partial l}\,|_L=\psi \tag{2.4}$$

shartlarni qanoatlantirsa, bunda φ va ψ berilgan, mos ravishda ikki va bir marta uzluksiz differentsiallanuvchi funksiyalar, l esa L da berilgan yo'nalish bo'lib, L ning urinmasi bilan ustma-ust tushmaydi, u holda $\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial y_1}$ noma'lum funksiyalarni

$$\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial s}+\frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial s}=\frac{d\varphi}{ds},$$

$$\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial l}+\frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial l}=\psi$$

tengliklardan aniqlab olamiz, bu yerda s - L chiziq yoyining uzunligi.

Ma'lum u, $\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial y_1}$ miqdorlarni (2.3) tenglikning o'ng tomoniga qo'yib, (2.1) tenglamaning (2.4) shartlarni qanoatlantiruvchi yechimini hosil qilamiz. Demak, Koshi masalasi birdan-bir turg'un yechimga egadir.

Odatda (2.1), (2.4) masala tor tebranish tenglamasi uchun umumiy qo'yilgan Koshi masalasi deyiladi.

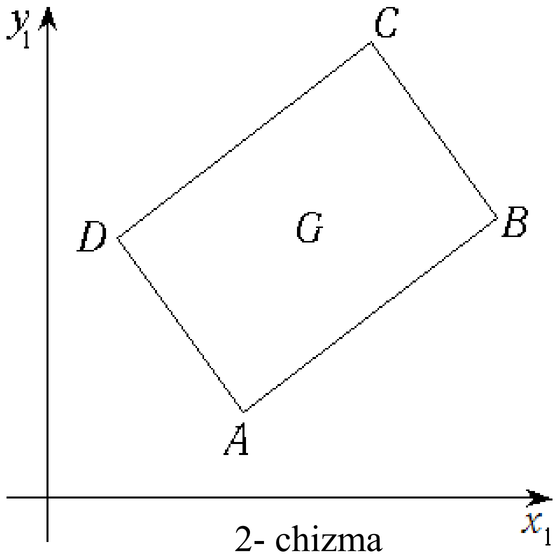
2.2-§ Gursa masalasi

(2.1) tenglamaning tayin $A(x_0, y_0)$ nuqtadan chiqadigan $AB: x_1 - x_0 = y_1 - y_0$, $AD: x_1 - x_0 = y_0 - y_1$ xarakteristikalari va $C(x, y)$ nuqtadan chiqadigan $CB: x_1 - x = y - y_1$, $CD: x_1 - x = y_1 - y$ xarakteristikalaridan tashkil topgan xarakteristik to'rtburchakni G orqali belgilab olamiz (2-chizma).

G soha uchun (2.2) formulani qo'llab,

$$\int\limits_{AB+BC+CD+DA} \frac{\partial u}{\partial x_1} dy_1 + \frac{\partial u}{\partial y_1} dx_1 = 0$$

tenglikka ega bo'lamiz.



2- chizma

AB va CD da $dx_1 = dy_1$, BC va DA da $dx_1 = -dy_1$ bo'lgani uchun avvalgi tenglikni quyidagi ko'rinishda yozib olamiz:

$$\int\limits_{AB} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 - \int\limits_{BC} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 + \int\limits_{CD} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 -$$

$$- \int\limits_{DA} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial y_1} dy_1 = \int\limits_{AB} du - \int\limits_{BC} du + \int\limits_{CD} du - \int\limits_{DA} du =$$

$$= 2u(B) - 2u(A) - 2u(C) + 2u(D) = 0.$$

Bundan

$$u(C) = u(B) + u(D) - u(A) \qquad (2.5)$$

yoki

$$u(A)+u(C)=u(B)+u(D).$$

Bu tenglik (2.1) tenglama uchun Asgeyrsson printsipi yoki o'rta qiymat to'g'risidagi teorema deb ataladi.

Bunga asosan (2.1) tenglama $u(x,y)$ yechimining xarakteristik to'rtburchak qarama-qarshi uchlaridagi qiymatlarining yig'indisi bir-biriga tengdir.

Asgeyrsson printsipi (2.1) tenglamaga masala qo'yishda va qo'yilgan tenglamalarni yechishda muhim rol o'ynaydi.

(2.1) tenglamaning $u(x,y)$ yechimi

$$u\,|_{AB}=\varphi(x_1), \qquad u\,|_{AD}=\psi(x_1), \quad \varphi(A)=\psi(A) \tag{2.6}$$

shartlarni qanoatlantirsin. U holda 2-chizmada B va D nuqtalarning koordinatalari mos ravishda $\left(\frac{x+x_0+y-y_0}{2},\frac{x-x_0+y+y_0}{2}\right)$ va $\left(\frac{x+x_0-y+y_0}{2},\frac{-x+x_0+y+y_0}{2}\right)$ lardan iboratligini e'tiborga olsak, (2.5) ga asosan, (2.1), (2.6) masalaning yechimi uchun

$$u(x,y)=\varphi\left(\frac{x+y+x_0-y_0}{2}\right)+\psi\left(\frac{x-y+x_0+y_0}{2}\right)-\varphi(x_0) \tag{2.7}$$

formula kelib chiqadi.

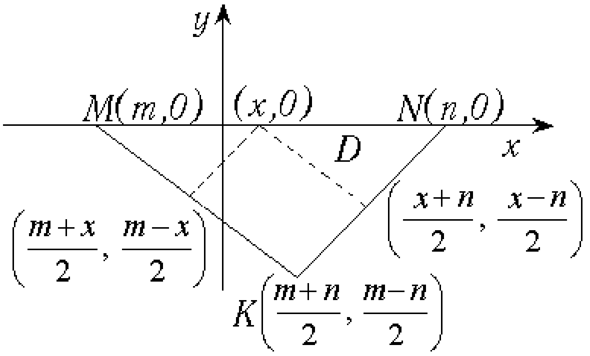
Agar $\varphi(x)$ va $\psi(x)$ funksiyalar ikki marta uzluksiz differentsiallanuvchi funksiyalar bo'lsa, (2.7) formula bilan aniqlangan $u(x,y)$ funksiya (2.1), (2.6) masalaning regulyar yechimidan iborat bo'ladi. Bu masala yechimining yagonaligi

Asgeyrsson printsipidan yoki (2.7) formulani hosil qilish usulidan darhol kelib chiqadi.

Odatda (2.1), (2.6) masala tor tebranish tenglamasi uchun umumiy qo'yilgan Gursa masalasi deyiladi.

2.3-§ Siljishli masalalar

Masalalarni bayon qilishda chegaraviy shartlar D soha chegarasining bir qismidagina berilib, qolgan qismi esa chegaraviy shartlardan ozod qolmoqda. Ikkinchi tomondan, Gursa va Darbu masalalarining qo'yilishidan (Asgeyrsson printsipidan ham) kelib chiqadiki, D



3- chizma

sohada (2.1) tenglama uchun Dirixle masalasi, ya'ni D da (2.1) tenglamaning $u|_{\overline{MN}\cup\overline{MK}\cup\overline{NK}}=\psi(x,y)$ shartni qanoatlantiruvchi yechimini topish haqidagi masala qaralsa, u ortiqcha shartli masala bo'lar ekan. Bular asosida 1960 yillarga kelib quyidagi savol paydo bo'ldi: gierbolik tipdagi tenglamalar uchun xarakteristik uchburchak chegarasining barcha qismida chegaraviy shartlar berilgan korrekt masala mavjudmi? Bu savolga 1960 yillarning oxiriga kelib A.M. Naxushev tomonidan ijobiy javob berildi [8]. Quyida biz ana shu masalalarning ba'zilarini bayon qilamiz.

Shunday qilib (2.1) tenglamani 3-chizmadagi $D\equiv MNK$ xarakteristik uchburchakda qaraymiz.

1-masala. (2.1) tenglamaning quyidagi shartlarni qanoatlantiruvchi $u(x,y)\in C(\overline{D})\cap C^2(D)$ yechimi to'ilsin:

$$u(x,0)=\tau(x),\ m\le x\le n; \tag{2.8}$$

$$\alpha(x)u\left(\frac{m+x}{2},\frac{m-x}{2}\right)+\beta(x)u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=\gamma(x),\qquad m\le x\le n, \tag{2.9}$$

bu yerda $\tau(x),\ \alpha(x),\ \beta(x),\ \gamma(x)\in C[m,n]\cap C^2(m,n)$-berilgan funksiyalar bo'lib,

$$\alpha(x)\neq\beta(x),\quad m\leq x\leq n,\quad \alpha(n)\beta(m)\neq 0,$$
$$\alpha(n)[\gamma(m)-\alpha(m)\tau(m)]=\beta(m)[\gamma(n)-\beta(n)\tau(n)] \tag{2.10}$$

shartlarni qanoatlantiradi.

Agar bu yerda ixtiyoriy $x\in[m,n]$ uchun

$$(x,0)\in\overline{MN},\qquad \left(\frac{m+x}{2},\frac{m-x}{2}\right)\in\overline{MK},\ \left(\frac{x+n}{2},\frac{x-n}{2}\right)\in\overline{NK}$$

ekanligini e'tiborga olsak, (2.8) shart noma'lum funksiyaning $\overline{MN}$ dagi qiymatini, (2.9) shart esa uning $\overline{MK}$ va $\overline{NK}$ dagi qiymatlari orasidagi munosabatni ifodalayotgani, va, demak, bu masalada D soha chegarasining barcha qismi chegaraviy shart bilan band ekanligi kelib chiqadi.

$$(x,0),\qquad \left(\frac{m+x}{2},\frac{m-x}{2}\right),\qquad \left(\frac{m+n}{2},\frac{m-n}{2}\right),\qquad \left(\frac{x+n}{2},\frac{x-n}{2}\right)$$

nuqtalar xarakteristik to'rtburchakning uchlaridan iboratligini inobatga olsak, Asgeyrsson printsipiga asosan

$$u\left(\frac{m+x}{2},\frac{m-x}{2}\right)+u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=u(K)+\tau(x) \tag{2.9_1}$$

kelib chiqadi, bu yerda $u(K)$ aniq son bo'lib, (2.10) dagi ikkinchi va uchinchi shartlarga asosan, (2.9) dan bir qiymatli aniqlanadi.

Bu tenglik (2.9) shart bilan birgalikda ikki noma'lumli ikki tenglamalar sistemasini tashkil etib, (2.10) dagi birinchi shartga asosan, bu sistemadan $u\left(\frac{m+x}{2},\frac{m-x}{2}\right)$, $u\left(\frac{x+n}{2},\frac{x-n}{2}\right)$ lar bir qiymatli topiladi, demak, berilgan masala Gursa masalasiga keladi.

Gursa masalasi korrekt qo'yilganligi uchun 1 -masala ham korrekt qo'yilgan bo'ladi.

Izoh. (2.10) dagi birinchi shart masala korrekt qo'yilgan bo'lishi uchun muhim rol o'ynaydi. Haqiqatdan ham, masalan, agar $\alpha(x)\equiv\beta(x)\equiv 1$ bo'lsa, masala faqat va faqat $\tau(x)-\tau(n)=\gamma(x)-\gamma(n)$ shart bajarilgandagina yechimga ega va bunda masala cheksiz ko'p chiziqli bog'liq bo'lmagan yechimlarga ega bo'ladi.

2-masala. (2.1) tenglamaning quyidagi shartlarni qanoatlantiruvchi $u(x,y)\in C(\overline{D})\cap C^1(D\cup MN)\cap C^2(D)$ yechimi topilsin:

$$u(x,0)=\tau(x), \qquad m\le x\le n; \tag{2.8}$$

$$\alpha(x)\frac{d}{dx}u\left(\frac{m+x}{2},\frac{m-x}{2}\right)+\beta(x)\frac{d}{dx}u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=\gamma(x),\ m<x<n, \tag{2.11}$$

bu yerda $\tau(x)\in C[m,n]\cap C^2(m,n)$, $\alpha(x),\beta(x),\gamma(x)\in C[m,n]\cap$ $\cap C^1(m,n)$ - berilgan funksiyalar bo'lib, $\alpha(x)\neq\beta(x)$, $x\in[m,n]$ shartni qanoatlantiradi.

Koshi masalasi korrekt qo'yilganligiga asoslanib, masala yechimini

$$u(x,y)=\frac{1}{2}[\tau(x-y)+\tau(x+y)]+\frac{1}{2}\int_{x-y}^{x+y}\nu(t)dt \tag{2.12}$$

ko'rinishda qidiramiz, bu yerda $\nu(x)\in C^1(m,n)$ - hozircha noma'lum funksiya.

(2.12) formula bilan aniqlanuvchi $u(x,y)$ funksiya (2.1) tenglamani va (2.8) shartni qanoatlantiradi. $\nu(x)$ funksiyani shunday tanlaymizki, (2.12) funksiya (2.11) shartni ham qanoatlantirsin.

(2.12) ga asosan

$$u\left(\frac{m+x}{2},\frac{m-x}{2}\right)=\frac{1}{2}[\tau(x)+\tau(m)]-\frac{1}{2}\int_m^x \nu(t)\,dt,$$

$$u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=\frac{1}{2}[\tau(n)+\tau(x)]-\frac{1}{2}\int_x^n \nu(t)\,dt.$$

Bu tengliklardan x bo'yicha hosila olib, so'ngra (2.11) ga qo'yib, $\alpha(x)\neq\beta(x),\ m\leq x\leq n$ ekanligini inobatga olsak, $\nu(x)$ funksiya

$$\nu(x)=\{[\alpha(x)+\beta(x)]\cdot\tau\ '(x)-2\gamma(x)\}/[\alpha(x)-\beta(x)]$$

tenglik bilan bir qiymatli topiladi. Bundan va berilgan funksiyalarga qo'yilgan shartlardan $\nu(x)\in C^1(m,n)$ kelib chiqadi.

Bu yerda ham masala korrekt qo'yilgan bo'lishi uchun $\alpha(x)\neq\beta(x)$, $m\leq x\leq n$ shart muhimdir. Aks holda masala faqat va faqat $[\alpha(x)+\beta(x)]\cdot\tau\ '(x)=2\gamma(x)$ tenglik bajarilganda yechimga ega bo'ladi. Agar bu shart bajarilgan bo'lsa, (2.12) formula ixtiyoriy $\nu(x)\in C^1(m,n)$ funksiyada masalaning yechimi bo'laveradi.

3-masala. (2.1) tenglamaning quyidagi shartlarni qanoatlantiruvchi $u(x,y)\in C(\bar{D})\cap C^1(D\cup MN)\cap C^2(D)$ yechimi to'ilsin:

$$u_y(x,0)=\nu(x), \qquad m<x<n; \tag{2.13}$$

$$u\left(\frac{m+x}{2},\frac{m-x}{2}\right)+\beta(x)\,u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=\gamma(x), \quad m\le x\le n, \tag{2.14}$$

$$\beta(m)\,u(n,0)=\beta_1,$$

bu yerda $\nu(x)\in C^1(m,n)-$berilgan funksiya $x\to n,\ x\to m$ da birdan kichik tartibda cheksizga intilishi mumkin; $\beta(x),\gamma(x)$-berilgan funksiyalar bo'lib, $C[m,n]\cap C^2(m,n)$ sinfga tegishli va $\beta(x)\ne -1,\ x\in[m,n]$; β_1- berilgan son bo'lib, $\beta(m)=0$ bo'lganda nol deb olinadi.

Koshi - Gursa masalasi korrekt qo'yilganligiga asoslanib, masala yechimini

$$u(x,y)=\varphi(x+y)+\varphi(x-y)-\varphi(m)+\int_m^{x+y}\nu(t)\,dt \tag{2.15}$$

ko'rinishda qidiramiz, bu yerda $\varphi(x)\in C[m,n]\cap C^2(m,n)$ -- noma'lum funksiya.

(2.15) funksiya (2.1) tenglamani va (2.13) shartni qanoatlantiradi. $\varphi(x)$ funksiyani shunday tanlaylikki, (2.15) funksiya (2.14) shartlarni ham qanoatlantirsin.

(2.15) ga asosan

$$u\left(\frac{m+x}{2},\frac{m-x}{2}\right)=\varphi(x), \tag{2.16}$$

$$u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=\varphi(x)+\varphi(n)-\varphi(m)+\int_m^x \nu(t)dt.$$

Bularni (2.14) shartlarning birinchisiga qo'ysak,

$$[1+\beta(x)]\varphi(x)=\gamma(x)-\beta(x)[\varphi(n)-\varphi(m)]-\beta(x)\int_m^x \nu(t)dt \quad (2.17)$$

tenglik kelib chiqadi.

(2.14) shartlar va (2.16) tenglikdan $x=n$ va $x=m$ da

$$u(K)=\gamma(n)-\beta_1=\varphi(n),$$

$$u(M)=\gamma(m)-\beta(m)\,[\gamma(n)-\beta_1]=\varphi(m),$$

ya'ni $\varphi(m),\varphi(n)$ kattaliklar berilganlar orqali to'la aniqlanishi kelib chiqadi.

Buni va $\beta(x)\neq -1,\ x\in[m,n]$ ekanligini inobatga olsak, $\varphi(x)$ funksiya (2.17) tenglikdan bir qiymatli aniqlanadi. Berilgan funksiyalarga qo'yilgan shartlarga asosan, topilgan $\varphi(x)$ funksiya $C[m,n]\cap C^2(m,n)$ sinfga tegishli bo'ladi.

Bu masalada ham $\beta(x)\neq -1,\ x\in[m,n]$ shart muhimdir. Aks holda, masala faqat va faqat $\gamma\,'(x)=-\nu(x)$ shart bajarilganda yechimga ega bo'ladi va bu shart bajarilgan bo'lsa, (2.15) formula ixtiyoriy $\varphi(x)$ funksiya olinganda ham masala yechimini ifodalaydi.

A.M.Naxushev tomonidan (2.1) tenglama uchun qo’yilgan va o’rganilgan masalalar hozirgi kungacha giperbolik tipdagi ancha murakkab tenglamalar uchun ham o’rganildi va matematik adabiyotlarda «siljishli masalalar» deb nomlandi.

Ma’lumki

$$u_{xx} - u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$$

tenglamaning koeffitsientlari qaralayotgan soha yopig’ida yetarlicha silliq bo’lsa, bu tenglama uchun klassik masalalar (Koshi, Gursa, Darbu, Koshi-Gursa masalalari) xuddi tor tebranish tenglamasidagidek qo’yiladi va korrekt bo’ladi, ya’ni tenglama kichik hadlarining koeffitsientlari klassik masalalar korrektligiga ta’sir ko’rsatmaydi. Siljishli masalalarni o’rganishda esa tenglama kichik hadlarining koeffitsientlari masala qo’yilishiga va korrektligiga jiddiy ta’sir ko’rsatadi, ya’ni har bir tenglama uchun o’ziga mos (agar mavjud bo’lsa) siljishli masala qo’yiladi.

Masalan, agar $n - m \neq 2k\pi,\ k = 0,1,2,...$ bo’lsa,

$$u_{xx} - u_{yy} = 0$$

tenglama uchun

$$u(x,0) = 0, \quad m \le x \le n; \tag{2.8$_1$}$$

$$\cos\frac{n-x}{2}\frac{d}{dx}u\left(\frac{m+x}{2},\frac{m-x}{2}\right) + \cos\frac{x-m}{2}\frac{d}{dx}u\left(\frac{x+n}{2},\frac{x-n}{2}\right) = 0 \tag{2.11$_1$}$$

bir jinsli masala faqat trivial yechimga ega.

Haqiqatdan ham, (2.16_1) shart va Asgeyrsson printsipini ifodalovchi (2.14_1) tenglikdan kelib chiquvchi ushbu

$$\frac{d}{dx}u\left(\frac{m+x}{2},\frac{m-x}{2}\right)+\frac{d}{dx}u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=\tau'(x)$$

tenglikdan

$$u\left(\frac{m+x}{2},\frac{m-x}{2}\right)=u\,|_{\overline{MK}}=0,\qquad u\left(\frac{x+n}{2},\frac{x-n}{2}\right)=u\,|_{\overline{NK}}=0$$

kelib chiqadi. Bu esa (2.8_1), (2.11_1) masala bir jinsli Gursa masalasiga ekvivalentligini ko'rsatadi. Oxirgi masala esa faqat trivial yechimga ega.

Lekin (2.8_1), (2.11_1) masala $u_{xx}-u_{yy}-u=0$ tenglama uchun trivial bo'lmagan $u(x,y)=\sin y$ yechimga ega. Bu tenglama uchun esa (2.8_1) va

$$\cos\frac{n-x}{2}A_{mx}^{1,1}\left[\frac{d}{dx}u\left(\frac{m+x}{2},\frac{m-x}{2}\right)\right]+$$
$$+\cos\frac{x-m}{2}A_{nx}^{1,1}\left[\frac{d}{dx}u\left(\frac{x+n}{2},\frac{x-n}{2}\right)\right]=0$$

shartlar bilan berilgan masala korrekt qo'yilgan bo'ladi , bu yerda

$$A_{kx}^{1,1}[f(x)]\equiv f(x)-\int_k^x f(t)\frac{t-k}{x-k}\frac{\partial}{\partial t}J_0[\sqrt{(x-k)(x-t)}]\,dt,$$

$J_o(x)-$birinchi turdagi nolinchi tartibli Bessel funksiyasi.

Izoh. Yuqorida ko’rib o’tilgan siljishli masalalardan ilgari o’rganilgan chegaraviy masalalar kelib chiqadi. Haqiqatdan ham:

1. 1 - va 2 - siljishli masalalar Gursa masalasiga ekvivalent. Buni 1 - masala misolida ko’rsatdik.

2. 1 - va 2 - siljishli masalalardan $\beta(x)\equiv 0, \alpha(x)\neq 0$ $[\alpha(x)\equiv 0, \beta(x)\neq 0]$ da Darbuning birinchi masalasi kelib chiqadi.

3. $\beta(x)\equiv 0$ da 3-siljishli masaladan Koshi-Gursa masalasi kelib chiqadi.

III BOB. CHEGARAVIY SHARTIDA KASR TARTIBLI INTEGRODIFFERENTSIAL OPERATORLAR QATNASHGAN SILJISHLI MASALALAR.

3.1-§ Tor tebranish tenglamasi uchun chegaraviy masalalar.

(x,y)-o'zgaruvchilar tekisligida

$$U_{xx} - U_{yy} = 0 \tag{3.1}$$

tenglamaning $AC: x+y=0, \quad BC: x-y=1$ xarakteristikalari va $y=0$ to'g'ri chiziqning $\overline{J}=[0,1]$ kesmasi bilan chegaralangan D sohani ko'raylik.

Masala. Quyidagi shartlarni qanoatlantiruvchi $U(x,y)$ funksiya topilsin.

1) $U(x,y) \in C(\overline{D}) \cap C^2(D)$,

2) D sohada (3.1) tenglamani qanoatlantiradi.

3) D soha chegarasida

$$U(x,0) = \tau(x), \qquad x \in \overline{J} \tag{3.2}$$

$$a(x)D_{0x}^{\alpha}U\left(\frac{x}{2},-\frac{x}{2}\right) + b(x)D_{x1}^{\beta}U\left(\frac{x+1}{2},\frac{x-1}{2}\right) =$$

$$= p(x)U_y(x,0) + q(x), \qquad x \in J \tag{3.3}$$

shartlarni qanoatlantiradi. Bu yerda $\alpha, \beta -$ haqiqiy sonlar; $\tau(x)$, $a(x)$, $b(x)$, $p(x)$, $q(x)$-berilgan funksiyalar, $D_{0x}^{\alpha}, D_{x1}^{\beta}$ -kasr tartibli integrodifferentsial operatorlar [8].

1-teorema. Agar $U(x,y)$ funksiya (3.1)-(3.3) masalaning regulyar yechimi bo'lsa, u holda $U_y(x,0)=\nu(x)$ funksiya

$$2p(x)\nu(x) + a(x)D_{0x}^{\alpha-1}\nu(x) + b(x)D_{x1}^{\beta-1}\nu(x) = \gamma(x), \tag{3.4}$$

tenglik bilan topiladi.

Bu yerda

$$\gamma(x) = a(x)D_{0x}^{\alpha}\left[\tau(0)+\tau(x)\right]+b(x)D_{x1}^{\beta}\left[\tau(x)+\tau(1)\right]-2q(x) \qquad (3.5)$$

Isboti. Berilgan tenglama uchun Koshi masalasi korrekt qo'yilganligiga asoslanib, masala yechimini

$$U(x,y)=\frac{1}{2}\left[\tau(x-y)+\tau(x+y)\right]+\frac{1}{2}\int_{x-y}^{x+y}\nu(t)dt, \qquad (3.6)$$

ko'rinishda qidiramiz, bu yerda $\nu(x)\in C^2(J)$ - noma'lum funksiya[9].

$U(x,y)$ funksiya (3.1) tenglama va (3.2) shartni qanoatlantiradi. Biz $\nu(x)$ funksiyani shunday tanlaymizki, (3.6) funksiya (3.3) shartni ham qanoatlantirsin.

Berilgan (3.6) formulaga asosan,

$$U(\frac{x}{2},-\frac{x}{2})=\frac{1}{2}\left[\tau(x)+\tau(0)\right]-\frac{1}{2}\int_{0}^{x}\nu(t)dt=\frac{1}{2}\left[\tau(x)+\tau(0)\right]-\frac{1}{2}D_{0x}^{-1}\nu(x)\ , \qquad (3.7)$$

$$U(\frac{x+1}{2},-\frac{x-1}{2})=\frac{1}{2}\left[\tau(1)+\tau(x)\right]-\frac{1}{2}\int_{x}^{1}\nu(t)dt=\frac{1}{2}\left[\tau(1)+\tau(x)\right]-\frac{1}{2}D_{x1}^{-1}\nu(x)\, . \qquad (3.8)$$

Bu tengliklarni (3.3) shartga qo'yib, (3.4) tenglikni hosil qilamiz.

$$a(x)D_{0x}^{\alpha}[\frac{1}{2}(\tau(0)+\tau(x))-\frac{1}{2}D_{0x}^{-1}\nu(x)]+b(x)D_{x1}^{\beta}[(\tau(x)+\tau(1))-\frac{1}{2}D_{x1}^{-1}\nu(x)]=$$

$$=\frac{1}{2}a(x)D_{0x}^{\alpha}(\tau(0)+\tau(1))-\frac{1}{2}a(x)D_{0x}^{\alpha-1}\nu(x)+\frac{1}{2}b(x)D_{x1}^{\beta}(\tau(x)+\tau(1))-$$

$$-\frac{1}{2}b(x)D_{x1}^{\beta-1}\nu(x)=p(x)\nu(x)+q(x)$$

$$2p(x)\nu(x)+a(x)D_{0x}^{\alpha-1}\nu(x)+b(x)D_{x1}^{\beta-1}\nu(x)=\gamma(x), \qquad (3.4)$$

$$\gamma(x)=a(x)D_{0x}^{\alpha}\left[\tau(0)+\tau(x)\right]+b(x)D_{x1}^{\beta}\left[\tau(x)+\tau(1)\right]-2q(x) \tag{3.5}$$

Teorema isbotlandi.

1-hol: $\alpha<1$, $\beta=1$ bo'lsin.

2-teorema. Agar $\tau(x)\in C^1(\overline{AB})\cap C^3(AB)$, $a(x)$, $b(x)$, $p(x)$, $q(x)\in C(\overline{AB})$, $2p(x)+b(x)\neq 0$ bo'lsa. U holda (3.1) – (3.3) masalaning yechimi mavjud va yagona bo'ladi.

Isbot. Bu holda (3.4) tenglikni quyidagi ko'rinishda yozish mumkin.

$$2p(x)\nu(x)+a(x)D_{0x}^{\alpha-1}\nu(x)+b(x)\nu(x)=\gamma(x), \tag{3.9}$$

$$[2p(x)+b(x)]\nu(x)+\frac{a(x)}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}\nu(t)dt=\gamma(x), \tag{3.10}$$

Integral operator ta'rifidan

$$\nu(x)+\int_0^x K_1(x,t)\nu(t)dt=\gamma_1(x), \quad 0<x<1 \tag{3.11}$$

tenglik kelib chiqadi.

Bu yerda

$$K_1(x,t)=\frac{a(x)(x-t)^{-\alpha}}{\Gamma(1-\alpha)(2p(x)+b(x))}, \qquad \gamma_1(x)=\frac{\gamma(x)}{2p(x)+b(x)}, \tag{3.12}$$

Hosil bo'lgan (3.11) tenglama ikkinchi tur Volterra integral tenglamasi bo'lganligi uchun (3.1)-(3.3) masala yechimi mavjud va yagonadir.

2-hol: $\alpha=1$, $\beta<1$ bo'lsin.

3-teorema. Agar $\tau(x)\in C^1(\overline{AB})\cap C^3(AB)$, $a(x)$, $b(x)$, $p(x)$, $q(x)\in C(\overline{AB})$, $2p(x)+a(x)\neq 0$ bo'lsa. U holda (3.1) – (3.3) masalaning yechimi mavjud va yagona bo'ladi.

Isbot. Bu holda (3.4) tenglikni quyidagi ko'rinishda yozish mumkin.

$$2p(x)\nu(x)+b(x)D_{x1}^{\beta-1}\nu(x)+a(x)\nu(x)=\gamma(x), \qquad (3.13)$$

$$[2p(x)+a(x)]\nu(x)+\frac{b(x)}{\Gamma(1-\beta)}\int_x^1(t-x)^{-\beta}\nu(t)dt=\gamma(x), \qquad (3.14)$$

Integral operator ta'rifidan

$$\nu(x)+\int_x^1 K_2(x,t)\nu(t)dt=\gamma_2(x), \qquad 0<x<1 \qquad (3.15)$$

tenglik kelib chiqadi. Bu yerda

$$K_2(x,t)=\frac{b(x)(t-x)^{-\beta}}{\Gamma(1-\beta)(2p(x)+a(x))}, \qquad \gamma_2(x)=\frac{\gamma(x)}{2p(x)+a(x)}, \qquad (3.16)$$

Hosil bo'lgan (3.15) tenglama ikkinchi tur Volterra integral tenglamasi bo'lganligi uchun (3.1)-(3.3) masala yechimi mavjud va yagonadir.

3-hol: $1<\alpha<2, \quad \beta=1$ bo'lsin.

4-teorema. Agar quyidagi shartlar bajarilgan bo'lsa,

1) $\nu(x)=x^{\alpha-2}\nu_1(x), \quad \nu_1(x)\in C^1(\overline{J}), \quad \nu_1(0)\neq 0.$

2) $\tau(x)=x^{\sigma}\tau_1(x), \quad \tau_1(x)\in C^1(\overline{J})\cap C^3(J)$, $\alpha-1<\sigma<\alpha$

3) $a(x)$, $b(x)$, $p(x)$, $q(x)\in C^1(\overline{J})$, $2p(x)+b(x)\neq 0$

u holda (3.1)-(3.3) masala cheksiz ko'p yechimga ega bo'ladi.

Isboti. Bu holda (3.4) tenglik quyidagi ko'rinishga o'tadi

$$[2p(x)+b(x)]\nu(x)+a(x)\frac{d}{dx}D_{0x}^{\alpha-2}\nu(x)=\gamma(x), \quad 0<x<1 \qquad (3.17)$$

Quyidagi belgilashni kiritamiz,

$$\varphi(x)=D_{0x}^{\alpha-2}\nu(x), \qquad (3.18)$$

Hosil qilingan (3.18) tenglikning har ikki tomoniga $D_{0x}^{2-\alpha}$ teskari operatorni qo'llaymiz. Natijada

$$\nu(x) = D_{0x}^{2-\alpha}\varphi(x), \tag{3.19}$$

tenglikni olamiz. Bundan (3.18) va (3.19) tengliklarni (3.17) qo'llab

$$a(x)\frac{d}{dx}\varphi(x) + [2p(x) + b(x)]\frac{d}{dx}D_{0x}^{1-\alpha}\varphi(x) = \gamma(x), \tag{3.20}$$

tenglikni hosil qilamiz.

Ba'zi sodda almashtirishlarni bajarib, (3.20) tenglikni

$$a(x)\frac{d}{dx}\varphi(x) + \frac{[2p(x)+b(x)](\alpha-1)}{\Gamma(\alpha-1)}x^{-1}\int_0^x (x-t)^{\alpha-2}\varphi(t)dt +$$

$$+\frac{2p(x)+b(x)}{\Gamma(\alpha-1)}x^{-1}\int_0^x (x-t)^{\alpha-2}t\varphi'(t)dt = \gamma(x), \tag{3.21}$$

ko'rinishda yozish mumkin.

$$\varphi(x) = D_{0x}^{\alpha-2}\nu(x) = \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}\nu(t)dt = \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}t^{\alpha-2}\nu_1(t)dt =$$

$$= \left\langle \begin{array}{c} t = xz \\ (x-t) = (1-z)x \end{array} \right\rangle = \frac{1}{\Gamma(2-\alpha)}x^{1-\alpha+\alpha-1}\int_0^1 z^{\alpha-2}(1-z)^{1-\alpha}\nu_1(xz)dz,$$

Bundan

$$\varphi(0) = \frac{\nu_1(0)}{\Gamma(1-\alpha)}B(\alpha-1, 2-\alpha) \neq 0$$

(3.18) shartdan hamda 4-teoremaning (1) shartidan quyidagi kelib chiqadi.

$$\varphi(0) = \frac{\nu_1(0)}{\Gamma(1-\alpha)}B(\alpha-1, 2-\alpha) \neq C_0 \neq 0 \tag{3.22}$$

U holda, agar

$$\psi(x) = \frac{d}{dx}\varphi(x) \tag{3.23}$$

belgilash kiritsak (3.22) ga asosan

$$\varphi(x) = C_0 + \int_0^x \psi(t)dt, \tag{3.24}$$

tenglikni olamiz.

Agar (3.23),(3.24) tengliklarni (3.21) ga qo'ysak,

$$a(x)\psi(x) + \frac{[2p(x)+b(x)](\alpha-1)}{\Gamma(\alpha-1)x}\int_0^x (x-t)^{\alpha-2}dt\int_0^t \psi(t_1)dt_1 +$$

$$+\frac{2p(x)+b(x)}{\Gamma(\alpha-1)x}\int_0^x (x-t)^{\alpha-2}t\psi(t)dt = g(x)+\gamma(x), \tag{3.25}$$

$$g(x) = \frac{[2p(x)+b(x)](\alpha-1)}{\Gamma(\alpha-1)x}C_0\int_0^x (x-t)^{\alpha-2}dt \tag{3.26}$$

tenglik hosil bo'ladi.

Ba'zi almashtirishlardan keyin,

$$\psi(x) + \int_0^x K_3(x,t)\psi(t)dt = \gamma_3(x), \tag{3.27}$$

$$K_3(x,t) = \frac{[2p(x)+b(x)]x^{\alpha-2}}{\Gamma(\alpha-1)a(x)} + \frac{2p(x)+b(x)(x-t)^{\alpha-2}}{\Gamma(\alpha-1)a(x)x}t\ ,\quad \gamma_3(x) = \frac{\gamma(x)-g(x)}{a(x)} \tag{3.28}$$

Hosil bo'lgan (3.27) tenglikdan ikkinchi tur Volterra integral tenglamasi kelib chiqadi. U holda (3.4) va (3.27) tenglamalar ekvivalent bo'lganligi uchun (3.4) tenglama ham yechimga ega bo'ladi. Bundan (3.1)-(3.3) masala ham yechimga ega ekanligi va uning yechimi (3.6) formula bilan aniqlanganligi kelib chiqadi.

Qo'yilgan masala yechimining yagona emasligini isbotlash uchun (3.4) tenglamaga mos bir jinsli tenglama

$$2p(x)\nu(x) + a(x)D_{0x}^{\alpha-1}\nu(x) + b(x)D_{x1}^{\beta-1}\nu(x) = 0, \tag{3.29}$$

trivial bo'lmagan yechimga ega ekanligini ko'rsatish yetarli bo'ladi.

Bir qator almashtirishlardan keyin o'ng tomoni (3.28) tenglik bilan aniqlanadigan (3.27) ko'rinishdagi ikkinchi tur Volterra integral tenglamasini hosil qilamiz.

$$\psi(x) + \int_0^x K_3(x,t)\psi(t)dt = g_1(x), \tag{3.30}$$

Bu yerda $g_1(x) \neq 0, \quad x \in \overline{J}$

Demak, (3.29) tenglamaning trivial bo'lmagan yechimi mavjud. Shuning uchun ham bir jinsli bo'lmagan (3.4) tenglama cheksiz ko'p yechimga ega bo'ladi. Teorema isbotlandi.

3.2-§. Yuqori ko'rsatkichli integrodifferentsial operatorlar yordamida siljishli masalalar yechish.

(x,y)-o'zgaruvchilar tekisligida

$$U_{xx} - U_{yy} = 0 \qquad (3.31)$$

tenglamaning $AC: x+y=0, \quad BC: x-y=1$ xarakteristikalari va $y=0$ to'g'ri chiziqning $\overline{J}=[0,1]$ kesmasi bilan chegaralangan D sohani ko'raylik.

Masala. Quyidagi shartlarni qanoatlantiruvchi $U(x,y)$ funksiya topilsin.

1) $U(x,y) \in C(\overline{D}) \cap C^2(D)$,

2) D sohada (3.31) tenglamani qanoatlantiradi.

3) D soha chegarasida

$$U(x,0) = \tau(x), \qquad x \in \overline{J} \qquad (3.32)$$

$$a(x)D_{0x^k}^{\alpha}U\left(\frac{x}{2},-\frac{x}{2}\right)+b(x)D_{x^k 1}^{\beta}U\left(\frac{x+1}{2},\frac{x-1}{2}\right)=$$

$$= p(x)U_y(x,0)+q(x), \qquad x \in J \qquad (3.33)$$

shartlarni qanoatlantiradi. Bu yerda $\alpha,\beta-$ haqiqiy sonlar; $\tau(x)$, $a(x)$, $b(x)$, $p(x)$, $q(x)$-berilgan funksiyalar, $D_{0x^k}^{\alpha}$,$D_{x^k 1}^{\beta}$ -kasr tartibli integrodifferentsial operatorlar [8].

Biz $k=2$ bo'lgan holni ko'rib chiqamiz.

1-teorema. Agar $U(x,y)$ funksiya (3.31)-(3.33) masalaning regulyar yechimi bo'lsa, u holda $U_y(x,0)=\nu(x)$ funksiya

$$4p(x)\nu(x)+a(x)D_{0x^2}^{\alpha-1}\frac{\nu(x)}{x}+b(x)D_{x^2 1}^{\beta-1}\frac{\nu(x)}{x}=\gamma(x), \qquad (3.34)$$

tenglik bilan topiladi.

Bu yerda

$$\gamma(x)=2a(x)D_{0x^2}^{\alpha}\left[\tau(0)+\tau(x)\right]+2b(x)D_{x^2 1}^{\beta}\left[\tau(x)+\tau(1)\right]-4q(x) \tag{3.35}$$

Isboti. Berilgan tenglama uchun Koshi masalasi korrekt qo'yilganligiga asoslanib, masala yechimini

$$U(x,y)=\frac{1}{2}\left[\tau(x-y)+\tau(x+y)\right]+\frac{1}{2}\int_{x-y}^{x+y}\nu(t)dt, \tag{3.36}$$

ko'rinishda qidiramiz, bu yerda $\nu(x)\in C^2(J)$ - noma'lum funksiya[9].

$U(x,y)$ funksiya (3.31) tenglama va (3.32) shartni qanoatlantiradi. Biz $\nu(x)$ funksiyani shunday tanlaymizki, (3.36) funksiya (3.33) shartni ham qanoatlantirsin.

Berilgan (3.36) formulaga asosan,

$$U(\frac{x}{2},-\frac{x}{2})=\frac{1}{2}\left[\tau(x)+\tau(0)\right]-\frac{1}{2}\int_0^x\nu(t)dt=\frac{1}{2}\left[\tau(x)+\tau(0)\right]-\frac{1}{4}D_{0x^2}^{-1}\frac{\nu(x)}{x}, \tag{3.37}$$

$$U(\frac{x+1}{2},-\frac{x-1}{2})=\frac{1}{2}\left[\tau(1)+\tau(x)\right]-\frac{1}{2}\int_x^1\nu(t)dt=\frac{1}{2}\left[\tau(1)+\tau(x)\right]-\frac{1}{4}D_{x^2 1}^{-1}\frac{\nu(x)}{x}. \tag{3.38}$$

Bu tengliklarni (3.33) shartga qo'yib, (3.34) tenglikni hosil qilamiz.

Teorema isbotlandi.

Quyidagi belgilashlarni kiritamiz

$$a_1(x)=a(\sqrt{x}),\qquad b_1(x)=b(\sqrt{x}),\qquad p_1(x)=\sqrt{x}\,p(\sqrt{x}),\qquad q_1(x)=q(\sqrt{x}),$$

$$\tau_1(x)=\tau(\sqrt{x}),\qquad \gamma_1(x)=\gamma(\sqrt{x}),\qquad \nu_1(x)=\nu(\sqrt{x}),\qquad \tilde{\nu}(x)=\frac{\nu_1(x)}{\sqrt{x}}.$$

1-hol: $2\alpha < 1, \quad \beta = 1$ bo'lsin.

2-teorema. Agar $\tau_1(x) \in C^1(\overline{J}) \cap C^3(J)$, $a_1(x)$, $b_1(x)$, $p_1(x)$, $q_1(x) \in C(\overline{J}) \cap C^2(J)$, $4p_1(x) + b_1(x) \neq 0$ bo'lsa. U holda (3.31) – (3.33) masalaning yechimi mavjud va yagona bo'ladi.

Isbot. Bu holda (3.34) tenglikni quyidagi ko'rinishda yozish mumkin.

$$4p_1(x)\widetilde{\nu}(x) + a_1(x)D_{0x}^{\alpha-1}\widetilde{\nu}(x) + b_1(x)\widetilde{\nu}(x) = \gamma_1(x), \tag{3.39}$$

$$[4p_1(x) + b_1(x)]\widetilde{\nu}(x) + \frac{a_1(x)}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}\,\widetilde{\nu}(t)dt = \gamma_1(x), \tag{3.40}$$

Integral operator ta'rifidan

$$\widetilde{\nu}(x) + \int_0^x K_1(x,t)\widetilde{\nu}(t)dt = \widetilde{\gamma}_1(x), \quad 0<x<1 \tag{3.41}$$

tenglik kelib chiqadi.

Bu yerda

$$K_1(x,t) = \frac{a_1(x)(x-t)^{-\alpha}}{\Gamma(1-\alpha)(4p_1(x) + b_1(x))}, \qquad \widetilde{\gamma}_1(x) = \frac{\gamma_1(x)}{4p_1(x) + b_1(x)}, \tag{3.42}$$

Hosil bo'lgan (3.41) tenglama ikkinchi tur Volterra integral tenglamasi bo'lganligi uchun (3.31)-(3.33) masala yechimi mavjud va yagonadir.

2-hol: $\alpha = 1, \quad 2\beta < 1$ bo'lsin.

3-teorema. Agar $\tau_1(x) \in C^1(\overline{J}) \cap C^3(J)$, $a_1(x)$, $b_1(x)$, $p_1(x)$, $q_1(x) \in C(\overline{J}) \cap C^2(J)$, $4p_1(x) + a_1(x) \neq 0$ bo'lsa. U holda (3.31) – (3.33) masalaning yechimi mavjud va yagona bo'ladi.

Isbot. Bu holda (3.34) tenglikni quyidagi ko'rinishda yozish mumkin.

$$4p_1(x)\widetilde{\nu}(x) + b_1(x)D_{x1}^{\beta-1}\widetilde{\nu}(x) + a_1(x)\widetilde{\nu}(x) = \gamma_2(x), \tag{3.43}$$

$$[4p_1(x)+a_1(x)]\widetilde{\nu}(x)+\frac{b_1(x)}{\Gamma(1-\beta)}\int_x^1 (t-x)^{-\beta}\widetilde{\nu}(t)dt=\gamma_2(x), \qquad (3.44)$$

Integral operator ta'rifidan

$$\widetilde{\nu}(x)+\int_0^x K_2(x,t)\widetilde{\nu}(t)dt=\widetilde{\gamma}_2(x), \quad , \quad 0<x<1 \qquad (3.45)$$

tenglik kelib chiqadi. Bu yerda

$$K_2(x,t)=\frac{b_1(x)(x-t)^{-\alpha}}{\Gamma(1-\alpha)(4p_1(x)+a_1(x))}, \qquad \widetilde{\gamma}_2(x)=\frac{\gamma_1(x)}{4p_1(x)+a_1(x)}, \qquad (3.46)$$

Hosil bo'lgan (3.45) tenglama ikkinchi tur Volterra integral tenglamasi bo'lganligi uchun (3.31)-(3.33) masala yechimi mavjud va yagonadir.

3-hol: $2<2\alpha<4, \quad \beta=1$ bo'lsin.

4-teorema. Agar quyidagi shartlar bajarilgan bo'lsa,

1) $\widetilde{\nu}(x)=x^{\alpha-2}\nu_{11}(x), \quad \nu_{11}(x)\in C^1(\overline{J}), \quad \nu_{11}(0)\neq 0.$

2) $\tau_1(x)=x^{\sigma}\tau_{11}(x), \quad \tau_{11}(x)\in C^1(\overline{J})\cap C^3(J)$, $2\alpha-1<\sigma<2\alpha$

3) $a_1(x)$, $b_1(x)$, $p_1(x)$, $q_1(x)\in C^1(\overline{J})$, $4p_1(x)+b_1(x)\neq 0$

u holda (3.31)-(3.33) masala cheksiz ko'p yechimga ega bo'ladi.

Isboti. Bu holda (3.34) tenglik quyidagi ko'rinishga o'tadi

$$[4p_1(x)+b_1(x)]\widetilde{\nu}(x)+a_1(x)\frac{d}{dx}D_{0x}^{\alpha-2}\widetilde{\nu}(x)=\gamma_1(x), \quad 0<x<1 \qquad (3.47)$$

Quyidagi belgilashni kiritamiz,

$$\varphi_1(x)=D_{0x}^{\alpha-2}\widetilde{\nu}(x), \qquad (3.48)$$

Hosil qilingan (3.48) tenglikning har ikki tomoniga $D_{0x}^{2-\alpha}$ teskari operatorni qo'llaymiz. Natijada

$$\widetilde{\nu}(x) = D_{0x}^{2-\alpha}\varphi_1(x), \tag{3.49}$$

tenglikni olamiz. Bundan (3.48) va (3.49) tengliklarni (3.47) qo'llab

$$a_1(x)\frac{d}{dx}\varphi(x) + [4p_1(x) + b_1(x)]\frac{d}{dx}D_{0x}^{2-\alpha}\varphi_1(x) = \gamma_1(x), \tag{3.50}$$

tenglikni hosil qilamiz.

Ba'zi sodda almashtirishlarni bajarib, (3.50) tenglikni

$$a_1(x)\frac{d}{dx}\varphi_1(x) + \frac{[4p_1(x) + b_1(x)](\alpha-1)}{\Gamma(\alpha-1)}x^{-1}\int_0^x (x-t)^{\alpha-2}\varphi_1(t)dt +$$

$$+\frac{4p_1(x)+b_1(x)}{\Gamma(\alpha-1)}x^{-1}\int_0^x (x-t)^{\alpha-2}t\varphi_1'(t)dt = \gamma_1(x), \tag{3.51}$$

ko'rinishda yozish mumkin.

$$\varphi_1(x) = D_{0x}^{\alpha-2}\widetilde{\nu}(x) = \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}\widetilde{\nu}(t)dt = \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}t^{\alpha-2}\widetilde{\nu}_{11}(t)dt =$$

$$= \left\langle \begin{matrix} t = xz \\ (x-t) = (1-z)x \end{matrix} \right\rangle = \frac{1}{\Gamma(2-\alpha)}x^{1-\alpha+\alpha-1}\int_0^1 z^{\alpha-2}(1-z)^{1-\alpha}\widetilde{\nu}_{11}(xz)dz,$$

Bundan

$$\varphi(0) = \frac{\nu_{11}(0)}{\Gamma(1-\alpha)}B(\alpha-1, 2-\alpha) \neq 0$$

(3.48) shartdan hamda 4-teoremaning (1) shartidan quyidagi kelib chiqadi.

$$\varphi(0) = \frac{\nu_1(0)}{\Gamma(1-\alpha)}B(\alpha-1, 2-\alpha) \neq C_0 \neq 0 \tag{3.52}$$

U holda, agar

$$\psi(x) = \frac{d}{dx}\varphi_1(x) \tag{3.53}$$

belgilash kiritsak (3.52) ga asosan

$$\varphi_1(x) = C_0 + \int_0^x \psi(t)dt, \tag{3.54}$$

tenglikni olamiz.

Agar (3.53),(3.54) tengliklarni (3.51) ga qo'ysak,

$$a_1(x)\psi(x)+\frac{[4p_1(x)+b_1(x)](\alpha-1)}{\Gamma(\alpha-1)x}\int_0^x(x-t)^{\alpha-2}dt\int_0^t\psi(t_1)dt_1+$$

$$+\frac{4p_1(x)+b_1(x)}{\Gamma(\alpha-1)x}\int_0^x(x-t)^{\alpha-2}t\psi(t)dt=g(x)+\gamma(x), \tag{3.55}$$

$$g(x)=\frac{[4p_1(x)+b_1(x)](\alpha-1)}{\Gamma(\alpha-1)x}C_0\int_0^x(x-t)^{\alpha-2}dt \tag{3.56}$$

tenglik hosil bo'ladi.

Ba'zi almashtirishlardan keyin,

$$\psi(x)+\int_0^x K_3(x,t)\psi(t)dt=\gamma_3(x), \tag{3.57}$$

$$K_3(x,t)=\frac{[4p_1(x)+b_1(x)]x^{\alpha-2}}{\Gamma(\alpha-1)a_1(x)}+\frac{[4p(x)+b_1(x)](x-t)^{\alpha-2}}{\Gamma(\alpha-1)a_1(x)x}t,\ \gamma_3(x)=\frac{\gamma(x)-g(x)}{a\ (x)}, \tag{3.58}$$

(3.57) tenglikdan ikkinchi tur Volterra integral tenglamasi kelib chiqadi. U holda (3.34) va (3.57) tenglamalar ekvivalent bo'lganligi uchun (3.34) tenglama ham yechimga ega bo'ladi. Bundan (3.31)-(3.33) masala ham yechimga ega ekanligi va uning yechimi (3.36) formula bilan aniqlanganligi kelib chiqadi.

Qo'yilgan masala yechimining yagona emasligini isbotlash uchun (3.34) tenglamaga mos bir jinsli tenglama

$$4p_1(x)\widetilde{\nu}(x)+a_1(x)D_{0x}^{\alpha-1}\widetilde{\nu}(x)+b_1(x)D_{x1}^{\beta-1}\widetilde{\nu}(x)=0, \tag{3.59}$$

trivial bo'lmagan yechimga ega ekanligini ko'rsatish yetarli bo'ladi.

Bir qator almashtirishlardan keyin o'ng tomoni (3.58) tenglik bilan aniqlanadigan (3.57) ko'rinishdagi ikkinchi tur Volterra integral tenglamasini hosil qilamiz.

$$\psi(x) + \int_0^x K_3(x,t)\psi(t)dt = g_1(x), \tag{3.60}$$

Bu yerda $g_1(x) \neq 0, \quad x \in \overline{J}$

Demak, (3.59) tenglamaning trivial bo'lmagan yechimi mavjud. Shuning uchun ham bir jinsli bo'lmagan (3.34) tenglama cheksiz ko'p yechimga ega bo'ladi. Teorema isbotlandi.

Xulosa

Magistrlik dissertatsiya ishida kasr tartibli integrodifferentsial operatorlarning xossalari keltirilib, qo’yilgan masalalarga tadbiq qilindi.

1. Birinchi bobda kasr tartibli integrodifferentsial operatorlar va ularning xossalari, birinchi va ikkinchi tur Eyler integrallari, Abel integral tenglamasi tatbiqlari, kasr tartibli integral operatorlar, kasr tartibli differentsial operatorlar o’rganilgan. Uzluksiz ta’limning har bir bo’g’inida uzluksiz va uzviylikni ta’minlash maqsadida bu bobda integrodifferentsial tushunchasi kiritilgan.

2. Ikkinchi bobda tor tebranish tenglamasi uchun Koshi va Gursa masalasi, tor tebranish tenglamasi uchun siljishli masalalar o’rganishga bag’ishlangan.

3. Uchinchi bobda tor tebranish tenglamasi uchun integrodifferentsial operatorlar yordamida yangi siljishli masalalar qo’yilgan va o’rganilgan. Bunda qo’yilgan masalalar Volterra ikkinchi tur integral tenglamalariga keltirilgan. Integral tenglamalar nazariyasidan foydalanib qo’yilgan masala yechimining mavjudligi va yagonaligi isbotlangan.

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